Counting Tulips: Three Combinatorial Proofs

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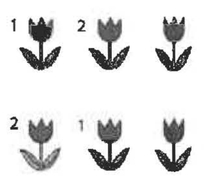
Sometimes a movie starts at the end, then does a flashback to show how the ending situation arose.[1] I will take the same approach in this presentation. The end result is interesting, but the process that led to that result is even more interesting.

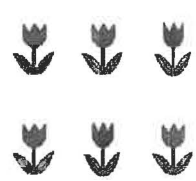
The Problem

A gardener has $r \geq 1$ red tulips and $b \geq 1$ blue tulips, each in its own pot. She plans to plant them in a line along the edge of her driveway. In how many visually distinguishable ways can she arrange them?

The meaning of *visually distinguishable* in this context can be illustrated by the following examples. Even though red tulips 1 and 2 have changed places in the left-hand diagram, the two rows are considered visually indistinguishable because it is the patterns of colors that are important. In the right-hand example, moving the blue tulip from the right end to the middle causes the two rows to be visually distinguishable.



visually indistinguishable



visually distinguishable

The answer to this question certainly depends upon the numbers of each color that must be planted. The solution is summarized in Theorem 1.

[1] I started the live presentation with a 50-second clip from the movie *Kamikazee Girls* (distributed by Viz Media). The clip starts right after the opening credits and arrives at the end of the movie in about 40 seconds. The clip does not include any tulips, but cabbages play a major role.

The theorem uses binomial coefficients. The binomial coefficient $\binom{n}{k}$ is defined as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. The easy-to-prove equality $\binom{n}{k} = \binom{n}{n-k}$ will be used several times. A standard result asserts that $\binom{n}{k}$ represents the number of distinct subsets of size k that can be chosen from a set of size n.

The Solution

Theorem 1 *Counting Tulip Patterns*

A gardener has $r \geq 1$ red tulips and $b \geq 1$ blue tulips, each in its own pot. She plans to plant them in a line along the edge of her driveway. The number of visually distinguishable ways she can arrange them is summarized in the following table.

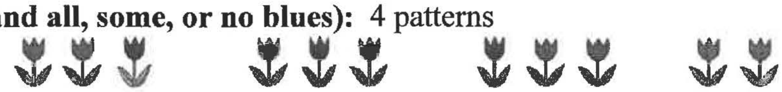
Number of Red Tulips Planted	Number of Blue Tulips Planted	Number of Visually Distinguishable Patterns
0in.2inr	b	$\binom{b+r}{r}$
r	$0 \leq$ number planted $\leq b$	$\binom{b+r+1}{r+1}$
$0 \leq$ number planted $\leq r$	$0 \leq$ number planted $\leq b$	$\binom{b+r+2}{r+1} - 1$

Example 1 *Illustrating Theorem 1*

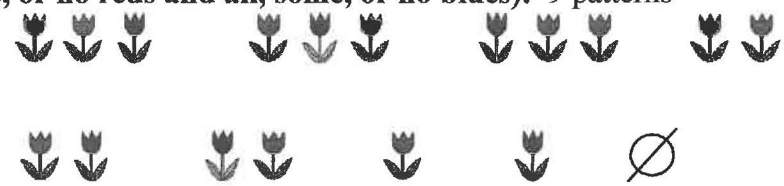
Let $r = 2$ and $b = 1$. The number of visually distinguishable patterns in each of the three cases is easy to calculate and to list. Notice that each case contains all patterns in the previous case, plus additional choices.

Case 1 (use all reds and all blues): 3 patterns

Case 2 (use all reds and all, some, or no blues): 4 patterns



Case 3 (use all, some, or no reds and all, some, or no blues): 9 patterns



The Origins of the Theorem

Math majors at Bethel University enroll in a senior seminar during our January Interim term. During the 2007 course, one of the math seniors (Kelly Kirkwood) came to me with a question about one of the source papers for her presentation. Her presentation was about Moessner's Theorem.

> **Moessner's Theorem**
> From the series of natural numbers strike out every k^{th} number. From this resulting series form the series of partial sums and from this series strike out every $(k-1)^{st}$ number. Again form the series of partial sums and now delete every $(k-2)^{nd}$ number. Repeat the process $k-1$ times, striking out every second number the last time, and from this forming the final series of partial sums. This final series of partial sums is the series of natural k^{th} powers. In other words, the final row will be n^k for each successive natural number n.

The following charts illustrate the theorem for $k = 2,3,4$.

Striking every second integer gives the squares																			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1+3		*4+5*		*9+7*		*16+9...*												
1		**4**		**9**		**16**		**25**		**36**		**49**		**64**		**81**		**100**	

Striking every third integer gives the cubics																				
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	3		7	12		19	27		37	48		61	75		91	108		127	147	
1			**8**			**27**			**64**			**125**			**216**			**343**		

Striking every fourth integer gives the fourth powers																			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	3	6		11	17	24		33	43	54		67	81	96		113	131	150	
1	4			15	32			65	108			175	256			369	500		
1				**16**				**81**				**256**				**625**			

Kelly's question was about the binomial identity

$$\sum_{k=0}^{n}\binom{k}{r}=\binom{n+1}{r+1}.$$

She recalled working with similar identities in the Discrete Mathematics course. We reviewed the meaning of the identity and looked at one of two proofs that I was aware of. The proof used algebraic manipulations of the identity

$$\sum_{k=0}^{b}\binom{r+k}{k}=\binom{b+r+1}{b}$$

which is a favorite identity from the Discrete Mathematics class, proved using a very clever combinatorial proof (see Corollary 1 on page 5).[2]

At this point, Kelly was quite content to go back to working on Moessner's Theorem. I on the other hand was not content. Although I could prove that $\sum_{k=0}^{n}\binom{k}{r}=\binom{n+1}{r+1}$ by using induction or by algebraically transforming $\sum_{k=0}^{b}\binom{r+k}{k}=\binom{b+r+1}{b}$, I always felt that it deserved its own combinatorial proof.

After spending some time attempting to create such a combinatorial proof, it occurred to me that I should try another ``tulip proof''. For that to make sense, consider the combinatorial proof for case 1 of the tulip problem.

All Tulips Must Be Planted

Suppose the gardener has decided to plant all $r+b$ tulips. In how many visually distinguishable ways can she arrange them?

Variations on the following pair of enumerations have been around for a while.

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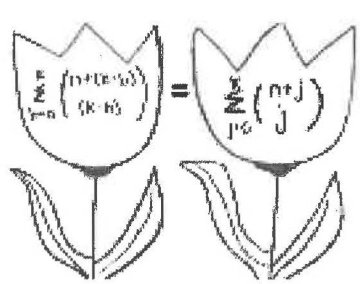
[2]Kelly was quite familiar with the combinatorial proof - it is quite famous at Bethel. In fact, Kelly and her colleague Liz O'Conner used construction paper at a Math Lab party to create the red and blue tulips containing the two sides of an intermediate identity from that proof. Their classmate Steven Yackel was a computer science major. After mastering the combinatorial proof of Theorem 2, he decided to add a second major in mathematics, which he completed.



The first enumeration

For this case of Theorem 1, the first enumeration is more complicated than the second enumeration. For the other two cases, the second enumerations will be more challenging.

This first enumeration uses a generalization of a standard principle from elementary counting theory. Suppose a task can be completed in one of two mutually exclusive ways. If the first option can be completed in n_1 ways and the second option can be completed in n_2 ways, then the task can be completed in $n_1 + n_2$ ways. For example, a teacher might be leading a seminar with 3 male and 4 female students. If one student is to be chosen to lead the seminar today, there are 3 ways to have a male lead the class and 4 ways to have a female lead the class. There are 3 + 4 = 7 ways to choose a student leader.

The gardener can group patterns by the number, i, of blue tulips planted on the left before the first red tulip is encountered. Different values of i will result in mutually exclusive sets of patterns. If there are i blue tulips and then a red tulip, then there are $(r-1)+(b-i)$ tulips that still must be arranged, with $b-i$ of them being blue. There are therefore $\binom{(r-1)+(b-i)}{b-i}$ patterns with exactly i blue tulips before the first red is encountered. Thus there are

$$\sum_{i=0}^{b}\binom{(r-1)+(b-i)}{b-i}=\sum_{k=0}^{b}\binom{(r-1)+k}{k}$$

distinct patterns. (The left-hand sum is

$$\binom{(r-1)+b}{b}+\binom{(r-1)+(b-1)}{b-1}+\cdots+\binom{(r-1)+1}{1}+\binom{(r-1)}{0}$$

whereas the right-hand sum is

$$\binom{(r-1)}{0}+\binom{(r-1)+1}{1}+\cdots+\binom{(r-1)+(b-1)}{b-1}+\binom{(r-1)+b}{b}.)$$

The second enumeration

Since there are $b+r$ tulips, and the pattern is determined by which of the $b+r$ positions contain red tulips, there are $\binom{b+r}{r}$ distinct color patterns.

Combining the enumerations

Since the two enumerations are both counting the number of visually distinguishable patterns of red and blue tulips, the results must be equal, establishing Theorem 2.

Theorem 2 *Plant all tulips*

For all positive integers b and r,

$$\sum_{k=0}^{b}\binom{r-1+k}{k}=\binom{b+r}{r}.$$

Theorem 2 is often presented in the following form.

Corollary 1

For all positive integers b and nonnegative integers r,

$$\sum_{k=0}^{b}\binom{r+k}{k}=\binom{b+r+1}{b}.$$

Proof:
Replace $r-1$ by s in the theorem, then rename s as r. Finally, observe that $\binom{b+r+1}{r+1}=\binom{b+r+1}{b}$.

A brief digression. A visualization of the Theorem 2/Corollary 1 proof can be found at
http://www.mathcs.bethel.edu/: gossett/vcp/ .
The visualization is the result of a summer project by Callie Wurtz (mathematics major) and Phil Kaasa (computer science major). The project created an extensible platform for illustrating combinatorial proofs. Callie wrote expositions (including biographical and background information) and presented the proofs. Phil created the platform. Together, they created illustrative examples for each proof that use animations with explanations. My students have found these to be very helpful for grasping the idea of a combinatorial proof.[3]

All Of The Red Tulips Must Be Planted

The missing combinatorial proof for the identity $\sum_{k=0}^{n}\binom{k}{r}=\binom{n+1}{r+1}$ was found by considering the following variation of the tulip problem.

Suppose now that the gardener intends to use all r of the red tulips, but the number of blue tulips can be any number from 0 to b, inclusive.

[3]If you attended the 2005 ACMS conference, you probably recall the proof, presented by Fernando Gouvêa, of a result by Nicole Oresme in the 1300s. It is a marvelous visual proof. Callie and Phil have a very nice animation for that proof.

The first enumeration

The gardener must first decide how many blue tulips to use. Suppose she decides to use a combined total of i tulips. Then there will be $i-r$ blue tulips. She can mark i evenly spaced positions along the driveway at which to plant the tulips. There are $\binom{i}{r}$ ways to place the r red tulips among the i positions. Each such set of positions leads to a different visually distinguishable pattern.

Since the patterns with j blue tulips and the patterns with $m \neq j$ blue tulips are visually distinguishable, the total number of visually distinguishable patterns is

$$\sum_{i=r}^{b+r} \binom{i}{r}.$$

The summation starts with $i=r$ because the minimum planting contains all of the red tulips. A simple expansion shows that

$$\sum_{i=r}^{b+r} \binom{i}{r} = \sum_{k=0}^{b} \binom{r+k}{r}.$$

The second enumeration

The gardener can also use the following procedure to choose the pattern. First, she places all the pots containing the red tulips along the edge of the driveway, leaving room between and outside the pots for her assistant to place pots of blue tulips.

She then generates a string of $b+r+1$ letters. The string will contain b copies of the letter ``B" and $r+1$ copies of the letter ``M". The string will be a prescription for placing a subset of blue tulips in the spaces that have been reserved.

The assistant starts by standing by the leftmost gap (that is, to the left of all the red tulips). The gardener starts reading the string. She hands one blue tulip to the assistant for each letter ``B" that occurs before an ``M" is encountered. The assistant places those pots in the current gap. Letters are crossed out once they are processed.

When an ``M" is encountered, the assistant moves right to the next gap. The gardener then starts handing a blue tulip to the assistant for each letter ``B" contained in the unprocessed portion of the string, stopping when a letter ``M" is encountered. As soon as an ``M" appears, the assistant moves to the next gap. If multiple ``M"s are adjacent, there will be one or more gaps with no blue tulips.

This process continues until all of the ``M"s have been processed. The remaining letter ``B"s (and the remaining pots of blue tulips) will be ignored.

At this point, the gardener can plant the tulips in evenly spaced holes along the driveway.

The key point is that every string of b ``B"s and $r+1$ ``M"s results in a visually distinguishable pattern of tulips and every visually distinguishable pattern corresponds to a unique string. The $r+1$ ``M"s allow the assistant to move past each of the $r+1$ positions that are between or outside the r red tulips.

Two examples might be helpful.

The pattern BBMMBM corresponds to .

The pattern MBBMMB corresponds to .

How many strings of letters are possible? Once the positions of the $r+1$ ``M"s have been chosen, the string is uniquely determined. There are thus $\binom{b+r+1}{r+1}$ distinct patterns.

Combining the enumerations

Since the two enumerations are counting the number of visually distinguishable patterns of red and blue tulips, the results must be equal, completing the proof of Theorem 3.

> **Theorem 3** *Plant all of the red tulips and* $0 \le$ *# blue tulips* $\le b$
>
> For all positive integers n and r,
>
> $$\sum_{k=0}^{b} \binom{r+k}{r} = \binom{b+r+1}{r+1}.$$

The following corollary does not involve red and blue tulips, but is a useful form that can be used in other contexts.

> **Corollary 2**
>
> For all positive integers n and r,
>
> $$\sum_{i=0}^{n} \binom{i}{r} = \binom{n+1}{r+1}.$$

Proof:

$$\binom{b+r+1}{r+1} = \sum_{k=0}^{b} \binom{r+k}{r} = \sum_{i=r}^{b+r} \binom{i}{r} = \sum_{i=0}^{b+r} \binom{i}{r}$$

The second equality can be seen to be true by expanding the two middle summations. The final equality is valid because of the convention that $\binom{i}{r} = 0$ whenever $i < r$. Now let $n = b + r$.

A Status Check

At this point, my objective of finding a combinatorial proof for the binomial identity

$\sum_{k=0}^{n}\binom{k}{r}=\binom{n+1}{r+1}$ had been reached. I looked at a summary of what had been accomplished.

A gardener has $r \geq 1$ red tulips and $b \geq 1$ blue tulips, each in its own pot. She plans to plant them in a line along the edge of her driveway. The number of visually distinguishable ways she can arrange them is summarized in the following table.

Number of Red Tulips Planted	Number of Blue Tulips Planted	Number of Visually Distinguishable Patterns
0in.2inr	b	$\binom{b+r}{r}$
r	$0 \leq$ number planted $\leq b$	$\binom{b+r+1}{r+1}$

Clearly, there was a missing case.[4]

The Lower Bound For Both Red And Blue Tulips Is Zero

Suppose, finally, that the number of red tulips actually planted can be any number from 0 to r and the number of blue tulips actually planted can be any number from 0 to b. In how many visually distinguishable ways can the gardener arrange the tulips?

The first enumeration

The gardener may first decide how many red tulips and how many blue tulips to use. Suppose she decides to use j red tulips and k blue tulips. Then there will be $j+k$ tulips to plant. Different choices for the numbers of red and blue tulips will lead to visually distinguishable patterns. For a specific choice of j and k, the number of visually distinct patterns will be $\binom{j+k}{k}$ because the positions of the blue tulips will determine the visual pattern.

The total number of visually distinguishable patterns is therefore

$$\sum_{j=0}^{r}\sum_{k=0}^{b}\binom{j+k}{k}.$$

Another brief digression

The double sum can be algebraically simplified (using other binomial identities) to

[4] Actually, two missing cases, but the symmetry of r and b meant that one missing case was just case 2 with r and b interchanged.

$$\binom{b+r+2}{r+1}-1.$$

The ``$-1$" seems suspicious. A re-check of the algebra and a few test cases indicates that the ``-1" really does belong in the solution. However, this algebraic approach does not give any hint as to why the ``-1" should be there. The second enumeration does provide a satisfying (at least for me) explanation.

The second enumeration

The gardener can use the following procedure for determining the pattern. She starts with $b+1$ letter ``B"s and $r+1$ letter ``R"s. These $b+r+2$ letters are randomly arranged in a string. Once the string has been formed, she discards the first ``R" and all ``B"s to the left of that ``R". If any ``B"s remain, she then discards the final ``B" and all ``R"s to the right of that ``B". Otherwise, she discards all the remaining ``R"s (leaving an empty string). The remaining string is a prescription for the pattern of tulips to plant (the empty string indicates that no tulips will be planted).

A few examples are appropriate.

The pattern R**RBRB**BR corresponds to  .

The pattern BR**RBRR**B corresponds to  .

The pattern R**BBRRR**B corresponds to  .

The pattern BBRBRRR corresponds to $\varnothing$.

The procedure (with one exception) produces distinct prescriptions. If that were not true, then there must be two distinct strings, X and Y, that form the same prescription after the discards. These two strings must either have a different number of ``B"s to the left of the first ``R" or a different number of ``R"s to the right of the final ``B" (or both). Suppose that string X has more ``B"s to the left of the first ``R" than does string Y.[5] Since both strings have $b+1$ ``B"s, string $Y$ (with one exception) must have more ``B"s than string X does between the first ``R" and the final ``B". But that means that the resulting prescriptions will be visually distinct, contradicting the assumption that the two strings produced the same prescription.

The previously mentioned exception occurs when string X consists of $b+1$ ``B"s followed by $r+1$ ``R"s and string Y consists of b ``B"s, followed by ``RB" and then r ``R"s. Both lead to the prescription that indicates that no tulips should be planted.

BBBBBBRRRRRR BBBBBRBRRRRR

The procedure will produce every possible pattern. For each possible pattern, write a sequence of ``B"s and ``R"s that matches the visual pattern. Denote that string by P. Suppose there are j red tulips and k blue tulips in the pattern. The string consisting of $b-k$ ``B"s, followed by an ``R",

[5]The other case can be handled in the same manner.

followed by the string P, followed by a ``B'' and then $r-j$ ``R''s will be a string of length $b+r+2$ that produces the desired prescription.

There are $\binom{b+r+2}{r+1}$ distinct strings and only two of them produce the same pattern, so there must be

$$\binom{b+r+2}{r+1}-1$$

visually distinct ways to plant the tulips.

Combining the enumerations

The two enumerations can be combined to establish Theorem 4.

Theorem 4 $0 \le \# \, red \, tulips \le r$ *and* $0 \le \# \, blue \, tulips \le b$

For all positive integers b and r,

$$\sum_{j=0}^{r}\sum_{k=0}^{b}\binom{j+k}{k}=\binom{b+r+2}{r+1}-1.$$

The inner summation in Theorem **4** can be expanded to produce another combinatorial identity.

Corollary 3

For all positive integers b and r,

$$\sum_{j=0}^{r}\binom{b+j+1}{j+1}=\binom{b+r+2}{r+1}-1.$$

Proof:

$$\binom{b+r+2}{r+1}-1=\sum_{j=0}^{r}\sum_{k=0}^{b}\binom{j+k}{k}=\sum_{j=0}^{r}\sum_{k=0}^{b}\binom{j+k}{j}=\sum_{j=0}^{r}\binom{b+j+1}{j+1}.$$

The final equality follows from Theorem 3.

Back To The End Of The Movie

Extracting the right-hand sides of Theorems 2, 3, and 4 proves Theorem 1.

Theorem 1 *Counting Tulip Patterns*

A gardener has $r \geq 1$ red tulips and $b \geq 1$ blue tulips, each in its own pot. She plans to plant them in a line along the edge of her driveway. The number of visually distinguishable ways she can arrange them is summarized in the following table.

Number of Red Tulips Planted	Number of Blue Tulips Planted	Number of Visually Distinguishable Patterns
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r	$0 \leq$ number planted $\leq b$	$\binom{b+r+1}{r+1}$
$0 \leq$ number planted $\leq r$	$0 \leq$ number planted $\leq b$	$\binom{b+r+2}{r+1}-1$