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De Poincaré a May: El nacimiento de la Dinámica Discreta

From Poincaré to May: The Genesis of Discrete Dynamics

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Resumen

En este artículo, se establece el origen de los sistemas dinámicos discretos desde una perspectiva histórica, así como sus ideas seminales: puntos fijos y periódicos, comportamiento caótico, bifurcaciones. Este viaje comienza con los trabajos de Poincaré y termina con la obra de May, uno de los artículos científicos más importantes del siglo XX.

Este artículo está basado en la tesis de Maestría "Simple Permutatio, Pasting and Reversing"([22]), presentada por el primer autor bajo la dirección del segundo autor.

Palabras claves:

Poincaré, Lorenz, Hénon, May, Mapa, Cuadrático, Sistemas, Dinámicos, Problema de los tres cuerpos, Atractor, Caótico, Conjunto, Cantor.

Abstract

In this paper, the origin of discrete dynamics is stated from a historical point of view, as well as its main ideas: fixed and periodic points, chaotic behaviour, bifurcations. This travel will begin with Poincaré's work and will finish with May's work, one of the most important scientific papers of 20th century.

This paper is based on the M.Sc. thesis "Simple Permutations, Pasting and Reversing"([22]), written by the first author under the guidance of the second author.

Keywords:

Poincaré, Lorenz, Hénon, May, Quadratic, Map, Dynamical, Systems, Three-body Problem, Chaotic, Attractor, Cantor, Set.

1. Introduction

This paper corresponds to the theoretical framework of the master thesis [22], in where were combined techniques of dynamical systems (some of them will be presented here) with *Pasting and Reversing* techniques (see [2, 3, 9, 10, 11, 12]). In particular, in this paper the origin of Discrete Dynamics will be presented. A meaningful context may be useful to understand the relevance and further applications of this branch of Dynamical Systems. In this way, a historical landscape will help the reader to answer the question: Why and what for do we study discrete dynamical systems?

2. Poincaré

Going back in time, the origin of this problem can be set in one name: Henri Poincaré. Born into a middle-upper class family, Poincaré was one of the last universal mathematicians. He entered the École Polytechnique in 1873, receiving his doctorate in 1879, after this, began a university career at University of Caen and then, in 1881, at University of Paris ([17]). Poincaré's work includes contributions to almost all areas of mathematics, even physics and theoretical astronomy.

In fact, The beginning of this story lies on an astronomical problem: The three-body problem. It is well known that a system of two orbiting masses, interacting through gravitational acceleration can be described by using a differential equation. In this case, the system is analytically solvable. However, a system with three or more masses under the same hypotheses is not analytically solvable.

Since there is no way to determine the significant number of state spaces analytically, A contest was held to produce the best research in celestial mechanics, related to the stability of the solar system (a particular case of the n -body problem). This contest, held in 1889 to commemorate the 60th birthday of King Oscar II of Sweden and Norway ([13]) declared Poincaré as winner.

In a field mainly dominated by quantitative methods so far (i.e. series expansions), Poincaré innovated by using new quantitative methods and simplifying assumptions. He assumed that the three bodies can be in a plane, since this decreases one degree of freedom. Furthermore, he assumed that one of the masses can be depreciated respect to the other two (e.g. Earth, Moon, and an artificial satellite orbiting among them). From this point of view, the problem turns into describing the third (small) body orbit using as reference the other two. Even when it was well known that two large masses would travel in ellipses, Poincaré considered once again a particular case: they would move in circles (with center in the center of mass of the system) at a constant speed. This simpler way to face the problem, took him into the ideas of stable and unstable manifolds, as well as homoclinic points.

2.1. Poincaré Sections

Even if all these ideas are revolutionary at that time, there is still one more, an idea that will create a whole new way to understand dynamic systems: The Poincaré section. This method is used to look complex trajectories at a simpler way. Instead of study (trace) the whole trajectory, Poincaré consider the intersection of it with a two-dimensional plane. This reduces the problem from n to $n - 1$ dimensions. Besides, in this particular case, the dynamic system is reduced into a discrete map.

Definition 2.1. *A map is a formula that describes the new state in terms of the previous state.*

Consider C as a trajectory (i.e. the solution of a differential equation with initial conditions A) and S as a Two-dimensional plane. The intersections of C and S determines a (discrete) set of points. This set is the Poincaré map.

Definition 2.2. Let G be a map defined by $C \cap S$, such that $G(A) = B$. G is a Poincaré map.

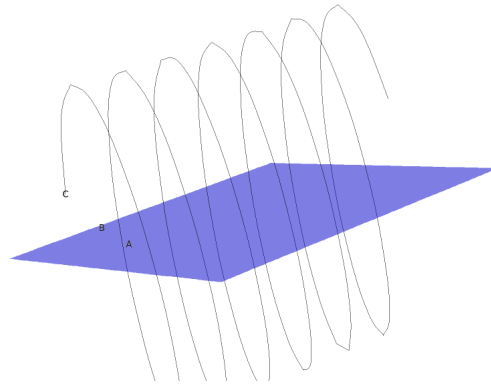


Figura 1. Poincaré section

The plane S is named surface of section. By using this technique, Chaotic behavior of differential equations can be studied by reduction to discrete dynamics. An orbit will be periodic if there exists n such that $G^{(n)}(A) = A$.

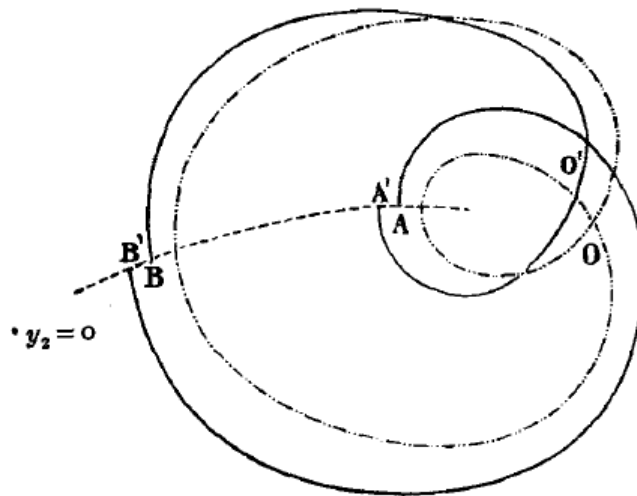


Figura 2. Poincaré section as in [25]

Poincaré sections were applied in problems related with the *integrability and non-integrability of hamiltonian dynamical systems*, see [4, 5, 6, 7, 8] Since Poincaré maps are simpler to evaluate, even by computer, they were used to study complex trajectories, such as the ones determined by Lorenz system in ([19]). Once the discrete dynamics is created, our path follows in two branches: Lorenz (and Hénon) and May.

3. Lorenz and Hénon

Edward Lorenz (1917 - 2008) began his career as mathematician, however, during WWII he moved into Atmospheric Sciences (Weather predictions) ([14]). He studied systems with forced dissipative hydrodynamic flow, whose solutions can be understood as trajectories in phase space, and based on these solutions, examine the feasibility of very-long-range weather predictions. If it is true that some hydrodynamical systems show either steady-state flow patterns or periodic oscillations, Lorenz found a case (cellular convection) in which all solutions are unstable and nonperiodic.



Figura 3. Edward Lorenz as in [14]

3.1. Lorenz Chaotic Attractor

In Deterministic Nonperiodic Flow ([19]) Lorenz stated instability conditions with respect to modifications of small amplitude. This instability is due to sensitive dependence on initial conditions. Let $P(t)$ a trajectory. The trajectory is unstable if $|P(t_1) - P(t_1 + \tau)| < \varepsilon$ for some t but $|P(t) - P(t + \tau)| \not< \varepsilon$ as $t \rightarrow \infty$.

Definition 3.1. *The Lorenz system*

$$\begin{cases} \dot{X} = -\sigma X + \sigma Y \\ \dot{Y} = -XZ + rX - Y \\ \dot{Z} = XY - bZ \end{cases}$$

The stability of the system can be understood by linearization

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ (r - Z) & -1 & -X \\ Y & X & -b \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

This system has a steady-state solution if $X = Y = Z = 0$. With this solution, the characteristic equation of this matrix is

$$[\lambda + b][\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)] = 0$$

This equation has three real roots if $r > 0$, and one of them is positive if $r > 1$. In this case the system has two more steady-state solutions $X = Y = \pm \sqrt{b(r - 1)}$, $Z = r - 1$. Taking these solutions into the linearization system the new characteristic equation is

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2\sigma b(r - 1) = 0$$

Once again, if $r > 1$ this equation has one real negative root. Besides, if

$$r = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

the complex conjugate roots are pure imaginary. Unfortunately, this information about Lorenz system obtained by linearization only applies over small perturbations, so numerical integration is required. Following Saltzman's work, Lorenz used $\sigma = 10, r = 28, b = \frac{8}{3}$, a double-approximation procedure, and a Royal McBee LGP-30 electronic computer to obtain numerical solutions (see Figure 4).

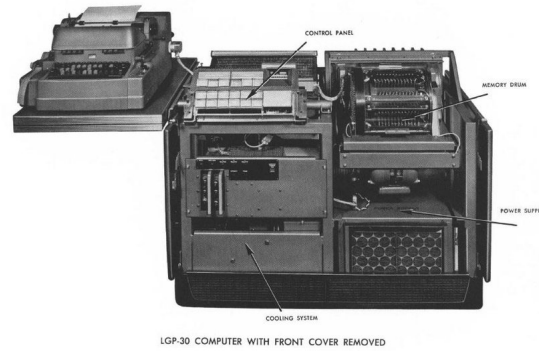


Figura 4. Royal McBee LGP-30 electronic computer

In this method, an initial time t_0 and an increment Δt are set, and let $X_{i,n} = X_i(t_0 + n\Delta t)$. Then we introduce the auxiliary approximations

$$\begin{aligned} X_{i,n+1} &= X_{i,n} + F_i(P_n)\Delta t \\ X_{i,n+2} &= X_{i,n+1} + F_i(P_{n+1})\Delta t \end{aligned}$$

where $P_n = (X_{1,n}, \dots, X_{M,n})$, with M as the number of variables of the system. The double-approximation procedure is defined by

$$X_{i,n+1} = X_{i,n} + \frac{1}{2}[F_i(P_n) + F_i(P_{n+1})]\Delta t$$

From the previous equations it follows that the double-approximation procedure can be rewritten as:

$$\begin{aligned} X_{i,n+1} &= X_{i,n} + \frac{1}{2}[F_i(P_n) + F_i(P_{n+1})]\Delta t \\ &= X_{i,n} + \frac{1}{2}\left[\frac{X_{i,n+1} - X_{i,n}}{\Delta t} + \frac{X_{i,n+2} - X_{i,n+1}}{\Delta t}\right]\Delta t \\ &= X_{i,n} + \frac{1}{2}[X_{i,n+1} - X_{i,n} + X_{i,n+2} - X_{i,n+1}] \\ &= \frac{1}{2}[X_{i,n+2} + X_{i,n}] \end{aligned}$$

Lorenz concluded that "when our results concerning the instability of nonperiodic flow are applied to the atmosphere, which is ostensibly nonperiodic, they indicate that prediction of the sufficiently distant future is impossible by any method"([19]). However, this is not the key point in our story. He found that all solutions are confined within the same bounds. This object that "traps" all solutions is known as Lorenz Attractor.

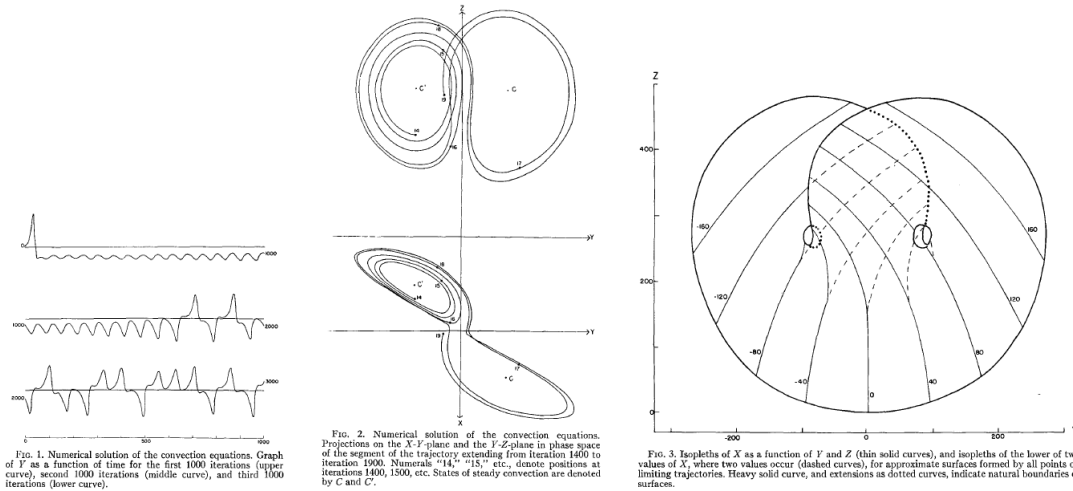


Figure 5. Numerical solutions of Lorenz System, as in [19]

As we can see in Lorenz paper, he could catch the idea of the "butterfly" trapping all trajectories. In fact, he did a remarkable work showing the geometry of the attractor with the tools at his reach, by using phase portraits and particular trajectories.

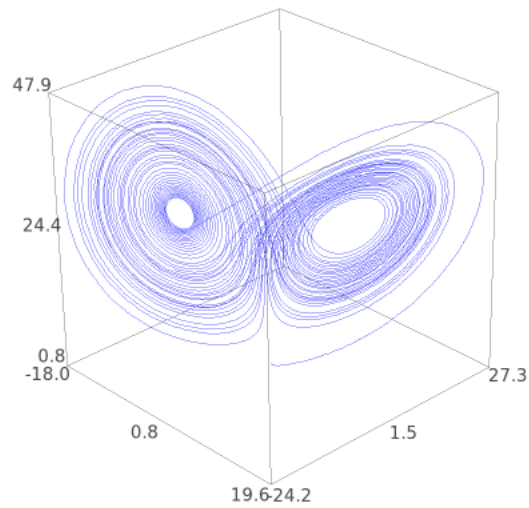


Figura 6. Lorenz attractor for $\sigma = 10, r = 28, b = \frac{8}{3}$

The attractor (as it can be seen in 3D plots) is a complex object, and due to its properties (explained previously), the use of Poincaré sections to study it seems to be a good idea. Besides, the use of computers favour this approach to the problem.

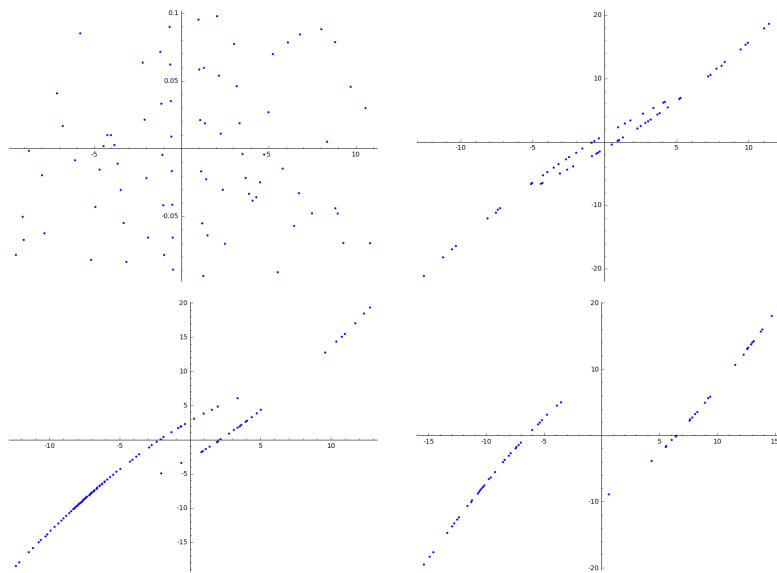


Figura 7. Some Poincaré sections of Lorenz Attractor

The idea of Poincaré sections of Lorenz attractor will bring the next starring of our story: the french mathematician and astronomer Michel Hénon (1931-2013). Despite his main research was in restricted-three-body and n-body problems, he also did something quite relevant in discrete dynamics: Poincaré sections of Lorenz system.

3.2. Hénon Map

In the paper A Two-dimensional Mapping with a Strange Attractor ([16]) Hénon shows that a discrete mapping of the plane holds the same properties of Lorenz attractor. In Lorenz attractor, the divergence of the flow has a constant negative value, so any volume shrinks exponentially with time, with a bounded region trapping all trajectories. Hénon pointed out that the strange attractor seems to be (locally) the product of a two-dimensional manifold by a Cantor set.

Definition 3.2. A set Λ is a Cantor Set if it is a closed, totally disconnected, and perfect subset of I . A set is totally disconnected if it contains no intervals; a set is perfect if every point in it is an accumulation point of limit point of other points in the set.

So finding a simpler model of Lorenz strange attractor allows better (qualitative and quantitative) explorations of it. As a first step, Poincaré mapping of Lorenz attractor will be considered, instead of the whole set. This will "decrease" the problem in one less dimension. However, it still requires the numerical integration of the differential equation so a new (explicit) map T will be defined. even if it is not Lorenz system, essential properties will be held.

Pomeau (1976) showed that, in Lorenz system, a volume is stretched in one direction, and simultaneously folded over itself, in the course of a revolution. Based on this Hénon took an elongated region and applied the following transformations:

$$\begin{aligned} T' : x' &= x, y' = y + 1 - ax^2 \\ T'' : x'' &= bx', y'' = y' \\ T''' : x''' &= y'', y''' = x'' \end{aligned}$$

The Hénon mapping is then defined as $T = T'''T''T'$.

Definition 3.3. Hénon Map

$$T : x_{i+1} = y_i + 1 - ax_i^2, y_{i+1} = bx_i$$

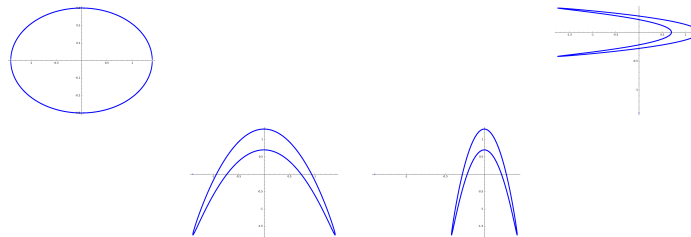


Figura 8. Transformations to obtain the Hénon map $T : x_{i+1} = y_i + 1 - ax_i^2, y_{i+1} = bx_i$

It is possible to state some main qualitative features of this map. The Jacobian of the Hénon map is constant ([26])

$$\frac{\partial(x_{i+1}, y_{i+1})}{\partial(x_i, y_i)} = -b$$

Besides, if $a^2x^2 + b \geq 0$ the (real) eigenvalues are

$$-ax \pm \sqrt{a^2x^2 + b}$$

An important element for its analysis is the fact that T is one to-one ([?]). Thus it admits an inverse transformation

$$T^{-1} : x_i = b^{-1}y_{i+1}, y_i = x_{i+1} - 1 + \frac{a}{b^2}y_{i+1}^2$$

Finally, T has two fixed points: $x = \frac{1}{2a} \pm \sqrt{(1-b)^2 + 4a}$, $y = bx$. The positive case is attracting if

$$a_0 = \frac{(1-b)^2}{4} < a < \frac{3(1-b)^2}{4}$$

This will be used to run numerical simulations.

In order to choose parameters, the author suggested $b = 0,3$. Once this value is fixed, the following values of a represent qualitative changes of behavior:

$$a_0 = \frac{(1-b)^2}{4}, a_1 = \frac{3(1-b)^2}{4}, a_2 \approx 1,06, a_3 \approx 1,55$$

Taking into account that the important case of study is the strange attractor, let be $a = 1,4, b = 0,3$. The invariant point is approximately $(0,631354477089505, 0,189406343126851)$, so the initial value is set as $(0,63135448, 0,18940634)$.

$a < a_0$ or $a > a_3$	points escape to infinity
$a_0 < a < a_1$	stable invariant point
$a_1 < a < a_2$	stable invariant set of p points
$a_2 < a < a_3$	strange attractor

Tabla 1. Behavior of T respect to a , with $b = 0,3$

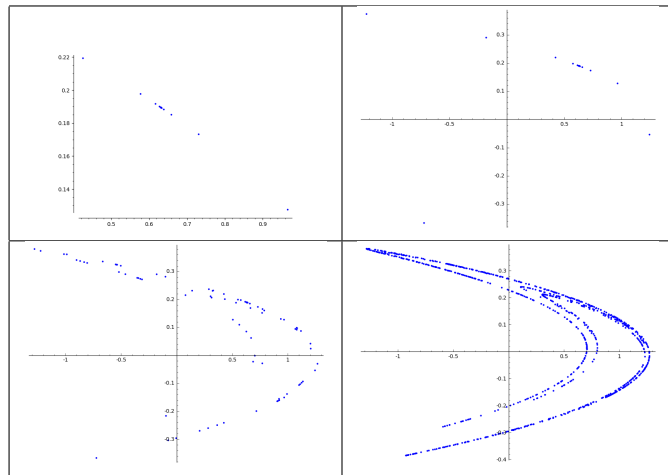


Figura 9. Iterates of Hénon map $n = 29, 33, 100, 1000$

In the first numerical simulation, Hénon used $(0, 0)$ as initial value. However, after finding the invariant points, He used approximations of that point to start simulations.

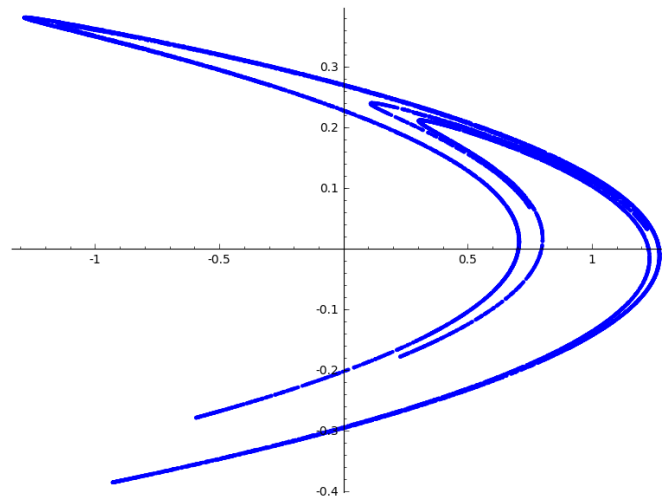


Figura 10. Hénon map $n = 10^4$

The eigenvalues associated to this invariant point are

$$\lambda_1 = 0,15594632\dots$$

$$\lambda_2 = -1,92373886\dots$$

The slopes given by the eigenvectors are

$$p_1 = 1,92373886...$$

$$p_2 = -0,15594632...$$

Due to λ_2 , this point is unstable. and the lines passing through that point with slopes p_1 and p_2 are the stable and unstable manifolds respectively. The lines seem to be continuous, but after an enlargement, it is evident that each curve is made of curves, as well as its similarity with a Cantor Set

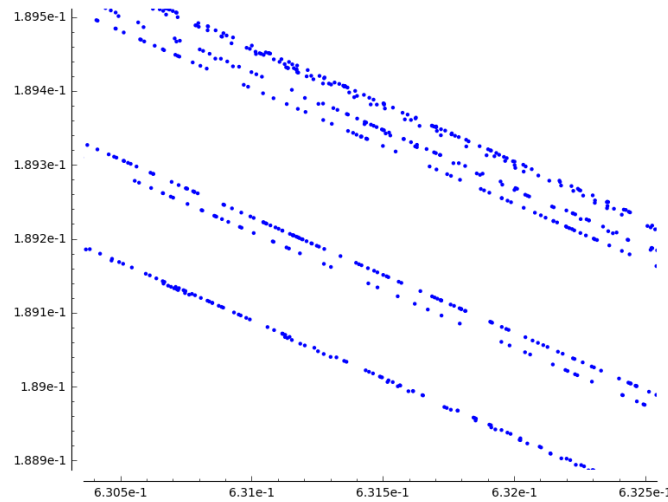


Figura 11. Hénon map $n = 5 \times 10^6$, CPU time: 707.81 s

Summarizing, Henón map shows a similar structure to Lorenz attractor, but its numerical exploration is simpler. Lorenz and Pomeau inferred the Cantor set, but due to the contracting ratio on Lorenz map (7×10^5) they couldn't observe it, while T has a smaller ratio ($b = 0,3$). Now it is time to simplify this problem one more time, going to one dimension and face the Cantor set in a simpler way.

4. May

Robert M. May (1936-2020) was, in the most general sense of the word, a scientist. Despite being educated in chemistry and physics, he develop significant research in population dynamics, specially in animal populations. This work was quite relevant to develop theoretical ecology, and also states a new context to explore discrete dynamic systems: Difference equations. Based in a particular equation (known as logistic model) he introduced chaotic behaviour by double-period bifurcation.

In Simple mathematical models with very complicated dynamics ([23]), the author summarized non-linear phenomena, showing the way in which those simple models, used to model dynamics of biological populations, move from stable points to chaos. In first-order difference equations, a variable X_{t+1} is related to its preceding value X_t . This is usually expressed as $X_{t+1} = F(X_t)$. A particular (and relevant) case is the logistic difference equation $N_{t+1} = N_t(a - bN_t)$. Taking $X = \frac{bN}{a}$, the equation turns into $X_{t+1} = aX_t(1 + X_t)$.

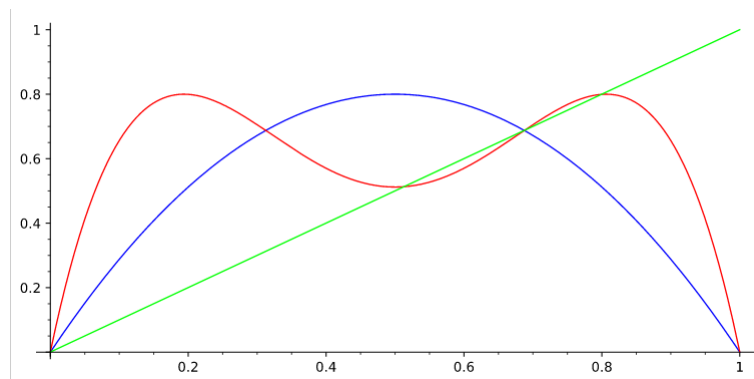


Figura 12. $X_t^{(2)}$ with $a = 3,2$

This last form was used for May to develop his paper. In order to keep all iterates in $[0, 1]$, the maximum value of f must be $\frac{a}{4} = 1$, so $a < 4$. This is interesting, because May developed bifurcation theory with this restriction over a , however, the case $a > 4$ is as much as interesting as the previous one, due to the emerging Cantor set.

4.1. Quadratic Map, $a < 4$

Definition 4.1. Let $f^n(x)$ be the $f \circ f \circ \dots \circ f$ n times.

A point x is fixed for $f(x)$ if $f(x) = x$. A point x is said to be periodic of period n if $f^n(x) = x$, where the least n holding this equation is named the prime period of x . ([15])

Fixed point are analogous to equilibrium values in (continuous) differential equations. In this case the fixed points are:

$$\begin{aligned} X^* &= aX^*(1 - X^*) \\ X^* &= aX^* - a(X^*)^2 \\ (1 - a)X^* &= -a(X^*)^2 \\ a(X^*)^2 + (1 - a)X^* &= 0 \\ X^*(aX^* + (1 - a)) &= 0 \\ X^* = 0; X^* &= \frac{a - 1}{a} \end{aligned}$$

Definition 4.2. A fixed point p is named hyperbolic if $|f'(p)| \neq 1$. This point will be an attracting point if $|f'(p)| < 1$, and a repelling point if $|f'(p)| > 1$

Since $X'(0) = a$ and $X'(\frac{a-1}{a}) = 2 - a$. So 0 is a repelling point for $a > 1$, and $\frac{a-1}{a}$ is an attracting point for $1 < a < 3$. If $a > 3$, the point $\frac{a-1}{a}$ is not hyperbolic, however, a pair of fixed point appear in $X^{(2)}$, it means, a period-2 point. This is known as a double-period bifurcation of the parameter a , and the points repelled by $\frac{a-1}{a}$ lie on the new periodic orbit.

The period-2 orbit is stable if $a < 3,449489742783179$. After this value of a the period-2 orbit turns into a period-4 orbit, showing again a double-period bifurcation. This bifurcation will appear as a increases,

generating period- 2^n up to the accumulation point at $a \approx 3,57$ ([23]). Beyond this point, infinite periodic points with different periods will appear. In fact, at $a \approx 3,8284$ appears a period-3 point, and after that, it is possible to find periodic points of all periods, as well as aperiodic points. This step to chaos was proofed simultaneously (and independently) by Li and Yorke in ([18]) and Sharkovskii ([?]). This last approach is one of the main referents of this work.

Still, there is one loose end: the relation between May's difference equation and Hénon map. These two objects will meet in a common place: the Cantor Set, and this is the reason why it makes sense "to go down" from 2 to 1 dimension. In this travel from Poincaré to May, it is possible to see how dynamic phenomena are quite similar in (apparently) different structures. This is the importance of Discrete Dynamics as a tool to understand continuous systems. In order to conclude this first part, the Cantor Set structure of the quadratic map will be studied.

4.2. Quadratic map, $a \geq 4$

The variation of a in the interval $1 < a < 4$ produces an amazing chain reaction of bifurcations from fixed points with stable orbits to chaos but, What if $a \geq 4$? If $x \in I = [0, 1]$ and $1 \leq a \leq 4$, the map $X_{t+1} = aX_t(1 + X_t)$ sends the interval I into a set $S \subset I$, keeping all iterates into I . On the other hand, if $a > 4$, the image of I is $[0, \frac{a}{4}] \supset I$, and the orbit of points in the interval $A_0 = [\frac{a - \sqrt{a(a-4)}}{2a}, \frac{a + \sqrt{a(a-4)}}{2a}]$ is unstable, since $X_1 > 1$, $X_2 < 0$ and $X_n \rightarrow -\infty$.

After the first iterate, only the points in $I \setminus A_0$ have an image in I . Let be $I_0 \cup I_1 = I \setminus A_0$ the disjoint intervals produced by subtracting A_0 . Analogously, the points of I with image in A_0 in the first iterate, will be part of an unstable orbit. The inverse image of A_0 is a pair of intervals $A_{1,0}, A_{1,1}$ such that $A_1 = A_{1,0} \cup A_{1,1}$ and $A_{1,0} \subset I_0, A_{1,1} \subset I_1$. So after this, the points remaining in I after 2 iterates are in four disjoint subsets of I , generated by $I_0 \setminus A_1$ and $I_1 \setminus A_2$, it means, taking a middle third of the intervals again.

Let be $A_n = \{x \in I / X_i \in I, X_{n+1} \notin I\}$. Points in A_n eventually will be part of an unstable orbit, so the points with full dynamics in I will be the points in

$$\Lambda = I - \bigcup_{i=0}^{\infty} A_i$$

This process remains the Cantor Middle-Thirds set, since each iterate takes a "middle interval" of the remaining subintervals of I .

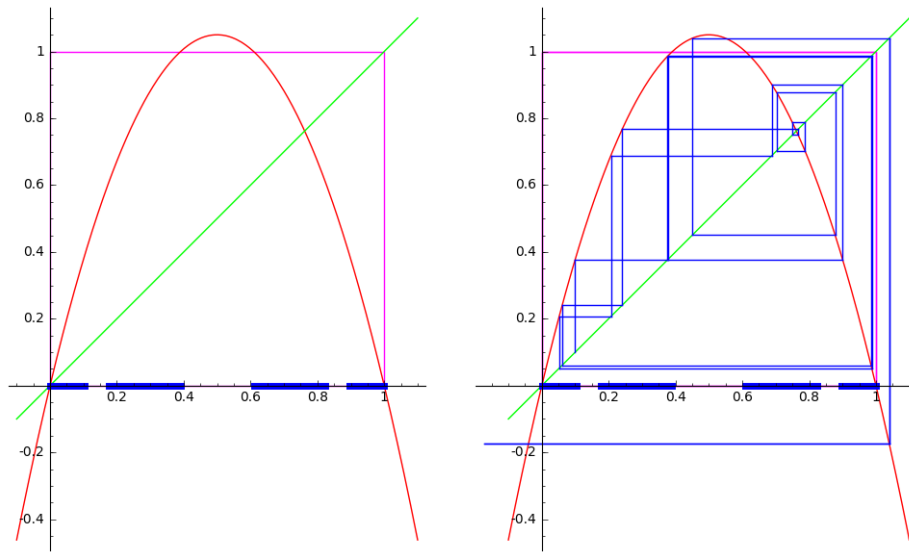


Figura 13. Iterates of X_n with $a > 4$

Theorem 4.3. *If a accomplishes that $(\forall x \in I \setminus A_0)(|x'| > 1)$ then Λ is a Cantor set ([15])*

Setting a proper a as above, there exists $\lambda > 1$ such that $(\forall x \in \Lambda)(|x'_n| > \lambda)$, and it implies that $|(x_n)'| > \lambda^n$. This is relevant to show Λ is totally disconnected.

Suppose $x \neq y, [x, y] \subset \Lambda$, if $c \in [x, y]$ then $|(c_n)'| > \lambda^n$. Due to archimedean property it is possible to choose n so that $\lambda^n|x - y| > 1$. According to the Mean Value Theorem $|y_n - x_n| \geq \lambda^n|y - x| > 1$ so either x_n or y_n is out of I , which is a contradiction, so Λ is totally disconnected. Besides, Λ is closed, since it is the intersection of nested closed intervals.

Finally, Λ must be perfect. All endpoints of A_n will have 0 as image, staying in I from that point and on. Consider now an isolated point $c \in \Lambda$. points in a neighbourhood of p should leave I after certain n^* , so these points belong to A_{n^*} . There are two possibilities: a sequence of endpoints converges to c , or all points in deleted neighbourhood of c leaves I eventually. In the first case, the points will stay in Λ . in the second one, X_n maps c to 0, and its neighbourhood out of I (into the negative real axis). X_n has a maximum at p since $(p_n)' = 0$, so $p_i = \frac{1}{2}$ for some $i < n$. This implies that $p_{i+1} \notin I$, which is again a contradiction.

5. Final Remarks

Dynamical systems are very important in the development of different areas, see for example [1] for applications in Quantum Mechanics and see also [5, 6, 7, 8] for applications in Classical Mechanics. In particular, discrete dynamical systems play an important role due to they are an useful tool to understand continuous systems by decreasing its complexity, either by diminish dimensions or passing from continuous to discrete time. Besides, they are a rich research field, with a significant number of unsolved problems, such as genealogy of permutations, a qualitative description of forcing relationships in bifurcations, and the extension of most of the 1-dimension methods to 2-dimensional systems.

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