Euler and the Ongoing Search for Odd Perfect Numbers

Brian D. Beasley
Presbyterian College, Clinton, SC

Abstract

Leonhard Euler, after proving that every even perfect number has the form given by Euclid, turned his attention to finding odd perfect numbers. Euler established a basic factorization pattern that every odd perfect number must have, and mathematicians have expanded upon this Eulerian form ever since. This talk will present a brief summary of Euler's result and some recent generalizations. It will also note connections between odd perfect numbers and the abundancy index (the abundancy index of a positive integer is the ratio of the sum of its positive divisors to itself). In particular, finding a positive integer with an abundancy index of 5/3 would finally produce that elusive odd perfect number.

1 Before Euler

"*As for me, I judge that one can find real odd perfect numbers.*"

– René Descartes

1.1 Euclid and Subsequent Conjectures

As noted in a number of historical accounts (including [1], [2], and [4]), the study of perfect numbers dates back even before the time of Euclid (ca. 300 B.C.). Euclid provided the following definition: A *perfect* number is equal to its parts. That is, a number is perfect if it equals the sum of its proper (aliquot) positive divisors. In addition, the first recorded result concerning perfect numbers is due to Euclid [5] (Book IX, Proposition 36).

Theorem 1. *If $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect.*

Using $n = 2, 3, 5,$ and 7 in Euclid's formula produces the first four perfect numbers, well known to the ancients:

6, 28, 496, 8128, ...

Based on this short list, two conjectures about perfect numbers proved irresistible. Nicomachus (ca. 60-120) believed that perfect numbers alternate ending in 6 and 8, while Iamblichus (ca. 283-330) claimed that for each positive integer k, there is exactly one perfect number with k decimal digits [2]. Alas, both conjectures are false, as the next two perfect numbers are 33,550,336 and 8,589,869,056. On the other hand, a more intriguing question, also from Nicomachus [2], remains open to this day.

Conjecture 1. *Euclid's rule gives* all *perfect numbers; in particular, no odd number is perfect.*

In addition to his conjecture about the nonexistence of odd perfect numbers, Nicomachus, along with Theon of Smyrna (ca. 70-135), distinguished between deficient and abundant numbers [2]. In modern notation, given a positive integer n, we let $s(n)$ be the sum of the proper positive divisors of n. Then Euclid's definition translates into: n is perfect if $s(n) = n$. Similarly, n is *deficient* if $s(n) < n$, while n is *abundant* if $s(n) > n$. In particular, we note that every prime power is deficient, since $1+p+p^2+\cdots+p^{k-1} < p^k$ for any prime p and any positive integer k; hence there are infinitely many deficient numbers. The smallest abundant numbers are 12 and 18, which happen to be nontrivial multiples of the first perfect number, although the next largest abundant number (20) is not a multiple of any perfect number.

Motivated by such examples, and the fact that every abundant number smaller than 900 is even, Jordanus Nemorarius (1225-1260) made the following claims [2].

Conjecture 2. *(a) Every (nontrivial) multiple of a perfect number is abundant.*

(b) Every (nontrivial) divisor of a perfect number is deficient.

(c) No odd number is abundant.

In fact, the first two of these conjectures are true, but the third is false, since 945 is abundant. Since every multiple of an abundant number is also abundant, we may conclude that there are infinitely many abundant numbers, including infinitely many odd ones. This observation at least provides us with hope that there are odd perfect numbers, but we still have no proof or disproof of the existence of such a number.

1.2 Setting the Stage for Euler: Descartes and Frenicle

Interest in studying perfect numbers increased in the 1600s, with a particular focus on disproving the conjecture of Nicomachus. For example, René Descartes (1596-1650) believed that he could prove the following claims [2].

Conjecture 3. *(a) Euclid's rule gives all even perfect numbers.*

(b) Every odd perfect number has the form ps^2 with p prime.

We continue to follow Dickson's account in [2], examining the exchange between Descartes and one of his frequent correspondents, Bernard Frenicle de Bessy (1605-1675). Frenicle agreed with Descartes' conjecture about odd perfect numbers and further claimed that this prime p must be congruent to 1 mod 4. However, neither Descartes nor Frenicle offered a proof of either claim. To demonstrate his conviction that an odd perfect number exists, Descartes noted that if $p = 22,021$ were prime, then taking $s = 3 \cdot 7 \cdot 11 \cdot 13$ would produce the odd perfect number ps^2; alas, as Descartes knew, we have the factorization $22,021 = 19^2 \cdot 61$. Undaunted, Descartes wrote to Frenicle to suggest that replacing 7 or 11 or 13 in s might eventually work instead. Indeed, both Descartes and Frenicle found several examples of *even* multiperfect numbers – but no *odd* perfect numbers ...

2 Euler

"*Whether ... there are any odd perfect numbers is a most difficult question.*"

– Leonhard Euler

2.1 Completing the Work of Euclid on Even Perfect Numbers

As Dunham notes in [4], a letter from Christian Goldbach in 1729 may have initiated Leonhard Euler's work in the field of number theory, inspiring Euler to tackle the following claim by Pierre de Fermat.

Conjecture 4. *For each nonnegative integer n, $F_n = 2^{2^n} + 1$ is prime.*

This sequence of so-called Fermat numbers begins 3, 5, 17, 257, 65537, etc. In particular, F_n is indeed prime for $0 \leq n \leq 4$. But Euler showed the claim is false when $n = 5$, as 641 divides $F_5 = 4,294,967,297$. In fact, the reality about Fermat numbers could ultimately prove to be the exact opposite of the original conjecture, as no other primes have been found among F_n for $n \geq 5$ [1].

It is indeed an understatement to say that Euler went on to make many contributions to number theory. One example was the search for amicable numbers, a pair of positive integers m and n with $s(m) = n$ and $s(n) = m$. Only three pairs of amicable numbers were known before Euler, yet he proceeded to discover 59 new pairs [4]. Another example was Euler's introduction of the function ϕ to enumerate the positive integers not exceeding a given natural number n which are relatively prime to n. Euler applied this new function to generalize Fermat's Little Theorem, which states that if p is prime and $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$. Following is Euler's generalization of Fermat's result [1].

Theorem 2. *Given the positive integer n, if a is any integer with $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$.*

In keeping with his usual creativity in applying new approaches to old problems, Euler reached a breakthrough in studying perfect numbers, a seemingly simple observation that nevertheless produced profound results. Instead of using $s(n)$ to sum the proper positive divisors of n, Euler introduced the notation $\int n$ to represent the sum of *all* positive divisors of n, including n itself. The modern notation for this function uses σ to replace $\int$, so that we now write $\sigma(n) = s(n) + n$.

Euler was able to show that σ is a multiplicative number theoretic function [1]; that is, if $\gcd(m, n) = 1$, then $\sigma(mn) = \sigma(m)\sigma(n)$. This property proved quite valuable for both computational and analytical use. For example, we may calculate $\sigma(360) = \sigma(2^3 \cdot 3^2 \cdot 5)$ as follows:

$$\begin{aligned} \sigma(360) &= \sigma(2^3)\sigma(3^2)\sigma(5) \\ &= (1 + 2 + 2^2 + 2^3)(1 + 3 + 3^2)(1 + 5) \\ &= (15)(13)(6) \\ &= 1170. \end{aligned}$$

We note that if the product of the three factors in the second line is expanded via distribution, then the resulting 24 terms are precisely the positive divisors of 360.

More importantly, Euler used the multiplicative property of σ to prove the long-awaited converse of Euclid's theorem on even perfect numbers.

Theorem 3. *Euclid's rule gives all even perfect numbers.*

In particular, Euler made use of the ratio $\sigma(n)/n$, which we now call the *abundancy index* $I(n)$ of n. We will return to the abundancy index later, but for now, we simply note that n is perfect if and only if $I(n) = 2$; similarly, n is deficient when $I(n) < 2$ and is abundant when $I(n) > 2$.

Proof. We follow Euler's proof as outlined in [1] and [4]. If N is an even perfect number, then write $N = 2^{k-1}b$ with b odd and $k > 1$. Then $2N = \sigma(N) = 2^k b$, while $\sigma(N) = \sigma(2^{k-1})\sigma(b) = (2^k - 1)\sigma(b)$. Thus

$$\frac{\sigma(b)}{b} = \frac{2^k}{2^k - 1}.$$

Since $2^k - 1$ and 2^k are relatively prime, we know that $2^k - 1$ divides b. (At first, Euler seems to have believed that this observation was enough to conclude $b = 2^k - 1$, but he later corrected the gap in his proof as follows; see [2] and [16].) Then $b = (2^k - 1)a$ for some positive integer a, which implies $\sigma(b) = 2^k a$. But both a and b are positive divisors of b, so $2^k a = \sigma(b) \geq a + b = 2^k a$ and thus $\sigma(b) = a + b$. This yields $a = 1$, so $\sigma(b) = 1 + b$ and hence b is prime. $\square$

2.2 The Eulerian Form of an Odd Perfect Number

In addition to establishing that Euclid's rule produces all even perfect numbers, Euler was able to prove that every odd perfect number must have the following form [1].

Theorem 4. *If an odd perfect number exists, then it has the form $p^k s^2$, where p is prime, $\gcd(p, s) = 1$, and $p \equiv k \equiv 1 \pmod 4$.*

Before proceeding with Euler's proof, we pause to note that his result was not quite what Descartes and Frenicle had conjectured, as they believed that $k = 1$, but it came very close. In fact, current research continues in an effort to prove $k = 1$. For example, Dris has made progress in this direction, although his paper refers to Descartes' and Frenicle's claim (that $k = 1$) as Sorli's conjecture [3]; Dickson has documented Descartes's conjecture as occurring in a letter to Marin Mersenne in 1638, with Frenicle's subsequent observation occurring in 1657 [2].

Proof. We outline the proof of Euler given in [1], noting that once again, Euler employed the σ function in his argument. If N is an odd perfect number, then write $N = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ with each p_i an odd prime and each $k_i > 0$. Thus

$$2N = \sigma(p_1^{k_1})\sigma(p_2^{k_2}) \cdots \sigma(p_r^{k_r}).$$

Since $2N \equiv 2 \pmod 4$, exactly one factor $\sigma(p_j^{k_j})$ is congruent to 2 modulo 4, with the other factors all being odd. Also, since $\sigma(p_i^{k_i}) = 1 + p_i + p_i^2 + \cdots + p_i^{k_i}$, we may consider cases for p_i modulo 4:

(a) If $p_i \equiv 1 \pmod 4$, then $\sigma(p_i^{k_i}) \equiv k_i + 1 \pmod 4$. Since $\sigma(p_j^{k_j}) \equiv 2 \pmod 4$, we conclude $k_j \equiv 1 \pmod 4$. But every other $\sigma(p_i^{k_i})$ is odd, so k_i must be even for $i \neq j$.

(b) If $p_i \equiv 3 \equiv -1 \pmod 4$, then $\sigma(p_i^{k_i})$ is 0 modulo 4 if k_i is odd and is 1 modulo 4 if k_i is even, which immediately implies that $p_j \equiv 1 \pmod 4$. But 4 cannot divide $\sigma(p_i^{k_i})$, so k_i must be even for any prime p_i which is congruent to 3 modulo 4. $\square$

With this theorem concerning the necessary form of any odd perfect number, Euler prepared the way for future mathematicians to refine his result and to continue progress toward a proof or disproof of the existence of odd perfect numbers. Just over one hundred years after Euler's death, another mathematician would indeed contribute significantly to the list of conditions needed for an odd perfect number.

3 After Euler

" ... a prolonged meditation on the subject has satisfied me that the existence of any one such [odd perfect number] – its escape, so to say, from the complex web of conditions which hem it in on all sides – would be little short of a miracle."

– James Joseph Sylvester

3.1 Sylvester's Web

In 1888, James Joseph Sylvester picked up where Euler left off, using the Eulerian form of an odd perfect number to establish a number of important results [6]. As Dickson [2] noted, in that year alone, Sylvester was able to prove:

- No odd perfect number is divisible by 105.
- An odd perfect number must have at least four distinct prime divisors. (Sylvester proved later that year that an odd perfect number must have at least five distinct prime divisors, and he conjectured that at least six were required.)
- If an odd perfect number is not divisible by 3, then it must have at least eight distinct prime divisors.

In addition to establishing this "web" of requirements for odd perfect numbers, Sylvester emphasized the importance of resolving such a question that dated back to ancient times. He referred to the issue as being a "problem of the ages comparable in difficulty to that which previously to the labors of Hermite and Lindemann environed the subject of the quadrature of the circle" [2].

Inspired by Sylvester's work, mathematicians have endeavored ever since to extend the web of conditions for odd perfect numbers. We outline a small sample of such conditions. For example, it took 37 years after Sylvester's claim before Gradstein proved that an odd perfect number must have at least six distinct prime divisors; that lower bound was subsequently improved to seven by Robbins and Pomerance (independently) in 1972 and to eight by Hagis in 1980 [19]. (Chein also proved the result for eight in 1979, but his dissertation was not published [6].) The current best known lower bound is nine, due to Nielsen in 2007 [12]. Nielsen also established that if an odd perfect number is not divisible by 3, then it has at least twelve distinct prime divisors [12].

Next, we follow the progress listed in [9] on finding lower bounds on the largest prime factor of an odd perfect number:

60 – Kanold, 1944

11,200 – Hagis and McDaniel, 1973

10^5 – Hagis and McDaniel, 1975

3×10^5 – Condict, 1978

5×10^5 – Brandstein, 1982

10^6 – Cohen and Hagis, 1998

10^7 – Jenkins, 2003

Since the publication of [9], the lower bound on the largest prime divisor of an odd perfect number has been improved, to 10^8 by Goto and Ohno in 2008 [7]. Iannucci has also shown that the second largest prime factor of an odd perfect number must be at least 10^4, improving on the previous results of 139 (Pomerance, 1975) and 10^3 (Hagis, 1981) [9]. In addition, Iannucci has proved that the third largest prime factor of an odd perfect number must be at least 100 [10].

We also note some of the congruence conditions that apply to odd perfect numbers. We have already encountered Euler's result that every odd perfect number must be congruent to 1 modulo 4. In 1953, Touchard established that an odd perfect number must be congruent to either 1 modulo 12 or 9 modulo 36 [18]. In 2008, Roberts refined this result, proving that every odd perfect number must be congruent to either 1 modulo 12, 117 modulo 468, or 81 modulo 324 [14].

Since Sylvester's time, mathematicians have woven more and more strands in the "web" which surrounds odd perfect numbers. Indeed, it would be quite difficult to list all of the additional conditions now known. Instead, we conclude this section by simply noting the recent contributions of Ochem and Rao: In 2012, they showed that an odd perfect number must be greater than 10^{1500}, must be the product of at least 101 (not necessarily distinct) prime factors, and must have a prime power divisor greater than 10^{62} [13].

3.2 Connections with the Abundancy Index

We recall that Euler applied the ratio $\sigma(n)/n$ in his proof that Euclid's rule gives all even perfect numbers. Using the modern notation and terminology $I(n) = \sigma(n)/n$ for the *abundancy index* of a positive integer n, we summarize in this section some of the connections between abundancy results and the search for odd perfect numbers.

To generalize an earlier definition, given an integer $k \geq 2$, we call the positive integer n *multiperfect* or *k-perfect* if $I(n) = k$. Descartes [2] established a number of results concerning multiperfect numbers, including the following connection between 3-perfect and 4-perfect numbers.

Theorem 5. *If n is 3-perfect and 3 does not divide n, then $3n$ is 4-perfect.*

Proof. We observe that since σ is a multiplicative function, so is I. Consequently, since 3 does not divide n, we have

$$I(3n) = I(3)I(n) = \tfrac{4}{3} \cdot 3 = 4.$$

$\square$

In 2000, Weiner [20] showed a remarkable connection between a specific abundancy index and the existence of an odd perfect number. We give his theorem and proof here, noting the similarity to the method used by Descartes in the previous result.

Theorem 6. *If there is a positive integer n with $I(n) = 5/3$, then $5n$ is an odd perfect number.*

Proof. We outline the key steps in Weiner's proof [20]. First, since $3\sigma(n) = 5n$, 3 must divide n. But then 2 cannot divide n, as otherwise, 6 would also, yielding the contradiction that n must be perfect or abundant. Hence n is odd, which implies that $\sigma(n)$ is also odd. In particular, this means that each prime factor of n must occur to an even power [1], so 3^2 must divide n. But then 5 cannot divide n, as otherwise, 45 would also, yielding the contradiction that $I(n) \geq I(45) = 26/15 > 5/3$. (Here, we have used the fact that $I(n) = \sum_{d|n}(1/d)$ (for example, see [1]), which in turn implies that if d is a positive divisor of n, then $I(d) \leq I(n)$.) Thus

$$I(5n) = I(5)I(n) = \tfrac{6}{5} \cdot \tfrac{5}{3} = 2.$$

$\square$

In 2003, Ryan [15] provided the following generalization of Weiner's theorem.

Theorem 7. *If there exist positive integers m and n such that m is odd, $2m-1$ is prime, $2m-1$ does not divide n, and $I(n) = (2m-1)/m$, then $n(2m-1)$ is an odd perfect number.*

Ryan also showed in [15] that if m is even but not a power of 2, then $I(n) = (2m-1)/m$ has no solution. Three years later, Holdener [8] proved another generalization of both Weiner's and Ryan's theorems, giving the following necessary and sufficient condition for the existence of an odd perfect number.

Theorem 8. *There is an odd perfect number if and only if there are positive integers p, n, and k such that p is prime, p does not divide n, $p \equiv k \equiv 1 \pmod 4$, and*

$$I(n) = \frac{2p^k(p-1)}{p^{k+1}-1}.$$

We note the direct connection between Holdener's result and the Eulerian form of an odd perfect number: If such positive integers p, n, and k do exist, then $p^k n$ is an odd perfect number, since

$$I(p^k n) = I(p^k)I(n) = \frac{p^{k+1}-1}{p^k(p-1)} \cdot \frac{2p^k(p-1)}{p^{k+1}-1} = 2.$$

In particular, n would equal s^2 in Euler's result.

In the opposite direction, since the non-existence of such a value of $I(n)$ would prove that no odd perfect numbers exist, Holdener and Stanton [17] have studied these rational numbers outside the range of I and have given them a descriptive name: Given a rational number $r > 1$, r is an *abundancy outlaw* if $I(n) \neq r$ for every positive integer n. Applying Holdener's theorem, we note for example that if 5/3, 13/7, or 17/9 is *not* an abundancy outlaw, then an odd perfect number exists. The "outlaw" status of these three rational numbers, along with others of the form in Holdener's result, remains unknown [17].

Such observations raise the natural question of the distribution of abundancy indices vs. the distribution of abundancy outlaws in $[1, \infty)$. On the one hand, Laatsch [11] proved in 1986 that $\{I(n) : n \in \mathbb{N}\}$ is dense in $[1, \infty)$, which raises the hope that perhaps 5/3 is in fact an abundancy index. On the other hand, Weiner [20] showed in 2000 that the set of abundancy outlaws is also dense in $[1, \infty)$.

While the question of the existence of odd perfect numbers remains open, we may apply Laatsch's result about the density of abundancy indices to create some interesting examples. In particular, we can make $I(n)$ arbitrarily close to certain "famous" numbers:

- $I(3 \cdot 17 \cdot 577 \cdot 665857) = 1.414213562371\ldots$
- $I(2 \cdot 3 \cdot 5 \cdot 11 \cdot 29 \cdot 277 \cdot 67927 \cdot 204109349) = 2.71828182845903\ldots$
- $I(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 163 \cdot 23509 \cdot 690120229) = 3.141592653589792\ldots$

In addition, we are able to make $I(n)$ arbitrarily close to 2 for odd n:

- $I(3 \cdot 5 \cdot 7 \cdot 11) = 1.9948\ldots$
- $I(3 \cdot 5 \cdot 7 \cdot 11 \cdot 389) = 1.99993\ldots$
- $I(3 \cdot 5 \cdot 7 \cdot 11 \cdot 383) = 2.00001\ldots$
- $I(3 \cdot 5 \cdot 7 \cdot 11 \cdot 389 \cdot 29959) = 1.99999998\ldots$

But the main question still remains, handed down to us by Euclid, Descartes, Euler, Sylvester, and many others: Can we make $I(n)$ *equal* to 2 for odd n???

4 Bibliography

4.1 References

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4.2 Helpful Web Sites

1. MacTutor History of Mathematics, J. J. O'Connor and E. F. Robertson:

   http://www-history.mcs.st-and.ac.uk/HistTopics/Perfect_numbers.html

2. OddPerfect.org:

   http://www.oddperfect.org

3. Wolfram/MathWorld, C. Greathouse and E. Weisstein:

   http://mathworld.wolfram.com/OddPerfectNumber.html