A Pre-Calculus Controversy: Infinitesimals and Why They Matter

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Abstract

In teaching calculus, it is not uncommon to mention the controversy over the role of infinitesimals with Newton's and Leibniz' calculus, including Berkeley's objections. In a history of mathematics course, it is a required topic! But rancor over infinitesimals and their role in mathematics predates calculus—so much so that a popular recent book is dedicated to this topic.

In this paper, I will discuss not just the relevant controversies between Cavalieri and the Jesuits, between Thomas Hobbes and John Wallis, but also why they matter to us today, both as mathematicians and as Christians. Perhaps most importantly, I will present the book in question on these controversies as an example of overreach in the history of science, along with suggestions for how to bring the whole story into the mathematics classroom.

1 Introduction

Students often repeat colorful stories we tell them to 'spice up' our classes. And why not? I have long used the non-mathematical anecdotes about John Napier which Eli Maor [28] recounts to give them a sense of the real—and often quirky—people in mathematics. Ramanujan recognizing the taxicab number 1729 as a sum of cubes in two different ways [21] is inspiring in a whole different way, and reinforces the mathematics. What's wrong with a good story?

But there can be pernicious aspects to telling a story. The oft-told meeting between Diderot and Euler where Euler 'proves' God's existence with a random equation seems to have existed (in its usually-told form) only in the playful mind of Augustus DeMorgan[1], and doesn't really do credit to Euler's faith or Diderot's mathematical ability. Some 'liberal arts' math courses still assert the golden rectangle is the basis for art through the centuries, including the Parthenon, even though it has been years since the notion was rightfully debunked (see e.g. [24]).

When there are good true stories to tell, everyone suffers when we tell the false ones. This is especially important when the stories directly impact the matters students are studying. If one tells the story of a man named Hippasus being tossed from a ship with regard to incommensurables, better to talk about what it says about worldviews that such a story seemed credible when first written down centuries later[2]. It's good to tell about Newton versus Leibniz; it's even better to use this as an opportunity to say just

[1] See [12] on some of the history around how this became popular.

[2] See [17], p. 46 for a particularly dry dismissal of this story.

how difficult the process of coming up with calculus was, and how remarkable it is that something so challenging and controversial is now taught to college freshmen as a matter of course.

Along those lines, in this paper, I will discuss some aspects of the story of infinitesimals, and why they matter. But I will also talk about why they don't matter in quite the way a recent popular book says they do, and why that matters. Along the way, we'll see why you've probably never heard of Thomas Hobbes the scientist, and why that matters. My goal is that you would be able to incorporate both the lesson about stories for your classes, as well as to see infinitesimals in a new light that could inspire them.

2 Infinitesimals

Usually an aspiring mathematician first encounters the idea of infinitesimals when studying calculus. In a typical 'modern' curriculum, they are cloaked in the language of limits, but as educators we are not amiss in mentioning Leibniz and how he came up with the $\frac{dy}{dx}$ notation[3]; in differential equations we might even teach students how to use differentials in an formal/infinitesimal fashion. In an ambitious class, or later in real analysis, we may be made aware of Berkeley's famous objections[4] to "ghosts of departed quantities" and his taking to task those who "submit to Authority, take things on Trust" in mathematics but refuse to do so in religion.

Later, in a math history or capstone course we may see fascinating revelations (see [34], [29]) that even Archimedes used infinitely small sections to originate many of his great theorems on volumes and areas, even if he had to hide their genesis in his formal proofs. I will assume that most of us did not profit from a mathematical logic course enabling us to learn 'modern' Robinsonian [31] infinitesimals, though there is once again a movement afoot to make it useful for the freshman audience[5], as the pre-Cauchy practice was in France [6].

But let us start at the beginning; what does it mean for something to be infinitely small? Right off, we can see that such an object must be at least partly notional, since we couldn't directly see it. Nonetheless, there are two ways to conceive of an object that small. A typical analogy, most often associated with the Italian Jesuat[6] brother and mathematician Bonaventura Cavalieri, is of infinitely thin pages of a book.

1. The pages could be so thin that, no matter how (finitely) many you stacked, all of them together would not be as thick as any actual book; you would need infinitely many. Such pages are *infinitesimal*.
2. The pages could be so thin that, no matter how you tried, you could not slice them any thinner; they are *indivisible*. They are parts of the book, but if infinite in number, whether they comprise the whole book was open to question.

[3] Though note that perhaps Leibniz did know what he was doing with regard to infinitesimals; see "Is Mathematical History Written by the Victors?" [4].

[4] See any source on the history of calculus for *The Analyst*; excerpts are in V.12 of [35], though the second quote is from Query 64 at the end, which is easy to find online.

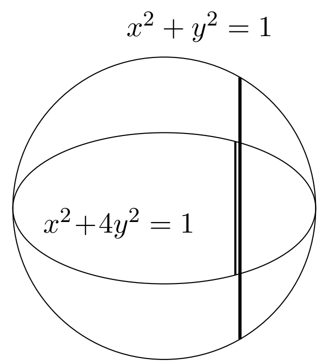
[5] There has always been a small cohort using the now-venerable Keisler text, which he has since retained copyright for and posted online [22]; and several authors of [4] have also advocated for this for some time. For a more recent venture, see publications by Bryan Dawson of Union University, such as [11].

[6] As always has to be explained regarding Cavalieri, he was a member of the Jesuat/Gesuat, not Jesuit, order. It flourished from 1361 until suppressed by papal decree in 1668.

An important distinction between these notions is that indivisibles are typically considered to have strictly lower dimension than the original object, so for instance in the book example the pages are only two-dimensional, even though the book itself is three-dimensional. In contrast, infinitesimals in principle do not have to be indivisible; one could imagine three-dimensional pages which, though already infinitely thin, could still always be sliced to be half as thin again. These would be homogeneous, of the same dimensionality as the page, just ever slimmer. We will return to this idea again.

To see how this might be used to solve 'pre-calculus' problems, consider Figure 1. Using this type of argument, one can relate the area of an ellipse to that of a circle by noting that each indivisible 'slice' of this particular ellipse is half the height of the circle. Hence the total area of the ellipse is half that of the circle as well. Cavalieri himself was both a master user of indivisibles (his book was called *Geometria Indivisibilibus Continuorum*; see [35], IV.5 for excerpts) and perforce a very careful one—cautious to the point of his work being very difficult to read—but this example would have passed muster.

Figure 1: Ellipse area via Cavalieri's principle.



Now we are in a position to state the main thesis of historian Amir Alexander's recent book [2], *Infinitesimal*. Namely, those who believed in such entities, and in using them without caution in mathematics and science, were at the forefront of "the ultimate victory ... [of] a new and dynamic science, [of] religious toleration, and [of] political freedoms." Those who put barriers in the way of their use are responsible for the general and grim intellectual darkness of post-Galilean Catholic Europe; the lack of such barriers made England a veritable beacon of prosperity and (relative) religious freedom. We will unpack this, and the infinitesimals themselves, in the succeeding sections.

3 The Jesuits and Indivisibles

In the first half of his book, Alexander tells a gripping story of Jesuits seeking to control the post-Reformation madness and anarchy in philosophy, theology, and civil order, using their effective network of schools to remove this pernicious and anti-Aristotelian notion[7] from mathematics in lands where they had the upper hand. Only Euclidean geometry in its unique perfection could be taught, and then only as a handmaiden to the notion that there is one truth, one right theology, and one approach to civil order

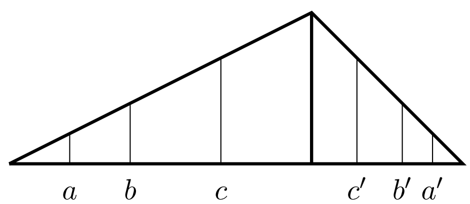
[7] An article [32] to appear in *Foundations of Science* argues that the Jesuits more objected to indivisibles (and not infinitesimals) due to implications for the nature of the Eucharist; Alexander has a response [3] which is presumably to appear with the article when it is officially published.

– headed by the Pope. Infinitesimals, which used sketchy logic, had to be suppressed, or political and religious chaos would result, because they were a "simple idea that punctured a great and beautiful dream: that the world is a perfectly rational place, governed by strict mathematical rules."

Now, there is plenty of truth in this story. First, there is no doubt as to the remarkable success of the Jesuit order in (among other things) establishing rigorous, desirable, 'safe' schools for the minor nobility[8] and nascent bourgeoisie – all in the name of the Catholic/Counter-Reformation. Their most prominent mathematician, Christopher Clavius, was a solid proponent of Euclidean geometry in these schools; under his aegis the church (and hence much of Europe) achieved the long-sought goal of calendrical reform in 1582, where October 15th followed October 4th to make up for the solar year not being evenly divisible by standard days.

Indivisibles did not seem to follow that same impeccable logic. Using similar ideas (indeed, this non-example goes back to Cavalieri), one could assert that two obviously non congruent triangles have the same area. See Figure 2.

Figure 2: Triangles which do not have the same area.



This should be thought of as two triangles, each with the same height but obviously different areas. It should be clear that any vertical segment (indivisible) in the left-hand triangle can be corresponded to a vertical segment of the same length in the right-hand triangle; three sample pairs (a and a', b and b', c and c') of segments are drawn in Figure 2. Cavalieri's principle would, improperly applied, imply that they have the same area.

Such deficiencies had no place in traditional mathematics. While some proponents found various ways to govern their use, for many geometers[9] it was better to simply avoid their use altogether. And it is certainly true that the Jesuits as an order repeatedly disallowed any use or teaching of these concepts in their schools, and many individual Jesuits wrote long screeds against those who used them for the first half of the seventeenth century. (In my view, Berkeley's later objections, though different, are analogous.)

But the overall story, alluring as it may be in our society of absolute freedom and intellectual inquiry, and as exciting as it might be to think of calculus (!) as having political importance so early on[10], is selling a bill of goods. The suppression of indivisibles[11] was surely part of a long fight over new ideas and to what extent the Church could direct or restrict many activities, but not the only one, and much of

[8]For example, Descartes attended the 'College' at La Flèche.

[9]See [25] for many interesting remarks not just on the use of the terms geometer/mathematician, but for the more subtle question of whether anyone before 1650 could properly be called a mathematician in the modern sense.

[10]This is completely separate from whether applications like the Gini coefficient or discussing unequal access to calculus might have contemporary political ramifications.

[11]Not infinitesimals, though Alexander constantly equates the terms; see again [4] as well as [15].

the hyperbolic verbiage used was par for the course at that time.

Despite Alexander's often deft handling of the distance between (post) modern secular readers and the worldview landscape fifty years on either side of Galileo, much of the political discussion is speculation, or simply wrong[12]. As just one example, while in such a political age it was possible that the Jesuits got the Pope to suppress the Jesuat order, and (barely) conceivable they would have been motivated because some of the most prominent advocates of indivisibles (notably Cavalieri) were from it, among his copious endnotes there is not one to be found about this topic.

We should teach about incommensurables, and that the Pythagoreans may have had strong feelings about this; but we should also make it clear that the death of Hippasus is probably just a story. Likewise, the example of the triangles is a good warning against setting up integrals without caution, and the very real fights waged over infinitesimals/indivisibles for some of these reasons (Galileo was an early advocate) is a wonderful topic in a Calculus II course. But let's not suggest it is really about preparing the West for a modern secularist worldview (as even his subtitle *How a Dangerous Mathematical Theory Shaped the Modern World* implies) that Alexander acknowledges no one in question was actually fighting for[13].

4 Hobbes versus Wallis

I have promised a positive statement of why infinitesimals matter, and why they should matter to mathematics educators. But to prepare us for this, we need to observe a second, somewhat more famous controversy, which the second half of Alexander's book focuses on.

Until I began teaching history of mathematics to pre-service teachers I was unaware, perhaps like many readers, that 'the Monster of Malmesbury,' Thomas Hobbes, wrote anything other than political philosophy. Yet his early fame rested as much on contributing to 'natural philosophy', in particular geometry, as anything else. This fact makes it less surprising that only three years after publishing *Leviathan*, Hobbes includes many geometric results, including no fewer than three 'squarings of the circle', in what was intended to be his magnum opus, *De Corpore* ('On Body').

In Hobbes' philosophical scheme everything was material, and all his accounts of the world followed as logically and inexorably from body in motion as geometry did from Euclid's definitions and postulates. This materialism includes mathematics, as demonstrated in the latter third of *De Corpore*. More specifically, Hobbes used methods very similar to Cavalieri's[14]. Cavalieri himself was very careful to leave open the question of whether these indivisibles actually comprise a figure. Consider the opening

[12] It is amazing how many reviewers came to this conclusion. The Chronicle of Higher Education review [14] confirms "[this] world of moribund Roman Catholic thought going nowhere" has little contemporary traction among historians of science. One blogger even went so far as to suggest the story should rather not be told than be told this way, though I hope this paper argues something more positive. In the *Mathematical Intelligencer* review (by the author of [20]) we read, "The errors in this account [are] . . . so great that it is difficult to know where to begin ..." Interestingly, Alexander then responds in the next issue that the review "is so misleading that it cries out for at least a brief correction," [1, 18]. Perhaps a war of their own over infinitesimals is brewing.

[13] In [1], he suggests that the book does not argue "the struggle over infinitesimal methods was 'actually' about politics" but rather that "one will search the pages of *Infinitesimal* in vain for any suggestion of such subterfuge." Perhaps, but even if "the implications of their stance[s] extended . . . to the proper order of the world" then one should stick to their story in their times, rather than veer toward Whiggish history.

[14] Alexander and Jesseph concur in one thing – that Hobbes may have been the only man in Europe to actually read Cavalieri's work all the way through in detail and understand it.

example; do all the leaves of such a book make up the book? Under Aristotelian principles, this would imply an actualized infinity, which is anathema. Hobbes takes a more direct, physicalist, view, that the pages are very small indeed – so small that we do not take their width into consideration – but they are still 'body' (as is everything in existence) so they do still have a width; this is decidedly not the usual viewpoint on indivisibles, but he requires it.

Now, Hobbes reasoned if he could achieve the long-desired (Euclidean) construction of a square with the same area as a circle using his techniques, then surely his entire philosophy would be accepted – including otherwise-distasteful-to-all outcomes such as the Leviathan totalitarian-yet-not-monarchist regime. (Here is where Alexander draws a direct comparison to the top-down regime promoted by the Jesuits, adherents of papal authority as their *raison d'etre*.) Hobbes alluded to his successful squaring on various occasions; when Oxford professor Seth Ward suggested he reveal it, Alexander rightly says "it was a trap" – one Hobbes was incapable of turning down, and the one that led to his downfall.

Philosopher Douglas Jesseph[15] tells the tale of the scientific controversy erupting from these attempts in the extremely scholarly *Squaring the Circle* [20]. It is a comprehensive (at times too comprehensive for the casual reader) description and analysis of the so-called 'Hobbes-Wallis controversy'. For the man primarily responsible for Hobbes' downfall was another Oxford don, erstwhile Presbyterian-party preacher and Parliamentarian-party cryptoanalyst John Wallis.

Through all the regime change of the English Civil War era, Wallis retained favor by adroit maneuvering; unlike many similar men, he seems to have been addicted to producing (and publishing) correspondence aimed at defeating anyone he disagreed with about anything. Since Hobbes was just as stubborn, and since Wallis disliked Hobbes' theology, views on the university, and (to him) inadequate mathematics, their dispute lasted through over twenty years' worth of letters, publication, and 'transactions'. They argued primarily over mathematics, but interspersed accusations of plagiarism, bad Latin[16], and proper authority of a minister of the Gospel.

Hobbes was wrong in his circle-squaring, though, and there is no way around it; even his friends found errors fairly easily, which he kept attempting to fix before publication. Still, this was not somehow an English version of the Italian controversy. In fact, Wallis was even more cavalier about his use of infinitesimals than Hobbes was about his indivisibles (see the next section for an example). In Alexander's telling, Wallis, not Hobbes, plays the role of Cavalieri's school, with repeated direct references to the Baconian ideals of experimental induction and free inquiry espoused by the Royal Society of London. All Wallis wanted, in the best experimentalist tradition, were "theorems that were sufficiently 'true' for the business at hand."

Hence the Society's (with Wallis) repudiation of Hobbes[17] (and so his philosophy) was an example of the key to English (and, by extension, later American?) pre-eminence in science being tolerance of differing opinion, where "a land of many voices. . . discover[ed] its path to wealth and power". This seems to be an awful lot to lay at the feet of the infinitesimals, or even the Royal Society, alone[18].

This is especially so since the technicalities of these questions looked so different in the dawning eigh-

[15] Who also wrote the Intelligencer review [18] of *Infinitesimal*.

[16] At which point it seems relevant to note that Alexander's translation of 'in Guldinum' as 'on Guldin' seems far more appropriately rendered 'against Guldin', like Cicero's 'In Catalinam' speeches.

[17] Despite repeated attempts, Hobbes was *not* a member of the Royal Society; according to both Jesseph and Alexander, this was 'overdetermined'. My favorite reason (Jesseph quoting Quentin Skinner) is his potential to be the 'club bore'.

[18] Economic historians might have some quibbles with it, for instance.

teenth century, where people abandoned Cavalieri or Wallis as dead ends to grapple with utilizing Leibniz' tools (and Newton's in England), with different controversies. Similarly, it is very hard to imagine the political development of Italy or England going very differently with or without infinitesimals. Even if the Royal Society's openness to work with the sort of deficiencies Wallis' had might perhaps be a token of a more general openness in English society to once-heretical ideas, it is not a main reason we should care about this dispute.

5 Taking Care

The first reason it matters is because we, too, should be circumspect with our claims, cautious like Cavalieri in our assertions. For those who believe in fallible humans and infallible God, that alone should cause us to reconsider any overreach; for those who don't, the ample history of human blundering should be sufficient – including Hobbes' 'delusions' with regard to geometry. This goes for grandiose teleological claims, whether by Hobbes or modern authors, as much as for mathematics. If even someone who really did understand geometry could mess up due to philosophical preconceptions, so can we.

Let us see how this works out in Wallis' work in a nice example from his *Arithmetica Infinitorum* (usually rendered 'Arithmetic of Infinitesimals' – see the 2004 translation [33]; Alexander gives this example on page 270) one can use in class. Here, Wallis is about to compare sizes (areas of infinitesimal parallelograms) which he needs in order to reconstruct the area under a parabola (in this particular case, already known by Archimedes). As a crucial step, he adds the perfect squares up to n^2, divided by n copies of that same square.

$$\frac{0+1}{1+1} = \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$
$$\frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$
$$\frac{0+1+4+9}{9+9+9+9} = \frac{14}{36} = \frac{1}{3} + \frac{1}{18}$$
$$\frac{0+1+4+9+16}{16+16+16+16+16} = \frac{30}{80} = \frac{1}{3} + \frac{1}{24}$$

At each step, we get one third plus one over $6n$. Wallis then asserts that this will hold forever, as well as in the infinite case, which is what he needs if he is adding up the infinitely many infinitesimally thin parallelograms he has chopped his area into. This would lead to a final answer of $\frac{1}{3} + \frac{1}{\infty} = \frac{1}{3}$, which should look familiar[19] to those who have integrated parabolas before. See [35], IV.13, for Wallis' computation of the same sums for n^3.

This is not by any means a traditional proof, and on the surface certainly has more in common with experimental induction than the mathematical variety. It is a fantastic exercise to ask students to determine what grade this argument might receive in work on infinite series today. However, even if Wallis claims this is more than sufficient, even he recognizes that the argument as stated is not complete, and he doesn't only care about sufficient truth. Wallis, on not showing every detail: "[Euclid] leaves you to supply ... and then infers his general conclusion. Yet I have not heard any man object ..." Here is a

[19] It is a relevant trivia tidbit that Wallis himself introduced the now-usual symbol for infinity.

perfect opportunity to lead discussion or ask for responses on what level of proof or detail is sufficient in different contexts[20]. It is interesting to note that Leibniz provides essentially the same justification for his own infinitesimal methods in replying to the less well-known criticisms of his work by Bernard Nieuwentijdt (see e.g. the end of [13], Ch. 9): "For they contain a handy means of reckoning, as can manifestly be verified in every case in a rigorous manner ..."

Wallis cared about this perception in much of his writing. In [19] Jesseph remarks that "Wallis was also intent upon defending the rigor of his approach" in a long correspondence with Leibniz about the relation of his calculus with Wallis' methods, going so far to claim that they are equivalent to Eudoxus' methods: "for it is shorter, nor is it less demonstrative, if it is applied with due caution." Late in his career he even took pains to critique the types of indivisibles which Clavius had used for discussing 'horn' angle measures for similar reasons (see [26]).

Now, there is no doubt that later users of calculus did often defend results partly because they just worked, especially in physical applications[21], while providing evidence of various kinds. As for Wallis, he almost certainly found proper proofs of these results tedious; he may have found that anyone who disagreed with him about it was backward-looking; he may even have been dismissive of those who would wish him to put in the time (all of which I agree with Alexander in his discussion of his correspondence with Fermat on these matters). But that is not the same as suggesting that his defenses were all ironic or crypto-heretical[22], or that he "dismissed thousands of years of tradition."

In any case, Jesseph demolishes the notion that the Hobbes-Wallis scientific controversy was solely about 'deeper sociopolitical differences', even though that may have been a contributing factor to its vehemence[23]. The dispute was largely coldly scientific in nature, even among the personal aspersions. Indeed, we can trace with students through writings of Berkeley, MacLaurin, Cauchy, and others how the same objections persisted or impacted thinking about calculus in one form or another in a slow, groping process. Books intended for classroom supplement about these matters such as [7] or [30] can be valuable in this regard. In any case, by the late 1800s persisting doubts finally led to modern treatment of calculus lacking infinitesimals entirely (and in the mid-1900s a logic-based versions thereof). As Jesseph points out, Hobbes' "objections were not the ravings of a madman."

6 The Creative Impulse

Aside from all controversy and teachable moments, I find there is a deeper reason infinitesimals matter. Now as mathematicians, we know the amazing victories of science they make possible, and teach them to our students. There is almost nothing our technological world touches that isn't somehow enabled by calculus, from modern engineering to predicting space weather to being able to download this very article. But this is not the reason.

I believe infinitesimals matter because they are an expression of the human urge to create, instilled as

[20] This is quoted in [20] in a footnote on pages 177-178, from one of Wallis' screeds, *Due correction for Mr Hobbes Or Schoole discipline, for not saying his lessons right.* Euclid's number theory proofs, including that of the infinitude of the primes, are good to ask about here.

[21] Modern physics essentially follows the same path, with mathematicians scrambling to unpack the often-prescient implications.

[22] Compare the question of whether Hobbes' statements about God were perhaps intended ironically.

[23] Again, one would have liked [2] to have read more like [1] on this point.

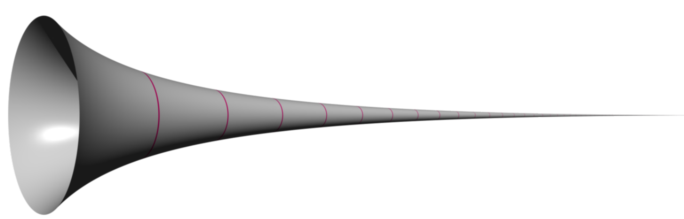
part of the 'imago Dei'. J.R.R. Tolkien expresses this beautifully in his notion of a 'sub-creation' [36]; he himself constructed magnificent worlds that captured his Roman Catholic faith as well as his love of epic saga. But computer programmers creatively make tools to analyze vast quantities of data pouring down on us daily (see 'The Tar Pit' in [8]). Musicians compose and perform and sample to make new pieces that can touch us in new ways while evoking the past (see the discussion on 'Liberating Constraint' in [5]).

Our notions of infinitesimals or indivisibles aren't physical realities, but are creatively brought into math by analogy[24]. We know we can't continually subdivide any real book forever, but we also know that someone more skilled than us could probably slice that slice of bread just once more then we can. These concepts are inherent in mathematics – you can take your pick as to whether that means the 'mind of God', as a shared social construct or something else, [16]. Exploring where those concepts go, if they can solve new problems, if they can lead our imagination to discover new worlds – that is what infinitesimals did and do. Mathematics can do this, whether in progressive situations like England under the Glorious Revolution, or in incredibly repressive ones like the Soviet Union, where mathematics was one of the sole refuges of the creative[25].

To illustrate this, let us consider one of the most beautiful and yet perplexing examples of a victory of the infinitesimal. Evangelista Torricelli was a disciple of Galileo's, like Cavalieri. He embodies the whole story here well, as one in the Baconian experimentalist mindset (he constructed the first true vacuum), devoted to indivisibles (far beyond Cavalieri's conservatism), and possibly a casualty of the Jesuits' distaste for indivisibles. The following result secured his mathematical fame, and figured prominently in the Hobbes-Wallis controversy (see [27]).

Consider the solid in Figure 3, constructed by rotating a hyperbola around its axis. This solid has infinite length but has finite volume. In fact, the volume is equal to that of a cylinder with base the same circle as the solid, and with height the radius of the base circle.

Figure 3: Gabriel's Horn – public domain graphic courtesy of Wikipedia



So something infinite can yet be contained by finitude. (Torricelli managed to give both a proof by indivisibles and one using 'conventional' methods.) Needless to say, both philosophers and mathematicians of the 17th century had a lot to say about this, including how it related to Aristotelian notions that one simply cannot compare finite and infinite magnitudes. When Hobbes discovered it[26] via Wallis' invective, he famously wrote (regarding whether Torricelli could have meant a completed infinitely long solid, rather than an indefinitely long one) that to understand it, "it is not required that a man should be a geometrician or a logician, but that he should be mad." Wallis, on the other hand, not only accepted it

[24] Though not only thus, contra [23].

[25] Just two books worth exploring about this are *Naming Infinity* and *Love and Math*.

[26] Or claims perhaps it isn't there ("I do not remember this of Toricellio"); see Hobbes' *Considerations upon the answer of Doctor Wallis to the three papers of Mr. Hobbes*, where all these quotes appear.

but posited that to understand its implications "requires more of Geometry and Logick than Mr. Hobs is Master of."

Who could have imagined such a prodigy? It does not end there. Neither author mentions one of the most amazing ultimate products of infinitary thinking. Two centuries later Georg Cantor[27] discovered that there might even be different sizes of infinity—perhaps ironically, a notion at first only supported by Catholic theologians, not other mathematicians. Any course discussing Cantor can take this opportunity to look back on the long history of fights over infinity—and whether we might not want to be more cautious in our own pronouncements about such a challenging concept.

Cantor's dictum, in the face of much opposition to his results, was that "the essence of mathematics lies in its freedom." Yet this is clearly not a complete liberty, since there are results which are wrong – like squaring the circle. This expansive freedom also means that we cannot always find a correspondence between mathematical objects like Torricelli's and 'physical' entities, and that we may yet find paradoxes we cannot resolve – notably regarding infinity, as Russell and Gödel helped us see.

Early 20th-century mathematician Hermann Weyl put it better in Section III, "Infinity" in Chapter 4 of [37]. "We stand in mathematics precisely at that point of intersection of limitation and freedom which is the essence of man himself." In my view, that is precisely the intersection of limitation and freedom imposed by God.

Infinitesimals cannot do everything, and they cannot by themselves lead to political freedom or scientific discovery. But they can bring joy and beauty of discovering all the things of creation, while submitting to our inability to ever fully comprehend their implications. Even, in the words of the edition of the Bible of the very king whose progeny sponsored both Hobbes and Wallis, "that leviathan, whom thou hast made to play therein."

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