Is Mathematical Truth Time Dependent?
Comments from a Paper by Judith Grabiner

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Abstract

Judith Grabiner's paper "Is Mathematical Truth Time Dependent?" appeared in the April, 1974 issue of the *Monthly*. Although the paper was published forty-five years ago, the topic is relevant today, especially as we consider questions about how to integrate ideas about mathematical truth with a Christian faith. Focusing on the work done in the eighteenth and nineteenth centuries to make the foundations of analysis more rigorous, Grabiner explores what it means to say that our ideas about truth in mathematics can change over time. In this paper we summarize and react to some of her ideas, including her intriguing suggestion about why this project in analysis was originally undertaken.

In his book *Soul Survivor*, Philip Yancey profiles thirteen people who, through their lives and written work, have mentored him on his Christian journey. One such mentor is the English writer G. K. Chesterton, who, as Yancey points out, was fascinated by both the problem of pain and its opposite, the problem of pleasure. As Yancey writes, "It struck me, after reading my umpteenth book on the problem of pain, that I have never even seen a book on 'the problem of pleasure.' Nor have I met a philosopher who goes around shaking his or her head in perplexity over the question of why we experience pleasure. Yet it looms a huge question; the philosophical equivalent, for atheists, to the problem of pain for Christians." [5, pp. 53, 54]. Yancey then recounts how finding an answer to the problem of pleasure helped bring Chesterton back to faith, realizing that a good God would want us to experience pleasure

As I read that passage I thought about things that have given me pleasure and mathematics is definitely on the list. For some of my students and most of my acquaintances, such a claim probably sounds very strange. They would likely include mathematics on the "pain" list rather than the "pleasure" list. That isn't to say that doing mathematics is without any pain whatever. Solving problems or understanding and proving a theorem can be accompanied by false starts, some head scratching, discouragement, and maybe even down time to just sit and ponder, any one of which is not necessarily pleasurable. But when the problem is solved or the theorem is proved, all the painful work often results in a profound sense of satisfaction and yes, pleasure. In the BBC video about Andrew Wiles and his efforts to prove Fermat's Last Theorem, Wiles shows genuine emotion as he describes the incredible beauty and satisfaction he experienced during his seven-year journey to find the proof, even as he went through some well-publicized painful times after discovering an error in the original argument. Although solving the problem made him famous, what seemed most rewarding was his pleasure in observing the intrinsic beauty of the subject matter he worked with during his seven-year journey.

Why does mathematics have this effect? Although finishing any project brings satisfaction, the pleasure I'm referring to goes deeper than that. What, for example, attracts us to continue studying mathematics long after we've learned all the algorithms we need for our daily lives? I doubt the answer is just because we can then solve more practical problems. After all, we rarely encounter practical situations where a knowledge of group theory or hyperbolic geometry is helpful. Perhaps we do it because understanding mathematics is deeply rewarding and brings a great deal of pleasure. And that pleasure comes, at least in part, from working with material that is not part of our normal experience. Doing mathematics gives us an opportunity to glimpse abstract and eternal truths. I recall having an epiphany one evening in graduate school, when it suddenly occurred to me that doing mathematics was like shoveling a sidewalk after a snowstorm. Confronted with a blanket of snow that covers everything, our job is to determine how to find and uncover the sidewalk that lies underneath the snow.

Over the years I've taken comfort that my sense of discovery is shared by others in the community. For instance, Andrew Wiles describes the process of doing mathematics with the metaphor of entering a dark room and searching for the light switch. Once that is found and he knows where all the furniture is, he moves on to the next room. Barry Mazur offers the following as part of his response to the big question of whether mathematics is invented or discovered. Mazur states: "The bizarre aspect of the mathematical experience—and this is what gives such fierce energy to The Question—is that one feels (I feel) that mathematical ideas can be hunted down, and in a way that is essentially different from, say, the way I am currently hunting the next word to write to finish this sentence" [3, p. 10]. At the conclusion of that paper Mazur discusses the morass of being caught up in philosophical arguments. But then returns to the practice of doing mathematics: "Happily I soon snap out of it and remember again the remarkable sense of independence—autonomy even—of mathematical concepts, and the transcendental quality, the uniqueness—and the passion—of doing mathematics. I resolve then that (Plato or Anti-Plato) whatever I come to believe about The Question, my belief must thoroughly respect and not ignore all this" [3, p. 21].

Seeing these and similar testimonies, I was convinced that my ideas were not out of the mainstream and saw myself as venturing forth, confident in the fact that my job was to understand and identify the truth. Of course, doing so requires an assumption that there is something to search for, some content that we refer to as truth.

Feeling secure about this search, I was shaken a bit some years ago when I encountered Judith Grabiner's paper *Is Mathematical Truth Time Dependent?*. Grabiner, an eminent mathematical historian, has written numerous papers in the history of mathematics and received both the Ford and Allendorfer Awards for her skill as an expository writer. Over the years I've come back to this paper on several occasions, always intrigued by the title. If there is such a thing as mathematical truth forming the basis for our work, how could it be dependent on anything, and how could it change with time? Even though this paper first appeared in the *Monthly* in April 1974, I think it is worth considering today, so I would like to share some of her ideas as well as remarks of my own. In doing so, I am assuming that even though her paper appeared some time ago, her ideas are still worth discussing and are not themselves time-dependent.

Grabiner begins by acknowledging that when confronted by the question of whether mathematical truth is time dependent, our first impulse is to answer no. So far so good! She describes how our discipline is unlike other sciences, which periodically undergo radical change as accommodations are made to theories. Although mathematicians generally do not tear down, but add to, structures that are already in place, there are upheavals in mathematics. Examples include the axiomatization of geometry, changing it from an experimental science into an intellectual pursuit, or the discovery of non-Euclidean geometries or non-commutative algebras, which Grabiner claims helped us realize that mathematics is mainly a study of abstract systems. She states "These were revolutions in thought which changed mathematicians' views about the nature of mathematical truth, and what could or should be proved" [2, p. 355].

In her paper Grabiner focuses on the revolution of developing a rigorous foundation for the calculus which occurred in the late eighteenth on through the nineteenth century. Describing this change as "a rejection of the mathematics of powerful techniques and novel results in favor of the mathematics of clear definitions and rigorous proofs" [2, p. 355], she begins by offering an eighteenth century example, showing how Euler derived the Maclaurin Series for the cosine function. Admittedly, using an example of Euler's work as typical of the eighteenth century might be a little unfair. Twenty years ago Bill Dunham was the keynote speaker at the 1999 ACMS conference, just after his book *Euler, the Master of Us All* was published. For three days he enchanted us with examples of Euler's creative and sometimes unorthodox methods. While we were delighted to see examples of the genius Euler at work, I doubt that any of us left that conference resolved to enhance our calculus classes by emulating the great master. Nevertheless, we'll begin with Grabiner's review of this example. Be warned, there are places in this example where you might cringe at the liberties Euler takes. Who knows, perhaps Euler cringed as well. This example and Euler's techniques serve to highlight the distinction between acceptable practices in his day and those of today.

Euler's quest to establish the series for the cosine function begins with the identity

$$(\cos z + i \sin z)^n = \cos nz + i \sin nz.$$

Expand the left side of the equation using the binomial theorem, then take the real part of that expansion and equate it to $\cos(nz)$. You have

$$\cos nz = (\cos z)^n - \frac{n(n-1)}{2!}(\cos z)^{n-2}(\sin z)^2 + \frac{n(n-1)(n-2)(n-3)}{4!}(\cos z)^{n-4}(\sin z)^4 - \dots.$$

Now let z be an infinitely small positive number and let n be infinitely large. Then

$$\cos z = 1, \quad \sin z = z, \quad n(n-1) = n^2, \quad n(n-1)(n-2)(n-3) = n^4, \quad \text{and so on.}$$

The equation now becomes recognizable as

$$\cos nz = 1 - \frac{n^2 z^2}{2!} + \frac{n^4 z^4}{4!} - \dots.$$

But since z is infinitely small and n is infinitely large, Euler concludes that nz is a finite quantity, so he lets $nz = v$. The result is that

$$\cos v = 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \dots.$$

While the result is correct, his techniques are unacceptable to us. If a student gave this argument we would likely not extend mercy. However, since this is the great Euler, we all smile, shake our heads in amazement, and move on. This change in attitude exemplifies what Grabiner means when she claims that mathematical truth is time dependent.

Of course, Euler wasn't the only one who took these liberties. Eighteenth century mathematicians wanted new results, to discover the truth, and they weren't about to stop for minor issues like rigor. As Grabiner states: "It is doubtful that Euler and his contemporaries would have been able to derive their results if they had been burdened by our sense of rigor" [2, p. 356]. Nevertheless, it is reasonable to conclude that eighteenth century mathematicians also felt some need to justify their results. After all, even Euler provides supporting arguments in the example above. However, comparing his method to an appropriate argument today, it's easy to agree that a revolution has taken place. But does that represent a change in the truth of the subject itself or is it a change in the standards for discovering that truth? I believe that these are different things.

Grabiner lists, then rejects, several reasons for the lack of rigor in the eighteenth century. These include the premise that eighteenth century mathematicians were primarily interested in getting results, that they placed heavy reliance on the power of symbolism, and that there was a purposeful disregard of the role of rigor. Her last point, that they simply disregarded the role of rigor was addressed by Professor Joan Richards in a talk she gave at the 1991 ACMS conference, The Rigorous and the Natural in Eighteenth Century Mathematics. Richards concludes that there appears to have been a great deal of discussion about the importance and role of rigor in eighteenth century mathematics, although the consensus was that "Rigor was a suspect value among the philosophe of the eighteenth century" [4, p. 15]. Some, such as D'Alembert thought that excessive rigor made mathematics unintelligible, that it is more important to be simple, or natural, than to be rigorous. She also describes several ways in which authors developed a geometry by taking a very "natural approach to the subject, including one by Clairaut where results are based on practical activities such as measurement and surveying. In short, questions about the role of rigor in mathematics were discussed, however, a "severe and inflexible conformity to some law," as rigor was thought to require, was rejected and other methods were found to support results. Of course, that attitude changed in the next century.

The nineteenth century saw a number of significant and important changes in mathematics; making calculus more rigorous was one of them. Starting with Lagrange and extending through the work of Cauchy, Bolzano, Weierstrass, and others, the challenge of providing a proper and rigorous foundation for the results of calculus was met. Concepts like continuity, limits, derivatives, and integrals were formalized, and the resulting theorems had rigorous proofs. What brought about this change? Clearly, it wasn't just because nineteenth century mathematicians were superior to their earlier counterparts. Who, after all, would consider Euler inferior to any of those just listed?

Grabiner then mentions several possible responses, reasons for the change in attitude toward rigor; the need to ensure errors were not made, a desire to generalize and unify results, or to meet the protests of critics, like Bishop Berkeley. She rejects these, offering instead an intriguing, plausible, and practical explanation. During the eighteenth century important mathematicians were typically supported by royal courts or wealthy benefactors. As the century came to a close, these opportunities dried up and mathematicians needed to get jobs, so many of them turned to teaching. In fact, since the time of the French Revolution, almost all mathematicians have also been teachers, a role that presents new challenges. Although mathematicians might be fascinated by Euler's techniques, a teacher is likely to feel that an example like Euler's development of the series for cosine will not win the day in a classroom. The need to provide students with a suitable explanation for aspects of the calculus forced mathematicians to concede that the foundations of the subject were inadequate and that new approaches to convey and support results had to be found. Thus Grabiner argues that the work on foundations was to a large extent based on the need for good explanations and points out that the work on the foundations of analysis of Lagrange, Cauchy, Weierstrass, and Dedekind all originated from lecture notes.

Finally, Grabiner mentions that more than just a new attitude was needed to affect a change in rigor. For instance, Lagrange's alternative definition of the derivative as a coefficient in the Taylor series is an example of an attempt that failed. This new revolution also required both the appropriate way to formulate definitions and the techniques of proof that allowed us to pass from these definitions to theorems. Fortunately, such techniques had recently become available. In an explanation that we will not pursue here, Grabiner gives examples of how eighteenth century work on approximations and perfecting the use of inequalities, were some of the tools needed to raise the standards of rigor in analysis.

Among the conclusions Grabiner draws from her description of these events is that while mathematical truth is perhaps eternal, our knowledge of it is not, and can change over time. Given that assumption, she suggests three possible reactions. One is to adopt a form of relativism, realizing that mathematical truth is just what the experts say it is. She considers this to be an invalid response, pointing out that if Cauchy had adopted this attitude, nothing would have changed. A second approach, which she also rejects, is to set the highest possible standard, to never give an argument unless we can completely and adequately explain every detail. Euler must have known that there were difficulties in dealing with the infinitely large and infinitely small. If he had waited for all the questions to be settled, there would have been no results for Cauchy or Weierstrass to work on. Sacrificing results to rigor is not a valid approach. She suggests a third alternative. We must recognize the "existential situations" that we find ourselves in, living with the realization that mathematics not only grows incrementally but, as is true of other sciences, also has occasional revolutions. By accepting the possibility of present error, we can hope that the future will bring a fundamental improvement in our knowledge.

I find the third response to be quite reasonable, especially as it applies to our own teaching. I believe that mathematical truth is eternal, however, our understanding of it is open to revision, perhaps even drastic change, as we acknowledge the need for such change. Assuming we agree with Grabiner that mathematical truth is eternal, or at least could be eternal, but that our knowledge of it changes over time, what implications might that have for us and for our work with students?

For one thing, I believe her conclusion supports the position on the discovery/invention question that is outlined in *Mathematics Through the Eyes of Faith*. As Christians, we may certainly believe that mathematics is part of God's creation. However, even if you aren't willing to go that far, you must admit that, as Mazur implies, mathematicians generally believe there is an underlying content to what we study. Our role is to discover that truth, finding appropriate theories to describe it. That part of the process is created by us and as such is subject to change over time. This is consistent with Grabiner's view. And, as mentioned earlier, I believe that the acceptance of an underlying content and our ability to interact with it, is consistent with the delight we have in mathematics.

I also appreciate Grabiner's willingness to elevate the role of the teacher/communicator of mathematics in her proposition that the revolution in analysis was at least partly the result of the need to provide legitimate explanations that can be passed on to students. As we teach our courses we must also decide the proper role played by both rigor and intuition in the development of the content. We are responsible for ensuring that our students not only have factual content, but also the necessary skills to communicate that content, that they can be skillful mathematicians. On the other hand, students also need to be inspired, to learn to use and trust their intuition, to have a little Euler in their lives. In some ways we should heed those eighteenth century arguments and allow for the simple and natural to come through, to appeal to intuition when possible. Our students need the opportunity to use their imagination, to discover on their own and to be given explicit opportunities to appreciate the beauty of our subject. They must be directed in a way that makes mathematics a pleasurable experience for them. If we don't do that, we have failed.

References

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