



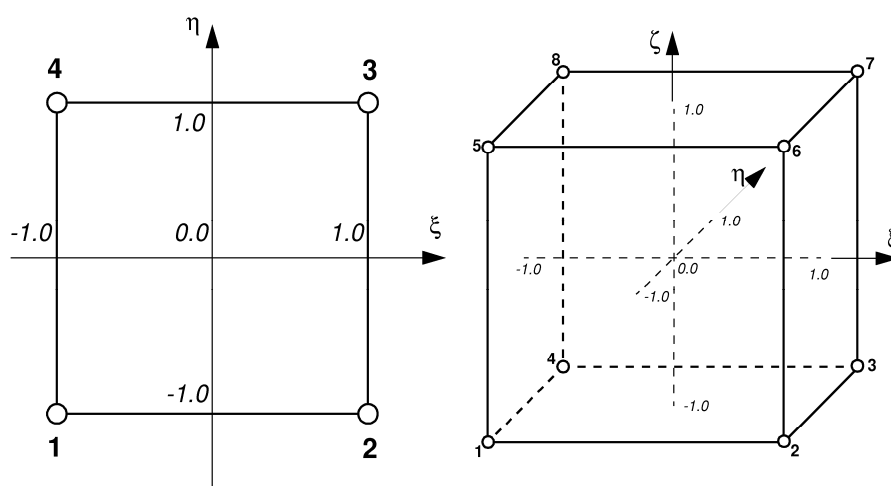
J R C T E C H N I C A L R E P O R T S

Implementation of Linear Lagrange 2D and 3D Continuum Elements for Solids in EUROPLEXUS

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12 January 2015

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1. Introduction

This report describes the implementation of 3D linear Lagrange continuum elements in EUROPLEXUS.

EUROPLEXUS [1] is a computer code for fast explicit transient dynamic analysis of fluid-structure systems jointly developed by the French Commissariat à l’Energie Atomique (CEA Saclay) and by the Joint Research Centre of the European Commission (JRC Ispra).

Parabolic Lagrange continuum elements in 2D (the Q9x 9-node element family, containing the Q92, Q92A, Q93 and Q95 elements) were among the first elements implemented in EUROPLEXUS predecessor codes at JRC in the late 70’s, see e.g. references [2-5]. These elements are relatively expensive computationally but offer outstanding numerical accuracy and are typically used to obtain reference solutions in complex elasto-plastic problems, for the validation of other more computationally efficient elements, such as shell or beam elements.

Recently a 3D version of the parabolic Lagrange elements, the C27x hexahedra with 27 nodes (named C272 and C273), have been developed in EUROPLEXUS and are described (together with the Q9x 2D elements) in reference [6].

In the late ‘80s and early ‘90s a linear version of the Q9x, the 4-node 2D quadrilaterals Q4xL (namely Q41L and Q42L), had been developed, but not documented. The final L in their names indicates that these elements are for Lagrangian calculations only, i.e. for solid materials. This was to distinguish them from similar versions of the 4-node 2D elements (Q41, Q42, Q41N, Q42N, Q42G) which can be used either in Lagrangian or in ALE/Eulerian calculations (such as for example in metal forming).

The present report shortly summarizes the implementation of the Q4xL 2D 4-node linear displacement quadrilateral elements and then introduces the implementation of their 3D equivalents, the C8xL 8-node linear hexahedra. The main issue in the 3D element formulation is the treatment of large strains and in particular of large rotations, see Section 3.7.

This document is organized as follows:

- Section 2 briefly recalls some aspects of the previous implementation of Q4xL elements in 2D.
- Section 3 presents the newly implemented C8xL 3D elements.
- Section 4 shows a series of numerical examples that check the performance of the 2D element and of the new 3D element.

The Appendix contains a listing of all the input files mentioned in the present report.

2. The 4-node linear displacement element in 2D

The 4-node Lagrange linear displacement element (Q4) for 2D plane and axisymmetric analysis in fast transient dynamics is shown in Figure 1 in the so-called parent element configuration, i.e. for normalized coordinates $-1 \leq \xi \leq 1$, $-1 \leq \eta \leq 1$. The local element numbering is anti-clockwise, as shown in the Figure.

2.1 Shape functions and their derivatives

The nodal shape functions $N_I(\xi, \eta)$, $I = 1, \dots, 4$ are obtained by blending of the 1-D linear shape functions $L_K(\delta)$, $K = 1, 2$, also shown in Figure 1:

$$\begin{aligned} L_1(\delta) &= \frac{1}{2}(1 - \delta) \\ L_2(\delta) &= \frac{1}{2}(1 + \delta) \end{aligned} \quad (1)$$

and result in the following expressions:

$$\begin{aligned} N_1(\xi, \eta) &= L_1(\xi)L_1(\eta) & N_3(\xi, \eta) &= L_1(\xi)L_2(\eta) \\ N_2(\xi, \eta) &= L_2(\xi)L_1(\eta) & N_4(\xi, \eta) &= L_2(\xi)L_2(\eta) \end{aligned} \quad (2)$$

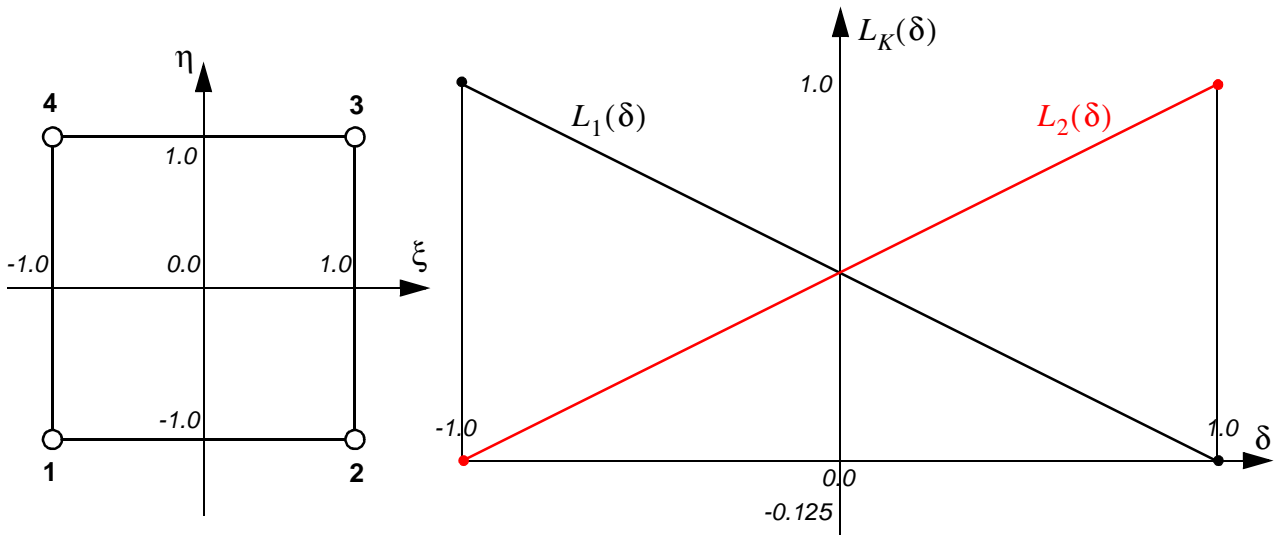


Figure 1 - The 4-node Lagrange linear element in 2D (Q4)

By replacing (1) into (2) one obtains:

$$\begin{aligned}
N_1 &= \frac{1}{4}(1 - \xi)(1 - \eta) \\
N_2 &= \frac{1}{4}(1 + \xi)(1 - \eta) \\
N_3 &= \frac{1}{4}(1 + \xi)(1 + \eta) \\
N_4 &= \frac{1}{4}(1 - \xi)(1 + \eta)
\end{aligned} \tag{3}$$

or, in compact form:

$$N_I = \frac{1}{4}(1 + \xi_I \xi)(1 + \eta_I \eta) \quad (\text{no sum on } I) \tag{4}$$

The shape function derivatives then result in:

$$\begin{aligned}
\frac{\partial N_1}{\partial \xi} &= -\frac{1}{4}(1 - \eta) & \frac{\partial N_1}{\partial \eta} &= -\frac{1}{4}(1 - \xi) \\
\frac{\partial N_2}{\partial \xi} &= \frac{1}{4}(1 - \eta) & \frac{\partial N_2}{\partial \eta} &= -\frac{1}{4}(1 + \xi) \\
\frac{\partial N_3}{\partial \xi} &= \frac{1}{4}(1 + \eta) & \frac{\partial N_3}{\partial \eta} &= \frac{1}{4}(1 + \xi) \\
\frac{\partial N_4}{\partial \xi} &= -\frac{1}{4}(1 + \eta) & \frac{\partial N_4}{\partial \eta} &= \frac{1}{4}(1 - \xi)
\end{aligned} \tag{5}$$

or, in compact form:

$$\frac{\partial N_I}{\partial \xi} = \frac{1}{4}\xi_I(1 + \eta_I \eta) \quad \frac{\partial N_I}{\partial \eta} = \frac{1}{4}\eta_I(1 + \xi_I \xi) \tag{6}$$

Any of the element's kinematic variables (coordinates x_i , displacements d_i , velocities v_i , accelerations a_i , $i = 1, 2$), say f_p , at a generic point $p = (\xi, \eta)$ in the parent element are obtained by interpolation from the corresponding nodal values using the shape functions (3):

$$f_p = f(\xi, \eta) = \sum_{I=1}^4 N_I(\xi, \eta) f_I. \tag{7}$$

Since the same interpolation (7) is used for the geometry ($f \equiv x$) as well as for the kinematic variables, the formulation is called isoparametric.

2.2 Lumped mass matrix

The consistent mass matrix for the generic element e with n nodes reads:

$$\underline{M}^c = [M_{IJ}^c] = \left[\int_{V^e} N_I \rho N_J dV \right] \quad I, J = 1, \dots, n. \quad (8)$$

The (diagonal) lumped mass matrix:

$$\underline{M}_{IJ}^l = [M_{IJ}^l] \quad (9)$$

is obtained by row summation:

$$\begin{aligned} M_I &= M_{II}^l = \sum_{J=1}^n M_{IJ}^l \\ M_{IJ}^l &= 0 \quad \text{for } I \neq J \end{aligned} \quad (10)$$

The generic diagonal term becomes therefore:

$$M_I = \sum_{J=1}^n M_{IJ}^l = \sum_{J=1}^n \left(\int_{V^e} N_I \rho N_J dV \right) = \int_{V^e} \left(\sum_{J=1}^n N_I \rho N_J \right) dV = \int_{V^e} N_I \rho \left(\sum_{J=1}^n N_J \right) dV. \quad (11)$$

But the sum of the shape functions is 1 in every point by definition:

$$\sum_{J=1}^n N_J = 1, \quad (12)$$

therefore (11) gives:

$$M_I = \int_{V^e} N_I \rho dV \quad (13)$$

and, if the density ρ is element-wise uniform:

$$M_I = \rho \int_{V^e} N_I dV. \quad (14)$$

The volume integral appearing in expressions (13) or (14) is then expressed in terms of the normalized coordinates. Then (14) becomes:

$$M_I = \rho \int_{V^e} N_I(x, y) dV = \rho \int_{-1}^1 \int_{-1}^1 N_I(\xi, \eta) \det \underline{J} d\xi d\eta, \quad (15)$$

where \underline{J} is the Jacobian matrix of the coordinates transformation between (ξ, η) and (x, y) :

$$\underline{J} = [J_{ij}] = \begin{bmatrix} \frac{\partial x_j}{\partial \xi_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}, \quad (16)$$

so that:

$$\det J = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta}. \quad (17)$$

The derivatives of the geometry with respect to the normalized coordinates $\frac{\partial x_i}{\partial \xi_j}$ appearing in (17) are obtained by means of the interpolation (7):

$$\underline{x} = \sum_{I=1}^4 N_I(\xi, \eta) \underline{x}_I, \quad (18)$$

so that:

$$\frac{\partial x_i}{\partial \xi_j} = \sum_{I=1}^4 \frac{\partial N_I}{\partial \xi_j} x_{iI}, \quad (19)$$

where $\frac{\partial N_I}{\partial \xi_j}$ are given by (5).

2.3 Spatial integration

Space integrals such as for example the one in (15) are computed numerically according to:

$$I = \int_{-1}^1 \int_{-1}^1 G(\xi, \eta) d\xi d\eta = \sum_{k=1}^m W_k G(\xi_k, \eta_k), \quad (20)$$

where (ξ_k, η_k) are appropriately chosen locations in the parent domain and W_k are the corresponding integration weights. In general the greater the number of sampling points m , the greater is the precision. For the present element, Gauss integration is adopted: the integration points are obtained by combining 1-D Gauss integration rules along each of the global spatial directions.

Either 1×1 (i.e., single-point) or 2×2 Gauss integration can be chosen. The first choice corresponds to element Q41L and the second one to element Q42L. The positions and numbering of the corresponding Gauss points is shown in Figure 2. The Gauss points are numbered first along the y (or η) direction, and then along the x (or ξ) direction.

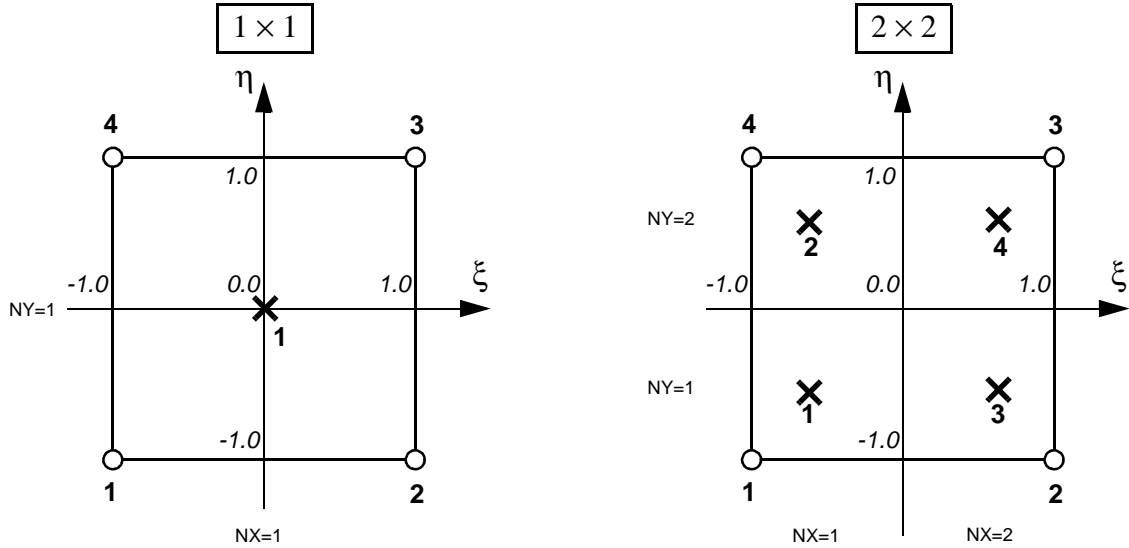


Figure 2 - Gauss point positions in the Q4 element

2.4 Treatment of large deformation

The *Lagrangian description of motion* reads:

$$\underline{x} = \underline{x}(\underline{X}, t), \quad (21)$$

where \underline{x} are so-called spatial coordinates denoting the current position of each particle and \underline{X} are so-called material coordinates, denoting the position of each particle in a reference configuration (usually at $t = t_0$, i.e. in the initial configuration). A particle occupying position \underline{X} at time t_0 occupies position \underline{x} at time t , i.e. the description (21) follows the particles.

Measures of the instantaneous (i.e. of the rate of) deformation and rotation can be obtained e.g. from the *spatial velocity gradient* \underline{L} , defined as:

$$\underline{L} = \frac{\partial \dot{\underline{x}}}{\partial \underline{x}}, \quad (22)$$

where a superposed dot denotes derivative with respect to time (particle velocity in this case):

$$\dot{\underline{x}} = \frac{\partial \underline{x}}{\partial t} = \underline{v}. \quad (23)$$

The \underline{L} tensor can be (like any tensor) decomposed into a symmetric and an antisymmetric component by the following additive decomposition:

$$\underline{L} = \underline{D} + \underline{W}, \quad (24)$$

where:

$$\underline{D} = \frac{1}{2}(\underline{L} + \underline{L}^T) \quad (25)$$

and

$$\underline{W} = \frac{1}{2}(\underline{L} - \underline{L}^T). \quad (26)$$

In the above:

- A superposed T denotes the transpose of a matrix or tensor.
- \underline{D} , i.e. the symmetric (see below) part of \underline{L} , is called the *stretching* tensor.
- \underline{W} , i.e. the anti-symmetric (see below) part of \underline{L} , is called the *spin* tensor.

From the definitions (25) and (26) it is immediate to verify that (24) holds:

$$\underline{L} = \underline{D} + \underline{W} = \frac{1}{2}(\underline{L} + \underline{L}^T) + \frac{1}{2}(\underline{L} - \underline{L}^T) = \frac{1}{2}(\underline{L} + \underline{L}^T + \underline{L} - \underline{L}^T) = \underline{L}. \quad (27)$$

Furthermore, one sees from (25) that:

$$D_{ij} = \frac{1}{2}(L_{ij} + L_{ji}) \quad D_{ji} = \frac{1}{2}(L_{ji} + L_{ij}) \quad (28)$$

so that:

$$D_{ij} = D_{ji}, \quad (29)$$

i.e. \underline{D} is symmetric. Moreover:

$$D_{ii} = \frac{1}{2}(L_{ii} + L_{ii}) = L_{ii}, \quad (30)$$

i.e., the diagonal of \underline{D} is equal to the diagonal of \underline{L} .

Finally, from (26):

$$W_{ij} = \frac{1}{2}(L_{ij} - L_{ji}) \quad W_{ji} = \frac{1}{2}(L_{ji} - L_{ij}), \quad (31)$$

so that:

$$W_{ij} = -W_{ji}, \quad (32)$$

i.e. \underline{W} is anti-symmetric. Note also that the diagonal of \underline{W} is zero (as for any anti-symmetric tensor):

$$W_{ii} = \frac{1}{2}(L_{ii} - L_{ii}) = 0. \quad (33)$$

Consider the following example of additive decomposition:

$$\underline{L} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \underline{L}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\underline{D} = \frac{1}{2}(\underline{L} + \underline{L}^T) = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}_{\text{sym.}} \quad \underline{W} = \frac{1}{2}(\underline{L} - \underline{L}^T) = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}_{\text{anti-sym.}}$$

$$\underline{D} + \underline{W} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \underline{L}.$$

2.5 Calculation of the spatial velocity gradient

From (22) one can write, in components[†]:

$$L_{ij} = \frac{\partial \dot{x}_j}{\partial x_i} = \frac{\partial v_j}{\partial x_i}, \quad (34)$$

where $\dot{\underline{x}} = \underline{v}$ denotes the particle velocity.

For an iso-parametric element formulation:

$$v_j(\underline{x}) = \sum_{I=1}^n N_I v_{jI}, \quad (35)$$

where N_I are the element's shape functions, n is the number of nodes of the element and v_{jI} is the j -th component of the velocity at node I . Thus, from (34)-(35):

$$L_{ij}(\underline{x}) = \frac{\partial v_j(\underline{x})}{\partial x_i} = \sum_{I=1}^n \frac{\partial N_I}{\partial x_i} v_{jI}. \quad (36)$$

Therefore, we need to compute spatial derivatives of the shape functions N_I with respect to the global coordinates x_i . These can be expressed as follows:

$$\left\{ \frac{\partial N_I}{\partial x_i} \right\} = |J|^{-1} \left\{ \frac{\partial N_I(\underline{\xi})}{\partial \xi_i} \right\}, \quad (37)$$

[†]. It seems that the expression of \underline{L} is the one given by (34) and *not* its transpose ($L_{ij} = \partial v_i / \partial x_j$). Otherwise, there are differences in sign in the formulas for Cauchy stress rotation implemented in the Q4 routine. But this should be further verified.

where $\{ \}$ denotes a column vector ($j = 1, 2$), ξ_i are normalized coordinates (parent element) ranging from -1 to 1 , and $|J|$ is the determinant of the Jacobian matrix \underline{J} of the coordinates transformation between $\underline{\xi}$ and \underline{x} .

The Jacobian matrix is given by:

$$\underline{J} = [J_{ij}] = \begin{bmatrix} \frac{\partial x_j}{\partial \xi_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}. \quad (38)$$

In two spatial dimensions, (37) becomes:

$$\begin{aligned} \frac{\partial N_I}{\partial x} &= \frac{1}{\det \underline{J}} \left(\frac{\partial y}{\partial \eta} \frac{\partial N_I}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial N_I}{\partial \eta} \right) \\ \frac{\partial N_I}{\partial y} &= \frac{1}{\det \underline{J}} \left(\frac{\partial x}{\partial \xi} \frac{\partial N_I}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial N_I}{\partial \xi} \right), \end{aligned} \quad (39)$$

where the determinant is given by:

$$\det \underline{J} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi}. \quad (40)$$

In the above expressions, terms of the form $\frac{\partial N_I}{\partial \xi_i}$ are obtained by direct differentiation of the shape functions, while those of the type $\frac{\partial x_j}{\partial \xi_i}$ are computed based on (35):

$$\frac{\partial x_j}{\partial \xi_i} = \sum_{I=1}^n \frac{\partial N_I}{\partial \xi_i} x_{jI}. \quad (41)$$

In two spatial dimensions, (37) becomes:

$$\begin{aligned} \frac{\partial N_I}{\partial x} &= \frac{1}{\det \underline{J}} \left(\frac{\partial y}{\partial \eta} \frac{\partial N_I}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial N_I}{\partial \eta} \right) \\ \frac{\partial N_I}{\partial y} &= \frac{1}{\det \underline{J}} \left(\frac{\partial x}{\partial \xi} \frac{\partial N_I}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial N_I}{\partial \xi} \right), \end{aligned} \quad (42)$$

where the determinant is given by:

$$\det \underline{J} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi}. \quad (43)$$

2.6 The Zaremba-Jaumann-Noll (ZJN) rate of Cauchy stress

A hypo-elastic material can be defined as behaving in the following way:

$$\hat{\underline{\underline{\sigma}}} = \underline{\underline{C}}(\underline{\underline{\sigma}})\underline{\underline{D}}, \quad (44)$$

i.e. the rate of Cauchy stress $\hat{\underline{\underline{\sigma}}}$ is proportional to the stretching $\underline{\underline{D}}$ via a material response function $\underline{\underline{C}}$ which depends only on the stress $\underline{\underline{\sigma}}$.

The stress rate $\hat{\underline{\underline{\sigma}}}$ is called the co-rotational or the Zaremba-Jaumann-Noll (ZJN) stress rate, and is given by:

$$\hat{\underline{\underline{\sigma}}} = \dot{\underline{\underline{\sigma}}} - \underline{\underline{W}}\underline{\underline{\sigma}} + \underline{\underline{\sigma}}\underline{\underline{W}}, \quad (45)$$

where $\underline{\underline{W}}$ is the spin tensor. It can be verified that this stress rate is (infinitesimally) objective: it preserves invariance under superposed rotation in an infinitesimal sense, i.e. in the limit of vanishingly small time steps.

The spatial velocity gradient $\underline{\underline{L}}$ given by (22) furnishes, as already stated above, the rate of deformation (stretching) in its symmetric part and the rate of rotation (spin) in its anti-symmetric part, as expressed by the additive decomposition (24)-(26):

$$\underline{\underline{L}} = \frac{\partial \underline{\underline{v}}}{\partial \underline{\underline{x}}} = \underline{\underline{D}} + \underline{\underline{W}}. \quad (46)$$

It may be noted that the ZJN stress rate $\hat{\underline{\underline{\sigma}}}$ is linear in the stretching $\underline{\underline{D}}$ and that the material response function $\underline{\underline{C}}$ depends on the stress $\underline{\underline{\sigma}}$ only. If $\underline{\underline{C}}$ is further restricted to be stress-independent, a hypo-elastic material of grade 0 (Jaumann type) is obtained. Then $\underline{\underline{C}}$ can be viewed as a generalized Hooke matrix depending on the elastic constants κ (bulk modulus) and μ (shear modulus) only. This offers a computationally tempting opportunity to include elasto-plastic material behavior by merely replacing the elastic matrix by the elasto-plastic matrix as classically used in small-strain analysis. As this depends on the stress state only, a form coherent with (44) is apparently obtained.

From a numerical point of view it is important to note that a straightforward use of (45) preserves objectivity only in an infinitesimal sense, i.e. objectivity is achieved only in the limit of vanishingly small time steps. From geometrical considerations, an elementary algorithm preserving objectivity for an arbitrarily large time step can be designed in 2-D. If the rotation angle during a time step is denoted by θ , a tensor $\underline{\underline{\tau}}$ (the stress for instance) transforms like: $\underline{\underline{T}}^T \underline{\underline{\tau}} \underline{\underline{T}}$, where the rotation matrix $\underline{\underline{T}}$ is given by:

$$\underline{\underline{T}} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}. \quad (47)$$

In the discrete process it is easily seen that the angle θ is related to the spin W_{12} at the mid-step (i.e., obtained from the mid-step velocity $\underline{v}^{n+1/2}$) by the expression:

$$\Delta t W_{12} = 2 \tan \frac{\theta}{2}. \quad (48)$$

This discrete procedure for preserving objectivity is equivalent in 2-D to a general procedure based on polar decomposition rather than on separation between the symmetric and anti-symmetric parts of the velocity gradient. It should be noted, however, that in practice direct use of (45) delivers satisfactory results for all applications where the rotation increments remain small. When a discretization process is applied, it is necessary to define how to compute the approximate strain rate (in fact, how strain increment is performed). It can be shown that the best approximation for strain rate is obtained by making use of the stretching obtained at $n + 1/2$ through the mid-step velocity $\underline{v}^{n+1/2}$. In particular, the accumulated strain in the absence of rotation then provides an excellent approximation to the natural logarithmic strain, which is of great advantage in the interpretation of large strain constitutive relations.

Going further into the discretization process and in order to preserve maximum accuracy in the application of the constitutive relations, a double transformation can be performed. First the stress is rotated by an angle $\theta/2$ to the configuration $n + 1/2$, then the constitutive relation is applied and finally the updated stress is transformed to configuration $n + 1$ by a second half-rotation of $\theta/2$. This procedure thus requires two isoparametric transformations at each time step: one for configuration $n + 1/2$ and one for configuration $n + 1$.

The half-angle transformation (rotation) matrix is constructed according to (48) as:

$$\cos \frac{\theta}{2} = \frac{1}{\sqrt{1 + \frac{W_{12}^2 \Delta t^2}{4}}} \quad \sin \frac{\theta}{2} = \frac{\frac{W_{12} \Delta t}{2}}{\sqrt{1 + \frac{W_{12}^2 \Delta t^2}{4}}}. \quad (49)$$

Using half-step approximations for the stretching and spin, and implementing the constitutive relations with care, nearly exact solutions can be obtained.

The expressions (49) are obtained as follows. From (48) one gets:

$$\tan \frac{\theta}{2} = \frac{\Delta t}{2} W_{12}. \quad (50)$$

However, we need $\sin(\theta/2)$ and $\cos(\theta/2)$ in order to build the half-rotation matrix (see 47):

$$\underline{T}_2 = \begin{bmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix}. \quad (51)$$

The following trigonometric identities hold:

$$\cos \alpha = \frac{1}{\sqrt{1 + (\tan \alpha)^2}} \quad \sin \alpha = \frac{\tan \alpha}{\sqrt{1 + (\tan \alpha)^2}}. \quad (52)$$

By replacing the expression (50) of $\tan(\theta/2)$ for $\tan \alpha$ in (52) one obtains the expressions (49).

It is readily verified that the first of (52) is an identity. In fact, by squaring each member and by recalling that $\tan \alpha = (\sin \alpha)/(\cos \alpha)$ one gets:

$$(\cos \alpha)^2 = \frac{1}{1 + (\tan \alpha)^2} = \frac{1}{1 + \frac{(\sin \alpha)^2}{(\cos \alpha)^2}} = \frac{(\cos \alpha)^2}{(\cos \alpha)^2 + (\sin \alpha)^2} \equiv (\cos \alpha)^2. \quad (53)$$

The second identity is then obtained from the first one as follows:

$$\sin \alpha = \frac{\sin \alpha}{\cos \alpha} \cos \alpha = \tan \alpha \cos \alpha = \frac{\tan \alpha}{\sqrt{1 + (\tan \alpha)^2}}. \quad (54)$$

Given a (2-D) stress tensor $\underline{\sigma}$, the rotated stress $\underline{\sigma}_{\text{rot}}$ according to a rigid-body rotation of angle $\theta/2$ is:

$$\underline{\sigma}_{\text{rot}} = \underline{T}_2^T \underline{\sigma} \underline{T}_2 \quad (55)$$

with \underline{T}_2 given by (51).

By expanding this expression one obtains:

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}_{\text{rot}} = \begin{bmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix}. \quad (56)$$

By performing the multiplications one obtains, first:

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}_{\text{rot}} = \begin{bmatrix} \left(\sigma_x \cos \frac{\theta}{2} - \tau_{xy} \sin \frac{\theta}{2} \right) & \left(\tau_{xy} \cos \frac{\theta}{2} - \sigma_y \sin \frac{\theta}{2} \right) \\ \left(\sigma_x \sin \frac{\theta}{2} + \tau_{xy} \cos \frac{\theta}{2} \right) & \left(\tau_{xy} \sin \frac{\theta}{2} + \sigma_y \cos \frac{\theta}{2} \right) \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix} \quad (57)$$

and finally:

$$\begin{aligned} \sigma_{x, \text{rot}} &= \sigma_x \left(\cos \frac{\theta}{2} \right)^2 - 2\tau_{xy} \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \sigma_y \left(\sin \frac{\theta}{2} \right)^2 \\ \tau_{xy, \text{rot}} &= \tau_{yx, \text{rot}} = (\sigma_x - \sigma_y) \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \tau_{xy} \left[\left(\cos \frac{\theta}{2} \right)^2 - \left(\sin \frac{\theta}{2} \right)^2 \right]. \\ \sigma_{y, \text{rot}} &= \sigma_x \left(\sin \frac{\theta}{2} \right)^2 + 2\tau_{xy} \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \sigma_y \left(\cos \frac{\theta}{2} \right)^2 \end{aligned} \quad (58)$$

2.7 Calculation of the strain increments and of the rotation increment over the step

The strain increments over a time step Δt are given by:

$$\underline{\Delta \varepsilon} = \underline{D} \Delta t, \quad (59)$$

where \underline{D} is the stretching tensor given by (25):

$$\underline{D} = \frac{1}{2} (\underline{L} + \underline{L}^T) \quad (60)$$

and \underline{L} is the spatial velocity gradient given by (22) or (34):

$$\underline{L} = \frac{\partial \dot{\underline{x}}}{\partial \underline{x}} = \frac{\partial \underline{v}}{\partial \underline{x}} = [L_{ij}] = \left[\frac{\partial \dot{x}_j}{\partial x_i} \right] = \left[\frac{\partial v_j}{\partial x_i} \right]. \quad (61)$$

By denoting $\underline{\Delta u} \equiv [\Delta u \ \Delta v \ \Delta w]^T$ the displacement increment over the time step:

$$\underline{\Delta u} = \underline{v} \Delta t = \underline{v}^{n+1/2} \Delta t, \quad (62)$$

one obtains from (61) and (59):

$$\underline{L} \Delta t = \frac{\partial(\underline{\Delta u})}{\partial \underline{x}} = \left[\frac{\partial(\Delta u_j)}{\partial x_i} \right] = \begin{bmatrix} \frac{\partial(\Delta u)}{\partial x} & \frac{\partial(\Delta v)}{\partial x} \\ \frac{\partial(\Delta u)}{\partial y} & \frac{\partial(\Delta v)}{\partial y} \end{bmatrix}. \quad (63)$$

Then the strain increments result in:

$$\begin{aligned}
\Delta\varepsilon_{11} &= D_{11}\Delta t = L_{11}\Delta t = \frac{\partial(\Delta u)}{\partial x} \\
\Delta\varepsilon_{12} &= D_{12}\Delta t = \frac{1}{2}(L_{12} + L_{21})\Delta t = \frac{1}{2}\left[\frac{\partial(\Delta v)}{\partial x} + \frac{\partial(\Delta u)}{\partial y}\right] \\
\Delta\varepsilon_{21} &= D_{21}\Delta t = \frac{1}{2}(L_{21} + L_{12})\Delta t = \frac{1}{2}\left[\frac{\partial(\Delta u)}{\partial y} + \frac{\partial(\Delta v)}{\partial x}\right] = \Delta\varepsilon_{12} \\
\Delta\varepsilon_{22} &= D_{22}\Delta t = L_{22}\Delta t = \frac{\partial(\Delta v)}{\partial y}
\end{aligned} \tag{64}$$

By recalling that, as concerns shear components:

$$\gamma_{ij} = 2\varepsilon_{ij} \quad \text{for } i \neq j \tag{65}$$

and by using the iso-parametric expression of the displacement increments one has:

$$\begin{aligned}
\Delta\varepsilon_{11} &= \frac{\partial(\Delta u)}{\partial x} = \sum_{I=1}^n \frac{\partial N_I}{\partial x} \Delta u_I \\
\Delta\gamma_{12} = \Delta\gamma_{21} &= \frac{\partial(\Delta v)}{\partial x} + \frac{\partial(\Delta u)}{\partial y} = \sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta v_I + \frac{\partial N_I}{\partial y} \Delta u_I \right) \\
\Delta\varepsilon_{22} &= \frac{\partial(\Delta v)}{\partial y} = \sum_{I=1}^n \frac{\partial N_I}{\partial y} \Delta v_I
\end{aligned} \tag{66}$$

One can proceed similarly to compute the rotation half-increment ($\theta/2$) over the step. From (48) one has:

$$\tan \frac{\theta}{2} = \frac{\Delta t}{2} \underline{W}_{12} \tag{67}$$

where \underline{W} is the spin tensor given by (26):

$$\underline{W} = \frac{1}{2}(\underline{L} - \underline{L}^T). \tag{68}$$

From (68) one has:

$$\Delta t W_{12} = \frac{1}{2}(L_{12} - L_{21})\Delta t \tag{69}$$

and by using the expression (63) of the spatial velocity gradient:

$$\Delta t W_{12} = \frac{1}{2}\left[\frac{\partial(\Delta v)}{\partial x} - \frac{\partial(\Delta u)}{\partial y}\right]. \tag{70}$$

Finally, by using the iso-parametric expression of the displacement increments one has:

$$\Delta t W_{12} = \frac{1}{2} \left[\sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta v_I - \frac{\partial N_I}{\partial y} \Delta u_I \right) \right] \quad (71)$$

or, from (67):

$$\tan \frac{\theta}{2} = \frac{1}{4} \left[\sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta v_I - \frac{\partial N_I}{\partial y} \Delta u_I \right) \right]. \quad (72)$$

2.8 Stress increment procedure

The stress increment procedure at a generic Gauss point can now be summarized as follows for the 2-D plane stress case.

Let $\underline{\sigma}^n$ denote the old stress, i.e. the stress at the beginning of the time step $n \rightarrow n + 1$:

$$\underline{\sigma}^n = \begin{bmatrix} \sigma_x^n & \tau_{xy}^n \\ \tau_{xy}^n & \sigma_y^n \end{bmatrix}. \quad (73)$$

Compute the partial derivatives of (x, y) with respect to (ξ, η) at the current Gauss point:

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \xi} x_I & \frac{\partial x}{\partial \eta} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \eta} x_I \\ \frac{\partial y}{\partial \xi} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \xi} y_I & \frac{\partial y}{\partial \eta} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \eta} y_I \end{aligned}, \quad (74)$$

on the configuration $n + 1/2$, i.e. by using as nodal coordinates:

$$\underline{x}_I = \underline{x}_I^{n+1/2} = \underline{x}_I^{n+1} - \frac{1}{2} \Delta \underline{u}_I. \quad (75)$$

Evaluate the determinant of the Jacobian at $n + 1/2$ with (43):

$$\det \underline{J} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi}. \quad (76)$$

Compute the partial derivatives of the shape functions at $n + 1/2$ with (42):

$$\begin{aligned} \frac{\partial N_I}{\partial x} &= \frac{1}{\det \underline{J}} \left(\frac{\partial y}{\partial \eta} \frac{\partial N_I}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial N_I}{\partial \eta} \right) \\ \frac{\partial N_I}{\partial y} &= \frac{1}{\det \underline{J}} \left(\frac{\partial x}{\partial \xi} \frac{\partial N_I}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial N_I}{\partial \xi} \right) \end{aligned}. \quad (77)$$

Compute the strain increments over the step by (66):

$$\begin{aligned}
\Delta \varepsilon_{11} &= \frac{\partial(\Delta u)}{\partial x} = \sum_{I=1}^n \frac{\partial N_I}{\partial x} \Delta u_I \\
\Delta \gamma_{12} = \Delta \gamma_{21} &= \frac{\partial(\Delta v)}{\partial x} + \frac{\partial(\Delta u)}{\partial y} = \sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta v_I + \frac{\partial N_I}{\partial y} \Delta u_I \right) \\
\Delta \varepsilon_{22} &= \frac{\partial(\Delta v)}{\partial y} = \sum_{I=1}^n \frac{\partial N_I}{\partial y} \Delta v_I
\end{aligned} \tag{78}$$

and the rotation increment over the step by (72):

$$\tan \frac{\theta}{2} = \frac{1}{4} \left[\sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta v_I - \frac{\partial N_I}{\partial y} \Delta u_I \right) \right]. \tag{79}$$

Now we can rotate the old stress by $\theta/2$. Compute first:

$$\begin{aligned}
\left(\tan \frac{\theta}{2} \right)^2 &= \tan \frac{\theta}{2} \tan \frac{\theta}{2} \\
\left(\cos \frac{\theta}{2} \right)^2 &= \frac{1}{1 + \left(\tan \frac{\theta}{2} \right)^2}, \\
\left(\sin \frac{\theta}{2} \right)^2 &= \left(\cos \frac{\theta}{2} \right)^2 \left(\tan \frac{\theta}{2} \right)^2 \\
\sin \frac{\theta}{2} \cos \frac{\theta}{2} &= \left(\cos \frac{\theta}{2} \right)^2 \tan \frac{\theta}{2}
\end{aligned} \tag{80}$$

then the rotated stresses from (58):

$$\begin{aligned}
\sigma_{x, \text{rot}} &= \sigma_x \left(\cos \frac{\theta}{2} \right)^2 - 2\tau_{xy} \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \sigma_y \left(\sin \frac{\theta}{2} \right)^2 \\
\tau_{xy, \text{rot}} = \tau_{yx, \text{rot}} &= (\sigma_x - \sigma_y) \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \tau_{xy} \left[\left(\cos \frac{\theta}{2} \right)^2 - \left(\sin \frac{\theta}{2} \right)^2 \right]. \\
\sigma_{y, \text{rot}} &= \sigma_x \left(\sin \frac{\theta}{2} \right)^2 + 2\tau_{xy} \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \sigma_y \left(\cos \frac{\theta}{2} \right)^2
\end{aligned} \tag{81}$$

At this point, the material's constitutive equation is used to compute the stress increments $\Delta \underline{\sigma}$ corresponding to the strain increments $\Delta \underline{\varepsilon}$ given by (78), and the new (rotated) stresses $\underline{\sigma}_{\text{rot}}^{n+1}$. Formally, this can be expressed by (44):

$$\Delta \underline{\sigma} = \underline{C}(\underline{\sigma}) \Delta \underline{\varepsilon} \tag{82}$$

and by:

$$\underline{\sigma}_{\text{rot}}^{n+1} = \underline{\sigma}_{\text{rot}} + \Delta \underline{\sigma} = \underline{\sigma}_{\text{rot}} + \underline{C}(\underline{\sigma}) \Delta \underline{\varepsilon} \tag{83}$$

Finally, the new stress $\underline{\sigma}^{n+1}$ at $n+1$ is obtained by rotating $\underline{\sigma}_{\text{rot}}^{n+1}$ by the angle $\theta/2$. By applying once more the expressions (81) one gets:

$$\begin{aligned}\sigma_x^{n+1} &= \sigma_{x, \text{rot}} \left(\cos \frac{\theta}{2} \right)^2 - 2\tau_{xy, \text{rot}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \sigma_{y, \text{rot}} \left(\sin \frac{\theta}{2} \right)^2 \\ \tau_{xy}^{n+1} &= \tau_{yx}^{n+1} = (\sigma_{x, \text{rot}} - \sigma_{y, \text{rot}}) \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \tau_{xy, \text{rot}} \left[\left(\cos \frac{\theta}{2} \right)^2 - \left(\sin \frac{\theta}{2} \right)^2 \right]. \\ \sigma_y^{n+1} &= \sigma_{x, \text{rot}} \left(\sin \frac{\theta}{2} \right)^2 + 2\tau_{xy, \text{rot}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \sigma_{y, \text{rot}} \left(\cos \frac{\theta}{2} \right)^2\end{aligned}\quad (84)$$

2.9 Calculation of internal forces

Once the stresses have been updated, the element's (e) contributions to internal forces can be computed. These read:

$$\underline{F}_e^{\text{int}} = \int_{V^e} \underline{B}^T \underline{\sigma} dV, \quad (85)$$

where V^e is the element's current volume (current configuration), \underline{B} is the matrix of shape function (N) derivatives (see below) and $\underline{\sigma}$ is Cauchy stress. At a node I of the element e , this can be particularized as follows (we drop the superscript "int" for simplicity):

$$\underline{F}_I = \int_{V^e} \underline{B}_I^T \underline{\sigma} dV, \quad (86)$$

where \underline{B}_I is the sub-matrix of \underline{B} relative to node I . Note that in this expression $\underline{\sigma}$ represents a column vector containing (in a certain order) all the significant components of the stress tensor.

In 2D plane stress or plane strain it is:

$$\underline{B}_I = \begin{bmatrix} \frac{\partial N_I}{\partial x} & 0 \\ 0 & \frac{\partial N_I}{\partial y} \\ \frac{\partial N_I}{\partial y} & \frac{\partial N_I}{\partial x} \end{bmatrix} \quad \underline{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}. \quad (87)$$

Recall that the expression of \underline{B}_I can be deduced from the strain-displacement relations:

$$\underline{\varepsilon} = \underline{B}\underline{u} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_y}{\partial y} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \end{bmatrix}, \quad (88)$$

with (for example):

$$\frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x} \left(\sum_{I=1}^n N_I u_{xI} \right) = \sum_{I=1}^n \frac{\partial N_I}{\partial x} u_{xI}. \quad (89)$$

By expanding (86) one gets:

$$\begin{aligned} F_{Ix} &= \int_{V^e} \left(\frac{\partial N_I}{\partial x} \sigma_x + \frac{\partial N_I}{\partial y} \tau_{xy} \right) dV \\ F_{Iy} &= \int_{V^e} \left(\frac{\partial N_I}{\partial y} \sigma_y + \frac{\partial N_I}{\partial x} \tau_{xy} \right) dV \end{aligned} \quad (90)$$

In 2D axisymmetric (r, z coordinates) it is:

$$\underline{B}_I = \begin{bmatrix} 0 & \frac{\partial N_I}{\partial z} \\ \frac{\partial N_I}{\partial r} & 0 \\ \frac{N_I}{r} & 0 \\ \frac{\partial N_I}{\partial z} & \frac{\partial N_I}{\partial r} \end{bmatrix} \quad \underline{\sigma} = \begin{bmatrix} \sigma_z \\ \sigma_r \\ \sigma_\theta \\ \tau_{rz} \end{bmatrix} \quad (91)$$

where σ_θ is the hoop (circumferential) stress.

By expanding (86) one gets:

$$\begin{aligned} F_{Ir} &= \int_{V^e} \left(\frac{\partial N_I}{\partial r} \sigma_r + \frac{N_I}{r} \sigma_\theta + \frac{\partial N_I}{\partial z} \tau_{rz} \right) dV \\ F_{Iz} &= \int_{V^e} \left(\frac{\partial N_I}{\partial z} \sigma_z + \frac{\partial N_I}{\partial r} \tau_{rz} \right) dV \end{aligned} \quad (92)$$

The spatial integrals in (90) or (92) are computed numerically by Gauss integration rule, and become, in the plane case:

$$\begin{aligned} F_{Ix} &= \sum_{i=1}^{n_G} \sum_j^{n_G} \left(\frac{\partial N_I}{\partial x} \sigma_x + \frac{\partial N_I}{\partial y} \tau_{xy} \right)_{ij} W_i W_j \det \underline{J} \\ F_{Iy} &= \sum_{i=1}^{n_G} \sum_j^{n_G} \left(\frac{\partial N_I}{\partial y} \sigma_y + \frac{\partial N_I}{\partial x} \tau_{xy} \right)_{ij} W_i W_j \det \underline{J} \end{aligned} \quad (93)$$

where i, j from 1 to 2 are the Gauss points along each spatial direction, $()_{ij}$ means that the expression must be evaluated at the corresponding Gauss point, W_i, W_j are the weights and $\det \underline{J}$ is the determinant of the Jacobian matrix for the transformation between local (ξ, η) and global (x, y) coordinates, given by (43).

In the axisymmetric case the expression is:

$$\begin{aligned} F_{Ix} &= \sum_{i=1}^{n_G} \sum_j^{n_G} \left(\frac{\partial N_I}{\partial r} \sigma_r + \frac{N_I}{r} \sigma_\theta + \frac{\partial N_I}{\partial z} \tau_{rz} \right)_{ij} W_i W_j r_{ij} \det \underline{J} \\ F_{Iy} &= \sum_{i=1}^{n_G} \sum_j^{n_G} \left(\frac{\partial N_I}{\partial z} \sigma_z + \frac{\partial N_I}{\partial r} \tau_{rz} \right)_{ij} W_i W_j r_{ij} \det \underline{J} \end{aligned} \quad (94)$$

where r_{ij} is the mean radius, i.e. the radius at the corresponding Gauss point.

2.10 External pressure forces

An external pressure can be applied to the side of a Q4 element by attaching to it a “boundary condition” element of type CL22 with a material of type IMPE PIMP. The CL22 element has 2 nodes and is shown in Figure 3.

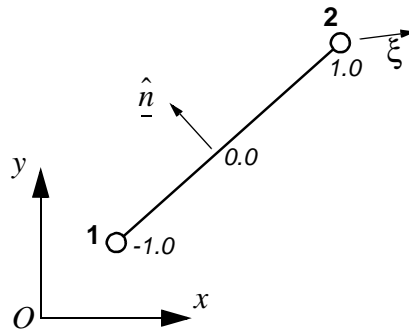


Figure 3 - The 2-node boundary condition element CL22

The element geometry is interpolated as:

$$\underline{x} = \sum_{I=1}^2 L_I \underline{x}_I \quad (95)$$

where L_I are the 1-D linear shape functions given by (1), function only of the curvilinear abscissa ξ .

If a normal pressure p acts along the side s (i.e. along the CL22 element), then the nodal pressure forces can be computed as:

$$F_{iI}^p = \int_s L_I p \hat{n}_i ds, \quad (96)$$

where \hat{n}_i are the global components of the *unit* normal vector $\underline{\hat{n}}$ to the side. If the pressure p is uniform over the element then:

$$F_{iI}^p = p \int_s L_I \hat{n}_i ds. \quad (97)$$

The tangent vector \underline{t} to the curve is given by:

$$\underline{t} = \frac{dx}{d\xi}, \quad (98)$$

or, in components:

$$t_x = \frac{dx}{d\xi} = \sum_{I=1}^2 \frac{dL_I}{d\xi} x_I \quad t_y = \frac{dy}{d\xi} = \sum_{I=1}^2 \frac{dL_I}{d\xi} y_I. \quad (99)$$

The normal vector \underline{n} is obtained by rotating \underline{t} anticlockwise by 90° , i.e.:

$$n_x = -t_y = -\frac{dy}{d\xi} \quad n_y = t_x = \frac{dx}{d\xi}. \quad (100)$$

The integral in (97) is normalized by noting that:

$$ds = \|\underline{t}\| d\xi. \quad (101)$$

From (100) one sees that:

$$\|\underline{n}\| = \|\underline{t}\| = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}, \quad (102)$$

so that the components of the *unit* normal vector $\underline{\hat{n}}$ can be written as:

$$\hat{n}_x = \frac{n_x}{\|\underline{n}\|} = \frac{n_x}{\|\underline{t}\|} \quad \hat{n}_y = \frac{n_y}{\|\underline{n}\|} = \frac{n_y}{\|\underline{t}\|}. \quad (103)$$

From (103) and (101):

$$\hat{n}_x ds = \frac{n_x}{\|\underline{t}\|} \|\underline{t}\| d\xi = n_x d\xi \quad \hat{n}_y ds = \frac{n_y}{\|\underline{t}\|} \|\underline{t}\| d\xi = n_y d\xi. \quad (104)$$

The pressure forces (97) become therefore:

$$F_{iI}^p = p \int_s L_I \hat{n}_i ds = p \int_{-1}^1 L_I n_i d\xi. \quad (105)$$

If one evaluates the integral in (105) numerically by Gauss integration by using G Gauss points along the CL22 element (with either $G = 1$ or $G = 2$) then:

$$F_{il}^p = p \int_{-1}^1 L_I n_i d\xi = p \sum_{g=1}^G W_g (L_I n_i)_g, \quad (106)$$

where W_g are the integration weights and $()_g$ indicates that the quantity has to be evaluated at the g -th Gauss point. By using (100) one obtains finally for the pressure force components:

$$\begin{aligned} F_{xl}^p &= -p \sum_{g=1}^G W_g \left(L_I \frac{dy}{d\xi} \right)_g \\ F_{yl}^p &= p \sum_{g=1}^G W_g \left(L_I \frac{dx}{d\xi} \right)_g \end{aligned} \quad (107)$$

2.11 Stability step

The element's stability step is computed as:

$$\Delta t^{\text{crit}} = L/c, \quad (108)$$

where L is the element's characteristic length and c the sound speed in the material. To compute L the following procedure is adopted, see Figure 4:

- Compute the four side lengths s_1 , s_2 , s_3 and s_4 to account for the element's "stretching". For example:

$$s_1 = \|\vec{12}\|. \quad (109)$$

- Build up the four mid-side points M_1 , M_2 , M_3 and M_4 . For example:

$$\vec{OM}_1 = \frac{1}{2}(\vec{O1} + \vec{O2}) \quad (110)$$

- Build up the two vectors \underline{a} and \underline{b} (which are only approximately perpendicular to each other):

$$\underline{a} = \vec{M_4M_2} \quad \underline{b} = \vec{M_1M_3} \quad (111)$$

- Build up two *unit* vectors \hat{n}_a , \hat{n}_b perpendicular to \underline{a} and \underline{b} , respectively:

$$\begin{aligned} \hat{n}_{ax} &= -a_y / \|\underline{a}\| & \hat{n}_{bx} &= -b_y / \|\underline{b}\| \\ \hat{n}_{ay} &= a_x / \|\underline{a}\| & \hat{n}_{by} &= b_x / \|\underline{b}\| \end{aligned} \quad (112)$$

- Compute the following two "heights", to take into account element's "shearing":

$$h_a = \underline{a} \cdot \hat{n}_b \quad h_b = \underline{b} \cdot \hat{n}_a. \quad (113)$$

- The characteristic length is the minimum of the computed lengths:

$$L = \min(s_1, s_2, s_3, s_4, h_a, h_b). \quad (114)$$

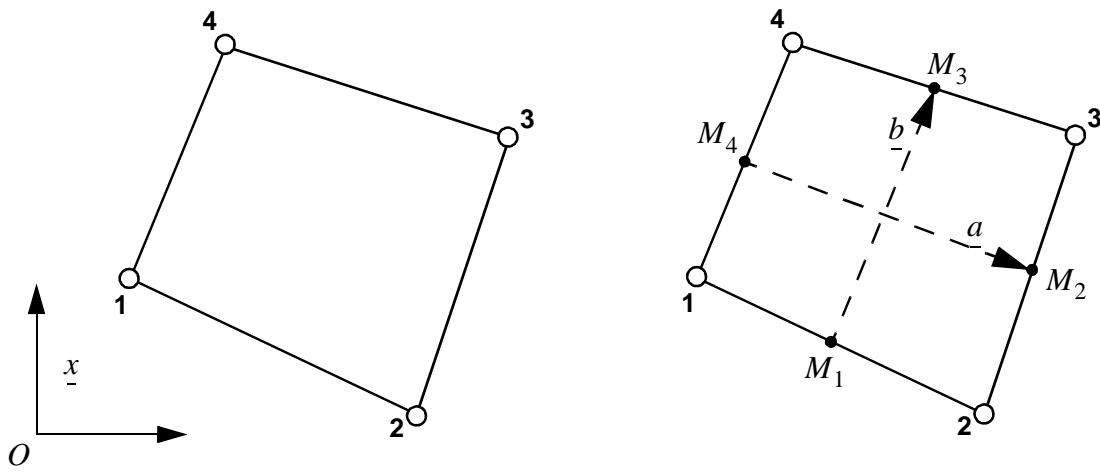


Figure 4 - Computing the characteristic length for the 4-node element

3. The 8-node linear element in 3D

The 8-node Lagrange linear element (C8) for 3D analysis in fast transient dynamics is shown in Figure 5 in the so-called parent element configuration, i.e. for normalized coordinates $-1 \leq \xi \leq 1$, $-1 \leq \eta \leq 1$, $-1 \leq \zeta \leq 1$. The local element numbering is as follows: first the 4 “lower” nodes, so that the oriented normal to the so-defined lower face points inside the element, then the 4 “upper” corresponding nodes.

3.1 Shape functions and their derivatives

The nodal shape functions $N_I(\xi, \eta, \zeta)$, $I = 1, \dots, 8$ are obtained by blending of the 1-D linear shape functions $L_K(\delta)$, $K = 1, 2$ given by (1) and shown in Figure 1:

$$\begin{aligned} L_1(\delta) &= \frac{1}{2}(1 - \delta) \\ L_2(\delta) &= \frac{1}{2}(1 + \delta) \end{aligned} \quad (115)$$

and result in the following expressions:

$$\begin{aligned} N_1(\xi, \eta, \zeta) &= L_1(\xi)L_1(\eta)L_1(\zeta) & N_5(\xi, \eta, \zeta) &= L_1(\xi)L_1(\eta)L_2(\zeta) \\ N_2(\xi, \eta, \zeta) &= L_2(\xi)L_1(\eta)L_1(\zeta) & N_6(\xi, \eta, \zeta) &= L_2(\xi)L_1(\eta)L_2(\zeta) \\ N_3(\xi, \eta, \zeta) &= L_2(\xi)L_2(\eta)L_1(\zeta) & N_7(\xi, \eta, \zeta) &= L_2(\xi)L_2(\eta)L_2(\zeta) \\ N_4(\xi, \eta, \zeta) &= L_1(\xi)L_2(\eta)L_1(\zeta) & N_8(\xi, \eta, \zeta) &= L_1(\xi)L_2(\eta)L_2(\zeta) \end{aligned} \quad (116)$$

The derivative of the I -th shape function N_I ($I = 1, \dots, 8$) with respect to the K -th normalized coordinate ξ_K ($K = 1, \dots, 3$) assumes the form:

$$\frac{\partial N_I}{\partial \xi_K} = L_\alpha(\xi_a)L_\beta(\xi_b) \frac{\partial L_\gamma(\xi_K)}{\partial \xi_K}, \quad (117)$$

where the coefficients $\alpha, \beta, \gamma, a, b$ (all varying between 1 and 2) can be deduced from (116). For example:

$$\frac{\partial N_6}{\partial \xi_2} = \frac{\partial N_6}{\partial \eta} = \frac{\partial}{\partial \eta} [L_2(\xi)L_1(\eta)L_2(\zeta)] = L_2(\xi)L_2(\zeta) \frac{\partial L_1(\eta)}{\partial \eta}.$$

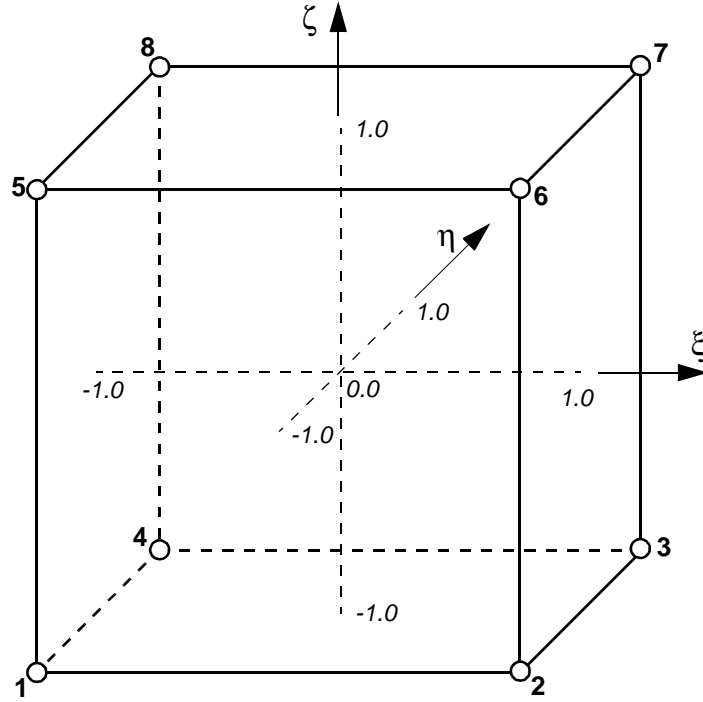


Figure 5 - The 8-node Lagrange linear element in 3D (C8)

Any of the element's kinematic variables (coordinates x_i , displacements d_i , velocities v_i , accelerations a_i , $i = 1, \dots, 3$), say f_p , at a generic point $p = (\xi, \eta, \zeta)$ in the parent element are obtained by interpolation from the corresponding nodal values using the shape functions (116):

$$f_p = f(\xi, \eta, \zeta) = \sum_{I=1}^8 N_I(\xi, \eta, \zeta) f_I. \quad (118)$$

Since the same interpolation (118) is used for the geometry ($f \equiv x$) as well as for the kinematic variables, the formulation is called isoparametric.

3.2 Lumped mass matrix

The consistent mass matrix for the generic element e with n nodes reads:

$$\underline{M}^c = [M_{IJ}^c] = \left[\int_{V^e} N_I \rho N_J dV \right] \quad I, J = 1, \dots, n. \quad (119)$$

The (diagonal) lumped mass matrix:

$$\underline{M}^l = [M_{IJ}^l] \quad (120)$$

is obtained by row summation:

$$\begin{aligned} M_I &= M_{II}^I = \sum_{J=1}^n M_{IJ}^I \\ M_{IJ}^I &= 0 \quad \text{for } I \neq J \end{aligned} \quad (121)$$

The generic diagonal term becomes therefore:

$$M_I = \sum_{J=1}^n M_{IJ}^I = \sum_{J=1}^n \left(\int_{V^e} N_I \rho N_J dV \right) = \int_{V^e} \left(\sum_{J=1}^n N_I \rho N_J \right) dV = \int_{V^e} N_I \rho \left(\sum_{J=1}^n N_J \right) dV \quad (122)$$

But the sum of the shape functions is 1 in every point by definition:

$$\sum_{J=1}^n N_J = 1, \quad (123)$$

therefore (122) gives:

$$M_I = \int_{V^e} N_I \rho dV \quad (124)$$

and, if the density ρ is element-wise uniform:

$$M_I = \rho \int_{V^e} N_I dV. \quad (125)$$

The volume integral appearing in expressions (124) or (125) is then expressed in terms of the normalized coordinates. Then (125) becomes:

$$M_I = \rho \int_{V^e} N_I(x, y, z) dV = \rho \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 N_I(\xi, \eta, \zeta) \det \underline{J} d\xi d\eta d\zeta, \quad (126)$$

where \underline{J} is the Jacobian matrix of the coordinates transformation between (ξ, η, ζ) and (x, y, z) :

$$\underline{J} = [J_{ij}] = \begin{bmatrix} \frac{\partial x_j}{\partial \xi_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}, \quad (127)$$

so that, for example:

$$\det \underline{J} = A_{11} \frac{\partial x}{\partial \xi} + A_{12} \frac{\partial y}{\partial \xi} + A_{13} \frac{\partial z}{\partial \xi}, \quad (128)$$

and A_{ij} is the (2×2) minor determinant of \underline{J} , obtained by eliminating the i -th row and the j -th column:

$$\begin{aligned}
A_{11} &= \frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \eta} \frac{\partial y}{\partial \zeta} & A_{21} &= \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \zeta} - \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \zeta} & A_{31} &= \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \eta} \\
A_{12} &= \frac{\partial z}{\partial \eta} \frac{\partial x}{\partial \zeta} - \frac{\partial x}{\partial \eta} \frac{\partial z}{\partial \zeta} & A_{22} &= \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \zeta} & A_{32} &= \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \eta} \\
A_{13} &= \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \zeta} - \frac{\partial y}{\partial \eta} \frac{\partial x}{\partial \zeta} & A_{23} &= \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \zeta} - \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \zeta} & A_{33} &= \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta}
\end{aligned} \tag{129}$$

The derivatives of the geometry with respect to the normalized coordinates $\frac{\partial x_i}{\partial \xi_j}$ appearing in (128) are obtained by means of the interpolation (118):

$$\underline{x} = \sum_{I=1}^8 N_I(\xi, \eta, \zeta) \underline{x}_I, \tag{130}$$

so that:

$$\frac{\partial x_i}{\partial \xi_j} = \sum_{I=1}^8 \frac{\partial N_I}{\partial \xi_j} x_{iI}, \tag{131}$$

where $\frac{\partial N_I}{\partial \xi_j}$ are given by (117).

3.3 Spatial integration

Space integrals such as for example the one in (126) are computed numerically according to:

$$I = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 G(\xi, \eta, \zeta) d\xi d\eta d\zeta = \sum_{k=1}^m W_k G(\xi_k, \eta_k, \zeta_k), \tag{132}$$

where (ξ_k, η_k, ζ_k) are appropriately chosen locations in the parent domain and W_k are the corresponding integration weights. In general the greater the number of sampling points m , the greater is the precision. For the present element, Gauss integration is adopted: the integration points are obtained by combining 1-D Gauss integration rules along each of the global spatial directions.

Either $1 \times 1 \times 1$ (i.e., single-point) or $2 \times 2 \times 2$ Gauss integration can be chosen. The first choice corresponds to element C81L and the second one to element C82L. The positions and numbering of the corresponding Gauss points is shown in Figure 6. The Gauss points are numbered first along the z (or ζ) direction, then along the y (or η) direction, and finally along the x (or ξ) direction. Nodes

are not shown in Figure 6 because it represents cross-sections of the element with planes normal to the ζ axis.

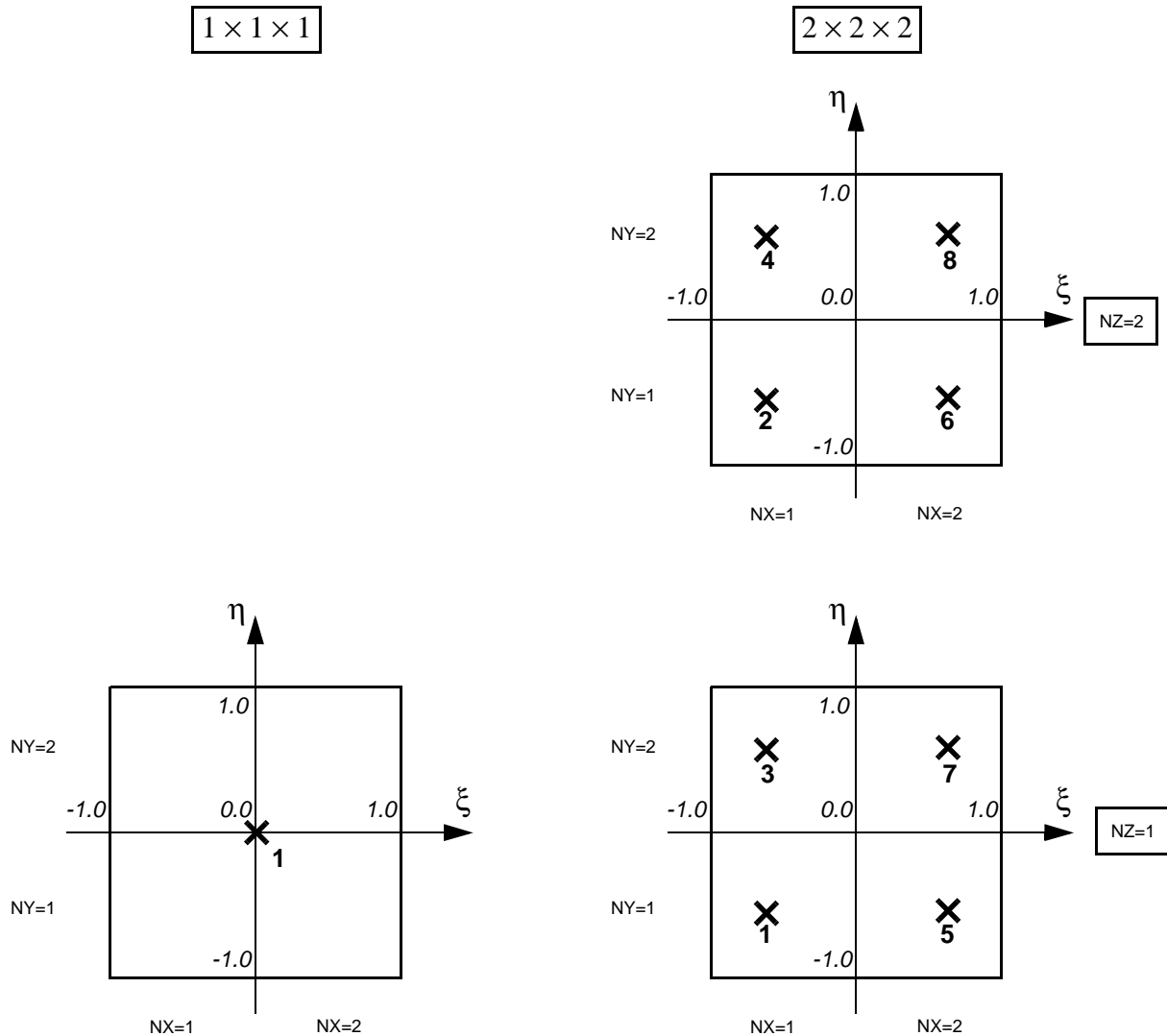


Figure 6 - Gauss point positions in the C8 element

3.4 Treatment of large deformation

The treatment of large deformation follows the lines presented in the previous Section for the 2D case, i.e. for the 4-node linear element. The key point is the separation of deformation from rigid-body rotation:

- We assume the *instantaneous rate of deformation* to be given by the stretching tensor \underline{D} , as given by (25), i.e. the symmetric part of the spatial velocity gradient \underline{L} expressed by (22), (34) or (61).
- Similarly, the *instantaneous rate of (rigid-body) rotation* is given by the spin tensor \underline{W} as given by (26), i.e. the anti-symmetric part of spatial velocity gradient \underline{L} .

3.5 Calculation of the spatial velocity gradient

From (22) one can write, in components:

$$L_{ij} = \frac{\partial \dot{x}_j}{\partial x_i} = \frac{\partial v_j}{\partial x_i}, \quad (133)$$

where $\dot{\underline{x}} = \underline{v}$ denotes the particle velocity.

For an iso-parametric element formulation:

$$v_j(\underline{x}) = \sum_{I=1}^n N_I v_{jI}, \quad (134)$$

where N_I are the element's shape functions, n is the number of nodes of the element and v_{jI} is the j -th component of the velocity at node I . Thus, from (133)-(134):

$$L_{ij}(\underline{x}) = \frac{\partial v_j(\underline{x})}{\partial x_i} = \sum_{I=1}^n \frac{\partial N_I}{\partial x_i} v_{jI}. \quad (135)$$

Therefore, we need to compute spatial derivatives of the shape functions N_I with respect to the global coordinates x_i . These can be expressed as follows:

$$\left\{ \frac{\partial N_I}{\partial x_i} \right\} = |J|^{-1} \left\{ \frac{\partial N_I(\underline{\xi})}{\partial \xi_i} \right\}, \quad (136)$$

where $\{ \}$ denotes a column vector ($j = 1, \dots, 3$), ξ_i are normalized coordinates (parent element) ranging from -1 to 1 , and $|J|$ is the determinant of the Jacobian matrix \underline{J} of the coordinates transformation between $\underline{\xi}$ and \underline{x} .

The Jacobian matrix is given by:

$$\underline{J} = [J_{ij}] = \begin{bmatrix} \frac{\partial x_j}{\partial \xi_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}. \quad (137)$$

In three spatial dimensions, (136) becomes:

$$\begin{aligned}\frac{\partial N_I}{\partial x} &= \frac{1}{\det \underline{J}} \left(A_{11} \frac{\partial N_I}{\partial \xi} + A_{21} \frac{\partial N_I}{\partial \eta} + A_{31} \frac{\partial N_I}{\partial \zeta} \right) \\ \frac{\partial N_I}{\partial y} &= \frac{1}{\det \underline{J}} \left(A_{12} \frac{\partial N_I}{\partial \xi} + A_{22} \frac{\partial N_I}{\partial \eta} + A_{32} \frac{\partial N_I}{\partial \zeta} \right), \\ \frac{\partial N_I}{\partial z} &= \frac{1}{\det \underline{J}} \left(A_{13} \frac{\partial N_I}{\partial \xi} + A_{23} \frac{\partial N_I}{\partial \eta} + A_{33} \frac{\partial N_I}{\partial \zeta} \right)\end{aligned}\quad (138)$$

where the determinant is given, for example, by:

$$\det \underline{J} = A_{11} \frac{\partial x}{\partial \xi} + A_{12} \frac{\partial y}{\partial \xi} + A_{13} \frac{\partial z}{\partial \xi} \quad (139)$$

and A_{ij} is the (2×2) minor determinant of \underline{J} , obtained by eliminating the i -th row and the j -th column:

$$\begin{aligned}A_{11} &= \frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \eta} \frac{\partial y}{\partial \zeta} & A_{21} &= \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \zeta} - \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \zeta} & A_{31} &= \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \eta} \\ A_{12} &= \frac{\partial z}{\partial \eta} \frac{\partial x}{\partial \zeta} - \frac{\partial x}{\partial \eta} \frac{\partial z}{\partial \zeta} & A_{22} &= \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \zeta} & A_{32} &= \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \eta} \\ A_{13} &= \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \zeta} - \frac{\partial y}{\partial \eta} \frac{\partial x}{\partial \zeta} & A_{23} &= \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \zeta} - \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \zeta} & A_{33} &= \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta}\end{aligned}\quad (140)$$

In the above expressions, terms of the form $\frac{\partial N_I}{\partial \xi_i}$ are obtained by direct differentiation of the shape functions, while those of the type $\frac{\partial x_j}{\partial \xi_i}$ are computed based on (134):

$$\frac{\partial x_j}{\partial \xi_i} = \sum_{I=1}^n \frac{\partial N_I}{\partial \xi_i} x_{jI}. \quad (141)$$

3.6 Calculation of the strain increments over the step

The strain increments over a time step Δt are given by:

$$\Delta \underline{\varepsilon} = \underline{D} \Delta t, \quad (142)$$

where \underline{D} is the stretching tensor given by (25):

$$\underline{D} = \frac{1}{2} (\underline{L} + \underline{L}^T) \quad (143)$$

and \underline{L} is the spatial velocity gradient given by (22) or (34):

$$\underline{L} = \frac{\partial \dot{\underline{x}}}{\partial \underline{x}} = \frac{\partial \underline{v}}{\partial \underline{x}} = [L_{ij}] = \left[\frac{\partial \dot{x}_j}{\partial x_i} \right] = \left[\frac{\partial v_j}{\partial x_i} \right]. \quad (144)$$

By denoting $\Delta \underline{u} \equiv [\Delta u \ \Delta v \ \Delta w]^T$ the displacement increment over the time step:

$$\Delta \underline{u} = \underline{v} \Delta t = \underline{v}^{n+1/2} \Delta t, \quad (145)$$

one obtains from (144) and (142):

$$\underline{L} \Delta t = \frac{\partial(\Delta \underline{u})}{\partial \underline{x}} = \left[\frac{\partial(\Delta u_j)}{\partial x_i} \right] = \begin{bmatrix} \frac{\partial(\Delta u)}{\partial x} & \frac{\partial(\Delta v)}{\partial x} & \frac{\partial(\Delta w)}{\partial x} \\ \frac{\partial(\Delta u)}{\partial y} & \frac{\partial(\Delta v)}{\partial y} & \frac{\partial(\Delta w)}{\partial y} \\ \frac{\partial(\Delta u)}{\partial z} & \frac{\partial(\Delta v)}{\partial z} & \frac{\partial(\Delta w)}{\partial z} \end{bmatrix}. \quad (146)$$

Then the strain increments result in:

$$\begin{aligned} \Delta \varepsilon_{11} &= D_{11} \Delta t = L_{11} \Delta t = \frac{\partial(\Delta u)}{\partial x} \\ \Delta \varepsilon_{12} &= D_{12} \Delta t = \frac{1}{2}(L_{12} + L_{21}) \Delta t = \frac{1}{2} \left[\frac{\partial(\Delta v)}{\partial x} + \frac{\partial(\Delta u)}{\partial y} \right] \\ \Delta \varepsilon_{13} &= D_{13} \Delta t = \frac{1}{2}(L_{13} + L_{31}) \Delta t = \frac{1}{2} \left[\frac{\partial(\Delta w)}{\partial x} + \frac{\partial(\Delta u)}{\partial z} \right] \\ \Delta \varepsilon_{21} &= D_{21} \Delta t = \frac{1}{2}(L_{21} + L_{12}) \Delta t = \frac{1}{2} \left[\frac{\partial(\Delta u)}{\partial y} + \frac{\partial(\Delta v)}{\partial x} \right] = \Delta \varepsilon_{12} \\ \Delta \varepsilon_{22} &= D_{22} \Delta t = L_{22} \Delta t = \frac{\partial(\Delta v)}{\partial y} \\ \Delta \varepsilon_{23} &= D_{23} \Delta t = \frac{1}{2}(L_{23} + L_{32}) \Delta t = \frac{1}{2} \left[\frac{\partial(\Delta w)}{\partial y} + \frac{\partial(\Delta v)}{\partial z} \right] \\ \Delta \varepsilon_{31} &= D_{31} \Delta t = \frac{1}{2}(L_{31} + L_{13}) \Delta t = \frac{1}{2} \left[\frac{\partial(\Delta u)}{\partial z} + \frac{\partial(\Delta w)}{\partial x} \right] = \Delta \varepsilon_{13} \\ \Delta \varepsilon_{32} &= D_{32} \Delta t = \frac{1}{2}(L_{32} + L_{23}) \Delta t = \frac{1}{2} \left[\frac{\partial(\Delta v)}{\partial z} + \frac{\partial(\Delta w)}{\partial y} \right] = \Delta \varepsilon_{23} \\ \Delta \varepsilon_{33} &= D_{33} \Delta t = L_{33} \Delta t = \frac{\partial(\Delta w)}{\partial z} \end{aligned} \quad (147)$$

By recalling that, as concerns shear components:

$$\gamma_{ij} = 2\varepsilon_{ij} \quad \text{for } i \neq j \quad (148)$$

and by using the iso-parametric expression of the displacement increments one has:

$$\begin{aligned}
\Delta \varepsilon_{11} &= \frac{\partial(\Delta u)}{\partial x} = \sum_{I=1}^n \frac{\partial N_I}{\partial x} \Delta u_I \\
\Delta \gamma_{12} &= \Delta \gamma_{21} = \frac{\partial(\Delta v)}{\partial x} + \frac{\partial(\Delta u)}{\partial y} = \sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta v_I + \frac{\partial N_I}{\partial y} \Delta u_I \right) \\
\Delta \gamma_{13} &= \Delta \gamma_{31} = \frac{\partial(\Delta w)}{\partial x} + \frac{\partial(\Delta u)}{\partial z} = \sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta w_I + \frac{\partial N_I}{\partial z} \Delta u_I \right) \\
\Delta \varepsilon_{22} &= \frac{\partial(\Delta v)}{\partial y} = \sum_{I=1}^n \frac{\partial N_I}{\partial y} \Delta v_I \\
\Delta \gamma_{23} &= \Delta \gamma_{32} = \frac{\partial(\Delta w)}{\partial y} + \frac{\partial(\Delta v)}{\partial z} = \sum_{I=1}^n \left(\frac{\partial N_I}{\partial y} \Delta w_I + \frac{\partial N_I}{\partial z} \Delta v_I \right) \\
\Delta \varepsilon_{33} &= \frac{\partial(\Delta w)}{\partial z} = \sum_{I=1}^n \frac{\partial N_I}{\partial z} \Delta w_I
\end{aligned} \tag{149}$$

3.7 Calculation of the rate of rotation and of the rotation increment over the step

The spin tensor \underline{W} expressed by (26) is a measure of the instantaneous rate of (rigid-body) rotation and results in:

$$\underline{W} = \frac{1}{2}(\underline{L} - \underline{L}^T) = \frac{1}{2} \begin{bmatrix} 0 & (L_{12} - L_{21}) & (L_{13} - L_{31}) \\ (L_{21} - L_{12}) & 0 & (L_{23} - L_{32}) \\ (L_{31} - L_{13}) & (L_{32} - L_{23}) & 0 \end{bmatrix}. \tag{150}$$

By using the expression (36) of the components of the spatial velocity gradient \underline{L} one gets:

$$\begin{aligned}
W_{12} &= \frac{1}{2}(L_{12} - L_{21}) = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = -W_{21} \\
W_{13} &= \frac{1}{2}(L_{13} - L_{31}) = \frac{1}{2} \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) = -W_{31} \\
W_{23} &= \frac{1}{2}(L_{23} - L_{32}) = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = -W_{32}
\end{aligned} \tag{151}$$

and by introducing the expansion of the velocities (134):

$$\begin{aligned}
W_{12} &= -W_{21} = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{2} \left[\sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} v_I - \frac{\partial N_I}{\partial y} u_I \right) \right] \\
W_{13} &= -W_{31} = \frac{1}{2} \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) = \frac{1}{2} \left[\sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} w_I - \frac{\partial N_I}{\partial z} u_I \right) \right] \\
W_{23} &= -W_{32} = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = \frac{1}{2} \left[\sum_{I=1}^n \left(\frac{\partial N_I}{\partial y} w_I - \frac{\partial N_I}{\partial z} v_I \right) \right]
\end{aligned} \tag{152}$$

The three (significant) components of the spin tensor W_{12} , W_{13} and W_{23} are a measure of the instantaneous rates of (rigid-body) rotation $\dot{\theta}_z$, $\dot{\theta}_y$, $\dot{\theta}_x$ around the (positive) global axes directions z , y and x , respectively:

$$\begin{aligned}\dot{\theta}_z &= W_{12} \\ \dot{\theta}_y &= W_{13} \\ \dot{\theta}_x &= W_{23}\end{aligned}\tag{153}$$

Thus the body is instantaneously rotating (rigidly) at a rate $\dot{\theta}$ given by:

$$\dot{\theta} = \sqrt{(\dot{\theta}_z)^2 + (\dot{\theta}_y)^2 + (\dot{\theta}_x)^2} = \sqrt{W_{12}^2 + W_{13}^2 + W_{23}^2}\tag{154}$$

around the axis defined by the unit vector \hat{r} of components:

$$\begin{aligned}\hat{r}_x &= \dot{\theta}_x / \dot{\theta} = W_{23} / \sqrt{W_{12}^2 + W_{13}^2 + W_{23}^2} \equiv \xi \\ \hat{r}_y &= \dot{\theta}_y / \dot{\theta} = W_{13} / \sqrt{W_{12}^2 + W_{13}^2 + W_{23}^2} \equiv \eta \\ \hat{r}_z &= \dot{\theta}_z / \dot{\theta} = W_{12} / \sqrt{W_{12}^2 + W_{13}^2 + W_{23}^2} \equiv \zeta\end{aligned}\tag{155}$$

By passing from the instantaneous rate of rotation to the time-discrete rotation increment θ (around the \hat{r} axis), one can write in analogy with the 2D case (48):

$$\Delta t \dot{\theta} = \Delta t \sqrt{W_{12}^2 + W_{13}^2 + W_{23}^2} = 2 \tan \frac{\theta}{2},\tag{156}$$

and from this equation one can obtain the half-rotation angle $\theta/2$.

Then the (half-)rotation matrix \underline{T}_2 is given, in analogy with (51), by the following expression [7]:

$$\underline{T}_2 = \begin{bmatrix} (\xi^2 M + C) & (\xi \eta M + \zeta S) & (\xi \zeta M - \eta S) \\ (\xi \eta M - \zeta S) & (\eta^2 M + C) & (\eta \zeta M + \xi S) \\ (\xi \zeta M + \eta S) & (\eta \zeta M - \xi S) & (\zeta^2 M + C) \end{bmatrix}\tag{157}$$

where ξ , η , ζ are the global components of \hat{r} given by (155) and:

$$C = \cos \frac{\theta}{2} \quad S = \sin \frac{\theta}{2} \quad M = 1 - C = 1 - \cos \frac{\theta}{2}.\tag{158}$$

3.8 Stress increment procedure

The stress increment procedure at a generic Gauss point can now be summarized as follows. Let $\underline{\sigma}^n$ denote the old stress, i.e. the stress at the beginning of the time step $n \rightarrow n + 1$:

$$\underline{\sigma}^n = \begin{bmatrix} \sigma_x^n & \tau_{xy}^n & \tau_{xz}^n \\ \tau_{xy}^n & \sigma_y^n & \tau_{yz}^n \\ \tau_{xz}^n & \tau_{yz}^n & \sigma_z^n \end{bmatrix}. \quad (159)$$

Compute the partial derivatives of (x, y, z) with respect to (ξ, η, ζ) at the current Gauss point:

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \xi} x_I & \frac{\partial x}{\partial \eta} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \eta} x_I & \frac{\partial x}{\partial \zeta} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \zeta} x_I \\ \frac{\partial y}{\partial \xi} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \xi} y_I & \frac{\partial y}{\partial \eta} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \eta} y_I & \frac{\partial y}{\partial \zeta} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \zeta} y_I, \\ \frac{\partial z}{\partial \xi} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \xi} z_I & \frac{\partial z}{\partial \eta} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \eta} z_I & \frac{\partial z}{\partial \zeta} &= \sum_{I=1}^n \frac{\partial N_I}{\partial \zeta} z_I \end{aligned} \quad (160)$$

on the configuration $n + 1/2$, i.e. by using as nodal coordinates:

$$\underline{x}_I = \underline{x}_I^{n+1/2} = \underline{x}_I^{n+1} - \frac{1}{2} \Delta u_I. \quad (161)$$

Evaluate the minors of the Jacobian at $n + 1/2$ with (140):

$$\begin{aligned} A_{11} &= \frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \eta} \frac{\partial y}{\partial \zeta} & A_{21} &= \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \zeta} - \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \zeta} & A_{31} &= \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \eta} \\ A_{12} &= \frac{\partial z}{\partial \eta} \frac{\partial x}{\partial \zeta} - \frac{\partial x}{\partial \eta} \frac{\partial z}{\partial \zeta} & A_{22} &= \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \zeta} & A_{32} &= \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \eta} \\ A_{13} &= \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \zeta} - \frac{\partial y}{\partial \eta} \frac{\partial x}{\partial \zeta} & A_{23} &= \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \zeta} - \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \zeta} & A_{33} &= \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta} \end{aligned} \quad (162)$$

and the determinant of the Jacobian at $n + 1/2$ with (139):

$$\det \underline{J} = A_{11} \frac{\partial x}{\partial \xi} + A_{12} \frac{\partial y}{\partial \xi} + A_{13} \frac{\partial z}{\partial \xi}. \quad (163)$$

Compute the partial derivatives of the shape functions at $n + 1/2$ with (138):

$$\begin{aligned} \frac{\partial N_I}{\partial x} &= \frac{1}{\det \underline{J}} \left(A_{11} \frac{\partial N_I}{\partial \xi} + A_{21} \frac{\partial N_I}{\partial \eta} + A_{31} \frac{\partial N_I}{\partial \zeta} \right) \\ \frac{\partial N_I}{\partial y} &= \frac{1}{\det \underline{J}} \left(A_{12} \frac{\partial N_I}{\partial \xi} + A_{22} \frac{\partial N_I}{\partial \eta} + A_{32} \frac{\partial N_I}{\partial \zeta} \right). \\ \frac{\partial N_I}{\partial z} &= \frac{1}{\det \underline{J}} \left(A_{13} \frac{\partial N_I}{\partial \xi} + A_{23} \frac{\partial N_I}{\partial \eta} + A_{33} \frac{\partial N_I}{\partial \zeta} \right) \end{aligned} \quad (164)$$

Compute the strain increments over the step by (149):

$$\begin{aligned}
\Delta \varepsilon_{11} &= \frac{\partial(\Delta u)}{\partial x} = \sum_{I=1}^n \frac{\partial N_I}{\partial x} \Delta u_I \\
\Delta \gamma_{12} &= \Delta \gamma_{21} = \frac{\partial(\Delta v)}{\partial x} + \frac{\partial(\Delta u)}{\partial y} = \sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta v_I + \frac{\partial N_I}{\partial y} \Delta u_I \right) \\
\Delta \gamma_{13} &= \Delta \gamma_{31} = \frac{\partial(\Delta w)}{\partial x} + \frac{\partial(\Delta u)}{\partial z} = \sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta w_I + \frac{\partial N_I}{\partial z} \Delta u_I \right) \\
\Delta \varepsilon_{22} &= \frac{\partial(\Delta v)}{\partial y} = \sum_{I=1}^n \frac{\partial N_I}{\partial y} \Delta v_I \\
\Delta \gamma_{23} &= \Delta \gamma_{32} = \frac{\partial(\Delta w)}{\partial y} + \frac{\partial(\Delta v)}{\partial z} = \sum_{I=1}^n \left(\frac{\partial N_I}{\partial y} \Delta w_I + \frac{\partial N_I}{\partial z} \Delta v_I \right) \\
\Delta \varepsilon_{33} &= \frac{\partial(\Delta w)}{\partial z} = \sum_{I=1}^n \frac{\partial N_I}{\partial z} \Delta w_I
\end{aligned} \tag{165}$$

Compute the rotation increments during the step around the three global axes, i.e. the components of the spin tensor multiplied by Δt (see 152 and 153), by using the displacement increments $\Delta u, \Delta v, \Delta w$:

$$\begin{aligned}
\theta_z &= \Delta t W_{12} = \frac{1}{2} \left[\sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta v_I - \frac{\partial N_I}{\partial y} \Delta u_I \right) \right] \\
\theta_y &= \Delta t W_{13} = \frac{1}{2} \left[\sum_{I=1}^n \left(\frac{\partial N_I}{\partial x} \Delta w_I - \frac{\partial N_I}{\partial z} \Delta u_I \right) \right] \\
\theta_x &= \Delta t W_{23} = \frac{1}{2} \left[\sum_{I=1}^n \left(\frac{\partial N_I}{\partial y} \Delta w_I - \frac{\partial N_I}{\partial z} \Delta v_I \right) \right]
\end{aligned} \tag{166}$$

Compute the direction of the axis of rotation, see (155):

$$\begin{aligned}
\hat{r}_x &= \Delta t W_{23} / \sqrt{(\Delta t W_{12})^2 + (\Delta t W_{13})^2 + (\Delta t W_{23})^2} \equiv \xi \\
\hat{r}_y &= \Delta t W_{13} / \sqrt{(\Delta t W_{12})^2 + (\Delta t W_{13})^2 + (\Delta t W_{23})^2} \equiv \eta \\
\hat{r}_z &= \Delta t W_{12} / \sqrt{(\Delta t W_{12})^2 + (\Delta t W_{13})^2 + (\Delta t W_{23})^2} \equiv \zeta
\end{aligned} \tag{167}$$

Compute the rotation increment over the step by (156):

$$\tan \frac{\theta}{2} = \frac{1}{2} \sqrt{(\Delta t W_{12})^2 + (\Delta t W_{13})^2 + (\Delta t W_{23})^2}. \tag{168}$$

Compute the half-rotation matrix \underline{T}_2 by (158), (157) and the identities (52):

$$C = \cos \frac{\theta}{2} = \frac{1}{\sqrt{1 + \left(\tan \frac{\theta}{2}\right)^2}} \quad S = \sin \frac{\theta}{2} = \frac{\tan \frac{\theta}{2}}{\sqrt{1 + \left(\tan \frac{\theta}{2}\right)^2}} \quad M = 1 - C = 1 - \cos \frac{\theta}{2} \quad (169)$$

$$\underline{T}_2 = \begin{bmatrix} (\xi^2 M + C) & (\xi \eta M + \zeta S) & (\xi \zeta M - \eta S) \\ (\xi \eta M - \zeta S) & (\eta^2 M + C) & (\eta \zeta M + \xi S) \\ (\xi \zeta M + \eta S) & (\eta \zeta M - \xi S) & (\zeta^2 M + C) \end{bmatrix}. \quad (170)$$

Compute the stress rotated by $\theta/2$ around the \hat{r} axis by:

$$\underline{\sigma}_{\text{rot}} = \underline{T}_2^T \underline{\sigma} \underline{T}_2. \quad (171)$$

At this point, the material's constitutive equation is used to compute the stress increments $\Delta \underline{\sigma}$ corresponding to the strain increments $\Delta \underline{\varepsilon}$ given by (149), and the new (rotated) stresses $\underline{\sigma}_{\text{rot}}^{n+1}$. Formally, this can be expressed by (44):

$$\Delta \underline{\sigma} = \underline{C}(\underline{\sigma}) \Delta \underline{\varepsilon} \quad (172)$$

and by:

$$\underline{\sigma}_{\text{rot}}^{n+1} = \underline{\sigma}_{\text{rot}} + \Delta \underline{\sigma} = \underline{\sigma}_{\text{rot}} + \underline{C}(\underline{\sigma}) \Delta \underline{\varepsilon} \quad (173)$$

Finally, the new stress $\underline{\sigma}^{n+1}$ at $n+1$ is obtained by rotating $\underline{\sigma}_{\text{rot}}^{n+1}$ by the angle $\theta/2$ around the \hat{r} axis. By applying once more (171) one gets:

$$\underline{\sigma}^{n+1} = \underline{T}_2^T \underline{\sigma}_{\text{rot}}^{n+1} \underline{T}_2. \quad (174)$$

3.9 Calculation of internal forces

Once the stresses have been updated, the element's (e) contributions to internal forces can be computed. These read:

$$\underline{F}_e^{\text{int}} = \int_{V^e} \underline{B}^T \underline{\sigma} dV, \quad (175)$$

where V^e is the element's current volume (current configuration), \underline{B} is the matrix of shape function (N) derivatives (see below) and $\underline{\sigma}$ is Cauchy stress. At a node I of the element e , this can be particularized as follows (we drop the superscript "int" for simplicity):

$$\underline{F}_I = \int_{V^e} \underline{B}_I^T \underline{\sigma} dV, \quad (176)$$

where \underline{B}_I is the sub-matrix of \underline{B} relative to node I . Note that in this expression $\underline{\sigma}$ represents a column vector containing (in a certain order) all the significant components of the stress tensor.

In 3D it is:

$$\underline{B}_I = \begin{bmatrix} \frac{\partial N_I}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_I}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_I}{\partial z} \\ \frac{\partial N_I}{\partial y} & \frac{\partial N_I}{\partial x} & 0 \\ 0 & \frac{\partial N_I}{\partial z} & \frac{\partial N_I}{\partial y} \\ \frac{\partial N_I}{\partial z} & 0 & \frac{\partial N_I}{\partial x} \end{bmatrix} \quad \underline{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}. \quad (177)$$

Recall that the expression of \underline{B}_I can be deduced from the strain-displacement relations:

$$\underline{\varepsilon} = \underline{B}\underline{u} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_y}{\partial y} \\ \frac{\partial u_x}{\partial z} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\ \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \end{bmatrix}, \quad (178)$$

with (for example):

$$\frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x} \left(\sum_{I=1}^n N_I u_{xI} \right) = \sum_{I=1}^n \frac{\partial N_I}{\partial x} u_{xI}. \quad (179)$$

By expanding (176) one gets:

$$\begin{aligned}
F_{Ix} &= \int_{V^e} \left(\frac{\partial N_I}{\partial x} \sigma_x + \frac{\partial N_I}{\partial y} \tau_{xy} + \frac{\partial N_I}{\partial z} \tau_{zx} \right) dV \\
F_{Iy} &= \int_{V^e} \left(\frac{\partial N_I}{\partial y} \sigma_y + \frac{\partial N_I}{\partial x} \tau_{xy} + \frac{\partial N_I}{\partial z} \tau_{yz} \right) dV. \\
F_{Iz} &= \int_{V^e} \left(\frac{\partial N_I}{\partial z} \sigma_z + \frac{\partial N_I}{\partial y} \tau_{yz} + \frac{\partial N_I}{\partial x} \tau_{zx} \right) dV
\end{aligned} \tag{180}$$

The spatial integrals in (180) are computed numerically by Gauss integration rule, and become:

$$\begin{aligned}
F_{Ix} &= \sum_{i=1}^{n_G} \sum_{j=1}^{n_G} \sum_{k=1}^{n_G} \left(\frac{\partial N_I}{\partial x} \sigma_x + \frac{\partial N_I}{\partial y} \tau_{xy} + \frac{\partial N_I}{\partial z} \tau_{zx} \right)_{ijk} W_i W_j W_k \det \underline{J} \\
F_{Iy} &= \sum_{i=1}^{n_G} \sum_{j=1}^{n_G} \sum_{k=1}^{n_G} \left(\frac{\partial N_I}{\partial y} \sigma_y + \frac{\partial N_I}{\partial x} \tau_{xy} + \frac{\partial N_I}{\partial z} \tau_{yz} \right)_{ijk} W_i W_j W_k \det \underline{J}, \\
F_{Iz} &= \sum_{i=1}^{n_G} \sum_{j=1}^{n_G} \sum_{k=1}^{n_G} \left(\frac{\partial N_I}{\partial z} \sigma_z + \frac{\partial N_I}{\partial y} \tau_{yz} + \frac{\partial N_I}{\partial x} \tau_{zx} \right)_{ijk} W_i W_j W_k \det \underline{J}
\end{aligned} \tag{181}$$

where i, j, k from 1 to 2 are the Gauss points along each spatial direction, $()_{ijk}$ means that the expression must be evaluated at the corresponding Gauss point, W_i, W_j, W_k are the weights and $\det \underline{J}$ is the determinant of the Jacobian matrix for the transformation between local (ξ, η, ζ) and global (x, y, z) coordinates, given by (40).

3.10 External pressure forces

An external pressure can be applied to the face of a C81L or C82L element by attaching to it a “boundary condition” element of type CL3Q with a material of type IMPE PIMP. The CL3Q element has the shape of a 4 node quadrilateral in 3D and is shown in Figure 7.

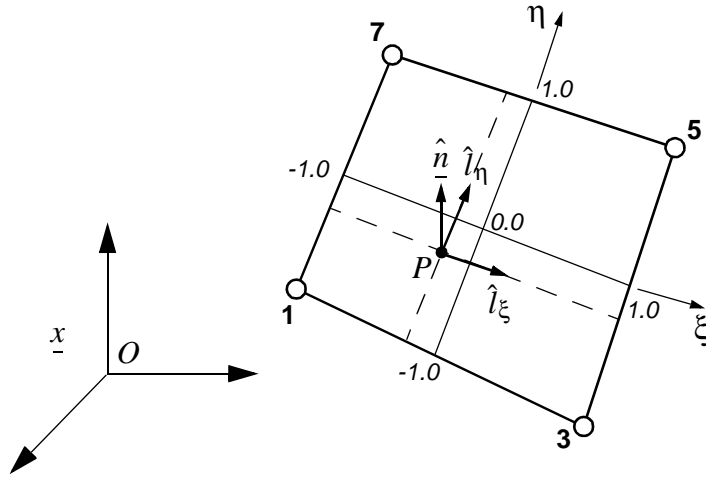


Figure 7 - The 4-node boundary condition element CL3Q

The element geometry is interpolated as:

$$\underline{x} = \sum_{I=1}^4 M_I \underline{x}_I, \quad (182)$$

where $M_I = M_I(\xi, \eta)$ are the 2-D linear shape functions (obtained by blending of the 1-D functions) given (under the name N_I) by (2) or (3), function of the curvilinear coordinates ξ and η . The symbol M is used instead of N to distinguish these shape functions from the (volumetric) shape functions (116) of the C8xL elements.

If a normal pressure p acts along the face S (i.e. along the CL3Q element), then the nodal pressure forces can be computed as:

$$F_{iI}^p = \int_S M_I p \hat{n}_i dS, \quad (183)$$

where \hat{n}_i are the global components of the *unit* normal vector $\hat{\underline{n}}$ to the face. If the pressure p is uniform over the element then:

$$F_{iI}^p = p \int_S M_I \hat{n}_i dS. \quad (184)$$

Following the paper by Hughes and Liu [8] (but see also references [9-10]) on degenerated shell elements (and by noting that the CL3Q element corresponds to a *lamina* of a degenerated shell), in each point $P = P(\xi, \eta)$ of the CL3Q element one can build up *unit tangent vectors* $\hat{l}_\xi, \hat{l}_\eta$ to the ξ and η coordinate directions, respectively:

$$\hat{l}_\xi = \frac{\partial \underline{x}}{\partial \xi} / \left\| \frac{\partial \underline{x}}{\partial \xi} \right\|, \quad \hat{l}_\eta = \frac{\partial \underline{x}}{\partial \eta} / \left\| \frac{\partial \underline{x}}{\partial \eta} \right\|, \quad (185)$$

or, from (182):

$$l_{\xi} = \frac{\partial x}{\partial \xi} = \sum_{I=1}^4 \frac{\partial M_I}{\partial \xi} x_I \quad l_{\eta} = \frac{\partial x}{\partial \eta} = \sum_{I=1}^4 \frac{\partial M_I}{\partial \eta} x_I. \quad (186)$$

Then, the *unit normal vector* $\hat{l}_3 \equiv \hat{n}$ is given by:

$$\hat{l}_3 \equiv \hat{n} = (l_{\xi} \times l_{\eta}) / \|l_{\xi} \times l_{\eta}\|, \quad (187)$$

where \times represents the vector product operator. The normal vector \underline{n} is given by:

$$\underline{n} = l_{\xi} \times l_{\eta}. \quad (188)$$

The integral in (184) is normalized by noting that:

$$dS = \|\underline{n}\| d\xi d\eta. \quad (189)$$

From (187-189) one sees that:

$$\hat{n} dS = \frac{\underline{n}}{\|\underline{n}\|} \|\underline{n}\| d\xi d\eta = \underline{n} d\xi d\eta. \quad (190)$$

The pressure forces (184) become therefore:

$$F_{ij}^p = p \int_S M_I \hat{n}_i dS = p \int_{-1}^1 \int_{-1}^1 M_I n_i d\xi d\eta. \quad (191)$$

If one evaluates the integral in (191) numerically by Gauss integration by using $G \times G$ Gauss points along the CL9x element (with either $G = 1$ or $G = 2$) then:

$$F_{ij}^p = p \int_{-1}^1 \int_{-1}^1 M_I n_i d\xi d\eta = p \sum_{g=1}^G \sum_{h=1}^G W_g W_h (M_I n_i)_{gh}, \quad (192)$$

where W_g, W_h are the integration weights and $()_{gh}$ indicates that the quantity has to be evaluated at the (gh) -th Gauss point.

3.11 Stability step

The element's stability step is computed as:

$$\Delta t^{\text{crit}} = L/c, \quad (193)$$

where L is the element's characteristic length and c the sound speed in the material. To compute L the following procedure is adopted, see Figure 8:

- Compute the twelve corner lengths $s_k, k = 1, \dots, 12$, to account for the element's "stretching".

For example:

$$s_1 = \|\vec{12}\|. \quad (194)$$

- Build up the six mid-face points M_f , $f = 1, \dots, 6$. For example:

$$\overrightarrow{OM_1} = \frac{1}{4}(\overrightarrow{O1} + \overrightarrow{O2} + \overrightarrow{O3} + \overrightarrow{O4}) \quad (195)$$

- Build up the three vectors \underline{a} , \underline{b} and \underline{c} (which are only approximately perpendicular to each other):

$$\underline{a} = \overrightarrow{M_3M_4} \quad \underline{b} = \overrightarrow{M_5M_6} \quad \underline{c} = \overrightarrow{M_1M_2} \quad (196)$$

- Compute the three “heights”, to take into account element’s “shearing”:

$$h_a = \left| \underline{a} \cdot \frac{(\underline{b} \times \underline{c})}{\|\underline{b} \times \underline{c}\|} \right| \quad h_b = \left| \underline{b} \cdot \frac{(\underline{c} \times \underline{a})}{\|\underline{c} \times \underline{a}\|} \right| \quad h_c = \left| \underline{c} \cdot \frac{(\underline{a} \times \underline{b})}{\|\underline{a} \times \underline{b}\|} \right|. \quad (197)$$

- The characteristic length is the minimum of the computed lengths:

$$L = \min(s_1, s_2, \dots, s_{12}, h_a, h_b, h_c). \quad (198)$$

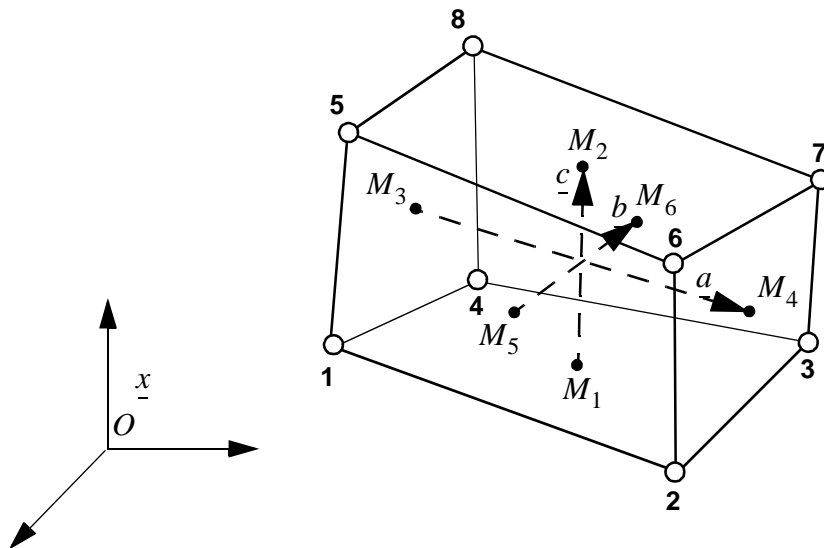


Figure 8 - Computing the characteristic length for the 27-node element

An alternative calculation of L as the minimum intra-nodal distance between any couple of nodes can be activated by the option OPTI DTML.

4. Numerical examples

Some numerical examples are presented illustrating the performance of the new elements introduced in the previous Sections.

4.1 Large-shear tests

We start by some one-element large-shear tests inspired by the calculations performed in references [2-5] when discussing the treatment of large strains in 2D. The tests performed are summarized in Table 1.

Name	Description	Steps	CPU (s)
SHEA11	Q41L	50000	5.44
SHEA12	Q42L	50000	5.10
SHEA13	C81L shear along x	50000	5.65
SHEA14	C82L shear along x	50000	5.37
SHEA15	C81L shear along y	50000	5.556
SHEA16	C82L shear along y	50000	5.46
SHEA17	C81L shear along z	50000	5.65
SHEA18	C82L shear along z	50000	6.15

Table 1 - Shear tests

SHEA11

This is a 2D plane strain calculation using the Q41L element, to be used as a reference for subsequent 3D calculations. The problem is shown in Figure 9. A square is subjected to a large shearing by blocking the base and by imposing horizontal displacements to the other nodes of the element as shown in the Figure. The material is assumed elastic.

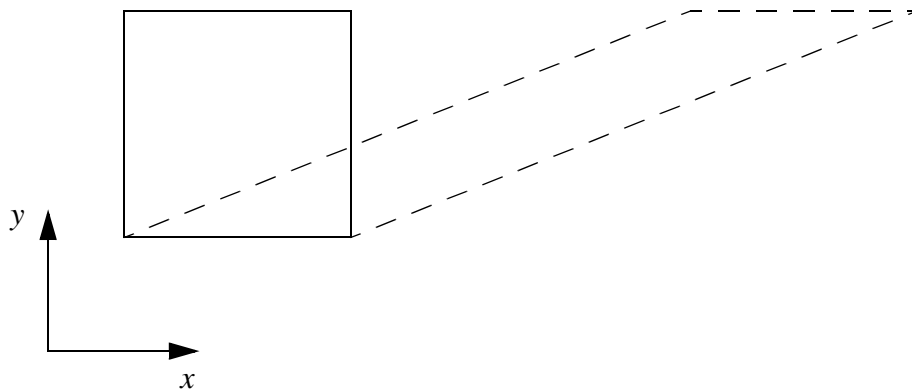


Figure 9 - The large shear problem in 2D.

The outcome of the calculation is the shear stress in the element which, according to the Zaremba-Jaumann-Noll formulation, has a periodic (sinusoidal) variation. In this case the maximum imposed horizontal displacement at the top of the element is 10 times the element length over a time span of 1 second.

The imposed displacements are shown in Figure 10 and the resulting stresses are shown in Figure 11. The stresses have a sinusoidal shape, as expected.

SHEA12

This is a repetition of test SHEA12 by using the Q42L (fully integrated) element instead of Q41L (reduced integrated). The results are practically identical as shown for example in Figure 12 for the shear stress.

SHEA13

This test is 3D and uses the C81L under-integrated hexahedron element. The base of the element (in the x - y plane) is blocked in all directions. All other nodes are blocked in the y and z directions and are assigned suitable displacements along x , corresponding to those of the previous 2D cases.

SHEA14

This test is similar to SHEA13 but uses the C82L fully-integrated element.

SHEA05

This test is similar to SHEA13 but the shearing occurs along the y -direction.

SHEA06

This test is similar to SHEA15 but uses the C82L fully-integrated element.

SHEA07

This test is similar to SHEA13 but the shearing occurs along the z -direction.

SHEA08

This test is similar to SHEA17 but uses the C82L fully-integrated element.

Comparisons

The results of all these 3D tests are all very similar and in very good agreement with the 2D reference. A comparison of the corresponding shear stresses is shown in Figure 13.

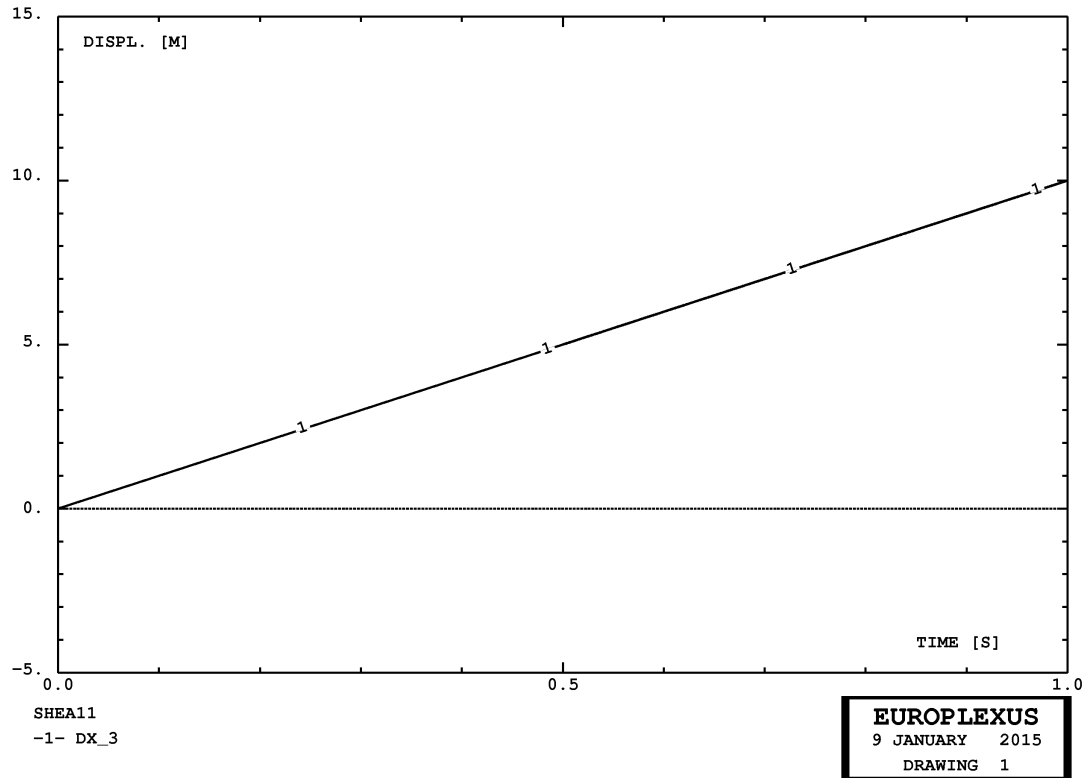


Figure 10 - Displacements in case SHEA11.

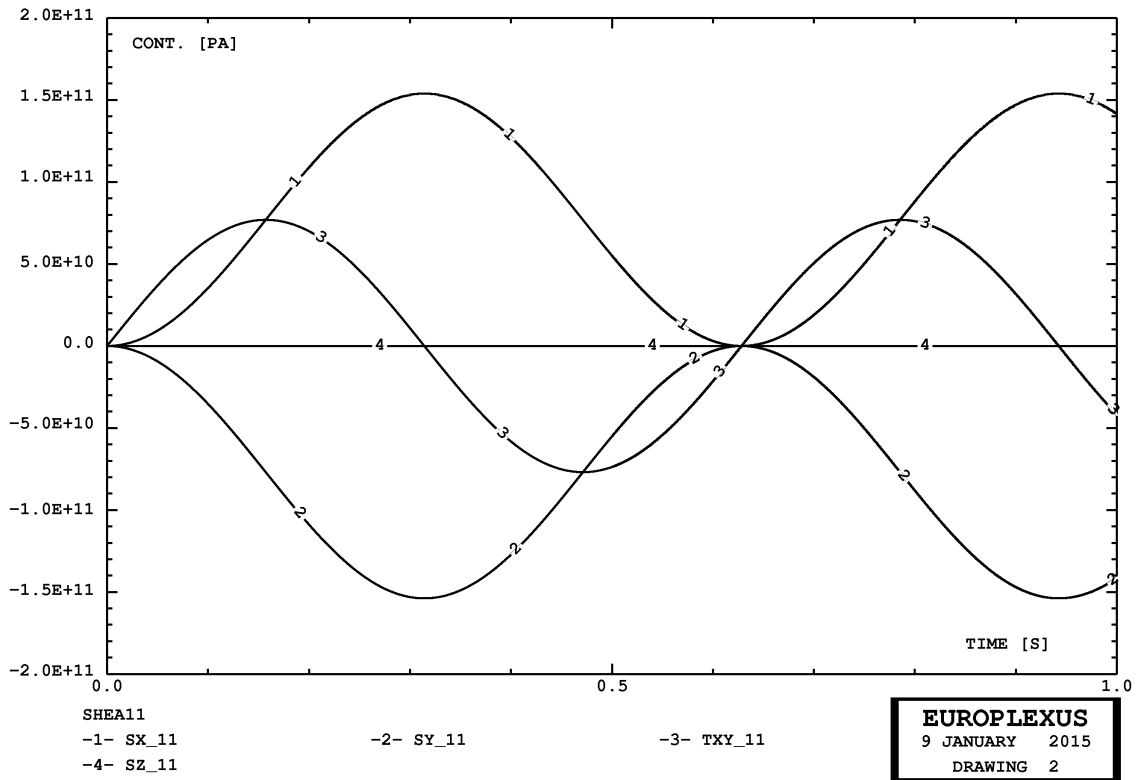


Figure 11 - Stresses in case SHEA11.

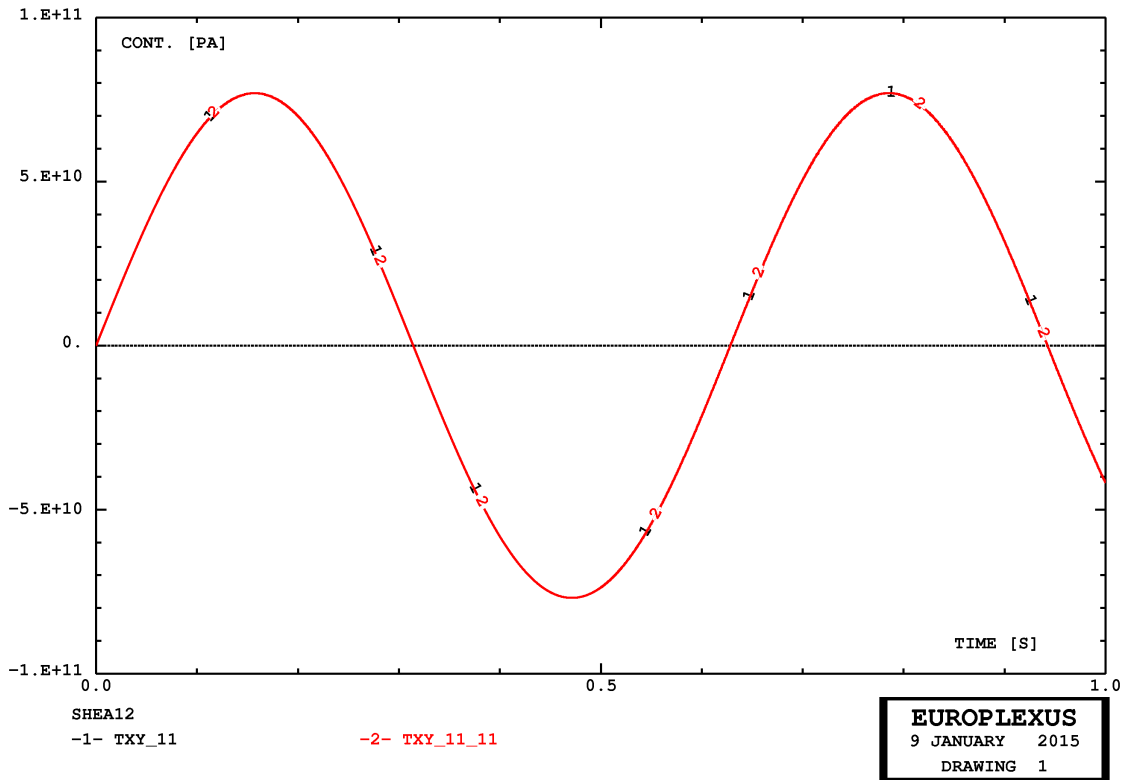


Figure 12 - Comparison of shear stresses in cases SHEA11 and SHEA12.

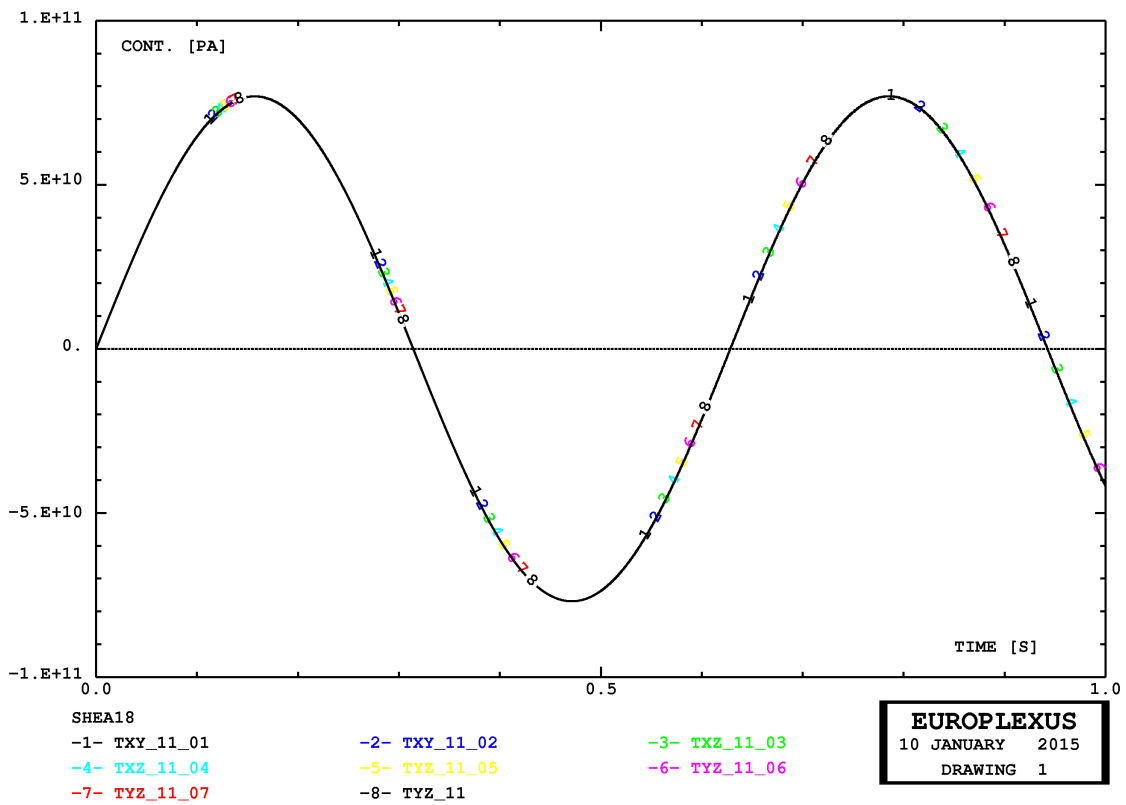


Figure 13 - Comparison of shear stresses in cases SHEA11 to SHEA18.

4.2 Large-rotation tests

The next series of tests is used to check large rotations. An elastic beam of square cross section and length equal to ten times the cross-section side is subjected to an applied pressure along the whole length of the beam and is clamped at one extremity. This test is inspired from a similar test reported in reference [11].

The tests performed are summarized in Table 2.

Name	Description	Steps	CPU (s)
POUT54	Q42L + CL22	5814	1.78
POU201	C81L + CL3Q	12034	12.57
POU202	C82L + CL3Q	11682	33.79
POU203	C81L + CL3Q rotated around x	11784	12.34
POU204	C81L + CL3Q rotated around y	12034	12.46

Table 2 - Pressurized beam tests

POUT54

This is a 2D plane strain calculation [11] using the Q42L linear quadrilateral and the CL22 boundary condition element to represent the applied pressure. This solution is then used as a reference for the subsequent 3D calculations.

The beam undergoes very large displacements and rotations. Figure 14 shows the initial beam configuration and the deformed configuration at 3 ms, corresponding approximately to the maximum deflection. Figure 15 shows the displacements of the nodes at the free beam end.

POU201

This is a 3D calculation using C81L and CL3Q (under-integrated) elements.

POU202

This test is similar to POU201 but uses C82L and CL3Q (fully-integrated) elements.

POU203

This test is similar to POU201 but the beam is rotated by 90° around the x -axis.

POU204

This test is similar to POU201 but the beam is rotated by 90° around the y -axis.

Comparisons

A comparison of the corresponding displacement of the mid-node at the free beam tip is shown in Figure 16 for all these tests. Two sets of curves are obtained: a set of slightly “stiffer” solutions is obtained by the fully-integrated elements, while a set of slightly “weaker” solutions is obtained by the reduced-integrated elements. The discrepancy is larger than in the case of parabolic elements (see Section 4.2 of reference [6]). In the case of reduced-integrated elements, some small mechanisms are observed.

POUT54 - 2D Q42L+CL22 DOUBLE LAYER
TIME: 3.00000E-03 STEP: 2933

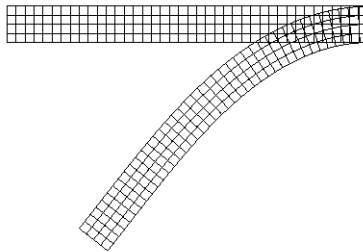


Figure 14 - Deformations in case POUT54.

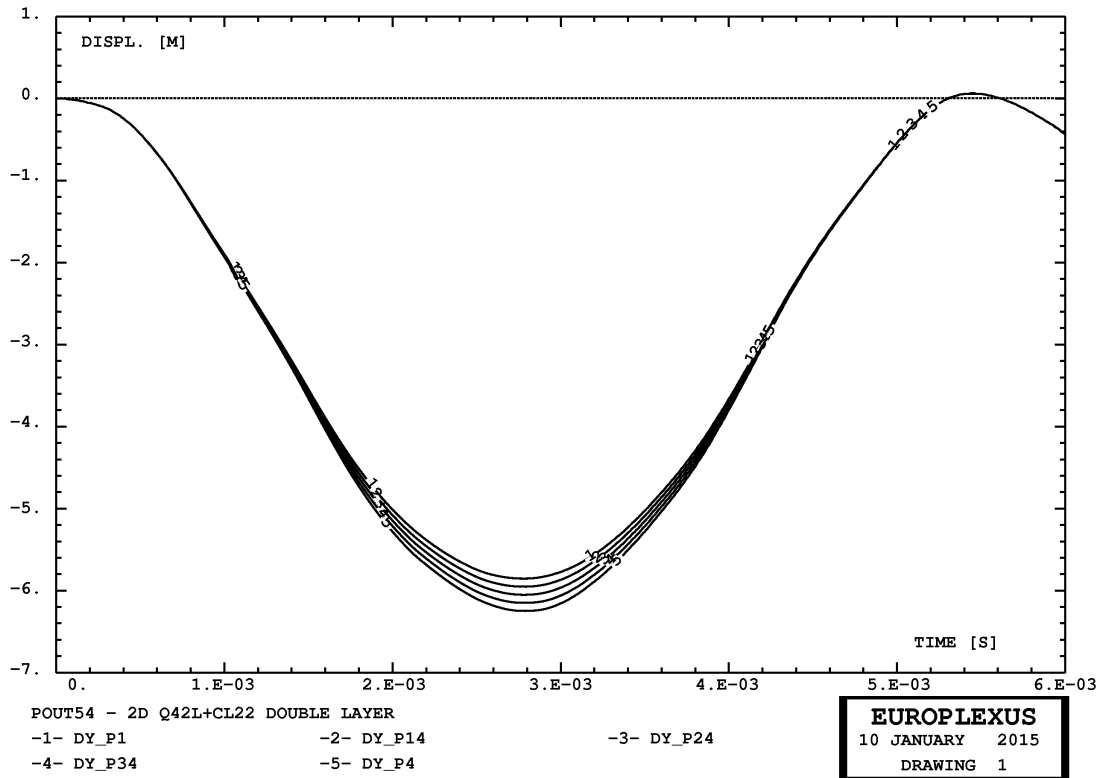


Figure 15 - Displacements in case POUT54.

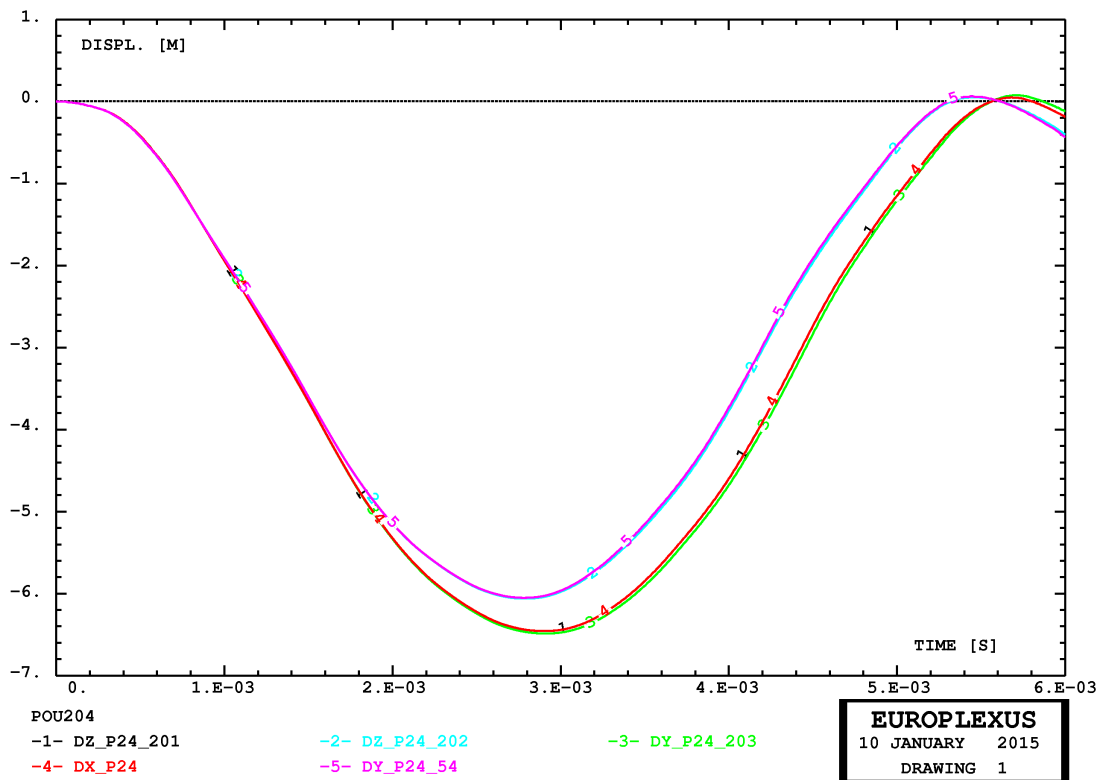


Figure 16 - Comparison of displacements in cases POUT54 to POU204.

4.3 Elastic wave propagation tests

The next series of tests is used to check elastic wave propagation. An elastic beam of square cross section and length equal to one hundred times the cross-section side is subjected to an initial velocity and is clamped at one extremity.

The tests performed are summarized in Table 3.

Name	Description	Steps	CPU (s)
BARE11	Q41L	800	0.59
BARE12	Q42L	800	0.69
BARE13	C81L	800	0.73
BARE14	C82L	800	1.05

Table 3 - Elastic wave propagation tests

BARE11

This is a 2D plane stress calculation using the Q41L linear quadrilateral. This solution is then used as a reference for the subsequent 3D calculations. The resulting stress at two points close to the bar mid-point are compared in Figure 17 against the analytical solution of the problem (a double step function).

BARE12

This test is similar to BARE11 but uses the Q42L element.

BARE03

This test is similar to BARE11 but uses the C81L element.

BARE04

This test is similar to BARE11 but uses the C82L element.

Comparison

The results of these tests are all very similar and in very good agreement with the 2D reference. A comparison of the corresponding stress at the mid-point of the bar is shown in Figures 18 (using the average stress in the element to the left of the bar mid-point) and 19 (using the average stress in the element to the right of the bar mid-point).

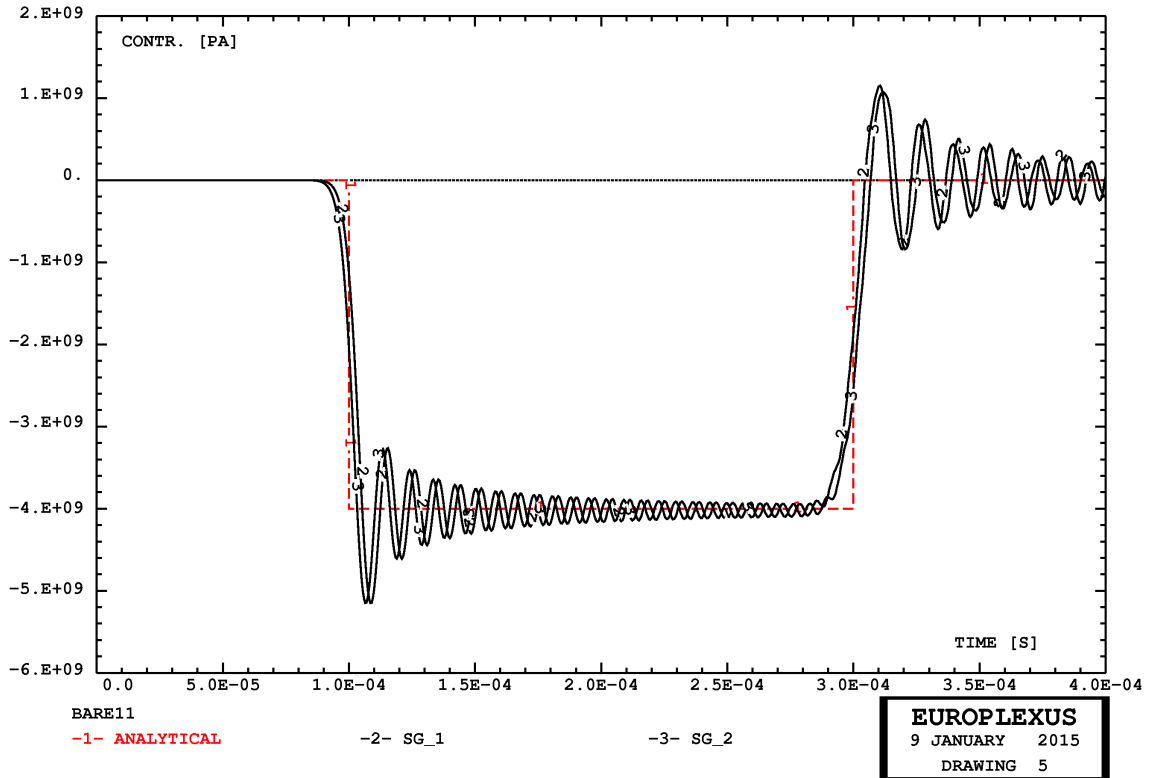


Figure 17 - Stress in case BARE11 compared with the analytical solution.

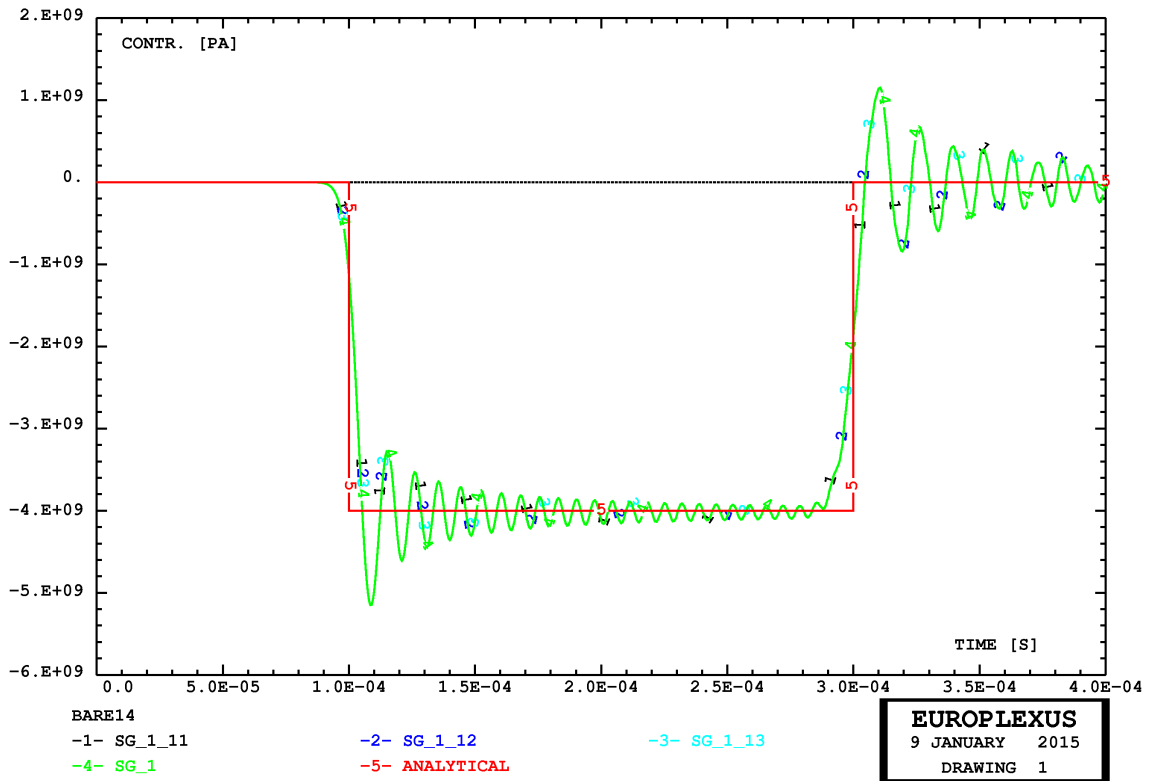


Figure 18 - Stress in test cases BARE11 to BARE14 compared with the analytical solution.

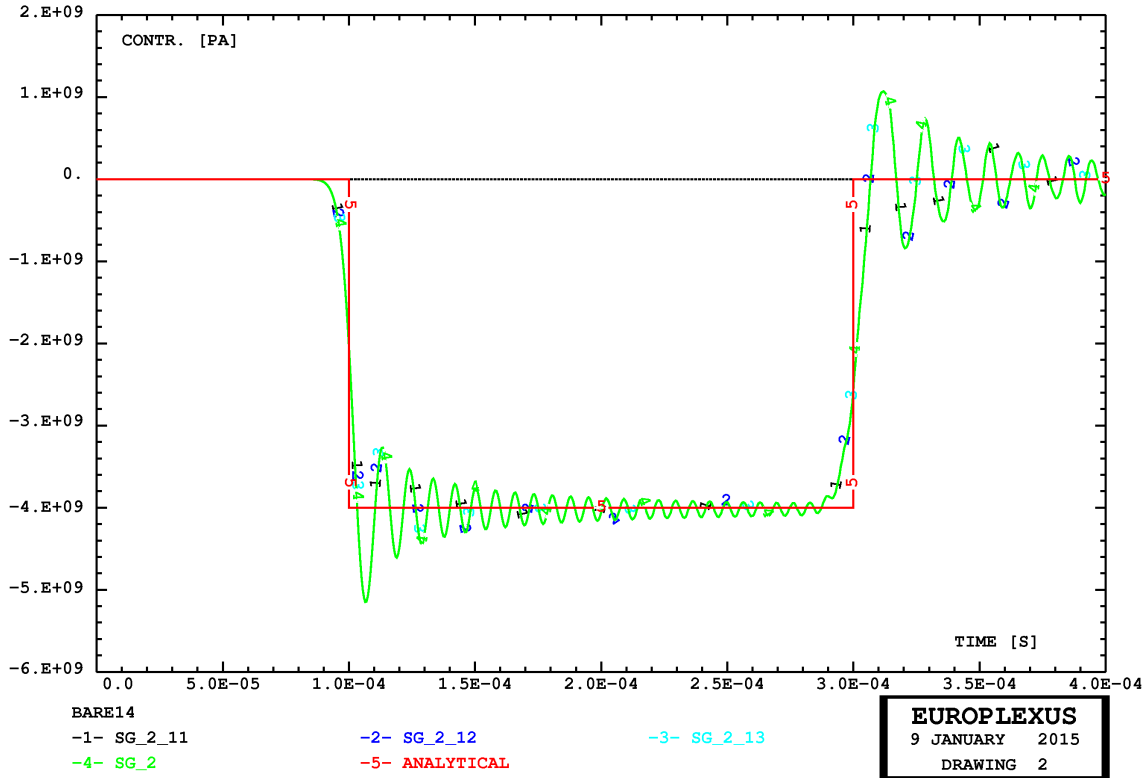


Figure 19 - Stress in test cases BARE11 to BARE14 compared with the analytical solution.

4.4 Elasto-plastic bar impact test

The next series of tests is used to check elasto-plasticity in large-strain deformation. The test is a classical elasto-plastic bar impact test, see e.g reference [12].

The tests performed are summarized in Table 4.

Name	Description	Steps	CPU (s)
BARI23	Q41L, OPTI PART	2,760 macro	13.8
BARI24	C81L	(17,225 for 26 μ s)	(11.8)
BARI25	C82L, 12 \times 40	4,115	8.9
BARI26	Q42L, OPTI PART	2,802 macro	20.5
BARI30	Q42L	34,088	51.3
BARI31	Q41L	38,998	32.9
BARI32	Q42L, OPTI PART PLIN	2,802 macro	13.5
BARI33	Q41L, OPTI PART PLIN	2,760 macro	9.2

Table 4 - Elasto-plastic bar impact tests

BARI23

This is a 2D plane axisymmetric calculation using the Q41L linear quadrilaterals. The mesh (1/2 of the bar cross-section by symmetry) uses 12×120 linear elements. This solution is then used as a reference for the subsequent 3D calculations. The resulting radial and axial maximum displacements are shown in Figure 20 and are in excellent agreement with the solutions obtained in reference [6] with the parabolic elements. This calculation uses time step partitioning (OPTI PART) to decrease the CPU time.

BARI24

This is a 3D calculation using the C81L linear hexahedron. The mesh (1/4 of the bar), shown in Figure 21, is relatively coarse and uses 12×40 elements. The calculation stops at about $26 \mu\text{s}$ because the element near the center of the impacting section becomes too squeezed and therefore the stability time step drops unacceptably.

Displacements are compared in Figure 22 with the reference solution, showing excellent agreement as long as the 3D simulation is capable of advancing in time. The deformed mesh at calculation stop is shown in Figure 23.

BARI25

This is a 3D calculation using the C82L linear hexahedron. The mesh is the same as in case BARI24. This calculation reaches without problems the final time of $80 \mu\text{s}$. However, the solution exhibits severe locking (the deformation of the bar is largely underestimated).

Displacements are compared in Figure 24 with the reference solution. The deformed mesh at calculation stop is shown in Figure 25.

Due to the presence of severe locking, no further solutions with C82L are attempted.

BARI26

This calculation is similar to BARI23 but uses the Q42L element instead of Q41L.

Displacements are compared in Figure 26 with the reference solution, showing poor agreement. The 2D fully integrated solution exhibits some locking, although less pronounced than in the 3D fully integrated case.

BARI30

This is a repetition of test BARI26 but without time step partitioning. Results are nearly identical to those of case BARI26 as shown in Figure 27, and the speed-up (in case BARI26) is about 2.5 times.

BARI31

This is a repetition of test BARI23 but without time step partitioning. Results are nearly identical to those of case BARI23 as shown in Figure 28, and the speed-up (in case BARI23) is about 2.4 times.

BARI32

This is a repetition of test BARI26 but using the LINK COUP directive instead of LIAI and the OPTI PART PLIN option for time step partitioning. Results are nearly identical to those of case BARI26 as shown in Figure 29, and the *further* speed-up (with respect to BARI26) is about 33%.

BARI33

This is a repetition of test BARI23 but using the LINK COUP directive instead of LIAI and the OPTI PART PLIN option for time step partitioning. Results are nearly identical to those of case BARI23 as shown in Figure 30, and the *further* speed-up (with respect to BARI23) is about 33%.

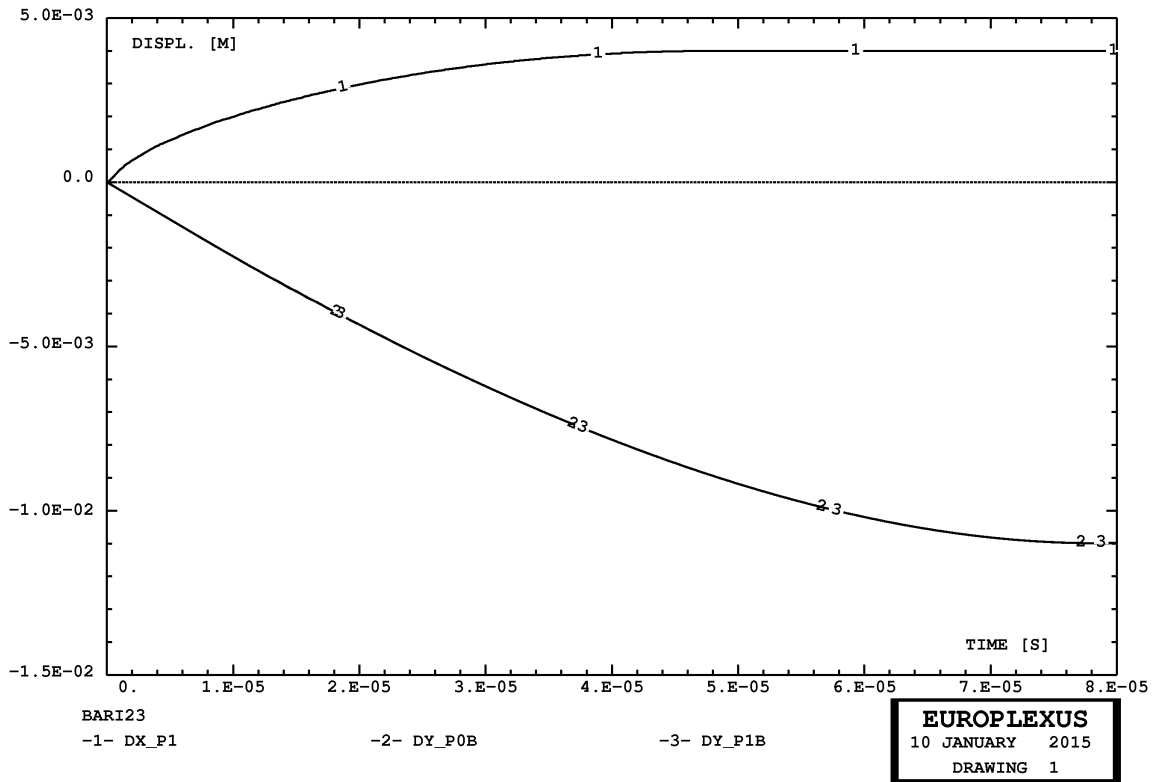


Figure 20 - Displacements in case BARI23.

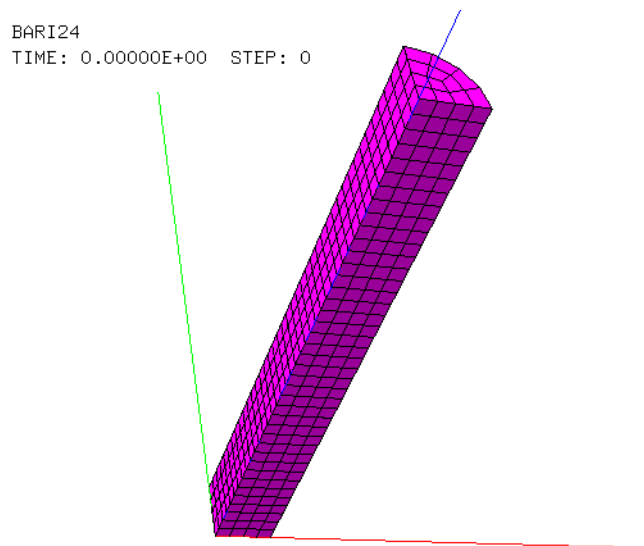


Figure 21 - 3D mesh of case BARI24.

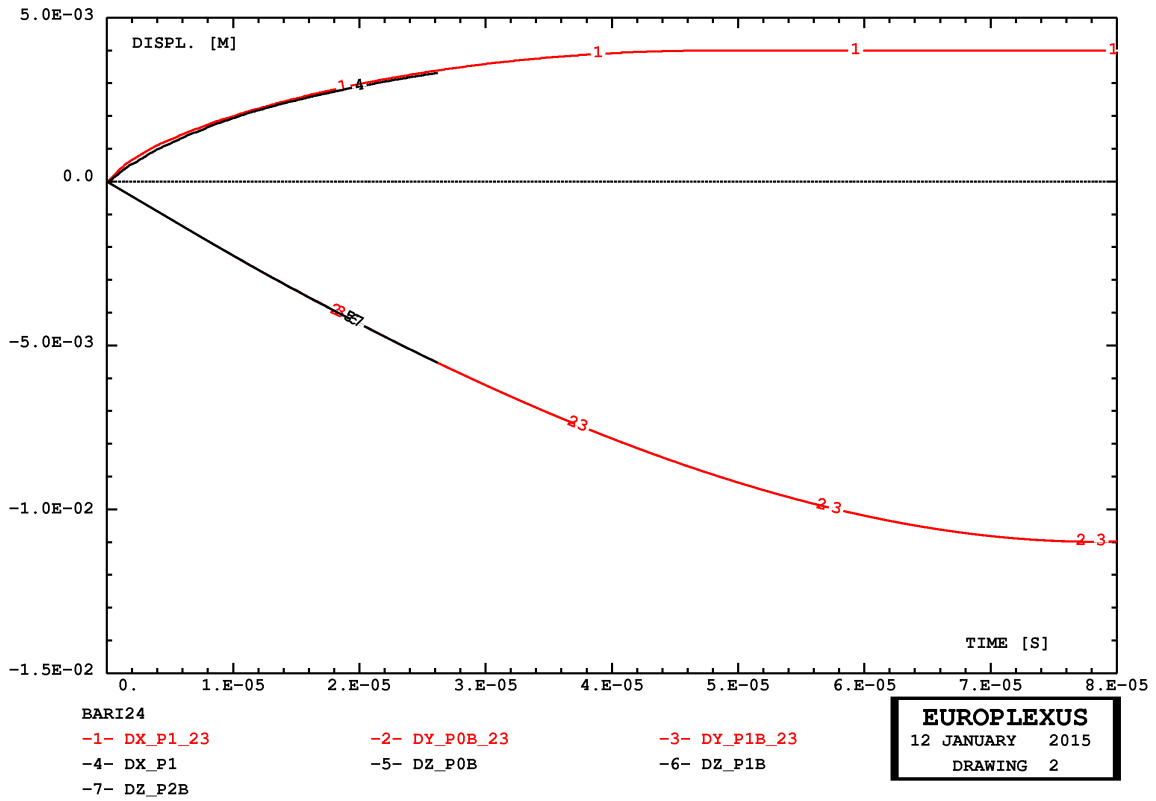
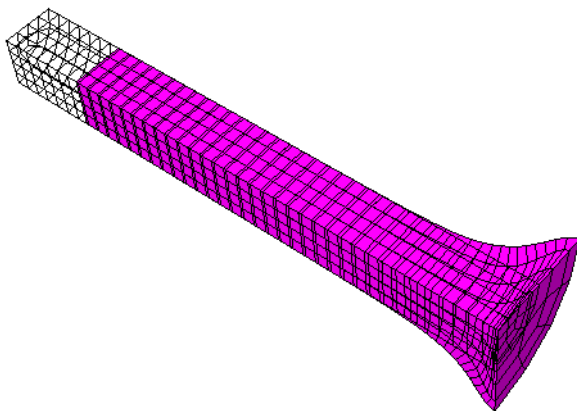


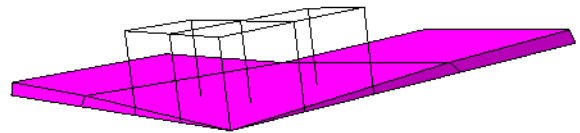
Figure 22 - Displacements in cases BARI23 and BARI24.

BARI24
TIME: 2.61832E-05 STEP: 17225

BARI24
TIME: 2.61832E-05 STEP: 17225



Deformed mesh



Three elements in the impact zone

Figure 23 - Mesh details of case BARI24 at calculation failure.

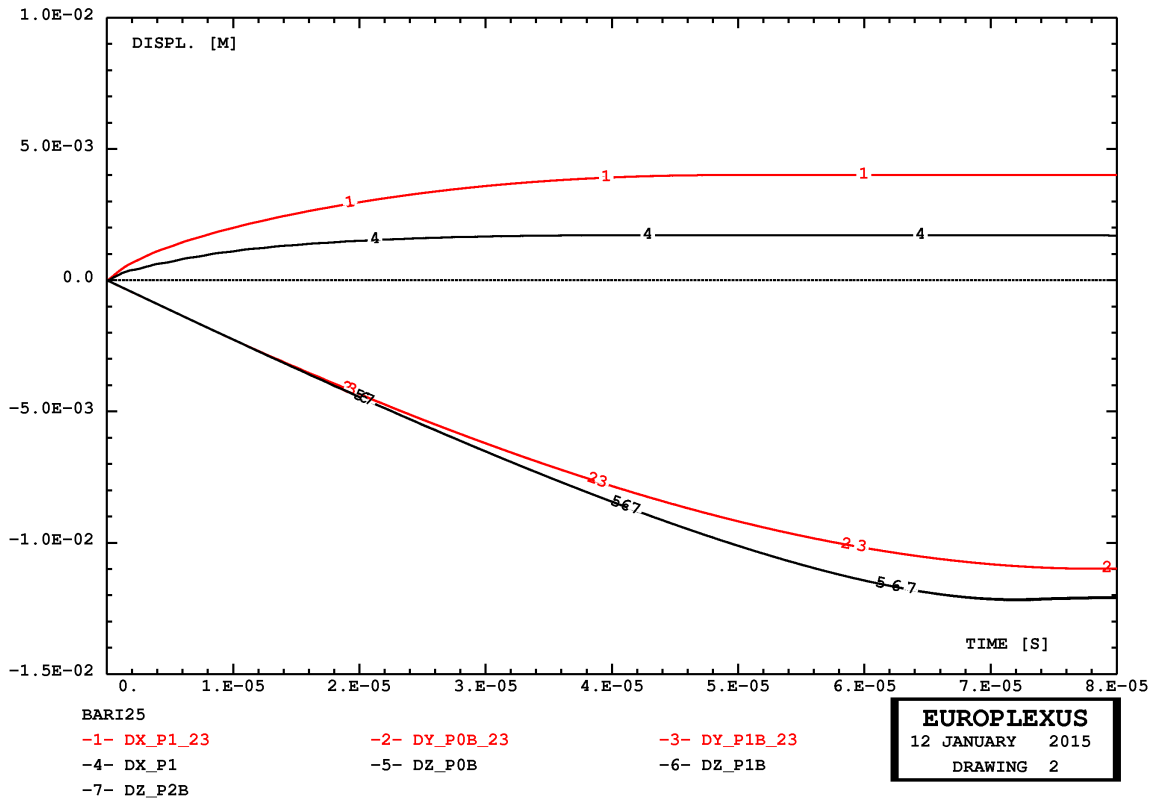
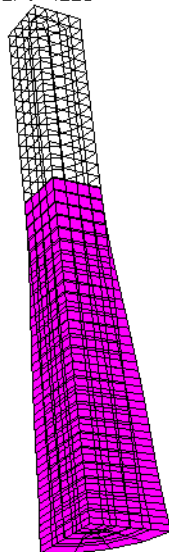


Figure 24 - Displacements in cases BARI23 and BARI25.

BARI25
TIME: 8.00000E-05 STEP: 4115



BARI25
TIME: 8.00000E-05 STEP: 4115



Figure 25 - Mesh details of case BARI25 at the final time.

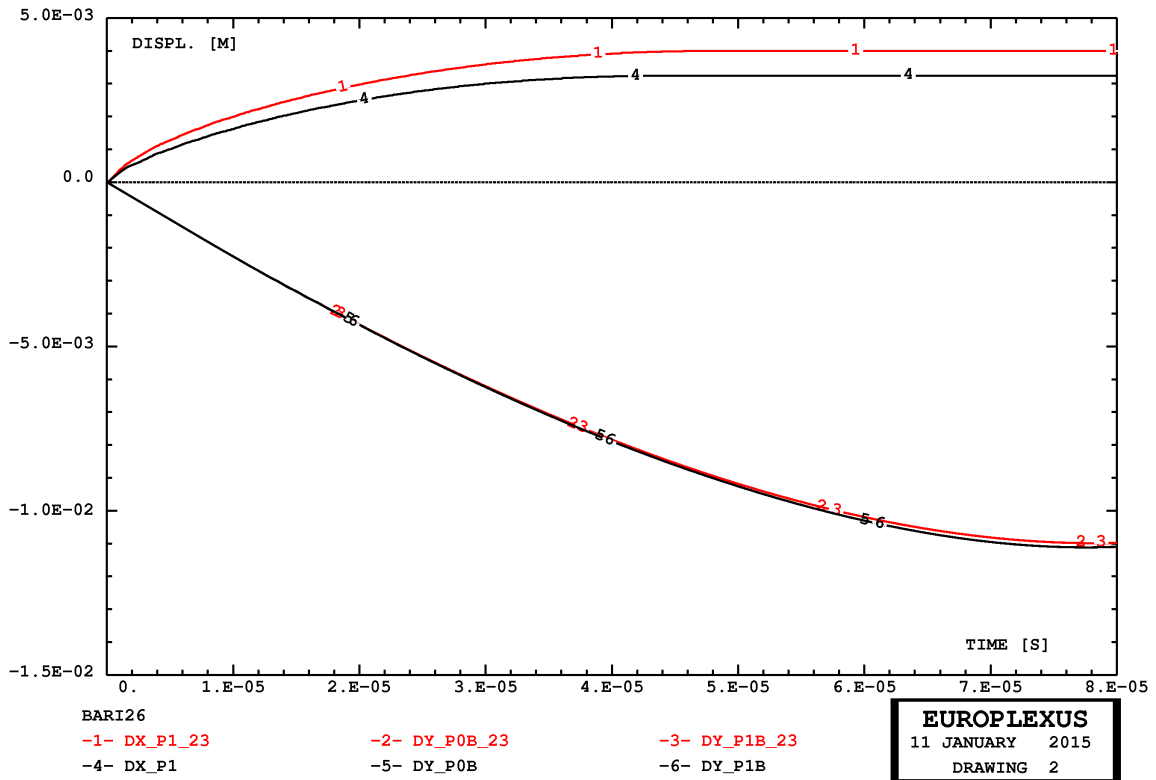


Figure 26 - Displacements in cases BARI23 and BARI26.

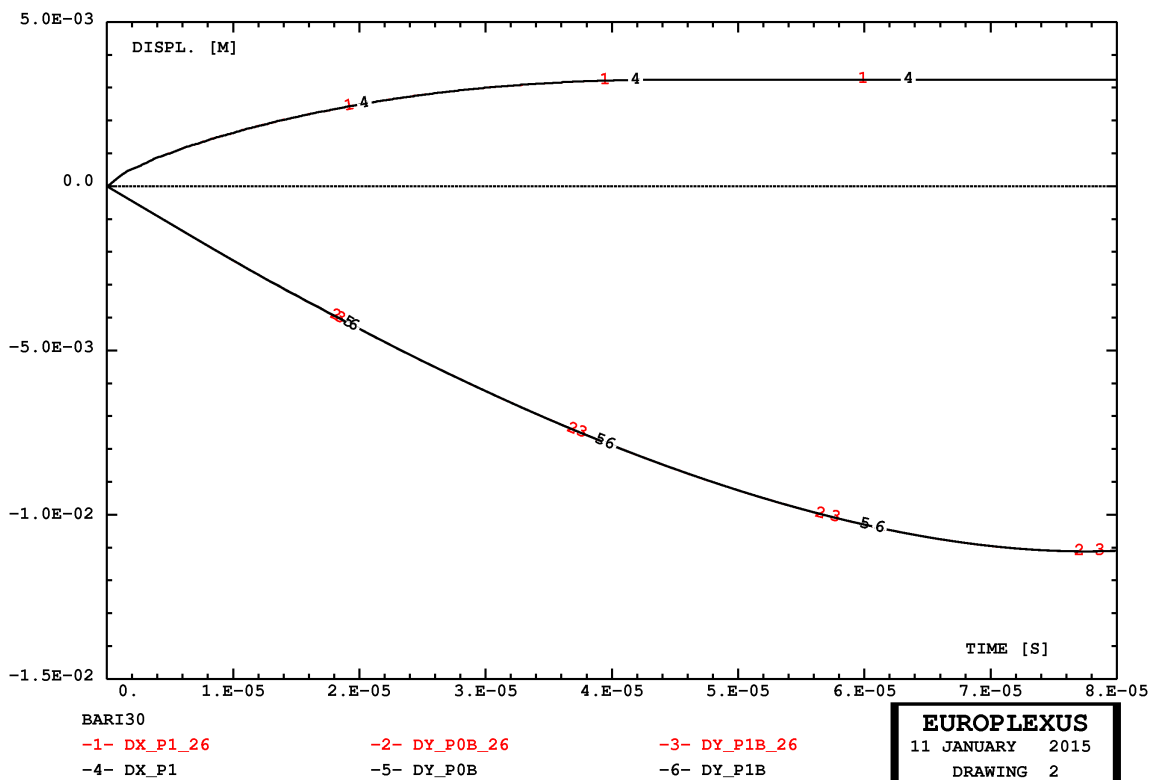


Figure 27 - Displacements in cases BARI26 and BARI30.

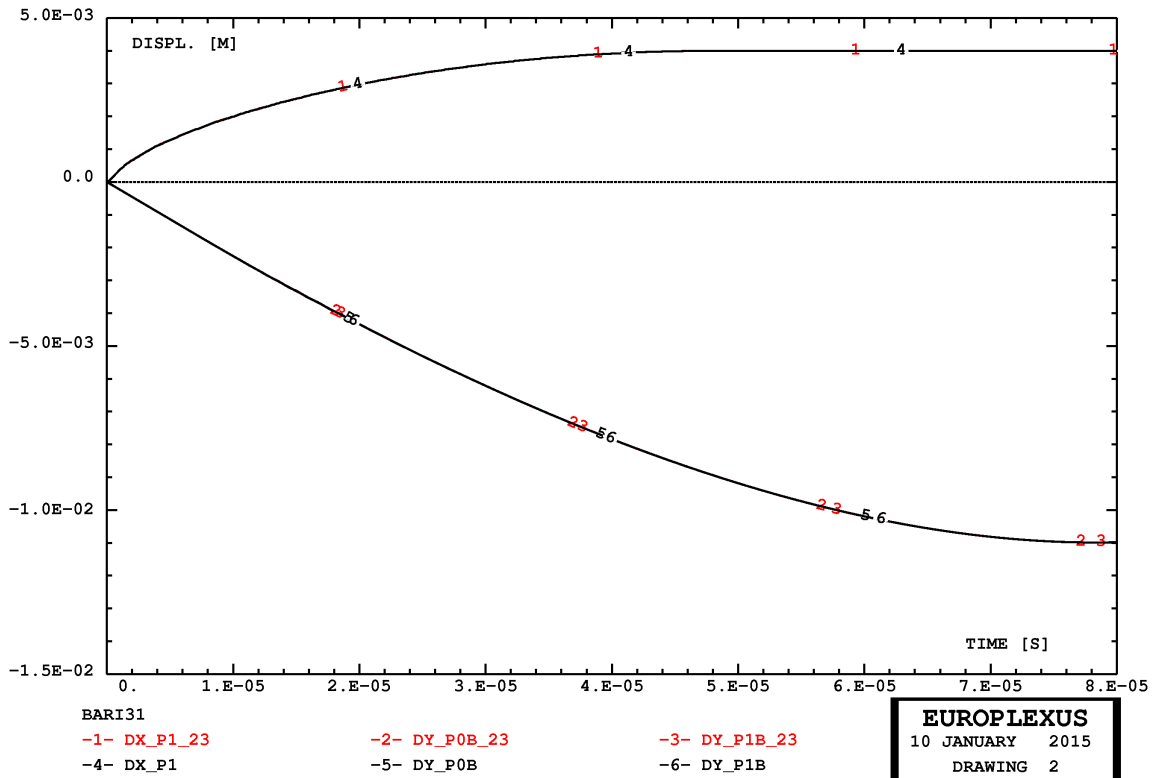


Figure 28 - Displacements in cases BARI23 and BARI31.

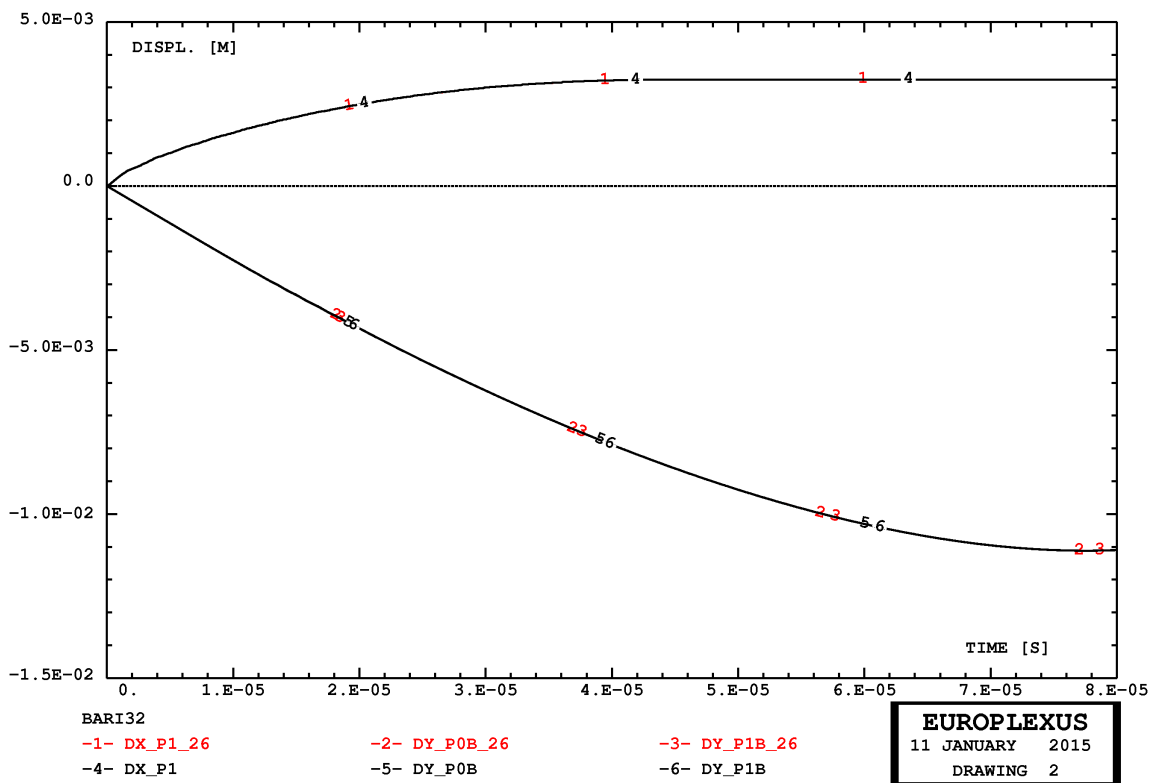


Figure 29 - Displacements in cases BARI26 and BARI32.

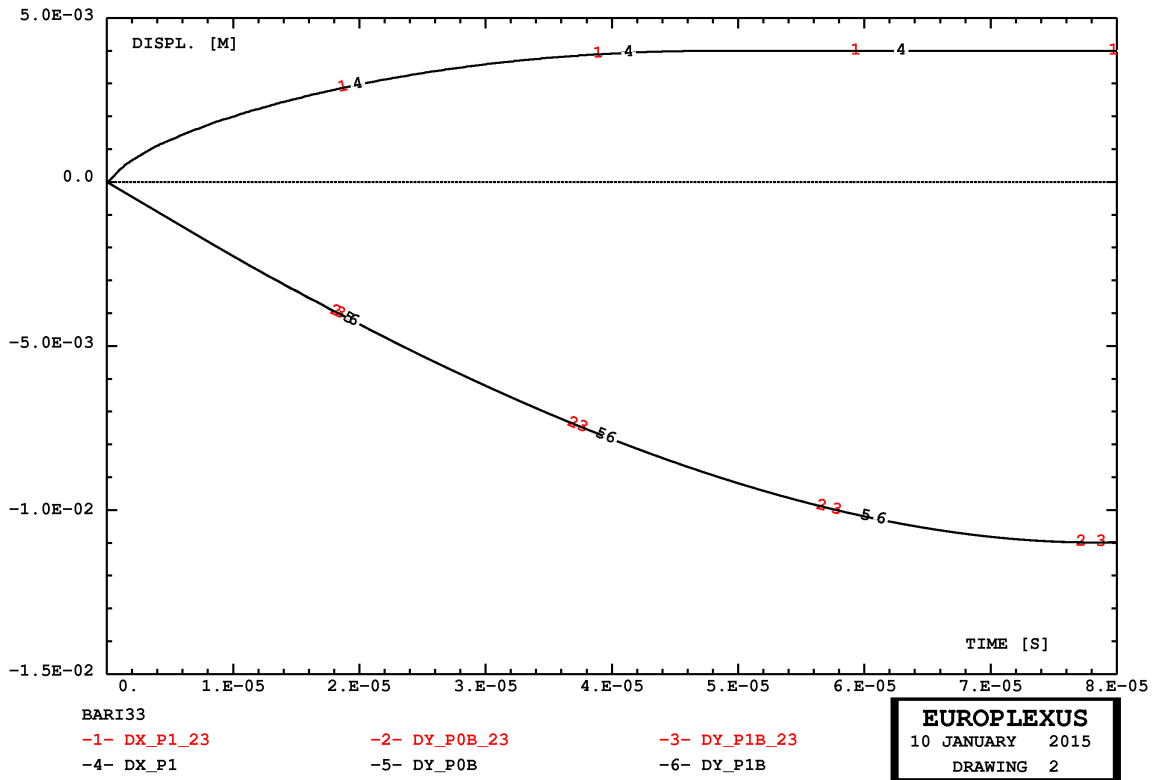


Figure 30 - Displacements in cases BARI23 and BARI33.

5. References

- [1] “*EUROPLEXUS User’s Manual*”, on-line version: <http://europlexus.jrc.ec.europa.eu>.
- [2] J. P. HALLEUX, F. CASADEI: “*Transient large strain analysis of plane and axisymmetric structures by means of biquadratic finite elements*”. 2nd Int. Conf. on Numerical Methods for Non-linear Problems, Barcelona, Apr. 9-13, 1984.
- [3] J. P. HALLEUX, F. CASADEI: “*Transient large strain analysis of plane and axisymmetric structures by means of biquadratic finite elements*”. *Engineering Computations*, Vol. 1, pp. 351-358, Dec. 1984.
- [4] F. CASADEI, J. DONEA, S. GIULIANI, J. P. HALLEUX, G. VERZELETTI: “*Recenti sviluppi nei metodi computazionali e sperimentali per l’ analisi dinamica di strutture nel campo nonlineare*”. XIII Convegno Nazionale dell’ Associazione Italiana per l’ Analisi delle Sollecitazioni, presso l’ I.S.M.E.S, Bergamo, 23-27 Settembre 1985.
- [5] J. P. HALLEUX, F. CASADEI: “*Transient large strain finite element analysis of solids*”. In ‘Computational Methods for Non-Linear Problems’, 2nd Volume in the Series: Recent Advances in Non-Linear Computational Mechanics, Eds. C. Taylor, D.R.J. Owen and E. Hinton, Pineridge Press, Swansea, 1987.
- [6] F. CASADEI: “*Implementation of Parabolic Lagrange 2D and 3D Continuum Elements in EUROPLEXUS*”. Technical Note Pubsy N. JRC66266, 2011.
- [7] F. CASADEI: “*The LOOM System Raster Graphics Model*”. Technical Note N. I.03.165, November 2003.
- [8] T.J.R. HUGHES, W.K. LIU: “*Nonlinear finite element analysis of shells: Part I - three-dimensional shells*”, *Comp. Meths. Appl. Mech. Eng.*, 26, pp. 331-362, 1981.
- [9] T.J.R. HUGHES, W.K. LIU, I. LEVIT: “*Nonlinear dynamic finite element analysis of shells*”, in Nonlinear Finite Element Analysis in Structural Mechanics, Eds. W. Wunderlich et al., Springer, Berlin, 1981.
- [10] F. CASADEI: “*A Nonlinear 3-D Shell Finite Element Implementation for Transient Problems*”, Technical Note N. I.93.41, April 1993.
- [11] F. CASADEI: “*Implementation of 3D Degenerated Shell Elements in PLEXIS-3C*”, Technical Note N. I.93.88, July 1993.
- [12] F. CASADEI, J. DONEA, A. HUERTA: “*Arbitrary Lagrangian-Eulerian Finite Elements in Non-Linear Fast Transient Continuum Mechanics*”, EUR 16327 EN, 1995.

6. Appendix

Sample input files

This Section contains, in alphabetical file order, the listings of all input files related to the examples which were proposed in the previous Sections.

bare11.dgibi

```
*$siz+100
opti echo 1;
opti dime 2 elem qua4;
p1 = 0 0;
p2 = 1 0;
p3 = 0.5 0;
tol = 0.001;
dy = 0.01;
n = 50;
bar11 = (p1 d n p3) coul vert;
bar21 = (p3 d n p2) coul vert;
bar1 = (bar11 tran 1 (0 dy)) coul vert;
bar2 = (bar21 tran 1 (0 dy)) coul vert;
elim tol (bar1 et bar2);
bar = (bar1 et bar2) coul vert;
e1 = bar1 elem cont p3;
e2 = bar2 elem cont p3;
bartip = bar2 elem cont p2;
mesh = bar et e1 et e2 et bartip;
tass mesh;
opti sauv form 'bare11.msh';
sauv form mesh;
opti trac psc ftra 'bare11_mesh.ps';
trac qual mesh;
fin;
```

bare11.epx

```
BARE11
ECHO
!CONV win
CAST mesh
CPLA LAGR
GEOM Q41L bar TERM
COMP NGRO 1 'end' LECT bar TERM COND X GT 0.999
EPAI 2.5E-8 LECT bar TERM
MATE VM23 RO 8000. YOUN 2.0E11 NU 0.0 ELAS 2.0E11
TRAC 1 2.0E11 1.E0
LECT bar TERM
LINK COUP BLOQ 1 LECT end TERM
INIT VITE 1 100.0 LECT bar DIFF end TERM
ECRI DEPL VITE CONT ECRO TFREQ 0.1E-3
FICH ALIC TEMP FREQ 1
POIN LECT p3 p2 TERM
ELEM LECT e1 e2 TERM
FICH ALIC FREQ 1
OPTI PAS UTIL NOTE LOG 1
CALC TINI 0. TEND 4.0E-4 PASF 5.0E-7
SUIT
Post-treatment
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_2' DEPL COMP 1 NOEU LECT p2 TERM
COUR 2 'dx_3' DEPL COMP 1 NOEU LECT p3 TERM
COUR 3 'sg_1' CONT COMP 1 ELEM LECT e1 TERM
COUR 4 'sg_2' CONT COMP 1 ELEM LECT e2 TERM
DCOU 5 'Analytical' 6
0.0 0.0
0.1E-3 0.0
0.1E-3 -4.E9
0.3E-3 -4.E9
0.3E-3 0.0
0.4E-3 0.0
TRAC 1 AXES 1.0 'DISPL. [M]'
TRAC 2 AXES 1.0 'DISPL. [M]'
TRAC 3 AXES 1.0 'CONTR. [PA]'
TRAC 4 AXES 1.0 'CONTR. [PA]'
TRAC 5 3 4 AXES 1.0 'CONTR. [PA]' YZER
COLO ROUG NOIR NOIR
DASH 2 0 0
LIST 1 AXES 1.0 'DISPL. [M]'
LIST 2 AXES 1.0 'DISPL. [M]'
LIST 3 AXES 1.0 'CONTR. [PA]'
LIST 4 AXES 1.0 'CONTR. [PA]'
FIN
```

bare12.dgibi

```
*$siz+100
opti echo 1;
opti dime 2 elem qua4;
p1 = 0 0;
p2 = 1 0;
p3 = 0.5 0;
tol = 0.001;
dy = 0.01;
n = 50;
bar11 = (p1 d n p3) coul vert;
bar21 = (p3 d n p2) coul vert;
bar1 = (bar11 tran 1 (0 dy)) coul vert;
bar2 = (bar21 tran 1 (0 dy)) coul vert;
```

```
elim tol (bar1 et bar2);
bar = (bar1 et bar2) coul vert;
e1 = bar1 elem cont p3;
e2 = bar2 elem cont p3;
bartip = bar2 elem cont p2;
mesh = bar et e1 et e2 et bartip;
tass mesh;
opti sauv form 'bare12.msh';
sauv form mesh;
opti trac psc ftra 'bare12_mesh.ps';
trac qual mesh;
fin;
```

bare12.epx

```
BARE12
ECHO
!CONV win
CAST mesh
CPLA LAGR
GEOM Q42L bar TERM
COMP NGRO 1 'end' LECT bar TERM COND X GT 0.999
EPAI 2.5E-8 LECT bar TERM
MATE VM23 RO 8000. YOUN 2.0E11 NU 0.0 ELAS 2.0E11
TRAC 1 2.0E11 1.E0
LECT bar TERM
LINK COUP BLOQ 1 LECT end TERM
INIT VITE 1 100.0 LECT bar DIFF end TERM
ECRI DEPL VITE CONT ECRO TFREQ 0.1E-3
FICH ALIC TEMP FREQ 1
POIN LECT p3 p2 TERM
ELEM LECT e1 e2 TERM
FICH ALIC FREQ 1
OPTI PAS UTIL NOTE LOG 1
CALC TINI 0. TEND 4.0E-4 PASF 5.0E-7
SUIT
Post-treatment
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_2' DEPL COMP 1 NOEU LECT p2 TERM
COUR 2 'dx_3' DEPL COMP 1 NOEU LECT p3 TERM
COUR 3 'sg_1' CONT COMP 1 ELEM LECT e1 TERM
COUR 4 'sg_2' CONT COMP 1 ELEM LECT e2 TERM
DCOU 5 'Analytical' 6
0.0 0.0
0.1E-3 0.0
0.1E-3 -4.E9
0.3E-3 -4.E9
0.3E-3 0.0
0.4E-3 0.0
TRAC 1 AXES 1.0 'DISPL. [M]'
TRAC 2 AXES 1.0 'DISPL. [M]'
TRAC 3 AXES 1.0 'CONTR. [PA]'
TRAC 4 AXES 1.0 'CONTR. [PA]'
TRAC 5 3 4 AXES 1.0 'CONTR. [PA]' YZER
COLO ROUG NOIR NOIR
DASH 2 0 0
LIST 1 AXES 1.0 'DISPL. [M]'
LIST 2 AXES 1.0 'DISPL. [M]'
LIST 3 AXES 1.0 'CONTR. [PA]'
LIST 4 AXES 1.0 'CONTR. [PA]'
FIN
```

bare13.dgibi

```
*$siz+100
opti echo 1;
opti dime 3 elem cub8;
p1 = 0 0 0;
p2 = 1 0 0;
p3 = 0.5 0 0;
tol = 0.001;
dy = 0.01;
dz = 0.01;
n = 50;
bas11 = (p1 d n p3) coul vert;
bas21 = (p3 d n p2) coul vert;
bas1 = (bas11 tran 1 (0 dy 0)) coul vert;
bas2 = (bas21 tran 1 (0 dy 0)) coul vert;
bar1 = bas1 volu tran 1 (0 0 dz);
bar2 = bas2 volu tran 1 (0 0 dz);
elim tol (bar1 et bar2);
bar = (bar1 et bar2) coul vert;
elim tol bar;
mesh = bar;
tass mesh;
opti sauv form 'bare13.msh';
sauv form mesh;
opti trac psc ftra 'bare13_mesh.ps';
trac cach qual (bar1 et bar2);
fin;
```

bare13.epx

```
BARE13
ECHO
!CONV win
CAST mesh
TRID LAGR
GEOM C81L bar TERM
COMP GROU 2 'e1' LECT bar TERM COND XB GT 0.49
COND XB LT 0.50
'e2' LECT bar TERM COND XB LT 0.51
COND XB GT 0.50
NGRO 1 'end' LECT bar TERM COND X GT 0.999
MATE VM23 RO 8000. YOUN 2.0E11 NU 0.0 ELAS 2.0E11
TRAC 1 2.0E11 1.E0
LECT bar TERM
LINK COUP BLOQ 1 LECT end TERM
```

```

INIT VITE 1 100.0 LECT bar DIFF end TERM
ECRI DEPL VITE CONT ECRO TFREQ 0.1E-3
  FICH ALIC TEMP FREQ 1
    POIN LECT p3 p2 TERM
    ELEM LECT e1 e2 TERM
  FICH ALIC FREQ 1
OPTI PAS UTIL NOTE LOG 1
CALC TINI 0. TEND 4.0E-4 PASP 5.0E-7
SUIT
Post-treatment
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_2' DEPL COMP 1 NOEU LECT p2 TERM
COUR 2 'dx_3' DEPL COMP 1 NOEU LECT p3 TERM
COUR 3 'sg_1' CONT COMP 1 ELEM LECT e1 TERM
COUR 4 'sg_2' CONT COMP 1 ELEM LECT e2 TERM
DCOU 5 'Analytical' 6
  0.0 0.0
  0.1E-3 0.0
  0.1E-3 -4.E9
  0.3E-3 -4.E9
  0.3E-3 0.0
  0.4E-3 0.0
TRAC 1 AXES 1.0 'DISPL. [M]'
TRAC 2 AXES 1.0 'DISPL. [M]'
TRAC 3 AXES 1.0 'CONTR. [PA]'
TRAC 4 AXES 1.0 'CONTR. [PA]'
TRAC 5 3 4 AXES 1.0 'CONTR. [PA]' YZER
  COLO ROUG NOIR NOIR
  DASH 2 0 0
LIST 1 AXES 1.0 'DISPL. [M]'
LIST 2 AXES 1.0 'DISPL. [M]'
LIST 3 AXES 1.0 'CONTR. [PA]'
LIST 4 AXES 1.0 'CONTR. [PA]'
FIN

```

bare14.dgibi

```

*%siz100
opti echo 1;
opti dime 3 elem cub8;
p1 = 0 0 0;
p2 = 1 0 0;
p3 = 0.5 0 0;
tol = 0.001;
dy = 0.01;
dz = 0.01;
n = 50;
bas11 = (p1 d n p3) coul vert;
bas21 = (p3 d n p2) coul vert;
bas1 = (bas11 tran 1 (0 dy 0)) coul vert;
bas2 = (bas21 tran 1 (0 dy 0)) coul vert;
bar1 = bas1 volu tran 1 (0 0 dz);
bar2 = bas2 volu tran 1 (0 0 dz);
elim tol (bar1 et bar2);
bar = (bar1 et bar2) coul vert;
elim tol bar;
mesh = bar;
tass mesh;
opti sauv form 'bare14.msh';
sauv form mesh;
opti trac psc ftra 'bare14_mesh.ps';
trac cach qual (bar1 et bar2);
fin;

```

bare14.epx

```

BARE14
ECHO
!CONV win
CAST mesh
TRID LAGR
GEOM C82L bar TERM
COMP GROU 2 'e1' LECT bar TERM COND XB GT 0.49
  COND XB LT 0.50
  'e2' LECT bar TERM COND XB LT 0.51
  COND XB GT 0.50
  NGRO 1 'end' LECT bar TERM COND X GT 0.999
MATE VM23 RO 8000. YOUN 2.0E11 NU 0.0 ELAS 2.0E11
  TRAC 1 2.0E11 1.E0
  LECT bar TERM
LINK COUP BLOQ 1 LECT end TERM
INIT VITE 1 100.0 LECT bar DIFF end TERM
ECRI DEPL VITE CONT ECRO TFREQ 0.1E-3
  FICH ALIC TEMP FREQ 1
    POIN LECT p3 p2 TERM
    ELEM LECT e1 e2 TERM
  FICH ALIC FREQ 1
OPTI PAS UTIL NOTE LOG 1
CALC TINI 0. TEND 4.0E-4 PASP 5.0E-7
SUIT
Post-treatment
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_2' DEPL COMP 1 NOEU LECT p2 TERM
COUR 2 'dx_3' DEPL COMP 1 NOEU LECT p3 TERM
COUR 3 'sg_1' CONT COMP 1 ELEM LECT e1 TERM
COUR 4 'sg_2' CONT COMP 1 ELEM LECT e2 TERM
DCOU 5 'Analytical' 6
  0.0 0.0
  0.1E-3 0.0
  0.1E-3 -4.E9
  0.3E-3 -4.E9
  0.3E-3 0.0
  0.4E-3 0.0
TRAC 1 AXES 1.0 'DISPL. [M]'
TRAC 2 AXES 1.0 'DISPL. [M]'
TRAC 3 AXES 1.0 'CONTR. [PA]'
TRAC 4 AXES 1.0 'CONTR. [PA]'

```

```

TRAC 5 3 4 AXES 1.0 'CONTR. [PA]' YZER
  COLO ROUG NOIR NOIR
  DASH 2 0 0
LIST 1 AXES 1.0 'DISPL. [M]'
LIST 2 AXES 1.0 'DISPL. [M]'
LIST 3 AXES 1.0 'CONTR. [PA]'
LIST 4 AXES 1.0 'CONTR. [PA]'
FIN

```

bare14c.epx

```

BARE14C
ECHO
RESU ALIC TEMP 'bare14.alt' GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 43 'sg_1' CONT COMP 1 ELEM LECT e1 TERM
COUR 44 'sg_2' CONT COMP 1 ELEM LECT e2 TERM
DCOU 45 'Analytical' 6
  0.0 0.0
  0.1E-3 0.0
  0.1E-3 -4.E9
  0.3E-3 -4.E9
  0.3E-3 0.0
  0.4E-3 0.0
RCOU 13 'sg_1' FICH 'bare11.pun' RENA 'sg_1_11'
RCOU 14 'sg_2' FICH 'bare11.pun' RENA 'sg_2_11'
RCOU 23 'sg_1' FICH 'bare12.pun' RENA 'sg_1_12'
RCOU 24 'sg_2' FICH 'bare12.pun' RENA 'sg_2_12'
RCOU 33 'sg_1' FICH 'bare13.pun' RENA 'sg_1_13'
RCOU 34 'sg_2' FICH 'bare13.pun' RENA 'sg_2_13'
TRAC 13 23 33 43 45 AXES 1.0 'CONTR. [PA]' YZER
  COLO NOIR BLEU TURQ VERT
TRAC 14 24 34 44 45 AXES 1.0 'CONTR. [PA]' YZER
  COLO NOIR BLEU TURQ VERT ROUG
FIN

```

bari23.dgibi

```

opti echo 1;
opti titr 'BARI23';
opti trac psc ftra 'bari23_mesh.ps';
opti dime 2 elem qua4;
p0=0 0;
p1=3.2E-3 0;
p2=0 32.4E-3;
tol=0.01E-3;
base=p0 d 12 p1;
stru=base TRAN 120 p2;
symax=stru poin droi p0 p2 tol;
top=stru poin droi p2 (p1 PLUS p2) tol;
elax=stru ELEM APPU LARG symax;
elot=diff stru elax;
strl=chan 'POI1' stru;
bas1=chan 'POI1' base;
viti=diff strl bas1;
mesh=stru et symax et viti;
tass mesh;
opti sauv form 'bari23.msh';
sauv form mesh;
trac qual mesh;
fin;

```

bari23.epx

```

BARI23
ECHO
!CONV WIN
CAST mesh
LAGR AXIS
GEOM Q41L elax Q41L elot TERM
COMP EPAI 1 LECT tous TERM
MATE VM23 RO 8930. YOUN 1.17D11 NU 0.35D0 ELAS 4.D8
  TRAC 2 4.D8 3.418803D-03 1.004D11 1000.003418803
  LECT tous TERM
LIAI BLOQ 2 LECT base TERM
  CONT SPLA NX 1 NY 0 LECT symax TERM
INIT VITE 2 -227. LECT viti TERM
ECRI DEPL VITE TFRE 10.E-6
  POIN LECT p1 top TERM
FICH ALIC TFRE 10.E-6
FICH ALIC TEMP TFRE 1.E-7
  POIN LECT p1 top TERM
OPTI PART NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0. TEND 80.D-6
FIN

```

bari23a.epx

```

BARI23A
ECHO
CONV WIN
RESU ALIC 'bari23.ali' GARD PSCR
SORT VISU NSTO 9
FIN

```

bari23b.epx

```

Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP 'bari23.alt' GARD PSCR
COMP NGRO 2 'p0b' LECT tous TERM COND NEAR POIN 0.0 3.24E-2
  'p1b' LECT tous TERM COND NEAR POIN 3.2E-3 3.24E-2
SORT GRAP
AXTE 1.0 'Time [s]'

```

```

COUR 1 'dx_p1' DEPL COMP 1 POIN LECT p1 TERM
COUR 2 'dy_p0b' DEPL COMP 2 POIN LECT p0b TERM
COUR 3 'dy_p1b' DEPL COMP 2 POIN LECT p1b TERM
TRAC 1 2 3 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 AXES 1.0 'DISPL. [M]' YZER
FIN

```

bari24.dgibi

```

opti echo 1;
opti titr 'BARI24';
opti trac psc ftra 'bari24_mesh.ps';
opti dime 3 elem cub8;
p0=0 0 0;
p0p=0 0 32.4E-3;
p1=3.2E-3 0 0;
p2=0 3.2E-3 0;
p3=1.6E-3 0 0;
p4=p1 tour 45 p0 p0p;
p5=0 1.6E-3 0;
x3 y3 z3 = coor p3;
x4 y4 z4 = coor p4;
x5 y5 z5 = coor p5;
x6 = (x3 + x4 + x5) / 3.0;
y6 = (y3 + y4 + y5) / 3.0;
z6 = (z3 + z4 + z5) / 3.0;
p6 = x6 y6 z6;
tol=0.01E-3;
n = 2;
nz = 40;
c1 = p0 d n p3;
c2 = p3 d n p6;
c3 = p6 d n p5;
c4 = p5 d n p0;
base1 = dall c1 c2 c3 c4 plan;
c1 = p3 d n p1;
c2 = p1 c n p0 p4;
c3 = p4 d n p6;
c4 = p6 d n p3;
base2 = dall c1 c2 c3 c4 plan;
c1 = p6 d n p4;
c2 = p4 c n p0 p2;
c3 = p2 d n p5;
c4 = p5 d n p6;
base3 = dall c1 c2 c3 c4 plan;
base=base1 et base2 et base3;
elim tol base;
list (nbel base);
list (nbno base);
stru=base volu tran nz p0p;
trac cach qual stru;
*
mesh=stru;
tass mesh;
opti sauv form 'bari24.msh';
sauv form mesh;
fin;

```

bari24.epx

```

BARI24
ECHO
!CONV WIN
CAST mesh
LAGR TRID
GEOM C81L stru TERM
COMP NGRO 5 'blox' LECT tous TERM COND X LT 1.E-5
      'bloy' LECT tous TERM COND Y LT 1.E-5
      'nbas' LECT tous TERM COND Z LT 1.E-5
      'top' LECT tous TERM COND Z GT 32.39E-3
      'viti' LECT tous DIFF nbas TERM
GROU 1 'ebas' LECT tous TERM COND ZB LT 5.E-4
COUL ROSE LECT stru TERM
MATE VM23 RO 8930. YOUN 1.17D11 NU 0.35D0 ELAS 4.D8
      TRAC 2 4.D8 3.418803D-03 1.004D11 1000.003418803
      LECT stru TERM
LIAI BLOQ 3 LECT nbas TERM
      BLOQ 1 LECT blox TERM
      BLOQ 2 LECT bloy TERM
INIT VITE 3 -227. LECT viti TERM
ECRI DEPL VITE TFRE 10.E-6
      POIN LECT p1 p2 top TERM
FICH ALIC TFRE 10.E-6
FICH ALIC TEMP TFRE 1.E-7
      POIN LECT p1 top TERM
OPTI PAS AUTO NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0. TEND 80.D-6
FIN

```

bari24a.epx

```

Post-treatment
ECHO
RESU ALIC 'bari24.ali' GARD PSCR
SORT VISU NSTO 5
FIN

```

bari24b.epx

```

Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP 'bari24.alt' GARD PSCR
COMP NGRO 3 'p0b' LECT tous TERM COND NEAR POIN 0.0 0.0 3.24E-2
      'p1b' LECT tous TERM COND NEAR POIN 3.2E-3 0.0 3.24E-2
      'p2b' LECT tous TERM COND NEAR POIN 0.0 3.2E-3 3.24E-2
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_p1' DEPL COMP 1 POIN LECT p1 TERM
COUR 2 'dz_p0b' DEPL COMP 3 POIN LECT p0b TERM

```

```

COUR 3 'dz_p1b' DEPL COMP 3 POIN LECT p1b TERM
COUR 4 'dz_p2b' DEPL COMP 3 POIN LECT p2b TERM
TRAC 1 2 3 4 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 4 AXES 1.0 'DISPL. [M]' YZER
RCOU 11 'dx_p1' FICH 'bari23b.pun' RENA 'dx_p1_23'
RCOU 12 'dy_p0b' FICH 'bari23b.pun' RENA 'dy_p0b_23'
RCOU 13 'dy_p1b' FICH 'bari23b.pun' RENA 'dy_p1b_23'
TRAC 11 12 13 1 2 3 4 AXES 1.0 'DISPL. [M]' YZER
COLO ROUG ROUG ROUG
      NOIR NOIR NOIR NOIR
FIN

```

bari24f.epx

```

BARI24F
ECHO
CONV WIN
RESU ALIC 'bari24.ali' GARD PSCR
SORT VISU NSTO 4
FIN

```

bari25.dgibi

```

opti echo 1;
opti titr 'BARI25';
opti trac psc ftra 'bari25_mesh.ps';
opti dime 3 elem cub8;
p0=0 0 0;
p0p=0 0 32.4E-3;
p1=3.2E-3 0 0;
p2=0 3.2E-3 0;
p3=1.6E-3 0 0;
p4=p1 tour 45 p0 p0p;
p5=0 1.6E-3 0;
x3 y3 z3 = coor p3;
x4 y4 z4 = coor p4;
x5 y5 z5 = coor p5;
x6 = (x3 + x4 + x5) / 3.0;
y6 = (y3 + y4 + y5) / 3.0;
z6 = (z3 + z4 + z5) / 3.0;
p6 = x6 y6 z6;
tol=0.01E-3;
n = 2;
nz = 40;
c1 = p0 d n p3;
c2 = p3 d n p6;
c3 = p6 d n p5;
c4 = p5 d n p0;
base1 = dall c1 c2 c3 c4 plan;
c1 = p3 d n p1;
c2 = p1 c n p0 p4;
c3 = p4 d n p6;
c4 = p6 d n p3;
base2 = dall c1 c2 c3 c4 plan;
c1 = p6 d n p4;
c2 = p4 c n p0 p2;
c3 = p2 d n p5;
c4 = p5 d n p6;
base3 = dall c1 c2 c3 c4 plan;
base=base1 et base2 et base3;
elim tol base;
list (nbel base);
list (nbno base);
stru=base volu tran nz p0p;
trac cach qual stru;
*
mesh=stru;
tass mesh;
opti sauv form 'bari25.msh';
sauv form mesh;
fin;

```

bari25.epx

```

BARI25
ECHO
!CONV WIN
CAST mesh
LAGR TRID
GEOM C82L stru TERM
COMP NGRO 5 'blox' LECT tous TERM COND X LT 1.E-5
      'bloy' LECT tous TERM COND Y LT 1.E-5
      'nbas' LECT tous TERM COND Z LT 1.E-5
      'top' LECT tous TERM COND Z GT 32.39E-3
      'viti' LECT tous DIFF nbas TERM
GROU 1 'ebas' LECT tous TERM COND ZB LT 5.E-4
COUL ROSE LECT stru TERM
MATE VM23 RO 8930. YOUN 1.17D11 NU 0.35D0 ELAS 4.D8
      TRAC 2 4.D8 3.418803D-03 1.004D11 1000.003418803
      LECT stru TERM
LIAI BLOQ 3 LECT nbas TERM
      BLOQ 1 LECT blox TERM
      BLOQ 2 LECT bloy TERM
INIT VITE 3 -227. LECT viti TERM
ECRI DEPL VITE TFRE 10.E-6
      POIN LECT p1 p2 top TERM
FICH ALIC TFRE 10.E-6
FICH ALIC TEMP TFRE 1.E-7
      POIN LECT p1 top TERM
OPTI PAS AUTO NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0. TEND 80.D-6
FIN

```

bari25a.epx

```

Post-treatment
ECHO
RESU ALIC 'bari25.ali' GARD PSCR
SORT VISU NSTO 9

```

FIN

bari25b.epx

```

Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP 'bari25.alt' GARD PSCR
COMP NGRO 3 'p0b' LECT tous TERM COND NEAR POIN 0.0 0.0 3.24E-2
          'p1b' LECT tous TERM COND NEAR POIN 3.2E-3 0.0 3.24E-2
          'p2b' LECT tous TERM COND NEAR POIN 0.0 3.2E-3 3.24E-2

SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_p1' DEPL COMP 1 POIN LECT p1 TERM
COUR 2 'dz_p0b' DEPL COMP 3 POIN LECT p0b TERM
COUR 3 'dz_p1b' DEPL COMP 3 POIN LECT p1b TERM
COUR 4 'dz_p2b' DEPL COMP 3 POIN LECT p2b TERM
TRAC 1 2 3 4 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 4 AXES 1.0 'DISPL. [M]' YZER
RCOU 11 'dx_p1' FICH 'bari23b.pun' RENA 'dx_p1_23'
RCOU 12 'dy_p0b' FICH 'bari23b.pun' RENA 'dy_p0b_23'
RCOU 13 'dy_p1b' FICH 'bari23b.pun' RENA 'dy_p1b_23'
TRAC 11 12 13 1 2 3 4 AXES 1.0 'DISPL. [M]' YZER
COLO ROUG ROUG ROUG
      NOIR NOIR NOIR NOIR
FIN

```

bari26.dgibi

```

opti echo 1;
opti titr 'BARI26';
opti trac psc ftra 'bari26_mesh.ps';
opti dime 2 elem qua4;
p0=0 0;
p1=3.2E-3 0;
p2=0 32.4E-3;
tol=0.01E-3;
base=p0 d 12 p1;
stru=base TRAN 120 p2;
symax=stru poin droi p0 p2 tol;
top=stru poin droi p2 (p1 PLUS p2) tol;
elax=stru ELEM APPU LARG symax;
elot=diff stru elax;
strl=chan 'POI1' stru;
basl=chan 'POI1' base;
viti=diff strl basl;
mesh=stru et symax et viti;
tass mesh;
opti sauv form 'bari26.msh';
sauv form mesh;
trac qual mesh;
fin;

```

bari26.epx

```

BARI26
ECHO
!CONV WIN
CAST mesh
LAGR AXIS
GEOM Q42L elax Q42L elot TERM
COMP EPAI 1 LECT tous TERM
MATE VM23 RO 8930. YOUN 1.17D11 NU 0.35D0 ELAS 4.D8
      TRAC 2 4.D8 3.418803D-03 1.004D11 1000.003418803
      LECT tous TERM
LIAI BLOQ 2 LECT base TERM
      CONT SPLA NX 1 NY 0 LECT symax TERM
INIT VITE 2 -227. LECT viti TERM
ECRI DEPL VITE TFRE 10.E-6
      POIN LECT p1 top TERM
FICH ALIC TFRE 10.E-6
FICH ALIC TEMP TFRE 1.E-7
      POIN LECT p1 top TERM
OPTI PART NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0. TEND 80.D-6
FIN

```

bari26b.epx

```

Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP 'bari26.alt' GARD PSCR
COMP NGRO 2 'p0b' LECT tous TERM COND NEAR POIN 0.0 3.24E-2
          'p1b' LECT tous TERM COND NEAR POIN 3.2E-3 3.24E-2

SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_p1' DEPL COMP 1 POIN LECT p1 TERM
COUR 2 'dy_p0b' DEPL COMP 2 POIN LECT p0b TERM
COUR 3 'dy_p1b' DEPL COMP 2 POIN LECT p1b TERM
TRAC 1 2 3 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 AXES 1.0 'DISPL. [M]' YZER
RCOU 11 'dx_p1' FICH 'bari23b.pun' RENA 'dx_p1_23'
RCOU 12 'dy_p0b' FICH 'bari23b.pun' RENA 'dy_p0b_23'
RCOU 13 'dy_p1b' FICH 'bari23b.pun' RENA 'dy_p1b_23'
TRAC 11 12 13 1 2 3 AXES 1.0 'DISPL. [M]' YZER
COLO ROUG ROUG ROUG
      NOIR NOIR NOIR
FIN

```

bari30.dgibi

```

opti echo 1;
opti titr 'BARI30';
opti trac psc ftra 'bari30_mesh.ps';
opti dime 2 elem qua4;
p0=0 0;
p1=3.2E-3 0;

```

```

p2=0 32.4E-3;
tol=0.01E-3;
base=p0 d 12 p1;
stru=base TRAN 120 p2;
symax=stru poin droi p0 p2 tol;
top=stru poin droi p2 (p1 PLUS p2) tol;
elax=stru ELEM APPU LARG symax;
elot=diff stru elax;
strl=chan 'POI1' stru;
basl=chan 'POI1' base;
viti=diff strl basl;
mesh=stru et symax et viti;
tass mesh;
opti sauv form 'bari30.msh';
sauv form mesh;
trac qual mesh;
fin;

```

bari30.epx

```

BARI30
ECHO
!CONV WIN
CAST mesh
LAGR AXIS
GEOM Q42L elax Q42L elot TERM
COMP EPAI 1 LECT tous TERM
MATE VM23 RO 8930. YOUN 1.17D11 NU 0.35D0 ELAS 4.D8
      TRAC 2 4.D8 3.418803D-03 1.004D11 1000.003418803
      LECT tous TERM
LIAI BLOQ 2 LECT base TERM
      CONT SPLA NX 1 NY 0 LECT symax TERM
INIT VITE 2 -227. LECT viti TERM
ECRI DEPL VITE TFRE 10.E-6
      POIN LECT p1 top TERM
FICH ALIC TFRE 10.E-6
FICH ALIC TEMP TFRE 1.E-7
      POIN LECT p1 top TERM
OPTI PAS AUTO NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0. TEND 80.D-6
FIN

```

bari30b.epx

```

Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP 'bari30.alt' GARD PSCR
COMP NGRO 2 'p0b' LECT tous TERM COND NEAR POIN 0.0 3.24E-2
          'p1b' LECT tous TERM COND NEAR POIN 3.2E-3 3.24E-2

SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_p1' DEPL COMP 1 POIN LECT p1 TERM
COUR 2 'dy_p0b' DEPL COMP 2 POIN LECT p0b TERM
COUR 3 'dy_p1b' DEPL COMP 2 POIN LECT p1b TERM
TRAC 1 2 3 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 AXES 1.0 'DISPL. [M]' YZER
RCOU 11 'dx_p1' FICH 'bari26b.pun' RENA 'dx_p1_26'
RCOU 12 'dy_p0b' FICH 'bari26b.pun' RENA 'dy_p0b_26'
RCOU 13 'dy_p1b' FICH 'bari26b.pun' RENA 'dy_p1b_26'
TRAC 11 12 13 1 2 3 AXES 1.0 'DISPL. [M]' YZER
COLO ROUG ROUG ROUG
      NOIR NOIR NOIR
FIN

```

bari31.dgibi

```

opti echo 1;
opti titr 'BARI31';
opti trac psc ftra 'bari31_mesh.ps';
opti dime 2 elem qua4;
p0=0 0;
p1=3.2E-3 0;
p2=0 32.4E-3;
tol=0.01E-3;
base=p0 d 12 p1;
stru=base TRAN 120 p2;
symax=stru poin droi p0 p2 tol;
top=stru poin droi p2 (p1 PLUS p2) tol;
elax=stru ELEM APPU LARG symax;
elot=diff stru elax;
strl=chan 'POI1' stru;
basl=chan 'POI1' base;
viti=diff strl basl;
mesh=stru et symax et viti;
tass mesh;
opti sauv form 'bari31.msh';
sauv form mesh;
trac qual mesh;
fin;

```

bari31.epx

```

BARI31
ECHO
!CONV WIN
CAST mesh
LAGR AXIS
GEOM Q41L elax Q41L elot TERM
COMP EPAI 1 LECT tous TERM
MATE VM23 RO 8930. YOUN 1.17D11 NU 0.35D0 ELAS 4.D8
      TRAC 2 4.D8 3.418803D-03 1.004D11 1000.003418803
      LECT tous TERM
LIAI BLOQ 2 LECT base TERM
      CONT SPLA NX 1 NY 0 LECT symax TERM
INIT VITE 2 -227. LECT viti TERM
ECRI DEPL VITE TFRE 10.E-6
      POIN LECT p1 top TERM
FICH ALIC TFRE 10.E-6
FICH ALIC TEMP TFRE 1.E-7

```

```

POIN LECT p1 top TERM
OPTI PAS AUTO NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0. TEND 80.D-6
FIN

```

bari31a.epx

```

BARI31A
ECHO
CONV WIN
RESU ALIC 'bari31.ali' GARD PSCR
SORT VISU NSTO 9
FIN

```

bari31b.epx

```

Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP 'bari31.alt' GARD PSCR
COMP NGRO 2 'p0b' LECT tous TERM COND NEAR POIN 0.0 3.24E-2
          'p1b' LECT tous TERM COND NEAR POIN 3.2E-3 3.24E-2
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_p1' DEPL COMP 1 POIN LECT p1 TERM
COUR 2 'dy_p0b' DEPL COMP 2 POIN LECT p0b TERM
COUR 3 'dy_p1b' DEPL COMP 2 POIN LECT p1b TERM
TRAC 1 2 3 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 AXES 1.0 'DISPL. [M]' YZER
RCOU 11 'dx_p1' FICH 'bari23b.pun' RENA 'dx_p1_23'
RCOU 12 'dy_p0b' FICH 'bari23b.pun' RENA 'dy_p0b_23'
RCOU 13 'dy_p1b' FICH 'bari23b.pun' RENA 'dy_p1b_23'
TRAC 11 12 13 1 2 3 AXES 1.0 'DISPL. [M]' YZER
COLO ROUG ROUG ROUG
      NOIR NOIR NOIR
FIN

```

bari32.dgibi

```

opti echo 1;
opti titr 'BARI32';
opti trac psc ftra 'bari32_mesh.ps';
opti dime 2 elem qua4;
p0=0 0;
p1=3.2E-3 0;
p2=0 32.4E-3;
tol=0.01E-3;
base=p0 d 12 p1;
stru=base TRAN 120 p2;
symax=stru poin droi p0 p2 tol;
top=stru poin droi p2 (p1 PLUS p2) tol;
elax=stru ELEM APPU LARG symax;
elot=diff stru elax;
str1=chan 'POI1' stru;
bas1=chan 'POI1' base;
viti=diff str1 bas1;
mesh=stru et symax et viti;
tass mesh;
opti sauv form 'bari32.msh';
sauv form mesh;
trac qual mesh;
fin;

```

bari32.epx

```

BARI32
ECHO
!CONV WIN
CAST mesh
LAGR AXIS
GEOM Q42L elax Q42L elot TERM
COMP EPAI 1 LECT tous TERM
MATE VM23 RO 8930. YOUN 1.17D11 NU 0.35D0 ELAS 4.D8
      TRAC 2 4.D8 3.418803D-03 1.004D11 1000.003418803
      LECT tous TERM
LINK COUP BLOQ 2 LECT base TERM
      CONT SPLA NX 1 NY 0 LECT symax TERM
INIT VITE 2 -227. LECT viti TERM
ECRI DEPL VITE TPRE 10.E-6
      POIN LECT p1 top TERM
FICH ALIC TPRE 10.E-6
FICH ALIC TEMP TPRE 1.E-7
      POIN LECT p1 top TERM
OPTI PART PLIN NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0. TEND 80.D-6
FIN

```

bari32b.epx

```

Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP 'bari32.alt' GARD PSCR
COMP NGRO 2 'p0b' LECT tous TERM COND NEAR POIN 0.0 3.24E-2
          'p1b' LECT tous TERM COND NEAR POIN 3.2E-3 3.24E-2
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_p1' DEPL COMP 1 POIN LECT p1 TERM
COUR 2 'dy_p0b' DEPL COMP 2 POIN LECT p0b TERM
COUR 3 'dy_p1b' DEPL COMP 2 POIN LECT p1b TERM
TRAC 1 2 3 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 AXES 1.0 'DISPL. [M]' YZER
RCOU 11 'dx_p1' FICH 'bari26b.pun' RENA 'dx_p1_26'
RCOU 12 'dy_p0b' FICH 'bari26b.pun' RENA 'dy_p0b_26'
RCOU 13 'dy_p1b' FICH 'bari26b.pun' RENA 'dy_p1b_26'
TRAC 11 12 13 1 2 3 AXES 1.0 'DISPL. [M]' YZER
COLO ROUG ROUG ROUG
      NOIR NOIR NOIR

```

FIN

bari33.dgibi

```

opti echo 1;
opti titr 'BARI33';
opti trac psc ftra 'bari33_mesh.ps';
opti dime 2 elem qua4;
p0=0 0;
p1=3.2E-3 0;
p2=0 32.4E-3;
tol=0.01E-3;
base=p0 d 12 p1;
stru=base TRAN 120 p2;
symax=stru poin droi p0 p2 tol;
top=stru poin droi p2 (p1 PLUS p2) tol;
elax=stru ELEM APPU LARG symax;
elot=diff stru elax;
str1=chan 'POI1' stru;
bas1=chan 'POI1' base;
viti=diff str1 bas1;
mesh=stru et symax et viti;
tass mesh;
opti sauv form 'bari33.msh';
sauv form mesh;
trac qual mesh;
fin;

```

bari33.epx

```

BARI33
ECHO
!CONV WIN
CAST mesh
LAGR AXIS
GEOM Q41L elax Q41L elot TERM
COMP EPAI 1 LECT tous TERM
MATE VM23 RO 8930. YOUN 1.17D11 NU 0.35D0 ELAS 4.D8
      TRAC 2 4.D8 3.418803D-03 1.004D11 1000.003418803
      LECT tous TERM
LINK COUP BLOQ 2 LECT base TERM
      CONT SPLA NX 1 NY 0 LECT symax TERM
INIT VITE 2 -227. LECT viti TERM
ECRI DEPL VITE TPRE 10.E-6
      POIN LECT p1 top TERM
FICH ALIC TPRE 10.E-6
FICH ALIC TEMP TPRE 1.E-7
      POIN LECT p1 top TERM
OPTI PART PLIN NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0. TEND 80.D-6
FIN

```

bari33a.epx

```

BARI33A
ECHO
CONV WIN
RESU ALIC 'bari33.ali' GARD PSCR
SORT VISU NSTO 9
FIN

```

bari33b.epx

```

Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP 'bari33.alt' GARD PSCR
COMP NGRO 2 'p0b' LECT tous TERM COND NEAR POIN 0.0 3.24E-2
          'p1b' LECT tous TERM COND NEAR POIN 3.2E-3 3.24E-2
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_p1' DEPL COMP 1 POIN LECT p1 TERM
COUR 2 'dy_p0b' DEPL COMP 2 POIN LECT p0b TERM
COUR 3 'dy_p1b' DEPL COMP 2 POIN LECT p1b TERM
TRAC 1 2 3 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 AXES 1.0 'DISPL. [M]' YZER
RCOU 11 'dx_p1' FICH 'bari23b.pun' RENA 'dx_p1_23'
RCOU 12 'dy_p0b' FICH 'bari23b.pun' RENA 'dy_p0b_23'
RCOU 13 'dy_p1b' FICH 'bari23b.pun' RENA 'dy_p1b_23'
TRAC 11 12 13 1 2 3 AXES 1.0 'DISPL. [M]' YZER
COLO ROUG ROUG ROUG
      NOIR NOIR NOIR
FIN

```

pou201.dgibi

```

opti titr 'pou201';
opti echo 1;
opti dime 3 elem cub8;
opti trac psc ftra 'pou201_mesh.ps';
p1=0 0 0;p2=10 0 0;
p3=10 1 0;p4=0 1 0;
p04 = 0 0.5 0.00;
p14 = 0 0.5 0.25;
p24 = 0 0.5 0.50;
p34 = 0 0.5 0.75;
p44 = 0 0.5 1.00;
tol=0.001;
*
* define structural mesh
*
c1=p1 d 40 p2;
c2=p2 d 4 p3;
c3=p3 d 40 p4;
c4=p4 d 4 p1;
base=daller c1 c2 c3 c4 plan;
pout=volu base tran 4 (0 0 1);

```

```

*
presslow = base;
baseup = base plus (0 0 1);
baseup = orie baseup poin (5 0.5 0);
*
pressup = baseup;
elim tol (pout et pressup);
elim tol (pout et p04 et p14 et p24 et p34 et p44);
*
pression=presslow et pressup;
*
* define total mesh object and eliminate double points
*
mesh = pout et pression;
elim tol mesh;
tass mesh;
*
* store the results
*
opti sauv form 'pou201.msh';
sauv form mesh;
trac cach qual pout;
trac cach qual (base et baseup);
fin;

```

pou201.epx

```

POU201
ECHO
!CONV WIN
CAST mesh
LAGR TRID
GEOM C81L pout CL3Q pression TERM
COMP NGRO 1 'bloxz' LECT tous TERM
COND X GT 9.99
MATE VM23 RO 1.030E-6 YOUN 1.2E4 NU 0.2 ELAS 1.32E4
TRAC 1 1.32E4 1.1
LECT pout TERM
IMPE PIMP RO 0 PREF 0 PRES -1.425
TABP 2 0.0 1.0 1.0 1.0
LECT pressup TERM
IMPE PIMP RO 0 PREF 0 PRES 1.425
TABP 2 0.0 1.0 1.0 1.0
LECT presslow TERM
LINK COUP BLOQ 2 LECT tous TERM
BLOQ 13 LECT bloxz TERM
ECRI DEPL VITE CONT TPRE 3.D-3
FICH ALIC TPRE 1.E-3
FICH ALIC TEMP TPRE 1.E-5
POIN LECT p04 p14 p24 p34 p44 TERM
OPTI PAS AUTO NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0.0 TEND 6.E-3
SUIT
Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dz_p04' DEPL COMP 3 NOEU LECT p04 TERM
COUR 2 'dz_p14' DEPL COMP 3 NOEU LECT p14 TERM
COUR 3 'dz_p24' DEPL COMP 3 NOEU LECT p24 TERM
COUR 4 'dz_p34' DEPL COMP 3 NOEU LECT p34 TERM
COUR 5 'dz_p44' DEPL COMP 3 NOEU LECT p44 TERM
TRAC 1 2 3 4 5 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 4 5 AXES 1.0 'DISPL. [M]'
FIN

```

pou201b.epx

```

POU201B
ECHO
CONV WIN
RESU ALIC 'pou201.ali' GARD PSCR
SORT VISU NSTO 4
FIN

```

pou202.dgibi

```

opti titr 'pou202';
opti echo 1;
opti dime 3 elem cub8;
opti trac psc ftra 'pou202_mesh.ps';
p1=0 0 0;p2=10 0 0;
p3=10 1 0;p4=0 1 0;
p04 = 0 0.5 0.00;
p14 = 0 0.5 0.25;
p24 = 0 0.5 0.50;
p34 = 0 0.5 0.75;
p44 = 0 0.5 1.00;
tol=0.001;
*
* define structural mesh
*
c1=p1 d 40 p2;
c2=p2 d 4 p3;
c3=p3 d 40 p4;
c4=p4 d 4 p1;
base=daller c1 c2 c3 c4 plan;
pout=volu base tran 4 (0 0 1);
*
presslow = base;
baseup = base plus (0 0 1);
baseup = orie baseup poin (5 0.5 0);
*
pressup = baseup;
elim tol (pout et pressup);
elim tol (pout et p04 et p14 et p24 et p34 et p44);
*
pression=presslow et pressup;
*
* define total mesh object and eliminate double points

```

```

*
mesh = pout et pression;
elim tol mesh;
tass mesh;
*
* store the results
*
opti sauv form 'pou202.msh';
sauv form mesh;
trac cach qual pout;
trac cach qual (base et baseup);
fin;

```

pou202.epx

```

POU202
ECHO
!CONV WIN
CAST mesh
LAGR TRID
GEOM C82L pout CL3Q pression TERM
COMP NGRO 1 'bloxz' LECT tous TERM
COND X GT 9.99
MATE VM23 RO 1.030E-6 YOUN 1.2E4 NU 0.2 ELAS 1.32E4
TRAC 1 1.32E4 1.1
LECT pout TERM
IMPE PIMP RO 0 PREF 0 PRES -1.425
TABP 2 0.0 1.0 1.0 1.0
LECT pressup TERM
IMPE PIMP RO 0 PREF 0 PRES 1.425
TABP 2 0.0 1.0 1.0 1.0
LECT presslow TERM
LINK COUP BLOQ 2 LECT tous TERM
BLOQ 13 LECT bloxz TERM
ECRI DEPL VITE CONT TPRE 3.D-3
FICH ALIC TPRE 1.E-3
FICH ALIC TEMP TPRE 1.E-5
POIN LECT p04 p14 p24 p34 p44 TERM
OPTI PAS AUTO NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0.0 TEND 6.E-3
SUIT
Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dz_p04' DEPL COMP 3 NOEU LECT p04 TERM
COUR 2 'dz_p14' DEPL COMP 3 NOEU LECT p14 TERM
COUR 3 'dz_p24' DEPL COMP 3 NOEU LECT p24 TERM
COUR 4 'dz_p34' DEPL COMP 3 NOEU LECT p34 TERM
COUR 5 'dz_p44' DEPL COMP 3 NOEU LECT p44 TERM
TRAC 1 2 3 4 5 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 4 5 AXES 1.0 'DISPL. [M]'
FIN

```

pou202b.epx

```

POU202B
ECHO
CONV WIN
RESU ALIC 'pou202.ali' GARD PSCR
SORT VISU NSTO 4
FIN

```

pou203.dgibi

```

opti titr 'pou203';
opti echo 1;
opti dime 3 elem cub8;
opti trac psc ftra 'pou203_mesh.ps';
p1=0 0 0;p2=10 0 0;
p3=10 1 0;p4=0 1 0;
p04 = 0 0.5 0.00;
p14 = 0 0.5 0.25;
p24 = 0 0.5 0.50;
p34 = 0 0.5 0.75;
p44 = 0 0.5 1.00;
tol=0.001;
*
* define structural mesh
*
c1=p1 d 40 p2;
c2=p2 d 4 p3;
c3=p3 d 40 p4;
c4=p4 d 4 p1;
base=daller c1 c2 c3 c4 plan;
pout=volu base tran 4 (0 0 1);
*
presslow = base;
baseup = base plus (0 0 1);
baseup = orie baseup poin (5 0.5 0);
*
pressup = baseup;
elim tol (pout et pressup);
elim tol (pout et p04 et p14 et p24 et p34 et p44);
*
pression=presslow et pressup;
*
* define total mesh object and eliminate double points
*
mesh = pout et pression;
depl mesh tour 90 p2 p1;
elim tol mesh;
tass mesh;
*
* store the results
*
opti sauv form 'pou203.msh';
sauv form mesh;
trac cach qual pout;
trac cach qual (base et baseup);

```

```
fin;
```

pou203 .epx

```
POU203
ECHO
!CONV WIN
CAST mesh
LAGR TRID
GEOM C81L pout CL3Q pression TERM
COMP NGRO 1 'bloxy' LECT tous TERM
COND X GT 9.99
MATE VM23 RO 1.030E-6 YOUN 1.2E4 NU 0.2 ELAS 1.32E4
TRAC 1 1.32E4 1.1
LECT pout TERM
IMPE PIMP RO 0 PREF 0 PRES -1.425
TABP 2 0.0 1.0 1.0 1.0
LECT pressup TERM
IMPE PIMP RO 0 PREF 0 PRES 1.425
TABP 2 0.0 1.0 1.0 1.0
LECT presslow TERM
LINK COUP BLOQ 3 LECT tous TERM
BLOQ 12 LECT bloxy TERM
ECRI DEPL VITE CONT TFRE 3.D-3
FICH ALIC TFRE 1.E-3
FICH ALIC TEMP TFRE 1.E-5
POIN LECT p04 p14 p24 p34 p44 TERM
OPTI PAS AUTO NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0.0 TEND 6.E-3
SUIT
Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dy_p04' DEPL COMP 2 NOEU LECT p04 TERM
COUR 2 'dy_p14' DEPL COMP 2 NOEU LECT p14 TERM
COUR 3 'dy_p24' DEPL COMP 2 NOEU LECT p24 TERM
COUR 4 'dy_p34' DEPL COMP 2 NOEU LECT p34 TERM
COUR 5 'dy_p44' DEPL COMP 2 NOEU LECT p44 TERM
TRAC 1 2 3 4 5 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 4 5 AXES 1.0 'DISPL. [M]'
FIN
```

pou203b.epx

```
POU203B
ECHO
CONV WIN
RESU ALIC 'pou203.ali' GARD PSCR
SORT VISU NSTO 4
FIN
```

pou204.dgibi

```
opti titr 'pou204';
opti echo 1;
opti dime 3 elem cub8;
opti trac psc ftra 'pou204_mesh.ps';
p1=0 0 0;p2=10 0 0;
p3=10 1 0;p4=0 1 0;
p04 = 0 0.5 0.00;
p14 = 0 0.5 0.25;
p24 = 0 0.5 0.50;
p34 = 0 0.5 0.75;
p44 = 0 0.5 1.00;
tol=0.001;
*
* define structural mesh
*
c1=p1 d 40 p2;
c2=p2 d 4 p3;
c3=p3 d 40 p4;
c4=p4 d 4 p1;
base=daller c1 c2 c3 c4 plan;
pout=volu base tran 4 (0 0 1);
*
presslow = base;
baseup = base plus (0 0 1);
baseup = orie baseup poin (5 0.5 0);
*
pressup = baseup;
elim tol (pout et pressup);
elim tol (pout et p04 et p14 et p24 et p34 et p44);
*
pression=presslow et pressup;
*
* define total mesh object and eliminate double points
*
mesh = pout et pression;
depl mesh tour 90 p1 p4;
elim tol mesh;
tass mesh;
*
* store the results
*
opti sauv form 'pou204.msh';
sauv form mesh;
trac cach qual pout;
trac cach qual (base et baseup);
fin;
```

pou204 .epx

```
POU204
ECHO
!CONV WIN
CAST mesh
LAGR TRID
```

```
GEOM C81L pout CL3Q pression TERM
COMP NGRO 1 'bloxz' LECT tous TERM
COND Z LT -9.99
MATE VM23 RO 1.030E-6 YOUN 1.2E4 NU 0.2 ELAS 1.32E4
TRAC 1 1.32E4 1.1
LECT pout TERM
IMPE PIMP RO 0 PREF 0 PRES -1.425
TABP 2 0.0 1.0 1.0 1.0
LECT pressup TERM
IMPE PIMP RO 0 PREF 0 PRES 1.425
TABP 2 0.0 1.0 1.0 1.0
LECT presslow TERM
LINK COUP BLOQ 2 LECT tous TERM
BLOQ 13 LECT bloxz TERM
ECRI DEPL VITE CONT TFRE 3.D-3
FICH ALIC TFRE 1.E-3
FICH ALIC TEMP TFRE 1.E-5
POIN LECT p04 p14 p24 p34 p44 TERM
OPTI PAS AUTO NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0.0 TEND 6.E-3
SUIT
Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_p04' DEPL COMP 1 NOEU LECT p04 TERM
COUR 2 'dx_p14' DEPL COMP 1 NOEU LECT p14 TERM
COUR 3 'dx_p24' DEPL COMP 1 NOEU LECT p24 TERM
COUR 4 'dx_p34' DEPL COMP 1 NOEU LECT p34 TERM
COUR 5 'dx_p44' DEPL COMP 1 NOEU LECT p44 TERM
TRAC 1 2 3 4 5 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 4 5 AXES 1.0 'DISPL. [M]'
FIN
```

pou204b.epx

```
POU204B
ECHO
CONV WIN
RESU ALIC 'pou204.ali' GARD PSCR
SORT VISU NSTO 4
FIN
```

pou204c.epx

```
POU204C
ECHO
RESU ALIC TEMP 'pou204.alt' GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 43 'dx_p24' DEPL COMP 1 NOEU LECT p24 TERM
RCOU 13 'dz_p24' FICH 'pou201.pun' RENA 'dz_p24_201'
RCOU 23 'dz_p24' FICH 'pou202.pun' RENA 'dz_p24_202'
RCOU 33 'dy_p24' FICH 'pou203.pun' RENA 'dy_p24_203'
RCOU 44 'dy_p24' FICH 'pout54.pun' RENA 'dy_p24_54'
TRAC 13 23 33 43 44 AXES 1.0 'DISPL. [M]' YZER
COLO NOIR TURQ VERT ROUG ROSE
FIN
```

pout54.dgibi

```
opti titr 'pout54';
opti echo 1;
opti dime 2 elem qua4;
opti trac psc ftra 'pout54_mesh.ps';
p1=0 0;p2=10 0;
p3=10 1;p4=0 1;
p14 = 0 0.25;
p24 = 0 0.50;
p34 = 0 0.75;
tol=0.001;
*
* define structural mesh
*
c1=p1 d 40 p2;
c2=p2 d 4 p3;
c3=p3 d 40 p4;
c4=p4 d 4 p1;
pout=daller c1 c2 c3 c4 plan;
elim tol (pout et p14 et p24 et p34);
*
* define nodes subjected to blockages or TPLOT
*
bloc=pout poin droi p2 p3 tol;
tpln=pout poin droi p1 p4 tol;
*
* define applied pressure elements
*
opti elem seg2;
presslow=p1 d 40 p2;
pressup=p3 d 40 p4;
pression=presslow et pressup;
*
* define total mesh object and eliminate double points
*
mesh=pout et pression et bloc et tpln;
elim tol mesh;
tass mesh;
*
* store the results
*
opti sauv form 'pout54.msh';
sauv form mesh;
trac qual mesh;
fin;
```


pout54.epx

```

POUT54 - 2D Q42L+CL22 DOUBLE LAYER
ECHO
!CONV WIN
CAST mesh
LAGR DPLA
GEOM Q42L pout CL22 pression TERM
COMP EPAI 1.0 LECT pout TERM
MATE VM23 RO 1.030E-6 YOUN 1.2E4 NU 0.2 ELAS 1.32E4
      TRAC 1 1.32E4 1.1
      LECT pout TERM
      IMPE PIMP RO 0 PREF 0 PRES -1.425
      TABP 2 0.0 1.0 1.0 1.0
      LECT pressup TERM
      IMPE PIMP RO 0 PREF 0 PRES 1.425
      TABP 2 0.0 1.0 1.0 1.0
      LECT presslow TERM
LIAI BLOQ 12 LECT bloc TERM
ECRI DEPL VITE CONT TPRE 3.D-3
FICH ALIC TPRE 1.E-3
FICH ALIC TEMP TPRE 1.E-5
      POIN LECT tpln TERM
OPTI PAS AUTO NOTE CSTA 0.5 STEP IO LOG 1
CALC TINI 0.0 TEND 6.D-3
SUIT
Post-treatment (time curves from alice temps file)
ECHO
RESU ALIC TEMP GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dy_p1' DEPL COMP 2 NOEU LECT p1 TERM
COUR 2 'dy_p14' DEPL COMP 2 NOEU LECT p14 TERM
COUR 3 'dy_p24' DEPL COMP 2 NOEU LECT p24 TERM
COUR 4 'dy_p34' DEPL COMP 2 NOEU LECT p34 TERM
COUR 5 'dy_p4' DEPL COMP 2 NOEU LECT p4 TERM
TRAC 1 2 3 4 5 AXES 1.0 'DISPL. [M]' YZER
LIST 1 2 3 4 5 AXES 1.0 'DISPL. [M]'
FIN

```

pout54b.epx

```

POUT54B
ECHO
CONV WIN
RESU ALIC 'pout54.ali' GARD PSCR
SORT VISU NSTO 4
FIN

```

shea11.epx

```

SHEA11
ECHO
!CONV WIN
LAGR DPLA
GEOM LIBR POIN 4 Q41L 1 TERM
      0 0 1 0 1 1 0 1
      1 2 3 4
COMP EPAI 1.0 LECT 1 TERM
MATE VM23 RO 8000 YOUN 2.E11 NU 0.3 ELAS 2.E12
      TRAC 1 2.E12 10
      LECT 1 TERM
LINK VITE 1 10 FONC 1 LECT 3 4 TERM
      BLOQ 2 LECT tous TERM
      BLOQ 1 LECT 1 2 TERM
FONC NUM 1 TABL 2 0 1 10 1
INIT VITE 1 10 LECT 3 4 TERM
ECRI DEPL VITE CONT ECRO TPRE 100.D-3
FICH ALIC FREQ 10
      POIN LECT tous TERM
      ELEM LECT tous TERM
OPTI PAS UTIL NOTE CSTA 0.5 LOG 1
CALC TINI 0.0 TEND 1000.D-3 PASF 2.E-5
SUIT
Post-treatment (time curves from alice file)
ECHO
RESU ALIC GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_3' DEPL COMP 1 NOEU LECT 3 TERM
COUR 3 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 5 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 7 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 9 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
TRAC 1 AXES 1.0 'DISPL. [M]' YZER
TRAC 3 5 7 9 AXES 1.0 'CONT. [PA]' YZER
LIST 1 AXES 1.0 'DISPL. [M]' YZER
LIST 3 5 7 9 AXES 1.0 'CONT. [PA]' YZER
FIN

```

shea12.epx

```

SHEA12
ECHO
!CONV WIN
LAGR DPLA
GEOM LIBR POIN 4 Q42L 1 TERM
      0 0 1 0 1 1 0 1
      1 2 3 4
COMP EPAI 1.0 LECT 1 TERM
MATE VM23 RO 8000 YOUN 2.E11 NU 0.3 ELAS 2.E12
      TRAC 1 2.E12 10
      LECT 1 TERM
LINK VITE 1 10 FONC 1 LECT 3 4 TERM
      BLOQ 2 LECT tous TERM
      BLOQ 1 LECT 1 2 TERM
FONC NUM 1 TABL 2 0 1 10 1
INIT VITE 1 10 LECT 3 4 TERM
ECRI DEPL VITE CONT ECRO TPRE 100.D-3

```

```

FICH ALIC FREQ 10
      POIN LECT tous TERM
      ELEM LECT tous TERM
OPTI PAS UTIL NOTE CSTA 0.5 LOG 1
CALC TINI 0.0 TEND 1000.D-3 PASF 2.E-5
SUIT
Post-treatment (time curves from alice file)
ECHO
RESU ALIC GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 1 'dx_3' DEPL COMP 1 NOEU LECT 3 TERM
COUR 3 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 4 'sx_14' CONT COMP 1 GAUS 4 ELEM LECT 1 TERM
COUR 5 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 6 'sy_14' CONT COMP 2 GAUS 4 ELEM LECT 1 TERM
COUR 7 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 8 'txy_14' CONT COMP 3 GAUS 4 ELEM LECT 1 TERM
COUR 9 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 10 'sz_14' CONT COMP 4 GAUS 4 ELEM LECT 1 TERM
TRAC 1 AXES 1.0 'DISPL. [M]' YZER
TRAC 3 4 5 6 7 8 9 10 AXES 1.0 'CONT. [PA]' YZER
LIST 1 AXES 1.0 'DISPL. [M]' YZER
LIST 3 4 5 6 7 8 9 10 AXES 1.0 'CONT. [PA]' YZER
FIN

```

shea12b.epx

```

Post-treatment (time curves from alice file)
ECHO
RESU ALIC 'shea12.ali' GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 33 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
RCOU 13 'txy_11' FICH 'shea11.pun' RENA 'txy_11_11'
TRAC 33 13 AXES 1.0 'CONT. [PA]' YZER
COLO NOIR ROUG
FIN

```

shea13.epx

```

SHEA13
ECHO
!CONV WIN
LAGR TRID
GEOM LIBR POIN 8 C81L 1 TERM
      0 0 0 1 0 0 1 1 0 0 1 0
      0 0 1 1 0 1 1 1 1 0 1 1
      1 2 3 4 5 6 7 8
MATE VM23 RO 8000 YOUN 2.E11 NU 0.3 ELAS 2.E12
      TRAC 1 2.E12 10
      LECT 1 TERM
LINK VITE 1 10 FONC 1 LECT 5 6 7 TERM
      BLOQ 23 LECT tous TERM
      BLOQ 1 LECT 1 2 4 3 TERM
FONC NUM 1 TABL 2 0 1 10 1
INIT VITE 1 10 LECT 5 6 7 TERM
ECRI DEPL VITE CONT ECRO TPRE 100.D-3
FICH ALIC FREQ 10
      POIN LECT tous TERM
      ELEM LECT tous TERM
OPTI PAS UTIL NOTE CSTA 0.5 LOG 1
CALC TINI 0.0 TEND 1000.D-3 PASF 2.E-5
SUIT
Post-treatment (time curves from alice file)
ECHO
RESU ALIC GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 2 'dx_5' DEPL COMP 1 NOEU LECT 5 TERM
COUR 3 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 5 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 7 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 9 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 11 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 13 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
TRAC 2 AXES 1.0 'DISPL. [M]' YZER
TRAC 3 5 7 9 11 13 AXES 1.0 'CONT. [PA]' YZER
LIST 2 AXES 1.0 'DISPL. [M]' YZER
LIST 3 5 7 9 11 13 AXES 1.0 'CONT. [PA]' YZER
FIN

```

shea13b.epx

```

Post-treatment (time curves from alice file)
ECHO
RESU ALIC 'shea13.ali' GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 31 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 32 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 33 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 34 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 35 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 36 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
TRAC 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
RCOU 11 'sx_11' FICH 'shea11.pun' RENA 'sx_11_11'
RCOU 12 'sy_11' FICH 'shea11.pun' RENA 'sy_11_11'
RCOU 13 'txy_11' FICH 'shea11.pun' RENA 'txy_11_11'
RCOU 14 'sz_11' FICH 'shea11.pun' RENA 'sz_11_11'
TRAC 11 12 13 14 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
COLO NOIR NOIR NOIR NOIR
      ROUG ROUG ROUG ROUG ROUG ROUG
FIN

```

shea14.epx

```

SHEA14
ECHO

```

```

!CONV WIN
LAGR TRID
GEOM LIBR POIN 8 C82L 1 TERM
  0 0 0 1 0 0 1 1 0 0 1 0
  0 0 1 1 0 1 1 1 1 0 1 1
  1 2 3 4 5 6 7 8
MATE VM23 RO 8000 YOUN 2.E11 NU 0.3 ELAS 2.E12
  TRAC 1 2.E12 10
  LECT 1 TERM
LINK VITE 1 10 PONC 1 LECT 5 6 8 7 TERM
  BLOQ 23 LECT tous TERM
  BLOQ 1 LECT 1 2 4 3 TERM
PONC NUM 1 TABL 2 0 1 10 1
INIT VITE 1 10 LECT 5 6 8 7 TERM
ECRI DEPL VITE CONT ECRO TPRE 100.D-3
FICH ALIC FREQ 10
  POIN LECT tous TERM
  ELEM LECT tous TERM
OPTI PAS UTIL NOTE CSTA 0.5 LOG 1
CALC TINI 0.0 TEND 1000.D-3 PASF 2.E-5
SUIT
Post-treatment (time curves from alice file)
ECHO
RESU ALIC GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 2 'dx_5' DEPL COMP 1 NOEU LECT 5 TERM
COUR 3 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 4 'sx_18' CONT COMP 1 GAUS 8 ELEM LECT 1 TERM
COUR 5 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 6 'sy_18' CONT COMP 2 GAUS 8 ELEM LECT 1 TERM
COUR 7 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 8 'txy_18' CONT COMP 3 GAUS 8 ELEM LECT 1 TERM
COUR 9 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 10 'sz_18' CONT COMP 4 GAUS 8 ELEM LECT 1 TERM
COUR 11 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 12 'tyz_18' CONT COMP 5 GAUS 8 ELEM LECT 1 TERM
COUR 13 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
COUR 14 'txz_18' CONT COMP 6 GAUS 8 ELEM LECT 1 TERM
TRAC 2 AXES 1.0 'DISPL. [M]' YZER
TRAC 3 4 5 6 7 8 9 10 11 12 13 AXES 1.0 'CONT. [PA]' YZER
LIST 2 AXES 1.0 'DISPL. [M]' YZER
LIST 3 4 5 6 7 8 9 10 11 12 13 AXES 1.0 'CONT. [PA]' YZER
FIN

```

sheal4b.epx

```

Post-treatment (time curves from alice file)
ECHO
RESU ALIC 'sheal4.ali' GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 31 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 32 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 33 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 34 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 35 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 36 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
TRAC 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
RCOU 11 'sx_11' FICH 'sheal3.pun' RENA 'sx_11_13'
RCOU 12 'sy_11' FICH 'sheal3.pun' RENA 'sy_11_13'
RCOU 13 'txy_11' FICH 'sheal3.pun' RENA 'txy_11_13'
RCOU 14 'sz_11' FICH 'sheal3.pun' RENA 'sz_11_13'
RCOU 15 'tyz_11' FICH 'sheal3.pun' RENA 'tyz_11_13'
RCOU 16 'txz_11' FICH 'sheal3.pun' RENA 'txz_11_13'
TRAC 11 12 13 14 15 16 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
COLO NOIR NOIR NOIR NOIR NOIR NOIR
  ROUG ROUG ROUG ROUG ROUG ROUG
FIN

```

sheal5.epx

```

SHEA15
ECHO
!CONV WIN
LAGR TRID
GEOM LIBR POIN 8 C81L 1 TERM
  0 0 0 1 0 0 1 1 0 0 1 0
  0 0 1 1 0 1 1 1 1 0 1 1
  1 2 3 4 5 6 7 8
MATE VM23 RO 8000 YOUN 2.E11 NU 0.3 ELAS 2.E12
  TRAC 1 2.E12 10
  LECT 1 TERM
LINK VITE 2 10 PONC 1 LECT 5 6 8 7 TERM
  BLOQ 13 LECT tous TERM
  BLOQ 2 LECT 1 2 4 3 TERM
PONC NUM 1 TABL 2 0 1 10 1
INIT VITE 2 10 LECT 5 6 8 7 TERM
ECRI DEPL VITE CONT ECRO TPRE 100.D-3
FICH ALIC FREQ 10
  POIN LECT tous TERM
  ELEM LECT tous TERM
OPTI PAS UTIL NOTE CSTA 0.5 LOG 1
CALC TINI 0.0 TEND 1000.D-3 PASF 2.E-5
SUIT
Post-treatment (time curves from alice file)
ECHO
RESU ALIC GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 2 'dy_5' DEPL COMP 2 NOEU LECT 5 TERM
COUR 3 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 5 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 7 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 9 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 11 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 13 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
TRAC 2 AXES 1.0 'DISPL. [M]' YZER
TRAC 3 5 7 9 11 13 AXES 1.0 'CONT. [PA]' YZER
LIST 2 AXES 1.0 'DISPL. [M]' YZER
LIST 3 5 7 9 11 13 AXES 1.0 'CONT. [PA]' YZER
FIN

```

sheal5b.epx

```

Post-treatment (time curves from alice file)
ECHO
RESU ALIC 'sheal5.ali' GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 31 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 32 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 33 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 34 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 35 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 36 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
TRAC 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
RCOU 11 'sx_11' FICH 'sheal3.pun' RENA 'sx_11_13'
RCOU 12 'sy_11' FICH 'sheal3.pun' RENA 'sy_11_13'
RCOU 13 'txy_11' FICH 'sheal3.pun' RENA 'txy_11_13'
RCOU 14 'sz_11' FICH 'sheal3.pun' RENA 'sz_11_13'
RCOU 15 'tyz_11' FICH 'sheal3.pun' RENA 'tyz_11_13'
RCOU 16 'txz_11' FICH 'sheal3.pun' RENA 'txz_11_13'
TRAC 11 12 13 14 15 16 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
COLO NOIR NOIR NOIR NOIR NOIR NOIR
  ROUG ROUG ROUG ROUG ROUG ROUG
FIN

```

sheal6.epx

```

SHEA16
ECHO
!CONV WIN
LAGR TRID
GEOM LIBR POIN 8 C82L 1 TERM
  0 0 0 1 0 0 1 1 0 0 1 0
  0 0 1 1 0 1 1 1 1 0 1 1
  1 2 3 4 5 6 7 8
MATE VM23 RO 8000 YOUN 2.E11 NU 0.3 ELAS 2.E12
  TRAC 1 2.E12 10
  LECT 1 TERM
LINK VITE 2 10 PONC 1 LECT 5 6 8 7 TERM
  BLOQ 13 LECT tous TERM
  BLOQ 2 LECT 1 2 4 3 TERM
PONC NUM 1 TABL 2 0 1 10 1
INIT VITE 2 10 LECT 5 6 8 7 TERM
ECRI DEPL VITE CONT ECRO TPRE 100.D-3
FICH ALIC FREQ 10
  POIN LECT tous TERM
  ELEM LECT tous TERM
OPTI PAS UTIL NOTE CSTA 0.5 LOG 1
CALC TINI 0.0 TEND 1000.D-3 PASF 2.E-5
SUIT
Post-treatment (time curves from alice file)
ECHO
RESU ALIC GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 2 'dy_5' DEPL COMP 2 NOEU LECT 5 TERM
COUR 3 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 4 'sx_18' CONT COMP 1 GAUS 8 ELEM LECT 1 TERM
COUR 5 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 6 'sy_18' CONT COMP 2 GAUS 8 ELEM LECT 1 TERM
COUR 7 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 8 'txy_18' CONT COMP 3 GAUS 8 ELEM LECT 1 TERM
COUR 9 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 10 'sz_18' CONT COMP 4 GAUS 8 ELEM LECT 1 TERM
COUR 11 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 12 'tyz_18' CONT COMP 5 GAUS 8 ELEM LECT 1 TERM
COUR 13 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
COUR 14 'txz_18' CONT COMP 6 GAUS 8 ELEM LECT 1 TERM
TRAC 2 AXES 1.0 'DISPL. [M]' YZER
TRAC 3 4 5 6 7 8 9 10 11 12 13 AXES 1.0 'CONT. [PA]' YZER
LIST 2 AXES 1.0 'DISPL. [M]' YZER
LIST 3 4 5 6 7 8 9 10 11 12 13 AXES 1.0 'CONT. [PA]' YZER
FIN

```

sheal6b.epx

```

Post-treatment (time curves from alice file)
ECHO
RESU ALIC 'sheal6.ali' GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 31 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 32 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 33 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 34 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 35 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 36 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
TRAC 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
RCOU 11 'sx_11' FICH 'sheal5.pun' RENA 'sx_11_15'
RCOU 12 'sy_11' FICH 'sheal5.pun' RENA 'sy_11_15'
RCOU 13 'txy_11' FICH 'sheal5.pun' RENA 'txy_11_15'
RCOU 14 'sz_11' FICH 'sheal5.pun' RENA 'sz_11_15'
RCOU 15 'tyz_11' FICH 'sheal5.pun' RENA 'tyz_11_15'
RCOU 16 'txz_11' FICH 'sheal5.pun' RENA 'txz_11_15'
TRAC 11 12 13 14 15 16 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
COLO NOIR NOIR NOIR NOIR NOIR NOIR
  ROUG ROUG ROUG ROUG ROUG ROUG
FIN

```

sheal7.epx

```

SHEA17
ECHO
!CONV WIN
LAGR TRID
GEOM LIBR POIN 8 C81L 1 TERM
  0 0 0 1 0 0 1 1 0 0 1 0
  0 0 1 1 0 1 1 1 1 0 1 1
  1 2 3 4 5 6 7 8

```

```

MATE VM23 RO 8000 YOUN 2.E11 NU 0.3 ELAS 2.E12
  TRAC 1 2.E12 10
  LECT 1 TERM
LINK VITE 3 10 FONC 1 LECT 4 3 8 7 TERM
  BLOQ 12 LECT tous TERM
  BLOQ 3 LECT 1 2 5 6 TERM
FONC NUM 1 TABL 2 0 1 10 1
INIT VITE 3 10 LECT 4 3 8 7 TERM
ECRI DEPL VITE CONT ECRO TPRE 100.D-3
FICH ALIC FREQ 10
  POIN LECT tous TERM
  ELEM LECT tous TERM
OPTI PAS UTIL NOTE CSTA 0.5 LOG 1
CALC TINI 0.0 TEND 1000.D-3 PASF 2.E-5
SUIT
Post-treatment (time curves from alice file)
ECHO
RESU ALIC GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 2 'dz_5' DEPL COMP 3 NOEU LECT 5 TERM
COUR 3 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 5 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 7 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 9 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 11 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 13 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
TRAC 2 AXES 1.0 'DISPL. [M]' YZER
TRAC 3 5 7 9 11 13 AXES 1.0 'CONT. [PA]' YZER
LIST 2 AXES 1.0 'DISPL. [M]' YZER
LIST 3 5 7 9 11 13 AXES 1.0 'CONT. [PA]' YZER
FIN

```

sheal7b.epx

```

Post-treatment (time curves from alice file)
ECHO
RESU ALIC 'sheal7.ali' GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 31 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 32 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 33 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 34 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 35 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 36 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
TRAC 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
RCOU 11 'sx_11' FICH 'sheal5.pun' RENA 'sx_11_15'
RCOU 12 'sy_11' FICH 'sheal5.pun' RENA 'sy_11_15'
RCOU 13 'txy_11' FICH 'sheal5.pun' RENA 'txy_11_15'
RCOU 14 'sz_11' FICH 'sheal5.pun' RENA 'sz_11_15'
RCOU 15 'tyz_11' FICH 'sheal5.pun' RENA 'tyz_11_15'
RCOU 16 'txz_11' FICH 'sheal5.pun' RENA 'txz_11_15'
TRAC 11 12 13 14 15 16 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
COLO NOIR NOIR NOIR NOIR NOIR NOIR
  ROUG ROUG ROUG ROUG ROUG ROUG
FIN

```

sheal8.epx

```

SHEA18
ECHO
!CONV WIN
LAGR TRID
GEOM LIBR POIN 8 C82L 1 TERM
  0 0 0 1 0 0 1 1 0 0 1 0
  0 0 1 1 0 1 1 1 1 0 1 1
  1 2 3 4 5 6 7 8
MATE VM23 RO 8000 YOUN 2.E11 NU 0.3 ELAS 2.E12
  TRAC 1 2.E12 10
  LECT 1 TERM
LINK VITE 3 10 FONC 1 LECT 4 3 8 7 TERM
  BLOQ 12 LECT tous TERM
  BLOQ 3 LECT 1 2 5 6 TERM
FONC NUM 1 TABL 2 0 1 10 1
INIT VITE 3 10 LECT 4 3 8 7 TERM
ECRI DEPL VITE CONT ECRO TPRE 100.D-3
FICH ALIC FREQ 10
  POIN LECT tous TERM
  ELEM LECT tous TERM
OPTI PAS UTIL NOTE CSTA 0.5 LOG 1
CALC TINI 0.0 TEND 1000.D-3 PASF 2.E-5
SUIT
Post-treatment (time curves from alice file)
ECHO
RESU ALIC GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 2 'dz_5' DEPL COMP 3 NOEU LECT 5 TERM
COUR 3 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 5 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 7 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM
COUR 9 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 11 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 13 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
TRAC 2 AXES 1.0 'DISPL. [M]' YZER
TRAC 3 5 7 9 11 13 AXES 1.0 'CONT. [PA]' YZER
LIST 2 AXES 1.0 'DISPL. [M]' YZER
LIST 3 5 7 9 11 13 AXES 1.0 'CONT. [PA]' YZER
FIN

```

sheal8b.epx

```

Post-treatment (time curves from alice file)
ECHO
RESU ALIC 'sheal8.ali' GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 31 'sx_11' CONT COMP 1 GAUS 1 ELEM LECT 1 TERM
COUR 32 'sy_11' CONT COMP 2 GAUS 1 ELEM LECT 1 TERM
COUR 33 'txy_11' CONT COMP 3 GAUS 1 ELEM LECT 1 TERM

```

```

COUR 34 'sz_11' CONT COMP 4 GAUS 1 ELEM LECT 1 TERM
COUR 35 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
COUR 36 'txz_11' CONT COMP 6 GAUS 1 ELEM LECT 1 TERM
TRAC 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
RCOU 11 'sx_11' FICH 'sheal7.pun' RENA 'sx_11_17'
RCOU 12 'sy_11' FICH 'sheal7.pun' RENA 'sy_11_17'
RCOU 13 'txy_11' FICH 'sheal7.pun' RENA 'txy_11_17'
RCOU 14 'sz_11' FICH 'sheal7.pun' RENA 'sz_11_17'
RCOU 15 'tyz_11' FICH 'sheal7.pun' RENA 'tyz_11_17'
RCOU 16 'txz_11' FICH 'sheal7.pun' RENA 'txz_11_17'
TRAC 11 12 13 14 15 16 31 32 33 34 35 36 AXES 1.0 'CONT. [PA]' YZER
COLO NOIR NOIR NOIR NOIR NOIR NOIR
  ROUG ROUG ROUG ROUG ROUG ROUG
FIN

```

sheal8c.epx

```

Post-treatment (time curves from alice file)
ECHO
RESU ALIC 'sheal8.ali' GARD PSCR
SORT GRAP
AXTE 1.0 'Time [s]'
COUR 81 'tyz_11' CONT COMP 5 GAUS 1 ELEM LECT 1 TERM
RCOU 11 'txy_11' FICH 'sheal1.pun' RENA 'txy_11_01'
RCOU 21 'txy_11' FICH 'sheal2.pun' RENA 'txy_11_02'
RCOU 31 'txz_11' FICH 'sheal3.pun' RENA 'txz_11_03'
RCOU 41 'txz_11' FICH 'sheal4.pun' RENA 'txz_11_04'
RCOU 51 'tyz_11' FICH 'sheal5.pun' RENA 'tyz_11_05'
RCOU 61 'tyz_11' FICH 'sheal6.pun' RENA 'tyz_11_06'
RCOU 71 'tyz_11' FICH 'sheal7.pun' RENA 'tyz_11_07'
TRAC 11 21 31 41 51 61 71 81 AXES 1.0 'CONT. [PA]' YZER
COLO NOIR BLEU VERT TURQ JAUN ROSE ROUG NOIR
FIN

```

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Abstract

The present report presents the implementation of the Q4xL 2D 4-node linear displacement quadrilateral elements and then introduces the implementation of their 3D equivalents, the C8xL 8-node linear hexahedra.

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