The Search for the Real Josephus Problem

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Many of the problems that mathematicians and computer scientists dearly love have been around for a long time. One such problem is known as *the Josephus Problem*, named after the first century Jewish historian Flavius Josephus. Josephus did not invent the problem. Instead, an event from his life served as the inspiration for the problem statement.

Many current books refer to *Mathematical Recreations and Essays* by W. W. Rouse Ball [1, originally published in 1892] for the problem statement:

> Another of these antique problems consists in placing men around a circle so that if every m^{th} man is killed, the remainder shall be certain specified individuals. Such problems can be easily solved empirically.
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> Hegesippus[a] says that Josephus saved his life by such a device. According to his account, after the Romans had captured Jotapat, Josephus and forty other Jews took refuge in a cave. Josephus, much to his disgust, found that all except himself and one other man were resolved to kill themselves, so as not to fall into the hands of their conquerors. Fearing to show his opposition too openly he consented, but declared that the operation must be carried out in an orderly way, and suggested that they should arrange themselves round a circle and that every third person should be killed until all but one man was left, who must then commit suicide. It is alleged that he placed himself and the other man in the 31[st] and 16[st] place respectively.
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> [a] *De Bello Judaico*, bk III, chaps. 16-18

The problem (which will be addressed eventually), is quite interesting. However, the story, as quoted above, is not completely accurate. In fact, Hegesippus never existed, and there is no evidence that Josephus and his allies ever sat in a circle and killed every third person.

I do not know when or where the mathematical version (with the circle and every third person) originated, but I have tracked down some partial answers.

The original event can be found in Josephus' book *The Jewish War*.[1] The Hegesippus that Ball cites was a fourth century translation of Josephus. Some anonymous translator got the author's name wrong.[2]

The story, as related by Josephus[3], is as follows:

Josephus was a general for the Jews in a war against the Romans, who were led by Vespacian. Josephus and his troops were surrounded in the city of Jotapata. Eventually the city fell, but Vespacian ordered his troops to capture Josephus (rather than kill him).

Before the city fell, Josephus and 40 others managed to hide in a cave. On the third day after the city fell, the Romans found out about the cave. Vespacian sent two men to offer Josephus safe passage if he

*This article is a modified version of section 7.5 in [2].

[1] A good translation into English is [4].

[2] Dr. Laurence Creider provided extensive help researching Hegesippus.

[3] With only one other surviving witness to verify the details.

would surrender. At first he refused, but eventually started to change his mind. His companions were not pleased when they saw he was starting to consider surrender. They told him he should kill himself instead of surrender, or, if he was not brave enough, they would take the matter into their own hands. Josephus then launched into an articulate speech about why suicide is morally wrong. His speech did not convince his allies. In fact, they were on the verge of killing him and then killing themselves. The story concludes[4] (with Josephus speaking in the third person):

> But in this predicament, his resourcefulness did not forsake him. Trusting in God's protection, he hazarded his life on one last throw, saying: "As we are resolved to die, come, let us draw lots and decide the order in which we are to kill each other in turn. Whoever draws the first lot shall die by the hand of him who comes next; luck will thus take its course down the whole line. In this way we shall be spared taking our lives in our own hands. For it would be unfair when the rest were gone if one man should change his mind and escape." This proposal inspired assurance; his advice was taken, and he drew lots with the rest. Each man in turn offered his throat for the next man to cut, in the belief that his general would immediately share his fate; they thought death together with Josephus sweeter than life. He, however – should we say by fortune or by divine providence – was left with one other man; and, anxious neither to be condemned by the lot, nor, if he were left as the last, to stain his hand with the blood of a fellow countryman, he persuaded this man also, under a pact, to remain alive.

The Search for the Current Josephus Problem's Origins Since the original accounts of this event do not contain any mention of a circle and every third man, when and how did the current version of the problem arise?

David Eugene Smith investigated this question in *On the Origin of Certain Typical Problems*, published in 1917 ([8]). He mentions accounts in Livy and Dionysius of an early Roman custom of decimating[5] a company in the army if that company was guilty of cowardice, mutiny, or other serious offenses. Smith states:

> In its semi-mathematical form it is first referred to in the work of an unknown author, possibly Ambrose of Milan, who wrote, under the nom de plume of Hegesippus, a work *De bello iudaico.*[a] In this work he refers to the fact that Josephus, the author of the well-known history of the wars of the Jews, was saved on the occasion of a choice of this kind.[b] Indeed, Joshephus himself refers to the matter of his being saved by lucky chance or by an act of God.[c]
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> [a]Edited by C. F. Weber and J. Caesar, Marburg, 1864. See Ahrens, Math. Unterh. u. Spiele, p 286.
> [b]"Itaque accidit ut interemtis reliquis Iosephus cum altero superesset neci." Quoted from Ahrens, l.c.
> [c]*Καταλείπεται δὲ ουτος, είτε υπὸ τύχ῀ης χρὴ λέγειν είτε υπὸ Θεοῦ προν οίας σὺν ετερω.*

My friend Laurence Creider has a Ph.D. in Medieval Church History and, at the time I was investigating this matter, was a reference librarian at the University of Pennsylvania. He is certain that Hegesippus was not Ambrose of Milan. He is the source of my previous claim that someone mangled Josephus' name while translating from the original Greek into Latin. Dr. Creider also looked at a copy of Hegesippus to see if the "every third man" story is actually in that translation. It is not. The translation follows the Greek account presented earlier in this paper.

[4]From chapter 8 of *The Jewish War*.
[5]Killing every tenth man.

Most of the other references I have located do not provide any additional information. The most interesting clue comes from some web pages by David Singmaster ([5], [6], and [7]). They mention a ninth century Irish version that is similar to the 15 Turks and 15 Christians variant mentioned later in this paper. Singmaster asserts that the first linkage of this problem with Josephus was by Girolamo Cardano (Cardan in Latin) in *Practica arithmetice* in 1539.

I have tried sending email to David Singmaster, but have never received a response.

Other Versions of the Problem A version of the problem, existing in published form at least as early as the 1500's or early 1600's, involves a ship with 15 Turks and 15 Christians. A storm has arisen and in order to save some, it is decided that half the passengers need to be thrown into the sea. The passengers are placed into a circle, and every ninth man is tossed overboard. The problem is to find an arrangement so that your favorite religio-ethnic group are all survivors and the other group are all fish food.

An Asian variant involves a man with two wives, each of whom is the mother of 15 children. The first wife has died and the man is getting old. The surviving wife convinces him that the estate is too small to divide among 30 children. In fact, it should go to just one child. The wife convinces him to arrange the children in a circle and eliminate (but not kill for a change!) every tenth child. The final child will inherit everything. The second wife arranges the children and the process begins. In an interesting twist, the first 14 to be eliminated are all children of the first wife. The father becomes alarmed, especially after he notices that the only remaining child from the first wife will be eliminated next. He suggests that they should start over, beginning with the sole remaining child of the first wife and travel around the circle in the opposite direction. The second wife cannot object without giving herself away, but she figures that the odds are 15 to 1 in her favor. The end result is that the child of the first wife is the final child, defeating the second wife's evil strategy. Your task, of course is to place the children around the circle to match the story.

Solving The Josephus Problem A solution to the original problem (with every third person being eliminated) can be found in *Concrete Mathematics* by Graham, Knuth and Patashnik [3].

Instead, imagine that n people are placed in a circle, and every second person is eliminated.[6] The value we want is the position (start counting at 1) of the final person. Call this position j_n.

A good place to begin is with a few small examples. Table 1 shows the order in which people are eliminated and the value of j_n, for several small n. You should draw a few of the circles and verify the numbers.

n	Elimination Sequence	j_n
1	-	1
2	2	1
3	2 1	3
4	2 4 3	1
5	2 4 1 5	3
6	2 4 6 3 1	5
7	2 4 6 1 5 3	7
8	2 4 6 8 3 7 5	1
9	2 4 6 8 1 5 9 7	3
⋮		

Table 1: Order of elimination with n people.

[6] The solution presented here is also from [3].

What can we observe from these examples? One trend (at least so far) is that j_n seems to be odd. Another trend seems to be that even numbered positions are eliminated first, in order. These two observations are actually related: since all even positions will be eliminated first (according to the "every second person" rule), the final position will always be an odd number.

It takes just a little bit of creativity (or else a few years worth of mathematical maturity and experience) to make the following observations:

> Since approximately half the people (those in even-numbered positions) are eliminated immediately, it may be profitable to write n in a form that involves the number 2. If n is even, we use up exactly half the people in this first phase, while if n is odd, there will still be one extra person left before wrapping back to the beginning.[7] Because even and odd are apparently significant characteristics, it may be useful to write n as either $n = 2k$ for even n, or $n = 2k+1$ for odd n.

Consider the case where $n = 2k$ is even. After phase one, only the odd numbered positions are left. There will be k such numbers and the next available position will be position 1. The problem has effectively been reduced to a problem of size k. There is one pesky detail: a problem of size k has the positions numbered as $1, 2, 3, \ldots, k$, but a problem of size $n = 2k$ has the remaining positions numbered $1, 3, 5, \ldots, 2k-1$. It is easy to see how the two sequences relate: the old sequence can be grouped in pairs (odd,even). We keep only the first member of each pair. Look at the table below as i goes from 1 to k:[8]

original sequence	1	2	3	4	5	$\cdots$	$(2i-1)$	$(2i)$	$\cdots$	$(2k-1)$	$(2k)$
re-labeled sequence	1	–	2	–	3	$\cdots$	i	–	$\cdots$	k	–

Suppose we already knew the final position number, j_k, for a circle of size k. Then a circle of size $n = 2k$ would end up in the same place, assuming we could suitably re-label the original odd positions after phase one eliminates the even positions. It should be clear from the table that re-labeled position i corresponds to original position $2i-1$.

This leads to a clever strategy: start with a circle of size $n = 2k$. After the even positions have been eliminated, re-label the positions as $1, 2, 3, \ldots, k$. The final position in this re-labeled circle will be j_k. This corresponds to position $2j_k - 1$ in the original circle.

We now have the recursive relations $j_1 = 1$, and $j_{2k} = 2j_k - 1$. What we need is a similar recursive reduction when n is odd.

If $n = 2k+1$ is odd, phase one leaves only the odd positions. There are now $k+1$ positions, so the reduced problem looks like a circle of size $k+1 = \frac{n+1}{2}$. The re-labeling is also a bit more complicated, since the next person is not in the original position 1, but in original position $2k+1$.[9]

original	1	2	3	4	5	$\cdots$	$(2i-2)$	$(2i-1)$	$(2i)$	$\cdots$	$(2k-1)$	$(2k)$	$(2k+1)$
re-labeled	2	–	3	–	4	$\cdots$	–	$(i+1)$	–	$\cdots$	$(k+1)$	–	1

The correspondence is a bit messy. Here is a revised idea: don't end phase one until the original position 1 is eliminated (that person will always be the next to go). If we re-label after this point, the table becomes:

original	1	2	3	4	5	$\cdots$	$(2i)$	$(2i+1)$	$(2i+2)$	$\cdots$	$(2k-1)$	$(2k)$	$(2k+1)$
re-labeled	–	–	1	–	2	$\cdots$	–	i	–	$\cdots$	$(k-1)$	–	k

[7] For example, when $n = 7$, positions 2, 4, and 6 are eliminated in phase one, but 7 still remains before getting back to 1.

[8] Look at the case $n = 8$ if you want something more concrete.

[9] Look at the case $n = 9$ for a concrete example.

That looks much better! In fact, after the revised phase one, there will be a circle of size k. The final person will be in re-labeled position j_k, corresponding to original position $2j_k+1$. This leads to the recursive relation $j_{2k+1} = 2j_k + 1$.

The recursive reduction formulas are:

- $j_1 = 1$
- $j_{2n} = 2j_n - 1$
- $j_{2n+1} = 2j_n + 1$

Can these recurrence relations be turned into a closed-form formula? If so, by what technique? Notice that they are not homogeneous, so the linear homogeneous recurrence relation with constant coefficients technique is out. Also, the characteristic equation for the associated homogeneous version will not have a fixed degree. If you try to do some back substitution, the need for two distinct relations (even vs odd) will quickly lead to something that is messy and quite awkward. You need to keep track of how many 2's are in the original n to keep this sorted out. Lets try this a bit just to see what happens. Let $n = 2^m q$, where q is odd and $m \geq 1$.

$$\begin{aligned}
j_{(2^m q)} &= 2j_{(2^{m-1}q)} - 1 \quad \text{substitute} \\
&= 2\left(2j_{(2^{m-2}q)} - 1\right) - 1 \quad \text{substitute} \\
&= 2^2 j_{(2^{m-2}q)} - (2+1) \quad \text{simplify} \\
&\vdots \\
&= 2^m j_q - \sum_{i=0}^{m-1} 2^i \\
&= 2^m j_q + 1 - 2^m
\end{aligned}$$

At this point, we know that q is odd, so there is an r with $q = 2r + 1$. Then $j_q = 2j_r + 1$. But what do we do about r? Is it even or odd?

We have reached an apparent dead end, but the experience may still provide some insight later on.

So, linear homogeneous recurrence relation with constant coefficients techniques don't work, back substitution seems to fail, and after some messing around, it seems that generating functions may also be difficult to apply. What can be done? One observation is that $j_{2k+1} - j_{2k} = 2$ in all cases. That is, for each odd n, subtracting j_{n-1} from j_n always equals 2. There is no similar constant difference if n is even and the same subtraction is done.

Perhaps a larger table of small cases will help, especially now that the recurrence relations help to reduce the work. For example, $j_{10} = 2j_5 - 1 = 2 \cdot 3 - 1 = 5$. (Note the duplication for $n = 16$ in the tables.)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
j_n	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
j_n	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	1

Notice the pattern in the second rows. In particular, notice that the pattern changes whenever $n = 2^m$. This should not be a big surprise if you consider what was learned in the attempt to use back substitution. The pattern seems to start at 1 when $n = 2^m$, then build by 2 until $n = 2^{m+1}$, where it returns to 1. A bit of thought and experimentation will lead to a simple formula, once the proper characterization of n is found.

The useful way to write n is: $n = 2^m + i$, where $0 \leq i < 2^m$. For example

n	4	5	6	7	8
$2^m + i$	$2^2 + 0$	$2^2 + 1$	$2^2 + 2$	$2^2 + 3$	$2^3 + 0$

Theorem 1 *The Modified Josephus Problem*

Suppose n people are seated around a circle, numbered from 1 to n. Start counting with the first person and eliminate every second person. Continue until only one person is left. Denote the final position by j_n. Let $n = 2^m + i$ with $0 \leq i < 2^m$. Then

$$j_n = j_{(2^m+i)} = 2i + 1.$$

Proof: The theorem can be proved using complete induction.

Base Step $n = 1 = 2^0 + 0$

Since $i = 0$, the theorem predicts $j_1 = 2 \cdot 0 + 1 = 1$, which is correct.

Inductive Step Assume that the theorem is true for all positive integers less than n.

Suppose first that n is even. Then $n = 2^m + i = 2^m + 2k = 2(2^{m-1} + k)$, for some $0 \leq k < 2^{m-1}$. The recurrence relation implies that

$$j_n = j_{(2(2^{m-1}+k))} = 2j_{(2^{m-1}+k)} - 1.$$

By the inductive hypothesis, $j_{(2^{m-1}+k)} = 2k + 1$. Thus

$$j_n = 2j_{(2^{m-1}+k)} - 1 = 2\,(2k + 1) - 1 = 2(2k) + 1 = 2i + 1.$$

Now suppose that n is odd. Then $n = 2^m + i = 2^m + 2k + 1 = 2(2^{m-1} + k) + 1$ where $0 \leq k < 2^{m-1}$. Using the recurrence relation, and then the inductive hypothesis

$$j_n = j_{(2(2^{m-1}+k)+1)} = 2j_{(2^{m-1}+k)} + 1 = 2\,(2k + 1) + 1 = 2i + 1.$$

The induction is finished: the theorem is true for $n = 1$, and whenever the theorem is true for all positive integers less than n, it is also true for n.

□

The following problem is left as an exercise for the reader:

Let p_n represent the position that Josephus' partner should be in so that he is the second-to-last to be selected for execution (in a circle with n people, and every second person executed).

1. Produce a formula for p_n.
2. Prove that your formula is correct.

References

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