Computation of Least-Conservative State-Constraint Sets for Decentralized MPC with Dynamic and Constraint Coupling

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***Abstract*—We address the problem of synthesizing state-constraint sets for a fully decentralized Model Predictive Control (MPC) scheme. We consider linear time-invariant discrete time systems, with subsystems possibly coupled in both dynamics and state constraints. For each individual subsystem we employ a set-based framework to compute the state-constraint sets, that are used to synthesize local tube-based MPC controllers. The offline problem that computes the constraint sets explicitly ensures that the feasible regions of the MPC controllers are non-empty, and whenever the controllers are feasible, the overall system constraints are satisfied with the least conservativeness possible. We demonstrate the closed-loop performance of the decentralized scheme, assessed with respect to centralized MPC, using a numerical example.**

***Index Terms*—List of keywords (from the L-CSS keyword list)**

I. Introduction

Model Predictive Control (MPC) of interconnected systems has been an active area of research, driven by practical requirements posed by communication and computation limitations [1]. Several control schemes satisfying these requirements have been proposed, which are based on decomposition methods of either the coupled system or of the centralized optimization problem [2]. These schemes are broadly divided into two categories: *distributed* MPC (DMPC) and *decentralized* MPC (DeMPC), with the division usually being defined based on the communication between the controllers. With respect to interconnection patterns, the two broad categories are systems with *dynamic coupling* and *constraint coupling*.

Dynamic couplings lead to interactions between the states of disparate constituent subsystems, thus requiring coordination between local controllers. Tube-based MPC [3] has been used as an effective framework to tackle this coordination problem. By modeling the state interactions as local disturbances, local controllers can be designed that explicitly take these disturbances into account to ensure robust constraint satisfaction. An example that uses this framework is the DMPC scheme proposed in [4], which accommodates both dynamic and constraint coupling. This scheme requires communication between the controllers of reference trajectories, and true states and inputs. On the DeMPC side, schemes that do not require communication between the controllers have been proposed. The lack of communication introduces unavoidable conservativeness, which should be tackled is a structured way. For example, in [5], local tube-based MPC controllers are synthesized using feedback gains, which are computed by solving an offline optimization problem that minimizes the conservativeness of the resulting control action. However, the scheme only accommodates dynamic coupling and not constraint coupling. A common theme among these approaches is the adoption of the method presented in [6] to compute tight outer approximations of the minimal Robust Positive Invariant (mRPI) set, which is an essential ingredient of tube-based MPC.

Recently, building on the work presented in [7], a one-step approach to compute outer approximations of the mRPI set has been presented in [8]. This approach, which allows for very quick online recomputation of a small RPI set, has been purposed in the development of a DMPC scheme in [9]. The recomputation leads to disturbance sets which reduce in size as the set-points are reached, therefore improving the performance of the overall distributed scheme.

In this paper we present a method to compute state-constraint sets for local tube-based MPC controllers [3] used within the DeMPC scheme of [5]. We consider linear time-invariant systems which can be coupled by both dynamics and constraints. To the best of the authors' knowledge, this interconnection pattern has not been considered previously in the DeMPC literature. The method is centered on the formulation of an offline optimization problem, which is developed using a set-based framework. The decoupled state-constraint sets are computed such that (a) the corresponding output set is the least conservative inner-approximation of the coupled constraint set, and (b) feasibility and stability of the local tube-based MPC controllers is ensured. The formulation of the optimization problem relies on some new results that were developed using the ideas presented in [10] to compute RPI sets.

Preliminaries: A compact set $\mathcal{X} \subset \mathbb{R}^n$ (which is bounded by definition) is *proper* if it contains the origin in its non-empty interior $\operatorname{int}(\mathcal{X})$. Under a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, the image $T\mathcal{X}$ of a set $\mathcal{X} \subset \mathbb{R}^n$ is given by $\{Tx : x \in \mathcal{X}\}$. The set $\mathcal{B}_p^n := \{x : \|x\|_p \leq 1\}$ is the p-norm ball in $\mathbb{R}^n$. A *polyhedron* is the intersection of a finite number of half-spaces, and a *polytope* is a compact polyhedron. A polytope $\mathcal{X} \subset \mathbb{R}^n$ is *full-dimensional* if there exists an $x \in \mathcal{X}$ and a scalar $\epsilon > 0$ such that $\{x\} \oplus \epsilon\mathcal{B}_p^n \subset \mathcal{X}$. The support function of a compact set $\mathcal{X}$ at a given $y \in \mathbb{R}^n$ is defined as $h_{\mathcal{X}}(y) := \max_{x \in \mathcal{X}} y^\top x$. The Minkowski set addition is defined as $\mathcal{X} \oplus \mathcal{Y} := \{x + y :$

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$x \in \mathcal{X}, y \in \mathcal{Y}\}$. If $\mathcal{Y} \subset \mathcal{X}$, then set subtraction $\mathcal{X} \ominus \mathcal{Y} := \{x : \{x\} \oplus \mathcal{Y} \subset \mathcal{X}\}$. If $\mathcal{X}$ and $\mathcal{Y}$ are compact, convex and non-empty, the Hausdorff distance metric is given by

$$d_{\mathrm{H}}(\mathcal{X}, \mathcal{Y}) = \min_{\epsilon \geq 0} \ \epsilon \text{ s.t. } \mathcal{X} \subseteq \mathcal{Y} \oplus \epsilon \mathcal{B}_{\infty}^{n},\ \mathcal{Y} \subseteq \mathcal{X} \oplus \epsilon \mathcal{B}_{\infty}^{n}.$$

Given two matrices $T, S \in \mathbb{R}^{n \times m}$, T_i denotes row i of matrix T, $T \leq S$ denotes element-wise inequality, $T \circ S$ denotes element-wise multiplication, and $\mathrm{diag}(T, S)$ represents a matrix with block-diagonal elements T and S. The symbols $\mathbf{1}$, $\mathbf{0}$, and $\mathbf{I}$ denote all-ones, all-zeros and identity matrix respectively, with dimensions inferred from context. Set $\mathbb{I}_m^n := \{m, \dots, n\}$ is the set of natural numbers between m and n. Given $v \in \mathbb{R}^n$, $S \in \mathbb{R}^{n \times n}$, we define $\|v\|_S^2 := v^\top S v$.

II. Decentralized Tube-Based MPC of Coupled Linear Systems

A. System Description

We consider a linear time-invariant system of the form

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{1}$$

with state $\mathbf{x} \in \mathbb{R}^{n_x}$, input $\mathbf{u} \in \mathbb{R}^{n_u}$. This system is subject to constraints

$$U := \{\mathbf{u} : G^{\mathrm{u}}\mathbf{u} \leq g^{\mathrm{u}}\}, \qquad g^{\mathrm{u}} \in \mathbb{R}^{m_U}, \tag{2a}$$

$$Y := \{\mathbf{y} \in \mathbb{R}^{n_y} : \mathbf{y} = \mathbf{C}\mathbf{x}, G^{\mathrm{y}}\mathbf{y} \leq g^{\mathrm{y}}\}, \qquad g^{\mathrm{y}} \in \mathbb{R}^{m_Y}. \tag{2b}$$

Assumption 1: The sets U and Y are full-dimensional polytopes containing the origin in their interior. □

We assume that the system in (1) can be partitioned into M subsystems, each with dynamics described by

$$x_{[i]}(t+1) = A_{[ii]}x_{[i]}(t) + B_{[i]}u_{[i]}(t) + \sum_{j \in L_i} A_{[ij]}x_{[j]}(t), \tag{3}$$

where i indicates the i^{th} subsystem with states $x_{[i]} \in \mathbb{R}^{n_x^i}$ and inputs $u_{[i]} \in \mathbb{R}^{n_u^i}$. The overall state and input vectors are then $\mathbf{x}(t) = [x_{[1]}(t)^\top, .., x_{[M]}(t)^\top]^\top$ and $\mathbf{u}(t) = [u_{[1]}(t)^\top, .., u_{[M]}(t)^\top]^\top$ respectively. The set L_i indicates the indices of the neighbors of i which are dynamically coupled to it. That is, $L_i := \{j \in M : i \neq j,\ A_{[ij]} \neq \mathbf{0}\}$. From (3), we have $\mathbf{B} = \mathrm{diag}(B_{[1]}, \cdots, B_{[M]})$. In addition, we assume that the input constraints are decoupled. That is, $U = \prod_{i \in M} U_i$, with $u_{[i]} \in U_i$ being the input constraint on individual subsystem i, where $\prod$ denotes the Cartesian product. Note that unlike in related literature [11], [5], [12], we do not assume Y to be decoupled between the subsystems. Our aim is to solve the following fully decentralized MPC problem:

Problem 1: Design M model predictive controllers $\mathcal{C}_i$, one per subsystem i, described by (3), such that (a) the state $\mathbf{x}$ is regulated to $\mathbf{0}$, (b) system constraints (2) are satisfied, (c) each controller $\mathcal{C}_i$ has access only to local states $x_{[i]}$, (d) there is no communication between the controllers. □

In order to solve Problem 1, we adopt the DeMPC scheme of [5], which uses the tube-based MPC approach [3] to design each controller $\mathcal{C}_i$. In the original approach, each state-constraint set X_i on the individual subsystem i is known a priori, while we only know the coupled constraint Y. Hence, in the next subsection, we recall the scheme in [5] for arbitrary state-constraint sets X_i, and use the properties of the scheme to derive requirements on X_i in order to satisfy the coupled constraint Y.

B. DeMPC Formulation

In order to formulate the controller as decentralized, we model all couplings as disturbances. Accordingly, we rewrite (3) as

$$x_{[i]}(t+1) = A_{[ii]}x_{[i]}(t) + B_{[i]}u_{[i]}(t) + w_{[i]}(t), \tag{4}$$

with $w_{[i]}(t) := \sum_{j \in L_i} A_{[ij]}x_{[j]}(t)$. As in standard tube-based MPC, we equip each subsystem i with a *pre-designed* feedback controller $K_{[i]} \in \mathbb{R}^{n_u^i \times n_x^i}$, satisfying the following stability assumption.

Assumption 2: (a) Each matrix pair $(A_{[ii]}, B_{[i]})$ is controllable, (b) each $K_{[i]}$ is designed such that $A_{[i]}^{\mathrm{K}} := A_{[ii]} - B_{[i]}K_{[i]}$ has all eigenvalues strictly within the unit circle, (c) defining $\mathbf{K} := \mathrm{diag}(K_{[1]}, \cdots, K_{[M]})$, the matrix $\mathbf{A}^{\mathrm{K}} := \mathbf{A} - \mathbf{B}\mathbf{K}$ has all eigenvalues strictly within the unit circle. □

Using $K_{[i]}$, we parameterize the control input as

$$u_{[i]}(t) = \hat{u}_{[i]}(t) - K_{[i]}\Delta x_{[i]}(t), \tag{5}$$

where $\Delta x_{[i]}(t) := x_{[i]}(t) - \hat{x}_{[i]}(t)$ is the state error with respect to the *nominal* system

$$\hat{x}_{[i]}(t+1) = A_{[ii]}\hat{x}_{[i]}(t) + B_{[i]}\hat{u}_{[i]}(t). \tag{6}$$

We also define the input error $\Delta u_{[i]}(t) := u_{[i]}(t) - \hat{u}_{[i]}(t)$. Using (4), (5) and (6), the dynamics of the *error* system for subsystem i can be derived as

$$\Delta x_{[i]}(t+1) = A_{[i]}^{\mathrm{K}}\Delta x_{[i]}(t) + w_{[i]}(t). \tag{7}$$

Since we assume that the state of each subsystem i is constrained to the set X_i, we have

$$w_{[i]}(t) \in W_i := \bigoplus_{j \in L_i} A_{[ij]}X_j. \tag{8}$$

Given the disturbance set W_i, the error state $\Delta x_{[i]}$ always belongs to the corresponding minimal Robust Positive Invariant (mRPI) set [13] $\Delta X_i(W_i)$:

$$\Delta x_{[i]}(t) \in \Delta X_i(W_i) := \bigoplus_{t=0}^{\infty} \left(A_{[i]}^{\mathrm{K}}\right)^t W_i. \tag{9}$$

In the following, we first formulate tube-based robust MPC controllers $\mathcal{C}_i$ by relying on sets X_i and $\Delta X_i(W_i)$ and afterwards discuss the properties that these sets must satisfy in order to guarantee that $\mathbf{y} \in Y$. Each $\mathcal{C}_i$ is based on solving

$$\min_{\mathbf{z}_i} \sum_{s=t}^{t+N_{[i]}-1} \|\hat{x}_{[i]}(s)\|_{Q_{[i]}}^2 + \|\hat{u}_{[i]}(s)\|_{R_{[i]}}^2 + \|\hat{x}_{[i]}(t+N_{[i]})\|_{P_{[i]}}^2 \tag{10a}$$

$$\text{s.t. } x_{[i]}(t) - \hat{x}_{[i]}(t) \in \Delta X_i(W_i), \tag{10b}$$

$$\hat{x}_{[i]}(s+1) = A_{[ii]}\hat{x}_{[i]}(s) + B_{[i]}\hat{u}_{[i]}(s), \quad s \in \mathbb{I}_t^{t+N_{[i]}-1}, \tag{10c}$$

$$\hat{x}_{[i]}(s) \in X_i \ominus \Delta X_i(W_i), \quad s \in \mathbb{I}_{t+1}^{t+N_{[i]}-1}, \tag{10d}$$

$$\hat{u}_{[i]}(s) \in U_i \ominus -K_{[i]}\Delta X_i(W_i), \quad s \in \mathbb{I}_t^{t+N_{[i]}-1}, \tag{10e}$$

$$\hat{x}_{[i]}(t+N_{[i]}) \in X_i^{\mathrm{terminal}}, \tag{10f}$$

with the optimization vector $\mathbf{z}_i := [\hat{x}_{[i]}(t : t+N_{[i]})^\top \ \hat{u}_{[i]}(t : t+N_{[i]}-1)^\top]$. The nominal model (6) is used to perform predictions of state evolutions, as indicated in (10c). The initial state is left as a free variable to be optimized through constraint (10b), and the predicted state and input constraints are tightened through constraints (10d) and (10e), such that the actual subsystem state $x_{[i]}(t) \in X_i$ and $u_{[i]}(t) \in U_i$ for all t. The feedback gain $K_{[i]}$ is chosen to be the terminal control law, and terminal set $X_i^{\text{terminal}} \subset X_i \ominus \Delta X_i(W_i)$ is chosen to be an invariant set for the system $x_{[i]}(t+1) = A^K_{[i]} x_{[i]}(t)$, such that $A^{\text{K}}_{[i]} X_i^{\text{terminal}} \subseteq X_i^{\text{terminal}}$ and $-K_{[i]} X_i^{\text{terminal}} \subseteq U_i \ominus -K_{[i]} \Delta X_i(W_i)$. The matrices $Q_{[i]} > 0$ and $R_{[i]} > 0$ are chosen such that $K_{[i]}$ is the associated LQ control gain for nominal system i, and $P_{[i]}$ is the solution of the corresponding Discrete Algebraic Riccati Equation. Upon solving (10), control input $u_{[i]}(t) = \hat{u}_{[i]}(t) - K_{[i]}(x_{[i]}(t) - \hat{x}_{[i]}(t))$ is applied to the plant.

We recall the properties of the DeMPC scheme from [5], and derive requirements on the sets X_i in the following result.

Proposition 1: Suppose Assumptions 1 and 2 hold, and for each $i \in \mathbb{I}_1^M$, sets X_i satisfy

$$\Delta X_i(W_i) \subset \text{int}(X_i), \tag{11a}$$

$$-K_{[i]} \Delta X_i(W_i) \subset \text{int}(U_i). \tag{11b}$$

(a) For each controller $\mathcal{C}_i$, we denote the feasible set $\mathcal{X}_i^{N_{[i]}} := \{x_{[i]} :$ (10b)-(10f) feasible for $x_{[i]}(t) = x_{[i]}\}$. Then, if $x_{[i]}(0) \in \mathcal{X}_i^{N_{[i]}}$ the controlled system in (3) satisfies $x_{[i]}(t) \in X_i$ and $u_{[i]}(t) \in U_i$ for all time t, and the origin is asymptotically stable; (b) Defining $\mathbf{C}_{[i]} \in \mathbb{R}^{n_y \times n_x^i}$ as the matrix composed of columns of matrix $\mathbf{C}$ multiplying states $x_{[i]}$ of subsystem i, if the inclusion

$$\bigoplus_{i=0}^{M} \mathbf{C}_{[i]} X_i \subseteq Y \tag{12}$$

is satisfied, then the controllers $\mathcal{C}_i$ solve Problem 1. □

Proof: (a) The conditions in (11) ensure that constraint sets in (10d) and (10e) are non-empty. This leads to non-empty feasible sets $\mathcal{X}_i^{N_{[i]}}$. The proof then follows from [5]. (b) The condition in (12) translates to $X_i = \{x_{[i]} : \forall\ x_{[j]} \in X_j,\ \mathbf{C}_{[i]} x_{[i]} + \sum_{j \in L_i} \mathbf{C}_{[j]} x_{[j]} \in Y\}$ for all $i \in \mathbb{I}_1^M$. This implies that if (12) holds, then $x_{[i]} \in X_i$ for all i ensures $y \in Y$. The fact that the former is guaranteed by Part (a) concludes the proof. ■

From Assumptions 1 and 2, we see that requirements (11) and (12) can be satisfied by compact sets X_i, which we compute in the next section.

Remark 1: Note that requirement (12) results in conservative sets X_i, since it enforces that the control applied to the subsystem must satisfy system constraints Y, for every possible control applied by the neighbors. This is unavoidable, unless communication is introduced. In case full state information of all neighbors were available to $\mathcal{C}_i$, one could formulate the local constraint set as $\{x_{[i]} :\ \exists x_{[j]} \in X_j,\ \mathbf{C}_{[i]} x_{[i]} + \sum_{j \in L_i} \mathbf{C}_{[j]} x_{[j]} \in Y\} \supseteq X_i$. □

III. Computation of State-Constraint Sets X_i

In this section, we present a formulation and a solution procedure to compute the sets X_i that satisfy requirements (11) and (12). To this end, we introduce the system

$$\Delta \mathbf{x}(t+1) = \tilde{\mathbf{A}} \Delta \mathbf{x}(t) + \tilde{\mathbf{B}} \mathbf{x}(t), \tag{13}$$

where the matrices $\tilde{\mathbf{A}} := \text{diag}(A^K_{[1]}, \cdots, A^K_{[M]})$ and

$$\tilde{\mathbf{B}} := \begin{bmatrix} \mathbf{0} & A_{[12]} & \cdots & A_{[1M]} \\ A_{[21]} & \mathbf{0} & \cdots & A_{[2M]} \\ \cdots & \cdots & \mathbf{0} & \cdots \\ A_{[M1]} & A_{[M2]} & \cdots & \mathbf{0} \end{bmatrix}$$

capture the dynamic coupling between the subsystems. For this system, we introduce the sets

$$X := \prod_{i \in M} X_i, \qquad \Delta X(X) := \prod_{i \in M} \Delta X_i(W_i).$$

The set $\Delta X(X)$ is the mRPI set of states corresponding to the system (13) when driven by *disturbances* $\mathbf{x}(t) \in X$, given by

$$\Delta X(X) := \bigoplus_{t=0}^{\infty} \tilde{\mathbf{A}}^t \tilde{\mathbf{B}} X. \tag{14}$$

We make the following assumption on the sets X_i.

Assumption 3: We assume that each X_i is a compact convex set containing the origin. □

Assumption 3 implies that the mRPI set $\Delta X(X)$ is a compact convex set containing the origin [13].

In order to encode inclusions (11), we introduce scalars $\phi_{\text{x}}, \phi_{\text{u}} \in [0, 1)$, and write the inclusions as

$$\Delta X(X) \subseteq \phi_{\text{x}} X, \qquad -\mathbf{K} \Delta X(X) \subseteq \phi_{\text{u}} U. \tag{15}$$

The values of ϕ_{x} and ϕ_{u} are tuning parameters which are related to the strength of dynamic coupling. Larger values correspond to increased permissible dynamic coupling, and hence increased size of sets X_j. However, this also corresponds to smaller terminal set X_i^{terminal}, thus requiring a longer horizon $N_{[i]}$ to ensure feasibility.

Considering requirement (12), we first note that

$$\mathbf{C} X = \bigoplus_{i=0}^{M} \mathbf{C}_{[i]} X_i \subseteq Y, \tag{16}$$

by definition of X. Ideally, one would like to satisfy the inclusion $\mathbf{C} X \subseteq Y$ with equality. This would imply that the sets X_i are chosen such that all the points in set Y are reachable by $\mathbf{C}\mathbf{x}$. This, however, might not be feasible given requirements (15). For further details, we refer the reader to [10]. Hence, we choose to minimize the Hausdorff distance $d_{\text{H}}(Y, \mathbf{C} X)$ between the sets, while enforcing the inclusion.

From these requirements, we obtain the following optimization problem for fixed scalars ϕ_{x} and ϕ_{u}:

$$\min_{X, \epsilon \geq 0}\ \epsilon \text{ s.t. } (14), (15), (16),\ Y \subseteq \mathbf{C} X \oplus \epsilon \mathcal{B}_\infty^{n_y}, \mathbf{0} \in X_i. \tag{17}$$

Our approach explicitly tackles the issue of conservativeness discussed in Remark 1: The sets X_i are computed such that $\mathbf{C} X$ is the largest feasible inner-approximation of Y, i.e., $d_{\text{H}}(Y, \mathbf{C} X)$ is minimized while ensuring feasibility and stability of $\mathcal{C}_i$. This implies that the system output $\mathbf{y} = \mathbf{C}\mathbf{x}$ is restricted to the least conservative subset of Y when controllers $\mathcal{C}_i$ safely regulate the system state $\mathbf{x}$ to the origin.

Remark 2: The structure of (13) follows from the assumption of a block-diagonal matrix $\mathbf{B}$, i.e., decoupled inputs. The approach can be extended to accommodate coupled inputs and input constraints through minor reformulations. □

A. Finite-Dimensional Parameterization

We parameterize each set X_i using a finite-dimensional vector $\epsilon^{x,i}$ as $X_i = \mathbb{X}_i(\epsilon^{x,i}) := \{x_{[i]} : F^i x_{[i]} \leq \epsilon^{x,i}\}$, where the rows of matrix $F^i \in \mathbb{R}^{m^i_X \times n^i_x}$ spanning $\mathbb{R}^{n^i_x}$ are fixed a priori. Since the corresponding mRPI sets $\Delta X_i(W_i)$ are in general not finitely determined [14], we rely on an outer RPI approximation, parameterized using a finite-dimensional vector $\epsilon^{\Delta x,i}$ as $\Delta X_i(W_i) \subseteq \Delta\mathbb{X}_i(\epsilon^{\Delta x,i}) := \{\Delta x_{[i]} : E^i \Delta x_{[i]} \leq \epsilon^{\Delta x,i}\}$, where the rows of matrix $E^i \in \mathbb{R}^{m^i_{\Delta X} \times n^i_x}$ spanning $\mathbb{R}^{n^i_x}$ are fixed a priori. These matrices should satisfy some requirements, that are formulated in the sequel. The overall state constraint set is hence $X = \mathbb{X}(\epsilon^x) := \{\mathbf{x} : F\mathbf{x} \leq \epsilon^x\}$, where $F := \mathrm{diag}(F^1, \cdots, F^M) \in \mathbb{R}^{m_X \times n_x}$ and $\epsilon^x := [\epsilon^{x,1^\top}, \cdots, \epsilon^{x,M^\top}]^\top$. The corresponding mRPI set is hence approximated as $\Delta X(\mathbb{X}(\epsilon^x)) \subseteq \Delta\mathbb{X}(\epsilon^{\Delta x}) := \{\Delta\mathbf{x} : E\Delta\mathbf{x} \leq \epsilon^{\Delta x}\}$, where $E := \mathrm{diag}(E^1, \cdots, E^M) \in \mathbb{R}^{m_{\Delta X} \times n_x}$ and $\epsilon^{\Delta x} := [\epsilon^{\Delta x,1^\top}, \cdots, \epsilon^{\Delta x,M^\top}]^\top$.

In order to obtain a close approximation of the equality constraint (14) for a given disturbance set $\mathbb{X}(\epsilon^x)$, we use the parameterized RPI set $\Delta\mathbb{X}(\epsilon^{\Delta x})$ that minimizes $d_{\mathrm{H}}(\Delta X(\mathbb{X}(\epsilon^x)), \Delta\mathbb{X}(\epsilon^{\Delta x}))$. Finally, since $\Delta X(\mathbb{X}(\epsilon^x)) \subseteq \Delta\mathbb{X}(\epsilon^{\Delta x})$, we replace $\Delta X(\mathbb{X}(\epsilon^x))$ by $\Delta\mathbb{X}(\epsilon^{\Delta x})$ in inclusions (15). Note that process noise can be accommodated in this framework through matrix $\tilde{\mathbf{B}}$ and parameterized sets X.

In terms of the above parameterized sets, (17) is approximated as the following bilevel optimization problem:

***Problem** 2:*

$$\min_{\epsilon^x \geq \mathbf{0}, \epsilon \geq 0} \quad \epsilon \tag{18a}$$

$$\text{s.t.} \quad \Delta\mathbb{X}(\epsilon^{\Delta x}) \subseteq \phi_{\mathrm{x}}\mathbb{X}(\epsilon^x), \tag{18b}$$

$$-\mathbf{K}\Delta\mathbb{X}(\epsilon^{\Delta x}) \subseteq \phi_{\mathrm{u}} U, \tag{18c}$$

$$\mathbf{C}\mathbb{X}(\epsilon^x) \subseteq Y, \tag{18d}$$

$$Y \subseteq \mathbf{C}\mathbb{X}(\epsilon^x) \oplus \epsilon\mathcal{B}^{n_y}_\infty, \tag{18e}$$

$$\begin{aligned} \epsilon^{\Delta x} = \arg\min_{\underline{\epsilon}^{\Delta x}} \quad & d_{\mathrm{H}}(\Delta X(\mathbb{X}(\epsilon^x)), \Delta\mathbb{X}(\underline{\epsilon}^{\Delta x})), \\ \text{s.t.} \quad & \tilde{\mathbf{A}}\Delta\mathbb{X}(\underline{\epsilon}^{\Delta x}) \oplus \tilde{\mathbf{B}}\mathbb{X}(\epsilon^x) \subseteq \Delta\mathbb{X}(\underline{\epsilon}^{\Delta x}). \end{aligned} \tag{18f}$$

We now discuss the implementation of Problem 2.

B. Implementation of RPI Constraint (18f)

The invariance condition $\tilde{\mathbf{A}}\Delta\mathbb{X}(\underline{\epsilon}^{\Delta x}) \oplus \tilde{\mathbf{B}}\mathbb{X}(\epsilon^x) \subseteq \Delta\mathbb{X}(\underline{\epsilon}^{\Delta x})$ can be equivalently written as

$$\boldsymbol{c}(\underline{\epsilon}^{\Delta x}) + \boldsymbol{d}(\epsilon^x) \leq \boldsymbol{b}(\underline{\epsilon}^{\Delta x}), \tag{19}$$

where, for all $i \in \mathbb{I}_1^{m_{\Delta X}}$, we use the support function h to define $\boldsymbol{c}_i\left(\underline{\epsilon}^{\Delta x}\right) := h_{\tilde{\mathbf{A}}\Delta\mathbb{X}(\underline{\epsilon}^{\Delta x})}\left(E_i^\top\right), \boldsymbol{d}_i\left(\epsilon^x\right) := h_{\tilde{\mathbf{B}}\mathbb{X}(\epsilon^x)}\left(E_i^\top\right)$ and $\boldsymbol{b}_i\left(\underline{\epsilon}^{\Delta x}\right) := h_{\Delta\mathbb{X}(\underline{\epsilon}^{\Delta x})}\left(E_i^\top\right)$. We make the following assumptions on matrices E and F parameterizing the sets $\Delta\mathbb{X}(\underline{\epsilon}^{\Delta x})$ and $\mathbb{X}(\epsilon^x)$:

Assumption 4: (a) Matrix F is chosen such that $\boldsymbol{d}(\mathbf{1})$ is bounded, i.e., $\mathbb{X}(\mathbf{1})$ is compact, (b) the set $\Delta\mathbb{X}(\mathbf{1})$ is a proper polytope, with $\boldsymbol{b}(\mathbf{1}) = \mathbf{1}$, (c) there exists a scalar $\beta \in [0, 1)$ such that $\tilde{\mathbf{A}}\Delta\mathbb{X}(\mathbf{1}) \subseteq \beta\Delta\mathbb{X}(\mathbf{1})$. □

These assumptions imply that the support functions $\boldsymbol{d}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are always bounded above. To satisfy Assumptions 4(b) and (c), the methods presented in [15], [16] can be used to compute β-contractive RPI sets that parameterize $\Delta\mathbb{X}(\mathbf{1})$. The assumptions also imply that there exists an RPI set parameterized as $\Delta\mathbb{X}(\epsilon^{\Delta x})$ for every compact disturbance set $\mathbb{X}(\epsilon^x)$.

Lemma 1 ([10]): Suppose Assumption 4 holds, then there exists $\hat{\epsilon}^{\Delta x} \geq \mathbf{0}$ satisfying the RPI relation $\boldsymbol{c}(\hat{\epsilon}^{\Delta x}) + \boldsymbol{d}(\epsilon^x) \leq \boldsymbol{b}(\hat{\epsilon}^{\Delta x})$ for all $\epsilon^x \geq \mathbf{0}$. □

Building on the results presented in [7] and [8], the following result shows that there exists an RPI set $\Delta\mathbb{X}(\epsilon^{\Delta x})$ that is minimal over all RPI sets parameterized with E, for every given disturbance set $\mathbb{X}(\epsilon^x)$.

Theorem 1 ([10]): Suppose Assumption 4 holds, then the value of $\epsilon^{\Delta x}$ that solves the equality relationship

$$\boldsymbol{c}(\epsilon^{\Delta x}) + \boldsymbol{d}(\epsilon^x) = \epsilon^{\Delta x} \tag{20}$$

for a given $\epsilon^x \geq \mathbf{0}$ is such that

$$d_{\mathrm{H}}(\Delta X(\mathbb{X}(\epsilon^x)), \Delta\mathbb{X}(\epsilon^{\Delta x})) \leq d_{\mathrm{H}}(\Delta X(\mathbb{X}(\epsilon^x)), \Delta\mathbb{X}(\underline{\epsilon}^{\Delta x}))$$

for all $\underline{\epsilon}^{\Delta x}$ such that $\boldsymbol{c}(\underline{\epsilon}^{\Delta x}) + \boldsymbol{d}(\epsilon^x) \leq \boldsymbol{b}(\underline{\epsilon}^{\Delta x})$, i.e., all $\epsilon^{\Delta x}$ satisfying the RPI condition (19). □

From this result, we conclude that the solution of (20) solves the lower level optimization problem (18f). Hence, we replace (18f) by the equality relationship in (20).

C. Implementation of Inclusions Constraints

We use two different encodings to implement the inclusion constraints in Problem 2. The first one is based on support functions, while the second uses sufficiency conditions presented in [17].

1) Support function encoding: Since all the sets involved in Problem 2 are polytopes, the inclusions (18b), (18c) and (18d) hold if and only if the inequality

$$\boldsymbol{g}(\epsilon^x, \epsilon^{\Delta x}) \leq \boldsymbol{f}(\epsilon^x) \tag{21}$$

is satisfied, where

$$\boldsymbol{g}(\epsilon^x, \epsilon^{\Delta x}) := \begin{bmatrix} \boldsymbol{g}^{\mathrm{x}}(\epsilon^{\Delta x}) \\ \boldsymbol{g}^{\mathrm{u}}(\epsilon^{\Delta x}) \\ \boldsymbol{g}^{\mathrm{y}}(\epsilon^x) \end{bmatrix}, \qquad \boldsymbol{f}(\epsilon^x) := \begin{bmatrix} \phi_{\mathrm{x}}\boldsymbol{f}^{\mathrm{x}}(\epsilon^x) \\ \phi_{\mathrm{u}} g^{\mathrm{u}} \\ g^{\mathrm{y}} \end{bmatrix}.$$

Functions $\boldsymbol{g}$ and $\boldsymbol{f}$ are defined using support functions as

- For each $i \in \mathbb{I}_1^{m_X}$, $\boldsymbol{g}_i^{\mathrm{x}}\left(\epsilon^{\Delta x}\right) := h_{\Delta\mathbb{X}(\epsilon^{\Delta x})}\left(E_i^\top\right)$ and $\boldsymbol{f}_i^{\mathrm{x}}\left(\epsilon^x\right) := h_{\mathbb{X}(\epsilon^x)}\left(E_i^\top\right)$,
- For each $i \in \mathbb{I}_1^{m_U}$, $\boldsymbol{g}_i^{\mathrm{u}}\left(\epsilon^{\Delta x}\right) := h_{-\mathbf{K}\Delta\mathbb{X}(\epsilon^{\Delta x})}\left(G_i^{\mathrm{u}^\top}\right)$,
- For each $i \in \mathbb{I}_1^{m_Y}$, $\boldsymbol{g}_i^{\mathrm{y}}\left(\epsilon^x\right) := h_{\mathbf{C}\mathbb{X}(\epsilon^x)}\left(G_i^{\mathrm{y}^\top}\right)$.

Hence, we replace (18b), (18c) and (18d) by the support function inequality in (21).

2) Sufficiency condition encoding: In order to encode inclusion (18e), we use the sufficiency condition presented in [17], which states that under Assumption 1, the inclusion holds if there exist matrices $z^{\mathrm{B}} := \{\Sigma, \Theta, \Pi\}$ of compatible dimensions such that

$$\begin{aligned} &\Theta \geq \mathbf{0}, \quad \begin{bmatrix}\mathbf{C} & \mathbf{I}\end{bmatrix}\Sigma = \mathbf{I}, \quad \begin{bmatrix}\mathbf{C} & \mathbf{I}\end{bmatrix}\Pi = \mathbf{0}, \\ &\Theta G^{\mathrm{y}} = \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & B^{\mathrm{y}} \end{bmatrix}\Sigma, \quad \Theta g^{\mathrm{y}} \leq \begin{bmatrix} \epsilon^x \\ \epsilon\mathbf{1} \end{bmatrix} + \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & B^{\mathrm{y}} \end{bmatrix}\Pi, \end{aligned} \tag{22}$$

where $B^{\mathrm{y}} \in \mathbb{R}^{2n_y \times n_y}$ defines the ∞-norm ball in $\mathbb{R}^{n_y}$ as $\mathcal{B}^{n_y}_\infty := \{y : B^{\mathrm{y}} y \leq \mathbf{1}\}$. The relations in (22) describe polyhedral constraints which we denote as Ξ^{B}, such that (22) $\Leftrightarrow$ $(\epsilon^x, \epsilon, z^{\mathrm{B}}) \in \Xi^{\mathrm{B}}$. Hence, we replace (18e) by (22). Since (22) is a sufficiency condition on the inclusion, we have $d_{\mathrm{H}}(Y, \mathbf{C}\mathbb{X}(\epsilon^x)) \leq \epsilon$. We refer to [17] for further details.

D. Problem Formulation and Solution Method

Using (20), (21) and (22), we formulate Problem 2 as

$$\min_{\epsilon^x, \epsilon^{\Delta x}, \epsilon, z^{\mathrm{B}}} \quad \epsilon \tag{23a}$$

$$\text{s.t.} \quad \boldsymbol{c}(\epsilon^{\Delta x}) + \boldsymbol{d}(\epsilon^x) = \epsilon^{\Delta x}, \tag{23b}$$

$$\boldsymbol{g}(\epsilon^x, \epsilon^{\Delta x}) \leq \boldsymbol{f}(\epsilon^x), \tag{23c}$$

$$(\epsilon^x, \epsilon, z^{\mathrm{B}}) \in \Xi^{\mathrm{B}}, \tag{23d}$$

$$\epsilon \geq 0, \quad \epsilon^x \geq \mathbf{0}. \tag{23e}$$

The above optimization problem is a bilevel programming problem. In order to solve it, we replace the lower-level problems with their corresponding Karush-Kuhn-Tucker (KKT) optimality conditions to obtain a Linear Program with Complementarity Constraints (LPCC) [18] of the form

$$\min_{x,\lambda,s} \quad \epsilon \quad \text{s.t.} \quad (x, \lambda, s) \in \mathbb{C}, \quad \lambda \geq \mathbf{0}, \ s \geq \mathbf{0}, \quad \lambda \circ s = \mathbf{0},$$

where $\mathbb{C}$ represents the set of all linear constraints, and λ and s represent the vectors of all dual and slack variables. We use the Sequential Quadratic Programing (SQP) algorithm presented in [19] to solve the LPCC. For all details on this formulation we refer the interested reader to [10]. Note that $\epsilon^x = \mathbf{0}$, $\epsilon^{\Delta x} = \mathbf{0}$ and corresponding values of ϵ and z^{B} are feasible solutions to (23). Note that $\epsilon^x = \mathbf{0}$, $\epsilon^{\Delta x} = \mathbf{0}$ and corresponding values of ϵ and z^{B} are feasible for (23).

E. Integration with Controllers $\mathcal{C}_i$

Upon solving (23), we recover constraint sets X_i from the solution $\mathbb{X}(\epsilon^x)$. Then, we compute the sets W_i given by (8). For each W_i, we compute RPI sets $\Delta\tilde{X}_i(W_i)$ by following the method presented in [20] to tightly approximate $\Delta X_i(W_i)$. By construction, we obtain $\Delta X_i(W_i) \subseteq \Delta\tilde{X}_i(W_i) \subseteq \Delta\mathbb{X}_i(\epsilon^{\Delta x,i})$ for a tight enough $\Delta\tilde{X}_i(W_i)$. Using X_i and $\Delta\tilde{X}_i(W_i)$, we construct the optimization problems in (10) solved by $\mathcal{C}_i$. We use Proposition 1 to check the validity of a given initial state.

Remark 3: One can directly use the RPI sets $\Delta\mathbb{X}_i(\epsilon^{\Delta x,i})$ in place of $\Delta X_i(W_i)$. However, this results in a smaller feasible region $\mathcal{X}_i^{N_{[i]}}$ and increases the conservativeness of $\mathcal{C}_i$. □

Remark 4: The proposed formulation allows one to introduce specific conditions to be satisfied by the parameterization of the sets X, e.g., symmetry constraints can be imposed; and the inclusion of a feasible region of the state-space in X can be imposed through the sufficiency conditions presented in [17]. □

Remark 5: The computed sets X_i can be used to synthesize local controllers $\mathcal{C}_i$ using other methods, e.g. [21]. □

IV. Numerical Example

We consider a system composed of three dynamically coupled double integrators given by

$$A_{[ii]} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B_{[i]} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$

with dynamic coupling matrix

$$\tilde{\mathbf{B}} = \begin{bmatrix} \multicolumn{2}{c}{\mathbf{0}} & 0 & 0 & 0 & 0 \\ & & 0.025 & -0.005 & 0.05 & -0.05 \\ 0 & 0 & \multicolumn{2}{c}{\mathbf{0}} & 0 & 0 \\ 0.025 & -0.025 & & & 0.01 & -0.01 \\ 0 & 0 & 0 & 0 & \multicolumn{2}{c}{\mathbf{0}} \\ 0.05 & -0.005 & 0.05 & -0.05 & & \end{bmatrix}$$

and coupled constraints

$$Y = \left\{ \begin{bmatrix} -1 \\ -1 \\ -5 \end{bmatrix} \leq \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \right\}.$$

The input constraints are $u_{[i]} \in [-0.5, 0.5]$ for each $i \in \mathbb{I}_1^3$ We equip the subsystems with LQR feedback gains $K_{[i]}$ corresponding to $R_{[1]} = 5$, $R_{[2]} = 1$ and $R_{[3]} = 10$, and $Q_{[i]} = \mathbf{I}$ for each of the subsystems. In order to synthesize state constraint sets X_i that satisfy the system constraints, we parameterize X_i as $\mathbb{X}_i(\epsilon^{x,i})$ with $m_X^1 = 12$, $m_X^2 = 8$ and $m_X^3 = 12$ hyperplanes. For the parameterized RPI sets $\Delta\mathbb{X}_i(\epsilon^{\Delta x,i})$, we choose sets defined by $m_{\Delta X}^1 = 12$, $m_{\Delta X}^2 = 8$ and $m_{\Delta X}^3 = 12$ hyperplanes respectively. We select the matrices E^i using the methods presented in [15], such that $\tilde{\mathbf{A}}\Delta\mathbb{X}(\mathbf{1}) \subseteq \beta\Delta\mathbb{X}(\mathbf{1})$ holds with $\beta = 0.7839$. We set $\phi_{\mathrm{x}}, \phi_{\mathrm{u}} = 0.5$. Larger values of these parameters correspond to larger X_i, but also smaller tightened constraints $X_i \ominus \Delta X_i(W_i)$, thus smaller feasible regions $\mathcal{X}_i^{N_{[i]}}$.

The results of formulating optimization problem (23) and solving it with the SQP algorithm presented in [19], is shown in Figure 1. The algorithm converges to $\epsilon = 0.6107$. Upon recovering the sets $X_i = \mathbb{X}_i(\epsilon^{x,i})$ from the solution, we recompute tight approximations $\Delta\tilde{X}_i(W_i)$ of the mRPI sets [20]. For the considered example we obtain the upper bound $d_{\mathrm{H}}(\Delta X_i(W_i), \Delta\mathbb{X}_i(\epsilon^{\Delta x,i})) \leq \delta_i$, with $\delta_1 = 0.0479$, $\delta_2 = 0.0219$, $\delta_3 = 0.069$.

Using the sets X_i and $\Delta\tilde{X}_i(W_i)$, we synthesize a tube-based MPC controller $\mathcal{C}_i$ for each $i \in \mathbb{I}_1^3$, which solves the optimization problem (10). We choose the terminal sets X_i^{terminal} to be the maximal positive invariant sets within $X_i \ominus \Delta\tilde{X}_i(W_i)$. We also synthesize a centralized MPC controller for the overall system, using the same control parameters. The results of the simulations from the same feasible initial point can be seen in Figures 1 and 2, for prediction horizon $N = 10$. *As pointed out in Remark 1, the state evolution with DeMPC controllers is restricted to X_i, while the centralized controller can violate this constraint but still satisfy the overall constraints*. The sum of quadratic stage costs is 27.3768 for DeMPC and 21.9298 for centralized MPC. As discussed in Remark 5, we use the sets X_i to also synthesize $\mathcal{C}_i$ to be the controllers proposed in [21]. This leads to a better closed loop performance with overall cost of 26.8348. The optimization problems were formulated with YALMIP [22] and solved using the Gurobi QP solver [23]. Set operations and plotting were performed using the Multi-Parametric Toolbox [24].

Note that the approximation of $\Delta X_i(W_i)$ by $\Delta\mathbb{X}_i(\epsilon^{\Delta x,i})$ can be improved by solving (23) iteratively and introducing additional hyperplanes defining E^i [10, Corollary 1]. One can, e.g., use the hyperplanes defining $\Delta\tilde{X}_i(W_i)$.

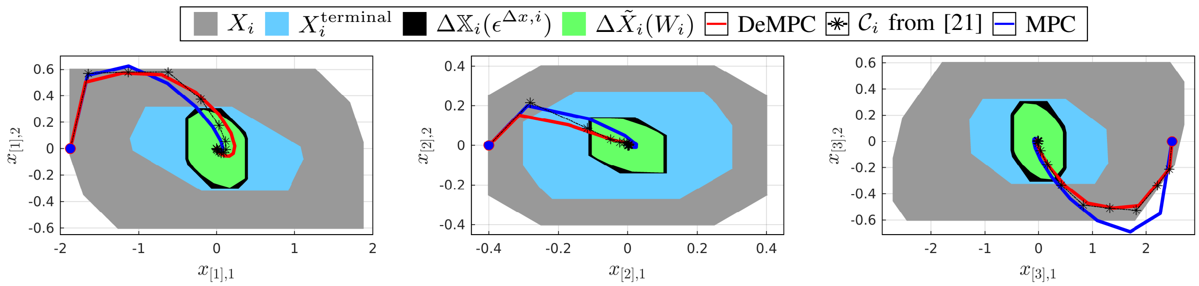

Fig. 1: Computed sets and simulation results in state space. Blue dots indicate initial states $x_{[i]}(0)$. Decentralized MPC restricts $x_{[i]} \in X_i$ to satisfy system constraints in (2), while centralized MPC does not require this restriction.

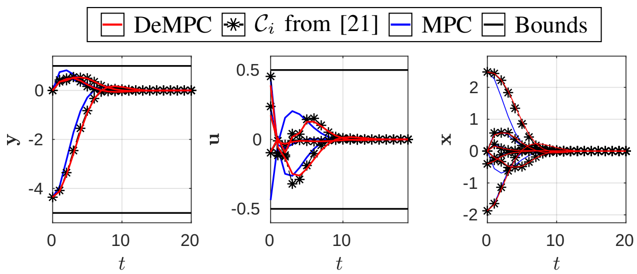

Fig. 2: System constraints and state regulation.

V. Conclusions and Future Work

We have presented a method to compute state-constraint sets for a fully decentralized MPC scheme to control a set of linear systems whose dynamics and state constraints can be coupled. We compute these sets by solving an offline optimization problem, which is formulated using a set-based framework. The problem explicitly ensures that conservativeness with respect to the coupled constraints in minimized, while guaranteeing feasibility and stability of the local tube-based MPC controllers. Future research will focus on (a) extensions to tracking problems; (b) co-synthesis of feedback controllers and constraint sets; (c) explicit enforcement of feasibility of a known state, based on the ideas presented in [17, Example 6]; (d) exploiting the possibility of partial communication between the controllers; e) efficient solution methods for problem (23), possibly avoiding the LPCC reformulation and introducing parallelized solution schemes.

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