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Kaplan classes of a certain family of functions

ABSTRACT. We give the complete characterization of members of Kaplan classes of products of power functions with all zeros symmetrically distributed in $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and weakly monotonic sequence of powers. In this way we extend Sheil-Small's theorem. We apply the obtained result to study univalence of antiderivative of these products of power functions.

Introduction. Let \mathcal{H}_d be the class of all analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ normalized by $f(0) = 1$ and such that $f \neq 0$ in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let \mathcal{S} be the class of all analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ normalized by $f(0) = f'(0) - 1 = 0$ which are univalent and \mathcal{C} be the class of functions in \mathcal{S} that are close-to-convex. For $\alpha, \beta \geq 0$ the Kaplan class $K(\alpha, \beta)$ is the set of all functions $f \in \mathcal{H}_d$ satisfying one of the two equivalent conditions:

$$(0.1) \quad \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \leq \beta\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2),$$

$$(0.2) \quad -\alpha\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2) \leq \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}).$$

for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$ (see [6, pp. 32–33]).

Let $\mathbb{N}_j := \mathbb{N} \cap [1; j]$ for $j \in \mathbb{N}$ and $\mathbb{R}^+ := (0; +\infty)$. Fix $n \in \mathbb{N}$ and a weakly monotonic sequence $m : \mathbb{N}_n \rightarrow \mathbb{R}^+$. Define the functions

$$(0.3) \quad \mathbb{D} \ni z \mapsto f_k(z) := 1 - ze^{-i\frac{2\pi(k-1)}{n}} \quad \text{for } k \in \mathbb{N}_n$$

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and

$$(0.4) \quad \mathbb{D} \ni z \mapsto P_n(z; m) := \prod_{k=1}^n f_k^{m_k}(z).$$

We denote the class of all such functions $P_n(\cdot; m)$ by \mathcal{P}_n . Let us notice that the function $P_n(\cdot; m)$ is a product of power functions with all zeros symmetrically distributed in \mathbb{T} . In particular if $m_k \in \mathbb{N}$ for all $k \in \mathbb{N}_n$, then $P_n(\cdot; m)$ is a polynomial of degree $\sum_{k=1}^n m_k$ with all zeros symmetrically distributed in \mathbb{T} . The functions of the form $\mathbb{D} \ni z \mapsto 1 - ze^{-it}$ for $t \in [0; 2\pi)$ play the central role in the univalent functions theory. Due to the result of Royster [5] they are used for example as an extremal functions in many articles (see [1, 4]).

The Kaplan classes were used as the universal tool for establishing many important subclasses of \mathcal{S} (see [6, p. 47]). Complete membership study even for the simplest functions from \mathcal{H}_d was not carried out. For a given function it can be difficult to check if it belongs to any Kaplan class. We deduce from [2, Theorem 1.1] that $f_k \in K(1, 0)$ for any $k \in \mathbb{N}_n$. Moreover, Sheil-Small proved the following theorem (see [7, p. 248]).

Theorem A (Sheil-Small). *For any polynomial $Q \in \mathcal{H}_d$ of the degree $n \in \mathbb{N} \setminus \{1\}$ with all zeros in \mathbb{T} , if λ is the minimal arclength between two consecutive zeros of Q , then $Q \in K(1, 2\pi/\lambda - n + 1)$.*

Theorem A can also be deduced from [3], where Jahangiri obtained a certain gap condition for polynomials with all zeros in \mathbb{T} . In [2], we extended the Jahangiri's result for all $\alpha, \beta \geq 0$ and effectively determined complete membership to Kaplan classes of polynomials with all zeros in \mathbb{T} . In this article, we extend the above results by describing complete membership to Kaplan classes of functions from the class \mathcal{P}_n for all $n \in \mathbb{N}$. To this end we recall some properties of Kaplan classes (see [7, p. 245]).

Lemma B. *For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ and $t > 0$ the following conditions hold:*

$$\begin{aligned} f \in K(\alpha_1, \beta_1) \text{ and } g \in K(\alpha_2, \beta_2) &\Rightarrow fg \in K(\alpha_1 + \alpha_2, \beta_1 + \beta_2), \\ f \in K(\alpha_1, \beta_1) &\Rightarrow f^0 \in K(0, 0), \\ f \in K(\alpha_1, \beta_1) &\iff f^t \in K(t\alpha_1, t\beta_1), \\ f \in K(\alpha_1, \beta_1) &\iff f^{-1} \in K(\beta_1, \alpha_1). \end{aligned}$$

1. Main theorems. Assume that $m_0 := 0$. For all $j \in \mathbb{N}$ and $k \in \mathbb{N}_n$ we define

$$\begin{aligned} t_j &:= \frac{2\pi(j-1)}{n}, \quad s := \sum_{l=1}^n m_l, \\ a_k &:= -\frac{n-k}{k}, \quad b_k := -s + \frac{n}{k} \sum_{l=n-k}^n m_l, \\ x_k &:= \sum_{l=n-k}^n m_l - km_{n-k}, \quad y_k := (n-k)m_{n-k} - \sum_{l=1}^{n-k-1} m_l, \end{aligned}$$

$$\begin{aligned} \Pi_0 &:= \{(x, y) \in \mathbb{R}^2 : x \geq m_n\}, \\ \Pi_k &:= \{(x, y) \in \mathbb{R}^2 : y \geq a_k x + b_k\}, \\ \Pi'_0 &:= \{(x, y) \in \mathbb{R}^2 : 0 \leq x < m_n\}, \\ \Pi'_k &:= \{(x, y) \in \mathbb{R}^2 : 0 \leq x, 0 \leq y < a_k x + b_k\}, \\ \Pi &:= \bigcap_{l=0}^n \Pi_l. \end{aligned}$$

Now we give the complete characterization of membership of $P_n(\cdot; m)$ to Kaplan classes.

Theorem 1.1. *If $m : \mathbb{N}_n \rightarrow \mathbb{R}^+$ is weakly monotonic, then for all $\alpha, \beta \geq 0$, $P_n(\cdot; m) \in K(\alpha, \beta)$ if and only if $(\alpha, \beta) \in \Pi$.*

Proof. Without loss of generality we assume that m is a nondecreasing sequence. Since $\prod_{k=1}^n f_k(z) = 1 - z^n$ and $1 - z^n$ has positive real part in \mathbb{D} , we have

$$(1.1) \quad \prod_{k=1}^n f_k \in K(1, 1).$$

First we prove that $P_n(\cdot; m) \in K(x_k, y_k)$ for $k \in \mathbb{N}_n$. Fix $k \in \mathbb{N}_n$. Therefore,

$$\begin{aligned} P_n(\cdot; m) &= \prod_{l=1}^n f_l^{m_{n-k}} \prod_{l=1}^n f_l^{m_l - m_{n-k}} \\ &= \prod_{l=1}^n f_l^{m_{n-k}} \prod_{l=1}^{n-k-1} f_l^{m_l - m_{n-k}} \prod_{l=n-k+1}^n f_l^{m_l - m_{n-k}} \\ &= \prod_{l=1}^n f_l^{m_{n-k}} \prod_{l=1}^{n-k-1} \left(\frac{1}{f_l}\right)^{m_{n-k} - m_l} \prod_{l=n-k+1}^n f_l^{m_l - m_{n-k}}. \end{aligned}$$

By (1.1) and Lemma B, we get

$$\prod_{l=1}^n f_l^{m_{n-k}} \in K(m_{n-k}, m_{n-k}),$$

$$(1/f_l)^{m_{n-k}-m_l} \in K(0, m_{n-k} - m_l) \text{ for } l \in \mathbb{N}_{n-k-1}$$

and

$$f_l^{m_l - m_{n-k}} \in K(m_l - m_{n-k}, 0) \text{ for } l \in \mathbb{N}_n \setminus \mathbb{N}_{n-k}.$$

Then

$$P_n(\cdot; m) \in K\left(m_{n-k} + \sum_{l=n-k+1}^n (m_l - m_{n-k}), m_{n-k} + \sum_{l=1}^{n-k-1} (m_{n-k} - m_l)\right)$$

and as a consequence

$$(1.2) \quad P_n(\cdot; m) \in K(x_k, y_k).$$

By Lemma B, we obtain $f \in \Pi$.

Now we prove the second part of the theorem. Fix $k \in \mathbb{N}_{n-1}$. Consider the left side of inequality (0.1) with $\mathbb{N} \ni j \mapsto \theta_1(j) := -2\pi/n + 1/j$, $\mathbb{N} \ni j \mapsto \theta_2(j) := 2\pi - 2\pi(k+1)/n - 1/j$ and $\mathbb{N} \ni j \mapsto r_j := 1 - 1/j^2$. Therefore,

$$\begin{aligned} & \arg(P_n(r_j e^{i\theta_2}; m)) - \arg(P_n(r_j e^{i\theta_1}; m)) \\ &= \sum_{l=1}^n m_l \left(\arctan \left(\frac{-r_j \sin(\theta_2(j) - \frac{2\pi}{n}(l-1))}{1 - r_j \cos(\theta_2(j) - \frac{2\pi}{n}(l-1))} \right) \right. \\ & \quad \left. - \arctan \left(\frac{-r_j \sin(\theta_1(j) - \frac{2\pi}{n}(l-1))}{1 - r_j \cos(\theta_1(j) - \frac{2\pi}{n}(l-1))} \right) \right) \\ &= \sum_{l=1}^n m_l \left(\arctan \left(\frac{r_j \sin\left(\frac{2\pi}{n}(k+l) + \frac{1}{j}\right)}{1 - r_j \cos\left(\frac{2\pi}{n}(k+l) + \frac{1}{j}\right)} \right) \right. \\ & \quad \left. - \arctan \left(\frac{r_j \sin\left(\frac{2\pi l}{n} - \frac{1}{j}\right)}{1 - r_j \cos\left(\frac{2\pi l}{n} - \frac{1}{j}\right)} \right) \right) \\ &= \sum_{l=1}^{n-k-1} m_l \left(\arctan \left(\frac{r_j \sin\left(\frac{2\pi}{n}(k+l) + \frac{1}{j}\right)}{1 - r_j \cos\left(\frac{2\pi}{n}(k+l) + \frac{1}{j}\right)} \right) \right. \\ & \quad \left. - \arctan \left(\frac{r_j \sin\left(\frac{2\pi l}{n} - \frac{1}{j}\right)}{1 - r_j \cos\left(\frac{2\pi l}{n} - \frac{1}{j}\right)} \right) \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{l=n-k+1}^{n-1} m_l \left(\arctan \left(\frac{r_j \sin \left(\frac{2\pi}{n} (k+l) + \frac{1}{j} \right)}{1 - r_j \cos \left(\frac{2\pi}{n} (k+l) + \frac{1}{j} \right)} \right) \right. \\
& \qquad \qquad \qquad \left. - \arctan \left(\frac{r_j \sin \left(\frac{2\pi l}{n} - \frac{1}{j} \right)}{1 - r_j \cos \left(\frac{2\pi l}{n} - \frac{1}{j} \right)} \right) \right) \\
& + m_{n-k} \left(\arctan \left(\frac{\left(1 - \frac{1}{j^2}\right) \sin \left(\frac{1}{j}\right)}{1 - \left(1 - \frac{1}{j^2}\right) \cos \left(\frac{1}{j}\right)} \right) \right. \\
& \qquad \qquad \qquad \left. + \arctan \left(\frac{r_j \sin \left(\frac{2\pi k}{n} + \frac{1}{j} \right)}{1 - r_j \cos \left(\frac{2\pi k}{n} + \frac{1}{j} \right)} \right) \right) \\
& + m_n \left(\arctan \left(\frac{r_j \sin \left(\frac{2\pi k}{n} + \frac{1}{j} \right)}{1 - r_j \cos \left(\frac{2\pi k}{n} + \frac{1}{j} \right)} \right) \right. \\
& \qquad \qquad \qquad \left. + \arctan \left(\frac{\left(1 - \frac{1}{j^2}\right) \sin \left(\frac{1}{j}\right)}{1 - \left(1 - \frac{1}{j^2}\right) \cos \left(\frac{1}{j}\right)} \right) \right)
\end{aligned}$$

and as a consequence

$$\begin{aligned}
& \lim_{j \rightarrow +\infty} (\arg(P_n(r_j e^{i\theta_2}; m)) - \arg(P_n(r_j e^{i\theta_1}; m))) \\
& = \sum_{l=1}^{n-k-1} m_l \left(\arctan \left(\frac{\sin \left(\frac{2\pi}{n} (k+l) \right)}{1 - \cos \left(\frac{2\pi}{n} (k+l) \right)} \right) - \arctan \left(\frac{\sin \left(\frac{2\pi l}{n} \right)}{1 - \cos \left(\frac{2\pi l}{n} \right)} \right) \right) \\
& + \sum_{l=n-k+1}^{n-1} m_l \left(\arctan \left(\frac{\sin \left(\frac{2\pi}{n} (k+l) \right)}{1 - \cos \left(\frac{2\pi}{n} (k+l) \right)} \right) - \arctan \left(\frac{\sin \left(\frac{2\pi l}{n} \right)}{1 - \cos \left(\frac{2\pi l}{n} \right)} \right) \right) \\
& + m_{n-k} \left(\frac{\pi}{2} + \arctan \left(\frac{\sin \left(\frac{2\pi k}{n} \right)}{1 - \cos \left(\frac{2\pi k}{n} \right)} \right) \right) \\
& + m_n \left(\arctan \left(\frac{\sin \left(\frac{2\pi k}{n} \right)}{1 - \cos \left(\frac{2\pi k}{n} \right)} \right) + \frac{\pi}{2} \right).
\end{aligned}$$

By the trigonometric identity:

$$\frac{\sin x}{1 - \cos x} = \tan \left(\frac{\pi}{2} - \frac{x}{2} \right) \quad \text{for } x \in \mathbb{R} \setminus \bigcup_{j \in \mathbb{Z}} \{2j\pi\}$$

we get

$$\begin{aligned}
& \lim_{j \rightarrow +\infty} (\arg(P_n(r_j e^{i\theta_2}; m)) - \arg(P_n(r_j e^{i\theta_1}; m))) \\
&= \sum_{l=1}^{n-k-1} m_l \left(\arctan \left(\tan \left(\frac{\pi}{2} - \frac{\pi}{n}(k+l) \right) \right) - \arctan \left(\tan \left(\frac{\pi}{2} - \frac{\pi l}{n} \right) \right) \right) \\
&+ \sum_{l=n-k+1}^{n-1} m_l \left(\arctan \left(\tan \left(\frac{\pi}{2} - \frac{\pi}{n}(k+l) \right) \right) - \arctan \left(\tan \left(\frac{\pi}{2} - \frac{\pi l}{n} \right) \right) \right) \\
&+ m_{n-k} \left(\frac{\pi}{2} + \arctan \left(\tan \left(\frac{\pi}{2} - \frac{\pi k}{n} \right) \right) \right) \\
&+ m_n \left(\arctan \left(\tan \left(\frac{\pi}{2} - \frac{\pi k}{n} \right) \right) + \frac{\pi}{2} \right).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\pi}{2} - \frac{\pi}{n}(k+l) &\in \left(-\frac{\pi}{2}; \frac{\pi}{2} \right) \text{ for } l \in \mathbb{N}_{n-k-1}, \\
\frac{\pi}{2} - \frac{\pi}{n}(k+l) &\in \left(-\frac{3\pi}{2}; -\frac{\pi}{2} \right) \text{ for } l \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-k}
\end{aligned}$$

and

$$\frac{\pi}{2} - \frac{\pi l}{n} \in \left(-\frac{\pi}{2}; \frac{\pi}{2} \right) \text{ for } l \in \mathbb{N}_{n-1},$$

we have

$$\begin{aligned}
& \lim_{j \rightarrow +\infty} (\arg(P_n(r_j e^{i\theta_2}; m)) - \arg(P_n(r_j e^{i\theta_1}; m))) \\
&= \sum_{l=1}^{n-k-1} m_l \left(\frac{\pi}{2} - \frac{\pi}{n}(k+l) - \frac{\pi}{2} + \frac{\pi l}{n} \right) \\
&+ \sum_{l=n-k+1}^{n-1} m_l \left(\frac{3\pi}{2} - \frac{\pi}{n}(k+l) - \frac{\pi}{2} + \frac{\pi l}{n} \right) \\
&+ m_{n-k} \left(\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi k}{n} \right) + m_n \left(\frac{\pi}{2} - \frac{\pi k}{n} + \frac{\pi}{2} \right) \\
&= -\frac{\pi k s}{n} + \pi \sum_{l=n-k}^n m_l.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{j \rightarrow +\infty} \left(\beta\pi + \frac{1}{2}(\alpha - \beta)(\theta_2(j) - \theta_1(j)) \right) &= \beta\pi + (\alpha - \beta) \left(\pi - \frac{\pi k}{n} \right) \\
&= \beta\pi \frac{k}{n} + \alpha\pi \frac{n-k}{n},
\end{aligned}$$

from which we deduce that inequality (0.1) does not hold for

$$\beta < -\frac{n-k}{k}\alpha - s + \frac{n}{k} \sum_{l=n-k}^n m_l$$

and as a consequence $P_n(\cdot; m) \notin K(\alpha, \beta)$ for $(\alpha, \beta) \in \Pi'_k$. Hence

$$(1.3) \quad P_n(\cdot; m) \notin \bigcup_{k=1}^{n-1} \Pi'_k.$$

Now we prove that $P_n(\cdot; m) \notin \Pi'_0$. Consider the right side of inequality (0.2) with $\mathbb{N} \ni j \mapsto \theta_1(j) := 2\pi(n-1)/n+1/j$, $\mathbb{N} \ni j \mapsto \theta_2(j) := 2\pi(n-1)/n-1/j$ and $\mathbb{N} \ni j \mapsto r_j := 1 - 1/j^2$. Therefore,

$$\begin{aligned} & \arg(P_n(r_j e^{i\theta_2}; m)) - \arg(P_n(r_j e^{i\theta_1}; m)) \\ &= -2m_n \arctan \left(\frac{\left(1 - \frac{1}{j^2}\right) \sin\left(\frac{1}{j}\right)}{1 - \left(1 - \frac{1}{j^2}\right) \cos\left(\frac{1}{j}\right)} \right) \\ &+ \sum_{l=1}^{n-1} m_l \left(\arctan \left(\frac{-r_j \sin\left(\frac{2\pi l}{n} - \frac{1}{j}\right)}{1 - r_j \cos\left(\frac{2\pi l}{n} - \frac{1}{j}\right)} \right) \right. \\ &\quad \left. - \arctan \left(\frac{-r_j \sin\left(\frac{2\pi l}{n} + \frac{1}{j}\right)}{1 - r_j \cos\left(\frac{2\pi l}{n} + \frac{1}{j}\right)} \right) \right) \end{aligned}$$

and as a consequence

$$\lim_{j \rightarrow +\infty} (\arg(P_n(r_j e^{i\theta_2}; m)) - \arg(P_n(r_j e^{i\theta_1}; m))) = -m_n \pi.$$

On the other hand, we have

$$\lim_{j \rightarrow +\infty} \left(-\alpha\pi + \frac{1}{2}(\alpha - \beta)(\theta_2(j) - \theta_1(j)) \right) = -\alpha\pi,$$

from which we deduce that inequality (0.2) does not hold for $\alpha < m_n$ and as a consequence $P_n(\cdot; m) \notin K(\alpha, \beta)$ for $(\alpha, \beta) \in \Pi'_0$. From this and (1.3) we obtain

$$P_n(\cdot; m) \notin \bigcup_{k=0}^n \Pi'_k. \quad \square$$

By Theorem A, if $m_k = 1$ for all $k \in \mathbb{N}_n$, then $P_n(\cdot; m) \in K(1, 1)$. Theorem 1.1 is an extension of Theorem A for functions from the class \mathcal{P}_n . Moreover, in the first part of the proof of Theorem 1.1 we obtain nontrivial, interesting factorization of $P_n(\cdot; m)$ (cf. [7, p. 246]).

Remark 1.2. Let us notice that for a nondecreasing sequence $m : \mathbb{N}_n \rightarrow \mathbb{R}^+$ points (x_k, y_k) for $k \in \mathbb{N}_n$ are all vertices of the set Π . Analogously we can effectively determine vertices of Π if m is nonincreasing.

Let $\varphi'_q := (P_n(\cdot; m))^q$ for any $q \in \mathbb{R}$ such that $\varphi_q(0) = 0$. The complete characterization of functions $P_n(\cdot; m)$ belonging to Kaplan classes obtained in Theorem 1.1 can be used to study univalence of φ_q .

Theorem 1.3. *If $m : \mathbb{N}_n \rightarrow \mathbb{R}^+$ is nondecreasing sequence, then for any $n \in \mathbb{N}$, $k \in \mathbb{N}_{n-1}$ and $q \geq 0$ the following implications hold:*

$$(1.4) \quad s \geq nm_{n-1} - 2m_n \implies \left(\varphi_q \in \mathcal{C} \iff q \in \left[0; \frac{1}{m_n} \right] \right),$$

$$(1.5) \quad s \in \left[(n+2k)m_{n-k-1} - 2 \sum_{l=n-k}^n m_l; (n+2k)m_{n-k} - 2 \sum_{l=n-k}^n m_l \right) \\ \implies \left(\varphi_q \in \mathcal{C} \iff q \in \left[0; \frac{n+2k}{n \sum_{l=n-k}^n m_l - ks} \right] \right).$$

Proof. Let m be a nondecreasing sequence. Fix $q \geq 0$. First we prove (1.4). If $s \geq nm_{n-1} - 2m_n$, then $y_1 \leq 3x_1$. This and Theorem 1.1 imply that $P_n(\cdot; m) \in K(m_n, 3m_n)$ and for any $\alpha \in [0; m_n]$, $P_n(\cdot; m) \notin K(\alpha, 3\alpha)$. Therefore, $(P_n(\cdot; m))^q \in K(1, 3)$ if and only if $q \in [0; 1/m_n]$.

Now we prove (1.5). Fix $k \in \mathbb{N}_{n-1}$. Assume that

$$s \in \left[(n+2k)m_{n-k-1} - 2 \sum_{l=n-k}^n m_l; (n+2k)m_{n-k} - 2 \sum_{l=n-k}^n m_l \right).$$

Then

$$\begin{cases} y_l > 3x_l & \text{for } l \in \mathbb{N}_k, \\ y_l \leq 3x_l & \text{for } l \in \mathbb{N}_n \setminus \mathbb{N}_k. \end{cases}$$

This and Theorem 1.1 imply that

$$P_n(\cdot; m) \in K \left(\frac{n}{n+2k} \sum_{l=n-k}^n m_l - ks, \frac{3n}{n+2k} \sum_{l=n-k}^n m_l - ks \right)$$

and for any

$$\alpha \in \left[0; \frac{n}{n+2k} \sum_{l=n-k}^n m_l - ks \right),$$

$P_n(\cdot; m) \notin K(\alpha, 3\alpha)$, which leads to (1.5). \square

Theorem 1.4. *If $m : \mathbb{N}_n \rightarrow \mathbb{R}^+$ is a nondecreasing sequence, then for any $n \in \mathbb{N}$, $k \in \mathbb{N}_{n-1}$ and $q < 0$ the following implications hold:*

$$(1.6) \quad s \geq \frac{2}{3}m_n + nm_{n-1} \implies \left(\varphi_q \in \mathcal{C} \iff q \in \left[-\frac{3}{m_n}; 0 \right) \right),$$

$$(1.7) \quad s \in \left[\left(n - \frac{2}{3}k \right) m_{n-k-1} + \frac{2}{3} \sum_{l=n-k}^n m_l; \left(n - \frac{2}{3}k \right) m_{n-k} + \frac{2}{3} \sum_{l=n-k}^n m_l \right) \\ \implies \left(\varphi_q \in \mathcal{C} \iff q \in \left[\frac{3n-2k}{ks - n \sum_{l=n-k}^n m_l}; 0 \right) \right).$$

Proof. Let m be a nondecreasing sequence. Fix $q < 0$. First we prove (1.6). If $s \geq 2/3m_n + nm_{n-1}$, then $3y_1 \leq x_1$. This and Theorem 1.1 imply that $P_n(\cdot; m) \in K(m_n, 1/3m_n)$ and for any $\alpha \in [0; m_n)$, $P_n(\cdot; m) \notin K(\alpha, 1/3\alpha)$. Therefore, $(P_n(\cdot; m)) \in K(1, 3)$ if and only if $q \in [-3/m_n; 0)$.

Now we prove (1.7). Fix $k \in \mathbb{N}_{n-1}$. Assume that

$$s \in \left[\left(n - \frac{2}{3}k \right) m_{n-k-1} + \frac{2}{3} \sum_{l=n-k}^n m_l; \left(n - \frac{2}{3}k \right) m_{n-k} + \frac{2}{3} \sum_{l=n-k}^n m_l \right).$$

Then

$$\begin{cases} 3y_l > x_l & \text{for } l \in \mathbb{N}_k, \\ 3y_l \leq x_l & \text{for } l \in \mathbb{N}_n \setminus \mathbb{N}_k. \end{cases}$$

This and Theorem 1.1 imply that

$$P_n(\cdot; m) \in K \left(\frac{3n}{3n-2k} \sum_{l=n-k}^n m_l - ks, \frac{n}{3n-2k} \sum_{l=n-k}^n m_l - ks \right)$$

and for any

$$\alpha \in \left[0; \frac{n}{3n-2k} \sum_{l=n-k}^n m_l - ks \right),$$

$P_n(\cdot; m) \notin K(3\alpha, \alpha)$, which leads to (1.7). \square

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