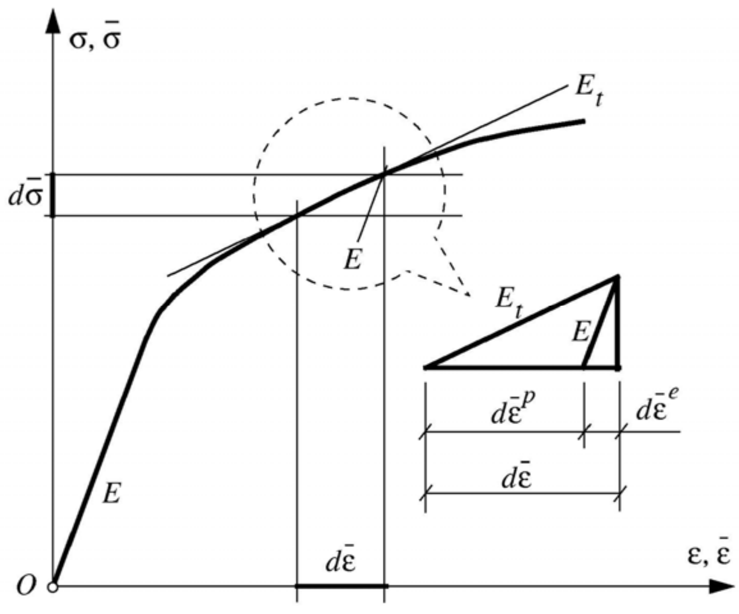
European Commission

JRC TECHNICAL REPORTS

Some notes on elasto-plasticity models in Europlexus ancestor codes

Folco Casadei
Georgios Valsamos
Martin Larcher

2015



EUR 27593 EN

Joint Research Centre

Some notes on elasto-plasticity models in Europlexus ancestor codes

JRC Science Hub
https://ec.europa.eu/jrc

JRC97564

EUR 27593 EN

ISBN 978-92-79-53988-6 (online)

ISSN 1831-9424 (online)

doi:10.2788/233591 (online)

Printed in Italy

How to cite: Authors; title; EUR; doi

Some notes on elasto-plasticity models in Europlexus ancestor codes

F. Casadei. G. Valsamos, M. Larcher

November 24, 2015

These notes are based upon the report by François Frey: *Le Calcul Elasto-plastique des Structures par la Méthode des Eléments Finis et son Application a l'Etat Plan de Contrainte.* Rapport N. 33, LMMSC, Université de Liège, July 1973.

1 Introduction

This report deals with the description of an elasto-plastic material model characterized by a stress-strain curve with hardening, i.e. monotonically increasing but irreversible beyond the elastic limit. The limit case of a material without hardening (i.e., elastic-perfectly plastic) is included. The model uses the differential elasto-plastic theory, the von Mises plasticity criterion and the isotropic hardening law.

2 Plasticity

2.1 Generalities

The **differential theory** of plasticity (also called flow or incremental theory) assumes that the increments of plastic deformation are function of the previous plastic deformations and are proportional to the stress deviator.

This theory fits the physical behaviour of metallic materials better than the the **finite theory** (also called deformation or total strain theory), which does not take into account the history of deformation.

The basic equations are due to Prandtl (1924) and Reuss (1930). Hardening and Bauschinger effect can be introduced. The plastic deformations are "memorized" by integrating the deformations along the loading time history. This theory therefore depends upon the "loading path" and requires a purely incremental resolution method.

The classical theory assumes that the material is initially isotropic and that time does not play a role, i.e., that there are no creep or relaxation effects ("inviscid" material). It neglects thermal effects (isothermal behaviour), dynamic effects (static or quasi-static behaviour) and large strains (but it may consider the large displacements due to finite rotations).

The primary goal of the theory is to define the **elasto-plastic constitutive equations**. The elasto-plastic problem is defined as follows. Given a structure (Figure 1), find

$$u_i \quad ; \quad \epsilon_{ij} \quad ; \quad \sigma_{ij} \quad ; \quad I$$

satisfying:

- equilibrium (within the domain D, on its contour C and across the elasto-plastic interface I)

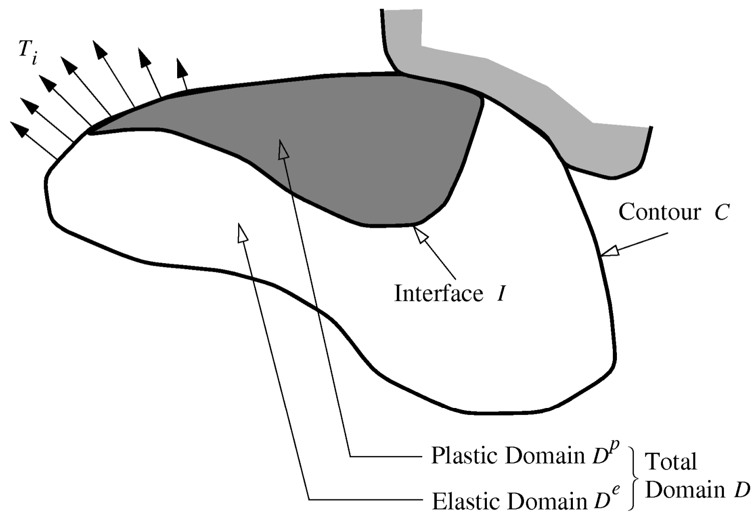


Figure 1: Elasto-plastic boundary value problem.

- continuity of displacements (within D, on C and across I)
- the linear elastic constitutive law (within the elastic sub-domain D^e)
- the elasto-plastic constitutive law (within the plastic sub-domain D^p)

The theory presented in the following is based on the fundamental book by Hill [1] and on the references [2] to [14].

2.2 Simple traction test

It is assumed, as proposed by Prandtl, that the stress-strain curve $\sigma - \epsilon$ resulting from a simple traction test can be idealized as presented in Figure 2, i.e. by supposing that:

a) The curve is linear up to point A, which represents the **initial elastic limit** (the physical notions of proportionality limit and of conventional elastic limit at 0.2% strain are supposed to coincide with point A);

b) Beyond point A, the material hardens by following a curve ABD monotonically increasing and irreversible; i.e., if the stress at a point B diminishes, the material follows a curve that:

  b1) does not present any hysteresis loop ($BC \equiv CB$),

  b2) is linear and has the same slope as OA,

  b3) touches the initial curve ABD in B for a new loading beyond point B, which represents a **new elastic limit** for the path CBD;

c) beyond point D, the deformations become too large to fit into the framework of the theory, so that the end of the curve ("large elongations" and necking) is not considered.

This curve, valid for uniaxial stress states, is fundamental because, in the following, any multi-axial stress state will be reconducted to the uniaxial case by using the notion of equivalent quantities.

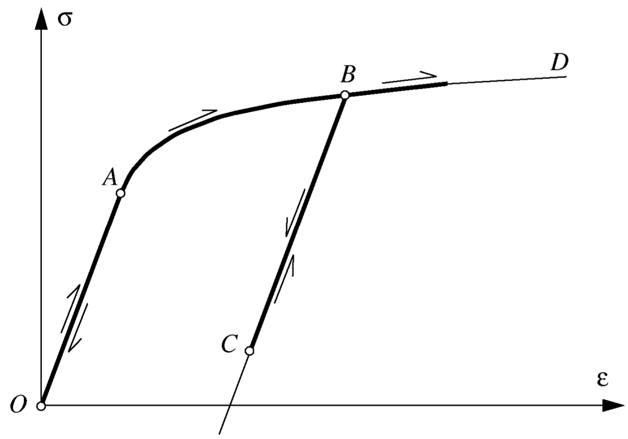


Figure 2: Stress-strain curve in pure traction.

Furthermore, the curve exhibits an initial elastic limit (point A), a current elastic limit (point B) and the possibility of elastic unloading starting from a plastic state (path BC). These phenomena are found also in multi-axial stress states, under the notion of initial and current yield surface and of elastic unloading starting from a generic plastic stress state.

2.3 Differential theory of plasticity

2.3.1 Basic hypotheses

The basic hypotheses are as follows:

A. **Decomposition of the total deformations.** It is assumed that the total strains ϵ_{ij} can be decomposed into an elastic part (suffix e), related to the the total stresses by Hooke's law, and a plastic part (suffix p). With Einstein's notation on repeated indices one has therefore:

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p \tag{1}$$

$$\epsilon_{ij}^e = H_{ijkl}\,\sigma_{kl} \quad \text{or} \quad \sigma_{ij} = D_{ijkl}\,\epsilon_{kl}^e \tag{2}$$

$$\text{(with } \mathbf{H} = \mathbf{D}^{-1}\text{)}$$

This hypothesis is directly inspired by, and is a generalization of, the uniaxial traction test. It is incorrect in large strains. Since

$$D_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{3}$$

the second of (2) can also be written:

$$\sigma_{ij} = 2\mu\epsilon_{ij}^e + \lambda\epsilon_{kk}^e\delta_{ij} \tag{4}$$

where δ_{ij} is Kronecker's delta and

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)} \equiv G \tag{5}$$

are Lamé's constants. Furthermore, one makes the complementary hypothesis that the plastic deformations are incompressible (they occur without changes of volume):

$$\epsilon^p_{ii} = \epsilon^p_{11} + \epsilon^p_{22} + \epsilon^p_{33} = 0 \tag{6}$$

This property is confirmed by experience as long as the hydrostatic pressure remains moderate (e.g., of the order of magnitude of the elastic limit).

B. **Existence of the yield surfaces**. The stress state can be represented by a point in the nine-dimensional space of stresses. Its origin is the initial configuration of the body (assumed stress-free) and, in its vicinity, there exists a zone where an increase of stresses $d\sigma_{ij}$ produces only an increase of elastic deformations $d\epsilon^e_{ij}$ (while $d\epsilon^p_{ij} = 0$). The boundary of this zone is the initial plasticity or **initial yield surface**. It is represented by the equation:

$$F_0(\sigma_{ij}) = 0 \tag{7}$$

When hardening occurs, this surface evolves as long as plastic deformation progresses. The mathematical expression of these successive plasticity surfaces is called the **loading function**. It defines the successive regions whose internal points represent elastic states, and whose boundary points can lead to plastic states. The loading function depends upon the reached stress state σ_{ij}, upon the history of plastic strains ϵ^p_{ij} and upon the hardening, through a parameter k. Summarizing:

$$F(\sigma_{ij}, \epsilon^p_{ij}, k) = 0 \tag{8}$$

exists and is such that

$$\begin{aligned} F < 0 \quad &; \quad \text{elastic state : } d\sigma_{ij} \text{ produces only } d\epsilon^e_{ij} \\ F = 0 \quad &; \quad \text{plastic state; } d\sigma_{ij} \text{ may produce } d\epsilon^p_{ij} \\ F > 0 \quad &; \quad \text{inadmissible state (meaningless)} \end{aligned} \tag{9}$$

The special value $F = 0$ represents the plasticity condition, starting from which one may define three different loading cases. For a given increment of the load, the corresponding variation dF of the the loading function is:

$$dF = \frac{\partial F}{\partial \sigma_{ij}}\, d\sigma_{ij} + \frac{\partial F}{\partial \epsilon^p_{ij}}\, d\epsilon^p_{ij} + \frac{\partial F}{\partial k}\, dk \tag{10}$$

It is recalled that, geometrically, $\partial F/\partial \sigma_{ij}$ represents the normal to the (current) yield surface in stress space. Starting from a point on the surface ($F = 0$), there are three cases:

a) $dF < 0$. Then $F + dF < 0$ and the reached state is elastic. Therefore, $d\epsilon^p_{ij} = 0$, $dk = 0$ and one has a process of unloading:

$$\frac{\partial F}{\partial \sigma_{ij}}\, d\sigma_{ij} < 0 \ , \ F = 0 \quad \text{unloading} \tag{11}$$

b) $dF = 0$ but $d\epsilon^p_{ij} = 0$ (hence $dk = 0$). This process, which passes from one plastic state to another without variation of the plastic deformations is called neutral loading:

$$\frac{\partial F}{\partial \sigma_{ij}}\, d\sigma_{ij} = 0 \ , \ F = 0 \quad \text{neutral loading} \tag{12}$$

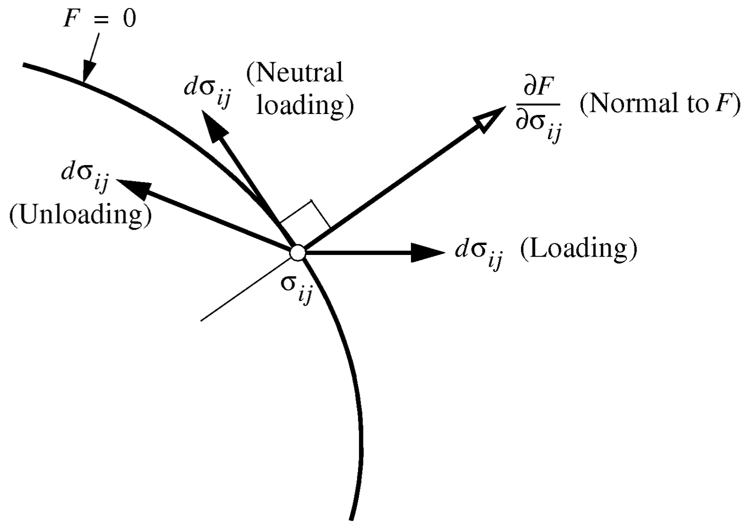


Figure 3: Loading, neutral loading and unloading.

c) $dF = 0$, $d\epsilon^p_{ij} \neq 0$, $dk \neq 0$. This process passes from one plastic state to another and is called loading. In this case it is:

$$\frac{\partial F}{\partial \sigma_{ij}} \, d\sigma_{ij} > 0 \ , \ F = 0 \quad \text{loading} \tag{13}$$

The Figure 3 illustrates graphically these three possibilities (scalar product).

C. **Drucker's existence postulate** [5, 6]. This postulate, which is a definition of hardening, can be expressed as follows. Let σ^*_{ij} be the stress state at a point of a body for a given load. Due to an external action independent from the previous load, an additional state of stress is applied and then gradually removed. Then, during the application only, or during the complete cycle (application then removal), the external agent produces a non-negative work.

Let σ^*_{ij} in Figure 4 denote the existing stress state. The external agent first brings this state onto the yield surface at point σ_{ij} by following an elastic path. Then, it causes an increase $d\sigma_{ij}$ of the stresses in the plastic domain by producing both elastic ($d\epsilon^e_{ij}$) and plastic ($d\epsilon^p_{ij}$) strain increments. Finally, it is removed and the stress state returns to the point σ^*_{ij} by following an elastic path. In this cycle, the elastic work is zero, so that the work produced by the external agent is (scalar product):

$$(\sigma_{ij} - \sigma^*_{ij}) \, d\epsilon^p_{ij} + d\sigma_{ij} \, d\epsilon^p_{ij} \geq 0 \tag{14}$$

Since one can choose $\sigma^*_{ij} \equiv \sigma_{ij}$, one has first that (Drucker's stability postulate)

$$d\sigma_{ij} \, d\epsilon^p_{ij} \geq 0 \tag{15}$$

and then, since this second term is an order of magnitude smaller than the first one, the latter must satisfy

$$(\sigma_{ij} - \sigma^*_{ij}) \, d\epsilon^p_{ij} \geq 0 \tag{16}$$

where the equalities hold in case of neutral loading. This postulate, and the inequalities to which it leads, are particularly evident in the uniaxial traction case.

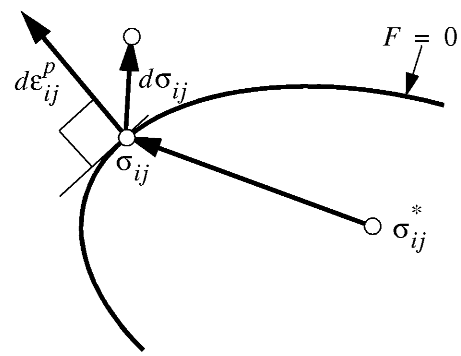


Figure 4: Drucker's postulate.

2.3.2 Consequences of Drucker's postulate and remarks

The inequalities (15) and (16), which are derived from Drucker's postulate, have some fundamental consequences on the theory of plasticity since, in practice, they determine the form of the sought constitutive equations. Two main properties stem from the postulate:

A. **Convexity.** Any yield surface $F = 0$ is convex. This is a consequence of (16), which states that the angle between $\sigma_{ij} - \sigma^*_{ij}$ and $d\epsilon^p_{ij}$ cannot exceed 90°, since the scalar product is zero or positive.

B. **Normality.** The vector $d\epsilon^p_{ij}$ representing the plastic strain increment is normal to the yield surface. In fact, since $\sigma_{ij} - \sigma^*_{ij}$ can only be located on one side of the hyperplane perpendicular to $d\epsilon^p_{ij}$ due to (16), this hyperplane is tangent to F, and therefore $d\epsilon^p_{ij}$ is normal to F. Furthermore, it follows that the plastic strain increment $d\epsilon^p_{ij}$ is independent of the stress increment $d\sigma_{ij}$. This normality property can be represented by the equation:

$$d\epsilon^p_{ij} = d\lambda \frac{\partial F}{\partial \sigma_{ij}} \quad \text{with} \quad d\lambda \geq 0 \tag{17}$$

where $d\lambda$ (or simply λ), called the plastic multiplier, is a proportionality factor which, because of (15), is non-negative. This fundamental relation is called the **flow rule**, or normality rule, or the law of plastic flow. One says also that F is the **plastic potential** function. We will also see that (17) is identical to Prandtl-Reuss equations if one adopts von Mises' yield criterion.

By replacing (17) into (15):

$$d\sigma_{ij} d\epsilon^p_{ij} = d\sigma_{ij} d\lambda \frac{\partial F}{\partial \sigma_{ij}} \geq 0$$

but since $d\lambda \geq 0$, then it must be $\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} \geq 0$, i.e., one proves the condition (13) for a loading (see also the first remark below).

From (1) and (2) one can write, using the expression (17) of $d\epsilon^p_{ij}$:

$$d\epsilon_{ij} = H_{ijkl}\, d\sigma_{kl} + d\lambda \, \frac{\partial F}{\partial \sigma_{ij}} \tag{18}$$

This is a general expression of the incremental constitutive equations. To particularize it, one must know F explicitly (see sections 2.3.3 to 2.3.5 below).

Remarks:

a) The inequality (13) is known under the name of Prager's condition or "consistency" condition: any change from a plastic state to another, accompanied by an increment of plastic deformation ($d\epsilon_{ij}^p \neq 0$) must satisfy this inequality.

b) Some authors do not accept Drucker's postulate, because it is not rigorous from the physical viewpoint (thermodynamics). This postulate can therefore be considered only an hypothesis, so that any other hypothesis which can be verified experimentally is also valid.

   One can thus assume, for example [7]:

   1°) The existence of yield surfaces F (identical to hypothesis B of section 2.3.1 above),

   2°) The existence of a plastic potential G, to which the normality rule is applied:

$$d\epsilon_{ij}^p = d\lambda \frac{\partial G}{\partial \sigma_{ij}}$$

   One says that this flow rule is **associated** to the yield (plasticity) surface, if one poses

$$F \equiv G$$

   and one obtains the "associated" laws of plasticity.

   Alternatively, one can assume valid the theory of plastic potential proposed by von Mises for an element of volume, extended by Prager [8] by introducing the notion of generalized stresses and strains, and demonstrated by Ziegler [9] for the whole body.

   Whaterver the starting hypotheses, all such theories produce the same relations, which will be shown in the following.

2.3.3 The initial yield surface

Let $F_0(\sigma_{ij})$ be the initial yield surface, which is function only of the stresses since no plastic deformation has occurred yet. By assuming that:

a) the material is isotropic in its initial configuration,

b) this surface (and also the following ones) is independent from a hydrostatic stress state (hypothesis A of section 2.3.1),

c) no Bauschinger effect is present initially, i.e. the initial elastic limits in traction and in compression are equal or, more generally, $F_0(\sigma_{ij}) = F_0(-\sigma_{ij})$.

the surface can be written as

$$F_0(J_2, J_3) = 0 \tag{19}$$

where J_2 and J_3 are the second and third invariants of the deviatoric stress tensor s_{ij}

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{20}$$

given by

$$J_2 = \frac{1}{2} s_{ij} s_{ij} \tag{21}$$

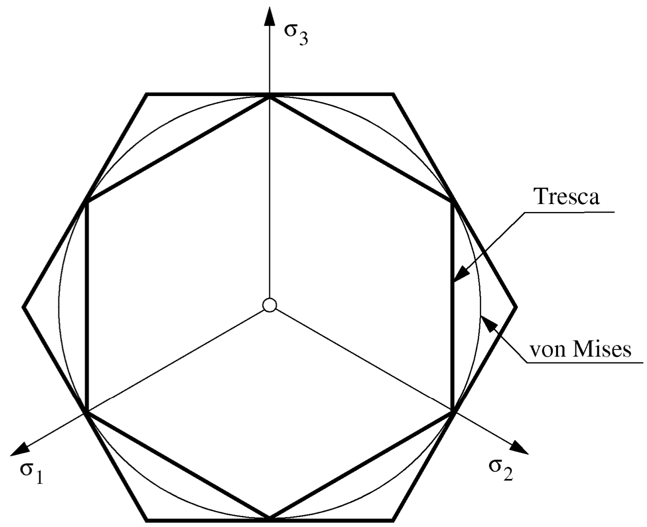


Figure 5: Axonometric view of the initial yield surface.

and

$$J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki} \tag{22}$$

This surface is cylindrical if represented in the principal stress space, so that it is completely defined by its normal cross-section. The curve representing such cross-section must be located between the two hexagons shown in Figure 5, and must be convex.

2.3.4 Yield criteria

One denotes yield (or plasticity) criterion a law which explicitly defines the form of the yield surface. The two most commonly employed criteria are:

a) the **Tresca criterion** (1864), that can be written (but this form is hardly usable):

$$4J_2^3 - 27J_3^2 - 9\overline{\tau}_e^2 J_2^2 + 6\overline{\tau}_e^4 J_2 - \overline{\tau}_e^6 = 0$$

where $\overline{\tau}_e$ is a constant representing the initial elastic limit in pure shear. If $\overline{\sigma}_e$ (also sometimes denoted σ_Y) is the initial elastic limit in traction, then:

$$\overline{\tau}_e = \overline{\sigma}_e / 2$$

This criterion is represented by the inscribed hexagon in Figure 5.

b) the **von Mises criterion** (1913), that can be written:

$$J_2 - \overline{\tau}_e^2 = 0 \tag{23}$$

with

$$\overline{\tau}_e = \overline{\sigma}_e / \sqrt{3} \tag{24}$$

and which is represented by the circle in Figure 5.

There are numerous interpretations of these criteria [10]. The second criterion, which better represents the bahaviour of metallic materials, will be adopted in the present work.

By replacing (24) into (23) one can write

$$\sqrt{J_2} = \frac{\overline{\sigma}_e}{\sqrt{3}}$$

which, by using the expression (21) of J_2 becomes

$$\sqrt{\frac{1}{2} s_{ij} s_{ij}} = \frac{\overline{\sigma}_e}{\sqrt{3}}$$

Finally, one obtains the initial yield criterion according to von Mises in its classical form:

$$F_0 = \sqrt{\frac{3}{2} s_{ij} s_{ij}} - \overline{\sigma}_e = \sigma_c - \overline{\sigma}_e = 0 \tag{25}$$

where the scalar σ_c, defined as

$$\sigma_c = \sqrt{(3/2) s_{ij} s_{ij}} \tag{26}$$

is called the von Mises equivalent stress. For any multiaxial stress state σ_{ij} the corresponding equivalent (uniaxial) von Mises stress $\sigma_c(\sigma_{ij})$ can be computed by (26) and the stress state is plastic whenever σ_c reaches the elastic yield limit $\overline{\sigma}_e$.

2.3.5 Hardening rule

One denotes hardening rule a law which describes explicitly the form of the load function or, in other terms, which defines the evolution of the successive yield surfaces. The two laws used in common practice are:

a) The **isotropic hardening** law (Hill, 1950 [1]), which essentially postulates that the load function is obtained via a uniform expansion in all directions of the initial yield surface. This law conserves the initial material isotropy (hence its name), but is in direct contradiction with the Bauschinger effect;

b) The linear **kinematic hardening** law (Prager, 1956 [11], Ziegler, 1959 [12]), which essentially assumes that the load function maintains the same shape as the initial yield surface, but it moves by translation in the stress space. This law partially accounts for the Bauschinger effect, as well as for the material anisotropy induced by plastic strains.

Although the first law can be subjected to criticism, it will be retained here by noting that the problems that we intend to solve (determination of "limit loads") do not present load cycles, for which the Bauschinger effect would be fundamental.

Isotropic hardening

The hypotheses for isotropic hardening are as follows:

a) The initial isotropy of the material is conserved;

b) Whatever the path followed in the space of strains in order to reach a certain stress state, the final load function is the same.

From these hypotheses one deduces that the load function has the same shape as the initial yield function, that only the constant (k parameter) appearing in it becomes function of a certain measure of the hardening, and that the experimental determination of this function is independent of the type of loading. In other words, it can be obtained, for example, from a simple traction test. Experimental confirmations of these hypotheses are scarce [10].

Geometrically, these hypotheses lead to an expansion (homothety) of the initial surface, which in mathematical terms corresponds to:

$$F(\sigma_{ij}, \epsilon_{ij}^p, k) = 0 \quad \longrightarrow \quad F(\sigma_{ij}, k) = f(\sigma_{ij}) - \overline{\sigma}(k) = 0 \tag{27}$$

where $f(\sigma_{ij})$ is the equivalent stress, also denoted σ_c and given by (26) for the von Mises yield criterion, and $\overline{\sigma}(k)$ is the current elastic limit in pure traction. Note that the fundamental dependency of F on ϵ_{ij}^p, which seems to have disappeared, is in fact maintained through k (see below), which depends upon the history of plastic strains.

In order to measure the hardening and use it to obtain the explicit form of the current yield stress $\overline{\sigma}(k)$, one can make two alternative assumptions ([1], [13]):

A. **Work-hardening** hypothesis (or hypothesis of the work of plastic strains). One defines:

$$dk \equiv dW^p = \sigma_{ij} d\epsilon_{ij}^p \quad \text{with} \quad W^p = \int_0^{\epsilon_{ij}^p} \sigma_{ij} d\epsilon_{ij}^p \qquad (W^p \geq 0) \tag{28}$$

and one poses

$$\overline{\sigma}(k) \equiv f_1(W^p)$$

B. **Strain-hardening** hypothesis (or hypothesis of the increment of equivalent plastic strain $d\overline{\epsilon}^p$). One defines (see Section 2.3.6 below for details):

$$dk \equiv d\overline{\epsilon}^p = \sqrt{\frac{2}{3} d\epsilon_{ij}^p d\epsilon_{ij}^p} \quad \text{with} \quad \overline{\epsilon}^p = \int_0^{\epsilon_{ij}^p} d\overline{\epsilon}^p \tag{29}$$

where the expression of $d\overline{\epsilon}^p$ suited for the von Mises criterion has been used, and one poses

$$\overline{\sigma}(k) \equiv f_2(\overline{\epsilon}^p)$$

It can be shown that, in the case of von Mises yield criterion, these two definitions are equivalent, i.e. that there exists a relation of the form

$$W^p = f_3(\overline{\epsilon}^p)$$

between the two measures W^p and $\overline{\epsilon}^p$ of the hardening.

In conclusion, the hardening parameter k can be eliminated in favour of the plastic deformations ϵ_{ij}^p in the load function (8) or (27), which assumes the form:

$$F(\sigma_{ij}, \epsilon_{ij}^p) = \sigma_c(\sigma_{ij}) - \overline{\sigma}(\epsilon_{ij}^p) = 0 \tag{30}$$

2.3.6 Definition of the equivalent plastic strain

A detailed description of the construction of the equivalent plastic strain increment has been given by Berg in reference [15], which is summarized hereafter.

We assume that the yield condition is given in the form (27), where $\overline{\sigma}(k)$ is the current or equivalent yield stress and depends upon the history of deformation so as to represent the hardening of the material. It is numerically equal to the yield stress in uniaxial tension. The function $f(\sigma_{ij})$ must, of course, have the physical dimensions of a stress.

We also assume that $f(\sigma_{ij})$ is a homogeneous function of degree unity in the stress components σ_{ij}. These assumptions imply that the yield locus merely expands without changing shape (isotropic hardening) as the material hardens, with the magnitude of $\overline{\sigma}(k)$ determining the current size of the yield locus.

If one assumes (as in [1]) that hardening is determined by the plastic work dW^p done on the material during each increment of plastic strain $d\epsilon_{ij}^p$, then one may construct an equivalent plastic strain increment to use in a theory of hardening as follows. The increment of plastic work dW^p for each increment of plastic strain $d\epsilon_{ij}^p$ is given by the first of (28), that is:

$$dW^p = \sigma_{ij} d\epsilon_{ij}^p \tag{31}$$

where the σ_{ij} are the stress components which satisfy the yield condition and produce the required plastic strain increment $d\epsilon_{ij}^p$. Now, the associated flow rule of plasticity theory requires that the plastic strain increment lie normal to the yield surface (in the appropriate space) so that

$$d\epsilon_{ij}^p = d\lambda \left(\frac{\partial f}{\partial \sigma_{ij}} \right)_{f-\overline{\sigma}=0} \tag{32}$$

where $d\lambda$ is a non-negative scalar multiplier.

In attempting to identify an equivalent plastic strain increment $d\epsilon^p$, one seeks a function of the plastic strain components $d\epsilon_{ij}^p$ with the property that the product of the equivalent plastic strain increment and the equivalent yield stress is always equal to the increment of plastic work:

$$d\epsilon^p \overline{\sigma} = dW^p = \sigma_{ij} d\epsilon_{ij}^p \tag{33}$$

By replacing (32) into equation (31)

$$dW^p = \sigma_{ij} d\epsilon_{ij}^p = d\lambda \sigma_{ij} \frac{\partial f}{\partial \sigma_{ij}} \tag{34}$$

and, since $f(\sigma_{ij})$ has been assumed to be a homogeneous function of degree one, Euler's theorem for homogeneous functions requires that:

$$\sigma_{ij} d\epsilon_{ij}^p = d\lambda f = d\lambda \overline{\sigma} \tag{35}$$

the last step coming from eq. (27). The desired relationship

$$\overline{\sigma} d\epsilon^p = dW^p = d\lambda \overline{\sigma} \tag{36}$$

is satisfied by setting

$$d\epsilon^p = d\lambda \tag{37}$$

That is, the equivalent plastic strain increment is just the nonnegative scalar multiplier $d\lambda$, which provides the generalized length of the plastic strain increment eq. (32).

If one uses the quantity

$$||d\epsilon|| = \left(d\epsilon_{ij}^p d\epsilon_{ij}^p\right)^{1/2} = \left(d\epsilon_1^2 + d\epsilon_2^2 + d\epsilon_3^2\right)^{1/2} \tag{38}$$

as a measure of the magnitude of the plastic strain increment, where $d\epsilon_1$, $d\epsilon_2$ and $d\epsilon_3$ are the principal values of plastic strain increment, then from equation (32)

$$||d\epsilon|| = d\lambda \left(\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}\right)^{1/2} \tag{39}$$

Thus the equivalent plastic strain increment can be given as

$$d\lambda = d\epsilon^p = \frac{\sqrt{d\epsilon_1^2 + d\epsilon_2^2 + d\epsilon_3^2}}{\sqrt{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}}} \tag{40}$$

for any yield locus $f(\sigma_{ij})$ with which one happens to be concerned. For example, in the case of the von Mises yield locus, eq. (27) becomes

$$\left(\frac{3}{2} s_{ij} s_{ij}\right)^{1/2} - \overline{\sigma} = 0 \tag{41}$$

For this case

$$\sqrt{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}} = \sqrt{\frac{3}{2}} \tag{42}$$

and the equivalent plastic strain is given by

$$d\epsilon^p = \sqrt{\frac{2}{3} d\epsilon_{ij}^p d\epsilon_{ij}^p} \tag{43}$$

which is the classical result.

Once the equivalent plastic strain has been constructued, it may be used as a variable to describe the history of deformation upon which hardening depends. With the total equivalent plastic strain ϵ^p given by

$$\epsilon^p = \int d\epsilon^p \tag{44}$$

(note: $d\epsilon^p > 0$ so that ϵ^p increases monotonically in any deformation process), one may rewrite eq. (27) as

$$f(\sigma_{ij}) - \overline{\sigma}(\epsilon^p) = 0 \tag{45}$$

2.4 Incremental elasto-plastic constitutive laws

In the case of loading ($d\epsilon_{ij}^p \neq 0$), $dF = 0$ can be written from (30) as:

$$\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \epsilon_{ij}^p} d\epsilon_{ij}^p = 0$$

By replacing the expression (17) of $d\epsilon_{ij}^p$ (normality rule) into the above equation, one gets:

$$\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \epsilon_{ij}^p} d\lambda \frac{\partial F}{\partial \sigma_{ij}} = 0$$

and then, solving for $d\lambda$:

$$d\lambda = -\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} / (\frac{\partial F}{\partial \epsilon^p_{mn}} \frac{\partial F}{\partial \sigma_{mn}}) \tag{46}$$

where for clarity the pair of dummy indexes ij has been replaced by mn in the term at the denominator.

From (30), by noting that $\partial \overline{\sigma}(\epsilon^p_{ij})/\partial \sigma_{ij} = 0$, one has

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial \sigma_c}{\partial \sigma_{ij}} \tag{47}$$

and by replacing here the expression of the von Mises equivalent stress σ_c, which can be deduced from equation (25), one has:

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}} \sqrt{\frac{3}{2} s_{ij} s_{ij}} = \sqrt{\frac{3}{2}} \frac{\partial}{\partial \sigma_{ij}} \sqrt{s_{ij} s_{ij}} \tag{48}$$

By successively expanding the derivative of the term under square root one obtains:

$$\begin{aligned} \frac{\partial F}{\partial \sigma_{ij}} &= \sqrt{\frac{3}{2}} \frac{1}{2\sqrt{s_{kl} s_{kl}}} \frac{\partial}{\partial \sigma_{ij}} (s_{kl} s_{kl}) \\ &= \sqrt{\frac{3}{2}} \frac{1}{2\sqrt{s_{kl} s_{kl}}} 2 s_{kl} \frac{\partial s_{kl}}{\partial \sigma_{ij}} \\ &= \frac{3}{2} \frac{1}{\sqrt{\frac{3}{2} s_{kl} s_{kl}}} s_{kl} \frac{\partial s_{kl}}{\partial \sigma_{ij}} \\ &= \frac{3}{2} \frac{1}{\sigma_c} s_{kl} \frac{\partial s_{kl}}{\partial \sigma_{ij}} \end{aligned} \tag{49}$$

where in the last passage the expression of σ_c from (25) has been used once more.

By using the definition (20) of the deviatoric stress tensor, the last term appearing in (49) becomes:

$$\begin{aligned} \frac{\partial s_{kl}}{\partial \sigma_{ij}} &= \frac{\partial}{\partial \sigma_{ij}} (\sigma_{kl} - \frac{1}{3} \sigma_{mm} \delta_{kl}) \\ &= \frac{\partial \sigma_{kl}}{\partial \sigma_{ij}} - \frac{1}{3} \frac{\partial}{\partial \sigma_{ij}} (\sigma_{mm} \delta_{kl}) \\ &= \delta_{ki} \delta_{lj} - \frac{1}{3} \delta_{ij} \delta_{kl} \end{aligned} \tag{50}$$

so that:

$$\begin{aligned} s_{kl} \frac{\partial s_{kl}}{\partial \sigma_{ij}} &= s_{kl} (\delta_{ki} \delta_{lj} - \frac{1}{3} \delta_{ij} \delta_{kl}) \\ &= s_{kl} \delta_{ki} \delta_{lj} - \frac{1}{3} s_{kl} \delta_{ij} \delta_{kl} \\ &= s_{ij} \end{aligned} \tag{51}$$

thanks to the following notable identity involving the deviatoric stress tensor:

$$\begin{aligned} s_{kl}\delta_{kl} = s_{mm} &= s_{11} + s_{22} + s_{33} \\ &= \sigma_{11} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \\ &\quad \sigma_{22} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \\ &\quad \sigma_{33} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ &= 0 \end{aligned} \tag{52}$$

By replacing (51) into (49) we obtain:

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial \sigma_c}{\partial \sigma_{ij}} = \frac{3}{2}\frac{s_{ij}}{\sigma_c} \tag{53}$$

which, by noting that for a stress state on the current yield surface ($F = 0$) the (von Mises) equivalent stress σ_c equals the current yield stress $\overline{\sigma}$, can also be finally written as:

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial \sigma_c}{\partial \sigma_{ij}} = \frac{\partial \overline{\sigma}}{\partial \sigma_{ij}} = \frac{3}{2}\frac{s_{ij}}{\overline{\sigma}} \tag{54}$$

By assuming as valid the hypothesis of work hardening (A, Section 2.3.5) one has from (28) and by using the normality rule (17):

$$dW^p = \sigma_{ij} d\epsilon^p_{ij} = \sigma_{ij} d\lambda \frac{\partial F}{\partial \sigma_{ij}} \tag{55}$$

which, by using the expression (54) of $\partial F/\partial \sigma_{ij}$ becomes:

$$dW^p = \sigma_{ij} d\lambda \frac{3}{2}\frac{s_{ij}}{\overline{\sigma}} \tag{56}$$

From the definition (20) of deviatoric stress tensor one has:

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{57}$$

so that:

$$\begin{aligned} \sigma_{ij} s_{ij} &= (s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}) s_{ij} \\ &= s_{ij} s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} s_{ij} \\ &= s_{ij} s_{ij} \end{aligned} \tag{58}$$

where the identity (52) has been used in the last passage.

From the expression (25) of von Mises' equivalent stress the previous expression becomes:

$$\sigma_{ij} s_{ij} = s_{ij} s_{ij} = \frac{2}{3}\sigma_c^2 = \frac{2}{3}\overline{\sigma}^2 \tag{59}$$

where, as previously, we have assumed that the stress state lies on the current yield surface ($F = 0$) so that $\sigma_c = \overline{\sigma}$. By using (59) the expression (56) becomes:

$$\begin{aligned} dW^p &= \frac{3}{2} d\lambda \frac{\sigma_{ij} s_{ij}}{\overline{\sigma}} \\ &= \frac{3}{2} d\lambda \frac{2}{3}\frac{\overline{\sigma}^2}{\overline{\sigma}} \\ &= \overline{\sigma} d\lambda \end{aligned} \tag{60}$$

The hypothesis of the increment of equivalent plastic strain, or strain hardening (B, section 2.3.5) allows us to write from (29):

$$d\overline{\epsilon}^p = \sqrt{\frac{2}{3} d\epsilon_{ij}^p d\epsilon_{ij}^p} \tag{61}$$

From the normality rule (17) and the expression (54) of $\partial F/\partial \sigma_{ij}$ we obtain:

$$d\epsilon_{ij}^p = d\lambda \frac{3}{2} \frac{s_{ij}}{\overline{\sigma}} \tag{62}$$

which, replaced into (61), gives:

$$\begin{aligned} d\overline{\epsilon}^p &= \sqrt{\frac{2}{3}(d\lambda)^2 \frac{9}{4} \frac{s_{ij}s_{ij}}{\overline{\sigma}^2}} \\ &= \sqrt{(d\lambda)^2 \frac{3}{2} \frac{s_{ij}s_{ij}}{\overline{\sigma}^2}} \end{aligned} \tag{63}$$

From the expression (25) of von Mises' equivalent stress one gets:

$$\overline{\sigma}^2 = \sigma_c^2 = \frac{3}{2} s_{ij} s_{ij} \tag{64}$$

where, as previously, we have assumed that the stress state lies on the current yield surface ($F = 0$) so that $\sigma_c = \overline{\sigma}$. By replacing this into (63) we have:

$$d\overline{\epsilon}^p = d\lambda \tag{65}$$

that is, the plastic multiplier $d\lambda$ is equal to the increment of equivalent plastic strain $d\overline{\epsilon}^p$. This, replaced into (62), gives the interesting relation

$$d\epsilon_{ij}^p = \frac{3}{2} \frac{d\overline{\epsilon}^p}{\overline{\sigma}} s_{ij} \tag{66}$$

In virtue of the equality (65), the increment of plastic work (55) can be written as

$$dW^p = \sigma_{ij} d\epsilon_{ij}^p = \overline{\sigma} d\overline{\epsilon}^p \tag{67}$$

Let us now search for an expression of $\partial F/\partial \epsilon_{ij}^p$. From (27)

$$\frac{\partial F}{\partial \epsilon_{ij}^p} = -\frac{\partial \overline{\sigma}}{\partial \epsilon_{ij}^p} \tag{68}$$

and this can be rewritten as

$$\frac{\partial F}{\partial \epsilon_{ij}^p} = -\frac{\partial \overline{\sigma}}{\partial W^p} \frac{\partial W^p}{\partial \epsilon_{ij}^p} = -\frac{d\overline{\sigma}}{dW^p} \frac{dW^p}{d\epsilon_{ij}^p} \tag{69}$$

From a uniaxial traction test, shown in Figure 6, one sees that

$$dW^p = \overline{\sigma}\left(d\overline{\epsilon} - d\overline{\epsilon}^e\right) = \overline{\sigma}\left(d\overline{\epsilon} - \frac{d\overline{\sigma}}{E}\right) \tag{70}$$

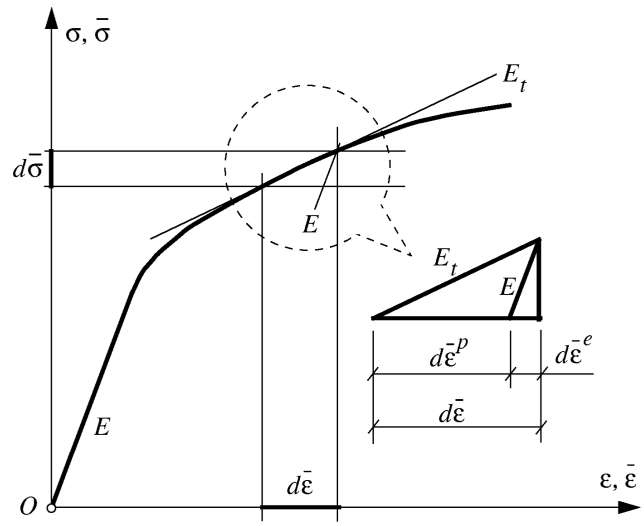


Figure 6: Uniaxial traction test.

so that

$$\begin{aligned} \frac{dW^p}{d\overline{\sigma}} &= \frac{d}{d\overline{\sigma}}\left(\overline{\sigma}d\overline{\epsilon} - \frac{\overline{\sigma}}{E}d\overline{\sigma}\right) \\ &= \overline{\sigma}\frac{d\overline{\epsilon}}{d\overline{\sigma}} - \frac{\overline{\sigma}}{E} \\ &= \overline{\sigma}\left(\frac{1}{\frac{d\overline{\sigma}}{d\overline{\epsilon}}} - \frac{1}{E}\right) \\ &= \overline{\sigma}\left(\frac{1}{E_t} - \frac{1}{E}\right) \\ &= \frac{\overline{\sigma}}{E'} \end{aligned} \tag{71}$$

having posed

$$E' = \frac{EE_t}{E - E_t} \tag{72}$$

so that

$$\frac{1}{E'} = \frac{E - E_t}{EE_t} = \frac{1}{E_t} - \frac{1}{E} \tag{73}$$

From (55), using (57) one can write

$$\begin{aligned} dW^p &= \sigma_{ij}d\epsilon^p_{ij} \\ &= \left(s_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}\right)d\epsilon^p_{ij} \\ &= s_{ij}d\epsilon^p_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}d\epsilon^p_{ij} \\ &= s_{ij}d\epsilon^p_{ij} \end{aligned} \tag{74}$$

thanks to the identity

$$d\epsilon^p_{ij}\delta_{ij} = d\epsilon^p_{kk} = 0 \tag{75}$$

where the last equality expresses the incompressibility condition (6) of Section 2.3.1.

From (74) one obtains

$$\frac{dW^p}{d\epsilon_{ij}^p} = s_{ij} \tag{76}$$

By replacing (the inverse of) (71) and (76) into (69) we obtain

$$\frac{\partial F}{\partial \epsilon_{ij}^p} = -\frac{E'}{\overline{\sigma}} s_{ij} \tag{77}$$

By introducing (54) and (77) into (46), one has

$$\begin{aligned} d\lambda &= -\frac{3}{2}\frac{s_{ij}}{\overline{\sigma}} d\sigma_{ij} \frac{1}{-E' \frac{s_{kl}}{\overline{\sigma}} \frac{3}{2} \frac{s_{kl}}{\overline{\sigma}}} \\ &= \frac{s_{ij} d\sigma_{ij} \overline{\sigma}}{E' s_{kl} s_{kl}} \end{aligned} \tag{78}$$

We can write

$$\begin{aligned} s_{ij} d\sigma_{ij} &= s_{ij} d\left(s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}\right) \\ &= s_{ij} ds_{ij} \\ &= \frac{1}{2} d\left(s_{ij}^2\right) \end{aligned} \tag{79}$$

since

$$d\left(\frac{1}{3}\sigma_{kk}\delta_{ij}\right) = 0 \tag{80}$$

From (26), assuming as usual that σ_c can be replaced by $\overline{\sigma}$, one has

$$\sqrt{\frac{3}{2} s_{ij} s_{ij}} = \overline{\sigma} \tag{81}$$

or

$$s_{ij} s_{ij} = s_{ij}^2 = \frac{2}{3}\overline{\sigma}^2 \tag{82}$$

and then

$$d\left(s_{ij}^2\right) = \frac{2}{3} d\left(\overline{\sigma}^2\right) = \frac{2}{3} 2\overline{\sigma} d\overline{\sigma} \tag{83}$$

With this result, we obtain from (79)

$$s_{ij} d\sigma_{ij} = \frac{2}{3}\overline{\sigma} d\overline{\sigma} \tag{84}$$

Finally, by using (26) and (84), the expression (78) of $d\lambda$ becomes

$$d\lambda = \frac{s_{ij} d\sigma_{ij}\overline{\sigma}}{E' s_{kl} s_{kl}} = \frac{\frac{2}{3}\overline{\sigma}^2 d\overline{\sigma}}{E' \frac{2}{3}\overline{\sigma}^2} = \frac{d\overline{\sigma}}{E'} \tag{85}$$

By using (82), (78) becomes

$$d\lambda = \frac{s_{ij} d\sigma_{ij}\overline{\sigma}}{E' s_{kl} s_{kl}} = \frac{s_{ij} d\sigma_{ij}\overline{\sigma}}{E' \frac{2}{3}\overline{\sigma}^2} = \frac{3}{2}\frac{s_{ij} d\sigma_{ij}}{E'\overline{\sigma}} \tag{86}$$

Let us now search a convenient expression of $d\sigma_{ij}$. From Hooke's law (2)

$$d\sigma_{ij} = D_{ijkl} d\epsilon^e_{kl} \tag{87}$$

or, with (1)

$$d\sigma_{ij} = D_{ijkl} \left(d\epsilon_{kl} - d\epsilon^p_{kl} \right) \tag{88}$$

With the normality rule (17) this becomes

$$d\sigma_{ij} = D_{ijkl} \left(d\epsilon_{kl} - d\lambda \frac{\partial F}{\partial \sigma_{kl}} \right) \tag{89}$$

and by means of (54)

$$d\sigma_{ij} = D_{ijkl} \left(d\epsilon_{kl} - d\lambda \frac{3}{2} \frac{s_{kl}}{\overline{\sigma}} \right) \tag{90}$$

Introducing this expression into (86) gives

$$\begin{aligned} d\lambda &= \frac{3}{2} \frac{s_{ij}}{E'\overline{\sigma}} D_{ijkl} \left(d\epsilon_{kl} - d\lambda \frac{3}{2} \frac{s_{kl}}{\overline{\sigma}} \right) \\ &= \frac{3}{2E'\overline{\sigma}} D_{ijkl} s_{ij} d\epsilon_{kl} - \frac{9}{4E'\overline{\sigma}^2} d\lambda D_{ijkl} s_{kl} s_{ij} \end{aligned} \tag{91}$$

Now, in analogy with (4) we can write

$$D_{ijkl} s_{kl} = 2\mu s_{ij} + \lambda s_{mm} \delta_{ij} = 2G s_{ij} \tag{92}$$

since $s_{mm} = 0$ according to the identity (52) and $\mu = G$ (shear modulus).

Furthermore, from (3) we obtain

$$D_{klij} = \lambda \delta_{kl} \delta_{ij} + \mu \left(\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li} \right) = D_{ijkl} \tag{93}$$

so that

$$D_{ijkl} s_{ij} = D_{klij} s_{ij} \tag{94}$$

and, for analogy with (92)

$$D_{ijkl} s_{ij} = 2G s_{kl} \tag{95}$$

By replacing (95) and (92) into (91)

$$\begin{aligned} d\lambda &= \frac{3}{2E'\overline{\sigma}} 2G s_{kl} d\epsilon_{kl} - \frac{9}{4E'\overline{\sigma}^2} d\lambda 2G s_{ij} s_{ij} \\ &= \frac{3G}{E'\overline{\sigma}} s_{kl} d\epsilon_{kl} - \frac{9G}{2E'\overline{\sigma}^2} d\lambda s_{ij} s_{ij} \end{aligned} \tag{96}$$

and by using the expression (82) of $s_{ij} s_{ij}$

$$\begin{aligned} d\lambda &= \frac{3G}{E'\overline{\sigma}} s_{kl} d\epsilon_{kl} - \frac{9G}{2E'\overline{\sigma}^2} d\lambda \frac{2}{3} \overline{\sigma}^2 \\ &= \frac{3G}{E'\overline{\sigma}} s_{kl} d\epsilon_{kl} - \frac{3G}{E'} d\lambda \end{aligned} \tag{97}$$

From (97) one obtains

$$d\lambda \left(1 + \frac{3G}{E'} \right) = \frac{3G}{E'\overline{\sigma}} s_{kl} d\epsilon_{kl} \tag{98}$$

and then

$$d\lambda = \frac{3G}{(E' + 3G)\,\overline{\sigma}} s_{kl} d\epsilon_{kl} \tag{99}$$

The normality rule (17) and (54) yield

$$d\epsilon^p_{ij} = \frac{3}{2} d\lambda \frac{s_{ij}}{\overline{\sigma}} \tag{100}$$

which, by using (99), becomes

$$\begin{aligned} d\epsilon^p_{ij} &= \frac{3}{2}\frac{s_{ij}}{\overline{\sigma}}\frac{3G}{(E' + 3G)\,\overline{\sigma}} s_{kl} d\epsilon_{kl} \\ &= \frac{9G}{2\,(E' + 3G)\,\overline{\sigma}^2} s_{ij} s_{kl} d\epsilon_{kl} \end{aligned} \tag{101}$$

By replacing this into (88)

$$\begin{aligned} d\sigma_{ij} &= D_{ijkl}\left(d\epsilon_{kl} - \frac{9G}{2\,(E' + 3G)\,\overline{\sigma}^2} s_{kl} s_{mn} d\epsilon_{mn}\right) \\ &= D_{ijkl}\left(\delta_{km}\delta_{ln} - \frac{9G}{2\,(E' + 3G)\,\overline{\sigma}^2} s_{kl} s_{mn}\right) d\epsilon_{mn} \end{aligned} \tag{102}$$

since

$$d\epsilon_{kl} = d\epsilon_{mn}\delta_{km}\delta_{ln} \tag{103}$$

By using (93) and the identity

$$D_{ijkl}\delta_{km}\delta_{ln} = D_{ijmn} \tag{104}$$

we obtain finally, from (102)

$$d\sigma_{ij} = \left[D_{ijmn} - \frac{9G^2}{(E' + 3G)\,\overline{\sigma}^2} s_{ij} s_{mn}\right] d\epsilon_{mn} \tag{105}$$

which is the relationship that we were trying to establish. It can be noted that, by using this relation in a finite increment procedure, the approximation introduced is on the term $(s_{ij}s_{mn})/\overline{\sigma}^2$, which is typically evaluated (only) at the beginning of each time step, while in reality it varies during the step.

By using the measure (29) of the hardening, one has:

$$\frac{\partial F}{\partial \epsilon^p_{ij}} = -\frac{\partial \overline{\sigma}}{\partial \epsilon^p_{ij}} = -\frac{\partial \overline{\sigma}}{\partial \overline{\epsilon}^p}\frac{\partial \overline{\epsilon}^p}{\partial \epsilon^p_{ij}} = -\frac{\partial \overline{\sigma}}{\partial \overline{\epsilon}^p}\frac{s_{ij}}{\overline{\sigma}} \tag{106}$$

The ratio $d\overline{\sigma}/d\overline{\epsilon}^p$ can be found from a traction test. In fact, in such a case $d\epsilon^p_{22} = d\epsilon^p_{33} = -d\epsilon^p_{11}/2$ (due to (6), plastic incompressibility), $d\epsilon^p_{ij} = 0$ (for $i \neq j$), so that:

$$d\overline{\epsilon}^p \equiv d\epsilon^p_{11}$$

Figure 6 shows that one has

$$d\overline{\sigma} = E_t\, d\overline{\epsilon}$$

where E_t is the tangent modulus of elasticity. Then, by (1):

$$d\overline{\sigma} = E_t(d\overline{\epsilon}^e + d\overline{\epsilon}^p) = E_t(d\overline{\sigma}/E + d\overline{\epsilon}^p)$$

from which one gets:

$$\frac{1}{d\overline{\sigma}/d\overline{\epsilon}^p} = \frac{1}{E_t} - \frac{1}{E} = \frac{1}{E'}$$

References

[1] Hill R. *The mathematical theory of plasticity.* Clarendon Press, Oxford, 1950.

[2] Naghdi P.M. *Stress-strain relations in plasticity and thermoplasticity.* Proc. second symposium on naval structural mechanics, Pergamon Press, 1960; p. 121.

[3] Koiter W.T. *General theorems for elastic-plastic solids.* Progress in solid mechanics; Vol. 1, Chap. 4, North-Holland, 1960; p. 167.

[4] Bogy D.B. *Theory of plasticity.* Lect. AM 286, University of California, Berkeley, 1968.

[5] Drucker D.C. *A more fundamental approach to plastic stress-strain relations.* Proc. first US national congress of appl. mech., Chicago, 1951; p. 487.

[6] Drucker D.C. *A definition of stable inelastic material.* Trans. ASME, J. appl. mech., Vol. 26, 1959; p. 101.

[7] Bland D.R. *The associated flow rule of plasticity.* J. mech. and physics of solids, Vol. 6, 1957; p. 71.

[8] Prager W. *An introduction to plasticity.* Addison-Wesley, 1959.

[9] Ziegler H. *On the theory of the plastic potential.* Quart. of appl. math., Vol. XIX, 1961, n. 1; p. 39.

[10] Massonnet Ch. *Résistance des Matériaux.* Volume 2, Sciences et Lettres, Liège, 1973.

[11] Prager W. *A new method of analysing stress and strains in work-hardening plastic solids.* Trans. ASME, J. appl. mech., Vol. 23, 1956; p. 493.

[12] Ziegler H. *A modification of Prager's hardening rule.* Quart. of appl. math., Vol. XVII, 1959, n. 1; p. 55.

[13] Bland D.R. *The two measures of workhardening.* IXème congrès intern. de méc. appl., Bruxelles, 1957; Actes, tome VIII, p. 45.

[14] Bergan P.G. *Non-linear analysis of plates considering geometric and material effects.* Report SESM 71-7, University of California, Berkeley, 1971.

[15] Berg C.A. *A note on construction of the equivalent plastic strain increment.* Journal of Research of the National Bureau of Standards – C. Engineering and Instrumentation, Vol. 76C, nos. 1 and 2, January-June, 1972.

LB-NA-27593-EN-N

JRC Mission

As the Commission's in-house science service, the Joint Research Centre's mission is to provide EU policies with independent, evidence-based scientific and technical support throughout the whole policy cycle.

Working in close cooperation with policy Directorates-General, the JRC addresses key societal challenges while stimulating innovation through developing new methods, tools and standards, and sharing its know-how with the Member States, the scientific community and international partners.

Serving society
Stimulating innovation
Supporting legislation

Publications Office

doi:10.2788/233591

ISBN 978-92-79-53988-6