

Start a Math Teacher Circle: Connect K-12 Teachers with Engaging, Approachable, and Meaningful Mathematical Problems

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Abstract

Many K-12 math teachers are not ready to teach from a conceptual and inquiry-oriented perspective because they have an algorithmic understanding of mathematics. One solution is to create a math teacher circle (MTC), which provides conceptual and inquiry-based learning activities and builds professionalism among the teachers. In this paper, we describe the origins of two such MTCs, highlighting the process of identifying leadership team members, submitting the grant proposal for seed money, and hosting launch events, intensive summer workshops, and monthly meetings during the academic year. We also share opportunities for professional development for college and university faculty, including research linked to shifts in in-service teacher attitudes. We finish the paper with several of this year's best activities used at our MTC meetings, including fair division, extensions and generalizations of numerical and algebraic patterns, and applications in cryptography.

1 Introduction

Close your eyes and imagine a typical middle school mathematics classroom. Are you picturing students seated at desks in rows and columns solving dozens of practice problems involving decimals, fractions, and percents? Or, were you envisioning students working collaboratively to solve mathematically rich, engaging, and open-ended tasks? While we might hope that teachers at these and similar grade levels would create classroom environments resembling the latter description, too many bear a striking resemblance to the former characterization. The result is procedure and algorithmic learning with little conceptual understanding or development of problem solving ability. To address this phenomenon, groups of K-12 mathematics teachers have collaborated with college and university faculty members to form math teachers' circles. In fact, during the past decade, nearly 100 MTCs have been formed in the United States, most with seed money grants from the American Institute of Mathematics (AIM). We hope to convince you to expand the reach of these MTCs by creating one in your local area.

2 How to Create a MTC

The necessary components of a MTC include people, problems, and pizza (or other food items). For people, it is important to form a leadership team that can apply for start-up funds often from the American Institute of Mathematics (AIM), form the circle, and hold regular meetings to make decisions about upcoming monthly meetings. To receive the seed money grant from AIM, the team must include at least two K-12 mathematics teachers along with multiple college and university faculty members. Typically, these faculty members will come from mathematics, mathematics education, or education departments. Once the team is in place, a chairperson and treasurer will need to be identified who can prepare regular reports and financial statements for AIM.

For problems, a great resource is the national website for MTCs listed in the reference section. Individual MTCs also maintain websites that include resources for activities used in recent meetings (see reference section for our two MTC websites). Leadership team members are also encouraged to develop their own mathematical activities or recruit local or national speakers to present their own activities at meetings. If your MTC agrees to host a single or multi-day workshop for local mathematics teachers, then separate funds from AIM may be available to cover the speaker costs, food costs, and materials costs for the workshops. In the case of the Southwest Chicago MTC, these funds proved effective to host an initial launch event prior to a later 3-day workshop and the start of monthly meetings during

the subsequent academic year. The AIM funding also helped to bring in two national MTC leaders to help with the workshop. These speakers initially led sessions on their own but then co-led sessions with members of our own more inexperienced leadership team before handing full responsibilities over to us. The Northwest Iowa MTC kicked off with a two-day summer workshop that also offered continuing education credits for the participants. Monthly sessions during the academic year followed including a fascinating investigation of the mathematics of bicycle tracks led by James Tanton with support from the AIM seed grant.

With regard to food and other logistics, it has worked well for the Southwest Chicago MTC to ask interested teachers to complete an RSVP on our website that includes information such as dietary restrictions. Some colleges and universities place restrictions on food brought onto campus by outside vendors, but most allow pizza or similar food for a monthly meeting or workshop. In fact, as part of the seed fund application, potential MTCs must provide letters of support from the department chairs of the institutions at which the monthly meetings are held. It is expected that the hosting departments are willing to provide basic resources and spaces for the meetings, though sometimes departments will work with local school districts so that meetings can be hosted in area schools.

Goals for monthly meetings include time for an initial math problem or two, a break for dinner, and the main activity with time for reflection at the end of the session. It is ideal if the initial problems transition well into the main activity although this is not necessary. Where appropriate, we provide the K-12 math teachers with links to state or national standards (e.g., Common Core State Standards for Math, [5]) for the main activity. In some cases, teachers must modify the activity to meet the education needs of students at specific grade levels. However, the overriding goal for MTC sessions is to offer the K-12 teachers mathematically rich and engaging activities, even if some of these activities are not usable in their classrooms without major modifications, if at all. The final 15-20 minutes of a monthly meeting is an opportunity for the teachers to reflect, first individually in writing and later in conversation with the entire group, on ways that their own strategies for solving math problems and willingness to try open-ended mathematical tasks may have grown or improved as a result of the activities from the session.

We recommend hosting meetings on a monthly basis. However, the Southwest Chicago MTC met only six times during the last academic year, skipping holidays and other conflicts in December as well as state testing preparation and other conflicts in March and April. The location for the MTC can be the same for all meetings (as with the NW Iowa MTC hosted at Dordt College) or can rotate among several colleges and universities (as with the Southwest Chicago MTC).

3 Benefits of MTCs

There are numerous potential benefits of a math teacher circle. First and foremost, the attitudes of K-12 mathematics teachers often shift toward a greater enjoyment of mathematically rich and engaging activities as well as a willingness to experiment with more open-ended math problems in their own classrooms. Angela Antonou reported in [1] on data from pre- and post-surveys completed by 13 such teachers at the 3-day workshop hosted at Trinity Christian College. Among the findings were shifts to more inventiveness and more confidence in creating opportunities to develop students' conceptual understanding of mathematics. Reflection journals were equally notable in the high praise from the teachers. Here's a sample:

“Each activity continues to open possibilities I hadn’t considered doing with my students.”

“I’ve noticed that I’m becoming less afraid to try different things.”

“It made me understand the vast usefulness of productive struggle. My students need to engage in this type of problem weekly! It would help them to persevere!!”

“I appreciate the opportunity to stretch my math skills. I don’t often have a chance to do this.”

Beyond the benefits for classroom teachers, current math education students at your own institution will benefit from participating at an MTC session. At the Southwest Chicago MTC meetings, the host institution typically invites up to five such students to attend. The NW Iowa MTC provides the opportunity for local undergraduate students to participate as well. We have found that a critical mass of classroom teachers is essential, but having a few undergraduates and/or university faculty members in attendance enhances the experience for everyone.

As illustrated by this paper, MTCs also provide opportunities for professional development among college and university faculty members. Several members of the Southwest Chicago MTC have spoken at local, regional, and national conferences. Clark recently published an article [4] in *MTCircular*, a national journal that solicits short articles describing original activities used at MTC meetings. Hendrickson devoted her entire dissertation [6] to an analysis of math teacher circles and published a short overview of her findings in *MTCircular*. Recent upcoming national and regional conferences, including the Joint Mathematical Meetings, MathFest, MAA Section meetings, and the Annual Convention of the National Council of Teachers of Mathematics, have included multiple sessions and even a full panel of presentations related to MTCs.

4 Exemplary Activities from our Recent MTC Meetings

We close the paper with brief descriptions of three original mathematical activities that were presented at our MTCs during the past year. Full descriptions and/or worksheets are available either on the MTC websites, [10] and [11], or directly from the authors. More examples of activities and related classroom resources can be found among the websites listed in the reference section.

4.1 Fair Division

In February of 2016, the NW Iowa Math Teachers Circle played with fair division questions. A fair division problem is one in which a number of participants (also called *players*) seek to divide a heterogeneous resource¹ in such a way that each player feels the resulting division is *fair*. There are at least two notions of fairness: proportionality, and envy free-ness. A proportional division is a division in which each of the N players feels they have received at least $1/N$ th of the value of the resource being divided. An envy-free division is a division in which each player values his/her resulting piece at least as much as every other piece.

¹For the purposes of the problem, it is important that the resource be heterogeneous. A homogeneous resource (such as a can of soda) can be split evenly by volume, but I may be willing to accept a slightly smaller (by volume) piece of a cookie if it has the majority of the chocolate chips.

Many are familiar with the simplest fair division problem: two people seek to divide a heterogeneous resource. A common example is two siblings seeking to divide a cookie fairly. The elegant solution is known as the “I cut, you choose” method: the first sibling cuts the cookie into two pieces she would be equally pleased with, and the second chooses the piece he wants. This leads to an *envy-free* division in which neither sibling values the other’s piece above the one s/he ended up with.

Our MTC activity began by exploring this two-player fair division problem. We discussed underlying assumptions and goals, and ensured that we could clearly articulate how the “I cut, you choose” method produces an envy-free division. We then asked the natural question: what about three players? For the purposes of experimentation, brownies were provided.

The solution to the three-player fair division problem is subtler. With some time and prompting, participants were able to construct the ‘moving knife’ procedure. In this procedure, player 1 moves a knife left-to-right across the brownie. As soon as one of the players believes that, in his/her opinion, $1/3$ of the value of the resource is to the left of the knife, that player yells ‘stop!’ and player 1 cuts the brownie. The remaining two players use the “I cut, you choose” method on the remaining part of the brownie.

While the moving knife solution is elegant, it produces a division of the brownie which is proportional but not necessarily envy-free. We considered why that is the case, and tried to come up with a way around this obstacle. However, we were unable to do so. This is not terribly surprising, as an envy-free algorithm was not discovered until the twentieth century by Selfridge and Conway [3].

The feedback from the session was fairly positive. The one negative comment was that the problem did not “feel like math”. This is true, but the author sees it as a feature and not a bug. While it is true that fair division problems are not generally a part of the K-12 curriculum, they are a rigorous application of mathematical/algorithmic thinking to solve real-world problems, and there are several points of contact with the Common Core State Standards for Mathematical Practice which can be made explicit.

4.2 Numerical and Algebraic Patterns

Exploring number patterns and sequences can provide interesting and accessible activities to actively engage students. Students are challenged to look for relationships and test hypotheses to determine if the patterns that they see are always true or if they describe only some but not all of the values. This exploration builds students’ abilities to organize and visualize information and enables them to uncover relationships through different representations and to make generalizations. This type of thinking resonates with the Common Core State Standards. The varying levels of complexity of sequences also allow these activities to be used with students at multiple grade levels.

The May 2017 Southwest Chicago MTC challenged teachers to explore multiple number patterns and representations. We started by considering the standard locker problem ² and modifications where only even numbered students changed lockers or only odd numbered students changed lockers.

We provided two-color disks as a means of representing closed and open lockers. It surprised us that few of the teachers chose to use the manipulatives as they solved the problems preferring instead to

²If, in a school of 500 lockers, one student opens every locker, a second student, beginning at the second locker, closes every second locker, a third student, beginning at locker three, changes every third locker and so on until the 500th student changes the 500th locker, which lockers are then open?

utilize a variety of table structures to organize their work. After some prompting, some of the groups of teachers incorporated these disks. Several interesting number theory results emerged in the final solutions including perfect squares for the original problem and two times a perfect square for the extension involving only the even-numbered students.

From there we moved to exploring sums of consecutive numbers. We had the teachers investigate the sum of the first one hundred natural numbers and then generalize the relationships that they discovered for n natural numbers. This problem is interesting because the teachers were able to utilize a variety of strategies and different representations offered different insights even though the solutions were equivalent. We found that the in-service teachers were able to explore the problem more freely and effectively than some of the college professors. Knowing that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ seemed to inhibit some of the professors from exploring the problem as a middle school student might. We provided linking cubes to enable tactile exploration and through discussion many were able to discover and justify the generalized relationship. Shown in figure 1.

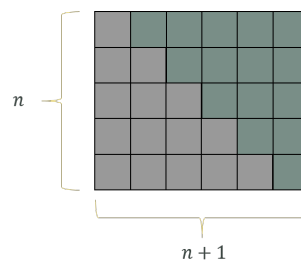


Figure 1: Number of blocks in one staircase = $\frac{n(n+1)}{2}$

The connection to visual representations provided the impetus for the remaining number patterns explored. The teachers initially were given a sequence of numbers, for example: 5, 11, 19, 29, 41, 55, . . . and were asked to find the next term, then the 10th term, and finally the n^{th} term for arbitrary n . The teachers were encouraged to use linking cubes to construct visual representations to supplement the numerical evidence they were generating. The representations shown in figure 2 were considered and a rich discussion of how a visual representation can provide new insight or justification for conjectured numerical relationships ensued. Teacher feedback was positive and some commented that having groups of students construct visual representations and translating them into numerical patterns to challenge their classmates could be a nice extension.

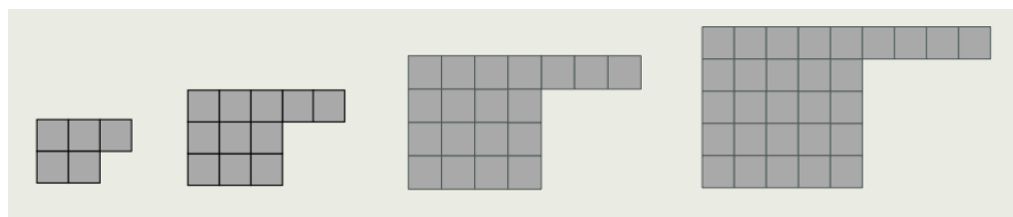


Figure 2: Visual representation of numerical patterns.

4.3 CryptoClue

Cryptography is a natural focus for mathematically rich, open-ended, and accessible activities that bridge academic disciplines and enhance professional experiences of K-12 mathematics teachers. Fun

puzzles designed using historical cryptosystems allow the mathematics classroom to become an interdisciplinary learning arena. In this arena, teachers can motivate the study of measurement, frequency analysis, statistics, and modular arithmetic using the historical context and political intrigue of secret codes.

At the February 2017 meeting of the Southwest Chicago MTC, participants solved CryptoClue puzzles using three historical cryptosystems. For the first, teachers deciphered a clue with letters written on a ribbon by identifying the particular diameter of tube from among six options the sequence of letters reformed into a sensible statement. This modern version of a fifth century BC Spartan scytale (rhymes with “Italy”) is an easy way to demonstrate how rod diameter can camouflage a message. The second clue was hidden using a shift cipher, where the alphabet is shifted uniformly by an unknown amount. For this variation on the Caesar cipher, teachers used a cipher wheel and were encouraged to experiment until the message was revealed. The final clue was hidden using a monoalphabetic substitution cipher where the denomination and suit of playing cards replaced letters in a random order. Since this message was designed to be too short to succumb to frequency analysis, teachers earned card-letter pairing by correctly solving a series of standard mathematical problems at the middle school level.

After the teachers solved each portion of the puzzle, the facilitator presented some cryptographic extensions, ideas for how to use cryptography in the individual classrooms of the teachers, and resources to aid in clue development. Examples included a rail fence cipher as a variation of the scytale and a Vigenère cipher as an extension of the shift cipher. A few excellent resources for those who wish to develop their own puzzles are books by Singh, Beissinger and Pless, and Lewand; a helpful online tool for both encryption and decryption is the Black Chamber website that serves as a companion to the Singh text.

The CryptoClue puzzles used in the 2017 MTC meeting were developed by Trinity Christian College students enrolled in a cryptography class for use in a junior high mathematics competition. Both MTC teacher participants and junior high competitors were initially reluctant to experiment in solving the puzzles, but all became more willing to explore and experiment as the respective events progressed. Perhaps by having the participants encode messages using each cryptosystem prior to working on decoding the puzzles, the participants might have felt more comfortable and jumped more quickly into the activities.

At the conclusion of the event, teachers were eager to bring tools home to develop more puzzles to enhance their own curriculum; at least one junior high student reported that he continued looking for cryptographic puzzles to solve the summer after the initial exposure to cryptography. Of the college students who designed the puzzles, one reported using cryptography as a basis for a mathematics classroom activity as a practicing teacher while several others reported how study of cryptography expanded their personal view of mathematics.

The puzzles used in the February 2017 MTC are available upon request.

5 Conclusion

Math Teachers’ Circles offer opportunities for collaborative solving of engaging, approachable, and worthwhile mathematical tasks. Some tasks are designed to impact K-12 mathematics classrooms by introducing novel and engaging mathematics to classroom teachers as have been described above, other

tasks allow teachers to explore typical content in new and interesting ways for example “Exploding Dots” [8] or “Conway’s Rational Tangles” [9]. MTCs also offer professional development options for college and university faculty members. For all of these reasons, and many more, we encourage others to start a MTC too!

References

- [1] Antonou, Angela. (2017). “Impact of the Southwest Chicago Math Teachers’ Circle on the Disposition of Teachers Toward Mathematics and Toward the Teaching and Learning of Mathematics.” Presentation at *Joint Mathematical Meetings*, Atlanta, Georgia.
- [2] Beissinger, Janet, and Pless, Vera. *The Cryptoclub Using Mathematics to Make and Break Secret Codes*. Wellesley, MA: AK Peters, Ltd., 2006.
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- [5] Common Core State Standards Initiative. 2011. *Common Core State Standards for Mathematics*.
- [6] Hendrickson, Katie. (2017): A Teacher and a Mathematician: The Identity Element. *MTCircular*, 4-7.
- [7] <http://www.mathteacherscircle.org/resources/math-sessions/> provides a wealth of engaging, approachable, and worthwhile activities for use at your MTC.
- [8] <https://www.explodingdots.org/> provides guides and worksheets to do exploding dots, an excellent exploration of place value and polynomials.
- [9] <http://www.geometer.org/mathcircles/tangle.pdf> provides a paper by Tom Davis describing how to use ropes to review fractions, functions, slopes of lines, and more.
- [10] <https://www.dordt.edu/events/math-teachers-circle> is the homepage for MTC sponsored by Dordt College.
- [11] <https://southwestchicagomathcircle.wordpress.com/> is the homepage for MTC sponsored by Lewis University, Saint Xavier University, Trinity Christian College, and University of St. Francis.
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