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Boundary Value Problems with Non-Local Conditions

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We describe a new algebra of boundary value problems which contains Lopatinskii elliptic as well as Toeplitz type conditions. These latter are necessary, if an analogue of the Atiyah-Bott obstruction does not vanish. Every elliptic operator is proved to admit up to a stabilisation elliptic conditions of such a kind. Corresponding boundary value problems are then Fredholm in adequate scales of spaces. The crucial novelty consists of the new type of weighted Sobolev spaces which fit well to the nature of pseudodifferential operators.

Keywords: pseudodifferential operators, boundary values problems, Toeplitz operators

Introduction

The boundary symbols of elliptic symbols with the transmission property on a manifold X with boundary Y are families of Fredholm operators acting in spaces normal to the boundary and parametrised by points of the cosphere bundle S^*Y. The situation for symbols without the transmission property is similar. To analyse the nature of associated boundary conditions, we investigate the associated index element.

If A is an elliptic differential operator then the boundary symbol $\sigma_\partial(A)(y,\eta)$ is surjective for all $(y,\eta) \in S^*Y$. Then the Lopatinskii condition entails that $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) = [s_Y^* \tilde{W}]$ is an element of $s_Y^* K(Y)$. In other words,

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in s_Y^* K(Y) \tag{0.1}$$

is a topological obstruction for A to possess boundary conditions T elliptic in the sense of Lopatinskii. The relation (0.1) goes at least as far as [1].

There are elliptic differential operators A on X which violate condition (0.1). It is well known that Dirac operators in even dimensions and other interesting geometric operators belong to this category, cf. [2]. Possible boundary conditions leading to associated Fredholm operators are then rather different from the Lopatinskii elliptic ones. In fact, after the works of Calderón [3], Seeley [4], Atiyah et al. [5] another kind of boundary conditions became a natural concept in the index theory of boundary value problems.

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There is now a stream of investigations in the literature to establish index formulas in terms of the so-called η-invariant of elliptic operators on the boundary, see for instance [6], [7], and the references there.

General elliptic boundary value problems for differential operators and boundary conditions in subspaces of Sobolev spaces that are ranges of pseudodifferential projections on the boundary were studied in [4]. It is natural to embed such problems into a pseudodifferential algebra, where arbitrary elliptic operators admit either Lopatinskii elliptic or global projection boundary conditions, and parametrices again belong to the algebra. Such a calculus for operators with the transmission property at the boundary has been introduced by Schulze [8] as a "Toeplitz extension" of Boutet de Monvel's calculus [9].

Elliptic operators in mixed, transmission or crack problems, or, more generally, on manifolds with edges also require additional conditions along the interfaces, crack boundaries, or edges, cf. [10]. The transmission property is not a reasonable assumption in such applications. In simplest cases the additional conditions satisfy an analogue of the Lopatinskii condition as a direct generalisation of ellipticity of boundary conditions in boundary value problems. However, for the existence of such conditions for an elliptic operator in the interior topological obstructions similar to those in boundary value problems are still to be overcome. Thus, it is again natural to ask whether there are Toeplitz extensions of the corresponding algebras which contain the genuine operator algebras and admit all interior elliptic symbols that are forbidden by the obstruction.

The paper [11] gives an answer for pseudodifferential boundary value problems with general interior symbols, i.e., without the condition of the transmission property at the boundary. This algebra may also be regarded as a model for operators on manifolds with edges, though the case of boundary value problems has certain properties which are not typical for edge operators in general.

The present paper contributes to the theory by new weighted Sobolev spaces which are invariant under local diffeomorphisms of X. Thus, the theory is carried over to manifolds with boundary while the approach of [11] seems to apply only in the case of half-space $\mathbb{R}^n_+$.

1. Weighted Sobolev Spaces

1.1. Cone Sobolev Spaces

The aim of this subsection is to fix some terminology for pseudodifferential analysis on manifolds with conical and edge singularities.

For $s = 0, 1, \ldots$ and $\gamma \in \mathbb{R}$, we let $H^{s,\gamma}(\mathbb{R}_+)$ be the Hilbert space of all distributions $u \in \mathcal{D}'(\mathbb{R}_+)$, such that

$$r^{-\gamma}\,(1+r)^{s-j}\,(rD_r)^j u(r) \in L^2(\mathbb{R}_+, dr)$$

for all $j \le s$.

By duality, the definition extends in a natural way to any negative integer s. Using complex interpolation, it then extends to arbitrary real s. The scalar product in $L^2(\mathbb{R}_+) = H^{0,0}(\mathbb{R}_+)$ induces a sesquilinear pairing $H^{-s,-\gamma}(\mathbb{R}_+) \times H^{s,\gamma}(\mathbb{R}_+) \to \mathbb{C}$ by

$(u,v) \mapsto (u,v)_{L^2(\mathbb{R}_+)}$, which allows one to identify the dual space of $H^{s,\gamma}(\mathbb{R}_+)$ with $H^{-s,-\gamma}(\mathbb{R}_+)$.

1.2. Edge Sobolev Spaces

Given a Hilbert space V endowed with a strongly continuous group of isomorphisms $(\kappa_\lambda)_{\lambda>0} \subset \mathcal{L}(V)$, we define the space $H^s(\mathbb{R}^q, \pi^* V)$ to be the completion of $\mathcal{S}(\mathbb{R}^q, V)$ with respect to the norm

$$u \mapsto \Big(\int \langle \eta \rangle^{2s} \| \kappa^{-1}_{\langle \eta \rangle} \hat{u}(\eta) \|^2_V d\eta \Big)^{1/2}.$$

If V is a Fréchet space written as a projective limit of Hilbert spaces V_j, $j \in \mathbb{N}$, and V is endowed with group action, we have the spaces $H^s(\mathbb{R}^q, \pi^* V_j)$ for all j. We then define $H^s(\mathbb{R}^q, \pi^* V)$ to be the projective limit of $H^s(\mathbb{R}^q, \pi^* V_j)$ over $j \in \mathbb{N}$.

Example 1.1. For $V = H^{s,\gamma}(\mathbb{R}_+)$ with the standard group action

$$(\kappa_\lambda u)(r) = \lambda^{s-\gamma+1/2} u(\lambda r)$$

we get a weighted Sobolev space $H^{s,\gamma}(\mathbb{R}^q \times \mathbb{R}_+)$ with the norm

$$\|u\| = \Big(\int_{\mathbb{R}^q} \int_0^\infty r^{-2\gamma} \sum_{|\beta|+j \le s} (1+r)^{2(s-|\beta|-j)} |(rD_y)^\beta (rD_r)^j u|^2 dy dr \Big)^{1/2}.$$

Let $\{O_1, \dots, O_N\}$ be a covering of the manifold X by coordinate neighbourhoods and $\{\phi_1, \dots, \phi_N\}$ a subordinate partition of unity on X. Suppose $O_j \cap \partial X \neq \emptyset$ for $j = 1, \dots, N'$ and $O_j \cap \partial X = \emptyset$ for $j = N'+1, \dots, N$. Fix charts $\delta_j : O_j \to \mathbb{R}^{n-1} \times \bar{\mathbb{R}}_+$ for $j = 1, \dots, N'$, and $\delta_j : O_j \to \mathbb{R}^n$ for $j = N'+1, \dots, N$. Then $H^{s,\gamma}(X)$ is defined to be the completion of C^∞ functions with compact support in $X \setminus Y$ with respect to the norm

$$\Big(\sum_{j=1}^{N'} \| \delta_j^{-1*} (\phi_j u) \|^2_{H^{s,\gamma}(\mathbb{R}^{n-1} \times \mathbb{R}_+)} + \sum_{j=N'+1}^{N} \| \delta_j^{-1*} (\phi_j u) \|^2_{H^s(\mathbb{R}^n)} \Big)^{1/2}. \tag{1.1}$$

Throughout this exposition we fix a Riemannian metric on X that induces a product metric of $Y \times [0,1]$ on a collar neighbourhood of Y. We then have a natural identification $H^{0,0}(X) = L^2(X)$ and, via the $L^2(X)$-scalar product, a non-degenerate sesquilinear pairing $H^{s,\gamma}(X) \times H^{-s,-\gamma}(X) \to \mathbb{C}$.

Analogous definitions make sense for the case of distributional sections of vector bundles. Given any smooth complex vector bundle V over X, we have an analogue $H^{s,\gamma}(X, V)$ of the above space of scalar-valued functions, locally modelled by $H^s(\mathbb{R}^{n-1}, \pi^* H^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^k))$, where $k \in \mathbb{Z}_{\ge 0}$ corresponds to the fibre dimension of V, cf. § 3.5.2 of [10].

For each V we fix a Hermitian metric. We thus obtain a Hilbert space $L^2(X,V)$ whose norm is clearly equivalent to that of $H^{0,0}(X,V)$.

2. The Transmission Property

2.1. Operators on a Manifold with Boundary

The study of ellipticity of operators A on a C^∞ manifold X with boundary Y gives rise to the question on proper algebras of pseudodifferential boundary value problems.

As mentioned, a particular answer is given in [8] in terms of an operator algebra $\Psi^\cdot_{\mathrm{gp}}(X)$ that contains Boutet de Monvel's algebra $\Psi^\cdot_{\mathrm{BdM}}(X)$ as well as an algebra $\Psi^\cdot_{\mathrm{T}}(Y)$ of Toeplitz operators on the boundary.

The transmission property suffices to generate an algebra that contains all differential boundary value problems together with the parametrices of elliptic elements. The transmission property has been imposed in $\Psi^\cdot_{\mathrm{BdM}}(X)$ as well as in $\Psi^\cdot_{\mathrm{gp}}(X)$. It is a natural condition if we prefer standard Sobolev spaces on X or scales of closed subspaces as a frame for Fredholm operators. On the other hand, in order to understand the structure of stable homotopies of elliptic boundary value problems, or to reach specific applications, the algebra $\Psi^\cdot_{\mathrm{BdM}}(X)$ appears too narrow. It is interesting to consider a larger algebra, namely, a suitable subalgebra $\Psi^\cdot_s(X)$ of the general edge algebra on X. In this interpretation X is regarded as a manifold with edge Y and $\bar{\mathbb{R}}_+$ as the model cone of the wedge $Y \times \bar{\mathbb{R}}_+$. The algebra $\Psi^\cdot_s(X)$ is adequate for studying mixed and transmission problems and consists of pseudodifferential boundary value problems not requiring the transmission property. All classical symbols on X that are smooth up to Y are admitted in $\Psi^\cdot_s(X)$.

Recall that the operators in $\Psi^\cdot_s(X)$ act in a certain scale $H^{s,\gamma}(X)$ of weighted edge Sobolev spaces which are different from the standard Sobolev spaces $H^s(X)$, except for $s=\gamma=0$ where we have $H^{0,0}(X)=L^2(X)=H^0(X)$.

To illustrate the idea of constructing our Toeplitz extension $\Psi^\cdot_{\mathrm{gp}}(X)$ of $\Psi^\cdot_s(X)$ we first discuss the corresponding construction for Boutet de Monvel's algebra $\Psi^\cdot_{\mathrm{BdM}}(X)$. The general case will be studied in Section 3.

Let X be a smooth compact manifold with boundary, V, $\tilde{V}$ smooth vector bundles over X, and W, $\tilde{W}$ smooth vector bundles over Y. Then $\Psi^{m,d}(X;v)$ for $m \in \mathbb{Z}$ and $d \in \mathbb{Z}_{\geq 0}$ is defined to be the space of all block matrix operators

$$\mathcal{A}: \begin{array}{c} C^\infty(X,V) \\ \oplus \\ C^\infty(Y,W) \end{array} \to \begin{array}{c} C^\infty(X,\tilde{V}) \\ \oplus \\ C^\infty(Y,\tilde{W}) \end{array} \tag{2.1}$$

of the form

$$\mathcal{A} = \begin{pmatrix} r^+Pe^+ & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G} + \mathcal{C}, \tag{2.2}$$

the components of (2.2) being given as follows.

By P is meant a classical pseudodifferential operator of order m on the double of X which has the transmission property at Y. As usual, e^+ is the operator of extension by zero from X to $2X$, and r^+ the restriction from $2X$ to the interior of X.

Recall that the transmission property of an operator P on $U \times \mathbb{R}$ with coordinates $x = (y, r)$, U being an open subset of $\mathbb{R}^{n-1}$, with respect to $r = 0$ is defined in terms of the homogeneous components $p_{m-j}(y, r, \eta, \varrho)$ of a symbol $p(y, r, \eta, \varrho)$ of P by the condition

$$D_r^k D_\eta^\beta \Big(p_{m-j}(y, r, \eta, \varrho) - (-1)^{m-j} p_{m-j}(y, r, -\eta, -\varrho) \Big)\big|_{\substack{r=0\\ \eta=0}} = 0$$

for all $y \in U$, $\varrho \in \mathbb{R} \setminus \{0\}$, and $k \in \mathbb{Z}_{\geq 0}$, $\beta \in \mathbb{Z}_{\geq 0}^{n-1}$ and all j. This condition is invariant under changes of coordinates which preserve the boundary.

Thus, for any vector bundles V and $\tilde{V}$ over $2X$, we have $\Psi_{\mathrm{tp}}^m(2X; V, \tilde{V})$, the space of all classical pseudodifferential operators of order m on $2X$ acting from sections of V to sections of $\tilde{V}$, whose symbols in local coordinates near Y possess the transmission property at Y. Set $\Psi_{\mathrm{tp}}^m(X; V, \tilde{V}) := \{r^+ P e^+ : P \in \Psi_{\mathrm{tp}}^m(2X; V, \tilde{V})\}$. In other words, the operator in the first summand on the right-hand side of (2.2) belongs to $\Psi_{\mathrm{tp}}^m(X; V, \tilde{V})$.

The operator $\mathcal{C}$ on the right side of (2.2) belongs to $\Psi^{-\infty,d}(X; v)$, i.e., it is smoothing and of type d.

Here, $\Psi^{-\infty,0}(X; v)$ is the space of all operators (2.1) whose Schwartz kernel is C^∞ up to the boundary. We fix Riemannian metrics on X and Y, such that a collar neighbourhood of Y has the product metric from $Y \times [0, 1)$. Then the entries of

$$\mathcal{C} = (C_{ij})_{\substack{i=1,2\\ j=1,2}}$$

are integral operators with C^∞ kernels over $X \times X$, $X \times Y$, $Y \times X$ and $Y \times Y$, respectively, which are sections of corresponding external tensor products of bundles on the respective Cartesian products. Now $\Psi^{-\infty,d}(X; v)$ is defined to be the space of all operators

$$\mathcal{C} = \mathcal{C}_0 + \sum_{j=1}^{d} \mathcal{C}_j \begin{pmatrix} D^j & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_d$ are arbitrary operators in $\Psi^{-\infty,0}(X; v)$ and D a first order differential operator which is equal to D_r in a collar neighbourhood of the boundary.

The operator $\mathcal{G}$ in (2.2) is a (2×2)-block matrix with entries G_{ij}, where G_{11} has a C^∞ kernel over $X^\circ \times X^\circ$, G_{12} has a C^∞ kernel over $X^\circ \times Y$, G_{21} has a C^∞ kernel over $Y \times X^\circ$ and G_{22} is a classical pseudodifferential operator of order m on Y, while $\mathcal{G}$ in local coordinates $(y, r) \in U \times \bar{\mathbb{R}}_+$ near Y is a pseudodifferential operator $\mathcal{G} = \mathrm{op}(g)$ with operator-valued symbol of the form

$$g(y, \eta) = g_0(y, \eta) + \sum_{j=1}^{d} g_j(y, \eta) \begin{pmatrix} D_r^j & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.3}$$

where $g_j \in \mathcal{S}_{\mathrm{cl}}^{m-j}(U \times \mathbb{R}^{n-1}, \Psi_G^{;0}(\mathbb{R}_+; \mathbb{C}^k, \mathbb{C}^{\tilde{k}}; \mathbb{C}^l, \mathbb{C}^{\tilde{l}}))$ and k, $\tilde{k}$, l, $\tilde{l}$ are the fibre dimensions of V, $\tilde{V}$, W, $\tilde{W}$, respectively.

The concept of a Green operator in Boutet de Monvel's algebra is slightly different from that in the edge algebra. Namely, by $\mathcal{S}_{\mathrm{cl}}^m(U \times \mathbb{R}^{n-1}, \Psi_G^{;0}(\mathbb{R}_+; \mathbb{C}^k, \mathbb{C}^{\tilde{k}}; \mathbb{C}^l, \mathbb{C}^{\tilde{l}}))$ is meant the space of all operator-valued symbols $g(y, \eta)$ on $U \times \mathbb{R}^{n-1}$ with the property that

$$\begin{array}{rcl} g(y, \eta) & \in & \mathcal{S}_{\mathrm{cl}}^m(U \times \mathbb{R}^{n-1}, \mathcal{L}(L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^l, \mathcal{S}(\bar{\mathbb{R}}_+, \mathbb{C}^{\tilde{k}}) \oplus \mathbb{C}^{\tilde{l}})), \\ g^*(y, \eta) & \in & \mathcal{S}_{\mathrm{cl}}^m(U \times \mathbb{R}^{n-1}, \mathcal{L}(L^2(\mathbb{R}_+, \mathbb{C}^{\tilde{k}}) \oplus \mathbb{C}^{\tilde{l}}, \mathcal{S}(\bar{\mathbb{R}}_+, \mathbb{C}^k) \oplus \mathbb{C}^l)). \end{array}$$

Symbols $g(y,\eta)$ of the form (2.3) are called Green symbols of order m and type d. The space of all such symbols is denoted by $\mathcal{S}^m_{\mathrm{cl}}(U \times \mathbb{R}^{n-1}, \Psi^{-\infty,d}_G(\mathbb{R}_+;\mathbb{C}^k,\mathbb{C}^{\tilde{k}};\mathbb{C}^l,\mathbb{C}^{\tilde{l}}))$.

To any pseudodifferential operator $\mathcal{A} \in \Psi^{m,d}(X;v)$ one assigns a pair of principal symbols $\sigma(\mathcal{A}) = (\sigma_\Psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$. Here,

$$\sigma_\Psi(\mathcal{A}):\ \pi_X^* V \to \pi_X^* \tilde{V}$$

is the interior symbol which is the restriction of the principal homogeneous symbol of P from $T^*(2X) \setminus \{0\}$ to $T^*X \setminus \{0\}$, cf. (2.2). Moreover,

$$\sigma_\partial(\mathcal{A}) : \pi_Y^* \begin{array}{c} H^s(\mathbb{R}_+) \otimes V_Y \\ \oplus \\ W \end{array} \to \pi_Y^* \begin{array}{c} H^{s-m}(\mathbb{R}_+) \otimes \tilde{V}_Y \\ \oplus \\ \tilde{W} \end{array} \tag{2.4}$$

is the boundary symbol of $\mathcal{A}$. It is defined for all $s > d - 1/2$. It is often convenient to think of it as a family of maps

$$\sigma_\partial(\mathcal{A}) : \pi_Y^* \begin{array}{c} \mathcal{S}(\bar{\mathbb{R}}_+) \otimes V_Y \\ \oplus \\ W \end{array} \to \pi_Y^* \begin{array}{c} \mathcal{S}(\bar{\mathbb{R}}_+) \otimes \tilde{V}_Y \\ \oplus \\ \tilde{W} \end{array}. \tag{2.5}$$

The boundary symbol is defined by

$$\sigma_\partial(\mathcal{A}) = \begin{pmatrix} \sigma_\partial(r^+Pe^+) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\partial(\mathcal{G}),$$

where $\sigma_\partial(r^+Pe^+)(y,\eta) = r^+\sigma_\Psi(\mathcal{A})(y,0,\eta,D_r)e^+$ and

$$\sigma_\partial(\mathcal{G})(y,\eta) = \sigma_\partial(g_0)(y,\eta) + \sum_{j=1}^{d} \sigma_\partial(g_j)(y,\eta) \begin{pmatrix} D_r^j & 0 \\ 0 & 0 \end{pmatrix},$$

$\sigma_\partial(g_j)$ being the principal homogeneous symbol of g_j. It is easy to verify that $\sigma_\partial(\mathcal{A})$ is twisted homogeneous of degree m, i.e.,

$$\sigma_\partial(\mathcal{A})(y,\lambda\eta) = \lambda^m \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & I \end{pmatrix} \sigma_\partial(\mathcal{A})(y,\eta) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & I \end{pmatrix}^{-1}$$

for all $\lambda \in \mathbb{R}_+$. It is worth emphasizing that the group action in $H^s(\mathbb{R}_+) \otimes V_Y$ is different from that in $H^{s,\gamma}(\mathbb{R}_+) \otimes V_Y$, namely, $(\kappa_\lambda u)(r) := \lambda^{1/2} u(\lambda r)$ for $\lambda > 0$, as if $s = \gamma$.

We systematically employ various facts on operators in $\Psi^{m,d}(X;v)$. In particular, any such operator $\mathcal{A}$ induces a continuous map

$$\mathcal{A} : \begin{array}{c} H^s(X,V) \\ \oplus \\ H^s(Y,W) \end{array} \to \begin{array}{c} H^{s-m}(X,\tilde{V}) \\ \oplus \\ H^{s-m}(Y,\tilde{W}) \end{array}$$

for all real $s > d - 1/2$, which is compact provided that $\sigma(\mathcal{A}) = 0$. Moreover, composition of operators induces a map

$$\Psi^{m_1,d_1}(X;v_1) \times \Psi^{m_2,d_2}(X;v_2) \hookrightarrow \Psi^{m,d}(X;v_2 \circ v_1)$$

where for

$$\begin{aligned} v_1 &= (V^1, V^2; W^1, W^2), \\ v_2 &= (V^2, V^3; W^2, W^3) \end{aligned}$$

we set $v_2 \circ v_1 = (V^1, V^3; W^1, W^3)$, while $m = m_1 + m_2$ and $d = \max\{d_1, m_1 + d_2\}$. On the level of principal symbols we get $\sigma(\mathcal{A}^2 \mathcal{A}^1) = \sigma(\mathcal{A}^2)\sigma(\mathcal{A}^1)$ with componentwise multiplication.

2.2. Conditions with Pseudodifferential Projections

As usual, an operator $\mathcal{A} \in \Psi^{m,d}(X; v)$ is called σ_Ψ-elliptic if the interior symbol $\sigma_\Psi(\mathcal{A})$ defines an isomorphism $\pi_X^* V \to \pi_X^* \tilde{V}$. In this case,

$$r^+ \sigma_\Psi(\mathcal{A})(y, 0, \eta, D_r) e^+ : \ H^s(\mathbb{R}_+) \otimes V_y \to H^{s-m}(\mathbb{R}_+) \otimes \tilde{V}_y \tag{2.6}$$

is known to be a family of Fredholm operators for all $(y, \eta) \in T^*Y \setminus \{0\}$ and all $s > \max\{m, d\} - 1/2$. The Fredholm property of (2.6) is in turn equivalent to that of

$$r^+ \sigma_\Psi(\mathcal{A})(y, 0, \eta, D_r) e^+ : \ \mathcal{S}(\bar{\mathbb{R}}_+) \otimes V_y \to \mathcal{S}(\bar{\mathbb{R}}_+) \otimes \tilde{V}_y$$

for all $(y, \eta) \in T^*Y \setminus \{0\}$.

An operator $\mathcal{A} \in \Psi^{m,d}(X; v)$ is called Lopatinskii elliptic if it is σ_Ψ-elliptic and if, in addition, $\sigma_\partial(\mathcal{A})$ induces an isomorphism (2.4) for any $s > \max\{m, d\} - 1/2$, or, equivalently, an isomorphism (2.5).

Let $\Psi^{m,d}(X; V, \tilde{V})$ stand for the space of upper left corners of operator block matrices in $\Psi^{m,d}(X; v)$, where $v = (v)$. The question whether or not a σ_Ψ-elliptic element $A \in \Psi^{m,d}(X; V, \tilde{V})$ may be interpreted as the upper left corner of a Lopatinskii elliptic operator $\mathcal{A} \in \Psi^{m,d}(X; v)$ gives rise to an operator algebra of boundary value problems that is different from Boutet de Monvel's algebra. A general answer is given in [8]. It consists of a new algebra with boundary conditions which in [8] are called global projection conditions. Operators in this algebra

$$\mathcal{A} : \begin{array}{c} H^s(X, V) \\ \oplus \\ \mathcal{H}^s(Y, Q) \end{array} \to \begin{array}{c} H^{s-m}(X, \tilde{V}) \\ \oplus \\ \mathcal{H}^{s-m}(Y, \tilde{Q}) \end{array} \tag{2.7}$$

are characterised by the following data.

The upper left corner A of the operator block matrix $\mathcal{A}$ is assumed to belong to $\Psi^{m,d}(X; V, \tilde{V})$.

By Q is meant a triple $Q = (F, W, P)$ consisting of a smooth vector bundle F over $T^*Y \setminus \{0\}$, a smooth vector bundle W over Y, and a pseudodifferential projection $P \in \Psi^0_{\mathrm{cl}}(Y; W)$ with the property that F just amounts to the range of the principal homogeneous symbol

$$p = \sigma_\Psi(P) : \ \pi_Y^* W \to \pi_Y^* W, \tag{2.8}$$

and similarly for $\tilde{Q} = (\tilde{F}, \tilde{W}, \tilde{P})$.

The spaces on the boundary in (2.7) are given by

$$\begin{aligned} \mathcal{H}^s(Y, Q) &= P H^s(Y, W), \\ \mathcal{H}^s(Y, \tilde{Q}) &= \tilde{P} H^s(Y, \tilde{W}), \end{aligned} \tag{2.9}$$

for $s \in \mathbb{R}$. It is obvious that these are closed subspaces of $H^s(Y,W)$ and $H^s(Y,\tilde{W})$, respectively.

The operator (2.7) is now defined to be a composition $\mathcal{A} = \tilde{\mathcal{P}}\tilde{\mathcal{A}}\mathcal{E}$ for an operator $\tilde{\mathcal{A}} \in \Psi^{m,d}(X;v)$ with $v = (V,\tilde{V};W,\tilde{W})$ and

$$\mathcal{E} = \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix}, \qquad \tilde{\mathcal{P}} = \begin{pmatrix} I & 0 \\ 0 & \tilde{P} \end{pmatrix},$$

where I stands for the identity operator in the corresponding Sobolev space on X and $E : \mathcal{H}^s(Y,Q) \hookrightarrow H^s(Y,W)$ for the canonical embedding.

For $v = (V,\tilde{V};Q,\tilde{Q})$, we denote by $\Psi^{m,d}_{\mathrm{gp}}(X;v)$ the set of all operators (2.7) described above. Continuity of (2.7) holds for all $s > d - 1/2$.

Remark 2.1. If $P \in \Psi^0_{\mathrm{cl}}(Y;W)$ is a pseudodifferential projection with principal homogeneous symbol p as above, then $p^2 = p$. Vice versa, given any smooth homomorphism $p : \pi^*_Y W \to \pi^*_Y W$ which is positively homogeneous of degree 0 and satisfies $p^2 = p$, there exists a projection $P \in \Psi^0_{\mathrm{cl}}(Y;W)$ with $\sigma_\Psi(P) = p$. This can be found in [8].

Ellipticity of an operator $\mathcal{A} \in \Psi^{m,d}_{\mathrm{gp}}(X;v)$ is defined by a pair of principal symbols $\sigma(\mathcal{A}) = (\sigma_\Psi(\mathcal{A}),\sigma_\partial(\mathcal{A}))$, where $\sigma_\Psi(\mathcal{A}) : \pi^*_X V \to \pi^*_X \tilde{V}$ is the interior symbol and $\sigma_\partial(\mathcal{A})$ the boundary symbol which is a bundle homomorphism

$$\sigma_\partial(\mathcal{A}) : \begin{array}{c} \pi^*_Y\, \mathcal{S}(\bar{\mathbb{R}}_+) \otimes V_Y \\ \oplus \\ F \end{array} \to \begin{array}{c} \pi^*_Y\, \mathcal{S}(\bar{\mathbb{R}}_+) \otimes \tilde{V}_Y \\ \oplus \\ \tilde{F} \end{array} \tag{2.10}$$

still satisfying

$$\sigma_\partial(\mathcal{A})(y,\lambda\eta) = \lambda^m \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & I_{\tilde{F}} \end{pmatrix} \sigma_\partial(\mathcal{A})(y,\eta) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & I_F \end{pmatrix}^{-1}.$$

The boundary value problem $\mathcal{A}$ is called elliptic if both $\sigma_\Psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$ are isomorphisms.

Instead of $\mathcal{S}(\bar{\mathbb{R}}_+)$ in (2.10) we could equivalently consider Sobolev spaces $H^s(\mathbb{R}_+)$ for arbitrary $s > \max\{m,d\} - 1/2$.

Recall, cf. [8], that if $\mathcal{A} \in \Psi^{m,d}_{\mathrm{gp}}(X;v)$ is elliptic then operator (2.7) is Fredholm for any $s > \max\{m,d\} - 1/2$. Moreover, this operator possesses a parametrix $\Pi \in \Psi^{-m,t}_{\mathrm{gp}}(X;v^{-1})$ with $t = \max\{d-m,0\}$ and $v^{-1} = (\tilde{V},V;\tilde{Q},Q)$ in the sense that

$$\begin{array}{rcl} \Pi\mathcal{A} - I & \in & \Psi^{-\infty,t_l}_{\mathrm{gp}}(X;V;Q), \\ \mathcal{A}\Pi - I & \in & \Psi^{-\infty,t_r}_{\mathrm{gp}}(X;\tilde{V};\tilde{Q}) \end{array} \tag{2.11}$$

for $t_l = \max\{m,d\}$ and $t_r = \max\{d-m,0\}$. Clearly, the remainders in (2.11) are compact in the respective spaces (2.7).

Notice that the index of $\mathcal{A}$ depends on the particular choice of the global pseudodifferential projections P and $\tilde{P}$. However, if we do not change the principal symbols (2.8), the freedom in the choice of the projections does not affect the Fredholm property. This is a general fact on operators in Hilbert spaces, as we shall discuss now.

To this end, let H and $\tilde{H}$ be Hilbert spaces, $P_1,P_2 \in \mathcal{L}(H)$ and $\tilde{P}_1,\tilde{P}_2 \in \mathcal{L}(\tilde{H})$ be projections, such that both $P_2 - P_1$ and $\tilde{P}_2 - \tilde{P}_1$ are compact. Then the following result holds.

Theorem 2.1. *Given $A \in \mathcal{L}(H, \tilde{H})$, assume that $A_1 = \tilde{P}_1 A : P_1 H \to \tilde{P}_1 \tilde{H}$ is a Fredholm operator. Then this is also true for $A_2 = \tilde{P}_2 A : P_2 H \to \tilde{P}_2 \tilde{H}$, and the relative index formula holds*

$$\operatorname{ind} A_2 - \operatorname{ind} A_1 = \operatorname{ind} \Big(P_1 : P_2 H \to P_1 H \Big) + \operatorname{ind} \Big(\tilde{P}_2 : \tilde{P}_1 \tilde{H} \to \tilde{P}_2 \tilde{H} \Big). \tag{2.12}$$

Proof. Let us first show that the operators on the right-hand side of (2.12) are Fredholm indeed. Since P_2 acts as the identity on $P_2 H$, the difference

$$\begin{aligned} P_2 P_1 - I &= P_2 P_1 - P_2^2 \\ &= P_2 \, (P_1 - P_2) \end{aligned}$$

is a compact operator on $P_2 H$. Therefore, P_2 is the Fredholm inverse for P_1, and $P_2 P_1 : P_2 H \to P_2 H$ is Fredholm of index 0. An analogous statement holds for the projections $\tilde{P}_2$ and $\tilde{P}_1$. It follows that the composition F given by

$$P_2 H \overset{P_1}{\to} P_1 H \overset{A_1}{\to} \tilde{P}_1 \tilde{H} \overset{\tilde{P}_2}{\to} \tilde{P}_2 \tilde{H}$$

is a Fredholm operator with index

$$\operatorname{ind} F = \operatorname{ind} A_1 + \operatorname{ind} \Big(P_1 : P_2 H \to P_1 H \Big) + \operatorname{ind} \Big(\tilde{P}_2 : \tilde{P}_1 \tilde{H} \to \tilde{P}_2 \tilde{H} \Big).$$

On the other hand, we get

$$F = (\tilde{P}_2 \tilde{P}_1) \, A_2 \, (P_2 P_1) - \tilde{P}_2 [\tilde{P}_1, \tilde{P}_2] A \, (P_2 P_1) + \tilde{P}_2 \tilde{P}_1 A \, (I - P_2) \, P_1$$

where $[\tilde{P}_1, \tilde{P}_2]$ is the commutator of $\tilde{P}_1$ and $\tilde{P}_2$ which is a compact operator on $\tilde{H}$, for

$$\begin{aligned} [\tilde{P}_1, \tilde{P}_2] &= \tilde{P}_1 \tilde{P}_2 - \tilde{P}_2 \tilde{P}_1 \\ &= (\tilde{P}_2 - \tilde{P}_1)(I - \tilde{P}_1 - \tilde{P}_2). \end{aligned}$$

Furthermore, $(I - P_2) P_1 = (P_1 - P_2) P_1$ is a compact operator on H. Hence, $(\tilde{P}_2 \tilde{P}_1) \, A_2 \, (P_2 P_1)$ differs from F by a compact remainder and thus is itself Fredholm with the same index $\operatorname{ind} F = \operatorname{ind} (\tilde{P}_2 \tilde{P}_1) \, A_2 \, (P_2 P_1)$. As we have already proved, $P_2 P_1$ and $\tilde{P}_2 \tilde{P}_1$ are Fredholm operators of index 0. □

It follows that A_2 itself is Fredholm and $\operatorname{ind} F = \operatorname{ind} A_2$, as desired.

3. Boundary Value Problems with Projection Conditions

3.1. Interior Operators

Let X be a smooth compact manifold of dimension n with smooth boundary $Y = \partial X$, and V, $\tilde{V}$ vector bundles over the double of X.

As defined above, $\Psi_s^m (X; V, \tilde{V})$ is the space of all pseudodifferential operators of the form

$$A = r^+ P e^+ + S$$

where $P \in \Psi_{\mathrm{cl}}^m (2X; V, \tilde{V})$ and $S \in \Psi^{-\infty} (X^\circ; V, \tilde{V})$.

Clearly, operators in $\Psi_s^m(X;V,\tilde V)$ are much more general than those in the subspace $\Psi_{s,\mathrm{tp}}^m(X;V,\tilde V)$ of operators with the transmission property.

If $\mathcal{S}_{\mathrm{hg}}^m(T^*X\setminus\{0\},\mathrm{Hom}(V,\tilde V))$ denotes the set of all smooth bundle homomorphisms $a_m:\pi_X^*V\to\pi_X^*\tilde V$ that are positively homogeneous of degree m in the covariable, every $A\in\Psi_s^m(X;V,\tilde V)$ has a well-defined principal homogeneous symbol

$$\sigma_\Psi(A):=\sigma_\Psi(P)\,|_{T^*X\setminus\{0\}},$$

where $P\in\Psi_{\mathrm{cl}}^m(2X;V,\tilde V)$ is any operator with the property that $A-r^+Pe^+$ belongs to $\Psi^{-\infty}(X^\circ;V,\tilde V)$. Moreover, there is a (non-canonical) linear map

$$\mathrm{op}:\ \mathcal{S}_{\mathrm{hg}}^m(T^*X\setminus\{0\},\mathrm{Hom}(V,\tilde V))\to\Psi_s^m(X;V,\tilde V) \tag{3.1}$$

with $\sigma_\Psi(\mathrm{op}(a_m))=a_m$. It can be generated by a standard procedure in terms of local charts and local representatives of operators with given principal symbols.

Using the spaces $H^s(\mathbb{R}^q,\pi^*H^{s,\gamma}(\mathbb{R}_+,\mathbb{C}^k))$ as a local model near the boundary, it is straightforward to introduce weighted Sobolev spaces $H^{s,\gamma}(X,V)$ on X for any vector bundle V over X. As mentioned, $H^{s,\gamma}(X,V)\hookrightarrow H_{\mathrm{loc}}^s(X^\circ,V)$ holds for all $s,\gamma\in\mathbb{R}$.

By [10], for every $A\in\Psi_s^m(X;V,\tilde V)$ and each $\gamma\in\mathbb{R}$ there is an operator $R_\gamma\in\Psi^{-\infty}(X^\circ;V,\tilde V)$ such that $A_\gamma:=A-R_\gamma$ induces a family of continuous operators

$$A_\gamma:\ H^{s,\gamma}(X,V)\to H^{s-m,\gamma-m}(X,\tilde V) \tag{3.2}$$

for all $s\in\mathbb{R}$.

There are many ways to find suitable operators R_γ. Any choice of a correspondence $A\mapsto A_\gamma$ may be regarded as an operator convention that maps a complete symbol of A, i.e., a system of local symbols corresponding to a covering of X by coordinate charts, to a continuous operator (3.2). Setting $\mathrm{op}_{,\gamma}(a_m):=(\mathrm{op}(a_m))_\gamma$, cf. (3.1), we get a map

$$\mathrm{op}_{,\gamma}:\ \mathcal{S}_{\mathrm{hg}}^m(T^*X\setminus\{0\},\mathrm{Hom}(V,\tilde V))\to\bigcap_{s\in\mathbb{R}}\mathcal{L}(H^{s,\gamma}(X,V),H^{s-m,\gamma-m}(X,\tilde V)).$$

In the rest of this paper we construct an operator algebra $\Psi_{\mathrm{gp}}^{\cdot}(X;v;w)$ of boundary value problems

$$\mathcal{A}=\begin{pmatrix}A_\gamma & P\\ T & Q\end{pmatrix}:\ \begin{array}{c}H^{s,\gamma}(X,V)\\ \oplus\\ \mathcal{H}^s(Y,Q)\end{array}\ \to\ \begin{array}{c}H^{s-m,\gamma-m}(X,\tilde V)\\ \oplus\\ \mathcal{H}^{s-m}(Y,\tilde Q)\end{array} \tag{3.3}$$

for arbitrary $A\in\Psi_s^m(X;V,\tilde V)$ and certain operators P, T and Q. The spaces $\mathcal{H}^s(Y,Q)$ and $\mathcal{H}^{s-m}(X,\tilde Q)$ are the same as in (2.9).

Every σ_Ψ-elliptic operator $A\in\Psi_s^m(X;V,\tilde V)$ occurs up to a stabilisation as an upper left corner of an elliptic (and then Fredholm) operator (3.3) for a suitable choice of P, T, Q and data Q, $\tilde Q$. The algebra $\Psi_{\mathrm{gp}}^{\cdot}(X;v;w)$ should contain parametrices of elliptic elements. We obtain $\Psi_{\mathrm{gp}}^{\cdot}(X;v;w)$ as an extension of the algebra $\Psi_s^{\cdot}(X;v;w)$ that plays a similar role as $\Psi_{\mathrm{BdM}}^{\cdot}(X;v)$ in connection with its Toeplitz extension $\Psi_{\mathrm{gp}}^{\cdot}(X;v)$.

3.2. The Edge Algebra Revisited

Recall the calculus of boundary value problems on X which need not satisfy the transmission property with respect to the boundary Y, cf. [12].

This algebra is denoted by $\Psi_s^{\cdot}(X;v;w)$ with $v = (V,\tilde{V};W,\tilde{W})$ and weight data $w = (\gamma,\gamma-m)$. It consists of block matrix operators

$$\mathcal{A}: \begin{array}{c} C^\infty_{\mathrm{comp}}(X^\circ,V) \\ \oplus \\ C^\infty(Y,W) \end{array} \to \begin{array}{c} C^\infty(X,\tilde{V}) \\ \oplus \\ C^\infty(Y,\tilde{W}) \end{array}$$

of the form

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G} + \mathcal{C}, \tag{3.4}$$

the components of (3.4) being as follows.

By A is meant a classical pseudodifferential operator of order m and type $V \to \tilde{V}$ in the interior of X. When localised to a coordinate chart at the boundary, A is the pull-back of an operator $\mathrm{op}(a)$ whose amplitude function a is a $(\tilde{k}\times k)$-matrix with entries from $\mathcal{S}^m_{\mathrm{cl}}(U\times\mathbb{R}^{n-1}, \Psi^m(\mathbb{R}_+;w))$, where k and $\tilde{k}$ are the fibre dimensions of V and $\tilde{V}$, respectively.

The operator $\mathcal{G}$ is a (2×2)-block matrix with entries G_{ij}, where G_{11} has a C^∞ kernel on $X^\circ\times X^\circ$, G_{12} has a C^∞ kernel on $X^\circ\times Y$, G_{21} has a C^∞ kernel on $Y\times X^\circ$ and G_{22} is a classical pseudodifferential operator of order m and type $W\to\tilde{W}$ on Y. When localised to a coordinate chart close to the boundary, $\mathcal{G}$ corresponds to an operator $\mathrm{op}(g)$ with a Green symbol $g\in\mathcal{S}^m_{\mathrm{cl}}(U\times\mathbb{R}^q, \Psi_G(\mathbb{R}_+;\mathbb{C}^k,\mathbb{C}^{\tilde{k}};\mathbb{C}^l,\mathbb{C}^{\tilde{l}};w))$.

Finally, the operator $\mathcal{C}$ on the right-hand side of (3.4) is assumed to belong to the space $\Psi_G^{-\infty}(X;v;w)$, i.e., it is a smoothing Green operator in the edge calculus over X. Such operators are globally characterised by the continuity properties

$$\begin{array}{lccc} \mathcal{C}: & \begin{array}{c} H^{s,\gamma}(X,V) \\ \oplus \\ H^s(Y,W) \end{array} & \to & \begin{array}{c} H^{\infty,\gamma-m+\varepsilon}(X,\tilde{V}) \\ \oplus \\ C^\infty(Y,\tilde{W}) \end{array}, \\ \mathcal{C}^*: & \begin{array}{c} H^{s,-\gamma+m}(X,\tilde{V}) \\ \oplus \\ H^s(Y,\tilde{W}) \end{array} & \to & \begin{array}{c} H^{\infty,-\gamma+\varepsilon}(X,V) \\ \oplus \\ C^\infty(Y,W) \end{array} \end{array}$$

for all $s\in\mathbb{R}$ and some $\varepsilon>0$ depending on $\mathcal{G}$. Here, $\mathcal{C}^*$ is the formal adjoint of $\mathcal{C}$ in the sense

$$(\mathcal{C}u,g)_{H^{0,0}(X,\tilde{V})\oplus H^0(Y,\tilde{W})} = (u,\mathcal{C}^*g)_{H^{0,0}(X,V)\oplus H^0(Y,W)}$$

for all

$$\begin{array}{rcl} u & \in & C^\infty_{\mathrm{comp}}(X^\circ,V)\oplus C^\infty(Y,W), \\ g & \in & C^\infty_{\mathrm{comp}}(X^\circ,\tilde{V})\oplus C^\infty(Y,\tilde{W}). \end{array}$$

Every operator $\mathcal{A}\in\Psi_s^m(X;v;w)$ is known to induce a family of continuous mappings

$$\mathcal{A}(\lambda): \begin{array}{c} H^{s,\gamma}(X,V) \\ \oplus \\ H^s(Y,W) \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(X,\tilde{V}) \\ \oplus \\ H^{s-m}(Y,\tilde{W}) \end{array}, \tag{3.5}$$

where $s \in \mathbb{R}$. If $\mathcal{A}$ is elliptic then the operator (3.5) is Fredholm for all $s \in \mathbb{R}$. In this case a parametrix $\mathcal{P} \in \Psi_s^{-m}(X; v^{-1}; w^{-1})$, can be chosen in such a way that the compact remainders are projections of finite rank. Namely, $\mathcal{P}\mathcal{A} - I$ projects onto the null-space of $\mathcal{A}$ while $\mathcal{A}\mathcal{P} - I$ onto a complement of the range of $\mathcal{A}$, for each fixed s. In fact, $\ker \mathcal{A}$ is independent of s as well as the dimension of $\operatorname{coker} \mathcal{A}$, i.e., the index of $\mathcal{A}$ does not depend on s.

The constructions of this section can easily be generalised to the case of lower order operators, i.e., one can introduce classes $\Psi_s^{m-j}(X; v; w)$ with $j \in \mathbb{Z}_{\geq 0}$ and weight data $w = (\gamma, \gamma - m)$. For $j \geq 1$, we require A to belong to $\Psi_{\mathrm{cl}}^{m-j}(X^\circ; V, \tilde{V})$ the local amplitude function a to $\mathcal{S}_{\mathrm{cl}}^{m-j}(U \times \mathbb{R}^{n-1}, \Psi^{m-j}(\mathbb{R}_+; v; w))$, and g to belong to $\mathcal{S}_{\mathrm{cl}}^{m-j}(U \times \mathbb{R}^q, \Psi_G(\mathbb{R}_+; v; w))$.

By $\mathcal{S}_{\mathrm{cl}}^{m-j}(U \times \mathbb{R}^{n-1}, \Psi^{m-j}(\mathbb{R}_+; v; w))$ is meant the set of all operator families of the form

$$a(y, \eta) = \begin{pmatrix} \sigma(y, \eta) & 0 \\ 0 & 0 \end{pmatrix} + c(y, \eta),$$

where σ is a $(\tilde{k} \times k)$-block matrix family with entries $\varphi\, (a_0(y, \eta) + a_\infty(y, \eta))\, \tilde{\varphi}$, and $c \in \mathcal{S}_{\mathrm{cl}}^{m-j}(U \times \mathbb{R}^{n-1}, \Psi_G(\mathbb{R}_+; v; w))$. The expressions a_0 and a_∞ stem from a Mellin quantisation, now related to a symbol $p \in \mathcal{S}_{\mathrm{cl}}^{m-j}((U \times \bar{\mathbb{R}}_+) \times \mathbb{R}^n, \mathcal{L}(\mathbb{C}^k, \mathbb{C}^{\tilde{k}}))$, and φ, $\tilde{\varphi}$ are cut-off functions.

The corresponding subclass of Green operators is denoted by $\Psi_{s,G}^{m-j}(X; v; w)$ and the spaces of upper left corners by $\Psi_s^{m-j}(X; V, \tilde{V}; w)$ and $\Psi_{s,G}^{m-j}(X; V, \tilde{V}; w)$, respectively. Instead of $\Psi_{s,M+G}^{m}(X; V, \tilde{V}; w) = \Psi_s^m(X; V, \tilde{V}; w) \cap \Psi^{-\infty}(X^\circ; V, \tilde{V}; w)$ we have

$$\Psi_{s,G}^{m-j}(X; V, \tilde{V}; w) = \Psi_s^{m-j}(X; V, \tilde{V}; w) \cap \Psi^{-\infty}(X^\circ; V, \tilde{V}; w)$$

for $j \geq 1$.

For $\mathcal{A} \in \Psi_s^{m-j}(X; v; w)$, we introduce the pair $\sigma^{m-j}(\mathcal{A}) = (\sigma_\Psi^{m-j}(\mathcal{A}), \sigma_\partial^{m-j}(\mathcal{A}))$ of principal interior symbol and boundary symbol. The scheme is the same as for $j = 0$. Then, $\Psi_s^{m-j-1}(X; v; w)$ just amounts to the space of all $\mathcal{A} \in \Psi_s^{m-j}(X; v; w)$ satisfying $\sigma^{m-j}(\mathcal{A}) = 0$.

Composition of operators induces a map

$$\Psi_s^{m_1-j}(X; v_1; w_1) \times \Psi_s^{m_2-k}(X; v_2; w_2) \hookrightarrow \Psi_s^{m_1+m_2-(j+k)}(X; v_2 \circ v_1; w_2 \circ w_1)$$

where for

$$\begin{array}{rclrcl} v_1 & = & (V^1, V^2; W^1, W^2), & w_1 & = & (\gamma_1, \gamma_1 - m_1), \\ v_2 & = & (V^2, V^3; W^2, W^3); & w_2 & = & (\gamma_1 - m_1, \gamma_1 - m_1 - m_2) \end{array}$$

we set $v_2 \circ v_1 = (V^1, V^3; W^1, W^3)$ and $w_2 \circ w_1 = (\gamma_1, \gamma_1 - m_1 - m_2)$. On the level of principal symbols we get

$$\sigma^{m_1+m_2-(j+k)}(\mathcal{A}^2 \mathcal{A}^1) = \sigma^{m_2-k}(\mathcal{A}^2) \sigma^{m_1-j}(\mathcal{A}^1)$$

with componentwise multiplication. For a thorough treatment we refer the reader to [13].

3.3. Constructions for Boundary Symbols

Let $\gamma \in \mathbb{R}$. Combining (3.1) with the operator convention of [10], we get a map

$$\mathrm{op}_{,\gamma} : \ \mathcal{S}_{\mathrm{hg}}^m(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V})) \to \Psi_s^m(X; V, \tilde{V}; w) \tag{3.6}$$

for $w = (\gamma, \gamma - m)$, such that $\sigma_\Psi(\mathrm{op}_{,\gamma}(a_m)) = a_m$. Clearly, such a construction is not canonical and not necessarily linear, but it yields a right inverse of the principal symbolic map σ_Ψ.

Denote by $\mathcal{S}^m_{\mathrm{hg}}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))$ the space of all principal homogeneous boundary symbols

$$\sigma_\partial(A) : \ \pi_Y^* H^{s,\gamma}(\mathbb{R}_+) \otimes V_Y \to \pi_Y^* H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_Y$$

belonging to elements $A \in \Psi^m_s(X; V, \tilde{V}; w)$.

Moreover, let $\mathcal{S}^m_{\mathrm{hg},M+G}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))$ be the space of all principal homogeneous boundary symbols $\sigma_\partial(A)$ of elements $A \in \Psi^m_{s,M+G}(X; V, \tilde{V}; w)$. In a similar manner we define $\mathcal{S}^m_{\mathrm{hg},G}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))$ in terms of the space of Green operators $\Psi^m_{s,G}(X; V, \tilde{V}; w)$.

Note that operators $\sigma_\partial(A)$ are pointwise elements of the cone algebra on $\mathbb{R}_+$ with weight control of breadth ε for some $\varepsilon > 0$ relative to the weights γ and $\gamma - m$, respectively. From the cone theory we have an interior symbolic structure in $(r, \varrho) \in T^*\mathbb{R}_+ \setminus \{0\}$ which is the standard one of classical pseudodifferential operators on $\mathbb{R}_+$, the exit symbolic structure that is responsible for $r \to +\infty$, and the principal conormal symbolic structure for $r \to 0$. This latter is given by the family

$$\sigma_M \sigma_\partial(A)(y, z) : \ V_y \to \tilde{V}_y$$

for $y \in Y$ and $z \in \Gamma_{1/2-\gamma}$.

Set $T^*_Y X := T^*X\,|_Y$ and write $\mathcal{S}^m_{\mathrm{hg}}(T^*_Y X \setminus \{0\}, \mathrm{Hom}(V_Y, \tilde{V}_Y))$ for the space of all restrictions of elements in $\mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V}))$ to $T^*_Y X \setminus \{0\}$. Given any $a_m \in \mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V}))$, we form $A = \mathrm{op}_{,\gamma}(a_m)$. The operator family $\sigma_\partial(A)(y, \eta)$ allows one to recover

$$a_m\,|_{T^*_Y X \setminus \{0\}} \in \mathcal{S}^m_{\mathrm{hg}}(T^*_Y X \setminus \{0\}, \mathrm{Hom}(V_Y, \tilde{V}_Y))$$

in a unique way, which yields a linear map

$$\sigma_{\Psi,Y} : \ \mathcal{S}^m_{\mathrm{hg}}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w)) \to \mathcal{S}^m_{\mathrm{hg}}(T^*_Y X \setminus \{0\}, \mathrm{Hom}(V_Y, \tilde{V}_Y))$$

with

$$\ker \sigma_{\Psi,Y} = \mathcal{S}^m_{\mathrm{hg},M+G}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w)). \tag{3.7}$$

Remark 3.1. For a pair

$$(p_\Psi, p_\partial) \in \mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V})) \times \mathcal{S}^m_{\mathrm{hg}}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))$$

there exists an $A \in \Psi^m_s(X; V, \tilde{V}; w)$ satisfying $\sigma(A) = (p_\Psi, p_\partial)$ if and only if $\sigma_{\Psi,Y}(p_\partial) = p_\Psi\,|_{T^*_Y X \setminus \{0\}}$.

It is worth pointing out that for every choice of $\mathrm{op}_{,\gamma}$ the composition $\sigma_\partial\, \mathrm{op}_{,\gamma}$ induces a linear map

$$\sigma_\partial\, \mathrm{op}_{,\gamma} : \ \mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V})) \to \frac{\mathcal{S}^m_{\mathrm{hg}}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))}{\mathcal{S}^m_{\mathrm{hg},M+G}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))}.$$

An element of $\mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V}))$ is called elliptic if it defines an isomorphism $\pi_X^* V \to \pi_X^* \tilde{V}$.

Theorem 3.1. *Assume that there exists a nowhere vanishing vector field on the boundary Y. Then, for every $\gamma \in \mathbb{R}$, the map $\mathrm{op}_{,\gamma}$, cf. (3.6), can be chosen in such a way that the ellipticity of $a_m \in \mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V}))$ entails the Fredholm property of*

$$\sigma_m(y,\eta) := \sigma_\partial \ \mathrm{op}_{,\gamma}(a_m)(y,\eta) : \ H^{s,\gamma}(\mathbb{R}_+) \otimes V_y \to H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_y \tag{3.8}$$

for all $(y,\eta) \in T^*Y \setminus \{0\}$.

For general X a similar result holds up to stabilisation. By this we mean an elliptic symbol $\tilde{a}_m \in \mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V \oplus B, \tilde{V} \oplus B))$ for some vector bundle B on X, such that

$$\tilde{a}_m = a_m \oplus I_{\pi_X^* B}$$

on S^*X.

Theorem 3.2. *Suppose $\gamma \in \mathbb{R}$. For any elliptic $a_m \in \mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V}))$ there is a smooth vector bundle B over X, such that for a suitable choice of the map $\mathrm{op}_{,\gamma}$*

$$\tilde{\sigma}_m(y,\eta) := \sigma_\partial \ \mathrm{op}_{,\gamma}(\tilde{a}_m)(y,\eta) : \ H^{s,\gamma}(\mathbb{R}_+) \otimes (V \oplus B)_y \to H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes (\tilde{V} \oplus B)_y$$

is a Fredholm operator for all $(y,\eta) \in T^*Y \setminus \{0\}$.

Theorems 3.1 and 3.2 will be proved in Section 3.7. If a_m is elliptic, the operator (3.8) is Fredholm for any $s = s_0 \in \mathbb{R}$ and $\eta \neq 0$ if and only if the principal conormal symbol

$$\sigma_M \, \sigma_\partial \ \mathrm{op}_{,\gamma}(a_m)\,(y,z) : \ V_y \to \tilde{V}_y$$

is a family of isomorphisms for all $z \in \Gamma_{1/2-\gamma}$. In this case $\sigma_m(y,\eta)$ is actually Fredholm for all $s \in \mathbb{R}$, the null-space of $\sigma_m(y,\eta)$ does not depend on s, and it is a finite-dimensional subspace of $\mathcal{S}^{\gamma+\varepsilon}(\mathbb{R}_+) \times V_y$ for some $\varepsilon > 0$. Moreover, there is a finite-dimensional subspace of $\mathcal{S}^{\gamma-m+\varepsilon}(\mathbb{R}_+) \times \tilde{V}_y$ for some $\varepsilon > 0$, which is a direct complement of the range of $\sigma_m(y,\eta)$ in $H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_y$ for all $s \in \mathbb{R}$. This is true for all $y \in Y$.

3.4. Lopatinskii Ellipticity

Let $\sigma_m \in \mathcal{S}^m_{\mathrm{hg}}(T^*Y \setminus \{0\}, \, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))$ be such that the operator

$$\sigma_m(y,\eta) : \ H^{s,\gamma}(\mathbb{R}_+) \otimes V_y \to H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_y$$

is Fredholm for every $s \in \mathbb{R}$ and $(y,\eta) \in T^*Y \setminus \{0\}$, cf. (3.8). Since σ_m is homogeneous, i.e., $\sigma_m(y,\lambda\eta) = \lambda^m \kappa_\lambda \sigma_m(y,\eta) \kappa_\lambda^{-1}$ for all $\lambda > 0$, it is often sufficient to consider σ_m on the unit cosphere bundle S^*Y. It will cause no confusion if we use the same letter to designate σ_m and its restriction to S^*Y. We then get an index element

$$\mathrm{ind}_{S^*Y} \, \sigma_m \in K(S^*Y).$$

If $\tau_m \in \mathcal{S}^m_{\mathrm{hg}}(T^*Y \setminus \{0\},\, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))$ is another element with the property that $\sigma_{\Psi,Y}(\tau_m) = \sigma_{\Psi,Y}(\sigma_m)$, then relation (3.7) gives

$$\sigma_m - \tau_m \in \mathcal{S}^m_{\mathrm{hg},M+G}(T^*Y \setminus \{0\},\, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w)).$$

Clearly, $\tau_m(y,\eta)$ is not necessarily a Fredholm family in the above setting, cf. (3.8). Moreover, if this is the case, it may happen that $\mathrm{ind}_{S^*Y}\, \sigma_m \neq \mathrm{ind}_{S^*Y}\, \tau_m$.

Fix $v = (V, \tilde{V}; W, \tilde{W})$. Let $\mathcal{A} \in \Psi^m_s(X; v; w)$ be a Lopatinskii elliptic boundary value problem with an upper left corner $A \in \Psi^m_s(X; V, \tilde{V}; w)$. If $\sigma_m = \sigma_\partial(A)$ we then have a Fredholm family (3.8) and

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) = [s^*_Y \tilde{W}] - [s^*_Y W], \tag{3.9}$$

where $s_Y : S^*Y \to Y$ is the canonical projection. Thus, as in the calculus of boundary value problems with the transmission property, we have

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in s^*_Y K(Y),$$

cf. relation (0.1). Hence, this is a necessary condition for $\mathcal{A}$ to be Lopatinskii elliptic.

Given an elliptic symbol $a_m \in \mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V}))$, we may ask whether to any $\gamma \in \mathbb{R}$ there corresponds a Lopatinskii elliptic operator $\mathcal{A} \in \Psi^m_s(X; v; w)$ for a suitable choice of bundles W and $\tilde{W}$ over Y, such that $a_m = \sigma_\Psi(\mathcal{A})$.

Theorem 3.3. *Let $\gamma \in \mathbb{R}$. Suppose $a_m \in \mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V}))$ is elliptic and $A := \mathrm{op}_{,\gamma}(a_m)$ is chosen in such a way that (3.8) is a family of Fredholm operators. Then the following are equivalent:*

1) *there is a Lopatinskii elliptic boundary value problem $\mathcal{A} \in \Psi^m_s(X; v; w)$ such that $a_m = \sigma_\Psi(\mathcal{A})$;*

2) $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in s^*_Y K(Y)$.

Proof. It remains to show the implication 2) $\Rightarrow$ 1). By assumption, there are vector bundles W and $\tilde{W}$ on Y, such that (3.9) holds. It is actually a general property of Fredholm families that there exists a $g_m \in \mathcal{S}^m_{\mathrm{hg},G}(T^*Y \setminus \{0\},\, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))$ with the property that under notation (3.8)

$$\begin{array}{rcl} \ker\,(\sigma_m + g_m)\,(y,\eta) & \cong & \tilde{W}_y, \\ \mathrm{coker}\,(\sigma_m + g_m)\,(y,\eta) & \cong & W_y \end{array}$$

for all $(y,\eta) \in T^*Y \setminus \{0\}$, independently of the specific choice of s. We can fill up the family of Fredholm operators $(\sigma_m + g_m)(y,\eta)$ to a smooth family of isomorphisms

$$\begin{pmatrix} \sigma_m + g_m & k_m \\ t_m & 0 \end{pmatrix}(y,\eta) : \begin{array}{c} H^{s,\gamma}(\mathbb{R}_+) \otimes V_y \\ \oplus \\ W_y \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_y \\ \oplus \\ \tilde{W}_y \end{array}, \tag{3.10}$$

first for all $(y,\eta) \in S^*Y$ and then for all $(y,\eta) \in T^*Y$ by twisted homogeneity of order m. In addition, since $C^\infty_{\mathrm{comp}}(\mathbb{R}_+)$ is dense in $H^{s,\gamma}(\mathbb{R}_+)$ for all $s, \gamma \in \mathbb{R}$, the potential part k_m

can be chosen to be a map $\pi_Y^* W \to \pi_Y^* C^\infty_{\rm comp}(\mathbb{R}_+) \otimes \tilde{V}_Y$, while the trace part t_m may be represented by an element in $\pi_Y^* \tilde{W} \otimes (C^\infty_{\rm comp}(\mathbb{R}_+) \otimes \tilde{V}_Y^*)$ through integration

$$u \mapsto \int_0^\infty \langle k_{t_m(y,\eta)}(r), u(r)\rangle_{V_y} dr$$

for all $u \in H^{s,\gamma}(\mathbb{R}_+) \otimes V_y$. Here, $\langle \cdot, \cdot \rangle_{V_y}$ denotes the pairing between V_y and its dual V_y^*. Let us now restrict g_m, k_m and t_m to a coordinate neighbourhood Ω_j on Y and interpret the variables y as local coordinates in $U \subset \mathbb{R}^{n-1}$ with respect to a chart $\Omega_j \to U$. Choosing a zero excision function $\chi(\eta)$ we obtain operator-valued symbols

$$\begin{array}{rcl} g = \chi g_m & \in & \mathcal{S}^m_{\rm cl}(U \times \mathbb{R}^{n-1}, \mathcal{L}(H^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^k), H^{\infty,\gamma-m}(\mathbb{R}_+, \mathbb{C}^{\tilde{k}}))), \\ k = \chi k_m & \in & \mathcal{S}^m_{\rm cl}(U \times \mathbb{R}^{n-1}, \mathcal{L}(\mathbb{C}^l, H^{\infty,\gamma-m}(\mathbb{R}_+, \mathbb{C}^{\tilde{k}}))), \\ t = \chi t_m & \in & \mathcal{S}^m_{\rm cl}(U \times \mathbb{R}^{n-1}, \mathcal{L}(H^{s,\gamma}(\mathbb{R}_+, \mathbb{C}^k), \mathbb{C}^{\tilde{l}})) \end{array}$$

for all $s \in \mathbb{R}$, where $k = \tilde{k}$ and l, $\tilde{l}$ are the fibre dimensions of the bundles V, $\tilde{V}$ and W, $\tilde{W}$, respectively. Denote by G_j, K_j and T_j the pull-backs of op(g), op(k) and op(t) from U to Ω_j with respect to the charts and trivialisations of the bundles involved. Pick a covering $\{\Omega_1, \ldots, \Omega_N\}$ of Y by such coordinate neighbourhoods, a subordinate partition of unity $\{\phi_1, \ldots, \phi_N\}$, and a family $\{\psi_1, \ldots, \psi_N\}$ of functions $\psi_j \in C^\infty_{\rm comp}(\Omega_j)$ satisfying $\phi_j \psi_j = \phi_j$. We can then pass in a familiar way to an operator

$$\begin{pmatrix} G & K \\ T & 0 \end{pmatrix} = \sum_{j=0}^N \begin{pmatrix} \varphi_b \phi_j & 0 \\ 0 & \phi_j \end{pmatrix} \begin{pmatrix} G_j & K_j \\ T_j & 0 \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_b \psi_j & 0 \\ 0 & \psi_j \end{pmatrix},$$

where φ_b and $\tilde{\varphi}_b$ are cut-off functions supported close to the boundary. It follows that

$$\mathcal{A} := \begin{pmatrix} {\rm op}_{,\gamma}(a_m) + G & K \\ T & 0 \end{pmatrix}$$

belongs to $\Psi^m_s(X; v; w)$ for $v = (V, \tilde{V}; W, \tilde{W})$ and $\sigma_\Psi(\mathcal{A})$ is equal to (3.10), while $\sigma_\Psi(\mathcal{A}) = \sigma_\Psi({\rm op}_{,\gamma}(a_m) + G)$ just amounts to a_m. □

Remark 3.2. Under the hypotheses 2) of Theorem 3.3 it is even possible to construct $\mathcal{A} \in \Psi^m_s(X; v; w)$ in such a way that $A = {\rm op}_{,\gamma}(a_m)$ is equal to the upper left corner of $\mathcal{A}$.

To verify this, it is sufficient to set $W = Y \times \mathbb{C}^l$ for $l \in \mathbb{N}$ large enough, and to choose some homogeneous potential symbol $k_m : \pi_Y^* W \to \pi_Y^* H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_Y$ such that

$$(\sigma_m \;\; k_m) : \; \pi_Y^* \begin{array}{c} H^{s,\gamma}(\mathbb{R}_+) \otimes V_Y \\ \oplus \\ W \end{array} \to \pi_Y^* H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_Y \qquad (3.11)$$

is surjective. For sufficiently large l this is possible, and then the null-space of $(\sigma_m \;\; k_m)$ can be taken as a copy of $\tilde{W}$. Finally, (3.11) can be filled up by a second row $(t_m \;\; q_m)$ to a block matrix isomorphism which plays the role of $\sigma_\partial(\mathcal{A})$. Then we can pass to a desired boundary value problem $\mathcal{A}$ just as in the proof of Theorem 3.3.

The following lemma states that the topological obstruction for the existence of a Lopatinskii elliptic boundary value problem is not affected by the choice of the operator convention ${\rm op}_{,\gamma}$.

Lemma 3.1. *Assume that* $a_m \in \mathcal{S}^m_{\rm hg}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V}))$ *is an elliptic symbol and let* $\tilde{\mathrm{op}}_{,\gamma} : \mathcal{S}^m_{\rm hg}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V})) \to \Psi^m_s(X; V, \tilde{V}; w)$ *be another choice of operator convention (3.6). If for* $A = \mathrm{op}_{,\gamma}(a_m)$ *and* $\tilde{A} = \tilde{\mathrm{op}}_{,\gamma}(a_m)$ *both* $\sigma_\partial(A)$ *and* $\sigma_\partial(\tilde{A})$ *are families of Fredholm operators* $H^{s,\gamma}(\mathbb{R}_+) \otimes V_y \to H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_y$ *for all* $(y, \eta) \in T^*Y \setminus \{0\}$*, then* $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A)$ *belongs to* $s^*_Y K(Y)$ *if and only if* $\mathrm{ind}_{S^*Y}\, \sigma_\partial(\tilde{A})$ *does.*

Proof. The symbols $\sigma_\partial(A)$ and $\sigma_\partial(\tilde{A})$ can be written in the form

$$\begin{array}{rcl} \sigma_\partial(A) & = & \sigma_\partial(a) + \sigma_\partial(m) + \sigma_\partial(g), \\ \sigma_\partial(\tilde{A}) & = & \sigma_\partial(\tilde{a}) + \sigma_\partial(\tilde{m}) + \sigma_\partial(\tilde{g}), \end{array}$$

the terms on the right-hand side having standard meaning in the cone theory. Since $\sigma_\partial(a) = \sigma_\partial(\tilde{a})$ modulo $\mathcal{S}^m_{\mathrm{hg},M+G}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))$, we may assume without loss of generality that $\sigma_\partial(a) = \sigma_\partial(\tilde{a})$. Furthermore, since the elements of $\mathcal{S}^m_{\mathrm{hg},G}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w))$ are families of compact operators, the property of $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A)$ or $\mathrm{ind}_{S^*Y}\, \sigma_\partial(\tilde{A})$ to belong to $s^*_Y K(Y)$ is not affected by a Green summand. Therefore, $\sigma_\partial(g)$ and $\sigma_\partial(\tilde{g})$ may be ignored.

There is $l \in \mathbb{N}$ and a monomorphism $k_m : s^*_Y(Y \times \mathbb{C}^l) \to s^*_Y\, H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_Y$ pointwise mapping to $C^\infty_{\mathrm{comp}}(\mathbb{R}_+) \otimes V_y$, such that both

$$(\sigma_\partial(A) \;\; k_m) : \; s^*_Y \begin{array}{c} H^{s,\gamma}(\mathbb{R}_+) \otimes V_Y \\ \oplus \\ Y \times \mathbb{C}^l \end{array} \to s^*_Y\, H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_Y$$

and

$$\Big(\sigma_\partial(\tilde{A}) \;\; k_m\Big) : \; s^*_Y \begin{array}{c} H^{s,\gamma}(\mathbb{R}_+) \otimes V_Y \\ \oplus \\ Y \times \mathbb{C}^l \end{array} \to s^*_Y\, H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_Y$$

are surjective. As usual, the choice of s is unessential.

Set $b_m = (\sigma_\partial(A) \;\; k_m)$ and $\tilde{b}_m = (\sigma_\partial(\tilde{A}) \;\; k_m)$. Observe that the property $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in s^*_Y K(Y)$ is equivalent to saying that for l large enough the bundle $\ker b_m$ over S^*Y may be represented by a system of trivialisations with transitions isomorphisms depending only on y, not on the covariable η. Clearly, we have $\mathrm{ind}_{S^*Y}\, \sigma_\partial(A) \in s^*_Y K(Y)$ if and only if $\mathrm{ind}_{S^*Y}\, b_m \in s^*_Y K(Y)$, and similarly for the operator families with tilde.

Let $\tilde{b}_m^{-1}$ be a right inverse of $\tilde{b}_m$. It can be calculated within our class of boundary symbols. In fact, in the case $m = \gamma = 0$ the right inverse is equal to $\tilde{b}^*_m(\tilde{b}_m \tilde{b}^*_m)^{-1}$ which possesses the required structure due to the algebra property of boundary symbols. The general case can then be treated by using order reducing operators, cf. [13].

Since $b_m - \tilde{b}_m = (\sigma_\partial(m - \tilde{m}) \;\; 0)$, it follows that

$$\begin{array}{rcl} b_m \tilde{b}_m^{-1} & = & I + (\sigma_\partial(m - \tilde{m}) \;\; 0)\, \tilde{b}_m^{-1} \\ & = & I + \sigma_\partial(m_0) + g_0 \end{array}$$

belongs to $\mathcal{S}^0_{\mathrm{hg},M+G}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w^{-1} \circ w))$ restricted to S^*Y. Here m_0 is a smoothing Mellin family which consists of a single term containing the zero power of r, and the family g_0 belongs to $\mathcal{S}^0_{\mathrm{hg},G}(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w^{-1} \circ w))$ restricted to S^*Y. Since

$\sigma_\partial(m_0)$ is actually independent of η on S^*Y and g_0 takes values in compact operators, we get

$$\begin{aligned}\operatorname{ind}_{S^*Y}(I+\sigma_\partial(m_0)+g_0) &= \operatorname{ind}_{S^*Y}(I+\sigma_\partial(m_0))\\ &\in \pi_Y^* K(Y).\end{aligned}$$

From

$$\operatorname{ind}_{S^*Y}\tilde{b}_m = \operatorname{ind}_{S^*Y} b_m - \operatorname{ind}_{S^*Y}(I+\sigma_\partial(m_0)+g_0)$$

we then immediately obtain the assertion.

The obstruction for the existence of Lopatinskii elliptic conditions is also not affected by the choice of the parameter $\gamma \in \mathbb{R}$ in the operator convention $\mathrm{op}_{,\gamma}$. □

Lemma 3.2. *Let $a_m \in \mathcal{S}^m_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V,\tilde{V}))$ be elliptic. If for $A_\gamma = \mathrm{op}_{,\gamma}(a_m)$ and $A_\delta = \mathrm{op}_{,\delta}(a_m)$ both*

$$\begin{aligned}\sigma_\partial(A_\gamma)(y,\eta): &\quad H^{s,\gamma}(\mathbb{R}_+)\otimes V_y \to H^{s-m,\gamma-m}(\mathbb{R}_+)\otimes \tilde{V}_y \quad \text{and}\\ \sigma_\partial(A_\delta)(y,\eta): &\quad H^{s,\delta}(\mathbb{R}_+)\otimes V_y \to H^{s-m,\delta-m}(\mathbb{R}_+)\otimes \tilde{V}_y\end{aligned}$$

are Fredholm operators for all $(y,\eta)\in T^*Y\setminus\{0\}$, then $\operatorname{ind}_{S^*Y}\sigma_\partial(A_\gamma)$ belongs to $s_Y^*K(Y)$ if and only if $\operatorname{ind}_{S^*Y}\sigma_\partial(A_\delta)$ does.

Proof. Starting with the operators

$$\begin{aligned}A_\gamma: &\quad H^{s,\gamma}(X,V)\to H^{s-m,\gamma-m}(X,\tilde{V}),\\ A_\delta: &\quad H^{s-\gamma+\delta,\delta}(X,V)\to H^{s-\gamma+\delta-m,\delta-m}(X,\tilde{V})\end{aligned}$$

which are continuous for all $s\in\mathbb{R}$, we pass to

$$\begin{aligned}\tilde{A}_\gamma &= \left(D_{\tilde{V}}^{\gamma-\delta}\right)^{-1} A_\delta\, D_V^{\gamma-\delta}\\ &\in \Psi_s^m(X;V,\tilde{V};w)\end{aligned}$$

by using the order reducing operators from [13]. We then obviously obtain $\sigma_\Psi(A_\gamma) = \sigma_\Psi(\tilde{A}_\gamma) = a_m$, and so the boundary symbols of $A = A_\gamma$ and $\tilde{A} = \tilde{A}_\gamma$ satisfy the assumptions of Lemma 3.1. In order to complete the proof it is now sufficient to observe that $\operatorname{ind}_{S^*Y}\sigma_\partial(\tilde{A}_\gamma)\in s_Y^*K(Y)$ is equivalent to saying that $\operatorname{ind}_{S^*Y}\sigma_\partial(A_\delta)\in s_Y^*K(Y)$, since both $\operatorname{ind}_{S^*Y}\sigma_\partial(D_{\tilde{V}}^{\gamma-\delta})^{-1}$ and $\operatorname{ind}_{S^*Y}\sigma_\partial(D_V^{\gamma-\delta})$ are equal to zero. □

3.5. Boundary Value Problems with Projection Data

In the previous section we have seen that Lopatinskii elliptic conditions for a given operator A of $\Psi_s^m(X;v;w)$ may only exists under condition 2) of Theorem 3.3. If this is not the case, one might pass to another kind conditions that we call global projection conditions.

Let us fix some vector space data $v=(V,\tilde{V};Q,\tilde{Q})$ with $Q=(F,W,P)$ and $\tilde{Q}=(\tilde{F},\tilde{W},\tilde{P})$ as in § 2.2.

Definition 3.1. For $w = (\gamma, \gamma - m)$, the space $\Psi^m_{\mathrm{gp}}(X; v; w)$ is defined to consist of all operators

$$\mathcal{A} : \begin{array}{c} H^{s,\gamma}(X, V) \\ \oplus \\ \mathcal{H}^s(Y, Q) \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(X, \tilde{V}) \\ \oplus \\ \mathcal{H}^{s-m}(Y, \tilde{Q}) \end{array}, \tag{3.12}$$

$s \in \mathbb{R}$, such that

1) the upper left corner A of the operator block matrix $\mathcal{A}$ is assumed to be in $\Psi^m_s(X; V, \tilde{V}; w)$;

2) there is an $\tilde{\mathcal{A}} \in \Psi^m_s(X; V, \tilde{V}; W, \tilde{W}; w)$ such that $\mathcal{A} = \tilde{\mathcal{P}}\tilde{\mathcal{A}}\mathcal{E}$, where $\tilde{\mathcal{P}}$ and $\mathcal{E}$ have the same meaning as in § 2.2.

Denote by $\Psi^m_{\mathrm{gp},M+G}(X; v; w)$ the subspace of $\Psi^m_{\mathrm{gp}}(X; v; w)$ consisting of all $\mathcal{A}$ such that $\mathcal{A} = \tilde{\mathcal{P}}\tilde{\mathcal{A}}\mathcal{E}$ for some $\tilde{\mathcal{A}} \in \Psi^m_{M+G}(X; V, \tilde{V}; W, \tilde{W}; w)$. In a similar way we introduce $\Psi^m_{\mathrm{gp},G}(X; v; w)$.

It is now straightforward that the principal symbolic structure of $\Psi^m_{\mathrm{gp}}(X; v; w)$ consists of pairs $\sigma(\mathcal{A}) = (\sigma_\Psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$, where $\sigma_\Psi(\mathcal{A}) : \pi^*_X V \to \pi^*_X \tilde{V}$ is the principal interior symbol and $\sigma_\partial(\mathcal{A})$ the principal boundary symbol which is a bundle homomorphism

$$\sigma_\partial(\mathcal{A}) : \begin{array}{c} \pi^*_Y\, H^{s,\gamma}(\mathbb{R}_+) \otimes V_Y \\ \oplus \\ F \end{array} \to \begin{array}{c} \pi^*_Y\, H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_Y \\ \oplus \\ \tilde{F} \end{array} \tag{3.13}$$

given by

$$\sigma_\partial(\mathcal{A})(y, \eta) := \begin{pmatrix} I & 0 \\ 0 & \tilde{p}(y, \eta) \end{pmatrix} \sigma_\partial(\tilde{\mathcal{A}})(y, \eta) \begin{pmatrix} I & 0 \\ 0 & e(y, \eta) \end{pmatrix},$$

where $e : F \hookrightarrow \pi^*_Y W$ is the canonical embedding and $\tilde{p}$ the principal homogeneous symbol of $\tilde{P} \in \Psi^0_{\mathrm{cl}}(Y; \tilde{W})$.

Theorem 3.4. *Composition of operators induces a map*

$$\Psi^{m_1}_{\mathrm{gp}}(X; v_1; w_1) \times \Psi^{m_2}_{\mathrm{gp}}(X; v_2; w_2) \hookrightarrow \Psi^{m_1+m_2}_{\mathrm{gp}}(X; v_2 \circ v_1; w_2 \circ w_1)$$

where for

$$\begin{array}{lll} v_1 = (V^1, V^2; Q^1, Q^2), & & w_1 = (\gamma_1, \gamma_1 - m_1), \\ v_2 = (V^2, V^3; Q^2, Q^3), & & w_2 = (\gamma_1 - m_1, \gamma_1 - m_1 - m_2) \end{array}$$

we set $v_2 \circ v_1 = (V^1, V^3; Q^1, Q^3)$ *and* $w_2 \circ w_1 = (\gamma_1, \gamma_1 - m_1 - m_2)$.

For the principal symbols we get

$$\sigma^{m_1+m_2-(j+k)}(\mathcal{A}^2\mathcal{A}^1) = \sigma^{m_2-k}(\mathcal{A}^2)\sigma^{m_1-j}(\mathcal{A}^1)$$

with componentwise multiplication.

If $\mathcal{A}^1$ or $\mathcal{A}^2$ belongs to one of the subspaces with subscript $M + G$ or G, the same is true for the composition.

PROOF. This assertion is an immediate consequence of Definition 3.1 and of what has been proved in § 3.2. □

Note that $\Psi^m_{\rm gp}(X;v;w)$ can be identified with the set of compositions $\mathcal{A} = \tilde{\mathcal{P}}\tilde{\mathcal{A}}\mathcal{P}$ with $\tilde{\mathcal{A}} \in \Psi^m_s(X;V,\tilde{V};W,\tilde{W};w)$ as in Definition 3.1. Hence $\Psi^m_{\rm gp}(X;v;w)$ survives under taking the formal adjoint $\mathcal{A}^*$ with respect to the scalar products in $H^{0,0}(X,V) \oplus L^2(X,W)$ and $H^{0,0}(X,\tilde{V}) \oplus L^2(X,\tilde{W})$, for the larger class $\Psi^m_s(X;\cdot;w)$ does.

Theorem 3.5. *Assume that $\mathcal{A} \in \Psi^m_{\rm gp}(X;v;w)$. Then, $\mathcal{A}^* \in \Psi^m_{\rm gp}(X;v^*;w^*)$ where $v^* = (\tilde{V},V;Q^*,\tilde{Q}^*)$ for $Q^* = (\sigma_\Psi(P^*)\pi^*_Y W, W, P^*)$ and $\tilde{Q}^*$ of a similar form, and $w^* = (-\gamma + m,\gamma)$.*

Let

$$\begin{array}{rcl} \mathcal{A} & \in & \Psi^m_{\rm gp}(X;v_{\mathcal{A}};w), \\ \mathcal{B} & \in & \Psi^m_{\rm gp}(X;v_{\mathcal{B}};w) \end{array}$$

for

$$\begin{array}{rclcrcl} v_{\mathcal{A}} & = & (V_{\mathcal{A}},\tilde{V}_{\mathcal{A}};Q_{\mathcal{A}},\tilde{Q}_{\mathcal{A}}), & \quad & Q_{\mathcal{A}} & = & (F_{\mathcal{A}},W_{\mathcal{A}},P_{\mathcal{A}}), \\ v_{\mathcal{B}} & = & (V_{\mathcal{B}},\tilde{V}_{\mathcal{B}};Q_{\mathcal{B}},\tilde{Q}_{\mathcal{B}}); & & \tilde{Q}_{\mathcal{A}} & = & (\tilde{F}_{\mathcal{A}},\tilde{W}_{\mathcal{A}},\tilde{P}_{\mathcal{A}}), \end{array}$$

and similarly $Q_{\mathcal{B}}$, $\tilde{Q}_{\mathcal{B}}$. Then one defines the direct sum $\mathcal{A} \oplus \mathcal{B} \in \Psi^m_{\rm gp}(X;v_{\mathcal{A}} \oplus v_{\mathcal{B}};w)$ of $\mathcal{A}$ and $\mathcal{B}$ in a canonical way, where

$$\begin{array}{rcl} v_{\mathcal{A}} \oplus v_{\mathcal{B}} & = & \left(V_{\mathcal{A}} \oplus V_{\mathcal{B}},\tilde{V}_{\mathcal{A}} \oplus \tilde{V}_{\mathcal{B}};Q_{\mathcal{A}} \oplus Q_{\mathcal{B}},\tilde{Q}_{\mathcal{A}} \oplus \tilde{Q}_{\mathcal{B}}\right), \\ Q_{\mathcal{A}} \oplus Q_{\mathcal{B}} & = & (F_{\mathcal{A}} \oplus F_{\mathcal{B}},W_{\mathcal{A}} \oplus W_{\mathcal{B}},P_{\mathcal{A}} \oplus P_{\mathcal{B}}) \end{array}$$

and, similarly, $\tilde{Q}_{\mathcal{A}} \oplus \tilde{Q}_{\mathcal{B}}$. For all $s \in \mathbb{R}$, the direct sum induces a continuous linear operator

$$\mathcal{A} \oplus \mathcal{B} : \begin{array}{c} H^{s,\gamma}(X,V_{\mathcal{A}} \oplus V_{\mathcal{B}}) \\ \oplus \\ \mathcal{H}^s(Y,Q_{\mathcal{A}} \oplus Q_{\mathcal{B}}) \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(X,\tilde{V}_{\mathcal{A}} \oplus \tilde{V}_{\mathcal{B}}) \\ \oplus \\ \mathcal{H}^{s-m}(Y,\tilde{Q}_{\mathcal{A}} \oplus \tilde{Q}_{\mathcal{B}}) \end{array}.$$

Using in Definition 3.1 the classes $\Psi^{m-j}_s(X;v;w)$ defined at the end of § 3.2, we also introduce the subspaces $\Psi^{m-j}_{\rm gp}(X;v;w)$ with $j \in \mathbb{Z}_{\geq 0}$. For any operator $\mathcal{A} \in \Psi^{m-j}_{\rm gp}(X;v;w)$, we have a corresponding pair $\sigma^{m-j}(\mathcal{A}) = (\sigma^{m-j}_\Psi(\mathcal{A}),\sigma^{m-j}_\partial(\mathcal{A}))$ of principal interior and boundary symbols of order $m-j$. Then, $\Psi^{m-j-1}_{\rm gp}(X;v;w)$ is easily seen to coincide with the space of all $\mathcal{A} \in \Psi^{m-j}_{\rm gp}(X;v;w)$ satisfying $\sigma^{m-j}(\mathcal{A}) = 0$.

Theorem 3.6. *Let $\mathcal{A} \in \Psi^m_{\rm gp}(X;v;w)$ and $\sigma(\mathcal{A}) = 0$. Then, $\mathcal{A} \in \Psi^{m-1}_{\rm gp}(X;v;w)$ and the operator (3.12) is compact for all $s \in \mathbb{R}$.*

PROOF. Let us write $\mathcal{A}$ in the form $\mathcal{A} = \tilde{\mathcal{P}}\tilde{\mathcal{A}}\mathcal{E}$ for an $\tilde{\mathcal{A}} \in \Psi^m_s(X;V,\tilde{V};W,\tilde{W};w)$. If we set

$$\tilde{\tilde{\mathcal{A}}} := \begin{pmatrix} I & 0 \\ 0 & \tilde{P} \end{pmatrix} \tilde{\mathcal{A}} \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix},$$

we also get $\mathcal{A} = \tilde{\mathcal{P}}\tilde{\tilde{\mathcal{A}}}\mathcal{E}$, and $\sigma(\mathcal{A}) = 0$ implies $\sigma(\tilde{\tilde{\mathcal{A}}}) = 0$, the latter symbol refers to $\Psi^m_s(X;V,\tilde{V};W,\tilde{W};w)$. This gives us

$$\tilde{\tilde{\mathcal{A}}} \in \Psi^{m-1}_s(X;V,\tilde{V};W,\tilde{W};w),$$

which entails $\mathcal{A} \in \Psi^{m-1}_{\rm gp}(X;v;w)$. The compactness of (3.12) follows from the compactness of $\tilde{\tilde{\mathcal{A}}}$ in usual Sobolev spaces. □

Theorem 3.7. *Let $\mathcal{A}_j \in \Psi_{\rm gp}^{m-j}(X;v;w)$ be a sequence of boundary value problems, such that the ε-weight in the Green operators involved in $\mathcal{A}_j$ does not depend on j. Then there exists an $\mathcal{A} \in \Psi_{\rm gp}^{m}(X;v;w)$, which is unique modulo $\Psi_{{\rm gp},G}^{-\infty}(X;v;w)$, such that*

$$\mathcal{A} \sim \sum_{j=0}^{\infty} \mathcal{A}_j,$$

i.e., $\mathcal{A} - \sum_{j=0}^{N-1} \mathcal{A}_j \in \Psi_{\rm gp}^{m-N}(X;v;w)$ *for all* $N \in \mathbb{N}$.

The proof is an easy consequence of a corresponding result for the operator space $\Psi_s^m(X;V,\tilde{V};W,\tilde{W};w)$.

3.6. Ellipticity under Projection Data

As usual, a boundary value problem $\mathcal{A} \in \Psi_{\rm gp}^{m}(X;v;w)$ is called elliptic if both $\sigma_\Psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$ are isomorphisms.

The condition that (3.13) is an isomorphism does not depend on s. If it is satisfied for an $s_0 \in \mathbb{R}$ then so is for all $s \in \mathbb{R}$.

Let us now show that in contrast to Lopatinskii conditions there is no obstruction for the existence of elliptic global projection conditions.

Theorem 3.8. *Let $a_m \in \mathcal{S}_{\rm hg}^m(T^*X \setminus \{0\}, {\rm Hom}(V,\tilde{V}))$ be an arbitrary elliptic element. Then there is a vector bundle B over X, such that for each $\gamma \in \mathbb{R}$ there are triples $Q = (F,W,P)$, $\tilde{Q} = (\tilde{F},\tilde{W},\tilde{P})$ depending on γ, and an elliptic operator $\mathcal{A} \in \Psi_{\rm gp}^{m}(X;\tilde{v};w)$ with $\tilde{v} = (V \oplus B, \tilde{V} \oplus B; Q, \tilde{Q})$ and $w = (\gamma, \gamma - m)$, satisfying $\sigma_\Psi(\mathcal{A}) = \tilde{a}_m$ in the notation of Theorem 3.2.*

Proof. For notational convenience let us assume that $B = 0$. The construction in the general case with a_m replaced by $\tilde{a}_m$ is completely analogous. According to Theorem 3.2 we find an operator $A_\gamma = {\rm op}_{,\gamma}(a_m)$ in $\Psi_s^m(X;V,\tilde{V};w)$ with the property that

$$\sigma_m(y,\eta) := \sigma_\partial(A_\gamma)(y,\eta) : \; H^{s,\gamma}(\mathbb{R}_+) \otimes V_y \to H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_y$$

is a family of Fredholm operators parametrised by $(y,\eta) \in T^*Y \setminus \{0\}$.

Choose vector bundles F and $\tilde{F}$ over S^*Y, such that $[\tilde{F}] - [F] = {\rm ind}_{S^*Y}\, \sigma_m$. By a familiar property of Fredholm families, there is a

$$g_m \in \mathcal{S}_{{\rm hg},G}^m(T^*Y \setminus \{0\}, \Psi^m(\mathbb{R}_+; V_Y, \tilde{V}_Y; w)),$$

such that under notation (3.8)

$$\begin{array}{rcl} \ker\,(\sigma_m + g_m)\,(y,\eta) & \cong & \tilde{F}_{(y,\eta)}, \\ {\rm coker}\,(\sigma_m + g_m)\,(y,\eta) & \cong & F_{(y,\eta)} \end{array}$$

for all $(y,\eta) \in T^*Y \setminus \{0\}$, independently of the specific choice of s. As usual, we can fill up the family of Fredholm operators $(\sigma_m + g_m)(y,\eta)$ to a family of isomorphisms

$$\begin{pmatrix} \sigma_m + g_m & k_m \\ t_m & 0 \end{pmatrix}(y,\eta) : \begin{array}{c} H^{s,\gamma}(\mathbb{R}_+) \otimes V_y \\ \oplus \\ F_{(y,\eta)} \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_y \\ \oplus \\ \tilde{F}_{(y,\eta)} \end{array}, \qquad (3.14)$$

first for all $(y,\eta) \in S^*Y$ and then for all $(y,\eta) \in T^*Y$ by twisted homogeneity of order m.

To shorten notation, the bundles F and $\tilde{F}$ over S^*Y will be identified with their pull-backs over $T^*Y \setminus \{0\}$ under the canonical projection $(y,\eta) \mapsto (y, \eta/|\eta|)$. Choose any bundles W and $\tilde{W}$ over Y, such that F and $\tilde{F}$ are subbundles of $\pi_Y^* W$ and $\pi_Y^* \tilde{W}$, respectively. From (3.14) we can pass to a homomorphism

$$\begin{pmatrix} \sigma_m + g_m & \tilde{k}_m \\ \tilde{t}_m & 0 \end{pmatrix} : \pi_Y^* \begin{array}{c} H^{s,\gamma}(\mathbb{R}_+) \otimes V_Y \\ \oplus \\ W \end{array} \to \pi_Y^* \begin{array}{c} H^{s-m,\gamma-m}(\mathbb{R}_+) \otimes \tilde{V}_Y \\ \oplus \\ \tilde{W} \end{array} \tag{3.15}$$

by extending k_m to $\tilde{k}_m$ by zero on a complementary bundle $F^\perp$ to F in $\pi_Y^* W$, while $\tilde{t}_m$ is defined by composing t_m with the embedding $\tilde{F} \to \pi_Y^* \tilde{W}$.

In the same way as in the proof of Theorem 3.3 we construct an operator

$$\tilde{\mathcal{A}} \in \Psi_s^m(X; V, \tilde{V}; W, \tilde{W}; w)$$

whose principal boundary symbol just amounts to (3.15). In addition, the projections $\pi_Y^* W \to F$ and $\pi_Y^* \tilde{W} \to \tilde{F}$ along complementary bundles $F^\perp$ of F in $\pi_Y^* W$ and $\tilde{F}^\perp$ of $\tilde{F}$ in $\pi_Y^* \tilde{W}$ can be interpreted as principal symbols of certain projections $P \in \Psi_{\mathrm{cl}}^0(Y, W)$ and $\tilde{P} \in \Psi_{\mathrm{cl}}^0(Y, \tilde{W})$, respectively, cf. Remark 2.1. Then, forming $\mathcal{A}$ by formula $\mathcal{A} = \tilde{\mathcal{P}} \tilde{\mathcal{A}} \mathcal{E}$ yields an elliptic boundary value problem $\mathcal{A} \in \Psi_{\mathrm{gp}}^m(X; v; w)$ for $v = (V, \tilde{V}; Q, \tilde{Q})$ and $Q = (F, W, P)$, $\tilde{Q} = (\tilde{F}, \tilde{W}, \tilde{P})$, satisfying $\sigma_\Psi(\mathcal{A}) = a_m$. □

To some extent, elliptic problems with global projection conditions are complemented to Lopatinskii elliptic boundary value problems.

Theorem 3.9. *For any elliptic boundary value problem $\mathcal{A} \in \Psi_{\mathrm{gp}}^m(X; v_\mathcal{A}; w)$ with $v_\mathcal{A} = (V, \tilde{V}; Q_\mathcal{A}, \tilde{Q}_\mathcal{A})$ there is an elliptic boundary value problem $\mathcal{B} \in \Psi_{\mathrm{gp}}^m(X; v_\mathcal{B}; w)$ with $v_\mathcal{B} = (\tilde{V}, V; Q_\mathcal{B}, \tilde{Q}_\mathcal{B})$, such that $\mathcal{A} \oplus \mathcal{B} \in \Psi_s^m(X; v; w)$ for $v = (V \oplus \tilde{V}; \mathbb{C}^N)$ is Lopatinskii elliptic.*

Proof. The upper left corner A of $\mathcal{A}$ belongs to $\Psi_s^m(X; V, \tilde{V}; w)$. Its formal adjoint A^* is an element of $\Psi_s^m(X; \tilde{V}, V; w^*)$ for $w^* = (-\gamma + m, -\gamma)$. The definition of A^* is based on the relation

$$(Au, g)_{H^{0,0}(X,\tilde{V})} = (u, A^* g)_{H^{0,0}(X,V)}$$

for all $u \in C^\infty(X, V)$ and $g \in C^\infty(X, \tilde{V})$ of compact support in the interior of X. This is compatible with the pointwise formal adjoint on the level of principal boundary symbols

$$(\sigma_\partial(A)(y,\eta)u, g)_{H^{0,0}(\mathbb{R}_+, \mathbb{C}^{\tilde{k}})} = (u, \sigma_\partial(A^*)(y,\eta)g)_{H^{0,0}(\mathbb{R}_+, \mathbb{C}^k)},$$

k and $\tilde{k}$ being the ranks of V and $\tilde{V}$, respectively. The symbol $\sigma_\partial(A^*)$ defines a bundle homomorphism $\pi_Y^* H^{s,-\gamma+m}(\mathbb{R}_+) \otimes \tilde{V}_Y \to \pi_Y^* H^{s-m,-\gamma}(\mathbb{R}_+) \otimes V_Y$ which is Fredholm for all $s \in \mathbb{R}$, and

$$\mathrm{ind}_{S^*Y}\, \sigma_\partial(A^*) = - \mathrm{ind}_{S^*Y}\, \sigma_\partial(A).$$

Pick a sufficiently large $N \in \mathbb{N}$, such that both $F_\mathcal{A}$ and $\tilde{F}_\mathcal{A}$ have complementary bundles $F_\mathcal{B}$ and $\tilde{F}_\mathcal{B}$ in $S^*Y \times \mathbb{C}^N$, i.e.,

$$\begin{array}{rcl} F_\mathcal{A} \oplus F_\mathcal{B} &=& S^*Y \times \mathbb{C}^N, \\ \tilde{F}_\mathcal{A} \oplus \tilde{F}_\mathcal{B} &=& S^*Y \times \mathbb{C}^N. \end{array}$$

Then,

$$\operatorname{ind}_{S^*Y} \sigma_\partial(A^*) = [\tilde{F}_\mathcal{B}] - [F_\mathcal{B}].$$

By [13], we have order and weight reducing isomorphisms

$$\begin{array}{rrcl} D_V^{m-2\gamma}: & H^{s,-\gamma}(X,V) & \to & H^{s-m+2\gamma,\gamma-m}(X,V), \\ D_{\tilde{V}}^{m-2\gamma}: & H^{s,-\gamma+m}(X,\tilde{V}) & \to & H^{s-m+2\gamma,\gamma}(X,\tilde{V}). \end{array}$$

which are continuous for all $s \in \mathbb{R}$. Using them we pass from A^* to the operator $B := D_V^{m-2\gamma} A^* \, (D_{\tilde{V}}^{m-2\gamma})^{-1}$ which obviously belongs to $\Psi_s^m(X;\tilde{V},V;w)$ and has the property

$$\operatorname{ind}_{S^*Y} \sigma_\partial(B) = \operatorname{ind}_{S^*Y} \sigma_\partial(A^*).$$

As in Theorem 3.8 we find an element $g_m \in \mathcal{S}^m_{\mathrm{hg},G}(T^*Y\setminus\{0\}, \Psi^m(\mathbb{R}_+;\tilde{V}_Y,V_Y;w))$, such that

$$\begin{array}{rcl} \ker\left(\sigma_\partial(B) + g_m\right)(y,\eta) & \cong & \tilde{F}_{\mathcal{B},(y,\eta)}, \\ \operatorname{coker}\left(\sigma_\partial(B) + g_m\right)(y,\eta) & \cong & F_{\mathcal{B},(y,\eta)} \end{array}$$

for all $(y,\eta) \in T^*Y \setminus \{0\}$. Set

$$\begin{array}{rcl} Q_\mathcal{B} & = & (F_\mathcal{B}, Y\times\mathbb{C}^N, P_\mathcal{B}), \\ \tilde{Q}_\mathcal{B} & = & (\tilde{F}_\mathcal{B}, Y\times\mathbb{C}^N, \tilde{P}_\mathcal{B}), \end{array}$$

where $P_\mathcal{B}$ and $\tilde{P}_\mathcal{B}$ are pseudodifferential projections of $\Psi^0_{\mathrm{cl}}(Y;\mathbb{C}^N)$, whose principal symbols are the projections $Y\times\mathbb{C}^N \to F_\mathcal{B}$ and $Y\times\mathbb{C}^N \to \tilde{F}_\mathcal{B}$ along $F_\mathcal{A}$ and $\tilde{F}_\mathcal{A}$, respectively. Analysis similar to that in the proof of Theorem 3.8 then gives us an elliptic operator $\mathcal{B} \in \Psi^m_{\mathrm{gp}}(X;v_\mathcal{B};w)$ with the desired properties. □

The boundary value problem $\mathcal{A}$ can be recovered from $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathcal{B}$ by the formula $\mathcal{A} = \tilde{\mathcal{P}}_\mathcal{A}\,\tilde{\mathcal{A}}\,\mathcal{E}_\mathcal{A}$ with

$$\mathcal{E}_\mathcal{A} = \begin{pmatrix} I & 0 \\ 0 & E_\mathcal{A} \end{pmatrix}, \qquad \tilde{\mathcal{P}}_\mathcal{A} = \begin{pmatrix} I & 0 \\ 0 & \tilde{P}_\mathcal{A} \end{pmatrix},$$

where $E_\mathcal{A}$ is the canonical embedding $\mathcal{H}^s(Y,Q_\mathcal{A}) \hookrightarrow H^s(Y,\mathbb{C}^N)$, and similarly for $\mathcal{B}$.

Let $\mathcal{A} \in \Psi^m_{\mathrm{gp}}(X;v;w)$ where $v = (V,\tilde{V};Q,\tilde{Q})$ and $w = (\gamma,\gamma-m)$. An operator $\Pi \in \Psi^{-m}_{\mathrm{gp}}(X;v^{-1};w^{-1})$ with $v^{-1} = (\tilde{V},V;\tilde{Q},Q)$ and $w^{-1} = (\gamma-m,\gamma)$ is called a parametrix of $\mathcal{A}$ if

$$\begin{array}{rcl} \Pi\mathcal{A} - I & \in & \Psi^{-\infty}_{\mathrm{gp},G}\left(X;v^{-1}\circ v;w^{-1}\circ w\right), \\ \mathcal{A}\Pi - I & \in & \Psi^{-\infty}_{\mathrm{gp},G}\left(X;v\circ v^{-1};w\circ w^{-1}\right). \end{array} \tag{3.16}$$

Theorem 3.10. *Every elliptic boundary value problem $\mathcal{A} \in \Psi^m_{\mathrm{gp}}(X;v;w)$ possesses a parametrix $\Pi \in \Psi^{-m}_{\mathrm{gp}}(X;v^{-1};w^{-1})$.*

Proof. Let us apply Theorem 3.9 to $\mathcal{A}$ and form $\tilde{\mathcal{A}} = \mathcal{A}\oplus\mathcal{B} \in \Psi^m_s(X;\tilde{v};w)$ with some complementary elliptic operator $\mathcal{B}$. By [13], $\tilde{\mathcal{A}}$ has a parametrix $\tilde{\mathcal{P}} \in \Psi^{-m}_s(X;\tilde{v}^{-1};w^{-1})$, where $\sigma(\tilde{\mathcal{P}}) = \sigma(\tilde{\mathcal{A}})^{-1}$. Define a soft left parametrix for $\mathcal{A}$ by

$$\Pi_0 = \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} \tilde{\mathcal{P}} \begin{pmatrix} I & 0 \\ 0 & \tilde{E} \end{pmatrix}$$

where $\tilde{E} : \mathcal{H}^{s-m}(Y,\tilde{Q}) \hookrightarrow H^{s-m}(Y,\tilde{W})$ is the canonical embedding involved in $\tilde{Q}$, and $P \in \Psi^0_{\mathrm{cl}}(Y,W)$ the projection involved in Q. Then we get

$$\varPi_0 \mathcal{A} = \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} \tilde{\mathcal{P}} \begin{pmatrix} I & 0 \\ 0 & \tilde{P} \end{pmatrix} \tilde{\mathcal{A}} \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix}.$$

It follows that the remainder $\mathcal{S}_l = I - \varPi_0 \mathcal{A}$ belongs to $\Psi^0_{\mathrm{gp}}(X; v^{-1} \circ v; w^{-1} \circ w)$ and satisfies $\sigma(\mathcal{S}_l) = 0$. By Theorem 3.6 we deduce that $\mathcal{S}_l \in \Psi^{-1}_{\mathrm{gp}}(X; v^{-1} \circ v; w^{-1} \circ w)$. Applying Theorem 3.7 we find an operator $\mathcal{C}_l \in \Psi^{-1}_{\mathrm{gp}}(X; v^{-1} \circ v; w^{-1} \circ w)$ satisfying $(I + \mathcal{C}_l)(I - \mathcal{S}_l) = I$ modulo $\Psi^{-\infty}_{\mathrm{gp},G}(X; v^{-1} \circ v; w^{-1} \circ w)$. To do this, it suffices to form the asymptotic sum

$$\mathcal{C}_l := \sum_{j=1}^{\infty} \mathcal{S}_l^j.$$

This immediately yields $(I + \mathcal{C}_l)\varPi_0 \mathcal{A} = 1$ modulo $\Psi^{-\infty}_{\mathrm{gp},G}(X; v^{-1} \circ v; w^{-1} \circ w)$, and therefore

$$\begin{array}{rcl} \varPi_l & := & (I + \mathcal{C}_l)\, \varPi_0 \\ & \in & \Psi^{-m}_{\mathrm{gp}}(X; v^{-1}; w^{-1}) \end{array}$$

is a left parametrix of $\mathcal{A}$. In a similar manner we find a right parametrix, and so we may take $\varPi = \varPi_l$. □

As usual, the existence of a parametrix implies the Fredholm property of elliptic problems with global projection conditions.

Theorem 3.11. *Let $\mathcal{A} \in \mathcal{S}^m_{\mathrm{gp}}(X; v; w)$ be elliptic. Then*

$$\mathcal{A} : \begin{array}{c} H^{s,\gamma}(X,V) \\ \oplus \\ \mathcal{H}^s(Y,Q) \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(X,\tilde{V}) \\ \oplus \\ \mathcal{H}^{s-m}(Y,\tilde{Q}) \end{array}$$

is a Fredholm operator for all $s \in \mathbb{R}$, cf. (2.7). Moreover, the null-space of $\mathcal{A}$ is independent of s as well as the codimension of the range of $\mathcal{A}$, i.e., $\operatorname{ind} \mathcal{A}$ *is independent of s.*

The parametrix $\varPi$ of Theorem 3.10 can be chosen in such a way that the smoothing remainders are projections of finite rank. In fact, $I - \varPi\mathcal{A}$ projects onto $\ker \mathcal{A}$ while $I - \mathcal{A}\varPi$ projects onto a complement of $\operatorname{im} \mathcal{A}$, for every s.

Proof. The Fredholm property is a direct consequence of the fact that the remainders $I - \varPi\mathcal{A}$ and $I - \mathcal{A}\varPi$ in (3.16) are compact operators, which is due to Theorem 3.6. The second part of Theorem 3.11 is a consequence of general facts on elliptic operators that are always satisfied when we have elliptic regularity in the respective scales of spaces. □

As a converse statement for Theorem 3.11 we prove that ellipticity is not only sufficient but also necessary for the Fredholm property.

Theorem 3.12. *Suppose $\mathcal{A} \in \Psi^0_{\mathrm{gp}}(X; v; w)$ for $v = (V, \tilde{V}; Q, \tilde{Q})$ and $w = (0, 0)$. If the operator*

$$\mathcal{A} : \begin{array}{c} L^2(X,V) \\ \oplus \\ \mathcal{H}^0(Y,Q) \end{array} \to \begin{array}{c} L^2(X,\tilde{V}) \\ \oplus \\ \mathcal{H}^0(Y,\tilde{Q}) \end{array} \tag{3.17}$$

is Fredholm, then $\mathcal{A}$ is elliptic.

Proof. Write

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix}$$

in (3.17) and set $Q^{\perp} = (\sigma_{\Psi}(I-P)W, W, I-P)$. Then

$$L^2(Y,W) = \mathcal{H}^0(Y,Q) \oplus \mathcal{H}^0(Y,Q^{\perp})$$

and we define $\mathcal{B} \in \Psi_s^0(X;V,\tilde{V};W,\tilde{W}\oplus W;(0,0))$ by

$$\mathcal{B}: \begin{array}{c} L^2(X,V) \\ \oplus \\ L^2(Y,W) \end{array} \cong \begin{array}{c} L^2(X,V) \\ \oplus \\ \mathcal{H}^0(Y,Q) \\ \oplus \\ \mathcal{H}^0(Y,Q^{\perp}) \end{array} \xrightarrow{\mathcal{C}} \begin{array}{c} L^2(X,\tilde{V}) \\ \oplus \\ \mathcal{H}^0(Y,\tilde{Q}) \\ \oplus \\ \mathcal{H}^0(Y,Q^{\perp}) \end{array} \hookrightarrow \begin{array}{c} L^2(X,\tilde{V}) \\ \oplus \\ L^2(Y,\tilde{W}) \\ \oplus \\ L^2(Y,W) \end{array} \cong \begin{array}{c} L^2(X,\tilde{V}) \\ \oplus \\ L^2(Y,\tilde{W}\oplus W) \end{array},$$

where the mapping $\mathcal{C}$ is given by

$$\mathcal{C} = \begin{pmatrix} A & K & 0 \\ T & Q & 0 \\ 0 & 0 & I \end{pmatrix}.$$

It is clear that $\dim\ker\mathcal{B} = \dim\ker\mathcal{A} < \infty$. Moreover,

$$\begin{aligned} \ker\mathcal{B}^*\mathcal{B} &= \ker\mathcal{B} \\ &= (\operatorname{im}\mathcal{B}^*\mathcal{B})^{\perp} \end{aligned}$$

and $\mathcal{B}^*\mathcal{B}$ has closed range, for $\mathcal{C}^*\mathcal{C}$ has. It follows that $\mathcal{B}^*\mathcal{B} \in \Psi_s^0(X;V;W;(0,0))$ is a Fredholm operator. By the above, $\mathcal{B}^*\mathcal{B}$ is an elliptic element of the calculus. This implies that both $\sigma_{\Psi}(\mathcal{A})$ and $\sigma_{\partial}(\mathcal{A})$ are injective. By passing to adjoint operators we can show in an analogous manner that the symbols $\sigma_{\Psi}(\mathcal{A})$ and $\sigma_{\partial}(\mathcal{A})$ are also surjective. □

3.7. Operators of Order Zero

Here we study operators $A \in \Psi_s^0(X;V,\tilde{V};(0,0))$ and associated boundary symbols in more detail and prove Theorems 3.1 and 3.2. Note that by setting

$$A \mapsto D_{\tilde{V}}^{\gamma-m} A D_V^{-\gamma}$$

one obtains an isomorphism $\Psi_s^m(X;V,\tilde{V};(\gamma,\gamma-m)) \to \Psi_s^0(X;V,\tilde{V};(0,0))$.

A direct computation shows that for every $\tilde{A} \in \Psi_{\mathrm{cl}}^0(2X;V,\tilde{V})$ the operator $r^+\tilde{A}e^+$ belongs to $\Psi_s^0(X;V,\tilde{V};(0,0))$. Moreover, for any $A \in \Psi_s^0(X;V,\tilde{V};(0,0))$ there exists an operator $\tilde{A} \in \Psi_{\mathrm{cl}}^0(2X;V,\tilde{V})$, such that $A = r^+\tilde{A}e^+ + M + G$ holds for suitable $M+G \in \Psi_{M+G}^0(X;V,\tilde{V};(0,0))$. For the principal boundary symbol of A we actually have

$$\sigma_{\partial}(A)(y,\eta) = r^+\tilde{a}_0(y,0,\eta,D_r)e^+ + \sigma_{\partial}(M+G)(y,\eta):\ L^2(\mathbb{R}_+)\otimes V_y \to L^2(\mathbb{R}_+)\otimes\tilde{V}_y, \quad (3.18)$$

where $\tilde{a}_0$ is the principal homogeneous symbol of $\tilde{A}$.

Note that in contrast to the usual domain of $\sigma_\partial(A)$ we now prefer L^2-spaces, because in the case of violated transmission property the standard Sobolev spaces or Schwartz spaces with smoothness up to the boundary do not survive under the action of pseudodifferential operators.

Set $S^*_Y X := S^*X\,|_Y$ and denote by $\mathcal{S}^0_{\rm hg}(S^*_Y X, \mathrm{Hom}(V_Y, \tilde{V}_Y))$ the space of all restrictions $a\,|_{S^*_Y X}$ for $a \in \mathcal{S}^0_{\rm hg}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V}))$.

Given any $A \in \Psi^0_s(X; V, \tilde{V}; (0,0))$, such that $\sigma_\Psi(A) \in \mathcal{S}^0_{\rm hg}(T^*X \setminus \{0\}, \mathrm{Hom}(V, \tilde{V}))$ is elliptic, we consider

$$a := \sigma_\Psi(A)\,|_{S^*_Y X}$$

and ask whether the family

$$\mathrm{op}^+(a)(y,\eta) = r^+ a(y,\eta,D_r) e^+ :\ L^2(\mathbb{R}_+) \otimes V_y \to L^2(\mathbb{R}_+) \otimes \tilde{V}_y \tag{3.19}$$

is Fredholm for all $(y,\eta) \in S^*Y$.

Write N for the $[-1,1]$-bundle over Y induced by the conormal bundle of Y, i.e., N is a trivial bundle whose fibres are intervals $[-1,1]$ connecting the south pole $(\eta,\varrho) = (0,-1)$ with the north pole $(\eta,\varrho) = (0,1)$ of $S^*_y X$, where y varies over all of Y.

Let us recall a criterion for the Fredholm property of (3.19) in terms of Mellin symbols

$$g^\pm(z) = \frac{1}{1 - e^{\mp 2\pi z}},$$

the functions $g^\pm(z)$ being meromorphic in $z \in \mathbb{C}$ with simple poles at the points $\imath j$, where $j \in \mathbb{Z}$. Thus the lines $\Gamma_\gamma = \{z \in \mathbb{C} : \Im z = \gamma\}$ do not contain poles provided that $\gamma \notin \mathbb{Z}$.

Choose a diffeomorphism $z : (-1,1) \to \Gamma_{1/2}$ with the property that $\Re z(\varrho) \to \pm\infty$ for $\varrho \to \pm 1$. Setting $a^\pm(y) := a(y,0,\pm 1)$ we introduce a family of homomorphisms $V_Y \to \tilde{V}_Y$ by

$$\tilde{a}(y,\varrho) := a^+(y) g^+(z(\varrho)) + a^-(y) g^-(z(\varrho)). \tag{3.20}$$

This is well defined for all $-1 \le \varrho \le 1$, since $g^+(z) + g^-(z) = 1$ and $g^\pm(z)$ tends to 1 when $\Re z \to \pm\infty$ along the line $\Gamma_{1/2}$.

More precisely, the family (3.20) is a convex combination of the homomorphisms $a^\pm(y) : V_y \to \tilde{V}_y$.

Theorem 3.13. *The operators (3.19) are Fredholm for all $(y,\eta) \in S^*Y$ if and only if*

$$\tilde{a}(y,\eta,\varrho) = \begin{cases} a^+(y) g^+(z(\varrho)) + a^-(y) g^-(z(\varrho)), & \text{for} \quad \eta = 0,\ \varrho \in [-1,1], \\ \sigma_\Psi(A)\,|_{S^*_Y X}, & \text{for} \quad |\eta,\varrho| = 1, \end{cases} \tag{3.21}$$

is a family of isomorphisms $V_y \to \tilde{V}_y$ for all $(y,\eta,\varrho) \in S^*_Y X \cup N$.

Theorem 3.13 is known from the theory of singular integral operators, cf. [14]. An explicit proof of the necessity may be found in [15].

Mention that when $\mathrm{op}^+(a)$ stems from a symbol $\sigma_\Psi(A)$ with the transmission property, we have $a^+(y) = a^-(y)$, and hence the criterion of Theorem 3.13 is automatically satisfied as soon as $\sigma_\Psi(A)$ is elliptic.

In general, each family of isomorphisms (3.21) represents an element $\sigma(\tilde{a})$ in the relative K-group of the pair $(B^*_Y X, S^*_Y X \cup N)$, where B^*X is the unit coball bundle of X and $B^*_Y X = B^*X\,|_Y$.

By $K(B^*_Y X, S^*_Y X \cup N) \cong K(\mathbb{R}^2 \times S^*Y)$ and the Bott periodicity theorem there is an isomorphism

$$\iota:\ K(B^*_Y X, S^*_Y X \cup N) \to K(S^*Y).$$

Theorem 3.14. *Let $\sigma_\Psi(A)$ be elliptic of order 0. Suppose $\sigma_\Psi(A)\,|_{S^*_Y X}$ extends to a family of isomorphisms (3.21) on $S^*_Y X \cup N$, and $\sigma(\tilde{a}) \in K(B^*_Y X, S^*_Y X \cup N)$ is the associated element. Then, the equality $\operatorname{ind}_{S^*Y} \operatorname{op}^+(a) = \iota(\sigma(\tilde{a}))$ holds for $a = \sigma_\Psi(A)\,|_{T^*_Y X \setminus \{0\}}$.*

For symbols with the transmission property Theorem 3.14 goes at least as far as [9]. A related statement for symbols of elliptic differential operators is owed to [1]. The general case not assuming the transmission property is treated in [15].

It is clear that any other extension $\tilde{\tilde{a}} : V_Y \to \tilde{V}_Y$ of the symbol $\sigma_\Psi(A)\,|_{S^*_Y X}$ to $S^*_Y X \cup N$ also represents an element $\sigma(\tilde{\tilde{a}}) \in K(B^*_Y X, S^*_Y X \cup N)$ and hence a certain $\iota(\sigma(\tilde{\tilde{a}})) \in K(S^*Y)$. It is not obvious at first glance how $\iota(\sigma(\tilde{\tilde{a}}))$ can be interpreted as $\operatorname{ind}_{S^*Y} \sigma$ for a family $\sigma(y,\eta) :\ L^2(\mathbb{R}_+) \otimes V_y \to L^2(\mathbb{R}_+) \otimes \tilde{V}_y$ of Fredholm operators parametrised by $(y,\eta) \in S^*Y$. But the pointwise analytic information from [14] combined with that on the structure of pseudodifferential boundary value problems not requiring the transmission property from [15] gives us the following scenario. Let $\mathcal{F}(V_Y, \tilde{V}_Y)$ denote the set of all families of homomorphisms $V_y \to \tilde{V}_y$, continuously parametrised by $(y,\eta,\varrho) \in S^*_Y X \cup N$, that vanish on $S^*_Y X$. Every element of $\mathcal{F}(V_Y, \tilde{V}_Y)$ can be canonically identified with a continuous family of homomorphisms, parametrised by $(y,\varrho) \in N = Y \times [-1,1]$, vanishing on $Y \times \partial[-1,1]$. We then have $\tilde{a}^{-1}\tilde{\tilde{a}}\,(y,\eta,\varrho) = 1 + f(y,\varrho)$ for some $f \in \mathcal{F}(V_Y, \tilde{V}_Y)$, or

$$\begin{aligned} \tilde{\tilde{a}}(y,\eta,\varrho) &= \tilde{a}(y,\eta,\varrho)\,(1 + f(y,\varrho)) \\ &= \tilde{a}(y,\eta,\varrho) + \tilde{f}(y,\varrho) \end{aligned}$$

for an $\tilde{f} \in \mathcal{F}(V_Y, \tilde{V}_Y)$. It suffices to consider elements $\tilde{\tilde{a}}$ of the above kind, such that the pull-back of $\tilde{f}(y,\varrho)$ under $\varrho = \varrho(z)$ is a Schwartz function of $z \in \Gamma_{1/2}$. In fact, we can obviously construct such an $\tilde{\tilde{a}}$ starting with an arbitrary family $\tilde{\tilde{\tilde{a}}}$ of isomorphisms, satisfying $\tilde{a} - \tilde{\tilde{\tilde{a}}} \in \mathcal{F}(V_Y, \tilde{V}_Y)$, by a small change of $\tilde{\tilde{\tilde{a}}}\,|_N$ near $Y \times \partial[-1,1]$ within the homotopy class of families of isomorphisms represented by $\tilde{\tilde{\tilde{a}}}$. We then obtain $\sigma(\tilde{\tilde{\tilde{a}}}) = \sigma(\tilde{\tilde{a}})$ and hence $\iota\,\sigma(\tilde{\tilde{\tilde{a}}}) = \iota\,\sigma(\tilde{\tilde{a}})$.

Using the spaces $C^\infty(U, M^m(\Gamma_{1/2}, \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^{\tilde{k}})))$ as local models, it is straightforward to define spaces $M^m(Y \times \Gamma_{1/2}, \operatorname{Hom}(V_Y, \tilde{V}_Y))$ for vector bundles V_Y and $\tilde{V}_Y$ over Y.

Theorem 3.15. *Let $\sigma_\Psi(A)$ be elliptic of order zero and $a(y,\xi)$ the restriction of $\sigma_\Psi(A)$ to $T^*_Y X \setminus \{0\}$. Suppose m is an element of $M^{-\infty}(Y \times \Gamma_{1/2}, \operatorname{Hom}(V_Y, \tilde{V}_Y))$, such that*

$$\tilde{\tilde{a}}(y,\eta,\varrho) = \begin{cases} a^+(y)g^+(z(\varrho)) + a^-(y)g^-(z(\varrho)) + m(y, z(\varrho)), & \text{if } \eta = 0,\ \varrho \in [-1,1], \\ \sigma_\Psi(A)\,|_{S^*_Y X}, & \text{if } |\eta,\varrho| = 1, \end{cases} \tag{3.22}$$

defines a family of isomorphisms $V_y \to \tilde{V}_y$ for all $(y,\eta,\varrho) \in S^*_Y X \cup N$. Then, for arbitrary cut-off functions $\omega(r)$ and $\tilde{\omega}(r)$,

$$\operatorname{op}^+(a)(y,\eta) + \omega(r|\eta|)\operatorname{op}_M(m)\tilde{\omega}(r|\eta|) :\ L^2(\mathbb{R}_+) \otimes V_y \to L^2(\mathbb{R}_+) \otimes \tilde{V}_y$$

is a family of Fredholm operators parametrised by $(y,\eta) \in T^*Y \setminus \{0\}$*, and for its restriction to* S^*Y *we have* $\mathrm{ind}_{S^*Y}(\cdot) = \imath\, \sigma(\tilde{\tilde{a}})$.

This theorem generalises Theorems 3.13 and 3.14. The Fredholm property is shown in [14] in a slightly modified form without $\tilde{\omega}$. The present formulation is given in [12].

Proof of Theorem 3.1. It suffices to treat the case $m = \gamma = 0$. Indeed, the reduction to order and weight zero as at the beginning of § 3.7 can also be done on the level of interior and boundary symbols. In other words, we can first pass to a symbol of order zero by setting $a_0 = \sigma_\Psi(D_{\tilde{V}}^{\gamma-m}) a_m \sigma_\Psi(D_V^{-\gamma})$, carry out our construction that yields a Fredholm family $\sigma_0(y,\eta)$ as asserted in (3.8), where it is sufficient to consider

$$\sigma_0(y,\eta): \; L^2(\mathbb{R}_+) \otimes V_y \to L^2(\mathbb{R}_+) \otimes \tilde{V}_y.$$

Then we may set $\sigma_m(y,\eta) := \sigma_\partial(D_{\tilde{V}}^{-\gamma+m})(y,\eta)\, \sigma_0(y,\eta)\, \sigma_\partial(D_V^{\gamma})(y,\eta)$. Since the boundary symbol can be represented in the form (3.18), it suffices to show that $a_0(x,\xi)\,|_{S_Y^*X}$ for an elliptic principal symbol $a_0 : \pi_X^* V \to \pi_X^* \tilde{V}$ admits an extension to an isomorphism

$$\tilde{a}: \; \pi^*_{S_Y^*(X) \cup N} V_Y \to \pi^*_{S_Y^*(X) \cup N} \tilde{V}_Y, \tag{3.23}$$

where $\pi_{S_Y^*(X) \cup N}: \; S_Y^*(X) \cup N \to Y$ stands for the canonical projection. In fact, having granted this, we apply an approximation argument of [15] to obtain an element

$$m(y,z) \in M^{-\infty}(Y \times \Gamma_{1/2}, \mathrm{Hom}(V_Y, \tilde{V}_Y)),$$

such that (3.22) with $\sigma_\Psi(A)\,|_{S_Y^*X}$ replaced by $a_0\,|_{S_Y^*X}$ is also an extension of $a_0\,|_{S_Y^*X}$ to an isomorphism over all of $S_Y^*X \cup N$, which is homotopic to $\tilde{a}$ through isomorphisms. By assumption, there is a nowhere vanishing vector field v on Y. Without loss of generality we can assume that $|v(y)| = 1$ for all $y \in Y$. Pick an isomorphism $TY \to T^*Y$. It induces a diffeomorphism $\Delta : SY \to S^*Y$ between the respective unit sphere bundles. Consider the composition $\Delta \circ v : Y \to S^*Y$. For every $y \in Y$ there is a unique half-circle $\tilde{N}_y$ on S_y^*X containing the points $\Delta \circ v\,(y)$ and $(y,0,0,\pm 1)$, north and south poles of the sphere. This yields a trivial bundle $\tilde{N}$ on Y with fibre $\tilde{N}_y$ over y. There is a projection of S_Y^*X to the conormal bundle N, given by $(y,0,\eta,\varrho) \mapsto (y,\varrho)$, which induces an isomorphism $h : \tilde{N} \to N$ as fibre bundles in the set-up of fibre homeomorphisms. To construct an extension of $a_0\,|_{S_Y^*X}$ to an isomorphism (3.23) it suffices to set $\tilde{a}(y,\varrho) := a_0(y,0,\tilde{\eta},\tilde{\varrho})$, for $h_y(\tilde{\eta},\tilde{\varrho}) = \varrho$. □

Proof of Theorem 3.2. Similarly to the preceding proof it suffices to consider the case of any fixed order $m \in \mathbb{R}$ and $\gamma = 0$. In the present case it is convenient to take $m = 1$. Let $a_1 \in \mathcal{S}^1_{\mathrm{hg}}(T^*X \setminus \{0\}, \mathrm{Hom}(V,\tilde{V}))$ be elliptic. Set $a_1' := a_1\,|_{T_Y^*X}$, thus obtaining a symbol in $\mathcal{S}^1_{\mathrm{hg}}(T_Y^*X \setminus \{0\}, \mathrm{Hom}(V_Y,\tilde{V}_Y))$. Using a familiar difference construction we get an element $[a_1'] \in K(T_Y^*X)$, the latter group just amounts to $K(T^*Y \times \mathbb{R})$. Every element in $K(T^*Y \times \mathbb{R})$ can be represented by a homomorphism

$$\sigma(y,\eta) + \imath\varrho : \; B \to B, \tag{3.24}$$

with B a smooth vector bundle on $T^*Y \times \mathbb{R}$ whose restriction to T^*Y is $\pi_Y^* B_Y$ for a vector bundle B_Y on Y, and $\sigma : \pi_Y^* B_Y \to \pi_Y^* B_Y$ a self-adjoint elliptic symbol of order 1 on Y, cf. [5, III]. Since $\sigma(y,\eta)$ is elliptic, (3.24) is an isomorphism between corresponding fibres for $\varrho = 0$. Moreover, since $\sigma(y,\eta)$ is self-adjoint, all eigenvalues are real. Hence, (3.24) is an isomorphism for all $\varrho \in \mathbb{R}$. Passing to stabilisations of a_1' and (3.24), we see that for a suitable $M \in \mathbb{N}$ the homomorphism $a_1' \oplus I_{\mathbb{C}^M}$ between the pull-backs of $V_Y \oplus \mathbb{C}^M$ and $\tilde{V}_Y \oplus \mathbb{C}^M$ to $S_Y^* X$ has an extension to an isomorphism $\tilde{a} : \pi^*_{S_Y^* X \cup N}(V_Y \oplus \mathbb{C}^M) \to \pi^*_{S_Y^* X \cup N}(\tilde{V}_Y \oplus \mathbb{C}^M)$. Similarly to the proof of Theorem 3.1 we find an element $m(y,z) \in M^{-\infty}(Y \times \Gamma_{1/2}, \mathrm{Hom}(V_Y \oplus \mathbb{C}^M, \tilde{V}_Y \oplus \mathbb{C}^M))$, such that (3.22) with $\sigma_\Psi(A)\,|_{S_Y^* X}$ replaced by $a_1' \oplus I_{\mathbb{C}^M}\,|_{S_Y^* X}$ defines an extension of $a_1' \oplus I_{\mathbb{C}^M}\,|_{S_Y^* X}$ to an isomorphism over all of $S_Y^* X \cup N$, homotopic to $\tilde{a}$ through isomorphisms. By analogy with Theorem 3.15 we now form

$$\begin{array}{rcl} \mathrm{op}^+(a_1)(y,\eta) + \omega(r|\eta|)r^{-1}\,\mathrm{op}_M(m)(y)\,\tilde{\omega}(r|\eta|) & : & H^{1,0}(\mathbb{R}_+) \otimes (V_y \oplus \mathbb{C}^M) \\ & \to & H^{0,-1}(\mathbb{R}_+) \otimes (\tilde{V}_y \oplus \mathbb{C}^M). \end{array}$$

To complete the proof, it suffices to apply a reduction of order and weight in much the same way as in the proof of Theorem 3.1. □

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