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Nuevos resultados sobre la derivada fraccionaria conformable de Khalil

Certain new results on the Khalil conformable fractional derivative

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Resumen

En el presente artículo se establecen ciertos resultados de importancia para el análisis matemático, específicamente relacionados con las derivadas conformables de orden fraccional, entre ellos se destacan: la regla de la cadena, el Teorema del valor medio de Cauchy y la Regla de L'Hopital. Se espera que estos resultados estimulen la investigación en esta área

Palabras claves: Derivada fraccionaria conformable, Cálculo fraccionario

Abstract

This article establishes certain important results for mathematical analysis, specifically related to conformable derivatives of fractional order, among them the following stand out: the chain rule, the Cauchy mean value theorem and L'Hopital's rule. These results are expected to stimulate research in this area.

Keywords: Conformable fractional derivative, Fractional Calculus

1. Introduction

Fractional calculus [2, 5] was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. In a wide range of publications dealing with this area [1, 3], several researchers have proposed various definitions for fractional derivatives, between them we find the Riemman-Liouville and Caputo fractional derivative as follows:

Riemman-Liouville fractional derivative. For $\alpha \in [n-1, n)$, the $\alpha-$derivative of f is defined by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

and

Caputo fractional derivative. For $\alpha \in [n-1, n)$, the $\alpha-$derivative of f is defined by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx$$

Recently Khalil et al. [4] introduced a new conformable fractional derivative as follows.

Definition 1.1. *Given a function $f : [0, \infty) \to R$. Then the conformable fractional derivative of f of order α is defined by*

$$T_{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{\alpha-1}) - f(t)}{\epsilon} \tag{1}$$

for all $t > 0$ and $\alpha \in (0, 1)$. If f is $\alpha-$differentiable in some $(0, a)$, $a > 0$, and $\lim_{t\to 0} f(t)$ exists then we define

$$f^{(\alpha)}(0) = \lim_{t \to 0} f^{(\alpha)}(t). \tag{2}$$

Here we denote $f^{(\alpha)}(t) = T_{\alpha}(f)(t)$.

In [4], certain important results were proved: algebraic properties, some conformable derivatives of basic functions, Rolle's theorem and mean value theorem for conformable fractional differentiable functions, a new conformable fractional integral operator and some applications to fractional differential equations.

Motivated by this work we present additional results concerning to the conformable fractional derivative of the composition of two functions, the Cauchy's mean value theorem and L'hopital rule. We hope that these results and applied method will serve to stimulate others research works.

2. Preliminaries

This section aims to summarize some of the results related to the conformable fractional derivative, which will serve as the basis for establishing the new results set out in section 3. In [4] we can find the following results.

Theorem 2.1. *If f is an $\alpha-$differentiable function in $t_0 > 0$, with $\alpha \in (0, 1)$, then f is a continuous function in t_0.*

Theorem 2.2. *Let $\alpha \in (0, 1)$ and f, g be an $\alpha-$differentiable functions in $t > 0$. Then*

1. $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g)$ *for all $a, b \in R$.*

2. $T_\alpha(\lambda) = 0$ *for a constant function* $f(t) = \lambda$.
3. $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in R$
4. $T_\alpha(f.g) = fT_\alpha(g) + gT_\alpha(f)$
5. $T_\alpha(f/g) = (fT_\alpha(g) - gT_\alpha(f))/g^2$
6. *If* f *is differentiable (*f' *exists) then* $T_\alpha(f)(t) = t^{1-\alpha}f'(t)$

Certain conformable fractional derivatives of some basic functions are:

1. $T_\alpha(e^{ct}) = ct^{1-\alpha}e^{cx}$, for all $c \in R$
2. $T_\alpha(\sin bt) = bt^{1-\alpha}\cos bt$, for all $b \in R$
3. $T_\alpha(\cos bt) = -bt^{1-\alpha}\sin bt$, for all $b \in R$
4. $T_\alpha((1/t)t^\alpha) = 1$

Also, the Rolle's theorem and the mean value theorem for conformable fractional differentiable functions are included.

Theorem 2.3. *Let* $\alpha \in (0,1)$ *and* $f : [a,b] \to R$ *be a function satisfying the following conditions:*

1. f *is continuous in* $[a,b]$
2. f *is* $\alpha-$*differentiable for some* $\alpha in(0,1)$
3. $f(a) = f(b)$.

Then, there exists $c \in (a,b)$ *such that* $f^{(\alpha)}(c) = 0$.

Theorem 2.4. *(Mean value Theorem) Let* $\alpha \in (0,1)$ *and* $f : [a,b] \to R$ *be a function satisfying the following conditions:*

1. f *is continuous in* $[a,b]$
2. f *is* $\alpha-$*differentiable for some* $\alpha \in (0,1)$

Then, there exists $c \in (a,b)$ *such that*

$$f^{(\alpha)}(c) = \frac{\alpha(f(b) - f(a))}{b^\alpha - a^\alpha}$$

Additionally, a result related with the $\alpha-$differentiability and the uniform continuity property of a function was proved.

Proposition 2.5. *Let* $f : [a,b] \to R$ *be a* $\alpha-$*differentiable function for some* $\alpha \in (0,1)$.

(a) *If* f^α *is bounded on* $[a,b], a > 0$. *Then* f *is uniformly continuous on* $[a,b]$, *and hence* f *is bounded*

(b) *If* f^α *is bounded on* $[a,b], a > 0$ *and continuous in* $t = a$. *Then* f *is uniformly continuous on* $[a,b]$, *and hence* f *is bounded*

With the results mentioned above, we proceed in the next section to present some new results associated with the definition of conformable fractional derivative of order α.

3. Main Results

Theorem 3.1. *If g is an $\alpha-$differentiable function in t and f is differentiable in $g(t)$ then $(f \circ g)$ is $\alpha-$differentiable and*

$$T_\alpha(f \circ g)(t) = f'(g(t))T_\alpha(g(t))$$

Proof. Using Definition 1.1 we have

$$\begin{aligned} T_\alpha(f \circ g)(t) &= \lim_{\epsilon \to 0} \frac{(f \circ g)(t + \epsilon t^{\alpha-1}) - (f \circ g)(t)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{f(g(t + \epsilon t^{\alpha-1})) - f(g(t))}{\epsilon}. \end{aligned}$$

Setting $h = g(t+\epsilon t^{\alpha-1}) - g(t)$ and $y = g(t)$, we have $g(t+\epsilon t^{\alpha-1}) = y+h$. By Theorem 2.1, if g is $\alpha-$differentiable in t then it is continuous in t, hence $g(t + \epsilon t^{\alpha-1}) \to g(t)$ when $\epsilon \to 0$, therefore $h \to 0$. So, we can write

$$\begin{aligned} T_\alpha(f \circ g)(t) &= \lim_{\epsilon \to 0} \frac{f(y+h)) - f(y)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{f(y+h)) - f(y)}{h} \frac{h}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{f(y+h)) - f(y)}{h} \frac{g(t + \epsilon t^{\alpha-1}) - g(t)}{\epsilon} \\ &= \lim_{h \to 0} \frac{f(y+h)) - f(y)}{h} . \lim_{\epsilon \to 0} \frac{g(t + \epsilon t^{\alpha-1}) - g(t)}{\epsilon} \\ &= f'(y)T_\alpha(g)(t) \\ &= f'(g(t))T_\alpha(g)(t). \end{aligned}$$

The proof is complete. ■

The following are some examples that illustrate the use of the previous theorem and some properties of the conformable fractional derivative.

Example 3.2. *If $h(t) = \cos^2(t) = (f \circ g)(t)$, where $f(t) = t^2$ and $g(t) = \cos(t)$, then, using Theorem 3.1 we have*

$$\begin{aligned} T_\alpha(f)(t) &= f'(\cos(t))T_\alpha(\cos(t)) \\ &= 2\cos(t)T_\alpha(\cos(t)) \\ &= -2t^{1-\alpha}\cos(t)\sin(t) \end{aligned}$$

and using property 6 in Theorem 2.2 we have

$$\begin{aligned} T_\alpha(\cos^2(t)) &= t^{1-\alpha}\left(\cos^2(t)\right)' \\ &= -2t^{1-\alpha}\cos(t)\sin(t) \end{aligned}$$

Also, using the property 1 in Theorem 2.2 and the conformable fractional derivative of the exponential *function we have*

$$
\begin{aligned}
T_\alpha(\cos^2(t)) &= T_\alpha\left(\frac{e^{2it}+e^{-2it}+2}{4}\right)\\
&= \frac{1}{4}T_\alpha(e^{2it})+\frac{1}{4}T_\alpha(e^{-2it})+\frac{1}{4}T_\alpha(2)\\
&= \frac{1}{4}\left(2it^{1-\alpha}e^{2it}-2it^{1-\alpha}e^{-2it}+0\right)\\
&= \frac{it^{1-\alpha}}{2}\left(e^{2it}-e^{-2it}\right)\\
&= \frac{it^{1-\alpha}}{2}\left(\cos(2t)+i\sin(2t)-\cos(-2t)-i\sin(-2t)\right)\\
&= \frac{t^{1-\alpha}}{2}\left(-\sin(2t)-\sin(2t)\right)\\
&= -t^{1-\alpha}\sin(2t)\\
&= -2t^{1-\alpha}\sin(t)cos(t)
\end{aligned}
$$

Example 3.3. *If $h(t) = \cos(t^3) = (f \circ g)(t)$, where $f(t) = \cos(t)$ and $g(t) = t^3$, then, using Theorem 3.1 we have*

$$
\begin{aligned}
T_\alpha(\cos(t^3)) &= f'(t^3)T_\alpha(t^3)\\
&= -\sin(t^3)3t^{3-\alpha}\\
&= -3t^{3-\alpha}\sin(t^3)
\end{aligned}
$$

and using property 6 in Theorem 2.2 we have

$$
\begin{aligned}
T_\alpha(\cos(t^3)) &= t^{1-\alpha}(h(t))'\\
&= t^{1-\alpha}(-\sin(t^3)3t^2\\
&= -t^{3-\alpha}\sin(t^3)
\end{aligned}
$$

Theorem 3.4. *Let f, g be continuous functions on $[a, b]$ and $\alpha-$differentiable on (a, b). If $g^{(\alpha)}(x) \neq 0$ for all $x \in 0$, then there exists $c \in (a, b)$ such that*

$$\frac{f^{(\alpha)}(c)}{g^{(\alpha)}\left(\frac{1}{\alpha}c^\alpha\right)} = \frac{f(b)-f(a)}{g(b^\alpha)-g(a^\alpha)}.$$

Proof. Let h be the function defined by

$$h(x) = f(x)\left[g\left(\frac{1}{\alpha}b^\alpha\right)-g\left(\frac{1}{\alpha}a^\alpha\right)\right]-g\left(\frac{1}{\alpha}x^\alpha\right)(f(b)-f(a))$$

Since f and g are continuous functions on $[a, b]$ then h is continuous too. Additionally, if $x = a$, we have

$$h(a) = f(a)g\left(\frac{1}{\alpha}b^\alpha\right)-f(b)g\left(\frac{1}{\alpha}a^\alpha\right)$$

and if $x = b$ then

$$h(b) = -f(b)g\left(\frac{1}{\alpha}a^{\alpha}\right) + f(a)g\left(\frac{1}{\alpha}b^{\alpha}\right),$$

so, $h(a) = h(b)$.

Also, since f and g are $\alpha-$differentiable functions then h is too, hence we have

$$h^{(\alpha)}(x) = f^{(\alpha)}(x)\left[g\left(\frac{1}{\alpha}b^{\alpha}\right) - g\left(\frac{1}{\alpha}a^{\alpha}\right)\right] - g^{(\alpha)}\left(\frac{1}{\alpha}x\right)(f(b) - f(a)).$$

Now, the function h satisfy the hypothesis of the Rolle's theorem for $\alpha-$differentiable functions, then there exists $c \in (a, b)$ such that $h^{(\alpha)}(c) = 0$, so we have

$$f^{(\alpha)}(c)\left[g\left(\frac{1}{\alpha}b^{\alpha}\right) - g\left(\frac{1}{\alpha}a^{\alpha}\right)\right] - g^{(\alpha)}\left(\frac{1}{\alpha}c\right)(f(b) - f(a)) = 0$$

and from this last expression we get

$$\frac{f^{(\alpha}(c)}{g^{(\alpha)}\left(\frac{1}{\alpha}c^{\alpha}\right)} = \frac{f(b) - f(a)}{g(b^{\alpha}) - g(a^{\alpha})}.$$

The proof is complete. ■

Remark 3.5. *If $g(x) = x$ in Theorem 3.4 we obtain Theorem 2.4.*

Theorem 3.6. *(L'Hopital type Theorem) Let f, g be an $\alpha-$differentiable functions on (a, b) except, possibly, in some $c \in (a, b)$, and $g^{(\alpha)}(x) \neq 0$ for all $x \neq c$ in (a, b). If*

$$\lim_{x\to c} f(x) = \lim_{x\to c} g\left(\frac{1}{\alpha}x^{\alpha}\right) = 0 \ \ and \ \ \lim_{x\to c}\frac{f^{(\alpha)}(x)}{g^{(\alpha)}\left(\frac{1}{\alpha}x^{\alpha}\right)} = L$$

then

$$\lim_{x\to c}\frac{f(x)}{g\left(\frac{1}{\alpha}x^{\alpha}\right)} = L$$

Proof. Since we have no definition of the value of $f(a)$ and $g(a)$ then we proceed to define the following functions:

$$F(x) = \begin{cases} f(x) & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases}$$

and

$$G(x) = \begin{cases} g\left(\frac{1}{\alpha}x^{\alpha}\right) & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases} .$$

Also, we consider three cases, when: i) $x \to c^{+}$, ii) $x \to c^{-}$ and iii) $x \to c$.

Case i):

Since f and g are differentiable on (a, b) except, possibly, in $c \in (a, b)$ then we have that F and G are differentiable functions on $(c, x]$, therefore F and G are continuous on $(c, x]$. Also, F and G are right continuous in c because $\lim_{x\to c^{+}} F(x) = \lim_{x\to c^{+}} f(x)$ and $\lim_{x\to c^{+}} f(x) = 0$, which is $F(c)$; similarly $\lim_{x\to c^{+}} G(x) = G(c)$.

Therefore, F and G are continuous on $[c, x]$. So we can use the Cauchy's mean value theorem for conformable fractional differentiable functions (Theorem 3.4) which assures us that there exists $z \in (c, x)$ such that

$$\frac{F^{(\alpha)}(z)}{G^{(\alpha)}\left(\frac{1}{\alpha}z^{\alpha}\right)} = \frac{f(x) - f(c)}{g\left(\frac{1}{\alpha}x^{\alpha}\right) - g\left(\frac{1}{\alpha}c^{\alpha}\right)}. \tag{3}$$

From the definition of F and G we have the following values

$$F(a) = G\left(\frac{1}{\alpha}c^{\alpha}\right) = 0 \ , \ F(x) = f(x) \ , \ G\left(\frac{1}{\alpha}x^{\alpha}\right) = g\left(\frac{1}{\alpha}x^{\alpha}\right) \ ,$$

$$F^{(\alpha)}(z) = f^{(\alpha)}(z) \text{ and } G^{(\alpha)}\left(\frac{1}{\alpha}z^{\alpha}\right) = g^{(\alpha)}\left(\frac{1}{\alpha}z^{\alpha}\right)$$

because f and g are $\alpha-$differentiable in c, then we can write (3) as follows

$$\frac{f^{(\alpha}(z)}{g^{(\alpha)}\left(\frac{1}{\alpha}z^{\alpha}\right)} = \frac{f(x) - 0}{g\left(\frac{1}{\alpha}x^{\alpha}\right) - 0}.$$

Since $x \to c^{+}$ implies that $z \to c^{+}$, we have

$$\lim_{x \to c^{+}} \frac{f^{(\alpha}(z)}{g^{(\alpha)}\left(\frac{1}{\alpha}z^{\alpha}\right)} = \lim_{x \to c^{+}} \frac{f(x)}{g\left(\frac{1}{\alpha}x^{\alpha}\right)}.$$

Since $z \in (c, x)$ and we have that $\lim_{x \to c^{+}} \frac{f^{(\alpha)}(x)}{g^{(\alpha)}\left(\frac{1}{\alpha}x^{\alpha}\right)} = L$ then

$$\lim_{x \to c^{+}} \frac{f(x)}{g\left(\frac{1}{\alpha}x^{\alpha}\right)} = L.$$

Case ii) follows similarly to the case i), and case iii) follows using case i) and ii).

The proof is complete. ■

Example 3.7.

Given the functions $f(x) = \sin(x) - x$ and $g(x) = x^3$, by the classical L'Hopital Theorem we have

$$\lim_{x \to 0} \frac{\sin(x) - x}{x^3} = \frac{1}{6}$$

Now, using the L'Hopital Theorem for the conformable fractional derivative with $\alpha = 3$ we observe that

$$\frac{\sin(x) - x}{x^3} = \frac{1}{3}\frac{\sin(x) - x}{\frac{1}{3}x^3}$$

and

$$\lim_{x \to 0}(\sin(x) - x) = \lim_{x \to 0} x^3 = 0$$

then we proceed to find the $\alpha-$derivatives

$$T_3(\sin(x) - x) = x^{-2}(\cos(x) - 1) \text{ and } T_3(\frac{1}{3}x^3) = 1$$

so

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \lim_{x\to 0} \frac{1}{3} \frac{\sin(x) - x}{\frac{1}{3}x^3} = \lim_{x\to 0} \frac{1}{3} \frac{\cos(x) - 1}{x^2},$$

but

$$\lim_{x\to 0}(\cos(x) - 1) = \lim_{x\to 0} x^2 = 0$$

then with the corresponding $\alpha-$derivatives we have

$$\lim_{x\to 0} \frac{1}{3} \frac{\cos(x) - 1}{x^2} = \lim_{x\to 0} -\frac{1}{6} \frac{\sin(x)}{x},$$

again

$$\lim_{x\to 0}(\sin(x)) = \lim_{x\to 0} x = 0$$

so with the corresponding $\alpha-$derivatives we obtain

$$\lim_{x\to 0} -\frac{1}{6} \frac{\sin(x)}{x} = \lim_{x\to 0} -\frac{1}{6} \cos(x) = -\frac{1}{6}$$

Conclusion

In the present work we have presented certain theorems related to the chain rule, Cauchy mean value and L'Hopital for fractional conformable derivatives, additionally some examples have been proposed that illustrate the use of the same, including forms compared with the theorems corresponding to the classical derivative. These results are expected to motivate research in this area.

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