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A Hybrid of Tense Logic $S4_T$ and Multi-Agent Logic with Interacting Agents

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In this paper we introduce a temporal multi-agent logic $S4_T^{\mathcal{IA}}$, which implements interacting agents. Logic $S4_T^{\mathcal{IA}}$ is defined semantically as the set of all formulas of the appropriate propositional language that are valid in special Kripke models. The models are based on S4-like time frames, i.e., with reflexive and transitive time-accessibility relations. Agents knowledge-accessibility relations R_i, defined independently for each individual agent, are S5-relations on R-time clusters, and interaction of the agents consists of passing knowledge along arbitrary paths of such relations. The key result of the paper is an algorithm for checking satisfiability and recognizing theorems of $S4_T^{\mathcal{IA}}$. We also prove the effective finite model property for the logic $S4_T^{\mathcal{IA}}$.

Keywords: multi-agent logics, tense logics, knowledge representation, satisfiability, decidability, inference rules.

Introduction

Multi-agent systems have attracted significant attention throughout recent years, due to various prospective applications. The latter include software distribution systems, computerized support of administrative tasks, modeling, routing, e-commerce, search engines and so on (cf. Bordini et al. [2], Dix et al. [3], Hoek et al. [9], Fisher [6], Hendler [12], Wooldridge [23, 24]). Based on informational aspect, multi-agent systems can be seen as communities of interacting entities that can perceive, act upon and transmit information to each other. The research on such systems necessarily combines methods and concepts from variety of sources, including artificial intelligence, computing and various areas of mathematical logic. One of the venues of the current research deals with specification, design and verification issues of such systems as well with logical and complexity analysis of these aspects.

Out of several viable approaches toward mathematical formalization of multi-agent systemы, modal and temporal propositional logics provides an impressive set of well-developed, mathematically effective and sound tools for logical and complexity analysis (cf. e.g. van Benthem [22, 21],

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Goldblatt [8], Gabbay and Hodkinson [7], Hodkinson [13]). However the use of these tools meets the problem of integration of agents paradigm into temporal and modal framework (cf. Fagin et al. [5, 4], Halpern and Shore [10]).

Although a particular evolution of a multi-agent system can be modeled on a discreet linear time frame (cf. Rybakov [16, 17, 18, 19, 20]), to allow for non-deterministic nature of interaction, more appropriate non-linear modal and temporal tools are required. While modal logics appear adequate for design and implementation purposes (since no agent is allowed to act by "fixing"past state of affairs), logical analysis calls for temporal logics framework (see Eds. Barringer, Fisher, Gabbay and Gough [1] for related applications of temporal logics).

In general, modal-like operations K_i (operators $\Diamond_i$ in this paper) are used to model the knowledge accessibility relations of individual agents. However, because they are related to individual agents, these operations are of relatively low logical level and do not permit by themselves adequate logical description of a situation of a practical value. To repair this deficiency, several higher-level operators were introduced (cf. e.g. Halpern [11]). We will augment our language with one of them.

In many cases, either it is a new record in a database or a changed data profile, the only available for an agent information about the source of new entry has the implicit form "obtained through interaction with (other) agents". An attempt to make this information more specific may fail, for a variety of reasons, the major are: information nodes often don't preserve all the necessary information about transactions, or because the investigation about the source might take more than a time-cycle of the system. One possible solution to reflect this situation is to consider an operator $\Diamond_{\mathfrak{K}}$ — "knowledge through agents which is the existential version of the "Common Knowledgeoperator from [5]. Formally,

$$(\mathcal{F}, a) \Vdash_V \Diamond_{\mathfrak{K}} \alpha \iff \exists a_1, \ldots, a_t \in W : aR_{j_1} a_1 R_{j_2} a_2 \ldots R_{j_t} a_t \,\&\, (\mathcal{F}, a_t) \Vdash_V \alpha.$$

This operator cannot be expressed by finitary means in terms of usual agent-knowledge operators, in fact, $\Diamond_{\mathfrak{K}} \alpha$ is equivalent to an infinite disjunction [5]

$$\Diamond_{\mathfrak{K}} \alpha = \bigvee_{\bar{n} \in \{1,\ldots,n\}^+} \Diamond_{\bar{n}} \alpha, \text{ where } \Diamond_{\langle i_1,\ldots,i_t \rangle} := \Diamond_{i_1} \ldots \Diamond_{i_t}.$$

This approach has been recently implemented in Rybakov [20] to handle interacting agents in the linear temporal logic (LTL). Our paper attempts to extend this technique to more general temporal logics, those which do not suppose linearity of time.

We introduce and study a tense multi-agent logic $S4_T^{\mathcal{IA}}$ with interacting agents (i.e. with the operator $\Diamond_{\mathfrak{K}}$) based on arbitrary reflexive and transitive time flows. The logic $S4_T^{\mathcal{IA}}$ is defined semantically, as the set of all propositional formulas valid in special Kripke-like models. These models are based on S4-like time frames, i.e., with reflexive and transitive time-accessibility relations. Agents knowledge-accessibility relations R_i are $S5$-relations on R-time clusters, and interaction of the agents consists in passing knowledge along arbitrary paths of knowledge-accessibility relations.

The language of $S4_T^{\mathcal{IA}}$ employs operations $\Diamond^+$ and $\Diamond^-$ related to future and past time-accessibility relations, knowledge-accessibility relations $\Diamond_j$, $1 \leqslant j \leqslant m$, — for each present agent. The number m of participating agents is assumed to be fixed. In addition, we introduce a higher-level operation $\Diamond_{\mathfrak{K}}$ to model a situation when information about an interaction path is absent or lost — $\Diamond_{\mathfrak{K}} \phi$ means that "ϕ is known by interaction with (other) agents".

The main result of the paper is an algorithm, which checks satisfiability in $S4_T^{\mathcal{IA}}$ and recognizes the theorems of $S4_T^{\mathcal{IA}}$. Thus we show that $S4_T^{\mathcal{IA}}$ is decidable. The algorithm is based on the effective finite model property for $S4_T^{\mathcal{IA}}$, also proved in this paper. Our approach uses representation of formulas by inference rules, and reduction these rules to special normal reduced forms, and further verification of satisfiability of these rules in Kripke models.

1. Preliminaries and Notation

A *language* of our tense multi-agent logic with interacting agents is a set of operations $L = \langle \wedge, \vee, \rightarrow, \neg, \Diamond^+, \Diamond^-, \Diamond_1, \dots, \Diamond_m, \Diamond_{\mathfrak{K}} \rangle$, where m is fixed. We fix an enumerable set Var := $\{x_1, x_2, x_3, \dots\}$ of *propositional* variables. Well-formed formulas of the language $\mathcal{L}$ ($\mathcal{L}$-*formulas*) are defined by the following grammar

$$\alpha ::= x_i \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \mid \alpha_1 \rightarrow \alpha_2 \mid \neg\alpha \mid \Diamond^{\pm}\alpha \mid \Diamond_j \alpha \mid \Diamond_{\mathfrak{K}}\, \alpha \mid$$

For a formula α, the *set of variables* $\mathrm{Var}(\alpha)$ *of* α is defined inductively:

$$\mathrm{Var}(x_i) := \{x_i\},\ \mathrm{Var}(*\beta) := \mathrm{Var}(\beta),\ \mathrm{Var}(\beta * \gamma) := \mathrm{Var}(\beta) \cup \mathrm{Var}(\gamma).$$

Suppose W is a non-empty set and $R \subseteq W \times W$ is a *preorder* on W (i.e., a reflexive and transitive binary relation on W). An *R-cluster of W* is a maximal by inclusion subset C of W, such that relation R, in restriction to C, is a an equivalence relation.

Definition 1. *Suppose that the language $\mathcal{L} = \langle \wedge, \vee, \rightarrow, \neg, \Diamond^{\pm}, \Diamond_1, \dots, \Diamond_m, \Diamond_{\mathfrak{K}} \rangle$ is fixed. An $S4_T^{\mathcal{IA}}$-frame is a Kripke multi-frame, i.e., $m+2$-tuple $\langle W, R, R_1, \dots, R_m \rangle$, where*

- *W is a non-empty set of* states *or* worlds*,*
- *$R \subseteq W \times W$ is a binary reflexive and transitive relation on W,*
- *$R_1, \dots, R_m \subseteq R$ are equivalence relations on W.*

The relation R is intended to model the *time-accessibility* relation, $R_1, \dots, R_m$ are meant to represent *knowledge-accessibility* relations of individual agents. Informally, each R-cluster C is a moment in time (a computational step, inner cycle of a system, etc). The cluster contains all possible states available at this moment (completed data outputs in parallel computing, network nodes to retrieve information from, etc). Individual agents can only retrieve information from the states attainable through the corresponding relation. All agents act concurrently. To take into account random factors in agents interaction, we assume that there is a variety of possible time-configurations following the current time moment. Access rules are not preserved along time flow, and agents cannot know in what accessibility configuration they will operate.

For a set X we denote by $\mathcal{P}(X) := \{Y \mid Y \subseteq X\}$ — the *power-set* of X. A *valuation* V on a $S4_T^{\mathcal{IA}}$-frame $\mathcal{F} = \langle W, R, R_1, \dots, R_m \rangle$ is a function

$$V : W \rightarrow \mathcal{P}(\{x_1, \dots, x_n\}).$$

The fact that $x_i \in V(a)$ means that x_i is *true* or *satisfied at a state $a \in W$ under valuation V*. A pair $\langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a $S4_T^{\mathcal{IA}}$-frame and V is a valuation, is called an *$S4_T^{\mathcal{IA}}$-model.*

Satisfiability of $\mathcal{L}$-formulas on a $S4_T^{\mathcal{IA}}$-model $\langle \mathcal{F}, V \rangle$ is defined as follows (we write $(\mathcal{F}, a) \Vdash_V \alpha$ to say that a formula α is *true* in a model $\langle \mathcal{F}, V \rangle$ at a given state a):

$$(\mathcal{F}, a) \Vdash_V x_i \iff x_i \in V(a),$$
$$(\mathcal{F}, a) \Vdash_V \alpha \wedge \beta \iff (\mathcal{F}, a) \Vdash_V \alpha \text{ and } (\mathcal{F}, a) \Vdash_V \beta,$$
$$(\mathcal{F}, a) \Vdash_V \alpha \vee \beta \iff (\mathcal{F}, a) \Vdash_V \alpha \text{ or } (\mathcal{F}, a) \Vdash_V \beta,$$
$$(\mathcal{F}, a) \Vdash_V \alpha \rightarrow \beta \iff (F, a) \nVdash_V \alpha \text{ or } (\mathcal{F}, a) \Vdash_V \beta,$$
$$(\mathcal{F}, a) \Vdash_V \neg\alpha \iff (\mathcal{F}, a) \nVdash_V \alpha,$$
$$(\mathcal{F}, a) \Vdash_V \Diamond^+\alpha \iff \exists b \in W : aRb \,\&\, (\mathcal{F}, b) \Vdash_V \alpha,$$
$$(\mathcal{F}, a) \Vdash_V \Diamond^-\alpha \iff \exists b \in W : bRa \,\&\, (\mathcal{F}, b) \Vdash_V \alpha,$$
$$(\mathcal{F}, a) \Vdash_V \Diamond_i\alpha \iff \exists b \in W : aR_ib \,\&\, (\mathcal{F}, b) \Vdash_V \alpha,$$
$$(\mathcal{F}, a) \Vdash_V \Diamond_{\mathfrak{K}}\alpha \iff \exists a_1, \ldots, a_t \in W : aR_{j_1}a_1R_{j_2}a_2 \ldots R_{j_t}a_t \,\&\, (\mathcal{F}, a_t) \Vdash_V \alpha.$$

We say that a formula α is *valid in a model* $\mathcal{M} = \langle \mathcal{F}, V \rangle$ (we write $\mathcal{M} \Vdash \alpha$ or $\mathcal{F} \Vdash_V \alpha$) if α is valid at any state of $\mathcal{F}$. A formula α is *valid on a frame* $\mathcal{F}$ ($\mathcal{F} \Vdash \alpha$) if it is valid on every model $\langle \mathcal{F}, V \rangle$ for any valuation $V : W \to \mathcal{P}(\mathrm{Var}(\alpha))$. Symbolically

$$\mathcal{F} \Vdash_V \alpha \iff \forall a : (\mathcal{F}, a) \Vdash_V \alpha,$$
$$\mathcal{F} \Vdash \alpha \iff \forall V : W \to \mathcal{P}(\mathrm{Var}(\alpha)) : \mathcal{F} \Vdash_V \alpha.$$

The usual “universal"adjoint operators $\Box^+, \Box^-, \Box_1, \ldots, \Box_m$ (sometimes taken as basic operations of the language) for modalities $\Diamond^+, \Diamond^-, \Diamond_1, \ldots, \Diamond_m$ are defined as accustomed:

$$\Box^{\pm}\alpha := \neg\Diamond^{\pm}\neg\alpha, \quad \Box_j\alpha := \neg\Diamond_j\neg\alpha, \text{ for all } j = 1, \ldots, m.$$

Definition 2. *The logic $S4_T^{\mathcal{IA}}$ is the set of all formulas over the language $\mathcal{L}$ that are valid on all $S4_T^{\mathcal{IA}}$-frames.*

2. Main Results

We will be representing formulas by inference rules. A *rule* is a pair $\langle \alpha, \beta \rangle$ of $\mathcal{L}$-formulas. We will usually write a rule $\langle \alpha, \beta \rangle$ in the form α/β. For a rule $r = \alpha/\beta$: $\mathrm{Var}(r) := \mathrm{Var}(\alpha) \cup \mathrm{Var}(\beta)$.

We say that a rule $r = \alpha/\beta$ is *valid on a model* $\langle \mathcal{F}, V \rangle$ (symbolically $\mathcal{F} \Vdash_V r$), if $V : W \to \mathcal{P}(\mathrm{Var}(r))$ and

$$\forall a \in \mathcal{F} : (\mathcal{F}, a) \Vdash_V \alpha \implies \forall a \in \mathcal{F} : (\mathcal{F}, a) \Vdash_V \beta.$$

A rule r is *valid on a frame* $\mathcal{F}$, if, for any valuation V of variables $\mathrm{Var}(r)$, $\mathcal{F} \Vdash_V r$. If the rule r is not valid on $\mathcal{F}$, then there is a valuation V such that $\mathcal{F} \nVdash_V r$. In that case we say that r is *refuted* on $\mathcal{F}$ (by valuation V).

A rule r is said to be in a *reduced normal form* if

$$r = \bigvee_{1 \leqslant j \leqslant s} \theta_j / x_1, \tag{1}$$

and each disjunct θ_j has the form

$$\theta_j = \bigwedge_{1 \leqslant i \leqslant n} \Bigg(x_i^{t(i,j,0)} \wedge (\Diamond^+ x_i)^{t(i,j,1)} \wedge (\Diamond^- x_i)^{t(i,j,2)} \wedge$$
$$\wedge (\Diamond_{\mathfrak{K}}\, x_i)^{t(i,j,3)} \wedge \bigwedge_{1 \leqslant k \leqslant m} (\Diamond_k x_i)^{t(i,j,3+k)} \Bigg),$$

and all θ_j are different, letters x_i are variables, $t(i,j,z) \in \{0,1\}$, and for any formula α, $\alpha^0 := \neg\alpha$, $\alpha^1 := \alpha$.

Two rules r_1, r_2 are *equivalent* if for any $S4_T^{\mathcal{IA}}$-frame $\mathcal{F}$:

$$\mathcal{F} \Vdash r_1 \iff \mathcal{F} \Vdash r_2.$$

For a formula α, the *set of subformulas* $\mathrm{Sub}(\alpha)$ *of* α is defined inductively:

$$\mathrm{Sub}(x_i) := \{x_i\},$$
$$\mathrm{Sub}(*\beta) := \mathrm{Sub}(\beta) \cup \{*\beta\},$$
$$\mathrm{Sub}(\beta * \gamma) := \mathrm{Sub}(\beta) \cup \mathrm{Sub}(\gamma) \cup \{\beta * \gamma\}.$$

For a rule $r = \alpha/\beta$: $\mathrm{Sub}(r) := \mathrm{Sub}(\alpha) \cup \mathrm{Sub}(\beta)$.

It was shown in Rybakov [14] that any modal inference rule may be transformed into an equivalent rule in the reduced normal form. Using essentially the same technique we can transform to normal reduced forms our rules in language of $S4_T^{\mathcal{IA}}$.

Lemma 1. *Any rule $r = \alpha/\beta$ can be transformed in exponential time to an $S4_T^{\mathcal{IA}}$-equivalent rule r_{nf} in the reduced normal form.*

Proof. We will specify the general algorithm described in Lemma 3.1.3 and Theorem 3.1.11 [15] for the language of our logic. Let $r = \alpha/\beta$ be an inference rule.

We will need a set of new variables $Z = \{z_\gamma \mid \gamma \in \mathrm{Sub}(r)\}$. The first step is to replace $r = \alpha/\beta$ with $r_1 = \alpha \wedge (z_\beta \leftrightarrow \beta)/z_\beta$. It is straightforward to see that r and r_1 are equivalent (it is easier to proof it this way: show that r is refuted on $\mathcal{F}$ if and only if r_1 can be refuted on $\mathcal{F}$).

Inductive step: suppose we have obtained a rule $r_i = \gamma_i/z_\beta$ at the i-th step. We call a formula $\delta \in \mathrm{Sub}(\gamma_i) \cap \mathrm{Sub}(r)$ *terminal*, if it is not a variable and not a proper subformula of any other formula in $\mathrm{Sub}(\gamma_i) \cap \mathrm{Sub}(r)$. Let T_i be the set of all terminal formulas at the i-th step.

We replace the rule r_i with a new one $r_{i+1} = \gamma_{i+1}/z_\beta$, where

$$\gamma_{i+1} = t_i(\gamma_i) \wedge \bigwedge_{z_\gamma * z_\delta \in T_i} ((z_\gamma \leftrightarrow \gamma) \wedge (z_\delta \leftrightarrow \delta)) \wedge \bigwedge_{* z_\delta \in T_i} (z_\delta \leftrightarrow \delta),$$

and $t_i(\gamma_i)$ is the formula obtained from γ_i by replacing all terminal subformulas δ with z_δ. It is straightforward to check that r_i and r_{i+1} are equivalent.

Note that every inductive step reduces the maximal height of non-boolean subformulas of the rule. Therefore after a finite number of steps we come to a premise γ_N, which is a boolean combination of *literals* of the form x or $*x$, where x is a propositional variable and symbol “$*$"belongs to the set $\{\Diamond_{\mathfrak{K}}, \Diamond^+, \Diamond^-, \Diamond_1, \dots, \Diamond_m\}$.

Finally, we transform the premise of the obtained rule $r_N = \gamma_N/z_\beta$ into a fully disjunctive normal form over literals. This requires no more than exponential time on the number of variables, i.e., on the number of subformulas of the original rule (the same as for reduction of any boolean formula to the full disjunctive normal form). ☐

As seen from the proof of Lemma 1, the variables in the reduced form represent the subformulas of α and β, in particular, variable z_β stands for the rule's consequent β. From the definition of normal reduced form, it is clear that under any given valuation of variables only one θ_j can hold true at a current state.

Remark. *One advantage of reduced forms is that they effectively separate “modal"subformulas of the kind $*\beta$, where $* \in \{\Diamond_{\mathfrak{K}}, \Diamond^+, \Diamond^-, \Diamond_1, \dots, \Diamond_m\}$, satisfiability of which depends on other* *states, from "boolean"subformulas of the type $\alpha * \beta$ and $*\alpha$, where $* \in \{\vee, \wedge, \rightarrow, \neg\}$, satisfiability of which depends on truth-values of its immediate subformulas at the same state. This will become useful for computing complexity bounds for the size of refuting model.*

Thus, by Lemma 1, we have, that for any formula α and any $S4_T^{\mathcal{IA}}$-frame $\mathcal{F}$:

$$\mathcal{F} \Vdash \alpha \iff \mathcal{F} \Vdash x \rightarrow x/\alpha \iff \mathcal{F} \Vdash (x \rightarrow x/\alpha)_{\mathrm{nf}}.$$

Therefore the following lemma holds:

Lemma 2. *A formula α is a theorem of $S4_T^{\mathcal{IA}}$ iff the rule $(x \rightarrow x/\alpha)_{\mathrm{nf}}$ is valid on all $S4_T^{\mathcal{IA}}$-frames.*

To prove decidability of $S4_T^{\mathcal{IA}}$ we will prove that it has effective (doubly exponential on the summary length of the formulas of a rule) finite model property. To reduce an arbitrary model refuting r to a finite one, we use the fact that agent-related operations $\Diamond_j$ and $\Diamond_{\mathfrak{K}}$ are, in a sense, *local* with respect to the time-relation R, that allows us to employ some sort of a two-phase *contraction* technique. (Note, that these contractions, though similar, are not filtrations at each stage. In particular, a simple filtration will not work for the operation $\Diamond_{\mathfrak{K}}$, responsible for obtaining knowledge through combinations of agent's accessibility relations). Lemma 3 will deal with the first stage and Lemma 4 with the second.

The following lemma for the effective bound on the sizes of time clusters in $S4_T^{\mathcal{IA}}$-models is the base for our further technique. The proof of it is based on the proof of the similar statement about sizes of time clusters in a multi-agent logic based on LTL from Rybakov [20]. Representation of formulas by rules in the reduced normal form is essential for the proof.

Definition 3. *Suppose w is a state in a $S4_T^{\mathcal{IA}}$-frame $\mathcal{F}$. Let $\mathfrak{K}$ be the smallest equivalence relation on $\mathcal{F}$, containing all equivalence relations $R_1, \ldots, R_m$, i.e., the transitive closure of the union $\bigcup_{i=1}^m R_i$. We denote $C_{\mathfrak{K}}(w)$ a $\mathfrak{K}$-cluster of $\mathcal{F}$ containing w.*

In other words, $C_{\mathfrak{K}}(w)$ includes all states of the frame $\mathcal{F}$ that are accessible from the state w through various combinations of $R_1, \ldots, R_m$ relations.

Definition 4. *We say that two $\mathfrak{K}$-clusters C_1, C_2 of a $S4_T^{\mathcal{IA}}$-model $\langle \mathcal{F}, V\rangle$ are* isomorphic *(we indicate this as $C_1 \cong C_2$), if there exists a bijection $f : C_1 \rightarrow C_2$ such that for all $a, b \in C_1$*

1. *$(\mathcal{F}, a) \Vdash_V x_i \iff (\mathcal{F}, f(a)) \Vdash_V x_i$,*
2. *$aR_jb \iff f(a)R_jf(b)$, for all $j = 1, \ldots, m$.*

Further on, the rule r will be of the form (1).

Lemma 3. *If a rule $r = \bigvee_{1 \leqslant j \leqslant s} \theta_j/x_1$ is refuted on a $S4_T^{\mathcal{IA}}$-model $\mathcal{M}$, then*
(i) r is refuted on a $S4_T^{\mathcal{IA}}$-model with size of R-clusters at most $s \cdot 2^{m \cdot s^2}$,
(ii) the number of non-isomorphic $\mathfrak{K}$-clusters is at most $2^{m \cdot s^2}$.

Proof. Let C be a time cluster (i.e., R-cluster) of $\mathcal{M}$ and $\mathcal{N}$ be the $S5_m$-submodel of $\mathcal{M}$, which universe is C (i.e., model $\langle C, R_{1\restriction C}, \ldots, R_{m\restriction C}, V_{\restriction C}\rangle$, where $R_{i\restriction C} := R_i \cap (C \times C)$). Cluster C consists of possibly infinitely many disjoint $\mathfrak{K}$-clusters C_i, $i \in I$. For every disjunct θ_j of the form

$$\theta_j = \bigwedge_{1 \leqslant i \leqslant n} \Bigg(x_i^{t(i,j,0)} \wedge (\Diamond^{+} x_i)^{t(i,j,1)} \wedge (\Diamond^{-} x_i)^{t(i,j,2)} \wedge \\ \wedge (\Diamond_{\mathfrak{K}} x_i)^{t(i,j,3)} \wedge \bigwedge_{1 \leqslant k \leqslant m} (\Diamond_k x_i)^{t(i,j,3+k)} \Bigg),$$

we denote

$$\theta_j^* := \bigwedge_{1\leqslant i\leqslant n}\left(x_i^{t(i,j,0)} \wedge \bigwedge_{1\leqslant k\leqslant m}(\Diamond_k x_i)^{t(i,j,3+k)}\right).$$

We filter each of the $\mathfrak{K}$-clusters C_i through the set of disjuncts $\{\theta_j^* \mid j = 1,\dots,s\}$, thereby obtaining a finite $\mathfrak{K}$-cluster $C_i^* = \{[\theta_j^*] \mid j = 1,\dots,s\}$ of the size not more than s. There are not more than $2^{m\cdot s^2}$ non-isomorphic $\mathfrak{K}$-clusters of the size not exceeding s. Therefore, after identifying the isomorphic $\mathfrak{K}$-clusters inside $\mathcal{N}$, we obtain a finite $S5_m$-model $\mathcal{N}^*$ of size less or equal to $s\cdot 2^{m\cdot s^2}$. Thus we can replace the original cluster C of $\mathcal{M}$, that had model $\mathcal{N}$ embedded, with this finite submodel $\mathcal{N}^*$ to obtain a new $S4_T^{\mathcal{IA}}$-model $\mathcal{M}^*$. It is straightforward to check that

$$(\mathcal{M}^*, [\theta_j^*]) \Vdash \theta_j.$$

Thus we can replace all time clusters of $\mathcal{M}$ by finite clusters of the size no more that $s\cdot 2^{m\cdot s^2}$, meantime preserving the satisfiability of all disjuncts θ_j, $j = 1,\dots,s$. □

Now we will deal with the time structure of model $\mathcal{M}$, which was mainly left intact by Lemma 3 (except for replacing time clusters with smaller ones).

Lemma 4. *Let $\mathcal{M} = \langle \mathcal{F}, V\rangle$ be a Kripke-model satisfying the conclusion of Lemma 3. If rule r is refuted on the model $\mathcal{M}$, then it can be refuted on a finite $S4_T^{\mathcal{IA}}$-model of the size effectively bounded by the size of r.*

Proof. Suppose $\mathcal{F} = \langle W, R, R_1,\dots,R_m\rangle$. We use the following notation:

- $\dot{\mathcal{F}}$ stands for an auxiliary S4$_T$-frame, obtained from $\mathcal{F}$ by replacing $\mathfrak{K}$-clusters with "solid"temporal worlds,
- $\tilde{\mathcal{F}}$ is a result of temporal filtration of the model $\langle\dot{\mathcal{F}}, \dot{V}\rangle$ based on frame $\dot{\mathcal{F}}$, where $\dot{V}$ is a special valuation, which characterizes temporal worlds corresponding to isomorphic in $\mathcal{M}$ $\mathfrak{K}$-clusters,
- $\widehat{\mathcal{F}}$ is a $S4_T^{\mathcal{IA}}$-frame, obtained from $\tilde{\mathcal{F}}$, by replacing each temporal world (equivalence class) of $\tilde{\mathcal{F}}$ with one of the original $\mathfrak{K}$-clusters.

More precisely, consider an auxiliary $S4_T^{\mathcal{IA}}$-frame $\dot{\mathcal{F}} = \langle \dot{W}, \dot{R}\rangle$, where

$$\dot{W} = \{C_{\mathfrak{K}}(a) \mid a \in W\}, \qquad C_{\mathfrak{K}}(a)\dot{R}C_{\mathfrak{K}}(b) \iff aRb.$$

We know from Lemma 3, that there are only finitely many non-$\cong$-isomorphic $\mathfrak{K}$-clusters in $\mathcal{M}$ (see Def. 4 for definition of relation $\cong$).

Let $\mathcal{S} := \{[C]_{\cong} \mid C \in \dot{W}\}$, i.e., $\mathcal{S}$ is the set of isomorphic types of $\mathfrak{K}$-clusters in the model $\mathcal{M}$. We introduce a set of additional variables $\{p_k \mid k \in \mathcal{S}\}$, and define a valuation $\dot{V} : \dot{W} \to \mathcal{P}(\mathrm{Var}(r) \cup \{p_k\}_{k\in\mathcal{S}})$ as follows:

- $(\dot{\mathcal{F}}, C) \Vdash_{\dot{V}} p_k \iff C \in k$,
- $(\dot{\mathcal{F}}, C) \Vdash_{\dot{V}} x_i \iff \exists a \in C : (\mathcal{F}, a) \Vdash_V x_i$.

As a result, we have that

$$(\dot{\mathcal{F}}, C) \Vdash_{\dot{V}} \Diamond^{\pm} x_i \iff \forall a \in C : (\mathcal{F}, a) \Vdash_V \Diamond^{\pm} x_i.$$

Now we are ready to apply a filtration procedure. Suppose C is a $\mathfrak{K}$-cluster of the model $\mathcal{M}$. Let us define

$$\Theta^+(C) := \{x_i \mid \exists a \in C : (\mathcal{M}, a) \Vdash \Diamond^+ x_i\},$$

$$\Theta^-(C) := \{x_i \mid \exists a \in C : (\mathcal{M}, a) \Vdash \Diamond^- x_i\},$$

$$\Theta(C) := \langle \Theta^+(C), \Theta^-(C) \rangle,$$

$$\Theta(C_1) \preccurlyeq \Theta(C_2) \overset{\text{def}}{\iff} \Theta^+(C_2) \subseteq \Theta^+(C_1) \,\&\, \Theta^-(C_1) \subseteq \Theta^-(C_2),$$

$$\Theta(C_1) = \Theta(C_2) \iff \Theta(C_1) \preccurlyeq \Theta(C_2) \,\&\, \Theta(C_2) \preccurlyeq \Theta(C_1).$$

We define an equivalence relation $\approx$ on $\dot{W}$ as follows

$$C_1 \approx C_2 \iff C_1 \cong C_2 \ \&\ \Theta(C_1) = \Theta(C_2).$$

Let $\tilde{\mathcal{F}} = \langle \tilde{W}, \tilde{R} \rangle$, where

$$\tilde{W} = \{[C]_\approx \mid C \in \dot{W}\},$$

$$[C_1]_\approx \tilde{R} [C_2]_\approx \iff \Theta(C_1) \preccurlyeq \Theta(C_2).$$

From each $\approx$-equivalence class we can choose a unique representative. The fact that a cluster C was chosen from a class X we indicate by writing $C \in_1 X$.

Now, define $\widehat{\mathcal{F}} = \langle \widehat{W}, \widehat{R} \rangle$ and a valuation $\widehat{V}$ as follows

$$\widehat{W} = \bigcup \{C \mid C \in_1 X \in \dot{W}\},$$

$$a\widehat{R}b \iff [C_{\mathfrak{K}}(a)]_\approx \tilde{R} [C_{\mathfrak{K}}(b)]_\approx,$$

$$a\widehat{R}_j b \iff \exists C \in_1 X \in \dot{W} : a, b \in C \text{ and } aR_j b \text{ in model } \mathcal{M}, \quad j = 1, \dots, s,$$

$$(\widehat{\mathcal{F}}, a) \Vdash_{\widehat{V}} x_i \iff (\mathcal{F}, a) \Vdash_V x_i.$$

Thus we have that $\widehat{W} \subseteq W$ and $\widehat{R}_j = R_j \cap (\widehat{W} \times \widehat{W})$. We want to prove now that for all θ_j and $a \in \widehat{W}$

$$(\widehat{\mathcal{F}}, a) \Vdash_{\widehat{V}} \theta_j \iff (\mathcal{F}, a) \Vdash_V \theta_j.$$

Since $\widehat{W}$ consists of elements of the original model and the valuation $\widehat{V}$ on them is the same as V, the boolean part of θ_j holds at $a \in \widehat{W}$ as required.

Suppose now, $(\mathcal{F}, a) \Vdash_V \Diamond^+ x$ and $a \in \widehat{W}$. Then there exists $b \in W$ such that aRb and $(\mathcal{F}, b) \Vdash_V x$. Since aRb, then $\Theta(C_{\mathfrak{K}}(a)) \preccurlyeq \Theta(C_{\mathfrak{K}}(b))$. Let $C \in_1 [C_{\mathfrak{K}}(b)]_\equiv$, then since $C \cong C_{\mathfrak{K}}(b)$, there is a state $c \in C$, such that $(\mathcal{F}, c) \Vdash_V x$. Thus, we have $[C_{\mathfrak{K}}(a)]_\approx \tilde{R} [C_{\mathfrak{K}}(c)]_\approx$, and $a, c \in \widehat{W}$, therefore $a\widehat{R}c$. So $(\widehat{\mathcal{F}}, a) \Vdash_{\widehat{V}} \Diamond^+ x$, as needed. On the other hand, if $(\widehat{\mathcal{F}}, a) \Vdash \Diamond^+ x$, then there exists $b \in \widehat{W}$ such that $a\widehat{R}b$ and $(\widehat{\mathcal{F}}, b) \Vdash_{\widehat{V}} x$, and therefore $(\widehat{\mathcal{F}}, b) \Vdash_{\widehat{V}} \Diamond^+ x$. Then, by definition of $\widehat{R}$, $\Theta(C_{\mathfrak{K}}(a)) \preccurlyeq \Theta(C_{\mathfrak{K}}(b))$. In particular, $\Theta^+(C_{\mathfrak{K}}(b)) \subseteq \Theta^+(C_{\mathfrak{K}}(a))$, hence $(\mathcal{F}, a) \Vdash_V \Diamond^+ x$. The case of $\Diamond^- x$ is proved similarly.

Since the temporal filtration does not affect the inner structure of $\mathfrak{K}$-clusters, satisfiability of formulas of the type $\Diamond_j x$ and $\Diamond_{\mathfrak{K}}$ is preserved automatically.

From the proof of the lemma follows that the size of the final model $\langle \widehat{\mathcal{F}}, \widehat{V} \rangle$ is less or equal than $2^{2n+m\cdot s^2}$, where n is the number of variables in r, m is the number of agents, s is the number of disjuncts in the premise of r. $\square$

From Lemma 3 and Lemma 4 it immediately follows that

Theorem 1. *The logic $S4_T^{\mathcal{IA}}$ has the effective finite model property. Every non-theorem α of $S4_T^{\mathcal{IA}}$ is refuted on a $S4_T^{\mathcal{IA}}$-frame of the size effectively bounded on the size of the formula α.*

Evidently, using Theorem 1, we can determine for any given formula whether or not it is a theorem of $S4_T^{\mathcal{IA}}$, and therefore the existence of effective finite model property for $S4_T^{\mathcal{IA}}$ also solves the satisfiability problem for $S4_T^{\mathcal{IA}}$:

Corollary 1. *The logic $S4_T^{\mathcal{IA}}$ is decidable and the satisfiability problem in $S4_T^{\mathcal{IA}}$ is decidable.*

Proof. Since $S4_T^{\mathcal{IA}}$ has the effective finite model property, it suffices to prove that finite models for $S4_T^{\mathcal{IA}}$ are effectively recognizable. To check that a given frame $\mathcal{F} = \langle W, R, R_1, \ldots, R_m \rangle$ is a $S4_T^{\mathcal{IA}}$-frame we only need to check that

1. Frame $\langle W, R \rangle$ is a $S4_T$-frame;

2. Frame $\langle C, R_{1\restriction C}, \ldots, R_{m\restriction C} \rangle$ is a $S5_m$-frame, for every R-cluster C of $\mathcal{F}$,

where $R_{i\restriction C} := R_i \cap (C \times C)$. Both conditions can be checked effectively, and also the procedure of recognizing clusters in a finite frame is effective. Thus logic $S4_T^{\mathcal{IA}}$ is decidable. $\square$

3. Conclusions and Future Work

This paper introduces a temporal multi-agent logic $S4_T^{\mathcal{IA}}$ with interacting agents and tense operators “in the future"and “in the past". The logic $S4_T^{\mathcal{IA}}$ is defined semantically, as the set of all propositional formulas valid on special Kripke models. The models are based on S4-like time frames with multi-modal $S5_m$ frames embedded into time clusters. The additional $S5$-like relations R_i, $i = 1, \ldots, m$, are intended to represent agents' knowledge-accessibility relations. Interaction of the agents consists in passing knowledge along arbitrary paths of knowledge-accessibility relations. In addition, we introduce a higher-level operation $\Diamond_{\mathfrak{K}}$ to model a situation when information about an interaction path is absent or lost.

The main result of the paper is a decision algorithm for theorems of $S4_T^{\mathcal{IA}}$, which also resolves the satisfiability problem for this logic. Our approach employs representation of formulas by inference rules, and reduction of these rules to special normal reduced forms. The presented proof uses a procedure which looks like a two-phase contraction/filtration. This approach seems powerful enough to be applied to a wide set of hybrid propositional logics. In particular, it might allow to generalize the technique of the paper on a variety of other tense and modal logics with filtration property.

From the point of view of modeling agents' interaction, logic $S5_m$ represents the situation when all agents can be equally and totally trusted. Obviously this assumption is rather theoretical. In a real-life situation, a certain hierarchy is usually imposed on agents — systems with users/software providers and certifying agencies being a simple real-life example. Therefore, it deems useful to take a look at generalizations of the current technique to the case of families of agents with certain forms of hierarchy placed, like subordination, supervision, rights to interdict, and other hierarchial restrictions that can manifest themselves through special constraints on knowledge-accessibility relations.

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