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Multi-Logarithmic Differential Forms on Complete Intersections

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We construct a complex $\Omega_S^\bullet(\log C)$ of sheaves of multi-logarithmic differential forms on a complex analytic manifold S with respect to a reduced complete intersection $C \subset S$, and define the residue map as a natural morphism from this complex onto the Barlet complex ω_C^\bullet of regular meromorphic differential forms on C . It follows then that sections of the Barlet complex can be regarded as a generalization of the residue differential forms defined by Leray. Moreover, we show that the residue map can be described explicitly in terms of certain integration current.

Keywords: complete intersection, multi-logarithmic differential forms, regular meromorphic differential forms, Poincaré residue, logarithmic residue, Grothendieck duality, residue current.

Introduction

In 1887, Henri Poincaré [21] introduced the notion of residue 1-form associated with rational 2-forms on \mathbf{C}^2 . In the papers of G. de Rham, [23], and J. Leray, [18], this notion was extended to the class of d -closed meromorphic q -form ω having poles of the first order along a smooth divisor. They proved that the residue of such a form is holomorphic. Inspired by the ideas of Leray and P. Dolbeault, then M. Herrera, P. Liberman, N. Coleff and others have developed theory of the integral representation and residue currents (see [7], [8], [6]).

The next stage of the development of the theory started in early '70's. It is closely related with the general theory of logarithmic differential forms. In fact, forms of such type were first considered in 1972 in a work by J.-B. Poly [22] who defined and studied the Leray residue for *semi-meromorphic* differential forms. He had proved that the Leray residue is well defined for any (not necessarily d -closed) semi-meromorphic differential form ω , as soon as ω and $d\omega$ have a simple pole along a hypersurface. Somewhat later, K. Saito investigated the class of *meromorphic* differential forms ω satisfying these conditions; in a series of his works,

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these forms were called *logarithmic* along a hypersurface (see [24], [25]). Saito established the basic properties of logarithmic differential forms and their residues, and considered some important and interesting applications. It should be also remarked that the same notion has occurred also in the book by Ph. Griffiths and J. Harris [12], with no references to previous works; in this book, the authors use this notion only in the case of divisors which are normal crossings of smooth hypersurfaces. From another point of view this case has also been treated in earlier works by P. Deligne, J. Steenbrink, F. Pham, and many others.

Almost at the same time, in 1976, D. Barlet introduced and investigated a complex ω_X^\bullet of subsheaves of the sheaves of meromorphic differential forms on an analytic subvariety X of a complex manifold (see [3], [4]). If $\dim X = n$, then the sheaf ω_X^n is naturally isomorphic to the Grothendieck dualizing module. In a purely algebraic context, a similar notion also appeared in the works by E. Kunz [16], [17], M. Kersken [15], and others on general duality theory. They call ω_X^q the sheaf of *regular meromorphic* differential forms of degree q . In a joint paper by G. Henkin and M. Passare an analytic interpretation of regular meromorphic forms is given (see [14]). Some of their results are essentially deduced from the basic properties of weakly holomorphic functions, established in a series of earlier works of the second author and his collaborators (see [28], [29], [27]).

In the present paper, we discuss in detail the notion of a multi-logarithmic differential form introduced in [2]. In fact, a meromorphic multi-logarithmic differential form has poles along a divisor $D = \bigcup_{i=1}^k D_i$ such that $C = \bigcap_{i=1}^k D_i$ is a reduced complete intersection. Our main theorem asserts that there is a natural morphism which maps the complex of the multi-logarithmic differential forms onto the complex of the regular meromorphic forms. The latter complex was defined and studied by D. Barlet [3], [4]. We also define the residue map and residue form of a multi-logarithmic differential form, as a generalization of corresponding definitions due to J. Leray, J.-B. Poly and K. Saito. In 1990 the first author investigated similar constructions in the case of logarithmic differential forms along the divisor D . More precisely, it was proved [1] that the residue map defined by K. Saito [26] gives a natural *quasi-isomorphism* between the complexes of sheaves of the logarithmic differential forms along the divisor D and of the regular meromorphic forms on D . It should also be mentioned that an explicit description of residue map in terms of integration currents is also given here; it is mainly based on results obtained in [28], [29], [27], [6], and [19].

The present paper contains four sections. In the first one we give the basic notation and definitions, and establish some properties of multi-logarithmic differential forms with respect to a reduced complete intersection in a complex analytic manifold. Then we consider the notion of regular meromorphic differential forms and discuss their properties. In Section 3, our main result is proved. The final section contains an explicit description in terms of integration currents of the residue map in the case of multi-logarithmic differential forms.

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1. Multi-Logarithmic Differential Forms

Let U be an open subset of $S = \mathbf{C}^m$, and let D_1, \dots, D_k be a set of hypersurfaces defined by the equations $h_j(z) = 0$, $j = 1, \dots, k$, $k \geq 1$, respectively. Here $h_j(z) = h_j(z_1, \dots, z_m)$,

$j = 1, \dots, k$, are holomorphic functions on U , and $z = (z_1, \dots, z_m)$ is a system of coordinates. In what follows we shall suppose that h_1, \dots, h_k is a *regular sequence* in \mathcal{O}_U , and $I = (h_1, \dots, h_k)\mathcal{O}_U$ is a *radical ideal*, that is, $I = \sqrt{I}$. In other words, the complex analytic variety $C = D_1 \cap \dots \cap D_k$ is a *reduced complete intersection*. In particular, $\dim C = m - k$ and the differential k -form $dh_1 \wedge \dots \wedge dh_k$ vanishes identically on no irreducible component of C . Let $\Omega_U^\bullet = (\Omega_U^q, d)_{q=0,1,\dots}$ be the de Rham complex of germs of holomorphic differential forms on U whose terms, locally at the point $0 \in U$, are defined as follows:

$$\Omega_{U,0}^q = \mathcal{O}_{U,0} \langle \dots, dz_{i_1} \wedge \dots \wedge dz_{i_q}, \dots \rangle, \quad (i_1, \dots, i_q) \in [1, m].$$

Set $D = D_1 \cup \dots \cup D_k$, and $\widehat{D}_j = D \setminus D_j$, $j = 1, \dots, k$. Let $\Omega_U^q(\star\widehat{D}_j)$ be the \mathcal{O}_U -module of meromorphic differential forms of degree $q \geq 0$ consisting of all the differential q -forms with the polar divisor \widehat{D}_j . We write $\widehat{D}_1 = \emptyset$ for $k = 1$, so that $\Omega_U^q(\star\widehat{D}_1) = \Omega_U^q$.

Proposition 1.1. *Let ω be a meromorphic differential q -form on U , $q \geq k$, with poles along the divisor $D = D_1 \cup \dots \cup D_k$. The following conditions are equivalent:*

$$i) \quad h_j \omega \in \sum_{i=1}^k \Omega_U^q(\star\widehat{D}_i), \quad h_j d\omega \in \sum_{i=1}^k \Omega_U^{q+1}(\star\widehat{D}_i), \quad j = 1, \dots, k;$$

$$ii) \quad h_j \omega \in \sum_{i=1}^k \Omega_U^q(\star\widehat{D}_i), \quad dh_j \wedge \omega \in \sum_{i=1}^k \Omega_U^{q+1}(\star\widehat{D}_i), \quad j = 1, \dots, k;$$

iii) *There exists a holomorphic function g , not equal identically to zero on every irreducible component of C , a holomorphic $(q-k)$ -form ξ and a meromorphic q -form $\eta \in \sum_{i=1}^k \Omega_U^q(\star\widehat{D}_i)$ on U such that*

$$g\omega = \frac{dh_1}{h_1} \wedge \dots \wedge \frac{dh_k}{h_k} \wedge \xi + \eta;$$

iv) *There exist analytic subsets $A_j \subset D_j$, $j = 1, \dots, k$, of codimension at least 2 such that the germ of ω at any point $x \in \bigcup_{j=1}^k (D_j \setminus A_j)$ belongs to*

$$\frac{dh_1}{h_1} \wedge \dots \wedge \frac{dh_k}{h_k} \wedge \Omega_{U,x}^{q-k} + \sum_{i=1}^k \Omega_{U,x}^q(\star\widehat{D}_i),$$

where $\Omega_{U,x}^{q-k}$ is the module of germs of holomorphic q -forms on U at x .

PROOF. The equivalence of *i)* and *ii)* follows directly from the Leibnitz rule:

$$d(h_j \omega) = dh_j \wedge \omega + h_j d\omega.$$

Let us prove the implication *ii) \Rightarrow iii)*. Consider the following representation of the form ω :

$$\omega = \frac{\sum_{|I|=q} a_I(z) \cdot dz_I}{h_1 \cdots h_k},$$

where $I := I^q = (i_1, \dots, i_q)$, $1 \leq i_1, \dots, i_q \leq m$, is a multi-index, $dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_q}$, and $a_I(z)$ is a set of holomorphic functions on U , skew symmetric with respect to the index I .

It is easy to see that the condition

$$dh_j \wedge \omega \in \sum_{i=1}^k \Omega_U^{q+1}(\star \widehat{D}_i)$$

is equivalent to the condition

$$dh_j \wedge \sum_I a_I(z) \cdot dz_I \in \sum_{\ell=1}^k h_\ell \Omega_U^{q+1}.$$

For an ordered multi-index $I := I^{q+1} = (i_1, \dots, i_{q+1})$, the latter gives us the following system of relations between the coefficients a_I and the derivatives $\partial h_j / \partial z_i$:

$$\sum_{\ell=1}^{q+1} (-1)^{\ell-1} \frac{\partial h_j}{\partial z_{i_\ell}} a_{I \setminus i_\ell} = b_{jI}^1 h_1 + \dots + b_{jI}^k h_k, \quad j = 1, \dots, k, \quad (1)$$

with holomorphic coefficients $b_{jI}^1, \dots, b_{jI}^k \in \mathcal{O}_U$.

Let us fix a multi-index $J^p = (j_1, \dots, j_p) \subset [1, \dots, m]$, $1 \leq p \leq k$, and denote the corresponding minor of the Jacobian matrix $\text{Jac}(h_1, \dots, h_k) = \|\partial h_i / \partial z_j\|$ as follows:

$$\Delta_{J^p} = \Delta_{j_1 \dots j_p} = \det \left\| \frac{\partial h_i}{\partial z_{j_r}} \right\|_{1 \leq i, r \leq p}.$$

Let $\mathcal{I} \subseteq \mathcal{O}_U$ be the ideal generated by the sequence of germs (h_1, \dots, h_k) . We shall prove the following relations:

$$\Delta_{J^p} a_{I^q} \equiv \sum_{\substack{K \subset I^q \\ |K|=p}} \text{sgn} \left(\begin{matrix} I^q \\ K, I^q \setminus K \end{matrix} \right) \Delta_K a_{(J^p, I^q \setminus K)} \pmod{(\mathcal{I})}, \quad p = 1, \dots, k, \quad (2)$$

by induction on p . Suppose that $p = 1$, $J^1 = j_1$, and set $I = (j, I^q) = (j_1, i_1, \dots, i_q)$ in formula (1). One gets

$$\frac{\partial h_1}{\partial z_j} a_I \equiv \sum_{\ell=1}^{q+1} (-1)^\ell \frac{\partial h_1}{\partial z_{i_\ell}} a_{I \setminus i_\ell} \pmod{(\mathcal{I})},$$

and this coincides with relation (2) for $p = 1$.

Assuming relation (2) to be true for $p - 1$, one can prove it for p as follows. Making use of the cofactor expansion along the p -th row of the determinant Δ_{J^p} , one obtains:

$$\Delta_{J^p} a_I = \sum_{\ell=1}^p (-1)^{p-\ell} \frac{\partial h_p}{\partial z_{j_\ell}} \Delta_{j_1 \dots \widehat{j}_\ell \dots j_p}^{p-1} a_I.$$

We have

$$\Delta_{j_1 \dots \widehat{j}_\ell \dots j_p} a_I \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \text{sgn} \left(\begin{matrix} I^q \\ K', I^q \setminus K' \end{matrix} \right) \Delta_{K'} a_{(j_1 \dots \widehat{j}_\ell \dots j_p, I \setminus K')} \pmod{(\mathcal{I})},$$

by the induction hypothesis. By substituting this relation into the previous equation and changing the order of summation, one obtains

$$\Delta_{J^p} a_I \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn} \left(\begin{matrix} I^q \\ K', I^q \setminus K' \end{matrix} \right) \Delta_{K'} \sum_{\ell=1}^p (-1)^{p-\ell} \frac{\partial h_p}{\partial z_\ell} a_{(j_1 \dots \widehat{j_\ell} \dots j_p, I \setminus K')} \pmod{(\mathcal{I})}.$$

The second sum consists of p summands contained in formula (1) for $j = p$, $I = (i_1, \dots, i_p, I \setminus K')$. Hence one can rewrite it as a sum of other $q - p + 1$ summands with opposite signs and an element of the ideal $(h_1, \dots, h_k) \mathcal{O}_U$. This yields the following relation modulo \mathcal{I} :

$$\Delta_{J^p} a_I \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn} \left(\begin{matrix} I^q \\ K', I^q \setminus K' \end{matrix} \right) \Delta_{K'} (-1)^{p-1} \sum_{i \in I \setminus K'} (-1)^{\#(i; I \setminus K')} \frac{\partial h_p}{\partial z_i} a_{(j_1 \dots j_p, I \setminus K' \setminus i)}, \quad (3)$$

where $\#(i; I \setminus K')$ denotes the number of occurrences of the index i in the set $I \setminus K'$. Let us order all the pairs (K', i) in such a way that the multi-index $K' \cup \{i\}$ coincides with a given $K \subset I$. For any such pair, the corresponding coefficient $a_{(j_1 \dots j_p, I \setminus (K' \setminus i))}$ is equal to $a_{(J, I \setminus K)}$. Thus the contribution of the set ordered as above to relation (3) looks as follows:

$$\begin{aligned} a_{(J, I \setminus K)} (-1)^{p-1} \sum_{i \in K} \operatorname{sgn} \left(\begin{matrix} I^q \\ K \setminus i, I^q \setminus K, i \end{matrix} \right) (-1)^{\#(i; I \setminus (K \setminus i))} \frac{\partial h_p}{\partial z_i} \Delta_{K \setminus i} = \\ = \operatorname{sgn} \left(\begin{matrix} I^q \\ K, I^q \setminus K \end{matrix} \right) a_{J, I \setminus K} \Delta_K. \end{aligned}$$

This completes the proof of relation (2) for $p \geq 1$.

It is not difficult to see that relations (2) imply the following equality

$$\Delta_{i_1 \dots i_k} \cdot \sum_{|I|=q} a_I dz_I = dh_1 \wedge \dots \wedge dh_k \wedge \left(\sum_{|I'|=q-k} a_{i_1 \dots i_k I'} dz_{I'} \right) + \nu,$$

with $\nu \in \sum_{j=1}^k h_j \Omega_U^{q-k}$, so that *iii*) holds, with $g = \Delta_{i_1 \dots i_k}$, and with ξ, η of the required shape. It remains to show that one can choose the function g in *iii*) in such a way that $g \neq 0$ on each of the irreducible components C_j of C . Let $\{\Delta_{i_1 \dots i_k}\}$, $(i_1, \dots, i_k) \in [1, \dots, m]$, be the set of all $k \times k$ -minors, with elements $\Delta^1, \dots, \Delta^N$, where $N = \binom{m}{k}$. By assumption, for every non-singular point of C , there is a minor that does not vanish at this point. Let $z^{(j)} \in C_j$ and suppose that $z^{(j)}$ is non-singular, then the function

$$\Delta_t(z) = t\Delta^1 + t^2\Delta^2 + \dots + t^N\Delta^N$$

regarded as polynomial of t , does not vanish identically for every $z = z^{(j)}$. Hence, there is a value $t_0 \in \mathbf{C}$ such that $\Delta_{t_0}(z^{(j)}) \neq 0$ for all j ; thus one can take $g(z) = \Delta_{t_0}(z)$.

Here is a purely algebraic proof of this result. Consider the ideal \mathcal{G} of \mathcal{O}_U generated by all the minors $\Delta_{j_1 \dots j_k}$ of maximal order of the Jacobian matrix $\operatorname{Jac}(h_1, \dots, h_k)$. The condition that $dh_1 \wedge \dots \wedge dh_k$ does not vanish identically on every irreducible component of C implies that the image $\tilde{\mathcal{G}}$ of \mathcal{G} in the ring $\mathcal{O}_{C,0}$ is not equal to $\operatorname{Ann} \mathcal{O}_{C,0}$. Thus one can use [5],

Th.2.4. (1) which implies that the $\mathcal{O}_{C,0}$ -depth of the ideal $\widetilde{\mathcal{G}}$ is at least 1. Hence there is an element $g \in \mathcal{O}_C$ satisfying the condition in *iii*).

The implication *iii*) \Rightarrow *iv*) is obvious: take $A_j = D_j \cap \{g = 0\}$, $j = 1, \dots, k$.

Let us prove the implication *iv*) \Rightarrow *ii*). It is obvious that $h_j\omega$ can be expressed as a sum of meromorphic forms each of which has singularities along at most $k - 1$ divisors, for example, on \widehat{D}_i , and on the subset $A_i \subset D_i$ of codimension 2. Hence by the Riemann extension theorem, such a meromorphic form has singularities *only* on \widehat{D}_i . As a result, $h_j\omega \in \sum_{i=1}^k \Omega_U^q(\star\widehat{D}_i)$, $j = 1, \dots, k$. Similar considerations are valid for the differential form $dh_j \wedge \omega$. This completes the proof of the implication *iv*) \Rightarrow *ii*). \square

Remark 1.1. *In fact, if $k = 1$ and $C = D$ is a reduced hypersurface then Proposition 1.1 is a generalization of the basic Lemma-Definition due to K. Saito (see [26], (1.1)).*

Definition 1.1. *A meromorphic differential q -form ω , $q \geq k$, on U is called multi-logarithmic with respect to the reduced complete intersection $C = D_1 \cap \dots \cap D_k$ if ω satisfies one of the conditions *i*) - *iv*) of Proposition 1.1.*

The localization of this definition leads to the notion of multi-logarithmic differential forms with respect to the complete intersection C at a point $x \in S$. We denote the $\mathcal{O}_{S,x}$ -module of germs of multi-logarithmic q -forms at x and the corresponding *sheaf* of multi-logarithmic differential q -forms on S by $\Omega_{S,x}^q(\log C)$ and $\Omega_S^q(\log C)$, $q \geq k$, respectively. Thus the \mathcal{O}_S -module $\Omega_S^q(\log C)$ is a submodule of $\Omega_S^q(\star D)$, consisting of all the differential forms with polar singularities along the *divisor* D .

Corollary 1.1. *There are the following natural inclusions*

$$h\Omega_S^q(\log C) \subset \Omega_S^q, \quad q \geq k.$$

PROOF. Condition *i*) implies that $h\omega$ has poles along D , while condition *iv*) shows that this meromorphic form has poles along the hypersurface defined by the holomorphic function g . Hence, $h\omega$ is holomorphic outside a subset of S of codimension at least 2 since the functions h and g are in general position. It remains to apply the Riemann extension theorem. \square

The structure of $\Omega_S^\bullet(\log C)$ seems to be more clear from the following observations. Set $\widehat{C}_j = \bigcap_{i=1, i \neq j}^k D_i$, where $j = 1, \dots, k$. Then there are the following natural inclusions

$$\Omega_S^q(\log \widehat{C}_j) \subset \sum_{i=1}^k \Omega_S^q(\star\widehat{D}_i) \subset \Omega_S^q(\log C), \quad \frac{dh_j}{h_j} \wedge \Omega_S^{q-1}(\log \widehat{C}_j) \subset \Omega_S^q(\log C).$$

In similar notations one gets

$$\Omega_S^q(\log \widehat{C}_{ij}) \subset \sum_{i=1}^k \Omega_S^q(\star\widehat{D}_i) \subset \Omega_S^q(\log C), \quad \frac{dh_i}{h_i} \wedge \frac{dh_j}{h_j} \wedge \Omega_S^{q-2}(\log \widehat{C}_{ij}) \subset \Omega_S^q(\log C),$$

and so on.

The following statement shows that the definition of multi-logarithmic differential forms is quite functorial; in particular, it is compatible with restrictions of certain type.

Lemma 1.1. *The sheaves $\Omega_S^q(\log C)$ and Ω_S^q coincide outside D for all $q \geq k$. The stalks $\Omega_{S,x}^q(\log C)$ and $\Omega_{S,x}^q(\log D_i)$ are isomorphic at all points $x \in D_i \setminus C$. Further, set $C_{ij} = D_i \cap D_j$. Then $\Omega_{S,x}^q(\log C) \cong \Omega_{S,x}^q(\log C_{ij})$ at all points $x \in C_{ij} \setminus C$. A similar assertion is valid for triple intersections, and so on.*

At last, for the sake of convenience of notation we set

$$\Omega_S^q(\log C) = \sum_{i=1}^k \Omega_S^q(\star \widehat{D}_i), \quad q = 0, \dots, k-1. \quad (4)$$

It should also be remarked that $\Omega_S^q(\log C)$, $q \geq 0$, are *coherent* sheaves of \mathcal{O}_S -modules. Moreover, they are \mathcal{O}_S -modules of finite type, and the direct sum $\bigoplus_{q=0}^m \Omega_S^q(\log C)$ is an \mathcal{O}_S -exterior algebra closed under the standard action induced by the exterior de Rham differentiation d on Ω_S^\bullet .

In what follows, we let for simplicity $x = 0$ when considering the local problems in the neighbourhood of x . We shall also assume that U is an open subset of $S = \mathbf{C}^m$ containing the origin.

Definition 1.2. *In the notation of iii), the restriction to the complete intersection $C = D_1 \cap \dots \cap D_k$ of the form ξ/g is called the residue form of ω :*

$$\text{res. } \omega = \left. \frac{\xi}{g} \right|_C.$$

Proposition 1.2. *The residue morphism res. is well-defined.*

PROOF. Let $\omega \in \Omega_S^q(\log C)$, $q \geq 0$. If ω has two different local presentations

$$g_\ell \omega = \frac{dh_1}{h_1} \wedge \dots \wedge \frac{dh_k}{h_k} \wedge \xi_\ell + \eta_\ell, \quad \ell = 1, 2,$$

then

$$dh_1 \wedge \dots \wedge dh_k \wedge (g_1 \xi_2 - g_2 \xi_1) = h_1 \cdots h_k (g_1 \eta_2 - g_2 \eta_1) \in (h_1, \dots, h_k) \Omega_S^q.$$

Hence

$$dh_1 \wedge \dots \wedge dh_k \wedge (g_1 \xi_2 - g_2 \xi_1) \equiv 0 \pmod{(h_1, \dots, h_k)}.$$

Now one can apply the part *i*) of the Theorem from [24] for $R = \mathcal{O}_{C,0}$, $M = \Omega_{S,0}^q \otimes \mathcal{O}_{C,0}$. It follows then that

$$\mathcal{G}^e(g_1 \xi_2 - g_2 \xi_1) \in dh_1 \wedge \Omega_{S,0}^{q-k} + \dots + dh_k \wedge \Omega_{S,0}^{q-k}, \quad e \in \mathbf{Z}, \quad e \geq 1,$$

in $\Omega_{S,0}^q \otimes \mathcal{O}_{C,0}$, where the ideal \mathcal{G} of $\mathcal{O}_{S,0}$ is generated by all minors of maximal order $\Delta_{j_1 \dots j_k}$ of the Jacobian matrix $\text{Jac}(h_1, \dots, h_k)$. Since C is reduced, the image $\tilde{\mathcal{G}}$ of \mathcal{G} in the ring $\mathcal{O}_{C,0}$ is not equal to $\text{Ann } \mathcal{O}_{C,0}$. Thus one can use [5], Th.2.4. (1), which implies that the $\mathcal{O}_{C,0}$ -depth of the ideal $\tilde{\mathcal{G}}$ is at least 1. Hence there is a maximal minor $\Delta = \Delta_{i_1 \dots i_k}$ which is not a zero-divisor in $\mathcal{O}_{C,0}$, and

$$\Delta^e(g_1 \xi_2 - g_2 \xi_1) \in dh_1 \wedge \Omega_{S,0}^{q-1} + \dots + dh_k \wedge \Omega_{S,0}^{q-1}$$

in $\Omega_{S,0}^q \otimes \mathcal{O}_{C,0}$. Therefore the class of $\Delta^e(g_1 \xi_2 - g_2 \xi_1)$ in $\Omega_{C,0}^q$ is equal to zero. Thus the two elements $\frac{1}{g_1} \xi_1$ and $\frac{1}{g_2} \xi_2$ define the same class in $\mathcal{M}_{C,0} \otimes_{\mathcal{O}_{C,0}} \Omega_{C,0}^q$. \square

Lemma 1.2. *The map res. commutes with the exterior de Rham differentiation d and defines a homomorphism of \mathcal{O}_S -modules*

$$\text{res.} : \Omega_S^q(\log C) \longrightarrow \mathcal{M}_C \otimes_{\mathcal{O}_C} \Omega_C^{q-k}, \quad q \geq k,$$

so that $\sum_{i=1}^k \Omega_S^q(\star \widehat{D}_i) \subseteq \text{Ker}(\text{res.})$.

Remark 1.2. *In particular, for $q = k$ we have a short complex*

$$\sum_{i=1}^k \Omega_S^k(\star \widehat{D}_i) \longrightarrow \Omega_S^k(\log C) \xrightarrow{\text{res.}} \mathcal{M}_C \cong \mathcal{M}_{\widetilde{C}},$$

where $\pi: \widetilde{C} \longrightarrow C$ is the normalization of C . It was observed by K. Saito that if $k = 1$ and $C = D$ is a hypersurface which is a normal crossing outside a subvariety of codimension at least 2, then the image $\text{Im}(\text{res.})$ coincides with $\pi_*(\mathcal{O}_{\widetilde{D}})$; it consists of the so-called weakly holomorphic functions on D , that is, meromorphic functions whose preimage becomes holomorphic on the normalization (see [26], (2.8)). Some interesting properties of the class of weakly holomorphic functions have been discussed in [28].

Lemma 1.3. *The image $\text{res.} \Omega_S^q(\log C)$, $q \geq k$, of the residue map are \mathcal{O}_C -modules.*

PROOF. The definition of multi-logarithmic forms implies that

$$h_j \cdot \left(\Omega_{S,0}^q(\log C) / \sum_{i=1}^k \Omega_S^q(\star \widehat{D}_i) \right) = 0, \quad j = 1, \dots, k.$$

Hence, the ideal $(h_1, \dots, h_k) \subset \mathcal{O}_S$ annihilates $\text{Im}(\text{res.})$. \square

Remark 1.3. *The requirement iii) of Proposition 1.1 is equivalent to the following condition:*

$$\Delta_{j_1 \dots j_k} \cdot \omega \in \frac{dh_1}{h_1} \wedge \dots \wedge \frac{dh_k}{h_k} \wedge \Omega_U^{\bullet-k} + \sum_{i=1}^k \Omega_U^{\bullet}(\star \widehat{D}_i), \quad (j_1, \dots, j_k) \in [1, \dots, m],$$

for any minor $\Delta_{j_1 \dots j_k}$ of maximal order of the Jacobian matrix $\text{Jac}(h)$. That is, these minors can be considered as universal denominators for complete intersections in the usual sense.

2. Regular Meromorphic Differential Forms

Let X be an analytic reduced subspace of a complex manifold S , let $\dim X = n$, and $\dim S = n + k$. The $\mathcal{O}_{X,0}$ -module

$$\omega_{X,0}^n = \text{Ext}_{\mathcal{O}_{S,0}}^k(\mathcal{O}_{X,0}, \Omega_{S,0}^{n+k})$$

is said to be the Grothendieck dualizing module of the germ $(X, 0)$.

Definition 2.1. *The sheaf ω_X^q , $q \geq 0$, of regular meromorphic differential q -forms is locally given as follows:*

$$\omega_{X,0}^q = \text{Hom}_{\mathcal{O}_{X,0}}(\Omega_{X,0}^{n-q}, \omega_{X,0}^n);$$

equivalently, the sheaf $\omega_{X,0}^q$ consists of all the meromorphic forms ω of degree q on X such that $\omega \wedge \eta \in \omega_{X,0}^n$ for any $\eta \in \Omega_{X,0}^{n-q}$.

We shall study the case when X is taken to be the reduced germ of a complete intersection $C = D_1 \cap \dots \cap D_k$, where D_1, \dots, D_k is a set of hypersurfaces defined by the equations $h_j(z) = 0$, $j = 1, \dots, k$, $k \geq 1$, respectively (in the notation of Section 1). In this case,

$$\omega_{C,0}^n = \text{Ext}_{\mathcal{O}_{S,0}}^k(\mathcal{O}_{C,0}, \Omega_{S,0}^{n+k}) \cong \mathcal{O}_{C,0} \left(\frac{dz_1 \wedge \dots \wedge dz_{n+k}}{dh_1 \wedge \dots \wedge dh_k} \right)$$

is a free module of rank one, and

$$\omega_{C,0}^q \cong \text{Hom}_{\mathcal{O}_{C,0}}(\Omega_{C,0}^{n-q}, \mathcal{O}_{C,0}), \quad 0 \leq q \leq n.$$

Here are some properties of regular meromorphic differential forms:

- 1) $\omega_C^q = 0$ for $q < 0$ and for $q > \dim C$;
- 2) the de Rham differential d is extended to the modules ω_C^q , $0 \leq q \leq n$, so that (ω_C^\bullet, d) is a complex;
- 3) in the case when C is *normal*, the sheaves ω_C^q have the following well-known interpretation:

$$\omega_C^q \cong j_* j^* \Omega_C^q,$$

where $j: C \setminus \text{Sing } C \rightarrow C$ is the canonical embedding. More precisely, in such a case, for any $q \geq 0$ there is an exact sequence of local cohomology groups with supports in $Z = \text{Sing } C$:

$$0 \longrightarrow H_Z^0(\Omega_C^q) \longrightarrow \Omega_C^q \longrightarrow \omega_C^q \longrightarrow H_Z^1(\Omega_C^q) \longrightarrow 0.$$

Let $k = 1$, so that $C = D$ is a *hypersurface*. Then both the module $\omega_{D,0}^0$ and the \mathcal{O}_D -module $\text{res. } \Omega_{S,0}^1(\log D)$ contain all the germs of *locally bounded meromorphic* functions on D (cf. [4], §2, ex. *i*). Moreover, these two sets coincide if the local fundamental group of the complement $S \setminus D$ is *abelian* (see [24]).

Theorem 2.1 ([4]). *There is the following exact sequence of \mathcal{O}_C -modules*

$$0 \longrightarrow \omega_C^q \xrightarrow{\mathcal{C}} \text{Ext}_{\mathcal{O}_S}^k(\mathcal{O}_C, \Omega_S^{q+k}) \xrightarrow{\mathcal{E}} \left(\text{Ext}_{\mathcal{O}_S}^k(\mathcal{O}_C, \Omega_S^{q+k+1}) \right)^k, \quad q \geq 0,$$

where $\omega_C^q \subset j_* j^* \Omega_C^q$, the map \mathcal{C} is multiplication by the fundamental class of C in S , and $\mathcal{E}(e) = (e \wedge dh_1, \dots, e \wedge dh_k)$.

Thus $\mathcal{C}(v)$ corresponds to a Čech cocycle w/h such that $w = v \wedge dh$. It is easy to see that

$$\omega_C^n \cong \text{Ext}_{\mathcal{O}_S}^k(\mathcal{O}_C, \Omega_S^{n+k}).$$

In particular, ω_C^n is isomorphic to the Grothendieck dualizing module of the n -dimensional complete intersection $C \subset S$. In conclusion, we establish a remarkable property of regular meromorphic differential forms in terms of the trace map associated with the Noether normalization. In fact, this property in a slightly different context has been proved by D. Barlet (see [4]).

Theorem 2.2. *The $\mathcal{O}_{C,0}$ -module $\omega_{C,0}^q$, $q \geq 0$, consists of all the meromorphic forms ω of degree q on C satisfying the following condition: for any $\eta \in \Omega_{C,0}^{n-q}$, the trace*

$$\text{Trace}_{C/\mathbf{C}^n}(\omega \wedge \eta)(z) = \sum_{x \in \pi^{-1}(z)} (\omega \wedge \eta)(x)$$

is the germ of a holomorphic form on C .

PROOF. Taking the Noether normalization of the germ C , one can suppose that the germ $(S,0)$ is isomorphic to the direct product $(\mathbf{C}^n,0) \times (\mathbf{C}^k,0)$ with coordinates (z,w) so that the natural projection $\pi: (C,0) \rightarrow (\mathbf{C}^n,0)$ is a finite covering.

Let $\omega \in \omega_{C,0}^q$, so that for any $\eta \in \Omega_{C,0}^{n-q}$ one has

$$\omega \wedge \eta = \frac{\alpha}{dh_1 \wedge \dots \wedge dh_k} \in \omega_{C,0}^n,$$

where $\alpha \in \Omega_{S,0}^{n+k}$. By definition, one can view the differential form $\omega \wedge \eta$ as an element of a Čech cochain in the open set $\{\Delta(z,w) \neq 0\}$:

$$\omega \wedge \eta = \frac{\varphi(z,w)}{\Delta} dz,$$

where φ is a holomorphic function and Δ is the determinant of the Jacobian matrix of the sequence of functions $h_1(z,w), \dots, h_k(z,w)$ with respect to w , that is,

$$\Delta = \det \left\| \frac{\partial h_j}{\partial w_i} \right\|_{1 \leq i, j \leq k}.$$

Thus

$$\text{Trace}_{C/\mathbf{C}^n}(\omega \wedge \eta)(z) = \sum_{\ell} \frac{\varphi}{\Delta}(z, w^\ell(z)) dz.$$

It is well-known (see [29], Ch.1) that the trace of the meromorphic function φ/Δ , whose denominator is equal to the Jacobian of the map π , is holomorphic. The proof of this fact is based on the integral representation of the trace as

$$\left(\frac{1}{2\pi\sqrt{-1}} \right)^k \int_{\Gamma} \frac{\varphi(z,w)dw}{h_1(z,w) \cdots h_k(z,w)},$$

where the integrand is holomorphic on the parameters z and

$$\Gamma = \{w \in \mathbf{C}^k : |h_1(0,w)| = \dots = |h_k(0,w)| = \varepsilon\}$$

is a compact cycle.

To prove the converse, for simplicity, we consider the case $n = 1$, so that C is a complete intersection curve. The general case can be studied in the same way. Let $\nu \in \Omega_{S,0}^1$, and suppose that $\text{Trace} \left(\varphi \frac{\nu}{g} \right)$ is holomorphic, for any holomorphic function $\varphi \in \mathcal{O}_{C,0}$. Then one has to prove

$$\frac{\nu}{g} \Big|_C = \frac{\varphi}{dh_1 \wedge \dots \wedge dh_k} \Big|_C,$$

that is,

$$\nu \wedge dh = \varphi g + h_1 f_1 + \dots + h_k f_k, \quad (5)$$

where $f_1, \dots, f_k \in \mathcal{O}_{S,0}$ are holomorphic functions. By assumption,

$$\text{Trace} \left(\varphi \frac{\nu}{g} \right) (z) = \sum_{\ell} \varphi \frac{\nu}{g} (z, w^{\ell}(z))$$

is holomorphic for any φ . Let us consider the Grothendieck residue

$$R_{h,g}[\varphi \nu \wedge dh_1 \wedge \dots \wedge dh_k] = \int_{\Gamma_{\varepsilon,\delta}} \frac{\varphi \nu \wedge dh_1 \wedge \dots \wedge dh_k}{gh_1 \dots h_k},$$

where $\Gamma_{\varepsilon,\delta} = \{(z, w) \in (S, 0) : |h_1| = \varepsilon, \dots, |h_k| = \varepsilon, |g| = \delta\}$. The local variant of the Hilbert Nullstellensatz shows that for some $N > 1$, one can write

$$z^N = q_1 h_1 + \dots + q_k h_k + Qg.$$

Therefore after applying the residue transformation formula one obtains

$$R_{h,g}[\varphi \nu \wedge dh_1 \wedge \dots \wedge dh_k] = R_{h,r}[Q\varphi \nu \wedge dh_1 \wedge \dots \wedge dh_k],$$

where $r = z^N$. The contour of integration in the latter integral

$$\gamma(\varepsilon, \delta) = \{(z, w) \in (S, 0) : |h_1| = \varepsilon, \dots, |h_k| = \varepsilon, |z| = \delta\}$$

is fibered under the projection π over the circle $|z| = \delta$. Hence, the integration with respect to w along the classical formula of the logarithmic residue yields

$$\begin{aligned} R_{h,g}[\varphi \nu \wedge dh_1 \wedge \dots \wedge dh_k] &= \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{|z|=\delta} \text{Trace} \left(\varphi \nu \frac{q}{r} \right) (z) = \frac{1}{2\pi\sqrt{-1}} \int_{|z|=\delta} \text{Trace} \left(\varphi \frac{\nu}{g} \right) (z), \end{aligned}$$

since the restrictions of q/r and $1/g$ to C are equal. By our assumption the last integral is equal to zero, hence the local duality theorem of Grothendieck implies (5). This completes the proof in the case $n = 1$. \square

3. The Poincaré Residue Map

First recall the definition of the classical Poincaré residue, following [11]. Let ω be a meromorphic differential m -form on an m -dimensional complex analytic manifold S with a polar divisor $D \subset S$. Thus, locally we have a representation:

$$\omega = \frac{f(z) dz_1 \wedge \dots \wedge dz_m}{h(z)}.$$

By definition, the Poincaré residue $\text{rés}_D(\omega)$ is a meromorphic $(m-1)$ -form on D whose singularities are contained in the singular locus $\text{Sing } D \subset D$. To define this form explicitly, let us note that at each point $x \in D \setminus \text{Sing } D$ one of the derivatives does not vanish:

$$\frac{\partial h}{\partial z_i}(x) \neq 0.$$

Hence, locally one can set

$$\text{rés}_D(\omega) = (-1)^{m-i} \frac{f(x) dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_m}{\frac{\partial h}{\partial z_i}(x)} \Big|_D.$$

It is not difficult to verify that this expression does not depend on the index i , on the local coordinates, and on the defining equation of D . Moreover, the Poincaré residue $\text{rés}_D(\omega)$ is holomorphic on the complement $S \setminus D$. When D is *smooth*, one can take $h(z) = z_m$, then

$$\text{rés}_D \left(\frac{f(x) dz_1 \wedge \dots \wedge dz_m}{z_m} \right) = f(x) dz_1 \wedge \dots \wedge dz_{m-1}$$

coincides with the usual residue.

For brevity, the Poincaré residue $\text{rés}_D(\omega)$ will be denoted by $\text{rés}(\omega)$. When $D \subset S$ is a *nonsingular* hypersurface, the Poincaré residue induces an exact sequence

$$0 \longrightarrow \Omega_S^m \longrightarrow \Omega_S^m(D) \xrightarrow{\text{rés}} \Omega_D^{m-1} \longrightarrow 0,$$

where $\Omega_S^m(D)$ denotes the set of meromorphic forms on S having a pole of the first order along the divisor D . In particular, it follows that the germ of every holomorphic $(m-1)$ -form on D is a Poincaré residue. It is obvious that this is true globally when the first cohomology group vanishes: $H^1(S, \Omega_S^m) = 0$.

More generally, following J.Leray (see [13]) one can apply the above construction to meromorphic differential q -forms on S with a *smooth* polar divisor $D \subset S$ for any $1 \leq q \leq m$, and write locally for any d -closed differential q -form ω on $S \setminus D$, having poles of the first order along D , the following presentation:

$$\omega = \frac{dh}{h} \wedge \xi + \eta,$$

where ξ and η are holomorphic. The restriction of ξ to D is a holomorphic form independent of the choice of the local equation $h = 0$ of D ; we denote it by $\text{rés}(\omega)$ as before. As a result, one gets (see [20]) an exact sequence

$$0 \longrightarrow \Omega_S^q \longrightarrow \Omega_S^q(D) \xrightarrow{\text{rés}} \Omega_D^{q-1} \longrightarrow 0, \quad 1 \leq q \leq m,$$

which is equivalent to the sequence

$$0 \longrightarrow \Omega_S^q \longrightarrow \Omega_S^q(\log D) \xrightarrow{\text{rés}} \Omega_D^{q-1} \longrightarrow 0, \quad (6)$$

since the divisor D is a *smooth* hypersurface.

Remark 3.1. *As it has been already remarked by de Rham and Leray (cf. [13]), for a meromorphic q -form ω with poles of the first order along D the above definition gives us the holomorphic $(q-1)$ -form $\text{rés}(\omega)$ in cases when the singular locus $\text{Sing } D$ of the divisor D consists of isolated double quadratic points only and either $q < m$, or $q = m$, and the coefficient of m -form $h \cdot \omega$ vanishes on $\text{Sing } D$. This phenomenon can be readily explained by considerations which we use in the next and previous sections.*

An extension of the exact sequence (6) to the case when the divisor D has arbitrary singularities was described in [1]

$$0 \longrightarrow \Omega_S^q \longrightarrow \Omega_S^q(\log D) \xrightarrow{\text{res.}} \omega_D^{q-1} \longrightarrow 0, \quad q \geq 1. \quad (7)$$

The following theorem is devoted to further generalization of this construction to the case of complete intersection.

Theorem 3.1. *Let $C \subset S$ be a reduced complete intersection. Then under notations of Section 1 there is a short exact sequence*

$$\sum_{i=1}^k \Omega_S^q(\star \widehat{D}_i) \longrightarrow \Omega_S^q(\log C) \xrightarrow{\text{res.}} \omega_C^{q-k} \longrightarrow 0, \quad q \geq k,$$

and a natural isomorphism of \mathcal{O}_C -modules

$$\text{res. } \Omega_S^q(\log C) \cong \omega_C^{q-k}, \quad q \geq k.$$

PROOF. The proof goes similarly to [1], §4. Thus, it is sufficient to verify our statements locally. Remark 1.3 implies

$$\Delta_{j_1 \dots j_k} \cdot \text{res. } \Omega_S^q(\log C)|_U \subset \Omega_C^{q-k}|_{C \cap U},$$

for all maximal minors $\Delta_{j_1 \dots j_k}$, $(j_1, \dots, j_k) \in [1, \dots, m]$ of the Jacobian matrix $\text{Jac}(\underline{h})$. Since $\omega_{C,0}^n \cong \mathcal{O}_{C,0}(dz_1 \wedge \dots \wedge dz_{n+k}/dh_1 \wedge \dots \wedge dh_k)$, the definition of the modules of regular meromorphic differential forms implies immediately that

$$\text{res. } \Omega_S^q(\log C) \subseteq \omega_C^{q-k}.$$

Let now $\mathcal{K}_\bullet(\underline{h})$ be the usual Koszul complex associated with the sequence $\underline{h} = (h_1, \dots, h_k)$:

$$0 \rightarrow \mathcal{O}_{S,0}\langle e_0 \wedge \dots \wedge e_{k-1} \rangle \xrightarrow{d_{k-1}} \dots \xrightarrow{d_1} \mathcal{O}_{S,0}\langle e_0 \rangle + \dots + \mathcal{O}_{S,0}\langle e_{k-1} \rangle \rightarrow \xrightarrow{d_0} \mathcal{O}_{S,0} \xrightarrow{d_{-1}} \mathcal{O}_{C,0} \rightarrow 0,$$

where

$$\mathcal{K}_k(\underline{h}) = \mathcal{O}_{S,0}\langle e_0 \wedge \dots \wedge e_{k-1} \rangle, \dots, \mathcal{K}_1(\underline{h}) = \mathcal{O}_{S,0}\langle e_0 \rangle + \dots + \mathcal{O}_{S,0}\langle e_{k-1} \rangle,$$

and $\mathcal{K}_0(\underline{h}) = \mathcal{O}_{S,0}$, $d_0(e_i) = h_{i+1}$, $i = 0, \dots, k-1$, $d_{-1}(1) = 1$. From the dual exact sequence one obtains, by definition,

$$\text{Ext}_{\mathcal{O}_{S,0}}^k(\mathcal{O}_{C,0}, \Omega_{S,0}^{q+1}) \cong \text{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_k(\underline{h}), \Omega_{S,0}^{q+1})/d^{k-1}(\text{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_{k-1}(\underline{h}), \Omega_{S,0}^{q+1})).$$

This means that any element of $\text{Ext}_{\mathcal{O}_{S,0}}^k(\mathcal{O}_{C,0}, \Omega_{S,0}^{q+1})$ can be represented as a Čech $(k-1)$ -cochain (more explicitly, a $(k-1)$ -cocycle)

$$\frac{\nu}{h_1 \dots h_k} \in \text{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_k(\underline{h}), \Omega_{S,0}^{q+1}) \cong C_{(k)}^{k-1}(\Omega_{S,0}^{q+1}),$$

where $\nu \in \Omega_{S,0}^{q+1}$. Choose now an element $\nu \in \Omega_{S,0}^{q+1}$ such that

$$\frac{\nu}{h_1 \cdots h_k} \wedge dh_j \in \text{Ext}_{\mathcal{O}_{S,0}}^k(\mathcal{O}_{C,0}, \Omega_{S,0}^{q+2}), \quad j = 1, \dots, k,$$

corresponds to the *trivial* element. This means that for any $j = 1, \dots, k$ the form $\nu \wedge dh_j/h_1 \cdots h_k$, is defined by an element of $d^{k-1}(\text{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_{k-1}(\underline{h}), \Omega_{S,0}^{q+2}))$. This yields that

$$\nu \wedge dh_j \in \sum_{i=1}^k h_i \Omega_S^{q+2}, \quad j = 1, \dots, k,$$

or, equivalently,

$$\omega \wedge dh_j \in \sum_{i=1}^k \Omega_S^{q+1}(\star \widehat{D}_i), \quad j = 1, \dots, k, \quad \text{where } \omega = \frac{\nu}{h_1 \cdots h_k}.$$

In view of Proposition 1.1 this means that $\omega \in \Omega_{S,0}^q(\log C)$. Now we shall use the notation and exact sequence of Theorem 2.1. Set $\tilde{\nu} = \mathcal{C}^{-1}(\nu/h_1 \cdots h_k)$. Then $\mathcal{C}(\tilde{\nu})$ corresponds to the Čech cocycle $\nu/h_1 \cdots h_k$ such that $\nu = \tilde{\nu} \wedge dh_1 \wedge \cdots \wedge dh_k$ (take $v = \tilde{\nu}$, $w = \nu$ in the description of ω_C^q in terms of multiplication by the fundamental class of C in S). This implies

$$\frac{\nu}{h_1 \cdots h_k} = \tilde{\nu} \wedge \frac{dh_1}{h_1} \wedge \cdots \wedge \frac{dh_k}{h_k}, \quad \text{and} \quad \text{res.} \left(\frac{\nu}{h_1 \cdots h_k} \right) = \tilde{\nu}.$$

Thus, for any element $\tilde{\nu} \in \omega_C^{q-k}$ there is a preimage under the multi-logarithmic residue map represented by $\nu/h_1 \cdots h_k$. \square

4. On Residues of Multi-Logarithmic Forms and $\bar{\partial}$ -closed Currents

We first remark, that in view of Theorem 3.1, the sections of ω_C^\bullet can be considered as “regular meromorphic forms.” More precisely, in complex analysis “regular” or “holomorphic” differential forms are usually described as $\bar{\partial}$ -closed forms (cf. [14]).

The key idea and further considerations in this section are closely related to the methods developed in [28], where the theory of weakly holomorphic functions is studied. Moreover, they lead to another proof of the statement of Theorem 3.1 concerning the representation of sections of Barlet sheaves ω_C^\bullet as residues of multi-logarithmic differential forms. The main result of this section is the following theorem.

Theorem 4.1. *There is a natural one to one correspondence between the sets of residues of multi-logarithmic forms and meromorphic $\bar{\partial}$ -closed currents on the complete intersection $C = D_1 \cap \dots \cap D_k$.*

The proof will be given below. We start with the definition of the integration current associated with a meromorphic differential form, and recall the basic properties of residue currents.

Let $\psi = \xi/g$ be a meromorphic differential form on a reduced analytic subset C of a domain $U \subset \mathbf{C}^m$; here ξ is a holomorphic differential form and g is a holomorphic function. Such a form defines the following current

$$\left\langle [\psi], \varphi \right\rangle = \left\langle \left[\frac{\xi}{g} \right] \Big|_C, \varphi \right\rangle = \lim_{\varepsilon \rightarrow 0} \int_{C \cap \{|g| > \varepsilon\}} \frac{\xi}{g} \wedge \varphi, \quad \varphi \in \mathcal{D}^\bullet(U), \quad (8)$$

where φ is a test C^∞ -differential form with compact support on U , and the real dimension of ξ is equal to the sum $(\deg \varphi + \deg \psi)$ of degrees of φ and ψ . This is an integration current on C in the principal value sense.

A residue current R_h associated with a holomorphic map $h = (h_1, \dots, h_k) : U \rightarrow \mathbf{C}^k$ is a functional on the space $\mathcal{D}^\bullet(U)$ of C^∞ -differential forms with compact supports on U . It is defined as follows (see [6] or [29]):

$$R_h(\varphi) = \lim_{\varepsilon \rightarrow 0} \int_{T_\varepsilon(h)} \frac{\varphi}{h_1 \cdots h_k}, \quad \varphi \in \mathcal{D}^{2m-k}(U),$$

where the integral is taken over the tube

$$T_\varepsilon(h) = \{z \in S : |h_j(z)| = \varepsilon_j, j = 1, \dots, k\}$$

with a fixed orientation. Here the approach of the radii $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ to zero is required to be along a so-called admissible path: a path $\varepsilon = \varepsilon(\tau)$ is called *admissible*, if $\varepsilon_k \rightarrow 0$ as $\tau \rightarrow 0$ and for each $j = 1, \dots, k-1$, the coordinate $\varepsilon_j(\tau)$ approaches zero faster than any power of the next coordinate $\varepsilon_{j+1}(\tau)$.

The *principal value* of the residue current R_h with respect to a function $g : U \rightarrow \mathbf{C}$ is defined as the limit

$$P_g R_h(\varphi) = \lim_{\tau \rightarrow 0} \int_{T_\varepsilon(h) \cap \{|g| > \delta\}} \frac{\varphi}{gh_1 \cdots h_k}$$

along the admissible path $(\varepsilon(\tau), \delta(\tau)) : [0, 1] \rightarrow \mathbf{R}^{k+1}$.

When the set $h^{-1}(0) = \{h_1 = \dots = h_k = 0\}$ is a complete intersection of codimension k then

$$R_h = \bar{\partial} \left[\frac{1}{h_1} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{h_k} \right].$$

Respectively, when $g \not\equiv 0$ on every irreducible component of the analytic set $h^{-1}(0)$ then

$$P_g R_h = \left[\frac{1}{g} \right] \bar{\partial} \left[\frac{1}{h_1} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{h_k} \right].$$

When $\omega = \alpha/h_1 \cdots h_k$ is a meromorphic differential form of degree ℓ then there is the current

$$R_h[\omega](\varphi) := R_h(\alpha \wedge \varphi), \quad \varphi \in \mathcal{D}^{2m-k-\ell}(U).$$

Analogously for $\omega = \alpha/g \cdot h_1 \cdots h_k$ we set

$$P_g R_h[\omega](\varphi) = P_g R_h(\alpha \wedge \varphi).$$

Let us assume that the zero set defined by the system of equations

$$\{g = h_1 = \dots = h_k = 0\}$$

is a *complete intersection*. Then the residue currents R_h and their principal values P_g with respect to g have the following properties:

$$1) R_h[\omega] = P_g R_h[g\omega];$$

$$2) P_g R_h \left[\frac{dh_1}{h_1} \wedge \dots \wedge \frac{dh_k}{h_k} \wedge \frac{\xi}{g} \right] = \left[\frac{\xi}{g} \right] \Big|_C = \left[\frac{\xi}{g} \right] \wedge [C];$$

where $[C]$ is the integration current (a holomorphic chain) on C . In particular, when ξ and g are identically equal to the constant 1, then

$$R_h \left[\frac{dh_1}{h_1} \wedge \dots \wedge \frac{dh_k}{h_k} \right] = dh_1 \wedge \dots \wedge dh_k \wedge \bar{\partial} \left[\frac{1}{h_1} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{h_k} \right] = [C];$$

$$3) h_j P_g R_h = 0, \quad j = 1, \dots, k;$$

4) If U is a domain of holomorphy (or a Stein manifold) then for $v \in \mathcal{O}_U$ one has

$$v R_h = 0 \iff v \in (h_1, \dots, h_k) \mathcal{O}_U.$$

In particular, this property is valid in the local rings $\mathcal{O}_{U,x}$, $x \in U$.

Properties 1) and 3) are proved in [6], Theorem 4.4.1 and Theorem 1.7.6, respectively, property 2) is obtained in [27], Theorem 2.1; property 4) is obtained in [19], Corollary 6.1.2.

PROOF of Theorem 4.1. Let $\omega \in \Omega_{S,0}^{q+k}(\log C)$, so that

$$\omega = \frac{\alpha}{h_1 \cdots h_k}, \quad \text{and} \quad g\omega = \frac{dh_1}{h_1} \wedge \dots \wedge \frac{dh_k}{h_k} \wedge \xi + \eta,$$

where α, ξ are holomorphic and η satisfies the condition *iii)* of Proposition 1.1. In view of property 1) one has

$$R_h[\omega] = P_g R_h[g\omega] = P_g R_h \left[\frac{dh_1}{h_1} \wedge \dots \wedge \frac{dh_k}{h_k} \wedge \frac{\xi}{g} \right] + P_g R_h \left[\frac{\eta}{g} \right].$$

The second summand corresponds to the current which is trivial by property 3). Property 2) implies

$$R_h[\omega] = \left[\frac{\xi}{g} \right] \Big|_C.$$

Hence, the current $\left[\frac{\xi}{g} \right] \Big|_C$ defined by the residue $\text{res. } \omega = \frac{\xi}{g} \Big|_C$ of the multi-logarithmic form ω coincides with the residue current $R_h[\omega]$ which is $\bar{\partial}$ -closed (see [6]).

Conversely, under the same assumptions, let ξ/g be a meromorphic form on C , and the corresponding current (8)

$$\langle [\psi], \varphi \rangle = \left\langle \left[\frac{\xi}{g} \right] \Big|_C, \varphi \right\rangle = \lim_{\varepsilon \rightarrow 0} \int_{C \cap \{|g| > \varepsilon\}} \frac{\xi}{g} \wedge \varphi, \quad \varphi \in \mathcal{D}^\bullet(U)$$

is $\bar{\partial}$ -closed. Then in virtue of property 2)

$$0 = \bar{\partial} \left[\frac{\xi}{g} \right] \Big|_C = \bar{\partial} \left(\left[\frac{\xi}{g} \right] \wedge [C] \right).$$

Since $\bar{\partial}[C] = 0$, applying the Leibnitz rule and property 2), one obtains

$$0 = \bar{\partial} \left[\frac{\xi}{g} \right] \wedge [C] = \bar{\partial} \left[\frac{\xi}{g} \right] \wedge dh_1 \wedge \dots \wedge dh_k \wedge \bar{\partial} \left[\frac{1}{h_1} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{h_k} \right].$$

Property 4) yields

$$dh_1 \wedge \dots \wedge dh_k \wedge \xi \in (g, h_1, \dots, h_k) \Omega_{U,0}^{q+k},$$

that is,

$$dh_1 \wedge \dots \wedge dh_k \wedge \xi = \alpha g - h_1 \eta_1 - \dots - h_k \eta_k, \quad (9)$$

where $\alpha, \eta_1, \dots, \eta_k \in \Omega_U^{q+k}$ are holomorphic forms in the domain $U \subset S$. Here property 4) is used in computing the coefficients of the form $\xi \wedge dh$. Let us consider the meromorphic form $\omega = \alpha/h_1 \cdots h_k$. We have

$$g \cdot \omega = g \cdot \frac{\alpha}{h_1 \cdots h_k} = \frac{dh \wedge \xi + h_1 \eta_1 + \dots + h_k \eta_k}{h_1 \cdots h_k} = \frac{dh_1}{h_1} \wedge \dots \wedge \frac{dh_k}{h_k} \wedge \xi + \eta,$$

where

$$\eta = \sum_{i=1}^k \frac{\eta_i}{h_1 \cdots h_k} \in \sum_{i=1}^k \Omega_{U,0}^{q+k}(\star \widehat{D}_i).$$

This completes the proof of Theorem 4.1. \square

One can deduce the main statement of Theorem 3.1 from the following theorem.

Theorem 4.2. *A meromorphic differential q -form on the complete intersection $C = D_1 \cap \dots \cap D_k$ defines a meromorphic $\bar{\partial}$ -closed current on C if and only if it is a section of the Barlet sheaf ω_C^q of regular meromorphic forms.*

PROOF. Let $\omega = \xi/g$ be a meromorphic q -form C . It is not difficult to see that

$$\bar{\partial}(\omega) = 0 \iff \bar{\partial}(\omega \wedge \nu) = 0 \quad \text{for any } \nu \in \Omega_C^{n-q}.$$

Making use of property 4) and of similar considerations of the converse part of the proof of Theorem 4.1 we conclude that the property of $\bar{\partial}$ -closedness of ω is equivalent to the existence of the relation $dh \wedge (\xi \wedge \nu) = \alpha g - h_1 \eta_1 - \dots - h_k \eta_k$, where $\alpha, \eta_1, \dots, \eta_k \in \Omega_U^m$ are holomorphic forms in the domain $U \subset S$. This is equivalent to the condition

$$\left. \frac{\xi}{g} \wedge \nu \right|_C = \left. \frac{\alpha}{dh_1 \wedge \dots \wedge dh_k} \right|_C \in \omega_C^n.$$

Hence, by definition, $\omega = \frac{\xi}{g} \in \omega_C^q$ as required. \square

Remark 4.1. *This proof is based on a similar idea from [14], Theorem 2.*

Remark 4.2. *Thus, any section of the Barlet sheaf ω_C^q is the residue of a multi-logarithmic form. Such meromorphic forms on the complete intersection C are characterized by the following two conditions:*

a) *they are holomorphic on the set of non-singular points of C (cf. [13]),*

b) they have no residues with respect to the singular locus of C (this is the so-called $\bar{\partial}$ -closedness property).

A differential form ψ which satisfies condition a) is not a priori meromorphic. Nevertheless it should define the following integration current (see [27]):

$$\langle [\psi], \varphi \rangle = \lim_{\delta \rightarrow 0} \int_{C \cap \{|dh| > \delta\}} \psi \wedge \varphi, \quad (10)$$

where $dh = dh_1 \wedge \cdots \wedge dh_k$. This integration current differs from (8) by the choice of the "principal value" of the integral. Recall that under our assumptions the singular locus $\text{Sing } C$ of C is defined by the condition $dh_1 \wedge \cdots \wedge dh_k|_C = 0$. Then a natural question arises: Whether the class of differential q -forms ψ which are holomorphic on the non-singular part of C and define the $\bar{\partial}$ -closed currents by means of formula (10), and the class of sections of Barlet sheaf ω_C^q coincide?

In connection with this problem it should be noted that there are non-exact $\bar{\partial}$ -closed currents corresponding to non-holomorphic differential forms (see [9]).

If one supposes that ψ is meromorphic then Proposition 1 of [27] asserts that it is true. Thus one gets the following elegant description of the Grothendieck dualizing module on a reduced complete intersection in terms of $\bar{\partial}$ -closed currents. We remark that *a priori* all sections of the dualizing module are meromorphic on C and holomorphic on the non-singular part of C .

Corollary 4.1. *Any element ψ of the Grothendieck dualizing module of the reduced complete intersection $C = D_1 \cap \dots \cap D_k$ defines a meromorphic $\bar{\partial}$ -closed current on C by formula (10) and vice versa.*

In conclusion we pose a problem which is a stronger version of the above question. Let Ψ^q be the set of differential q -forms ψ on U such that they are holomorphic on the non-singular part of C and satisfy the following condition (cf. formula (10)) :

$$\langle [\psi], \varphi \rangle = \lim_{\delta \rightarrow 0} \int_{C \cap \{|dh| = \delta\}} \psi \wedge \varphi = 0. \quad (11)$$

Conjecture 4.1. *The classes of differential q -forms defined by the set Ψ^q and by the sections of the Barlet sheaf ω_C^q coincide.*

Remark 4.3. *It is well-known that any pure dimensional analytic set can be regarded locally as a component of some reduced complete intersection (see [10], p.52). In fact, this leads to a generalization of the notion of the multi-logarithmic differential form with respect to an arbitrary pure dimensional analytic set.*

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