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Unannounced Interim Inspections

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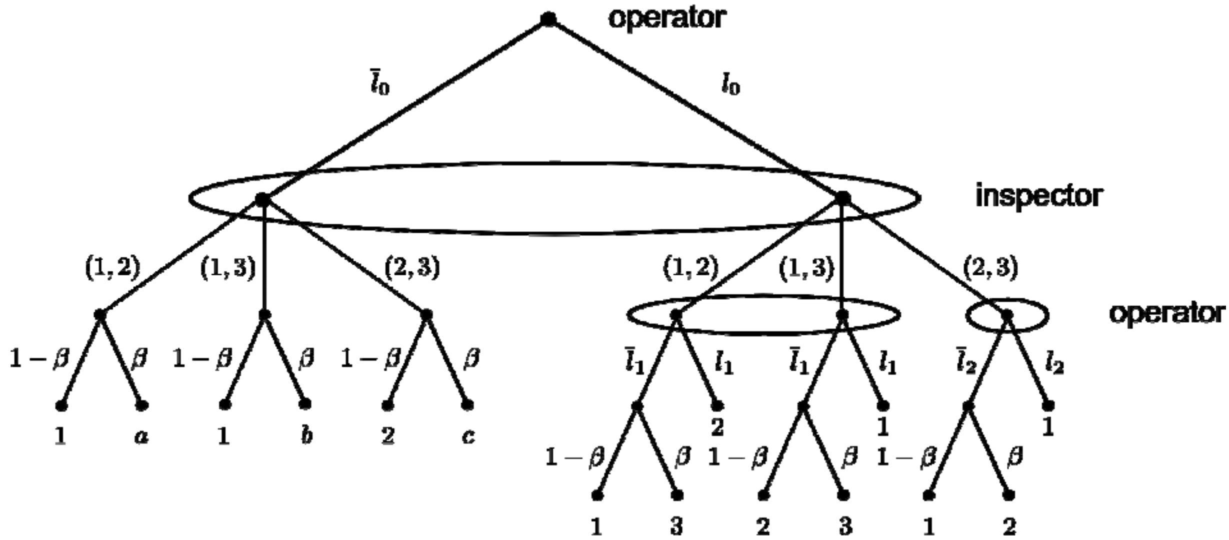

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Executive Summary

The appropriate number and timing of unannounced interim inspections in nuclear facilities in the framework of nuclear material safeguards has been discussed in the safeguards community already for a long time and for good reasons. The matter is relevant both for EURATOM and IAEA safeguards authorities.

For IAEA safeguards, the implementation is presently shifting from a system mostly focused on traditional safeguards to the so called "Integrated Safeguards", where the verification system is more holistic and and State level based approach. Consequently, at least in some cases (i.e. for some facilities in some States), there will be a decrease in the yearly number of fixed scheduled interim inspections by substituting some of them with unannounced ones.

For EURATOM safeguards there is also an evolution of the way to implement inspections in the European Union (EU) together with IAEA activities. Most of the IAEA inspections in EU will continue to be carried out in presence of EURATOM inspectors. On an other side, like the IAEA, EURATOM may also carry out unannounced inspection by its own.

Modelling

The analysis of this general problem is in some cases mathematically demanding, and concrete solutions, i.e., advices on numbers and points of time for specific facilities as well as effectiveness and efficiency considerations, depend crucially on special assumptions, some of the most important ones are:

- Whether or not the inspections have to be planned at the beginning of the reference time interval, e.g., one calendar year, and whether or not they can be observed by the facility operators;
- If the safeguards authority uses a concept of the kind the earlier an illegal activity is detected the better, or a critical time concept which means that any illegal activity has to be detected within a specific time;
- If statistical errors of the first (false alarms) and second kind (failing to detect illegal activities) have to be taken into account.

Objectives

In this study unannounced interim inspections are analyzed in general, and specifically in view of the recently defined IAEA/EURATOM Partnership Approach. In particular the following aspects are addressed:

- A general discussion of the problem of unannounced interim inspections which includes an identification and description of all the types of assumptions which are necessary to be made for a quantitative treatment. Although the discussion will be of general validity, the context of application will be that of nuclear safeguards. Prototypical situations arising in safeguards applications are identified with link to specific practical applications, e.g., On-Site Interim Storage Facilities and Fuel Element Fabrication Facilities, by recognizing which set of assumptions characterizes which facility.
- An annotated bibliography of applicable work on unannounced interim inspections done so far and publicly available.
- An identification of a reduced number of prototypical examples and link with specific safeguards inspections and applications.
- An outline of the general methods identified and proposed for the solution of problems of that kind.
- The identification of two examples for a detailed analysis to be carried out with the general method identified in the previous steps.
- Lessons learned and reflections aiming at identifying also practical recommendations on the matter of unannounced interim inspections.

Organization

This study is organized as follows: In order to be concrete and specific, in the second chapter two nuclear facilities are described, and the safeguards measures applied by EURATOM and IAEA in these two facilities are discussed.

The third and fourth chapter are central from the analytical point of view. In the former, it is assumed, that both antagonists, operator and inspector, plan their activities simultaneously, i.e., without knowing the strategy of the other one. In the latter, it is assumed, that the operator uses the knowledge of the inspections performed in the course of the game. In both chapters both discrete and continuous time versions are considered.

In the fifth chapter the analyses of the two foregoing chapters are complemented: The aspect of deterrence is discussed, a critical time game is presented in some detail, and the global sampling problem, i.e., the number and distribution of unannounced interim inspections of the IAEA in the States of the EU, is analyzed in some preliminary way.

In the concluding sixth chapter an attempt is made to summarize the results and to formulate, with all due care, some recommendations.

In the Annexes additional explanations and proofs are given which complement the main text.

Selected Findings

In order to give an idea of the kind of results of this work, and without going into mathematical details, three observations are presented subsequently.

First, a classification is developed of those assumptions which are necessary for uniquely identifying a quantitative model for unannounced interim inspections in nuclear facilities. This classification results in 36 different models, four of which are analyzed in detail. They are applied to two concrete nuclear facilities, namely an On-Site Interim Storage Facility and a Fuel Element Fabrication Facility.

Second, for a specific facility and a given number of inspections per reference time, e.g., one year, optimal inspection strategies are determined. Whereas for the case that inspection are perfect the results can be guessed, this does not hold anymore for imperfect inspections, e.g., limited sample sizes of seals to be checked.

Third, together with the optimal inspection strategies, optimal expected detection times are determined as functions of the parameters of the model, namely number k of inspections per reference time interval and probability β of not detecting an illegal activity during an inspection. These results, in particular, permit two uses in practice: Either the values of the parameters k and β are given and the resulting optimal expected detection time is determined – which then turns out to be sufficient or not. Or a value of the optimal expected detection time is postulated and the necessary values of k and β are determined. This postulated optimal detection time then may be guaranteed by a small number k of inspections and a small non-detection probability β or vice versa.

Contents

Chapter 1

Introduction

International nuclear material safeguards as carried through by the International Atomic Energy Agency (IAEA) in partial fulfillment of the Non-Proliferation Treaty (NPT) for Nuclear Weapons, has undergone considerable changes since its first codification in the Model Agreement in 1972, see [21]. Due to the experience in the years following this codification, safeguards was modified and extended considerably, the new provisions were laid down in the Additional Protocol in 1996, see [22].

In the course of these changes, the relations between safeguards procedures of the IAEA and those of the European (regional) safeguards authority EURATOM, which originally were laid down in the so-called Verification Agreement in 1976, see [15], were reorganized, see [16]. In particular the problem of interim inspections to be carried through by both safeguards organizations has been discussed for many years; it has been settled only recently in the so-called IAEA/EURATOM Partnership Approach.

The subject of this study is the analysis of unannounced interim inspections in nuclear facilities in quite a broad sense. Beyond that its subject is also to demonstrate in which way advanced methods of applied mathematics, in particular game theory, have to be used in order that these inspections are organized in an as efficient and effective way as possible in the spirit of NPT safeguards.

1.1 Objectives

In the course of the discussions leading to this study, three kinds of objectives were identified. They represent the guidelines and will be described in some more detail now.

First, unannounced interim inspections are just a special problem of the more general problem of nuclear material safeguards, i.e., the verification of the provisions of the Non-Proliferation Treaty (NPT) for Nuclear Weapons to be met by the (non nuclear weapons) States parties to the treaty. Since by definition it cannot be excluded that States do not meet these provisions – otherwise no safeguards measures would be required – and since, more than that, it has to be assumed that a State may plan eventual illegal behavior strategically, standard methods of Statistics and Decision Theory are not sufficient. Therefore, methods of Game Theory have to be applied, thus, a first objective of this study is to demonstrate at the hand of our specific problem how this may be achieved. Second, unannounced interim inspections pose some specific problems different from other safeguards measures and tools like material accountancy and data verification. While the latter ones provide very detailed information at the end of a reference time interval, e.g., one calendar year, and while they are characterized by the use of advanced statistical techniques for the evaluation and compilation of measurement data, the former ones are aiming at the immediate detection of illegal activities or, positively formulated, confirmation of legal behavior. Therefore, primarily simple techniques for the checking of seals, or comparing installations in facilities with the design information provided by the facility attachments, are used, and above all, time is important: Time available for the inspector in the facility, and time elapsed between the start of an illegal activity and its detection.

Third and finally, the concrete situation in the (non-nuclear weapons) States of the European Union (EU) has to be considered. In all nuclear facilities of the EU, the European Commission performs regular inspections, under Chapter VII of the EURATOM Treaty, their frequency depending on the type of facility. Now, according to the IAEA/EURATOM Partnership Approach, unannounced interim inspections are performed in addition to the regular interim inspections planned by EURATOM.

Beyond the latter remark, it turns out that for a substantive analysis of unannounced interim inspections many assumptions have to be made which may be disputed since they are either not explicitly formulated in the documents of EURATOM and IAEA, or deal with the behavior of States in case they might start an illegal activity. A classification of the more important assumptions will be given now.

1.2 Classification of Assumptions

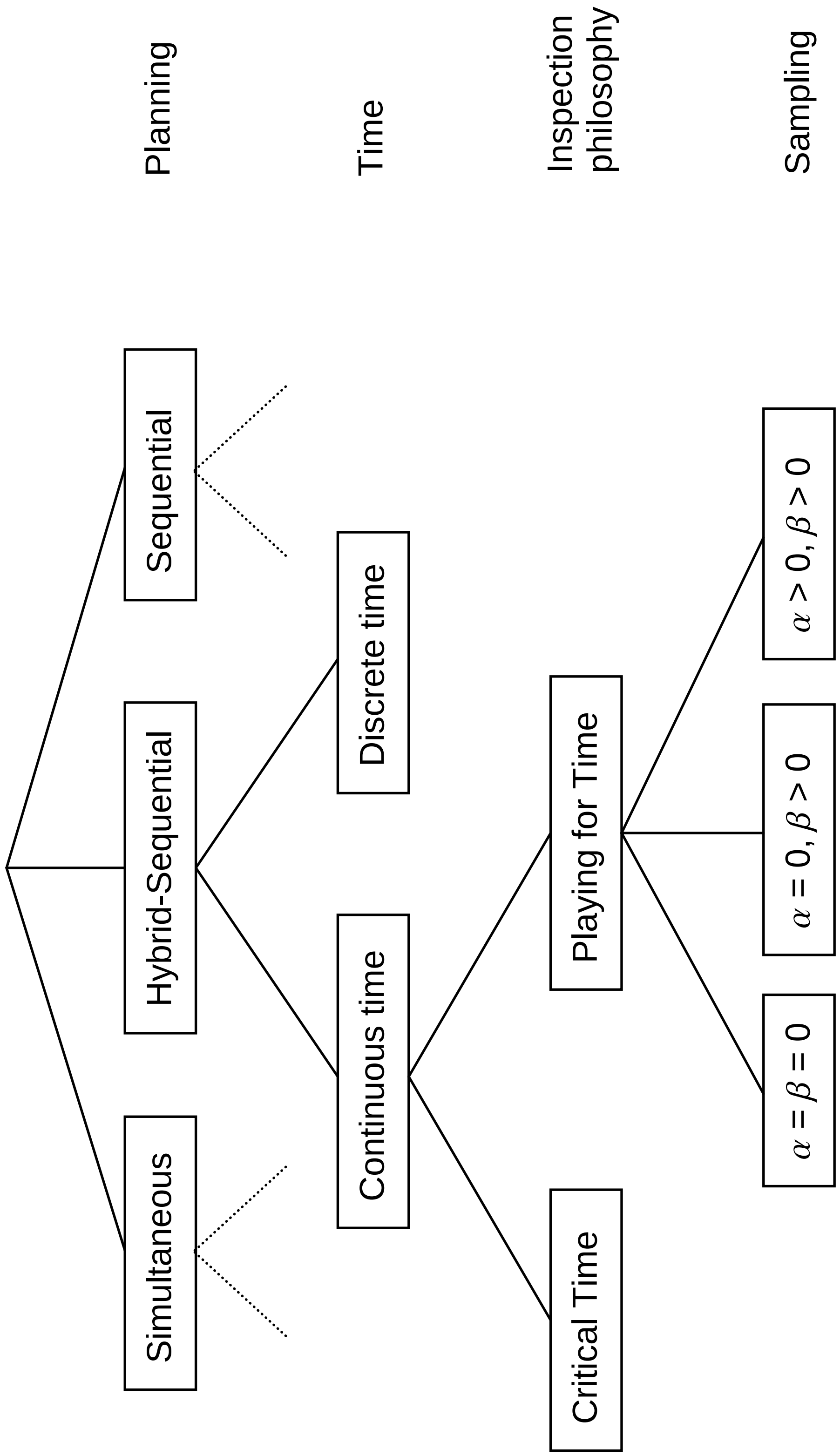
Quite in the sense of the general objective of this study it is an essential part of the work to carefully formulate all assumptions necessary for a mathematical model of unannounced interim inspections. Since, as already mentioned, the procedures for unannounced interim inspections are not laid down in every detail and therefore, alternative assumptions are possible, a whole tree of assumptions results, the more important ones of which are represented graphically in Figure 1.1.

Let us list them here:

- *Planning:* Does the facility operator plan his illegal activity – if at all – at the beginning of the reference time interval, or sequentially, during the reference time interval? The same question holds for the inspections[1].

[1] In this study we omit the case that the inspector plans sequentially, since the inspections authorities would have considerable organizational problems to implement such procedures.

Figure 1.1 Classifications of Assumptions. α and β are the error first and second kind probabilities.



- *Time:* Are interim inspections possible at any time point, or are they possible only at discrete time points? The significance of these two alternatives will be discussed in the forthcoming chapters.

- *Inspection philosophy:* Is the objective of the inspection authority to detect an illegal activity within a *critical time*[2] or alternatively, to detect it as soon as possible, which we call *playing for time*?

- *Sampling:* Are there no errors, only errors of the second kind, typical for *attribute sampling*, or errors of the first and second kind, typical for *variable sampling*, when inspections are performed?[3]

If all combinations are possible, then we have

$$2 \times 2 \times 3 \times 3 = 36$$

different sets of assumptions, i.e., 36 different mathematical models, the analysis of which require at least in part different analytical and numerical techniques.

Of course, it is neither possible with reasonable effort, nor interesting from a practical point of view to consider all 36 models. Instead four of them are selected with arguments given in the next three chapters, namely simultaneous and hybrid-sequential playing for time games, both time discrete and continuous. In addition, and for the sake of completeness, in the fifth chapter critical time games are considered with the help of one example.

1.3 Analytical Methods

It is stated already in the IAEA Model agreement, see [21] $\S$ 28, that the objective of safeguards is "... the timely detection of diversion of significant quantities of nuclear material ... and deterrence of such diversion by the risk of early detection."

Deterring the facility operator (or the State as partner to IAEA safeguards) from illegal behavior means that detected illegal behavior must be worse for him than legal behavior. Thus, illegal behavior must be considered in order that these two alternatives can be compared. More than that, it has to be assumed, in the sense of worst case considerations, that the operator - if at all - will plan and execute his illegal activity such that it is best for him, i.e., that it is detected, depending on the concrete situation, after as long a time as possible. This means that we have to formulate the problem of unannounced

[2] The critical time concept has its origin in the so-called conversion time introduced by the IAEA, see [23]. It says that for each type of fissile material some time, namely the conversion time, is needed to manufacture with its help a nuclear explosive device. Using this definition it appears to be quite natural to assume that the inspection authority has "won" the game if any illegal activity is detected within some "critical" time, otherwise it has lost it.

[3] For further details see section 1.4. Since in all three cases different analytical techniques are applied the first two are not just special cases of the last one.

interim inspections as a conflict situation between two antagonists both of which have several strategies at their disposal, and both of which want to use their best strategy. But what does this mean?

Quite generally is the appropriate tool for analyzing conflict situations the theory of non-cooperative games, also called theory of rational behavior. It requires the description of the full set of (pure) strategies of the players, and the payoffs to both players for all strategy combinations. If more detailed information, e.g., the timing of moves or some complicated information structure has to be taken into account, the resulting games are presented best in *extensive form*. If this is not the case then games are given in *normal form*. In particular, if the numbers of (pure) strategies are finite one arrives at so-called *matrix games*. All these forms are used in this study.

In any case the best strategy combination in non-cooperative games is represented by the Nash equilibrium, see [32], which is defined by the property that any unilateral deviation from that equilibrium does not improve the deviator´s payoff. It should be mentioned in passing that Nash equilibria need not be unique, however, non-uniqueness will not pose a problem in this study.

Applying this definition to safeguards one may say that implementing the concept of deterrence means to look for solutions of the safeguards problem which are Nash equilibria with the property that the equilibrium strategy of the State is legal behavior.

This convincing but somewhat abstract concept requires the definition of payoff parameters of the operator which describe his gains and losses in case of undetected and detected illegal activity. Since they have not yet been estimated by practitioners, a second-best approach is to take the detection time – time elapsing between the start of an illegal activity and its detection – as the payoff to the operator, and its negative value as that of the inspector. In other words, we assume that the operator will maximize his (expected) detection time, whereas the inspector wants to minimize it. This, by the way, corresponds to the "... risk of early detection", see [21].

This will be the basic approach of this study. We will have to consider these zero-sum games both in normal and in extensive forms, and we will determine Nash equilibrium strategies and payoffs, i.e., optimal expected detection times, which then can be used for the planning and implementation of inspections.

1.4 Previous Work

The aspect of timely detection of illegal behavior, see again [21], has been considered from the very beginning of safeguards analyses in the seventies. In the framework of the material balance concept the number of intermediate physical inventory takings has been discussed extensively. Whereas in the beginning, however, their number – and therefore the detection time – was considered more a boundary condition than an objective, later in the eighties, it became an important issue, an objective. The term Near Real Time Accountancy, see [28], characterizes this development best. It was around that time that the *conversion time* concept was introduced, see [23]. Depending on the physical and chemical form of the fissile material, detection times were postulated, ranging from 7 to 10 days for highly enriched uranium or plutonium to three months for spent fuel reactor fuels.

With the new safeguards approach, see [22], the detection time became even more important for several reasons. One of them was to create more flexibility in safeguards procedures in order to be able to distribute the limited inspection resources in a more effective and efficient way – even though it was never clearly defined what this meant in quantitative terms. Thus, unannounced interim inspections were proposed for those facilities in which a continuous presence of inspections was not considered necessary.

Since that time several major studies have been performed which cover some of the afore mentioned sets of assumptions. Even though they will also be quoted in the subsequent chapters, the more important ones will be listed here. Before, let us mention sampling procedures, which are so important for safeguards.

If only items are counted on a random sampling basis, i.e., if *attribute sampling* procedures are used, then falsified items may not be detected with the so-called error of the second kind or non-detection probability β. If quantitative measurements are performed, i.e., if *variable sampling* procedures are used, then in addition correct data or behavior may be declared as wrong with the so-called error of the first kind or false alarm probability α.

Taking the detection time as the payoff to the operator – in the following called *Playing for Time Game* – several studies have been published which cover some of the sets of assumptions given before.

Simultaneous discrete time models have been studied at various occasions without considering errors of the first and second kind, see, e.g., [1]. Only recently, Krieger, see [26] and [27], has investigated the case of one unannounced interim inspection and any number of possible time points; in particular he has shown the transition to the time continuous case studied by Diamond, see [13]. Attribute sampling procedures, i.e., $\beta > 0$, have been taken into account in a systematic way only in this study, both for the discrete and continuous time models.

Sequential discrete time models have to our best knowledge, not yet been studied. Sequential continuous time models have been studied under very general assumptions, inter alia taking into account errors of the first and second kind, i.e., $\alpha > 0, \beta > 0$, by Avenhaus and Canty, see [2]. Hybrid-sequential continuous time models with $\beta > 0$ are studied only in this study; interesting enough, the results coincide with those of the sequential model for attribute sampling procedures, i.e., $\alpha = 0$, and it remains to be shown if this still holds for variable sampling procedures, i.e., $\alpha > 0$.

Just for the sake of completeness let us mention some studies dealing with *critical time* games, i.e., games in which the facility operator has "won" the game if his illegal activity is not detected within some critical time, whereas he has "lost" it, if his illegal activity is detected within that time, and vice versa for the inspector. The work by Dresher, see [14], represents one the oldest and very influential inspection games; it analyses a sequential model and has because of its solution technique become one of the most well-known textbook examples of so-called recursive games, see, e.g., [34].

Both simultaneous and sequential models with $\alpha > 0$ and $\beta > 0$ have been studied by Canty, Rothenstein and Avenhaus, see [11]. Whereas for the simultaneous model very general solutions were obtained, for the sequential model for $\alpha > 0$ and $\beta > 0$, i.e., the generalization of Dresher's work, solutions were obtained only for special cases.

Finally, let us mention some attempt to consider both the playing for time and the critical time concept in one game. One first model has been published by Avenhaus and Krieger, see [5], it will be presented in some detail in the fifth chapter of this study.

1.5 Applications

In section 1.1 of this chapter we presented the objectives of this study. We identified three objectives, starting with the general modelling problem, and arriving at the concrete situation in the EU. Turning to this last problem one may ask what the European safeguards authority expects from quantitative analyses outlined above. Quite concretely it wants to get answers to two questions, namely

- How many unannounced interim inspections shall be performed in one nuclear facility in a State of the EU in a reference time interval, e.g., one calendar year?
- How shall these unannounced interim inspections in a nuclear facility be distributed in the reference time interval?

In principle the first question has to be answered in the way which was outlined before, taking the EU as a whole as antagonist of the IAEA. However, such an approach requires the estimation of payoff parameters which was not possible so far. Instead a pragmatic solution was found which is laid down in the documents of the IAEA/EURATOM Partnership Approach. Nevertheless, at the end of our study this problem will be considered again.

The second question will be answered in great detail in the following chapters. Not only the optimal distribution over time of a given number of unannounced interim inspections in one nuclear facility is determined; it will also be demonstrated what it means for two different types of nuclear facilities, namely on-site interim storage facilities, and fuel element fabrication facilities.

1.6 Organization of the Study

Following the broad outline of this study given so far, it is organized as follows: In order to be concrete and specific, in the second chapter two nuclear facilities are described, and the safeguards measures applied by EURATOM and IAEA in these two facilities are discussed.

The third and fourth chapter are central from the analytical point of view. In the former, it is assumed, that both antagonists, operator and inspector, plan their activities simultaneously, i.e., without knowing the strategy of the other one. In the latter, it is assumed, that the operator uses the knowledge of the inspections performed in the course of the game. In both chapters both discrete and continuous time versions are considered.

In all these four variants, i.e., throughout this study, errors of the second kind are taken into account but not errors of the first kind, in formula $\alpha = 0$ and $\beta > 0$. There are several reasons for these assumptions: First, the modelling effort for taking into account $\alpha > 0$ would be too large, let us just mention that the zero-sum assumption would no longer hold. Second, important results like optimal expected detection times and inspection time points would depend only weakly on α, as previous studies demonstrate, see, e.g., [2]. Third, in our applications primarily attribute sampling procedures were considered, for which $\beta > 0$ and $\alpha = 0$ holds. Also false alarms may happen which can be clarified immediately and therefore need not be modelled formally. It should be mentioned that the non-detection probability β can also be seen as a global parameter and therefore as a function of safeguard measures. Then β is the probability of not detecting some illegal activity and not only, e.g., the probability of not detecting at least one broken seal of a cask. If β_1 is the non-detection probability for checking of seals and β_2 the non-detection probability for other inspection activities and if these data are independent, then the total non-detection probability β is given by $\beta = \beta_1 \cdot \beta_2$.

In the fifth chapter the analyses of the two foregoing chapters are complemented: The aspect of deterrence is discussed, the critical time objective is presented in some detail, and the global sampling problem, i.e., the number and distribution of unannounced interim inspections of the IAEA in the States of the EU, is analyzed in some preliminary way.

In the concluding sixth chapter an attempt is made to summarize the results and to formulate, with all due care, some recommendations.

In the Annexes additional explanations and proofs are given which complement the main text.

Chapter 2

Examples of facility types considered in this Report

In order to emphasize this study´s orientation towards applications we present first, in this chapter, the facilities and the safeguards measures in these facilities, to which the results of our theoretical findings on unannounced interim inspections will be applied.

The first example is an on-site interim storage facility for spent fuel elements and the second one a fabrication facility for fuel elements for light water power reactors. The concrete facilities (Emsland Nuclear Power Plant and the Advanced Nuclear Fuels (ANF) Company, both located at Lingen) were selected since published information was easily available and furthermore, the operators were kindly enough to give personal interviews.

2.1 First example: On-Site Interim Storage Facility

As a first example, on-site interim storage facilities for spent nuclear fuel elements are chosen. In the following these facilities as well as the safeguards measures applied in these facilities are described in some detail. Based on this information, unannounced interim inspections will be discussed in the next section with a discrete time non-sequential game theoretical model.

The origin and function of on-site interim storage facilities have been described at various occasions. We refer to the articles by Behrens et al. [7], Rudolf et al. [41] and Rezniczek et al. [39] as a basis for the safeguards criteria and safeguards measures commonly used in dry storage facilities until today. An outline of the Integrated Safeguards concept and the IAEA/EUTATOM Partnership Approach papers intended as a guideline how to implement this new regime in Germany and the role of Unannounced Interim Inspections conclude this section.

2.1.1 Facility Features

Germany initially planned to store spent nuclear fuel in the two away-from-reactor interim storage facilities at Ahaus and Gorleben. The current approach for spent fuel management is on-site interim storage in transport and storage casks as part of a political agreement between the German government and the operators of nuclear facilities on the future use of nuclear energy. A reason for this agreement was to avoid near term transportation of spent fuel determined for direct disposal via public road or rail systems to away-from-reactor storage facilities. Recent legislation has triggered the construction of on-site dry storage facilities at twelve nuclear power plants. The first license was received by the Emsland Nuclear Power Plant located at Lingen, Lower Saxony, Northern Germany, see Figure 2.1. It was taken into operation in December 2002 and has the following features.

Figure 2.1 Emsland Nuclear Power Plant with on-site interim storage facility [43].

The facility consists of two buildings, namely storage building with storage area and reception area for spent fuel casks, and control building in which plant operations are controlled. The permitted storage period is limited to 40 years beginning with the emplacement of the first spent fuel cask in the storage building. There are 130 cask positions, five of which being reserved for empty casks only. The Lingen interim storage facility has a length of about 110 m, a width of about 30 m, and a height of about 20 m. The wall thickness is about 1.2 m, while the monolithic roof is about 1.3 m thick. The floor is made from armoured concrete.

In the reactor containment, spent fuel elements will be loaded into shielding casks, e.g., of the CASTOR®-type (cask for storage and transport of radioactive material), and then transported from the reactor building into the associated on-site dry storage facility.

2.1.2 Safeguards Measures

All parties involved - German plant operators and State authorities, EURATOM and IAEA - agree in keeping safeguards as simple as possible and furthermore, consistent with all on-site interim storage facilities. Nevertheless, there are technical and organisational differences between those individual facilities that have to be taken into account. From the State authorities´ point of view, safeguards have to comply with requirements related to operational safety, radiation protection, and physical protection. Furthermore, they have to take into account the political and technical boundary conditions as well as the time schedule for spent fuel transfers coordinated between all nuclear power plant operators in Germany. Also, for reasons of keeping persons´ exposure to radiation *as low as reasonably achievable* (ALARA principle) the storage area is not intended for frequent access. This has to be taken into account when designing an adequate safeguards concept for a dry storage facility usually licenced for a period of 40 years.

There is another aspect: Once the spent fuel has been loaded into casks the inventory is no longer accessible and respectively cannot be verified directly. Therefore, the safeguards measures applied to the storage facilities should be capable of maintaining the continuity of knowledge on the cask inventory. This requirement is met by using containment/surveillance (C/S) measures which is a combination of optical surveillance and sealing as well as additional Non-Destructive Analysis (NDA) techniques. Seals attached to individual casks and camera surveillance of the areas, where the casks are handled and stored, play a major role in proving the non-diversion of nuclear material thus providing the basis for a safeguarded dry storage facility.

The sealing of casks, which is on one hand an effective safeguards measure from the inspector´s point of view, on the other hand raises difficulties for the operator in general and in particular. Generally, according to the technical concept of a storage facility a regular visit of the storage hall is not necessary and should be avoided for radiation protection reasons. The operator will enter this area on a need-to-do-basis only. Seals on casks violate the above mentioned ALARA principle. In particular, if metal cap-and-wire seals have to be verified the inspector has to reach the top of the CASTOR®-cask at a height of about 6 m by means of a lifting platform and replace the seals to be verified by new ones. The length of stay in close proximity of the casks, which increases with the number of casks to be verified, may lead to unacceptably high radiation doses for inspectors and staff. In view of maximum of 192 storage positions in the largest on-site dry storage facility the metal seal makes sense rather as a backup seal that will be only verified if other Safeguards measures fail than a regularly verified seal.

To reduce radiation exposure different types of seals should be used ensuring a short-term presence of inspectors and staff in the storage hall for interrogation purposes. The nowadays commonly used COBRA fibre optical seals offer the advantage that the sealing body is at the inspector´s eye level. Therefore COBRA seals can easily be verified without the time-consuming climbing to the cask top. Due to a special screw cap the unauthorized removal of the screw and the seal is excluded. The COBRA seals constitute definitely a progress in comparison to metallic seals but still require a close contact to the casks. In on-site facilities with compact cask storage the inspector even has to slip into the narrow space between the casks where he is not only exposed to radiation but to high temperatures as well. Here, a better solution would be the use of seals with remote interrogation capability like the new generation of the electronic seal type EOSS. This kind of seal is equipped with interfaces allowing seal interrogation remotely from the outside of the storage hall. EOSS seals are planned to be used for the sealing of a group of casks. Although the attachment of the group seals needs also a close contact to the casks the advantage of this sealing mode is paramount.

In the previously mentioned publications by Behrens et al. and Rudolf et al. more details were given regarding camera surveillance and remote data transmission. Since these measures, however, are not relevant for the quantitative analysis to be performed in the next section we do not include them here, but refer to the original papers [7] and [41].

2.1.3 Inspections

The IAEA *Safeguards Criteria* [23], as currently defined, are the set of nuclear material verification activities considered by the IAEA as necessary for fulfilling its responsibilities under safeguards agreements. The Criteria are established for each facility type and location outside facilities (LOF), and specify the scope, the normal frequency and the extent of the verification activities required to meet the quantity and the timeliness components of the inspection goal at facilities and LOFs.

Without going into the details of further definitions in that context, e.g. IAEA inspection goals, quantity and timeliness components of the IAEA inspection goal, it should be mentioned here only that the basis of all quantity definitions is the so-called *significant quantity* (SQ), i.e., the approximate amount of nuclear material for which the possibility of manufacturing a nuclear explosive device cannot be excluded, and the so-called *conversion time*, i.e, the time required to convert different forms of nuclear material to the metallic components of a nuclear explosive device.

On the basis of these criteria and concepts it has been defined that in each of the on-site interim storage facilities once a year a physical inventory is taken, and that every three months a routine inspection is performed. The main purpose of the routine inspections is to check the seals at the casks on a random sampling basis.

For the subsequent quantitative analysis we consider a representative situation where there are N casks $(80$ to $190)$ with spent LWR fuel elements in the storage facility, and where each cask contains 19 spent fuel elements, see [37]. Without going into the details of the usability of the plutonium (Pu) for weapons in the fuel elements, see, e.g., [17] or [29], we assume that there are about 5 kg Pu in each fuel element thus, in order to illegally acquire one significant quantity, the seal of *at most two casks* need to be broken[1]. Here it is assumed that the seal of at most one cask needs to be broken, which represents the worst case for the inspector. In other words, during one routine inspection one broken seal has to be detected with sufficient probability $1-\beta$.

Also we mention that the inspector needs about five minutes net time in the storage to check one seal. There is, however, overhead work to be done by the inspector, primarily the evaluation of the findings outside the storage, and administrative work before and after the whole seal checking procedure. Therefore, during a one day visit only two to three hours may be available for checking seals in the storage.

Let us note in passing that we do not perform any diversion path analysis since this is not necessary for the purposes of this investigation.

Quite generally, let the total number of seals be N, the number of checked seals be n, and the number of broken seals be r. Then according to the hypergeometric distribution law the probability to detect at least one broken seal in case of drawing without replacement is, see, e.g., [1],

$$\begin{aligned} 1-\beta(N,n,r) &= \mathbf{P}(\{\text{ at least one broken seal in the sample }\}) \\ &= 1-\mathbf{P}(\{\text{ no broken seal in the sample }\}) \\ &= 1-\frac{\binom{r}{0}\cdot\binom{N-r}{n-0}}{\binom{N}{n}}, \end{aligned}$$

where the binomial coefficient $\binom{n}{m}$ for $0\le m\le n$, $n=1,2,\dots,$ is defined by

$$\binom{n}{m}=\frac{n!}{m!\cdot(n-m)!}=\frac{n\cdot(n-1)\cdot\ldots\cdot(n-m+1)}{m\cdot(m-1)\cdot\ldots\cdot 1}, \quad 0!=1\,.$$

Thus, for $r=1$ we get

$$1-\beta(N,n,1)=1-\frac{\frac{(N-1)!}{n!\cdot(N-n-1)!}}{\frac{N!}{n!\cdot(N-n)!}}=\frac{n}{N}, \tag{2.1}$$

which means that the probability of detection is proportional to the number of checked seals.

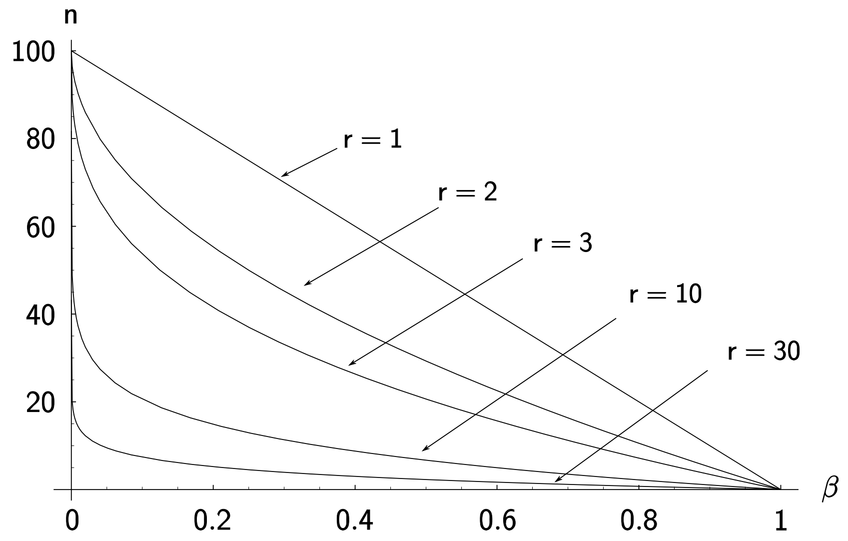
For small r, i.e. $r\ll N$, we get approximately

$$n(\beta)\approx N\cdot\left(1-\sqrt[r]{\beta}\right)=100\cdot\left(1-\sqrt[r]{\beta}\right)\,.$$

Note that in practice $n(\beta)$ has to be a natural number. Figure 2.2 shows for $N=100$ that the sample size n decreases with increasing number r of broken seals and fixed β, and it decreases with increasing β for fixed r, which is intuitive.

[1]For Pu the significant quantity is set to 8 kg.

Figure 2.2 Approximate sample sizes $n(\beta)$ as functions of the non-detection probability β for different numbers r of broken seals ($N = 100$).



For the purposes of illustrating the use of the probability of detection we determine the expected detection time, measured in numbers of quarters of years. We assume that at the occasion of an inventory taking a broken seal is detected with certainty, that the illegal activity starts immediately after the inventory verification - worst case for the inspector - and that EURATOM and IAEA check the seals every three months together. Then the expected detection time ET is given by

$$ET = 1 \cdot (1-\beta) + 2 \cdot \beta \cdot (1-\beta) + 3 \cdot \beta^2 \cdot (1-\beta) + 4 \cdot \beta^3 = 1 + \beta + \beta^2 + \beta^3 \,.$$

For example, we have for $\beta = 0$ resp. $\beta = 0.5$ the expected detection time $ET = 1$ resp. $ET = 1.88$ and $ET = 4$ for $\beta = 1$.

Finally, let us mention that there may be also errors of the first kind, e.g., when the inspector is looking for items which have been moved from the right location and have been put somewhere else. These errors, however, can be clarified immediately and therefore need not be taken into account formally.

2.1.4 Integrated Safeguards and the IAEA/EURATOM Partnership Approach

According to the IAEA Safeguards Glossary [23], *Integrated Safeguards* (IS) is the optimum combination of all safeguards measures available to the IAEA under comprehensive safeguards agreements and additional protocols to achieve maximum effectiveness and efficiency in meeting the IAEA´s safeguards obligations within available resources. There is also the IAEA/EURATOM Partnership Approach (PA), which is an approach for implementing safeguards in the non-nuclear-weapon States of EURATOM. It updates to

Integrated Safeguards (IS) the approach firstly agreed between the IAEA and EURATOM in 1992. The IAEA/EURATOM partnership approach provides for common use of safeguards equipment, joint scheduling of inspections and special arrangements for inspection work and data sharing by the two organizations. The PA enables the IAEA to economize on safeguards equipment and inspection efforts deployed in the relevant States while maintaining its ability to perform independent verification.

At present the traditional procedure is applied to on-site interim storages in Germany, i.e. both EURATOM and IAEA inspectors are present when the inventory is verified and when the three routine inspections per year are performed. In the framework of the above described concepts, IS and NPA, it is discussed [35] that in the future only EURATOM inspectors perform all routine inspections, and that IAEA inspectors perform *Unannounced Interim Inspections*. At this point it should be mentioned that two types of interim inspections are planned, namely *inspections with short notification (SNRI)*, e.g., from one to a few days, and *unannounced interim inspections* with no advance notification. To model the difference between these two types it is necessary to make assumptions about the operator´s possibilities to camouflage illegal activities within the advanced notification time. Since this goes beyond the scope of our study, we will, for the sake of simplicity, just use the term unannounced interim inspections. Since definite decisions have not yet been made, we discuss two alternatives, namely that IAEA inspectors join EURATOM inspectors while they perform routine inspections, or perform there unannounced interim inspections at any time, independent of the EURATOM inspections[2].

In the next chapter we will develop the simultaneous model, first the discrete time version and thereafter the continuous time one, and apply the results of both to the on-site interim storage as well as to the fuel element fabrication facility which will be described subsequently.

[2]The actual proposals (presented in May this year to the German government) of the IAEA for inspections schemes under Integrated Safeguards foresee one annual PIV and one annual Random Interim Inspection (RII) with 20% selection probability for Spent Fuel Storage Facilities (SFSF) in Germany. The RII will be carried out with an advance notification time of 24h to the operator and to EURATOM. This will allow EURATOM to join the inspection, but it is the IAEA who determines the time and the facility to inspect. The IAEA applies such short notice inspection instead of unannounced inspections in those cases where the time between notification of the inspection and arrival of the inspector at the facility is covered by surveillance. The inspector will check on his arrival by review of the surveillance records that no undeclared activities took place during that period. EURATOM will also adapt its safeguards approaches in future. The new EURATOM safeguards scheme for SFSF foresees also one annual PIV, planned and carried out jointly with the IAEA and one annual interim inspection to be carried out as an unannounced or a short notice inspection. The implementation of Integrated Safeguards in Germany will take time and be carried out stepwise. It is presently not yet to foresee when it will be fully implemented. How the inspection schemes will look like in detail in the transition period we do not know, [35].

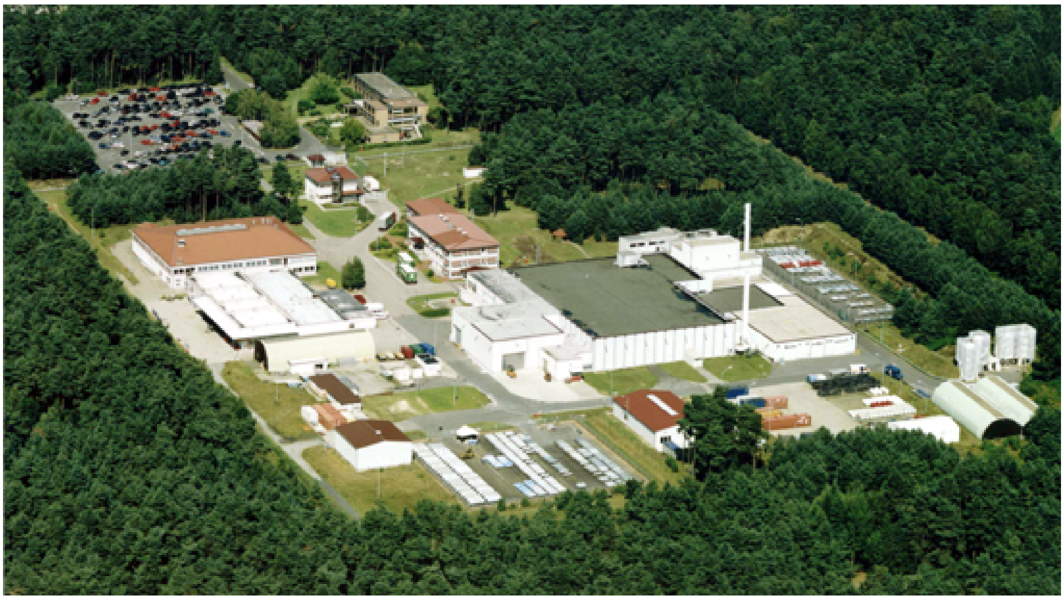
2.2 Second example: Fuel Element Fabrication Facility

As a second example, a fabrication facility for fuel elements for light water power reactors is chosen. As in the first example, the facility itself as well as the safeguards measures applied in this facility are described in some detail, see [36]. Based on this information, unannounced interim inspections will be discussed in the next section with the help of a continuous-time non-sequential game theoretical model.

2.2.1 Facility Features

The fuel element fabrication facility Lingen in Emsland, Germany, see Figure 2.3, is run by the Advanced Nuclear Fuels (ANF) Company (GmbH) which is a daughter of the German Regional Company of AREVA Nuclear Power (NP). Together with facilities for the fabrication of Uranium fuel elements in Romans (France), Dessel (Belgium) and Richland and Lynchburg (USA) the fuel element fabrication facility Lingen is part of the branch Fuel Element Fabrication of AREVA NP.

Figure 2.3 Fuel element fabrication facility Lingen in Emsland [20].



The fuel element fabrication facility Lingen of the ANF produces Uranium fuel elements for pressurized and boiling water power reactors for the German and beyond, for the European market and therefore, contributes to supply the nuclear power reactors with nuclear fuel. The facility is permitted to process up to 650 tons of Uranium per year with an enrichment of up to 5 % U-235.

Nuclear fuel elements are produced in the Lingen facility in four steps: First, in the conversion step gaseous uranium hexafluoride (UF_6) is converted to uranium dioxide (UO_2) powder. In the second step, fuel pellets are produced by pressing and sintering the (UO2) powder followed by a final grinding procedure. In the third step, the pellets are filled into cladding tubes which thereafter are closed using a careful welding procedure. In the fourth and final step, the fuel rods are assembled to fuel elements

The mechanical (non-nuclear) components of the elements like cladding tubes, end-pieces and others are produced in the ANF facilities Karlstein and Duisburg (Germany). Altogether about 800 persons are working in these three facilities.

2.2.2 Safeguards Measures

Basis of all safeguards measures of EURATOM and the IAEA in a bulk processing facility is the verification of the balance of the fissile material processed in the facility which is closed in regular intervals of time, e.g. once a year. For this purpose, an initial physical inventory (PIV) has to be taken, receipts and shipments during the reference time interval have to be determined, and the ending physical inventory has to be taken[3]. With these data, the so-called material unaccounted for (MUF), i.e. the difference between the book inventory (initial inventory plus receipts minus shipments) and the physical inventory at the end of the reference time interval is determined the expected value of which is just the missing material (loss or diversion). In principle, the two inspection authorities proceed in such a way that the operator of the facility measures all inventory and flow data and reports them to the safeguards authorities. The latter ones verify these data with the help of independent measurements on a random sampling basis and, if there were no significant differences, perform the MUF test at the end of the reference time interval with the help of the operator´s data.

Depending on size and other characteristics a facility may consist of one or more material balance areas, i.e. areas for which material balances are established. The Lingen facility consists of just one material balance area, since the storages of receipts and products are kept small, in particular when the PIV is taken. Also, it should be noted that in the Lingen facility the receipts (UF6) are not measured independently, instead the shipment data of those facilities are used which provide the Lingen facility with UF6.

Finally, let us mention - even though we will not use this information in our study - that in the Lingen facility material balances both for uranium and for uranium-235 are established.

In addition to these measures, design information of both the buildings and the production processes is reported and there are C/S measures which are regularly verified by the two inspection authorities.

[3]The acronym of the initial physical inventory taken by the operator is PIT to be distinguished from PIV as physical inventory verification carried out by the inspectorate, see [38]. For the sake of simplicity we use only the acronym PIV.

2.2.3 Inspections, Integrated Safeguards and the IAEA/EURATOM Partnership Approach

At present, in the Lingen facility once a year a physical inventory (PIV) is taken, and both EURATOM and IAEA inspectors are present at this occasion. They take samples of the fissile material at all process stages, measure them with their own instruments and compare the data with the corresponding ones reported by the plant operator. In addition, inspectors of both authorities visit the facility every six to eight weeks. Also at these occasions they take samples, measure these samples, perform the comparisons with the corresponding operator data, and verify design information and C/S measures.

There are some important differences to the situation in the on-site interim storage: First, in the storage a broken seal of a cask, detected at the occasion of an unannounced interim inspection is a strong hint for an illegal activity which in principle can be confirmed (or not) immediately by checking the content of the cask. In a bulk processing facility, however, the inspectors can infer the diversion of material, when they have detected an anomaly in form of a difference between reported and verified data, or some deviation from the design information, only at the end of the reference time interval, when the material balance is closed, i.e. the MUF test is performed.

Thus, the detection probability $1-\beta$, referring to an interim inspection, and introduced in section 2.1.3, has a different meaning here. It rather means the detection of an anomaly, but not necessarily the detection of an illegal activity, i.e. the diversion of fissile material. Thus, a detection probability may be determined with the help of the size of the samples of items verified, like in section 2.1.3, or it may be determined, e.g. in case of design verification, with the help of the fraction of parts of the facility which have been inspected. A final statement about missing material can only be made with the help of the material balance after a PIV has been taken.

Second, the verification of quantitative measurements poses new problems. Since here, as opposed to the checking of seals, measurement errors never can be avoided, errors of the first kind have to be taken into account. This is important if the MUF test is performed; in our case however - analysis of unannounced interim inspections - they may be neglected, at least in this study. There may be other sources for errors of the first kind, e.g., there are both bulk material and items like UF6 drums for which misdeclarations of location can happen. These errors, however, can be clarified immediately and therefore need not be taken into account formally.

What has been said in general in section 2.1.4 on Integrated Safeguards and the IAEA/-EURATOM partnership approach holds also here. In effect, IAEA inspectors will be present when the PIV is taken, and they will visit the facility in addition once or twice the year, together with EURATOM inspectors or, possibly in some cases independently. Again, we will consider both cases which means that we apply again both the results of the discrete time model and of the continuous time one to the fuel fabrication facility.

Chapter 3

Simultaneous (non-sequential) models

In this chapter we assume that the inspection authorities EURATOM and IAEA as well as the plant operator (here representing the State in the context of NPT safeguards) plan their activities - inspection and illegal activity, if at all - before the beginning of the reference time interval under consideration (e.g., a year). Also we assume that the objective of the inspection authorities is to minimize the time between the start and detection of the illegal activity, whereas the plant operator wants to maximize this time.

We consider two possibilities concerning the time points at which the inspections and the start of the illegal activity may take place, namely at discrete time points and at any time points. The models are applied to prototypes of an on-site interim storage and a fuel fabrication facility, which had been described in the forgoing chapter.

3.1 Discrete time models

For situations, where the inspector can only perform unannounced interim inspections at a finite number of well-defined time points we develop discrete time models. The analysis of these models differs considerably from that of continuous time models which will be considered subsequently.

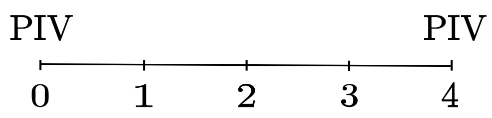
3.1.1 Mathematical analysis of unannounced interim inspections in an on-site interim storage facility

After having described on-site interim storages for spent fuel and EURATOM and IAEA inspection procedures we will now, based on the information given above, build a game theoretical model, which can be used for the optimization of IAEA inspections.

According to our assumptions we have only two players, namely the plant operator representing the State in the sense of the NPT and the IAEA inspector.

As already mentioned in section 2.1.3 it was decided that in each of the on-site interim storage facilities once a year a physical inventory verification (PIV), and that every three months a routine inspection is performed. Therefore, the IAEA inspector has besides the PIV three more possible time points for performing his inspection(s). This situation is graphically represented in Figure 3.1.

Figure 3.1 General inspection situation.



At time point 0 the PIV is performed and there remain the three time points 1, 2 and 3. Let k be the number of the unannounced interim inspection(s). k can only have the values $1, 2$ or 3. If $k = 3$, the inspector will make his inspection every three month and therefore the inspection are no more longer unannounced. For different values of k we obtain the following sets of pure strategies of the inspector:

- $k = 1$: the set of pure strategies is $\Phi_{Insp,1} = \{1, 2, 3\}$, i.e., the set of time points at which he can perform his inspection,
- $k = 2$: the set of pure strategies is $\Phi_{Insp,2} = \{(1, 2), (1, 3), (2, 3)\}$, i.e., the set of pairs of time points at which he can perform his two inspections, and
- $k = 3$: the only pure strategy is $\Phi_{Insp,3} = \{(1, 2, 3)\}$, i.e., performing his three inspections at the time points 1,2 and 3.

Before considering detailed strategies of the operator we assume that he will break seals in order to divert nuclear material: "This diversion hypothesis should not been understood - and in general is not understood - as an expression of distrust directed against States in general or any State in particular. Any misunderstanding might be dispelled by comparing diversion hypothesis with the philosophy of airport control. In order to be effective, airport control has to assume a priori and without any suspicion against a particular passenger that each handbag might contain prohibited goods", see [18]. Legal behavior will only be discussed in section 5.1.

The operator can start his illegal activity at *any* time point of the reference time interval. However, he will start his illegal activity at the time points $0, 1, 2$ and 3, since otherwise the time elapsed between the start of the illegal activity and its detection would become shorter. Therefore,

- The set of pure strategies of the operator is $\Phi_{Op} = \{0, 1, 2, 3\}$.

So far we have been defining the possible choices of each player but it is not yet clear in which way these choices are made by the players. We assume here that the operator needs time to prepare an illegal activity, since for the construction of a nuclear device more technical equipment is needed which for itself needs time for its preparation. Therefore, we assume in this chapter that the operator decides at the beginning of the year (or the reference time interval) at which time point he will start his illegal activity.

The IAEA on the other side has to plan the inspections over the Globe and also has to think in advance at which time a certain facility has to be inspected, i.e., the IAEA inspector has also to decide at the beginning of the year (or the reference time interval) at which time point he will perform his inspection(s). He also has to fix the number of unannounced interim inspections k ($k = 1, 2, 3$). The number k chosen by the inspector is also known to the operator[1].

Since neither the operator nor the inspector knows at the beginning of the year at which time point(s) the adversary will perform his inspection(s) resp. will start his illegal activity, we assume that they make their choices independently of each other (but of course following certain rules, which we explain in the following). One might think of situations in which the operator observes the behavior of the inspector and changes his own strategy appropriately. Such a decision making during the reference time interval is not possible in the models discussed in this chapter. It will be the subject of the next chapter.

The question is now, what the actors gain in case they choose a pure strategy independently of each other. For that purpose we have to consider the objectives of the operator and the inspector when a pure strategy combination is played. There are several possibilities to define such objectives. We will choose here the concept of *playing for time*, that is, the objective of the operator is to maximize the time between start and detection of the illegal activity, whereas the inspector wants to minimize this time. This means that we consider a *zero-sum game* with the detection time as payoff to the operator.

Let us mention that the playing for time concept is very intuitive, and that it meets one of the IAEA safeguards criteria "... by the risk of early detection", see [21]. There are, however, alternative important concepts; one of them will be discussed in chapter 5.

We assume that the inspector will commit an error of the second kind per inspection, i.e., an illegal activity is not detected with probability β although there is one.

In case of the coincidence of the start of the illegal activity and the inspection, the illegal activity is detected only at the occasion of the next inspection or the PIV.

Finally, the game ends either after the final PIV or after that interim inspection at which the illegal activity is detected. What happens in reality in the latter case is not discussed here.

Let us summarize the assumptions which we have made so far:

[1] This assumption deserves some justification: As long as the inspector knows the number k for one specific facility and year, the operator of that facility will know it after the first year. If, however, the number k is chosen randomly by the inspector, e.g., in the context of a larger number of facilities within a State, see section 5.3, then this assumption does not hold anymore. In this case one has to consider a larger game where the choice of the single facility and the number k of inspections in that facility represent just a part of a pure strategy of the inspector. As a result k may be randomized.

(i) There are two players: operator and inspector.

(ii) The inspector has three time points for inspections.

(iii) The operator behaves illegally, see page 20.

(iv) The operator has four time points for starting his illegal activity.

(v) The inspector commits an error of the second kind per inspection, i.e., an illegal activity is not detected with probability β although there is one.

(vi) The number of interim inspections is also known to the operator. At most two unannounced interim inspections are permitted in one facility and the reference time interval, see section 5.3.1[2].

(vii) Both players decide at the beginning of the reference time interval when to start the illegal activity and when to inspect.

(viii) Both players decide independently of each other.

(ix) The payoff to the operator is the time between start of the illegal activity and its detection. The payoff to the inspector is the negative one (zero-sum game).

(x) In case of the coincidence of the start of the illegal activity and the inspection, the illegal activity is detected only at the occasion of the next inspection or the PIV.

(xi) The game ends either after the final PIV or after that interim inspection at which the illegal activity is detected.

This verbal description of our inspection problem leads us to so-called matrix games, which are formally introduced in Appendix B.

The case of one unannounced interim inspection ($n = 3$ and $k = 1$)

In this section we consider the case $k = 1$, i.e., one unannounced interim inspection is performed. This conflict situation is depicted in Figure 3.2. In the first column the (pure) strategies of the operator are given, namely starting his illegal activity at time point 0, 1, 2 or 3. In the first row the (pure) strategies of the inspector are shown, i.e., the time at which he will perform his inspection. An entry in this payoff matrix means that if the operator starts his illegal activity at time point i and the inspector performs his inspection at time point j then the entry in the matrix gives us the expected time between start and detection of the illegal activity.

For two cases we explain the payoff and its computation. Let $i = 0$ and $j = 1$, i.e., the operators starts his illegal activity at time point 0 while the inspector performs his

[2] The arguments given in section 5.3.1 hold only for on-site-interim storage facilities. For fuel element fabrication facilities the documents by IAEA and EURATOM do not provide corresponding data, but we assume that similar arguments hold here as well.

Figure 3.2 The matrix game in case of one unannounced interim inspection ($n = 3$ and $k = 1$).

	1	2	3
0	$1 + 3 \cdot \beta$	$2 + 2 \cdot \beta$	$3 + \beta$
1	3	$1 + 2 \cdot \beta$	$2 + \beta$
2	2	2	$1 + \beta$
3	1	1	1

inspection at 1. Then this illegal activity is detected at 1 with probability $1 - \beta$ and not detected at 1 with β. If it is not detected at 1 then it will be detected at the end (PIV) with certainty. Therefore, we get

$$1 \cdot (1 - \beta) + 4 \cdot \beta = 1 + 3 \cdot \beta \,.$$

If $i = j = 1$, then - according to our model assumption - the illegal activity will be detected at the end and is therefore 3.

According to Appendix B a solution of this game is a so-called saddle point consisting of an optimal strategy for both the inspector and the operator. Therefore, we have to determine also the latter even though we are actually interested only in the former one.

Let $\mathbf{q}^T = (q_0, q_1, q_2, q_3)$ be a mixed strategy of the operator and $\mathbf{p}^T = (p_1, p_2, p_3)$ a mixed strategy of the inspector. Here, q_i is the probability to start the illegal activity at time point i and p_j the probability to perform the inspection at time point j. Then, depending on the value of the non-detection probability β, the optimal strategies $(\mathbf{q}^*, \mathbf{p}^*)$ as well as the optimal expected detection time $Op^*_{3,1}(\beta) = Op_{3,1}(\beta; \mathbf{q}^*, \mathbf{p}^*)$[3] are given as follows:

- For $0 \leq \beta < \dfrac{1}{6}$ we have the optimal strategies

  $$p_1^* = \frac{1}{1-\beta} \cdot \frac{1}{3}\,, \quad p_2^* = \frac{1}{1-\beta} \cdot \frac{1}{2}\,, \quad p_3^* = \frac{1}{1-\beta} \cdot \left(\frac{1}{6} - \beta\right) \tag{3.1}$$

  and

  $$q_0^* = \frac{1}{3}\,, \quad q_1^* = \frac{1}{6}\,, \quad q_2^* = \frac{1}{2}\,, \quad q_3^* = 0 \tag{3.2}$$

[3]Since for many different models and versions expected detection times will be determined, we use an appropriate notation which is explained in Appendix A.

with

$$Op^*_{3,1}(\beta) = \frac{11}{6} + \beta\,. \tag{3.3}$$

- For $\beta = \frac{1}{6}$ we have

$$p^*_1 = \frac{2}{5}\,, \quad p^*_2 = \frac{3}{5}\,, \quad p^*_3 = 0 \tag{3.4}$$

as optimal strategy for the inspector and the following optimal strategies for the operator

$$\begin{pmatrix} q^*_0 \\ q^*_1 \\ q^*_2 \\ q^*_3 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 1/3 \\ 1/6 \\ 1/2 \\ 0 \end{pmatrix} + (1-\lambda) \cdot \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} \,, \lambda \in [0,1]\,, \tag{3.5}$$

with

$$Op^*_{3,1}(\beta) = 2\,. \tag{3.6}$$

- For $\frac{1}{6} < \beta < \frac{2}{3}$ we have the optimal strategies

$$p^*_1 = \frac{1}{1-\beta} \cdot \frac{1}{3}\,, \quad p^*_2 = \frac{1}{1-\beta} \cdot \frac{2 - 3 \cdot \beta}{3}\,, \quad p^*_3 = 0 \tag{3.7}$$

and

$$q^*_0 = \frac{2}{3}\,, \quad q^*_1 = \frac{1}{3}\,, \quad q^*_2 = 0\,, \quad q^*_3 = 0 \tag{3.8}$$

with

$$Op^*_{3,1}(\beta) = \frac{5}{3} + 2 \cdot \beta = \frac{10}{6} + 2 \cdot \beta\,. \tag{3.9}$$

- For $\beta = \frac{2}{3}$ we have

$$p^*_1 = 1\,, \quad p^*_2 = 0\,, \quad p^*_3 = 0 \tag{3.10}$$

as optimal strategy for the inspector and the following optimal strategies for the operator

$$\begin{pmatrix} q^*_0 \\ q^*_1 \\ q^*_2 \\ q^*_3 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} + (1-\lambda) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \,, \lambda \in [0,1]\,, \tag{3.11}$$

with

$$Op^*_{3,1}(\beta) = 3\,. \tag{3.12}$$

- For $\frac{2}{3} < \beta \leq 1$ we have the optimal strategies

$$p_1^* = 1\,, \quad p_2^* = 0\,, \quad p_3^* = 0 \tag{3.13}$$

and

$$q_0^* = 1\,, \quad q_1^* = 0\,, \quad q_2^* = 0\,, \quad q_3^* = 0 \tag{3.14}$$

with

$$Op_{3,1}^*(\beta) = 1 + 3 \cdot \beta = \frac{6}{6} + 3 \cdot \beta\,. \tag{3.15}$$

The proof of these results is given in section D.1.1 of Appendix D. Figure 3.3 presents an overview of the optimal strategies of this game except those for the limiting cases of β since they will not be realized in practice.

Figure 3.3 Optimal strategies and corresponding payoffs in case of one unannounced interim inspection ($n = 3$ and $k = 1$). The limiting cases are omitted.

	$0 \leq \beta < \frac{1}{6}$	$\frac{1}{6} < \beta < \frac{2}{3}$	$\frac{2}{3} < \beta \leq 1$
q_0^*	$\frac{1}{3}$	$\frac{2}{3}$	1
q_1^*	$\frac{1}{6}$	$\frac{1}{3}$	0
q_2^*	$\frac{1}{2}$	0	0
q_3^*	0	0	0
p_1^*	$\frac{1}{1-\beta} \cdot \frac{1}{3}$	$\frac{1}{1-\beta} \cdot \frac{1}{3}$	1
p_2^*	$\frac{1}{1-\beta} \cdot \frac{1}{2}$	$\frac{1}{1-\beta} \cdot \frac{2-3\beta}{3}$	0
p_3^*	$\frac{1}{1-\beta} \cdot \left(\frac{1}{6} - \beta\right)$	0	0
$Op_{3,1}^*(\beta)$	$\frac{11}{6} + \beta$	$\frac{10}{6} + 2\beta$	$\frac{6}{6} + 3\beta$

Following our description of the inspections in the on-site interim storage, during a one day visit the inspector has two hours for checking seals. Since he needs 5 minutes to check one seal, we arrive for $N = 100$ using (2.1) at $\beta = 0.24$. Thus we deal with the optimal strategies of case $1/6 < \beta < 2/3$.

We continue with a few remarks on the solution of the game: First we look at this inspection problem from the common sense point of view. In that case the inspector should perform his inspection at time point 2, i.e., in the middle of the reference time interval. This way we would get the detection times as given in Figure 3.2. It turns out, that the game theoretical solution leads to slightly shorter detection times for all values of β. For $\beta = 0$, e.g., the game theoretical solution is $11/6$ which is smaller than 2.

Second, the cases $\beta = 2/3$ and $\beta = 1/6$ are not so important for practical reasons, since as already mentioned it is rather implausible to get for real applications exactly these two values. Third, solutions of most games seldom are intuitive. So it is also in our game. It is not trivial nor explainable that the inspector performs his inspection at time point 1 with the probability given here. It is rather a result. But what can be done in this game is to explain the structure of the optimal strategies. From the common sense of view it is clear that when the non-detection probability β is high ($\beta > 2/3$), the operator will start as early as possible and so the inspector will also perform his inspection as early as possible. Fourth, it is very interesting and surprising, that the operator´s optimal strategies are constant in given intervals of β, contrary to the inspector's optimal strategies.

The case of two unannounced interim inspections ($n = 3$ and $k = 2$)

In this game the inspector's set of pure strategy is given by $\Phi_{Insp,2} = \{(1,2),(1,3),(2,3)\}$ and therefore we obtain

$$Q_{Insp,2} = \{ (p_{(1,2)}, p_{(1,3)}, p_{(2,3)})^T \in \mathbb{R}^3 \;\; : \;\; p_{(1,2)} \geq 0, p_{(1,3)} \geq 0, p_{(2,3)} \geq 0 \;\; \text{and} \;\; p_{(1,2)} + p_{(1,3)} + p_{(2,3)} = 1 \}$$

as the set of mixed strategies. The operator starts again his illegal activity at 0, 1, 2 or 3, i.e., $\Phi_{Op} = \{ 0, 1, 2, 3 \}$ and his mixed strategy $\mathbf{q}^T$ is defined as before.

The simultaneous inspection game is depicted in Figure 3.4.

For the pure strategy combination $(0,(1,2))$, i.e., the operators starts at 0 with his illegal activity and the inspector performs his inspection at 1 and 2. Then the illegal activity is detected at 1 with probability $1 - \beta$ and not-detected with probability β. In the latter case at time point 2 the illegal activity is detected again with probability $1 - \beta$ and not detected with probability β[4]. In the latter case the inspector detects the illegal activity at the end of the year (because of the PIV). Using a kind of decision tree this situation can be illustrated graphically, see Figure 3.5.

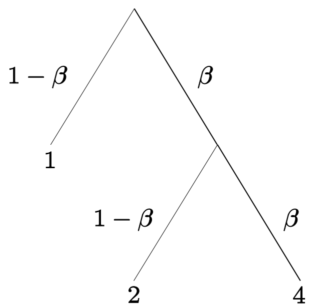
Therefore we get for the operator´s payoff in case of the pure strategy combination

[4]There are situations where inspectors tend to seal items together. In those cases they can be treated as a "static part" reducing the number of items to be checked. Therefore, the probability of detection for the second inspection may be different from that for the first one.

Figure 3.4 The matrix game in case of two unannounced interim inspections ($n = 3$ and $k = 2$).

	$(1,2)$	$(1,3)$	$(2,3)$
0	$1 + \beta + 2 \cdot \beta^2$	$1 + 2 \cdot \beta + \beta^2$	$2 + \beta + \beta^2$
1	$1 + 2 \cdot \beta$	$2 + \beta$	$1 + \beta + \beta^2$
2	2	$1 + \beta$	$1 + \beta$
3	1	1	1

Figure 3.5 Illustration of the computation of entry $(0, (1, 2))$ of the payoff matrix in Figure 3.4.



$(0, (1, 2))$

$$Op_{3,2}(\beta; 0, (1, 2)) = 1 \cdot (1 - \beta) + (2 \cdot (1 - \beta) + 4 \cdot \beta) \cdot \beta = 1 + \beta + 2 \cdot \beta^2 \,.$$

The remaining entries can be derived in a similar way. Let $\mathbf{q}^T = (q_0, q_1, q_2, q_3)$ be a mixed strategy of the operator and $\mathbf{p}^T = (p_{(1,2)}, p_{(1,3)}, p_{(2,3)})$ a mixed strategy of the inspector. Here, q_i is again the probability to start the illegal activity at time point i and $p_{(j_1,j_2)}$ the probability to perform the first inspection at time point j_1 and the second inspection at time point j_2. Then, depending on the value of the non-detection probability β, the optimal strategies $(\mathbf{q}^*, \mathbf{p}^*)$ as well as the optimal expected detection time $Op^*_{3,2}(\beta) = Op_{3,2}(\beta; \mathbf{q}^*, \mathbf{p}^*)$ are given as follows:

- For $0 \leq \beta < \dfrac{1}{2}$ we have the optimal strategies

$$p^*_{(1,2)} = \frac{1}{1-\beta} \cdot \frac{1+\beta+2\cdot\beta^2+\beta^3}{3+2\cdot\beta+\beta^2}\,, \tag{3.16}$$

$$p^*_{(1,3)} = \frac{1}{1-\beta} \cdot \frac{(1-2\cdot\beta)\cdot(1+\beta+\beta^2)}{3+2\cdot\beta+\beta^2}\,, \tag{3.17}$$

$$p^*_{(2,3)} = \frac{1}{1-\beta} \cdot \frac{(1-2\cdot\beta)\cdot(1+\beta)}{3+2\cdot\beta+\beta^2} \tag{3.18}$$

and

$$q^*_0 = \frac{1+\beta}{3+2\cdot\beta+\beta^2}\,, \quad q^*_1 = \frac{1}{3+2\cdot\beta+\beta^2} \quad \text{and} \tag{3.19}$$

$$q^*_2 = \frac{1+\beta+\beta^2}{3+2\cdot\beta+\beta^2}\,, \quad q^*_3 = 0 \tag{3.20}$$

with

$$Op^*_{3,2}(\beta) = \frac{4+6\cdot\beta+5\cdot\beta^2+2\cdot\beta^3}{3+2\cdot\beta+\beta^2}\,. \tag{3.21}$$

- For $\beta = \dfrac{1}{2}$ we have

$$p^*_{(1,2)} = 1\,, \quad p^*_{(1,3)} = 0\,, \quad p^*_{(2,3)} = 0 \tag{3.22}$$

as optimal strategy for the inspector and the following optimal strategies for the operator

$$\begin{pmatrix} q^*_0 \\ q^*_1 \\ q^*_2 \\ q^*_3 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 6/17 \\ 4/17 \\ 7/17 \\ 0 \end{pmatrix} + (1-\lambda) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \lambda \in [0,1]\,, \tag{3.23}$$

with

$$Op^*_{3,2}(\beta) = 2\,. \tag{3.24}$$

- For $\dfrac{1}{2} < \beta \leq 1$ we have the optimal strategies

$$p^*_{(1,2)} = 1\,, \quad p^*_{(1,3)} = 0\,, \quad p^*_{(2,3)} = 0 \tag{3.25}$$

and

$$q^*_0 = 1\,, \quad q^*_1 = 0\,, \quad q^*_2 = 0\,, \quad q^*_3 = 0 \tag{3.26}$$

with

$$Op^*_{3,2}(\beta) = 1+\beta+2\cdot\beta^2\,. \tag{3.27}$$

The proof of these results is given in section D.1.2 of Appendix D. Figure 3.6 presents an overview of the optimal strategies of this game except those for the limiting cases of β since the will not be realized in practice. It is surprising, that the operator´s and the inspector's optimal strategy is constant for all $1/2 < \beta < 1$.

Figure 3.6 Optimal strategies and corresponding payoffs in case of two unannounced interim inspections ($n = 3$ and $k = 2$). The limiting case is omitted.

	$0 \leq \beta < \frac{1}{2}$	$\frac{1}{2} < \beta \leq 1$
q_0^*	$\frac{1+\beta}{3+2\cdot\beta+\beta^2}$	1
q_1^*	$\frac{1}{3+2\cdot\beta+\beta^2}$	0
q_2^*	$\frac{1+\beta+\beta^2}{3+2\cdot\beta+\beta^2}$	0
q_3^*	0	0
$p_{(1,2)}^*$	$\frac{1}{1-\beta}\cdot\frac{1+\beta+2\cdot\beta^2+\beta^3}{3+2\cdot\beta+\beta^2}$	1
$p_{(1,3)}^*$	$\frac{1}{1-\beta}\cdot\frac{(1-2\cdot\beta)\cdot(1+\beta+\beta^2)}{3+2\cdot\beta+\beta^2}$	0
$p_{(2,3)}^*$	$\frac{1}{1-\beta}\cdot\frac{(1-2\cdot\beta)\cdot(1+\beta)}{3+2\cdot\beta+\beta^2}$	0
$Op_{3,2}^*(\beta)$	$\frac{4+6\cdot\beta+5\cdot\beta^2+2\cdot\beta^3}{3+2\cdot\beta+\beta^2}$	$1+\beta+2\cdot\beta^2$

The case of three (unannounced) interim inspections ($n = 3$ and $k = 3$)

For the sake of completeness we consider three (unannounced) interim inspections although they are not taken into account in our applications. The treatment of this case is simple, since the inspector has no real choice because he has only the pure strategy $(1, 2, 3)$, i.e., he has to perform his inspections at any possible time point. The game is depicted in Figure 3.7.

The entries in the payoff matrix can again be determined with help of a kind of decision tree like in Figure 3.5. If $\beta > 0$ then the operator will always choose time point $i = 0$ for the start of his illegal activity, since the expected detection time is there as large as possible. In case of $\beta = 0$ the operator may start at any time point he wish and the expected detection time is always 1. Formally we get:

- For $\beta = 0$ we have the optimal strategy

$$p_{(1,2,3)}^* = 1$$

Figure 3.7 The matrix game in case of three (unannounced) interim inspections ($n = 3$ and $k = 3$).

	$(1,2,3)$
0	$1+\beta+\beta^2+\beta^3$
1	$1+\beta+\beta^2$
2	$1+\beta$
3	1

for the inspector and each element $\mathbf{q} \in Q_{Op}$ is an optimal strategy for the operator. The optimal expected detection time is 1, i.e., $Op^*_{3,3}(\beta) = 1$.

- For $\beta > 0$ we have the optimal strategy

$$p^*_{(1,2,3)} = 1 \quad \text{and} \quad q^*_0 = 1\,, \quad q^*_1 = 0\,, \quad q^*_2 = 0\,, \quad q^*_3 = 0$$

with

$$Op^*_{3,3}(\beta) = 1+\beta+\beta^2+\beta^3\,.$$

Discussion of results

We have introduced simultaneous models for inspections in on-site interim storage facilities with different numbers of unannounced interim inspections. Now we discuss the results of their analysis and link them to the practice of inspections in those kind of facilities.

In Figure 3.8 we have drawn in the upper diagram the optimal expected detection times, i.e., those times which elapse between start and detection of the illegal activity, when the optimal strategies are played, for the three cases $k = 1, 2$ and 3. In the diagram below we draw relation (2.1) for the typical value $N = 100$ for an on-site interim storage facility, see section 2.1.3.

It can be seen that $Op^*_{3,3}(\beta) < Op^*_{3,2}(\beta) < Op^*_{3,1}(\beta)$ for $\beta \in [0,1)$ and $Op^*_{3,3}(1) = Op^*_{3,2}(1) = Op^*_{3,1}(1)$. This result is clear due to the fact, that more possible unannounced interim inspection(s) lead(s) to the a shorter optimal expected detection time. In case of $\beta = 1$. i.e., the detection probability $1-\beta$ is zero, any illegal activity is detected only at the end of the reference time interval and therefore the detection time is 4.

If the desired optimal expected detection time is about 1.5 quarters of a year we see that this expected detection time cannot be reached with $k = 1$, i.e., one unannounced interim inspection, since $Op^*_{3,1}(0) = 11/6 \approx 1.833$. An important question from the practical point of view is, if the number of unannounced interim inspections can be reduced assuring a desired optimal expected detection time. This question can be answered with the help of Figure 3.8.

Suppose the desired optimal expected detection time is about 2 quarters of a year. Then we see that

- In case of one unannounced interim inspection the non-detection probability has to be about 0.16 and therefore the sample size has to be about 84. The arrows with the solid lines illustrate this argumentation.
- In case of two unannounced interim inspections the non-detection probability has to be about 0.5 and therefore the sample size has to be about 50 per inspection. The arrows with the dotted-dashed lines illustrate this argumentation.
- In case of three (unannounced) interim inspections the non-detection probability has to be about 0.54 and therefore the sample size has to be about 46 per inspection. The arrows with the dotted lines illustrate this argumentation.

Thus, in the first case 7 hours net time are needed to check 84 seals. In the second case in total $2 \cdot 50 = 100$ seals have to be checked which needs about 8 hours and in the third case in total $3 \cdot 46 = 108$ seals have to be checked which needs about 9 hours net time. It depends on the overhead times which case is more economic for the inspection authority.

We can formalize this consideration with the help of a cost model: Let a be the overhead cost per inspection (travel and accommodation), and b the cost of checking one seal (inspector manhour cost). Then, for a postulated optimal expected detection time 2 the total cost of inspections are

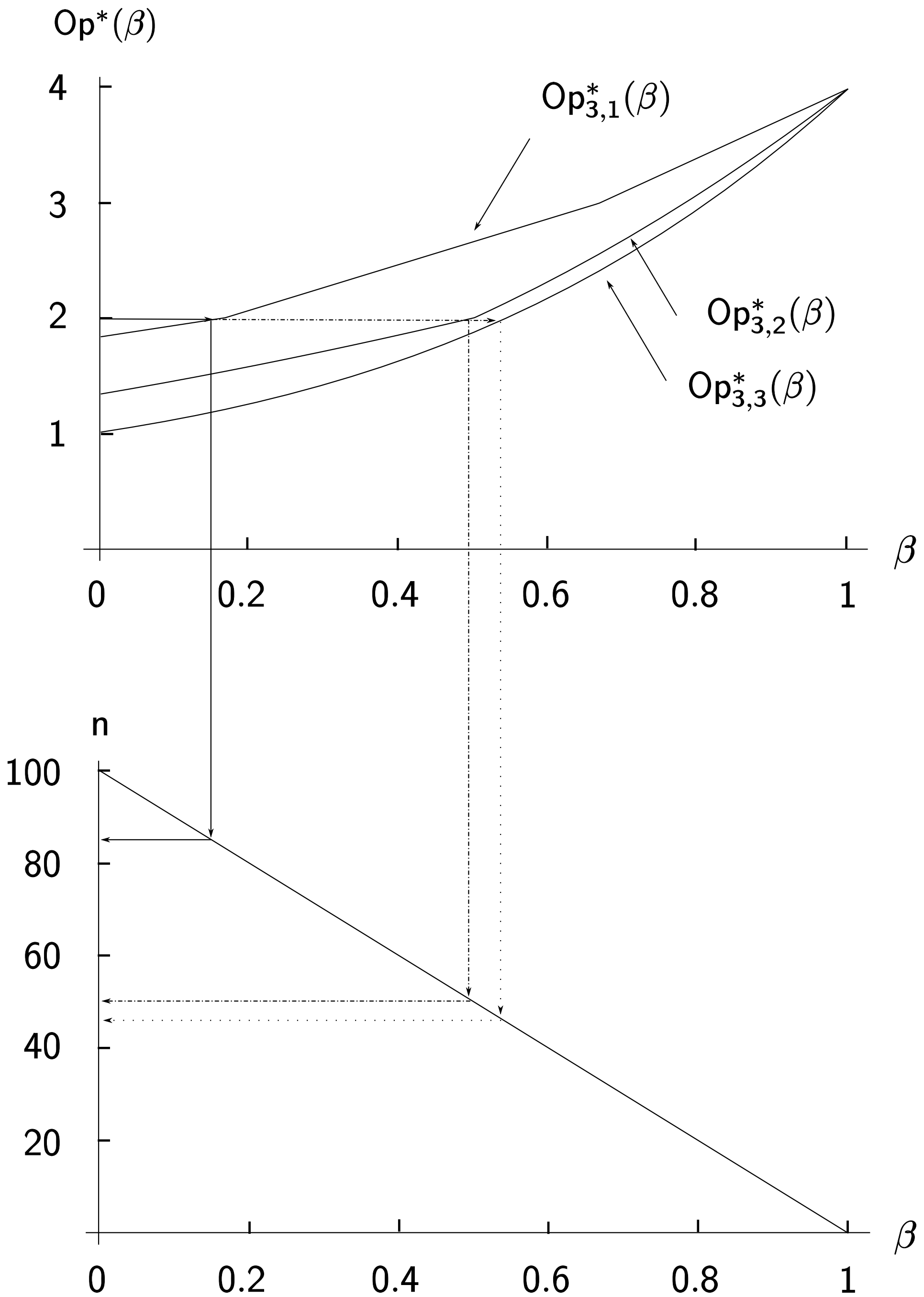
$$\begin{aligned} a + b \cdot 84 \quad & \text{for} \quad k = 1 \\ 2 \cdot a + b \cdot 2 \cdot 50 \quad & \text{for} \quad k = 2 \\ 3 \cdot a + b \cdot 3 \cdot 46 \quad & \text{for} \quad k = 3\,. \end{aligned}$$

We see immediately, that from this cost model point of view $k = 1$ inspection is the best choice, and this holds independently of the chosen optimal expected detection time (as long as the postulated expected detection time is larger than $Op^*_{3,1}(0)$, see Figure 3.8).

Of course, more complicated cost models could lead to different results. If, for example, the checking of 84 seals can not be achieved in one day, contrary to the checking of 46 seals, overhead costs may favor more than one inspection. Thus, a decision based on a cost model can only be made with the help of truly realistic cost data.

During the discussions in the course of this work JRC representatives forwarded the idea that the IAEA might accept models and solutions the easier the shorter the optimal expected detection time would be, ideally shorter than the conversion time. In our model we always have $Op^*_{3,k}(\beta) > 1$ for all β, i.e., $Op^*_{3,k}(\beta)$ is always greater than the conversion time. In order to meet the requirement that $Op^*_{3,k}(\beta) < 1$ for a k and β we have to consider continuous time models, which are the subject of section 3.2.

Figure 3.8 Upper graph: Optimal expected detection times as functions of β. Lower graph: inspection sample size n as a function of β. Further explanations are given in the text.



3.1.2 Mathematical analysis of unannounced interim inspections in a fuel element fabrication facility

In the same way as for the on-site interim storage facility we analyze unannounced interim inspections in the fuel element fabrication facility. The basic assumptions (i),(iii) and (v) - (xi) for the on-site interim storage formulated in section 3.1.1 hold here as well. Instead of (ii) and (iv) we have

(ii') The inspector has five time points for inspections.

(iv') The operator has six time points for starting his illegal activity.

The case of one unannounced interim inspection ($n = 5$ and $k = 1$)

In this case the operator has the time points $0, 1, \ldots, 5$ for starting his illegal activity and the inspector can perform his inspection at the time points $1, \ldots, 5$. The matrix of the matrix game is given in Figure 3.9.

Figure 3.9 The matrix game in case of one unannounced interim inspection ($n = 5$ and $k = 1$).

	1	2	3	4	5
0	$1 + 5 \cdot \beta$	$2 + 4 \cdot \beta$	$3 + 3 \cdot \beta$	$4 + 2 \cdot \beta$	$5 + \beta$
1	5	$1 + 4 \cdot \beta$	$2 + 3 \cdot \beta$	$3 + 2 \cdot \beta$	$4 + \beta$
2	4	4	$1 + 3 \cdot \beta$	$2 + 2 \cdot \beta$	$3 + \beta$
3	3	3	3	$1 + 2 \cdot \beta$	$2 + \beta$
4	2	2	2	2	$1 + \beta$
5	1	1	1	1	1

The solution of the game is given as follows[5]. For practical reasons the limiting cases are not represented completely.

[5]For the notation of the expected detection times see again Appendix A.

- For $0 \leq \beta < \frac{13}{60}$ we have the optimal strategies

$$p_1^* = \frac{1}{5} \cdot \frac{1}{1-\beta}, \quad p_2^* = \frac{1}{4} \cdot \frac{1}{1-\beta}, \quad p_3^* = \frac{1}{3} \cdot \frac{1}{1-\beta}, \tag{3.28}$$

$$p_4^* = \frac{13 - 60 \cdot \beta}{60} \cdot \frac{1}{1-\beta}, \quad p_5^* = 0 \tag{3.29}$$

and

$$q_0^* = \frac{2}{5}, \quad q_1^* = \frac{1}{10}, \quad q_2^* = \frac{1}{6}, \quad q_3^* = \frac{1}{3}, \quad q_4^* = q_5^* = 0 \tag{3.30}$$

with

$$Op_{5,1}^*(\beta) = \frac{77}{30} + 2 \cdot \beta\,. \tag{3.31}$$

- For $\frac{13}{60} \leq \beta < \frac{11}{20}$ we have the optimal strategies

$$p_1^* = \frac{1}{5} \cdot \frac{1}{1-\beta}, \quad p_2^* = \frac{1}{4} \cdot \frac{1}{1-\beta}, \quad p_3^* = \frac{11 - 20 \cdot \beta}{20} \cdot \frac{1}{1-\beta} \tag{3.32}$$

$$p_4^* = p_5^* = 0 \tag{3.33}$$

and

$$q_0^* = \frac{3}{5}, \quad q_1^* = \frac{3}{20}, \quad q_2^* = \frac{1}{4}, \quad q_3^* = q_4^* = q_5^* = 0 \tag{3.34}$$

with

$$Op_{5,1}^*(\beta) = \frac{47}{20} + 3 \cdot \beta\,. \tag{3.35}$$

- For $\frac{11}{20} \leq \beta < \frac{4}{5}$ we have the optimal strategies

$$p_1^* = \frac{1}{5} \cdot \frac{1}{1-\beta}, \quad p_2^* = \frac{4 - 5 \cdot \beta}{5} \cdot \frac{1}{1-\beta}, \quad p_3^* = p_4^* = p_5^* = 0 \tag{3.36}$$

and

$$q_0^* = \frac{4}{5}, \quad q_1^* = \frac{1}{5}, \quad q_2^* = q_3^* = q_4^* = q_5^* = 0 \tag{3.37}$$

with

$$Op_{5,1}^*(\beta) = \frac{9}{5} + 4 \cdot \beta\,. \tag{3.38}$$

- For $\frac{4}{5} \leq \beta \leq 1$ we have the optimal strategies

$$p_1^* = 1\,, \quad p_2^* = p_3^* = p_4^* = p_5^* = 0 \tag{3.39}$$

and

$$q_0^* = 1\,, \quad q_1^* = q_2^* = q_3^* = q_4^* = q_5^* = 0 \tag{3.40}$$

with

$$Op_{5,1}^*(\beta) = 1 + 5 \cdot \beta\,. \tag{3.41}$$

The proof of these results is given in Appendix D.1 section D.1.3. Since we will not discuss the structure of the optimal strategies in the following we do not present them here in tabular form. Let us just mention that for all values of β the operators optimal strategy is piecewise constant and that for $\beta > 4/5$ the inspector's optimal strategy is also constant.

The case of two unannounced interim inspections ($n = 5$ and $k = 2$)

The operator can start his illegal activity again at the time points $0, 1, \ldots, 5$ and the inspector performs his two inspections at the time points

$$(1,2)\,,(1,3)\,,(1,4)\,,(1,5)\,,(2,3)\,,(2,4)\,,(2,5)\,,(3,4)\,,(3,5)\,,(4,5)\,.$$

The payoff matrix of this situation is given in Figure 3.10.

The optimal strategies of this game do not give helpful insight, since they are too complicated. Therefore, in the following only the optimal expected detection times are given. The results are obtained with the help of a Mathematica® program by M. J. Canty [10]:

$$Op_{5,2}^*(\beta) = \begin{cases} \dfrac{59 + 133 \cdot \beta + 128 \cdot \beta^2 + 62 \cdot \beta^3 + 12 \cdot \beta^4}{34 + 48 \cdot \beta + 30 \cdot \beta^2 + 8 \cdot \beta^3} & \text{for} \quad 0 \leq \beta < 0.172965 \\[2ex] \dfrac{26 + 48 \cdot \beta + 23 \cdot \beta^2 + 2 \cdot \beta^3 - 14 \cdot \beta^4 - 12 \cdot \beta^5}{16 + 10 \cdot \beta - 5 \cdot \beta^2 - 4 \cdot \beta^3 - 2 \cdot \beta^4} & \text{for} \quad 0.179265 \leq \beta < 0.249989 \\[2ex] \dfrac{23 + 77 \cdot \beta + 102 \cdot \beta^2 + 72 \cdot \beta^3}{15 + 27 \cdot \beta + 18 \cdot \beta^2} & \text{for} \quad 0.249989 \leq \beta < 0.529234 \\[2ex] \dfrac{10 + 28 \cdot \beta + 33 \cdot \beta^2 + 36 \cdot \beta^3}{7 + 10 \cdot \beta + 2 \cdot \beta^2} & \text{for} \quad 0.529234 \leq \beta < 0.75 \\[2ex] 1 + \beta + 4 \cdot \beta^2 & \text{for} \quad 0.75 \leq \beta \leq 1 \end{cases} . \tag{3.42}$$

Let us note that we do not prove these results. Having checked so many times the results of this program with smaller games, we are confident that this program works here as well. A graphical representation of these results is given in section 3.3. This Figure shows by the way that the results are plausible.

Figure 3.10 The matrix game in case of two unannounced interim inspections ($n = 5$ and $k = 2$).

	(1,2)	(1,3)	(1,4)	(1,5)	(2,3)	(2,4)	(2,5)	(3,4)	(3,5)	(4,5)
0	$1+\beta+4\cdot\beta^2$	$1+2\cdot\beta+3\cdot\beta^2$	$1+3\cdot\beta+2\cdot\beta^2$	$1+4\cdot\beta+\beta^2$	$2+\beta+3\cdot\beta^2$	$2+2\cdot\beta+2\cdot\beta^2$	$2+3\cdot\beta+\beta^2$	$3+\beta+2\cdot\beta^2$	$3+2\cdot\beta+\beta^2$	$4+\beta+\beta^2$
1	$1+4\cdot\beta$	$2+3\cdot\beta$	$3+2\cdot\beta$	$4+\beta$	$1+\beta+3\cdot\beta^2$	$1+2\cdot\beta+2\cdot\beta^2$	$1+3\cdot\beta+\beta^2$	$2+\beta+2\cdot\beta^2$	$2+2\cdot\beta+\beta^2$	$3+\beta+\beta^2$
2	4	$1+3\cdot\beta$	$2+2\cdot\beta$	$3+\beta$	$1+3\cdot\beta$	$2+2\cdot\beta$	$3+\beta$	$1+\beta+\beta^2$	$1+2\cdot\beta+\beta^2$	$2+\beta+\beta^2$
3	3	3	$1+2\cdot\beta$	$2+\beta$	3	$1+2\cdot\beta$	$2+\beta$	$1+2\cdot\beta$	$2+\beta$	$1+\beta+\beta^2$
4	2	2	2	$1+\beta$	2	2	$1+\beta$	2	$1+\beta$	$1+\beta$
5	1	1	1	1	1	1	1	1	1	1

Should this model ($n = 5$ and $k = 2$) be implemented in practice, it would be possible, of course, to present also the optimal strategies.

The discussion of the results of this section goes along the same lines as those in the case $n = 3$ (see pages 30 - 32). Therefore we do not repeat it here.

3.2 Continuous time models

In this section we introduce another class of games which in the literature is called games over the unit square and which is adapted to our applications. For the reasons explained at the end of section 2.1.4 we will apply these models both to an on-site interim storage and a fuel element fabrication facility under the assumption that the IAEA inspectors can perform their unannounced interim inspections at any point of time of the reference time interval.

In the concluding section 3.3 we will discuss the application of both the discrete and continuous time models to both types of facilities, on-site interim storage and fuel element fabrication facility.

Most of the model assumptions made in section 3.1.1 are valid in this model as well. The main difference between these two models is that the operator resp. the inspector can start the illegal activity resp. perform his inspection(s) at *any* time point between 0 and t_0. t_0 is determined by the absolute length and scaling of the reference time interval. If this interval is one year, and time is measured in quarters of years, e.g., then we get $t_0 = 4$.

Let k be the number of the unannounced interim inspection(s), chosen by the inspector and also known to the operator. In section 5.3.1 we will show that it is sufficient to consider the cases of $k = 1$ and $k = 2$. Depending on k we have again different sets of pure strategies for the inspector:

- If $k = 1$: the set of pure strategies consists of all time points at which he can perform his inspection. If t is the time point for inspection we therefore get $\Phi_{Insp,1} = \{ t \in \mathbb{R} : 0 \leq t \leq t_0 \}$, and
- If $k = 2$: let t_1 and t_2 be the time points for the first resp. the second inspection. Then we have $\Phi_{Insp,2} = \{(t_1, t_2) \in \mathbb{R} \times \mathbb{R} : 0 \leq t_1 < t_2 \leq t_0\}$.

The operator may start his illegal activity at any time point between 0 and 4, therefore:

- The set of pure strategies of the operator is $\Phi_{Op} = \{ s \in \mathbb{R} : 0 \leq s \leq t_0 \}$.

Following the argumentation in section 3.1.1 we assume again that the operator decides at the beginning of the year (or the reference time interval) at which time point he will start his illegal activity and the IAEA (or inspector) decides at the beginning of the year (or the reference time interval) at which time point he will perform his inspection(s).

The time points at which the operator will start his illegal activity resp the inspector performs his inspection (s) are chosen independently of each other (at the beginning of the year).

We assume that the inspector will commit an error of the second kind per inspection, i.e., an illegal activity is not detected with probability β although there is one.

Let us summarize the assumptions which we have made so far:

(a) There are two players: operator and inspector.

(b) The inspector can perform his inspections at any time point between 0 and t_0.

(c) The operator behaves illegally.

(d) The operator can start his illegal activity at any time point between 0 and t_0.

(e) The inspector commits an error of the second kind per inspection, i.e., an illegal activity is not detected with probability β although there is one.

(f) The number of interim inspections is also known to the operator[6]. At most two unannounced interim inspections are permitted in one facility and the reference time interval, see section 5.3.1[7].

(g) Both players decide at the beginning of the reference time interval when to start the illegal activity and when to inspect.

(h) Both players decide independently of each other.

(i) The payoff to the operator is the time between the start of the illegal activity and its detection. The payoff to the inspector is the negative one (zero-sum game).

(j) In case of the coincidence of the start of the illegal activity and the inspection, the illegal activity is detected only at the occasion of the next inspection or the PIV.

(k) The game ends either after the final PIV or after that interim inspection at which the illegal activity is detected.

This verbal description of our inspection problem leads us to zero-sum games with infinite sets of pure strategies which requires a mathematical treatment different from the foregoing one.

[6]See footnote page 21.

[7]The arguments given in section 5.3.1 hold only for on-site-interim storage facilities. For fuel element fabrication facilities the documents by IAEA and EURATOM do not provide corresponding data, but we assume that similar arguments hold here as well.

The case of one unannounced interim inspection ($k = 1$)

Let s be the time at which the operator starts his illegal activity. At time point t the inspector will perform his inspection ($k = 1$). Then the payoff to the operator is given by[8]

$$Op_1(\beta; s, t) = \begin{cases} (t-s)\cdot(1-\beta) + (t_0 - s)\cdot\beta & \text{for} \quad 0 \le s < t \\ t_0 - s & \text{for} \quad t \le s \le t_0 \end{cases}, \tag{3.43}$$

which can be seen as a generalization of the payoff matrix in Figure 3.2. $Op_1(\beta; s, t)$ is called payoff kernel. It is important to remember that if the start of the illegal activity and the inspection time coincide, the illegal activity is detected at the end (PIV).

The saddle point conditions in terms of $Op_1(\beta; s, t)$ can now be written as

$$Op_1(\beta; s, t^*) \le Op_1(\beta; s^*, t^*) \le Op_1(\beta; s^*, t) \quad \text{for all} \quad s, t \in [0, t_0]\,.$$

It can be seen that these conditions cannot be satisfied, that is, there is no saddle point in pure strategies (like in the corresponding discrete time game in section 3.1.1). Therefore we must look for mixed strategies, which raises the question: What are mixed strategies for players with infinitely many pure strategies? The answer is that they can be represented, just as in matrix games, as probability distributions over the set of pure strategies. It is convenient to work with the cumulative distribution functions

$$Q(s) = Prob(S \le s) \quad \text{and} \quad P(t) = Prob(T \le t)$$

which are the probabilities that random variables S and T representing the violation and inspection times have values not exceeding s resp. t. The operator´s expected payoff for some mixed strategy combination (Q, P) is then given by

$$Op_1(\beta; Q, P) = \int_0^{t_0} \int_0^{t_0} Op_1(\beta; s, t)\, dQ(s)\, dP(t)\,,$$

where we are using Lebesque-Stieltjes integrales, see, e.g., [12]. We can assume here that the double integral exist.

For mixed strategies the saddle point conditions for our zero-sum game is

$$Op_1(\beta; Q, P^*) \le Op_1(\beta; Q^*, P^*) \le Op_1(\beta; Q^*, P) \quad \text{for all} \quad Q, P\,,$$

so we have to look for distribution functions Q^* and P^* which satisfies them. Success is by no means guaranteed, since the payoff kernel $Op_1(\beta; s, t)$ is discontinuous on $s = t$ and optimal strategies for those kind of games cannot be guaranteed without further assumptions. Fortunately, this game and the game discussed in the next paragraph possesses optimal strategies in mixed strategies. Finding these optimal strategies is in general a difficult task. The complete game theoretical solution of our continuous inspection game is given as follows:

[8] For the notation of the detection times and expected detection times see again Appendix A.

Let us define the parameter κ by

$$\kappa = t_0 \cdot \left(1 - e^{-(1-\beta)}\right) .$$

The zero-sum game with payoff kernel in (3.43) has the following solution: The operator chooses his start of the illegal activity s according to the distribution function $Q^*(s)$ with

$$Q^*(s) = \begin{cases} \dfrac{t_0}{e^{(1-\beta)}} \cdot \dfrac{1}{t_0 - s} & \text{for} \quad s \in [0, \kappa) \\ 1 & \text{for} \quad s \in [\kappa, t_0] \end{cases} ,$$

while the inspector chooses the inspection time t according to the distribution function $P^*(t)$ given by

$$P^*(t) = \begin{cases} -\dfrac{1}{1-\beta} \cdot \ln\left[1 - \dfrac{t}{t_0}\right] & \text{for} \quad t \in [0, \kappa) \\ 1 & \text{for} \quad t \in [\kappa, t_0] \end{cases} .$$

The optimal expected detection time is

$$Op_1^*(\beta) = t_0 - \kappa = t_0 \cdot e^{-(1-\beta)} . \tag{3.44}$$

The proof of this result can be found in [3] and – for $\beta = 0$ – in [6]. The surprising result is, that after time point κ neither an illegal activity is started nor an inspection is performed. This result makes sense since detection is guaranteed to occur at the end of the interval and the operator will not wish to wait too long before violating. We also see that even in the case of $\beta = 0$ and $t_0 = 4$ we have $Op_1^*(0) = 4 \cdot e^{-1} \approx 1.47$ (3.4 months) is still greater than the conversion time 1 (3 months) in our model, i.e., the conversion time of 3 month.

The optimal strategies of both players can also be formulated in another way - due to a brilliant idea of H. Diamond, see [13]. For the inspector's optimal strategy this looks like: The inspector realizes a uniformly distributed random variable $U = u$ on $[0, 1]$, i.e., the distribution function $F_U(u)$ of U is given by

$$F_U(u) = \begin{cases} 0 & \text{for} \quad u < 0 \\ u & \text{for} \quad u \in [0, 1] \\ 1 & \text{for} \quad u > 1 \end{cases} ,$$

and determines therewith his optimal inspection time point

$$t^* = t_0 \cdot \left(1 - \frac{h(1-u)}{h(1)}\right) \qquad \text{with} \qquad h(x) = e^{-(1-\beta)\cdot x} .$$

Therefore, (3.44) can also be written as $Op_1^*(\beta) = t_0 \cdot h(1)$.

This solution of our game theoretical problem renders its application very easy: The inspector uses a random number generator, realizes $U = u$ and inspects at time point $h(u)$.

The case of two unannounced interim inspection ($k = 2$)

Let s be again the time point for starting the illegal activity and let t_1 and t_2 with $t_1 < t_2$ be the time points for the interim inspections. Then the payoff kernel of this game is given by (see also footnote on page 26)

$$Op_2(\beta; s, (t_2, t_1)) = \begin{cases} (1-\beta)\cdot(t_1 - s) + & \\ \quad + \beta\cdot(1-\beta)\cdot(t_2 - s) & \\ \quad + \beta^2\cdot(t_0 - s) & 0 \leq s < t_1 < t_2 \leq t_0 \\ (1-\beta)\cdot(t_2 - s) + \beta\cdot(t_0 - s) & 0 \leq t_1 \leq s < t_2 \leq t_0 \\ (t_0 - s) & t_2 \leq s \leq t_0 \end{cases} .$$

In this game a mixed strategy of the inspector is a two-dimensional distribution function for the random vector (T_1, T_2), i.e., the random times T_1 and T_2 at which the inspector performs his inspection, see, e.g., [40], while a mixed strategy $Q(s)$ of the operator is defined as for the game $k = 1$. It turns out that the explicit formulae of the operator's resp. the inspector's optimal distribution function are complicated and are of less practical usage, see [1] in case of $\beta = 0$. Therefore, we again use Diamond's representation and describe the inspector's optimal strategy via a uniformly distributed random variable and - because of $k = 2$ - with two functions $h_1(x)$ and $h_2(x)$ fulfilling the following differential equation system

$$\begin{aligned} h_1'(x) &= (1-\beta)\cdot h_1(x) \\ h_2'(x) &= (1-\beta)\cdot h_2(x) - (1-\beta)^2\cdot h_1(x) \end{aligned}$$

with

$$h_1(0) = 1 \quad \text{and} \quad h_2(0) = h_1(1) .$$

This system has the unique solution

$$h_1(x) = e^{(1-\beta)\cdot x} \tag{3.45}$$
$$h_2(x) = e^{(1-\beta)\cdot x}\cdot\left(e^{1-\beta} - x\cdot(1-\beta)^2\right) , \tag{3.46}$$

see, e.g., [9]. Therewith the inspector has the following optimal strategy: The optimal time points for inspection (t_1^*, t_2^*) are

$$t_1^* = t_0\cdot\left(1 - \frac{h_2(1-u)}{h_2(1)}\right) \quad \text{and} \quad t_2^* = t_0\cdot\left(1 - \frac{h_1(1-u)}{h_2(1)}\right) ,$$

where u is the realization of a uniformly distributed random variable U on $[0, 1]$.

The optimal expected detection time is

$$Op_2^*(\beta) = \frac{t_0}{h_2(1)} = \frac{t_0\cdot e^{-2\cdot(1-\beta)}}{1-(1-\beta)^2\cdot e^{-(1-\beta)}} .$$

For $\beta = 0$ and $t_0 = 4$ we get

$$Op_2^*(0) = \frac{4}{e^2} \cdot \frac{1}{1 - e^{-1}} = \frac{4}{e} \cdot \frac{1}{e - 1} \approx 0.8563\,.$$

The proof of this result can be found in section D.2 Appendix D.

As mentioned before the optimal strategy of the operator has a much more complicated structure and is omitted here. As before, the operator resp. the inspector starts his illegal activity resp. performs his inspections not later than $t_0 \cdot (1 - 1/h_2(1))$.

We can now answer the question why we are only considering in this model the cases $k = 1$ and $k = 2$. Firstly, we see that in the case $k = 2$ the optimal strategy of the inspector is very complicated. For general k the optimal solution can only be represented recursively with the help of differential equations.

Secondly, the optimal strategy for the case $k = 2$ assures an optimal expected detection time which is smaller than the conversion time one which satisfies primary IAEA safeguards goals. It should be noted that this was not possible in the corresponding discrete time games, see section 3.1.1. Of course with increasing k this expected detection time is still decreasing.

Finally, it should be mentioned that in practice it may be difficult to plan and perform inspections with the continuous time model, since this may create too many problems for the joint performance between IAEA and EURATOM. A practical solution could be to take the nearest possible time point to the optimal time point(s) of inspection(s).

3.3 Presentation and evaluation of results

As already announced at the end of section 2.1.4 we can apply both the discrete and the continuous time models to both types of facilities considered in this study. For the purpose of illustration we demonstrate this first for the on-site interim storage.

3.3.1 On-site Interim Storage Facility

For one unannounced interim inspection we obtained with the help of the discrete time model the following optimal expected detection times:

$$Op_{3,1}^*(\beta) = \begin{cases} \dfrac{11}{6} + \beta & \text{for} \quad 0 \le \beta < 1/6 \\ \dfrac{10}{6} + 2 \cdot \beta & \text{for} \quad 1/6 \le \beta < 2/3 \\ \dfrac{6}{6} + 3 \cdot \beta & \text{for} \quad 2/3 \le \beta \le 1 \end{cases}\,,$$

whereas we got for the continuous time model

$$Op_1^*(\beta) = 4 \cdot e^{-(1-\beta)}\,.$$

It can be shown that the continuous time model gives a shorter optimal expected detection time for all values of β, starting with $\beta = 0$ $(1.47$ resp. $1.83)$. Of course, this does not mean that we recommend the use of the continuous time model because an additional burden is posed on the plant operators if IAEA inspectors visit the plant at different points of time than EURATOM inspectors.

The same relation holds for the case of two interim inspections. With the help of the discrete time model we obtained the following optimal expected detection times (again deleting the limiting cases)

$$Op^*_{3,2}(\beta) = \begin{cases} \dfrac{4 + 6 \cdot \beta + 5 \cdot \beta^2 + 2 \cdot \beta^3}{3 + 2 \cdot \beta + \beta^2} & \text{for} \quad 0 \leq \beta < 1/2 \\ 1 + \beta + 2 \cdot \beta^2 & \text{for} \quad 1/2 \leq \beta \leq 1 \end{cases},$$

whereas we got for the continuous time model

$$Op^*_2(\beta) = \frac{4 \cdot e^{-2 \cdot (1-\beta)}}{1 - (1-\beta)^2 \cdot e^{-(1-\beta)}} .$$

Here it has to be emphasized that for $k = 2$ unannounced interim inspections the continuous time model results in an optimal expected detection time that is *shorter* than the conversion time (see section 3.1.1). This was not possible in the discrete time model.

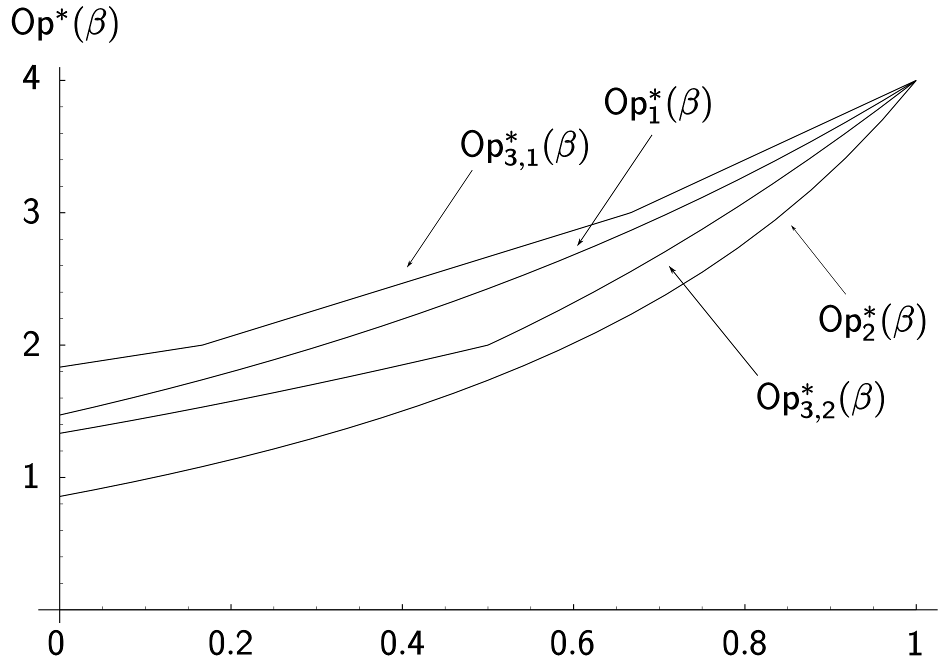
In Figure 3.11 all four cases are represented graphically. For the discrete time model the optimal expected detection times have already been presented in Figure 3.8. The discussion given there holds, of course, also for the continuous time model.

3.3.2 Fuel Element Fabrication Facility

Let us now turn to the fuel element fabrication facility example.

In our description of the facility and of the safeguards measures in section 2.2 we mentioned that every six to eight weeks EURATOM inspectors visit the facility. Here we assume intervals of two month. Then, according to our basic assumption for the discrete time model, there are five intermediate time points at which the IAEA inspectors may visit the facility. Thus, for one interim inspection $(k = 1)$ there are five pure strategies, whereas for two interim inspections $(k = 2)$ there are $\binom{5}{2} = 10$ pure inspection strategies. The operator has in both cases six pure strategies, namely to start his illegal activity at time point $0, 1, \ldots, 5$.

Figure 3.11 Graphical representation of the optimal expected detection times as functions of the probability of no detection β for the on-site interim storage facility.



For $k = 1$ the optimal expected detection time is

$$Op^*_{5,1}(\beta) = \begin{cases} \frac{77}{30} + 2 \cdot \beta & \text{for} \quad 0 \leq \beta < \frac{13}{60} \\ \frac{47}{20} + 3 \cdot \beta & \text{for} \quad \frac{13}{60} \leq \beta < \frac{11}{20} \\ \frac{9}{5} + 4 \cdot \beta & \text{for} \quad \frac{11}{20} \leq \beta < \frac{4}{5} \\ 1 + 5 \cdot \beta & \text{for} \quad \frac{4}{5} \leq \beta \leq 1 \end{cases}$$

whereas we got for the continuous time model

$$Op^*_1(\beta) = 6 \cdot e^{-(1-\beta)} \,.$$

In case of $k = 2$ we obtained

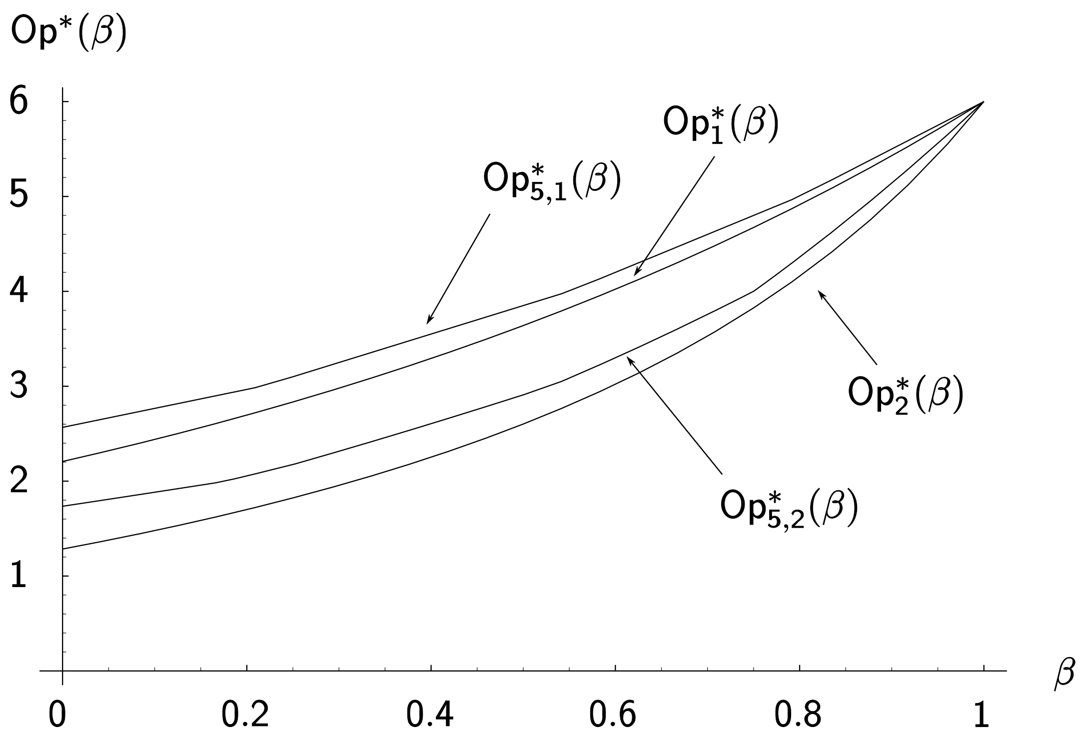
$$Op^*_{5,2}(\beta) = \begin{cases} \frac{59 + 133 \cdot \beta + 128 \cdot \beta^2 + 62 \cdot \beta^3 + 12 \cdot \beta^4}{34 + 48 \cdot \beta + 30 \cdot \beta^2 + 8 \cdot \beta^3} & \text{for} \quad 0 \leq \beta \leq 0.172965 \\ \frac{26 + 48 \cdot \beta + 23 \cdot \beta^2 + 2 \cdot \beta^3 - 14 \cdot \beta^4 - 12 \cdot \beta^5}{16 + 10 \cdot \beta - 5 \cdot \beta^2 - 4 \cdot \beta^3 - 2 \cdot \beta^4} & \text{for} \quad 0.179265 \leq \beta \leq 0.249989 \\ \frac{23 + 77 \cdot \beta + 102 \cdot \beta^2 + 72 \cdot \beta^3}{15 + 27 \cdot \beta + 18 \cdot \beta^2} & \text{for} \quad 0.249989 \leq \beta \leq 0.529234 \\ \frac{10 + 28 \cdot \beta + 33 \cdot \beta^2 + 36 \cdot \beta^3}{7 + 10 \cdot \beta + 2 \cdot \beta^2} & \text{for} \quad 0.529234 \leq \beta \leq 0.75 \\ 1 + \beta + 4 \cdot \beta^2 & \text{for} \quad 0.75 \leq \beta \leq 1 \end{cases}$$

whereas for the continuous time model

$$Op_2^*(\beta) = \frac{6 \cdot e^{-2\cdot(1-\beta)}}{1-(1-\beta)^2 \cdot e^{-(1-\beta)}} \,.$$

In Figure 3.12 all four cases are represented graphically.

Figure 3.12 Graphical representation of the optimal expected detection times as functions of the probability of no detection β for the fuel element fabrication facility.



Chapter 4

Hybrid-sequential models

There is just one assumption which differs from those in the previous chapter, but it leads to totally different game theoretical models: We assume here that the plant operator decides at the beginning of the reference time interval only whether to start an illegal activity immediately or not. In the latter case he decides after the first inspection whether to start an illegal activity immediately or not, and so on. Since we consider only illegal behavior during the reference time interval, we assume in addition that the operator has to start an illegal activity after the last inspection if he did not do so before.

Let us mention that we also could assume that the inspector acts in a similar way, namely deciding at the beginning of the reference time interval only when to perform the first inspection, after the first inspection deciding when to perform the second one, and so on. Since, however, the inspectorate has to plan the use of his resources for all plants and States, and furthermore, since he does not gain any information about the operator's behavior in the course of the game – except that he detects illegal behavior which finishes the game – we do not consider this possibility here. It should be mentioned that this variant has been analyzed in detail by Avenhaus and Canty, see [2].

Of course, for just one interim inspection during the reference time interval both variants are the same.

Contrary to the forgoing models, the ones to be analyzed now require quite different analytical tools: Instead of considering only games in normal form, we now have to deal primarily with so-called games in extensive form. A short introduction into extensive form games is given in Appendix C.

Again, the models are applied to prototypes of an on-site interim storage and a fuel fabrication facility, which had been described in chapter 2.

4.1 Discrete time models

We start again with the analysis of situations, where the inspector can only perform unannounced interim inspections at finite numbers of well-defined time points. The corresponding continuous time models are discussed subsequently.

4.1.1 Mathematical analysis of unannounced interim inspections in an on-site interim storage facility

Like in the previous chapter we assume that EURATOM inspectors visit the facility under consideration in regular intervals of time, and that IAEA inspectors join them once or twice during the reference time interval without announcing their visits in general. In case of the on-site interim storage facility there are three regular interim inspections performed by EURATOM, and in the case of the fuel element fabrication facility five ones.

For the subsequent models in this section we assume that the basic assumptions (i) - (vi) and (ix) - (xi) formulated for the on-site interim storage in section 3.1.1 hold. Assumption (viii) is deleted, since the operators moves depend on those of the inspector. Instead of (vii) we require

(vii") The inspector decides at the beginning of the reference time interval when to perform his inspection(s). The operator has to decide at the beginning of the reference time interval whether to start his illegal activity immediately or only after the inspection(s).

The meaning of this assumption, i.e., the strategic behavior of the operator, will become clearer when we discuss the case $n = 3$ and $k = 2$.

The case of one unannounced interim inspection (n arbitrary and $k = 1$)

Since the case of general number n of inspection points of time can be analyzed as easily as any special number n, we do this here. According to our assumptions (vii") the operator has to decide at the beginning of the reference time interval whether to start his illegal activity immediately or only after the inspection (by the IAEA). If the inspector performs his unannounced interim inspection at the j-th EURATOM inspection, then the expected detection time is in the first case given by

$$(1 - \beta) \cdot j + \beta \cdot (n + 1) \, ,$$

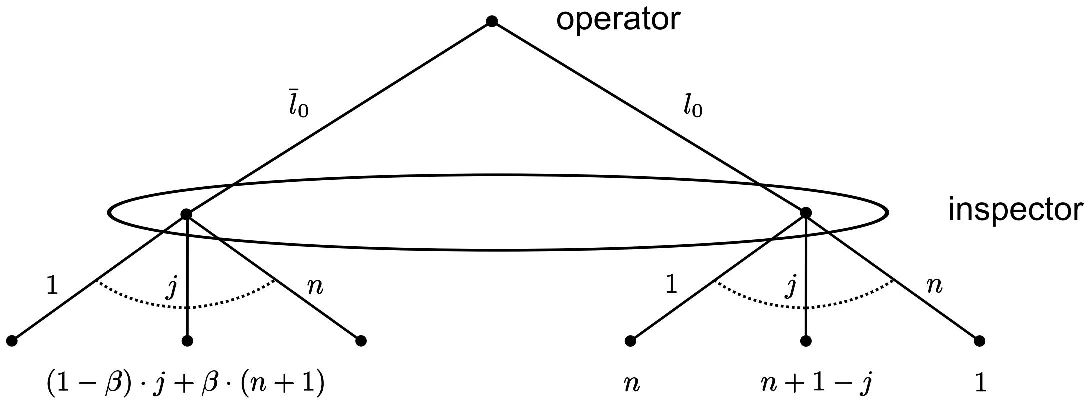
where n is the number of EURATOM interim inspections per reference time interval, and where β is again the non-detection probability for first IAEA inspection inspection after the beginning of the illegal activity. In the second case it is given by

$$n + 1 - j \, .$$

Even though this game can – and will – be represented subsequently in normal form, we present it now in so-called *extensive form*, since this representation gives a more illustrative idea of the information structure of the game and furthermore, since we have to do this anyhow for two IAEA interim inspections per reference time interval.

Thus, we introduce here a very simple form of an extensive form game, see Figure 4.1. The operator decides at the beginning of the reference time interval if to start the illegal activity immediately ($\bar{l}_0$) or not (l_0). In the latter case he has to start his illegal activity immediately after the inspection. The (IAEA) inspector decides at the beginning at which time point of the reference time interval he performs his inspection.

Figure 4.1 Extensive form of the discrete time hybrid-sequential inspection game with one interim inspection at one of the possible time points $1, \ldots, n$. $\bar{l}_0$ and l_0 denote illegal and legal behavior of the operator at the beginning of the game. The encircled area is the so-called information set of the inspector.



The important feature in Figure 4.1 is the information set of the inspector: He does not know, at which node in the game he stays, when the game arrives there and he has to make his decision.

The normal form of this extensive form game is given in Figure 4.2.

Figure 4.2 The matrix game of the hybrid-sequential inspection game represented graphically in Figure 4.1.

	1	$\ldots$	j	$\ldots$	n
$\bar{l}_0$	$(1-\beta)\cdot 1+\beta\cdot(n+1)$	$\ldots$	$(1-\beta)\cdot j+\beta\cdot(n+1)$	$\ldots$	$(1-\beta)\cdot n+\beta\cdot(n+1)$
l_0	n	$\ldots$	$n+1-j$	$\ldots$	1

Let $\mathbf{q}^T = (q_1, q_2)$ with $q_i \geq 0$ for $i = 1, 2$ and $q_1 + q_2 = 1$ be a mixed strategy of the operator, i.e., q_1 and q_2 denote the probabilities to choose $\bar{l}_0$ and l_0. Let $\mathbf{p}^T = (p_1, \ldots, p_n)$ with $p_j \geq 0$ for all $j = 1, \ldots, n$ and $\sum_{j=1}^{n} p_j = 1$ be a mixed strategy of the inspector, i.e., p_j denotes the probability to choose the time point j for the inspection. Then the expected detection time[1] is given by (see Appendix B):

$$Op_{n,1}(\beta;\mathbf{q},\mathbf{p}) = q_1 \cdot \sum_{j=1}^{n} ((1-\beta)\cdot j + \beta\cdot(n+1))\cdot p_j + q_2 \cdot \sum_{j=1}^{n} (n+1-j)\cdot p_j \,.$$

The solution of this game is given as follows: The (mixed) optimal strategy of the operator is given by

$$q_1^* = \frac{1}{2-\beta} \quad \text{and} \quad q_2^* = \frac{1-\beta}{2-\beta}\,. \tag{4.1}$$

The (not unique) optimal strategy $\mathbf{p}^* = (p_1^*, \ldots, p_n^*)^T$ of the inspector is given by

$$\sum_{j=1}^{n} j \cdot p_j^* = \frac{1-\beta}{2-\beta}\cdot(n+1) \quad \text{and} \quad \sum_{j=1}^{n} p_j^* = 1 \tag{4.2}$$

with the optimal expected detection time

$$Op_{n,1}^*(\beta) = Op_{n,1}(\beta;\mathbf{q}^*,\mathbf{p}^*) = \frac{n+1}{2-\beta}\,. \tag{4.3}$$

This result in proven in section D.3.1 of Appendix D.3 .

From a *theoretical* point of view it is interesting to note that in case the pure strategy j^* of the inspector fulfills the condition

$$(1-\beta)\cdot j^* + \beta\cdot(n+1) = n+1-j^*\,,$$

which is equivalent to

$$j^* = \frac{1-\beta}{2-\beta}\cdot(n+1)\,, \tag{4.4}$$

is an integer, then the inspector can use this *pure* strategy, i.e., he can announce this time point j^* in advance. The larger the error of the second kind probability β is, the smaller is this time point j^*.

For the on-site interim storage facility we have $n = 3$, that is with (4.4)

$$j^* = 4\cdot\frac{1-\beta}{2-\beta}\,,$$

which gives $j^* = 1$ for $\beta = 2/3$ and $j^* = 2$ for $\beta = 0$.

For the fuel element fabrication facility we have $n = 5$, that is with (4.4)

$$j^* = 6\cdot\frac{1-\beta}{2-\beta}\,,$$

which gives $j^* = 1$ for $\beta = 5/6$, $j^* = 2$ for $\beta = 3/5$ and $j^* = 3$ for $\beta = 1/4$.

For *practical* applications these results are not so interesting, since these special β-values will be hardly realized.

[1] For the notation of the expected detection times see again Appendix A.

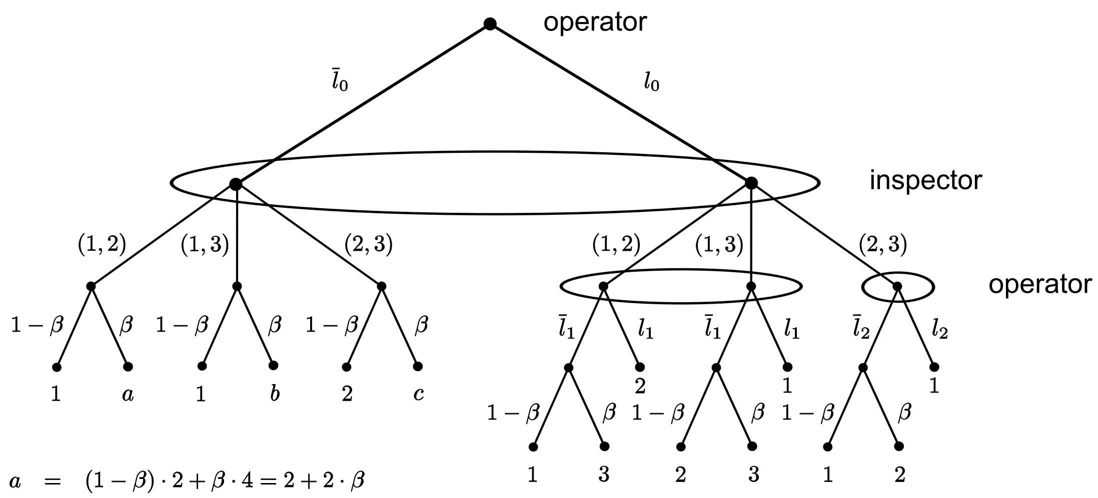
The case of two unannounced interim inspections ($n = 3$ and $k = 2$)

Since it is no longer possible to present the extensive form of the game describing two unannounced interim inspections for any number n inspections in a reasonable way, we have to specialize our analyses to the cases of our practical interest.

For $n = 3$ and $k = 2$ the inspector has the three possibilities $(1, 2)$, $(1, 3)$ and $(2, 3)$, whereas the operator decides at the beginning of the reference time interval and after the first inspection. The extensive form of this game is represented graphically in Figure 4.3. The operator decides at the beginning of the reference time interval whether to start an illegal activity immediately or not. The inspector in turn decides at the beginning where to place his two inspections. In case the operator did not start his illegal activity at the beginning, he decides after the first inspection whether to start his illegal activity now or to do this after the second inspection, see assumption (iii).

We see that here the information structure is much more complicated than in the previous case: Whereas the inspector has again just one information set the operator now has two information sets since after the first inspection at time point one he does not know when the second inspection will be performed. After the first inspection at time point two, however, he does know that the second one will be performed at time point three (see also footnote on page 26).

Figure 4.3 Extensive form of the discrete time hybrid-sequential inspection game with two interim inspections at two of the three time points $1, 2$ and 3. $\bar{l}_0$ and l_0 denote illegal and legal behavior of the operator at the beginning of the game. $\bar{l}_1$ and l_1 denote illegal and legal behavior of the operator if he behaved legally before, and if the first inspection takes place at time point 1. $\bar{l}_2$ and l_2 equivalently, if the first inspection takes place at time point 2. The encircled areas denote information sets.

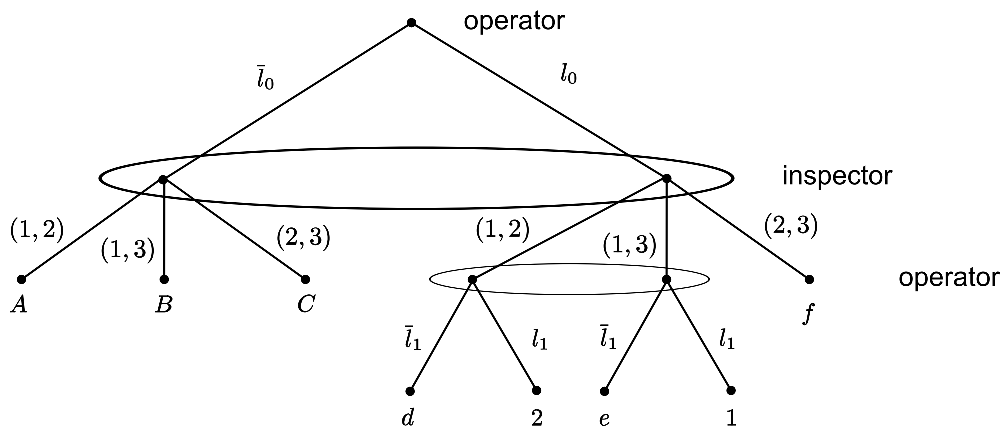


$$a = (1-\beta)\cdot 2 + \beta \cdot 4 = 2 + 2\cdot\beta$$
$$b = (1-\beta)\cdot 3 + \beta \cdot 4 = 3 + \beta$$
$$c = (1-\beta)\cdot 3 + \beta \cdot 4 = 3 + \beta$$

We see that we can simplify this game before trying to determine its solution. First, we can perform the expectation with respect to $(\beta, 1-\beta)$ at the end nodes. Second, comparing the alternatives $\bar{l}_2$ and l_2 we see immediately that $\bar{l}_2$ is better for the operator than l_2 thus, we can delete the latter alternative.

As a result, we arrive at the so-called reduced extensive form of our hybrid-sequential inspection game, the graphical representation of which is given in Figure 4.4.

Figure 4.4 Reduced form of the game represented graphically in Figure 4.3. Notation and information sets are the same as before.



$$
\begin{aligned}
d &= (1-\beta)\cdot 1 + \beta \cdot 3 = 1 + 2\cdot\beta \\
e &= (1-\beta)\cdot 2 + \beta \cdot 3 = 2 + \beta \\
f &= (1-\beta)\cdot 1 + \beta \cdot 2 = 1 + \beta \\
A &= (1-\beta)\cdot 1 + \beta \cdot a = 1 + \beta + 2\cdot\beta^2 \\
B &= (1-\beta)\cdot 1 + \beta \cdot b = 1 + 2\cdot\beta + \beta^2 \\
C &= (1-\beta)\cdot 2 + \beta \cdot c = 2 + \beta + \beta^2
\end{aligned}
$$

There are different ways to determine the solution of such an extensive form game. One way is to use so-called *behavioral strategies*, see also Appendix C: They are mixed strategies which assign probabilities to the choices of the players at all of their information sets. We will use them for the solution of the continuous time model in section 4.2, but demonstrate its use already now.

Let g_0 resp. g_1 be the operators probability to start his illegal activity at the beginning of the reference time interval resp. after the first inspection. Let $p_{(i,j)}$ denote the inspectors probability to inspect at time points (i,j), $(i < j \leq 3,\ i = 1, 2)$, and $p_{(1,2)} + p_{(1,3)} + p_{(2,3)} = 1$.

The expected detection time as function of the players strategies is

$$
\begin{aligned}
Op_{3,2}(\beta; \mathbf{g}, \mathbf{p}) \;=\;& g_0 \cdot [\, A \cdot p_{(1,2)} + B \cdot p_{(1,3)} + C \cdot p_{(2,3)} \,] \\
+\;& (1-g_0) \cdot [\, g_1 \cdot (d \cdot p_{(1,2)} + e \cdot p_{(1,3)}) \\
&\quad + (1-g_1)\cdot(2\cdot p_{(1,2)} + p_{(1,3)}) + f \cdot p_{(2,3)} \,]\,,
\end{aligned} \tag{4.5}
$$

and the saddle point criterion, accordingly, is given by

$$Op_{3,2}(\beta;\mathbf{g},\mathbf{p}^*) \le Op_{3,2}(\beta;\mathbf{g}^*,\mathbf{p}^*) \le Op_{3,2}(\beta;\mathbf{g}^*,\mathbf{p}) \quad \text{for all} \quad \mathbf{g}\,,\mathbf{p}\,.$$

Following the analysis as given in Appendix D.3, the optimal strategies $(g_0^*, 1-g_0^*)$ and $(g_1^*, 1-g_1^*)$, and $\mathbf{p}^* = (p^*_{(1,2)}, p^*_{(1,3)}, p^*_{(2,3)})^T$ are given as follows:

- $0 \le \beta < 0.5$: Then

$$g_0^* = \frac{1}{N}\cdot(1-\beta+\beta^2) \quad \text{and} \quad g_1^* = \frac{1+\beta^2}{2+\beta^2}\,, \tag{4.6}$$

where $N = 3 - 3\cdot\beta + 2\cdot\beta^2 - \beta^3$ and

$$\begin{aligned} p^*_{(1,2)} &= \frac{1}{N}\cdot(1+\beta+\beta^2+\beta^3)\,,\\ p^*_{(1,3)} &= \frac{1}{N}\cdot(1-2\cdot\beta+\beta^2-2\cdot\beta^3)\,,\\ p^*_{(2,3)} &= \frac{1}{N}\cdot(1-2\cdot\beta+\beta^2+\beta^3)\,. \end{aligned} \tag{4.7}$$

The optimal expected detection time is

$$Op^*_{3,2}(\beta) = Op_{3,2}(\beta;\mathbf{g}^*,\mathbf{p}^*) = \frac{4-\beta+\beta^2}{N}\,. \tag{4.8}$$

- $0.5 \le \beta \le 1$: Then

$$g_0^* = 1 \quad \text{and} \quad g_1^* \in [0,1]\,, \tag{4.9}$$

and

$$p^*_{(1,2)} = 1\,,\quad p^*_{(1,3)} = 0\,,\quad p^*_{(2,3)} = 0\,. \tag{4.10}$$

The optimal expected detection time is

$$Op^*_{3,2}(\beta) = 1+\beta+2\cdot\beta^2\,. \tag{4.11}$$

This result is proven in section D.3.2 of Appendix D.3.

Let us mention that for $\beta < 0.5$ the optimal strategy of the inspector is always mixed, contrary to the situation for just one interim inspection.

The case of three (unannounced) interim inspections ($n = 3$ and $k = 3$)

This case will not be analyzed here in detail, first, because only a maximum of two IAEA interim inspections is considered here and second, because its result, easily to be obtained, is again that the operator starts his illegal activity right at the beginning of the reference time interval. The optimal expected detection time is

$$Op^*_{3,3}(\beta) = 1 + \beta + \beta^2 + \beta^3 \,.$$

The discussion of the results of the section goes along the same lines as those in simultaneous case (see pages 30 - 32). Therefore we do not repeat it here. A graphical representation of the optimal expected detection times obtained so far is given in section 4.3.1.

4.1.2 Mathematical analysis of unannounced interim inspections in a fuel element fabrication facility

Like in the previous chapter we assume that EURATOM inspectors visit the facility under consideration in regular intervals of time, and that IAEA inspectors join them once or twice during the reference time interval without announcing their visits in general. In case of the on-site interim storage there are three regular interim inspections performed by EURATOM, and in the case of the fuel element fabrication facility five ones.

In the following models we maintain the basic assumptions (i),(iii), (v) and (ix)-(xi) from section 3.1.1, the assumptions (ii') and (iv') from section 3.1.2 and assumption (vi") from section 4.1.1.

The case of one unannounced interim inspection ($n = 5$ and $k = 1$)

This case was already treated in section 4.1.1, from (4.3) we get for the optimal expected detection time[2]

$$Op^*_{5,1}(\beta) = \frac{6}{2-\beta} \,.$$

The case of two unannounced interim inspections ($n = 5$ and $k = 2$)

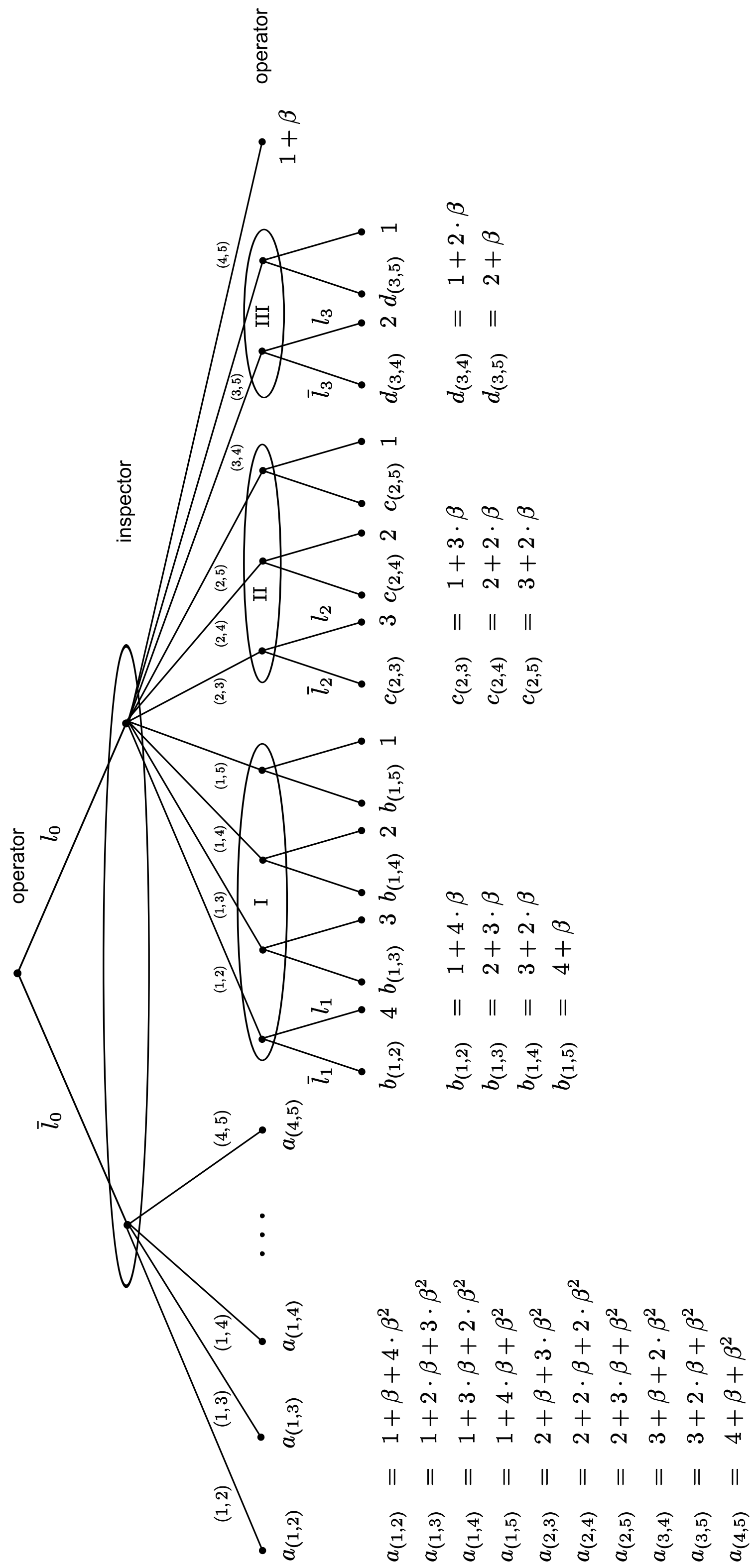
In this case the ten pure strategies of the inspector are

$$(1,2)\,,(1,3)\,,(1,4)\,,(1,5)\,,(2,3)\,,(2,4)\,,(2,5)\,,(3,4)\,,(3,5)\,,(4,5)\,.$$

The pure strategies of the operator cannot be determined so easily beforehand, thus, we consider first the extensive form of this game as given in Figure 4.5.

[2]For the notation of the expected detection times see again Appendix A.

Figure 4.5 Extensive form of the discrete time hybrid-sequential game with two unannounced interim inspections at two of the five possible time points $1, 2, \ldots, 5$. The encircled areas denote the information sets of the operator.



Of course, one can try to determine the equilibria of this game with the help of the behavioral strategies in the same way as before. It turns out, however, that it is much more convenient to transform this extensive form game into a normal form one, see also Appendix B. For this purpose we have to determine the set of pure strategies of the operator. According to the four non-trivial information sets of the operator, the pure strategies of the operator are given in Figure 4.6.

Figure 4.6 Set of pure strategies of the operator.

$\bar{l}_0$	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	l_0	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$
	$\bar{l}_1$	$\bar{l}_2$	l_3		$\bar{l}_1$	$\bar{l}_2$	l_3
	$\bar{l}_1$	l_2	$\bar{l}_3$		$\bar{l}_1$	l_2	$\bar{l}_3$
	$\bar{l}_1$	l_2	l_3		$\bar{l}_1$	l_2	l_3
$\bar{l}_0$	l_1	$\bar{l}_2$	$\bar{l}_3$	l_0	l_1	$\bar{l}_2$	$\bar{l}_3$
	l_1	$\bar{l}_2$	l_3		l_1	$\bar{l}_2$	l_3
	l_1	l_2	$\bar{l}_3$		l_1	l_2	$\bar{l}_3$
	l_1	l_2	l_3		l_1	l_2	l_3

Therefore, the normal form of this game is a 16×10 matrix game. Since, however, the first eight rows of the matrix are identical, we keep just the first of the eight rows. Thus we arrive at a 9×10 matrix game which is represented in Figure 4.7.

Of course it is no longer practically feasible to determine analytically the solution of this game for all values of β. Since, in addition, the optimal strategies of both players turn out to be numerous and complicated, we do not present them here, but just give the optimal expected detection as obtained by the Mathematica® program by M. J. Canty [10]

$$Op^*_{5,2}(\beta) = \begin{cases} \dfrac{6}{3 - 2 \cdot \beta} & \text{for} \quad 0 \leq \beta < 2/3 \\ \dfrac{8 - 5 \cdot \beta + 3 \cdot \beta^2}{5 - 9 \cdot \beta + 8 \cdot \beta^2 - 3 \cdot \beta^3} & \text{for} \quad 2/3 \leq \beta < 3/4 \\ 1 + \beta + 4 \cdot \beta^2 & \text{for} \quad 3/4 \leq \beta \leq 1 \end{cases} . \tag{4.12}$$

Again, we do not prove these results. Having checked so many times the results of this program with smaller games, we are confident that this program works here as well. A graphical representation of these results is given in section 4.3.2. This Figure shows by the way that the results are plausible.

Should this model ($n = 5$ and $k = 2$) be implemented in practice, it would be possible, of course, to present also the optimal strategies.

Figure 4.7 Matrix game of the extensive form game in Figure 4.5.

	(1,2)	(1,3)	(1,4)	(1,5)	(2,3)	(2,4)	(2,5)	(3,4)	(3,5)	(4,5)
$\bar{l}_0\,\bar{l}_1\,\bar{l}_2\,\bar{l}_3$	$1+\beta+4\cdot\beta^2$	$1+2\cdot\beta+3\cdot\beta^2$	$1+3\cdot\beta+2\cdot\beta^2$	$1+4\cdot\beta+\beta^2$	$2+\beta+3\cdot\beta^2$	$2+2\cdot\beta+2\cdot\beta^2$	$2+3\cdot\beta+\beta^2$	$3+\beta+2\cdot\beta^2$	$3+2\cdot\beta+\beta^2$	$4+\beta+\beta^2$
$l_0\,\bar{l}_1\,\bar{l}_2\,\bar{l}_3$	$1+4\cdot\beta$	$2+3\cdot\beta$	$3+2\cdot\beta$	$4+\beta$	$1+3\cdot\beta$	$2+2\cdot\beta$	$3+\beta$	$1+2\cdot\beta$	$2+\beta$	$1+\beta$
$l_0\,\bar{l}_1\,\bar{l}_2\,l_3$	$1+4\cdot\beta$	$2+3\cdot\beta$	$3+2\cdot\beta$	$4+\beta$	$1+3\cdot\beta$	$2+2\cdot\beta$	$3+\beta$	2	1	$1+\beta$
$l_0\,\bar{l}_1\,l_2\,\bar{l}_3$	$1+4\cdot\beta$	$2+3\cdot\beta$	$3+2\cdot\beta$	$4+\beta$	3	2	1	$1+2\cdot\beta$	$2+\beta$	$1+\beta$
$l_0\,\bar{l}_1\,l_2\,l_3$	$1+4\cdot\beta$	$2+3\cdot\beta$	$3+2\cdot\beta$	$4+\beta$	3	2	1	2	1	$1+\beta$
$l_0\,l_1\,\bar{l}_2\,\bar{l}_3$	4	3	2	1	$1+3\cdot\beta$	$2+2\cdot\beta$	$3+\beta$	$1+2\cdot\beta$	$2+\beta$	$1+\beta$
$l_0\,l_1\,\bar{l}_2\,l_3$	4	3	2	1	$1+3\cdot\beta$	$2+2\cdot\beta$	$3+\beta$	2	1	$1+\beta$
$l_0\,l_1\,l_2\,\bar{l}_3$	4	3	2	1	3	2	1	$1+2\cdot\beta$	$2+\beta$	$1+\beta$
$l_0\,l_1\,l_2\,l_3$	4	3	2	1	3	2	1	2	1	$1+\beta$

For answering questions which arise from the practical application of these results, e.g., those concerning the conversion time, this information is sufficient.

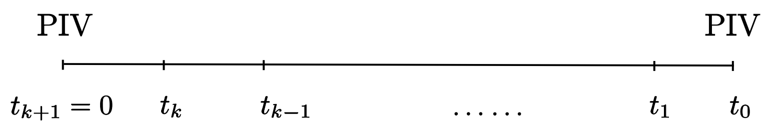
The discussion of the results of the section goes again along the same lines as those in simultaneous case (see pages 30-32). Therefore we do not repeat it here.

4.2 Continuous time models

Like in the case of the simultaneous models we consider now the time continuous variant of the hybrid-sequential model. This means that we assume, like in the forgoing section, that the (IAEA) inspector decides at the beginning of the reference time interval at which time point(s) he performs his inspection(s), whereas the operator decides at the beginning whether or not to start the illegal activity immediately or not, furthermore in the latter case, whether to start the illegal activity immediately after the first inspection and so on. Again, we assume that the operator will behave illegally during the reference time interval.

The following analysis holds both for application to the on-site interim storage and the fuel element fabrication facility. Quite generally, let us assume that in a facility k unannounced interim inspections will be performed in a reference time interval $[t_{k+1}, t_0]$, see Figure 4.8, at the beginning and end of which a physical inventory verification (PIV) is performed. The backward counting simplifies the mathematical analysis and the presentation of the solutions, also the use of t_{k+1} instead of zero. t_0 is determined by the absolute length and scaling of the reference time interval. If this interval is one year, and time is measured in quarters of years, e.g., then we get $t_0 = 4$.

Figure 4.8 Time line of k interim inspections.



In this section we require the basic assumptions (a) - (f) and (i) - (k) from section 3.2. Instead of (g) we assume

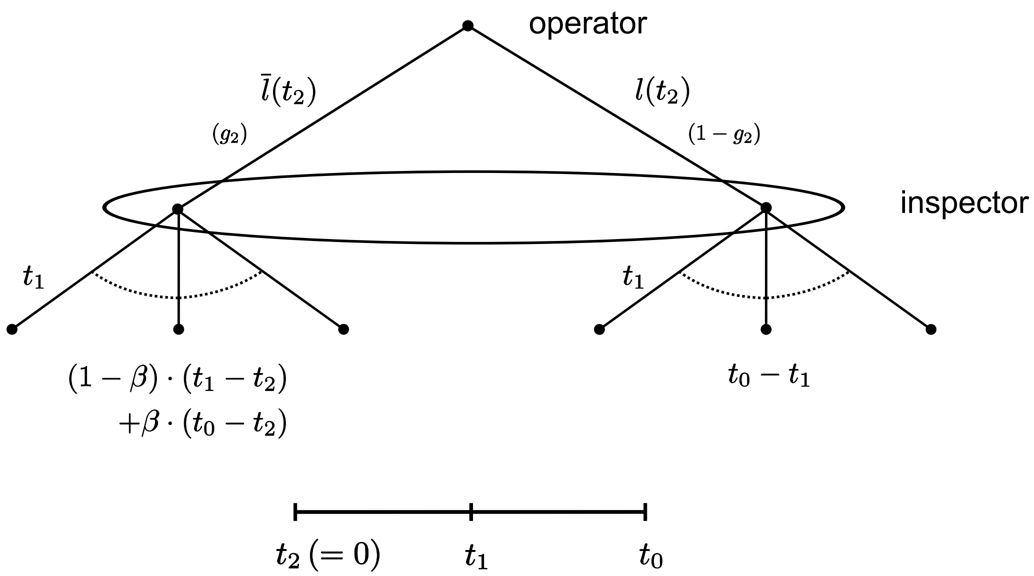
(g') The inspector decides at the beginning of the reference time interval when to perform his inspections. The operator has to decide at the beginning of the reference time interval whether to start his illegal activity immediately or only after the inspection(s).

The case of one unannounced interim inspection ($k = 1$)

As already mentioned, the operator decides at the beginning t_2 of the reference time interval $[t_2, t_0]$ whether to start his illegal activity immediately or only after the inspection.

If not, he has to do this at time point t_1. The graphical representation of this two-person zero-sum game in extensive form is given in Figure 4.9.

Figure 4.9 Extensive form of the continuous time hybrid-sequential inspection game with one interim inspection at any time point during the reference time interval. $\bar{l}(t_2)$ and $l(t_2)$ denote illegal and legal behavior of the operator at the beginning of the game. The encircled area is the information set of the operator. $(g_2, 1-g_2)$ denotes the behavioral strategies of the operator.



At the top of this figure it is indicated that at time point t_2 the operator either does start his illegal activity, $\bar{l}(t_2)$, or he does not, $l(t_2)$. The inspector chooses at t_2 a time point t_1 for his inspection without knowing the operator's decision at t_2. This is indicated by the oval which is called the information set of the inspector.

If the operator chooses $\bar{l}(t_2)$ and the inspector performs his unannounced interim inspection at time point t_1 , then the expected (conditional with respect β) detection time is given by

$$(1-\beta)\cdot(t_1-t_2)+\beta\cdot(t_0-t_2)\,,$$

(we maintain the general form for later purposes), whereas in the in case the operator chooses $l(t_2)$ it is given by t_0-t_1.

Let g_2 be the behavioral strategy of the operator, i.e., the probability to start the illegal activity at time point t_2. Then the (unconditional) expected detection time[3] is

$$Op_1(\beta;g_2,t_1)=g_2\cdot[\,(1-\beta)\cdot(t_1-t_2)+\beta\cdot(t_0-t_2)\,]+(1-g_2)\cdot(t_0-t_1)\,. \quad (4.13)$$

[3]For the notation of the expected detection times see again Appendix A.

The solution of this game is given as follows: The optimal inspection time point t_1^* is given by (remember $t_2 = 0$)

$$t_1^* - t_2 = \frac{1-\beta}{2-\beta} \cdot (t_0 - t_2) \tag{4.14}$$

and the optimal operator strategy by

$$g_2^* = \frac{1}{2-\beta} \,. \tag{4.15}$$

The optimal expected detection time is

$$Op_1^*(\beta) = t_0 - t_1^* = \frac{t_0 - t_2}{2-\beta} \,. \tag{4.16}$$

This result in proven in section D.4.1 of Appendix D.4 .

It should be emphasized that our analysis leads to an explicit dependence of the optimal time point for inspection t_1^* on β. Whereas for $\beta = 0$ the common sense point of view would lead to this result, for $\beta > 0$ one would hardly arrive at this result without quantitative analysis. The same holds for the operator's optimal strategy.

Also it is interesting to note that the optimal time point for inspection t_1^* depends on the length $t_0 - t_2$ of the reference time interval and β, while the optimal strategy of the operator g_2^* is only a function of β. It is intuitive, however, that both t_1^* decreases with increasing β whereas g_2^* increases with increasing β: for β close to 1 the detection probability is close to zero and therefore the operator starts with probability close to 1 at time point $t_2 = 0$. Consequently, the inspector will perform his inspections also very early.

Finally and most importantly, the optimal strategy of the inspector is a *pure* strategy, i.e., t_1^* is deterministic. In other words, the inspector can announce the time point of his interim inspection if he wishes so (and which the operator knows anyhow)[4].

Remember that in the discrete time version of this inspection problem this was the case for just one interim inspection ($k = 1$) only for special values of β, see section 4.1.1.

The case of two unannounced interim inspections ($k = 2$)

Again the operator decides at the beginning t_3 of the reference time interval $[t_3, t_0]$ whether to start his illegal activity immediately or later. In the latter case he decides

[4]Due to the linearity of the expected detection time in t_1, the inspector can also choose the time point t_1^* for inspection using an arbitrary distribution density $f(t_1)$ concentrated on $[t_2, t_0]$ such that his optimal expected time point t_1^* for inspection,

$$t_1^* = \int_{t_2}^{t_0} t_1 \cdot f(t_1)\, dt_1 \,,$$

is the same as the deterministic one given by (4.14). However, this way he does not gain anything.

after the first inspection at t_2, and finally if he does not start the illegal activity at t_2, he has to do this at t_1.

The extensive form of this game would have to show this information structure. It would be the time continuous version of Figure 4.3 and 4.5 and thus, difficult to be represented. Since, furthermore, the expected detection time can be written down in a straightforward way we do not show the graphical representation of this game.

Let g_3 and $g_2(t_2)$ be the probabilities that the operator either starts his illegal activity right at the beginning of the reference time interval, or after the first at t_2. Then the (unconditional) expected detection time is

$$\begin{aligned} Op_2(\beta; \mathbf{g}, \mathbf{t}) &= g_3 \cdot [\,(1-\beta)\cdot(t_2-t_3) + \beta\cdot(1-\beta)\cdot(t_1-t_3) + \beta^2\cdot(t_0-t_3)\,] \\ &+ (1-g_3)\cdot[\,g_2(t_2)\cdot((1-\beta)\cdot(t_1-t_2) + \beta\cdot(t_0-t_2)) \\ &\qquad + (1-g_2(t_2))\cdot(t_0-t_1)\,]\,. \end{aligned} \tag{4.17}$$

The solution of this game is recursively given by (remember $t_3 = 0$)

$$t_2^* - t_3 = \frac{1-\beta}{3-2\cdot\beta}\cdot(t_0-t_3)\,, \tag{4.18}$$

$$t_1^* - t_2^* = \frac{1-\beta}{2-\beta}\cdot(t_0-t_2^*) \tag{4.19}$$

and

$$g_3^* = \frac{1}{3-2\cdot\beta} \tag{4.20}$$

$$g_2^*(t_2) = \frac{1}{2} \quad \text{for all} \quad t_3 < t_2 < t_0\,. \tag{4.21}$$

The optimal expected detection time is

$$Op_2^*(\beta) = t_0 - t_1^* = \frac{t_0-t_3}{3-2\cdot\beta}\,. \tag{4.22}$$

This result in proven in section D.4.2 of Appendix D.4.

Technically speaking, these solutions are more simple then the corresponding ones for the discrete game, i.e., no distinction of cases with respect to β are necessary.

Since $t_1^* = 2\cdot t_2^*$, we obtain that for $\beta < 1$ the second inspection takes place after the double the time than the first one. For $\beta = 0$ we get

$$t_2^* = \frac{1}{3}\cdot t_0 \quad \text{and} \quad t_1^* = 2\cdot t_2^* = \frac{2}{3}\cdot t_0\,.$$

As in the case $k = 1$, for $\beta = 0$ the common sense point of view would lead to this result, for $\beta > 0$ one would hardly arrive at this result without quantitative analysis. The same holds for the operator's optimal strategy (g_3^*, g_2^*): Since the operator is confronted at t_3 with three inspection intervals of equal length he chooses g_3^*=1/3. After the first inspection however, only two intervals of equal length are left. Thus, he chooses $g_2^* = 1/2$.

Again, most importantly is the fact, that the inspector may announce the optimal time points of his inspections, if he wishes so, and the same arguments as given in the previous case hold as well.

The case of three unannounced interim inspection ($k = 3$)

Even though we do not need it for the purpose of this study, we present some remarks about extensions of our analysis to more than two interim inspections.

We see that (4.14) for $k = 1$ and (4.19) for $k = 2$ are identical – keeping in mind that in the first one $t_2 = 0$ is fixed. Thus we can guess in which way the determinants for the optimal strategies build up for increasing number of interim inspections. In fact, for $k = 3$ we get

$$t_3^* - t_4 = \frac{1-\beta}{4-3\cdot\beta} \cdot (t_0 - t_4)\,, \tag{4.23}$$

$$t_2^* - t_3^* = \frac{1-\beta}{3-2\cdot\beta} \cdot (t_0 - t_3^*)\,, \tag{4.24}$$

$$t_1^* - t_2^* = \frac{1-\beta}{2-\beta} \cdot (t_0 - t_2^*) \tag{4.25}$$

and

$$g_4^* = \frac{1}{4-3\cdot\beta}\,, \tag{4.26}$$

$$g_3^*(t_3) = \frac{1}{3} \quad \text{for all} \quad t_4 < t_3 < t_0 \tag{4.27}$$

$$g_2^*(t_3, t_2) = \frac{1}{2} \quad \text{for all} \quad t_4 < t_3 < t_2 < t_0\,. \tag{4.28}$$

The optimal expected detection time is

$$Op_3^*(\beta) = t_0 - t_1^* = \frac{t_0 - t_4}{4-3\cdot\beta}\,. \tag{4.29}$$

These results which are proven in section D.4.3 of Annex D are identical to those of a purely sequential variant of our model by Avenhaus and Canty, see [2], in the case of no errors of the first kind. In the Avenhaus-Canty model also the inspector behaves sequentially; this means that he decides at the beginning of the reference time interval at which time point he performs his *first* inspection, thereafter, when to perform his section inspection and so on. Of course, for just one inspection the hybrid-sequential model and the purely sequential model are identical.

One can explain the result that in case there are no errors of the first kind both models lead to the same solutions – ex post – by the fact that the inspector does not gain any information in the course of the game. Therefore, there is no difference between these two variants. Nevertheless it cannot be seen directly in view of the fact that the extensive forms of the two variants are so different.

It remains an open question – the answer to which is not subject of this study – whether or not the solutions of both variants are still the same if errors of the first kind are taken into account.

Again, it should be mentioned that in practice it may be difficult to plan and perform inspections with the continuous time model, since this may create too many problems for the joint performance between IAEA and EURATOM. A practical solution could be to take the nearest possible time point to the optimal time point(s) of inspection(s).

Discussion of results

Figure 4.10 combines all findings of this section and shows their dependencies and practical implications.

The optimal expected detection times are drawn in the upper diagram as functions of the non-detection probability β for the two cases $k = 1$ and $k = 2$. We have chosen $t_0 = 4$ according to the quarterly inspections of EURATOM in on-site interim storage facilities which means that the optimal expected detection times are measured in quarters of years. Choosing for instance $t_0 = 12$ would lead to a measurement in months.

If the desired optimal expected detection time is about 1.5 quarters of a year, we see that this optimal expected detection time cannot be reached with one unannounced interim inspection ($k = 1$), since with (4.16) we get

$$Op_1^*(0) = 2 > 1.5 \,.$$

The mid diagram in Figure 4.10 shows relation (2.1), i.e., the number of checked seals in case of $N = 100$, the total number of seals, as a function of β.

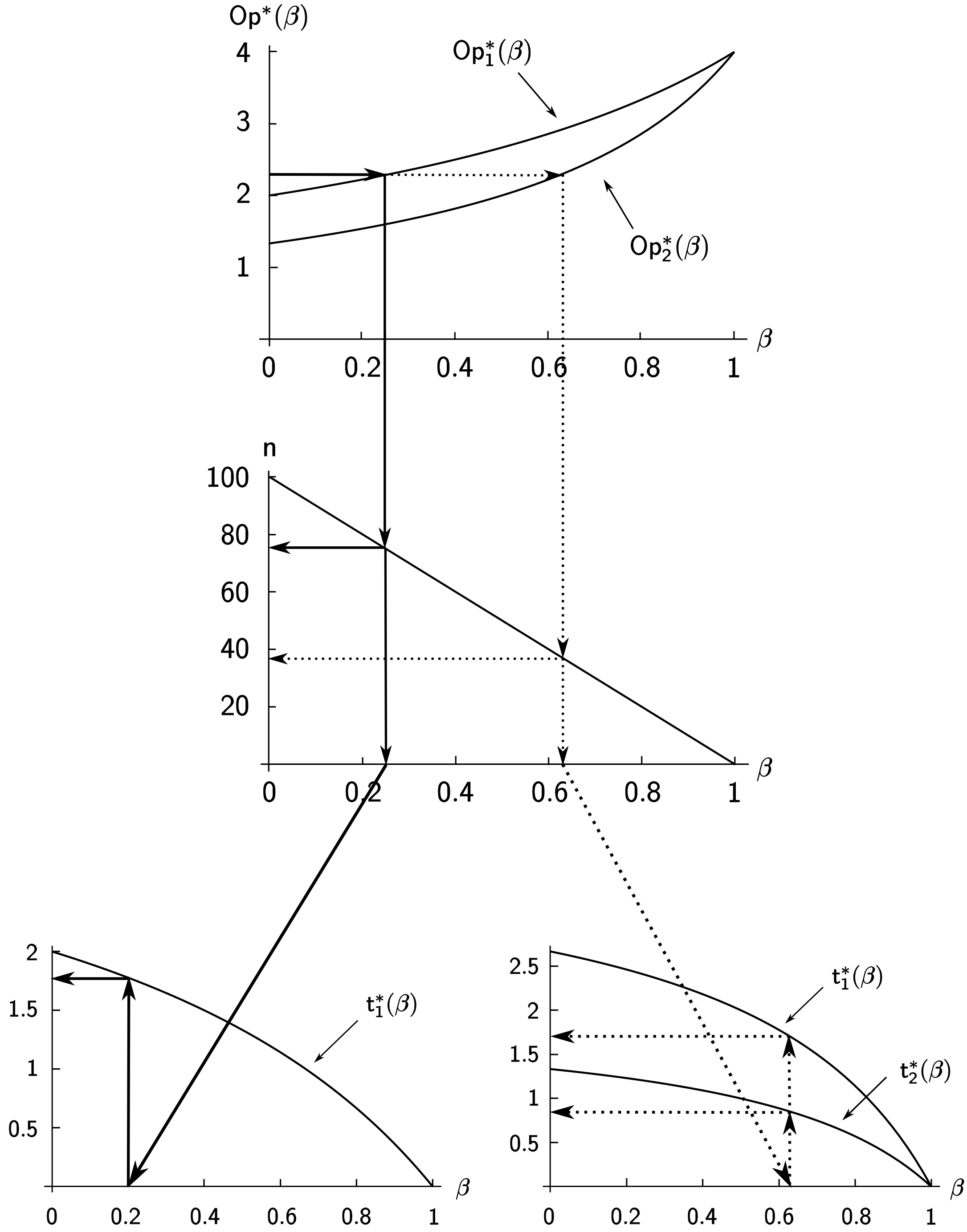
The two lower diagrams present the optimal time point(s) for inspection(s) as given by formulae (4.14) and (4.18) and (4.19): on the left side for $k = 1$ and on the right side for $k = 2$. It is interesting to note that for $k = 1$ and arbitrary β the optimal time point for inspection always lies between 0 and 2, whereas in case of $k = 2$ the first inspection time point lies between 0 and $2/3$ where for the second time point we have $1/3$ and $2/3$ quarters of a year.

All diagrams of Figure 4.10 can now be linked together as follows: Suppose the desired optimal expected detection time is about 2.25 quarters of a year. Then we see that

- In case of one unannounced interim inspection the non-detection probability has to be about 0.22 and therefore - using the mid diagram - the sample size has to be about 78. The corresponding optimal time point for the inspection is - using the left lower diagram - about 1.75 quarters of a year. The arrows with the solid lines illustrate this argumentation.

- In case of two unannounced interim inspections the non-detection probability has to be about 0.61 and therefore - using the mid diagram - the sample size has to be about 39 for each inspection. The corresponding optimal time points for the inspections are - using the right lower diagram - about 0.8 and 1.75 quarters of a year. The arrows with the dotted lines illustrate this argumentation.

Figure 4.10 Graphical presentation of results of section 4.2.



We see that we can assure the same optimal expected detection time of 2.25 quarters of a year with one or two unannounced interim inspections. In both cases we have to check in total the same number of seals, namely 78, which follows from the formulae (2.1), (4.16) and (4.22). It depends on the overhead costs which case is more economic for the inspection authority.

There is a second way in which Figure 4.10 can be interpreted (although not indicated with arrows). Starting with the mid diagram we assume that we can only check a small number of seals n. Then we see that we arrive at a quite high non-detection probability β and therefore - using the upper diagram - at quite high optimal expected detection times.

4.3 Presentation and evaluation of results

Let us apply the results presented so far in the same way to both types of nuclear facility considered in this study, i.e., represent the optimal expected detection times for $k = 1$ and $k = 2$ and the discrete and continuous time cases. For this purpose, we take $t_0 = 4$ for the on-site interim storage facility and $t_0 = 6$ for the fuel element fabrication facility.

4.3.1 On-site Interim Storage Facility

Here we obtained with $n = 3$ in the discrete time model for $k = 1$ with (4.3)

$$Op^*_{3,1}(\beta) = \frac{4}{2-\beta}$$

and we got for the continuous time model with (4.16) and $t_0 = 4$ the same formula

$$Op^*_1(\beta) = \frac{4}{2-\beta}\,.$$

For $k = 2$ we obtained for the discrete time model with (4.8) and (4.11)

$$Op^*_{3,2}(\beta) = \begin{cases} \dfrac{4-\beta+\beta^2}{3-3\cdot\beta+2\cdot\beta^2-\beta^3} & \text{for} \quad 0 \le \beta < 1/2 \\ 1+\beta+2\cdot\beta^2 & \text{for} \quad 1/2 \le \beta \le 1 \end{cases}$$

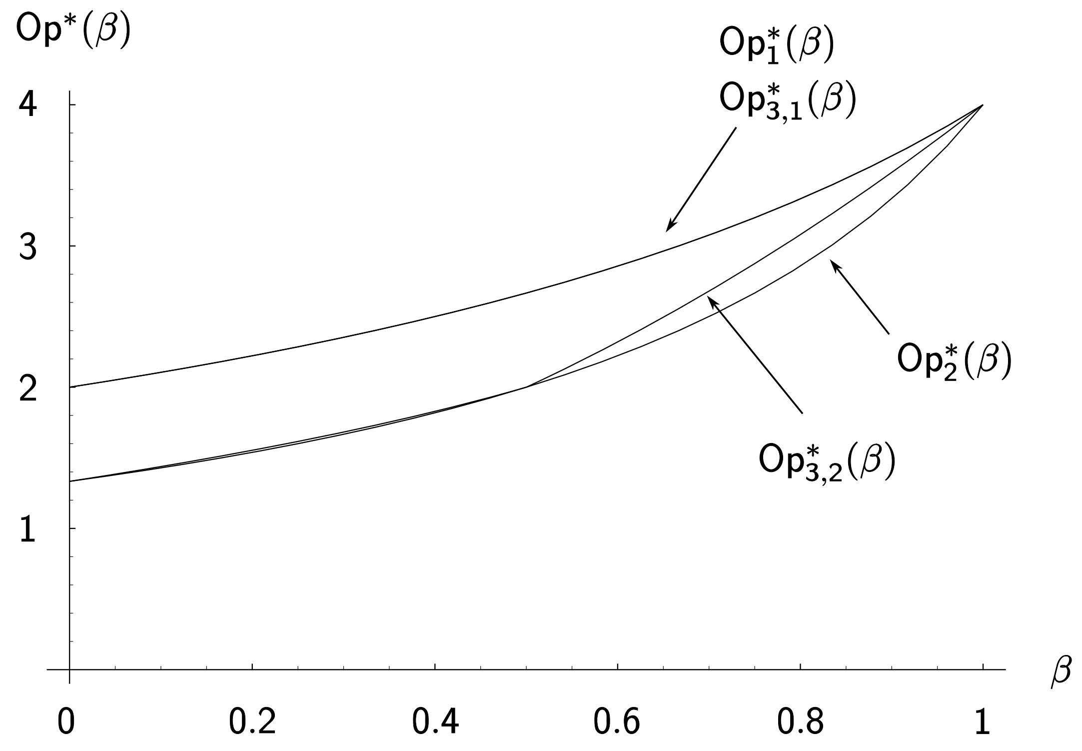
and for the continuous time model with (4.22) and $t_0 = 4$

$$Op^*_2(\beta) = \frac{4}{3-2\cdot\beta}\,.$$

We see that for just one unannounced interim inspection the optimal expected detection time is the same in both the discrete and continuous time cases – even though, of course, the optimal strategies of the inspector are not the same.

The optimal expected detection times for the various cases considered so far are represented graphically in Figure 4.11.

Figure 4.11 Optimal expected detection times as functions of the non-detection probability β for the hybrid-sequential models as applied to the on-site interim storage.



As mentioned, for $k = 1$ the discrete and the continuous time models give the same results. For $k = 2$ they are analytically different, but numerically very close.

It is important to realize that even for $\beta = 0$ and two interim inspections the expected detection time is longer than the conversion time 1, contrary to the situation in the simultaneous model, see Figure 3.11. The fact, that here the optimal expected detection times are longer than those for the corresponding simultaneous cases results from the information the operator can gain in the course of the game.

4.3.2 Fuel Element Fabrication Facility

Here we obtained with $n = 5$ for the discrete time model for $k = 1$ with (4.3)

$$Op^*_{5,1}(\beta) = \frac{6}{2-\beta},$$

and we got for the continuous time model with (4.16) and $t_0 = 6$ the same formula

$$Op^*_1(\beta) = \frac{6}{2-\beta}.$$

For $k = 2$ we obtained for the discrete time model with (4.12)

$$Op^*_{5,2}(\beta) = \begin{cases} \dfrac{6}{3 - 2 \cdot \beta} & \text{for} \quad 0 \le \beta < 2/3 \\ \dfrac{8 - 5 \cdot \beta + 3 \cdot \beta^2}{5 - 9 \cdot \beta + 8 \cdot \beta^2 - 3 \cdot \beta^3} & \text{for} \quad 2/3 \le \beta < 3/4 \\ 1 + \beta + 4 \cdot \beta^2 & \text{for} \quad 3/4 \le \beta \le 1 \end{cases}$$

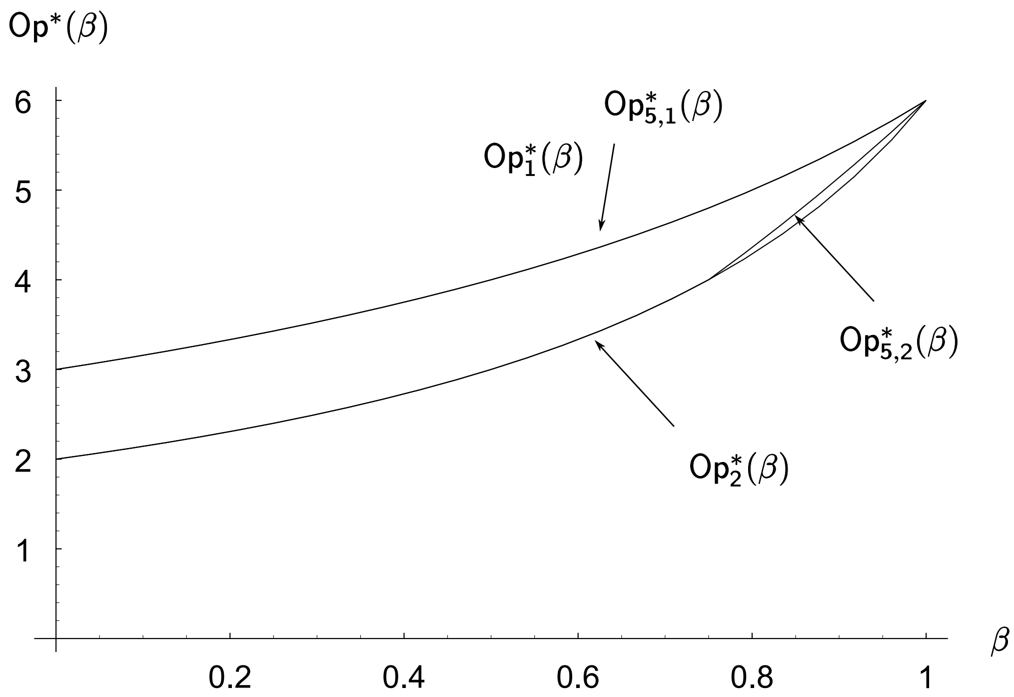
and for the continuous time model with (4.22) and $t_0 = 6$

$$Op^*_2(\beta) = \frac{6}{3 - 2 \cdot \beta} \, .$$

As before, the optimal expected detection times are the same for the time discrete and continuous cases for just one interim inspection. Here, contrary to the previous applications, also for $k = 2$ interim inspections and small values of the non-detection probability β the optimal expected detection times are the same. This is quite interesting and was not yet to be foreseen therefore, we dot not make any guesses about larger values of n.

The optimal expected detection times for the various cases considered so far are represented graphically in Figure 4.12. Again, for $k = 1$ the discrete and continuous time models give the same results, whereas for $k = 2$ they are numerically very close.

Figure 4.12 Optimal expected detection times as functions of the non-detection probability β for the hybrid-sequential models as applied to the fuel element fabrication facility.



Chapter 5

Extensions and further work

Three additional topics will be presented in this chapter; they complement the previous models and analyses and provide an outlook on possible future work.

First, the question of inducing the operator of a nuclear facility to legal behavior will be raised. Even though it is not relevant as long as the number of inspections per year, and the detection probabilities are fixed a priori, the analysis throws an interesting light on the optimal strategies of the inspector.

Second, so-called critical time games are considered, that is, games where the plant operator has "won" the game if his illegal activity is not detected within the critical time, and where otherwise the inspector has won it: Again, even though there were good reasons to consider "playing for time" games in the previous chapters, it is worthwhile to evaluate the consequences of the differences of these two types of objectives.

Finally, an outlook will be given as to the natural extension of our study. As mentioned, we considered so far fixed numbers of inspections per year in one plant. It is obvious to ask how optimal strategies will change if only the number of inspections per year in a State with more than one facility of a given type is fixed.

5.1 Legal behavior

In order to be able to determine inspection strategies which induce the plant operator to legal behavior one has to introduce utility functions which describe the losses and gains of the operator for all possible outcomes of the inspections.

By definition any illegal activity will be detected with certainty at the end of the year. Therefore, if we normalize the gain (or loss) of the operator in case of legal behavior to zero, any illegal activity causes the loss $b > 0$ (gain $-b < 0$). In addition, according to our playing for time criterion, the operator, starting an illegal activity, has a gain proportional to the time $\triangle t$ between the beginning of the illegal activity and its detection which we denote by $d \cdot \triangle t$, $d > 0$. If we normalize our reference time interval to one, the maximum gain of the operator in case of illegal behavior is $d - b$ thus, we assume

$d - b > 0$ otherwise the operator would never start an illegal activity. In sum, the payoff to the operator is

$$\begin{cases} d \cdot \triangle t - b & \text{for} \quad \text{illegal behavior and detection time } \triangle t \\ 0 & \text{for} \quad \text{legal behavior} \end{cases} .$$

Let us assume now that the operator decides before the beginning of the game whether to start or not to start an illegal activity at all. It will turn out that the operator never uses a mixed strategy in equilibrium thus, using our previous terminology, his optimal payoff will be

$$\begin{cases} d \cdot Op^*(\beta) - b & \text{for} \quad \text{illegal behavior and detection time } \triangle t \\ 0 & \text{for} \quad \text{legal behavior} \end{cases} ,$$

where $Op^*(\beta)$ stands for any of the optimal detection times determined in the previous chapters. Therefore, he will behave legally if

$$d \cdot Op^*(\beta) - b < 0$$

or equivalently

$$Op^*(\beta) < \frac{b}{d} .$$

This is a simple condition for legal behavior. For the case of the on-site interim storage facility, one unannounced interim inspection and the discrete time model, we get for $0 \leq \beta < \frac{1}{6}$ with (3.3)

$$\frac{11}{6} + \beta < \frac{b}{d} .$$

The question, however, remains what are the appropriate legal behavior Nash equilibrium inspection strategies. In order to answer this question we have to consider again our models.

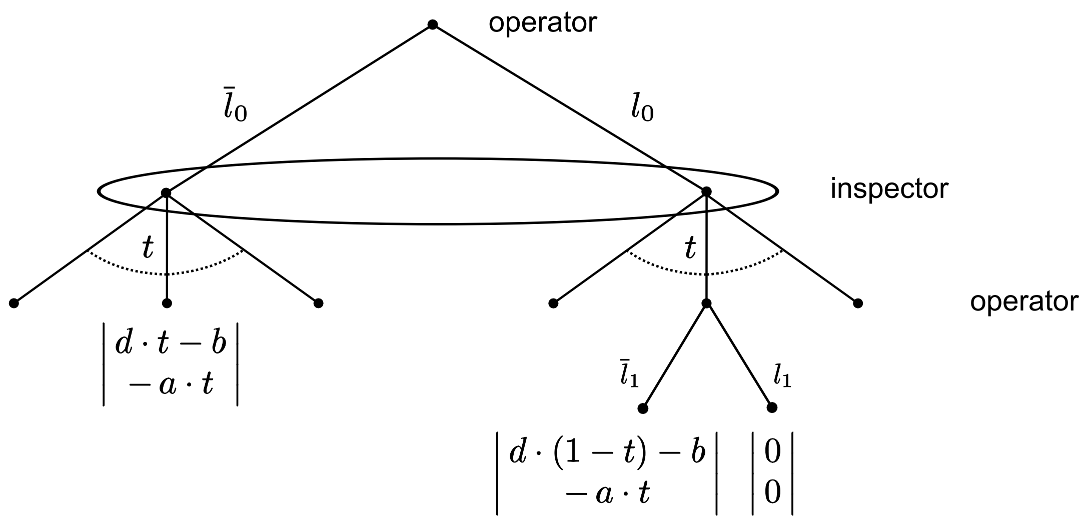
Since we present these considerations for principal purposes, not for immediate practical applications we limit our analyses to the cases $\beta = 0$ and $k = 1$ even though the generalization to $\beta > 0$ is straightforward. Also, we start with the time continuous hybrid-sequential model since here the analysis is very simple. Next we continue with the time continuous simultaneous model, and finally, we consider the time discrete models.

5.1.1 The time continuous hybrid-sequential model

Introducing utilities means that we do not deal any longer with zero sum games. Therefore, we have to define also the inspector's payoff as follows:

$$\begin{cases} -a \cdot \triangle t & \text{for} \quad \text{for illegal behavior and detection time } \triangle t \\ 0 & \text{for} \quad \text{for legal behavior} \end{cases} ,$$

Figure 5.1 Non-zero sum extensive form game including legal behavior of the operator (l_0, l_1), corresponding to the zero-sum extensive form game given in Figure 4.9. Note that $t_2 = 0$ and $t_1 = t$.



here, we have $a > 0$ since the highest priority of the inspector is, let us repeat, to induce the operator to legal behavior. With this definition the extensive form game of section 4.2 which includes legal behavior, is given in Figure 5.1.

Let W be the payoff to the operator, and V that of the inspector for this new game. The Nash equilibrium condition for the game including the legal behavior of the operator, $W_l^* = V_l^* = 0$, is given by

$$0 \geq W((\bar{l}_0, \bar{l}_1), t^*) \tag{5.1}$$

$$0 \geq W((l_0, \bar{l}_1), t^*) \tag{5.2}$$

$$0 \geq W((l_0, l_1), t^*) \tag{5.3}$$

$$0 \geq V((l_0, l_1), t) \quad \text{for all} \quad t\,. \tag{5.4}$$

Whereas condition (5.4) for the inspector is fulfilled as identity, like condition (5.3), condition (5.1) and (5.2) are equivalent to

$$0 \geq d \cdot t^* - b \quad \text{and} \quad 0 \geq d \cdot (1 - t^*) - b\,.$$

They can be combined to

$$1 - \frac{b}{d} \leq t^* \leq \frac{b}{d}\,. \tag{5.5}$$

In order that this interval for t^* is not empty, we need $b/d \geq 1/2$, otherwise the operator will behave illegally in equilibrium. Let us summarize: For $1/2 \leq b/d \leq 1$ there is a Nash equilibrium in which the operator behaves legally. The equilibrium strategy of the inspector is not unique, but given by (5.5).

One additional important remark has to be made here: Let $W_{\bar{l}}^*$ and $V_{\bar{l}}^*$ be the payoffs for the equilibrium in which the operator behaves illegally. Then the Nash condition for the operator is

$$\begin{aligned} W_{\bar{l}}^* &\geq W((\bar{l}_0, \bar{l}_1), t_{\bar{l}}^*) &(5.6)\\ W_{\bar{l}}^* &\geq W((l_0, \bar{l}_1), t_{\bar{l}}^*) &(5.7)\\ W_{\bar{l}}^* &\geq W((l_0, l_1), t_{\bar{l}}^*)\,. &(5.8)\end{aligned}$$

In order that legal behavior is the Nash equilibrium, $W_l^* > W_{\bar{l}}^*$ has to be fulfilled thus, with (5.6), (5.7) and (5.8) we can now write

$$\begin{aligned} W_l^* > W_{\bar{l}}^* &\geq W((\bar{l}_0, \bar{l}_1), t_{\bar{l}}^*) &(5.9)\\ W_l^* > W_{\bar{l}}^* &\geq W((l_0, \bar{l}_1), t_{\bar{l}}^*) &(5.10)\\ W_l^* \geq W_{\bar{l}}^* &\geq W((l_0, l_1), t_{\bar{l}}^*)\,. &(5.11)\end{aligned}$$

This means, however, that $t_{\bar{l}}^*$ is also a equilibrium strategy equilibrium in which the operator behaves legally. In other words, if the inspector uses his (unique) equilibrium strategy, which corresponds to the illegal equilibrium strategy of the operator, then it is also equilibrium strategy for the case the operator behaves legally provided $1/2 \leq b/d$ holds.

5.1.2 The time continuous simultaneous model

Let us now turn to the model analyzed in section 3.2, where the equilibrium strategy of the inspector is a mixed strategy, as we remember.The extensive form of the game which includes legal behavior, is given in Figure 5.2.

Let again be W and V the payoff to the operator and to the inspector, and let Q and P be a mixed strategy for the operator respectively the inspector. Furthermore, let q be the probability that the operator behaves illegally. Then the Nash condition for the equilibrium in which the operator behaves legally, $W_l^* = V_l^* = 0$, is

$$\begin{aligned} 0 &\geq W((q, Q), P^*) \quad \text{for all} \quad (q, Q)\\ 0 &\geq V((q^*, Q^*), P) \quad \text{for all} \quad P\end{aligned}$$

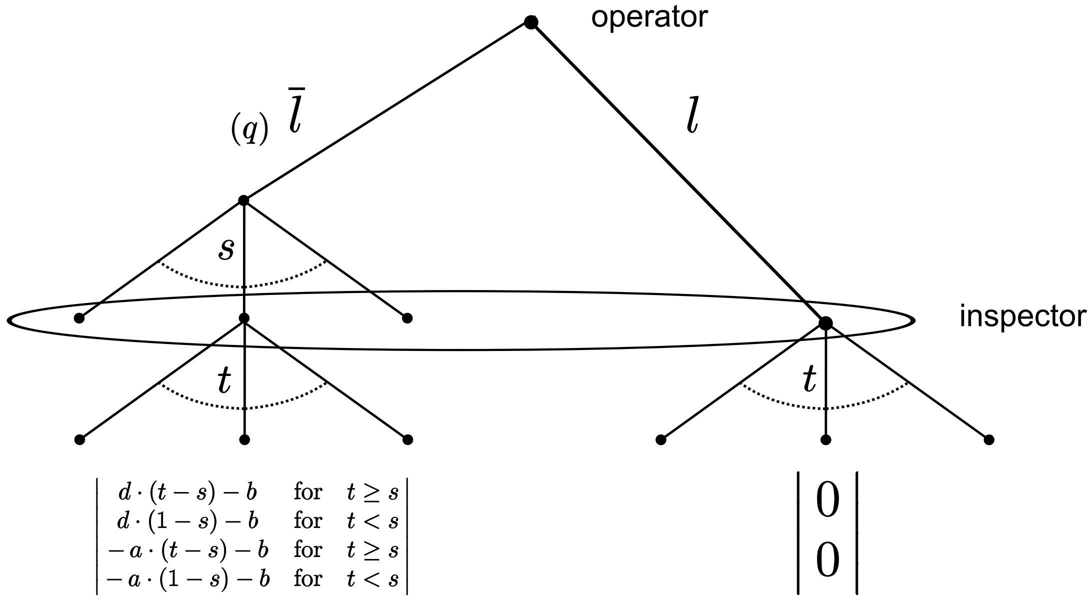
or equivalently

$$\begin{aligned} 0 &\geq W((\bar{l}, s), P^*) \quad \text{for all} \quad s &(5.12)\\ 0 &\geq W((l, s), P^*) \quad \text{for all} \quad s &(5.13)\\ 0 &\geq V((q^*, Q^*), t) \quad \text{for all} \quad t\,. &(5.14)\end{aligned}$$

Whereas (5.13) and (5.14) are again identically fulfilled, (5.12) is explicitly given by

$$0 \geq \int_0^s [\, d \cdot (1 - s) - b\,] \cdot p^*(t)\, dt + \int_s^1 [\, d \cdot (t - s) - b\,] \cdot p^*(t)\, dt \quad \text{for all} \quad s$$

Figure 5.2 Non zero sum extensive form game including legal behavior of the operator (l_0, l_1), corresponding to the zero sum game simultaneous game for one unannounced interim inspection ($k = 1$) described in section 3.2.



or equivalently, by

$$\frac{b}{d} \geq \int_0^s p^*(t)\,dt + \int_s^1 t \cdot p^*(t)\,dt - \int_0^1 s \cdot p^*(t)\,dt \quad \text{for all} \quad s$$

or, finally, by

$$\frac{b}{d} \geq \int_0^s p^*(t)\,dt + \int_s^1 t \cdot p^*(t)\,dt - s := H(s) \quad \text{for all} \quad s\,.$$

Thus, all mixed strategies densities $p^*(t)$ which fulfill the condition

$$\frac{b}{d} \geq \max_s H(s)$$

are equilibrium strategies for the legal behavior Nash equilibrium. With the same simple argument used in the previous section we can show that the equilibrium strategy of the inspector, which belongs to the illegal behavior Nash equilibrium, is also equilibrium strategy of the legal behavior Nash equilibrium. It is, however, interesting to show that there are also pure equilibrium strategies of the inspector belonging to the legal behavior Nash equilibrium.

Let t^* be such an equilibrium strategy. Then (5.12) is equivalent to

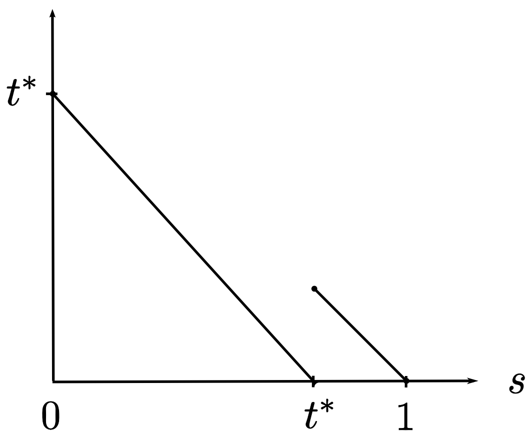
$$0 \geq \begin{cases} d \cdot (t^* - s) - b & \text{for} \quad t^* > s \\ d \cdot (1-s) - b & \text{for} \quad t^* \leq s \end{cases} \quad \text{for all} \quad s$$

or

$$\frac{b}{d} \geq \begin{cases} t^* - s & \text{for} \quad t^* > s \\ 1 - s & \text{for} \quad t^* \leq s \end{cases} \quad \text{for all} \quad s\,. \tag{5.15}$$

For $t^* > 1/2$ this is illustrated graphically in Figure 5.3. Therefore, we get

Figure 5.3 Graphical illustration of the right hand side of (5.15) for $t^* > 1/2$.



$$\max_s \begin{cases} t^* - s & \text{for} \quad t^* > s \\ 1 - s & \text{for} \quad t^* \leq s \end{cases} = \max\{\, t^*, 1 - t^* \,\}$$

that is, for $t^* > \frac{1}{2}$ we get

$$\max\{\, t^*, 1 - t^* \,\} = t^*\,, \quad \underset{s}{\text{argmax}} = 0\,,$$

and for $t^* < \frac{1}{2}$

$$\max\{\, t^*, 1 - t^* \,\} = 1 - t^*\,, \quad \underset{s}{\text{argmax}} = t^*\,.$$

Therefore, according to (5.15) we have

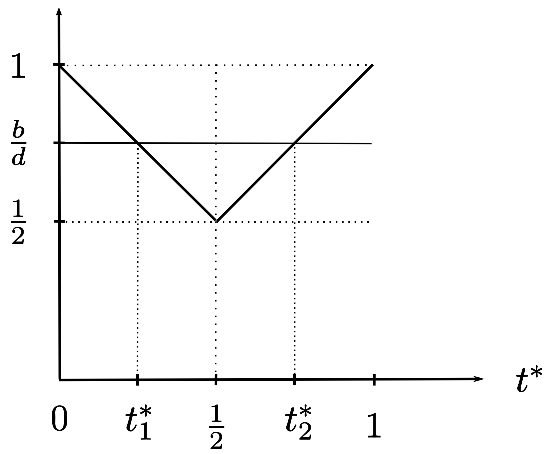
$$\begin{aligned} &\text{for} \quad t^* > \frac{1}{2} \quad : \quad \frac{b}{d} > t^*\,, \\ &\text{for} \quad t^* < \frac{1}{2} \quad : \quad \frac{b}{d} > 1 - t^*\,. \end{aligned} \tag{5.17}$$

This is represented graphically in Figure 5.4.

We see that the equilibrium strategy of the inspector has to satisfy the condition

$$t_1^* \leq t^* \leq t_2$$

Figure 5.4 Graphical representation of (5.17).



or, with $t_1^* = 1 - b/d$ and $t_2^* = b/d$, finally

$$1 - \frac{b}{d} \leq t^* \leq \frac{b}{d} \tag{5.17}$$

which is the same condition as in the previous model, as given by (5.5).

Let us mention that the illegal equilibrium leads to the payoff to the operator

$$W_i^* = d \cdot \frac{1}{e} - b$$

thus, the condition for legal behavior is

$$\frac{b}{d} > \frac{1}{e}\,.$$

A necessary condition for the pure equilibrium strategy t^* was $b/d > 1/2$, see Figure 5.4. Because of $1/2 > 1/e$ we conclude that for the pure strategy to be applied by the inspector the ratio b/d has to be a bit larger than for the mixed strategy $p^*(t)$ density as given in section 3.2.

5.1.3 Discrete time models

Since all discrete time models, both simultaneous and hybrid-sequential ones, can be transformed into matrix games, we demonstrate the concept of inducing the operator to legal behavior just at one simple case, namely the discrete time simultaneous model with $k = 1$ and $n = 3$ as analyzed in section 3.1.

Because of the maximum detection time $\triangle t = 4$, now the payoff to the operator in case he uses the illegal strategy i $(i = 0, 1, 2, 3)$, is

$$W(i) = d \cdot i - b\,,$$

where $4 \cdot d - b > 0$. His equilibrium payoff in case of illegal behavior is

$$W_{\bar{l}}(i) = d \cdot \frac{11}{6} - b\,,$$

thus, he will be induced to legal behavior if

$$\frac{b}{d} > \frac{11}{6}\,.$$

The normal form game, which includes the legal behavior strategy, is given in Figure 5.5.

Figure 5.5 Non zero sum normal form game including legal behavior of the operator and without his dominated strategy $i = 3$, corresponding to the zero sum normal form game given in Figure 3.2, and $\beta = 0$.

Op \ Insp	1	2	3
0	$-a$ / $d - b$	$-2 \cdot a$ / $2 \cdot d - b$	$-3 \cdot a$ / $3 \cdot d - b$
1	$-3 \cdot a$ / $3 \cdot d - b$	$-a$ / $d - b$	$-2 \cdot a$ / $2 \cdot d - b$
2	$-2 \cdot a$ / $2 \cdot d - b$	$-2 \cdot a$ / $2 \cdot d - b$	$-a$ / $d - b$
legal	0 / 0	0 / 0	0 / 0

It can be shown that this game has only an illegal or a legal equilibrium. The Nash condition for the operator is for the legal equilibrium

$$0 \geq (d - b) \cdot p_1^* + (2 \cdot d - b) \cdot p_2^* + (3 \cdot d - b) \cdot p_3^* \quad (5.18)$$
$$0 \geq (3 \cdot d - b) \cdot p_1^* + (d - b) \cdot p_2^* + (2 \cdot d - b) \cdot p_3^* \quad (5.19)$$
$$0 \geq (2 \cdot d - b) \cdot p_1^* + (2 \cdot d - b) \cdot p_2^* + (d - b) \cdot p_3^*\,, \quad (5.20)$$

where $p_1^* + p_2^* + p_3^* = 1$. The Nash condition for the legal equilibrium for the inspector is identically fulfilled.

With $p_3^* = 1 - p_1^* - p_2^*$ the inequalities (5.18) - (5.20) are equivalent to

$$\frac{b}{d} \geq -2 \cdot p_1^* - p_2^* + 3 \tag{5.21}$$

$$\frac{b}{d} \geq p_1^* - p_2^* + 2 \tag{5.22}$$

$$\frac{b}{d} \geq -p_1^* - p_2^* + 3\,. \tag{5.23}$$

For the purpose of illustration we take $b/d = 2$ which fulfills the condition for legal behavior of the operator,

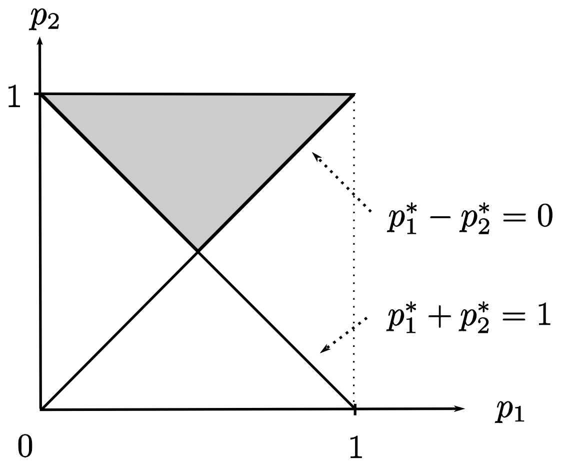
$$\frac{11}{6} < \frac{b}{d} < 4\,. \tag{5.24}$$

Then we get from (5.21) - (5.23)

$$\begin{aligned} 2 \cdot p_1^* + p_2^* &\geq 1 \\ p_1^* - p_2^* &\leq 0 \\ p_1^* + p_2^* &\geq 1\,. \end{aligned}$$

The first inequality is dominated by the third one. In Figure 5.6 we have represented these conditions graphically.

Figure 5.6 Set of mixed equilibrium strategies of the inspector which induce the operator to legal behavior.



The shaded are in Figure 5.6 represents the set of equilibrium strategies of the inspector provided (5.24) is fulfilled. Here we see why M. Kilgour [24] several years ago coined the term *cone of deterrence*.

Again, the illegal equilibrium strategy of the inspector is also legal equilibrium strategy provided (5.24) is fulfilled.

5.2 The Critical Time concept

Both four our simultaneous and hybrid-sequential models we used the "playing for time" concept which means that both the inspector and the operator choose the expected detection time as their optimization criterion.

In the following we will discuss an alternative concept which has its origin in the so-called conversion time introduced by the IAEA and mentioned already in the second chapter. It says that for each type of fissile material some time, namely the conversion time, is needed to manufacture with its help a nuclear explosive device. Using this definition it appears to be quite natural to assume that the inspection authority has "won" the game if any illegal activity is detected within some "critical" time, otherwise it has lost it, and vice versa for the operator of the plant.

Of course, the underlying assumption of this concept is that, if an illegal activity is detected within the critical time, then some action can be taken by the international community in order to stop the manufacture, or to impose sanctions, whereas in the other case the international community has to live with the fact that the State under consideration has the nuclear device and can use it, at least politically. The weak point of this concept is that, once an illegal activity is detected it may take weeks and months for a serious reaction of the international community, that is, a much longer time than defined by the conversion time.

It is mainly for this reason that we considered so far, only the "playing for time" concept. Nevertheless, we will now discuss the critical time concept with the help of one simple case which we analyzed already in the third chapter of this study, using the simultaneous approach. In the following we introduce the critical time concept. Thereafter, we compare the results obtained this way with those of the earlier approach. Finally, we consider both alternatives together which leads us to a normal form game with vector-valued payoffs.

The presentation is based on a paper delivered at the ESARDA conference 2007 in Aix en Provence, see [5]. The formulation has been adapted to what has been said already in earlier chapters of this study.

5.2.1 The Critical Time game

We consider once more the case of the on-site interim storage facility treated in section 3.1.1. The inspector has three points of time for his unannounced interim inspection at his disposal, and the operator has four points of time at which he can start an illegal activity. We select only the case of one unannounced interim inspection ($k = 1$), and only the case of perfect inspections ($\beta = 0$).

Now, as already stated, we assume that in case the inspection takes place within the critical time after the start of the illegal activity, the inspector has won and the operator has lost, otherwise vice versa. The payoff to the two players (operator, inspector) are

$(d_1, -c_1)$ for detection of an illegal activity outside the critical time, and
$(-b_1, -a_1)$ for detection of an illegal activity within the critical time.

Here we assume $b_1 > 0, d_1 > 0$ and $0 < a_1 < c_1$, since we normalize the payoffs for legal behavior to zero, and timely detection of an illegal activity still being worse then legal behavior of the operator. The normal form of this "illegal" game, i.e., the game where the legal behavior of the operator is excluded, is represented in Figure 5.7, where the rows resp. the columns represent the pure strategies of the operator resp. the inspector, and where in the lower left resp. upper right corner of each entry the payoffs to the operator resp. the inspector are given.

Figure 5.7 Normal form of the critical time game.

Op \ Insp	1	2	3
0	$-b_1$, $-a_1$	d_1, $-c_1$	d_1, $-c_1$
1	d_1, $-c_1$	$-b_1$, $-a_1$	d_1, $-c_1$
2	d_1, $-c_1$	d_1, $-c_1$	$-b_1$, $-a_1$
3	$-b_1$, $-a_1$	$-b_1$, $-a_1$	$-b_1$, $-a_1$

We realize immediately that the fourth pure strategy of the operator is dominated thus, we have to consider a quadratic 3×3 game. The Nash equilibrium of this game consists in mixed strategies, i.e., probabilities with which the pure strategies are played. Using the symmetry of the 3×3 matrix we obtain for the operator and for the inspector

$$q_0^* = q_1^* = q_2^* = \frac{1}{3}, q_3^* = 0 \quad \text{and} \quad p_1^* = p_2^* = p_3^* = \frac{1}{3},$$

independent of the payoff parameters; with the expected payoffs to the operator and inspector[1]

[1]For this section 5.2 we use special notations for the equilibrium payoffs of the operator, since it does interfere with those of the other chapters: Op_1^* for the critical time game, Op_2^* for the playing for time game and Op_3^* for the vector-valued game. For the inspector a corresponding notation is used.

$$Op_1^* = \frac{1}{3} \cdot (2 \cdot d_1 - b_1) \quad \text{and} \quad Insp_1^* = -\frac{1}{3} \cdot (2 \cdot c_1 + a_1)\,.$$

So far we discussed only the illegal game, i.e., the game in which the operator acts illegally with certainty. He will behave legally, if his expected payoff in this case is larger than in the other. Thus, if we add to the game in Figure 5.7 the legal behavior of the operator as his fifth pure strategy, and since the operator´s expected payoff in case of legal behavior is zero, for $2 \cdot d_1 - b_1 < 0$ the equilibrium strategy of the operator is legal behavior, i.e.,

$$q_0^* = q_1^* = q_2^* = q_3^* = 0 \quad \text{and} \quad q_4^* = 1\,.$$

The corresponding equilibrium strategy of the inspector is not unique, as the Nash condition for the operator show:

$$\begin{array}{rcl} 0 & \geq & -b_1 \cdot p_1^* + d_1 \cdot p_2^* + d_1 \cdot p_3^* \\ 0 & \geq & d_1 \cdot p_1^* - b_1 \cdot p_2^* + d_1 \cdot p_3^* \\ 0 & \geq & d_1 \cdot p_1^* + d_1 \cdot p_2^* - b_1 \cdot p_3^* \end{array}$$

or equivalently

$$p_j^* \geq \frac{1}{1 + b_1/d_1}\,, \quad j = 1, 2, 3\,, \quad p_1^* + p_2^* + p_3^* = 1\,.$$

Because of $2 \cdot d_1 - b_1 < 0$ we have $1 + b_1/d_1 > 3$, which means that the equilibrium strategy for the game in which the operator behaves only illegally is contained in the set of legal behavior equilibrium strategies.

5.2.2 The Playing for Time game

Now we consider once more the inspection problem treated before, but take like in section 3.1.1 the expected detection time as payoff to the operator. In order to be able to compare the results with those of the previous section, however, we have to generalize the payoffs as follows. Let i, $i = 1, 2, 3$, be the time between the start of the illegal activity and its detection. Then the payoff to the two players (operator, inspector) are

$$(d_2 \cdot i - b_2, -c_2 \cdot i - a_2)\,,$$

where $d_2 > 0, b_2 > 0, c_2 > 0$ and $a_2 > 0$. Let us mention in passing that for $d_2 = c_2 = 1$ and $b_2 = a_2 = 0$ we arrive again at the same game treated in section 3.1.1 of this study.

The normal form of this game is given in Figure 5.8; we see that the fourth pure strategy of the operator is again dominated.

Again there is a Nash equilibrium in mixed strategies. With the help of a Mathematica®-program developed by J. M. Canty, see [10], we obtain for the operator and for the inspector

$$q_0^* = \frac{1}{3}\,, q_1^* = \frac{1}{6}\,, q_2^* = \frac{1}{2}\,, q_3^* = 0 \quad \text{and} \quad p_1^* = \frac{1}{3}\,, p_2^* = \frac{1}{2}\,, p_3^* = \frac{1}{6}\,,$$

Figure 5.8 Normal form of the playing for time game.

Op \ Insp	1	2	3
0	$-c_2-a_2$ d_2-b_2	$-2\cdot c_2-a_2$ $2\cdot d_2-b_2$	$-3\cdot c_2-a_2$ $3\cdot d_2-b_2$
1	$-3\cdot c_2-a_2$ $3\cdot d_2-b_2$	$-c_2-a_2$ d_2-b_2	$-2\cdot c_2-a_2$ $2\cdot d_2-b_2$
2	$-2\cdot c_2-a_2$ $2\cdot d_2-b_2$	$-2\cdot c_2-a_2$ $2\cdot d_2-b_2$	$-c_2-a_2$ d_2-b_2
3	$-c_2-a_2$ d_2-b_2	$-c_2-a_2$ d_2-b_2	$-c_2-a_2$ d_2-b_2

again independent of the payoff parameters. They are the same as those given in section 3.1.1. The expected equilibrium payoffs are

$$Op_2^* = \frac{11}{6}\cdot d_2 - b_2 \quad \text{and} \quad Insp_2^* = -\frac{11}{6}\cdot c_2 - a_2\,.$$

With the same arguments as before we note that for $11\cdot d_2 - b_2 < 0$ the equilibrium strategy of the operator is legal behavior; the corresponding equilibrium strategy of the inspector has the same properties as that for the critical time game.

5.2.3 The vector-valued game

We now analyze the critical time game and the playing for time game simultaneously. Since in both games the pure strategies of both players are the same, we just have to consider the payoffs simultaneously which leads to a vector-valued game the normal form of which is given in Figure 5.9[2].

We can see that the components of the operator´s payoff if he plays his 3rd pure strategy are always greater or at least equal to the corresponding components if he plays his fourth pure strategy, i.e., the fourth pure strategy of the operator is again dominated. The analysis of this game still poses problems, since the payoffs of the two separate games cannot be compared directly. For example is d_1 an absolute gain of the operator, whereas d_2 is the rate of a gain. Therefore, we make the additional assumptions that the maximal and minimal payoffs to both players are the same. We justify this with the

[2]In an introductory textbook one would put an asterisk at the heading of this section meaning that this section is not necessary at first reading.

Figure 5.9 Vector valued normal form game.

Op \ Insp	1	2	3
0	$\begin{pmatrix}-b_1\\d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-a_1\\-c_2-a_2\end{pmatrix}$	$\begin{pmatrix}d_1\\2\cdot d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-c_1\\-2\cdot c_2-a_2\end{pmatrix}$	$\begin{pmatrix}d_1\\3\cdot d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-c_1\\-3\cdot c_2-a_2\end{pmatrix}$
1	$\begin{pmatrix}d_1\\3\cdot d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-c_1\\-3\cdot c_2-a_2\end{pmatrix}$	$\begin{pmatrix}-b_1\\d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-a_1\\-c_2-a_2\end{pmatrix}$	$\begin{pmatrix}d_1\\2\cdot d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-c_1\\-2\cdot c_2-a_2\end{pmatrix}$
2	$\begin{pmatrix}d_1\\2\cdot d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-c_1\\-2\cdot c_2-a_2\end{pmatrix}$	$\begin{pmatrix}d_1\\2\cdot d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-c_1\\-2\cdot c_2-a_2\end{pmatrix}$	$\begin{pmatrix}-b_1\\d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-a_1\\-c_2-a_2\end{pmatrix}$
3	$\begin{pmatrix}-b_1\\d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-a_1\\-c_2-a_2\end{pmatrix}$	$\begin{pmatrix}-b_1\\d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-a_1\\-c_2-a_2\end{pmatrix}$	$\begin{pmatrix}-b_1\\d_2-b_2\end{pmatrix}$ $\begin{pmatrix}-a_1\\-c_2-a_2\end{pmatrix}$

remark that both games describe the same principal conflict situation thus, the best and the worst outcome is evaluated in the same way by the two players. This leads to the following conditions

$$d_2 - b_2 = -b_1\,, \quad 3\cdot d_2 - b_2 = d_1\,, \quad -c_2 - a_2 = -a_1\,, \quad -3\cdot c_2 - a_2 = -c_1$$

which is equivalent to

$$d_2 = \frac{1}{2}\cdot(b_1+d_1)\,,\ b_2 = \frac{1}{2}\cdot(3\cdot b_1 + d_1)\,,\ c_2 = \frac{1}{2}\cdot(c_1 - a_1)\,,\ a_2 = \frac{1}{2}\cdot(3\cdot a_1 - c_1)\,.$$

Because of $b_1 > 0$, $d_1 > 0$ and $a_1 > 0$, $c_1 > 0$ the following conditions have to be fulfilled

$$0 < d_2 < b_2 < 3\cdot d_2 \qquad \text{and} \qquad 0 < a_2 < c_2 < 3\cdot a_2\,.$$

Taking these conditions into account we can model the conflict situation as the vector-valued normal form game given in Figure 5.10, where we have already deleted the fourth pure strategy of the operator, as argued above.

5.2.4 Vector-valued games in normal form

In order to analyse the vector-valued normal form game formulated in the previous section, we have to introduce the concept of the Pareto equilibrium, which is a natural generalization of the Nash equilibrium concept.

Figure 5.10 Modified and reduced vector valued normal form game.

Op \ Insp	1	2	3
0	$\begin{pmatrix}-a_1\\-a_1\end{pmatrix}$ $\begin{pmatrix}-b_1\\-b_1\end{pmatrix}$	$\begin{pmatrix}-c_1\\-1/2\cdot(c_1+a_1)\end{pmatrix}$ $\begin{pmatrix}d_1\\1/2\cdot(d_1-b_1)\end{pmatrix}$	$\begin{pmatrix}-c_1\\-c_1\end{pmatrix}$ $\begin{pmatrix}d_1\\d_1\end{pmatrix}$
1	$\begin{pmatrix}-c_1\\-c_1\end{pmatrix}$ $\begin{pmatrix}d_1\\d_1\end{pmatrix}$	$\begin{pmatrix}-a_1\\-a_1\end{pmatrix}$ $\begin{pmatrix}-b_1\\-b_1\end{pmatrix}$	$\begin{pmatrix}-c_1\\-1/2\cdot(c_1+a_1)\end{pmatrix}$ $\begin{pmatrix}d_1\\1/2\cdot(d_1-b_1)\end{pmatrix}$
2	$\begin{pmatrix}-c_1\\-1/2\cdot(c_1+a_1)\end{pmatrix}$ $\begin{pmatrix}d_1\\1/2\cdot(d_1-b_1)\end{pmatrix}$	$\begin{pmatrix}-c_1\\-1/2\cdot(c_1+a_1)\end{pmatrix}$ $\begin{pmatrix}d_1\\1/2\cdot(d_1-b_1)\end{pmatrix}$	$\begin{pmatrix}-a_1\\-a_1\end{pmatrix}$ $\begin{pmatrix}-b_1\\-b_1\end{pmatrix}$

For vectors $Z = (z_1, \dots, z_n)$ and $W = (w_1, \dots, w_n)$ we define the natural semi-order in $\mathbb{R}^n$ by $W \succ Z$, i.e., W dominates Z, if and only if $w_i \geq z_i$ for all i $(i = 1, \dots, n)$ and $w_i > z_i$ for at least one i $(i = 1, \dots, n)$, see, e.g., [44]. For a set $A \subseteq \mathbb{R}^n$ we define

$$M(A) = \{\, Z \in A : \text{there exists no} \quad W \in A \quad \text{with} \quad W \succ Z \,\}$$

as the set of *undominated* elements of A. It can be shown that $Z \in M(A)$ if and only if

$$A \cap \{\, W \in \mathbb{R}^n : w_i \geq z_i \quad \text{for all} \quad i = 1, \dots, n \,\} = Z\,.$$

which represents a geometrical interpretation of $M(A)$, see, e.g., [8].

The vector-valued normal form game considered in this section is a game with two players, namely the inspector and the operator, and each of the players has a finite set of pure strategies. The operator´s set of pure strategies consists of all time points, where he can start his illegal activity, i.e., $0, 1, 2, 3$. The inspectors´s set of pure strategies consists of all time points, where he can perform his inspection, i.e., $1, 2, 3$. Given a pure strategy combination (i, j), the operator will receive the payoff vector $Op_3(i, j) \in \mathbb{R}^2$ and the inspector the payoff vector $Insp_3(i, j) \in \mathbb{R}^2$.

The games in section 3.1 are special cases of vector-valued normal form games; the players receive a real number as payoffs instead of a vector.

A mixed strategy of the operator is again – like in section 3.1 page 22 –

$$\mathbf{q} := (q_0, q_1, q_2) \quad \text{with} \quad q_i \geq 0\,, \quad i = 0, \dots, 2 \quad \text{and} \quad \sum_{i=0}^{2} q_i = 1\,,$$

where q_i is the probability, that the operator starts his illegal activity at time point i.

For the inspector we get

$$\mathbf{p} := (p_1, p_1, p_2) \quad \text{with} \quad p_j \geq 0\,, \quad j = 1, 2, 3 \quad \text{and} \quad \sum_{j=1}^{3} p_j = 1\,,$$

where p_j is the probability, that the inspector performs his inspection at time point j.

Having introduced the concept of mixed strategies, the vector-valued payoff function of each player, originally defined on the set of pure strategy combinations $\{0, 1, 2\} \times \{1, 2, 3\}$, can be extended to the set of mixed strategies in the following way:

$$Op_3(\mathbf{q}, \mathbf{p}) = \sum_{i=0}^{2} \sum_{j=1}^{3} q_i \cdot p_j \cdot Op_3(i, j)$$

and

$$Insp_3(\mathbf{q}, \mathbf{p}) = \sum_{i=0}^{2} \sum_{j=1}^{3} q_i \cdot p_j \cdot Insp_3(i, j)\,.$$

$Op_3(\mathbf{q}, \mathbf{p})$ respectively $Insp_3(\mathbf{q}, \mathbf{p})$ is called the expected payoff vector to the operator respectively to the inspector.

Let us now introduce the Pareto equilibrium concept. We consider the mixed extension of a vector-valued normal form game. Then $(\mathbf{q}^*, \mathbf{p}^*)$ is a Pareto equilibrium of the game if and only if

$$Op_3(\mathbf{q}^*, \mathbf{p}^*) \in M(Op_3(\mathbf{q}, \mathbf{p}^*)) \quad \text{and} \quad Insp_3(\mathbf{q}^*, \mathbf{p}^*) \in M(Insp_3(\mathbf{q}^*, \mathbf{p}))\,.$$

This definition says, that any unilateral deviation of one player from the mixed strategy combination $(\mathbf{q}^*, \mathbf{p}^*)$ will not improve his expected payoff vector. In case of real-valued payoff´s to each player, the set of Pareto equilibria is equal to the set of Nash equilibria however, there are important differences between real- and vector-valued normal form games which are discussed in [25] and [26].

The Pareto equilibrium was first introduced by Shapley [42] - he called it strong equilibrium - for two-person vector-valued zero-sum normal form games. Borm et. al. [8] generalized Shapley´s idea considering strong equilibria for two-person vector-valued normal form games. A further generalization to strong equilibria for n-person vector-valued normal form games was made by Krieger in [25] and [26].

It is quite difficult to determine the Pareto equilibria of vector-valued normal form games by using the definition of these equilibria. Therefore, we present a Lemma which shows a very close relation between Pareto equilibria in vector-valued normal form games and Nash equilibria in certain induced real-valued normal form games.

Let us consider the mixed extension of a (in our case: two-component) vector-valued normal form game, and define

$$\Delta = \{\, a = (a_1, a_2)^T \in \mathbb{R}^2 : a_1 \geq 0\,, a_2 \geq 0 \quad \text{and} \quad a_1 + a_2 = 1 \,\}\,.$$

Then the following Lemma holds:

1. Consider arbitrary vectors $\lambda, \gamma \in \Delta$: If the mixed strategy combination $(\mathbf{q}^*, \mathbf{p}^*)$ is a Nash equilibrium of the (real-valued) normal form game with expected payoffs $\lambda^T \, Op_3(\mathbf{q}, \mathbf{p})$ for the operator and $\gamma^T \, Insp_3(\mathbf{q}, \mathbf{p})$ for the inspector, then $(\mathbf{q}^*, \mathbf{p}^*)$ is a Pareto equilibrium of the original game.

2. If the mixed strategy combination $(\mathbf{q}^*, \mathbf{p}^*)$ is a Pareto equilibrium of the original game, then there exist vectors $\lambda, \gamma \in \Delta$ such that $(\mathbf{q}^*, \mathbf{p}^*)$ is a Nash equilibrium of the (real-valued) normal form game with expected payoffs $\lambda^T \, Op_3(\mathbf{q}, \mathbf{p})$ for the operator and $\gamma^T \, Insp_3(\mathbf{q}, \mathbf{p})$ for the inspector.

The proof of this Lemma may be found in [8] or a generalization in [27]. Using this Lemma and Nash´s famous existence theorem, see [32], we get the result, that every vector-valued normal form game has at least one Pareto equilibrium in mixed strategies.

5.2.5 Solution of the vector-valued game

Using the payoff matrix in Figure 5.10 the operator´s payoff vectors are multiplied with the scalarization vector $(\gamma, 1-\gamma)^T$ $(\gamma \in [0,1])$ and the inspectors payoff vectors with the scalarization vector $(\lambda, 1-\lambda)^T$ $(\lambda \in [0,1])$. This leads with the abbreviations

$$a = \frac{1}{2} \cdot [\,(1+\gamma)\cdot d_1 - (1-\gamma)\cdot b_1\,] \quad b = -\frac{1}{2} \cdot [\,(1+\gamma)\cdot c_1 - (1-\gamma)\cdot a_1\,]$$

to the real-valued normal form game given in Figure 5.11.

Figure 5.11 Scalarized form of the vector-valued normal form game given in Figure 5.10.

Op \ Insp	1	2	3
0	$-a_1$ $-b_1$	b a	$-c_1$ d_1
1	$-c_1$ d_1	$-a_1$ $-b_1$	b a
2	b a	b a	$-a_1$ $-b_1$

Because of the many - in total six - parameters of this game it is not so easy to use Canty´s program, see [10], for the determination of its Nash equilibria. It is better instead, to use the method of making the adversary indifferent as regards to his strategies to be chosen. This way we obtain for fixed values of γ and λ the only Nash equilibrium in mixed strategies of this game as follows. The equilibrium strategies of the operator are

$$q_0^*(\lambda) = \frac{1+\lambda}{(1+\lambda)^2+2}\,, \quad q_1^*(\lambda) = \frac{1}{2}\cdot\frac{(1+\lambda)^2}{(1+\lambda)^2+2}\,, \quad q_2^*(\lambda) = \frac{1}{2}\cdot\frac{3+\lambda^2}{(1+\lambda)^2+2}\,,$$

and for the inspector

$$p_1^*(\gamma) = \frac{1+\gamma}{(1+\gamma)^2+2}\,, \quad p_2^*(\gamma) = \frac{1}{2}\cdot\frac{3+\gamma^2}{(1+\gamma)^2+2}\,, \quad p_3^*(\gamma) = \frac{1}{2}\cdot\frac{(1+\gamma)^2}{(1+\gamma)^2+2}\,,$$

and the expected equilibrium payoffs (of the scalarized game)

$$\begin{aligned} Op_3^*(\gamma) &= (d_1+b_1)\cdot\left(f(\gamma) - \frac{1}{1+d_1/b_1}\right) \quad \text{and} \\ Insp_3^*(\lambda) &= -(c_1-a_1)\cdot\left(f(\gamma) - \frac{1}{1-c_1/a_1}\right)\,, \end{aligned}$$

where $f(x)$ defined on $[0,1]$ is given by

$$f(x) = \frac{1+x}{2} - \frac{1}{4}\cdot\frac{(1+x)^2}{(1+x)^2+2}\,.$$

According to the Lemma in the previous section, the complete set of Pareto equilibria is given by the set of these Nash equilibria for all combinations $(\lambda,\gamma)\in[0,1]\times[0,1]$.

As we see the equilibrium strategies of both players depend only on γ respectively λ; more than that, $q_0^*(\lambda)$ and $p_1^*(\gamma)$ depend on λ respectively γ very slightly, see Figure 5.12. For $p_1^*(\gamma)$ we have for example

$$p_1^*(0) = p_1^*(1) = \frac{1}{3}$$

and the only maximum of $p_1^*(\gamma)$ in $[0,1]$ is given at

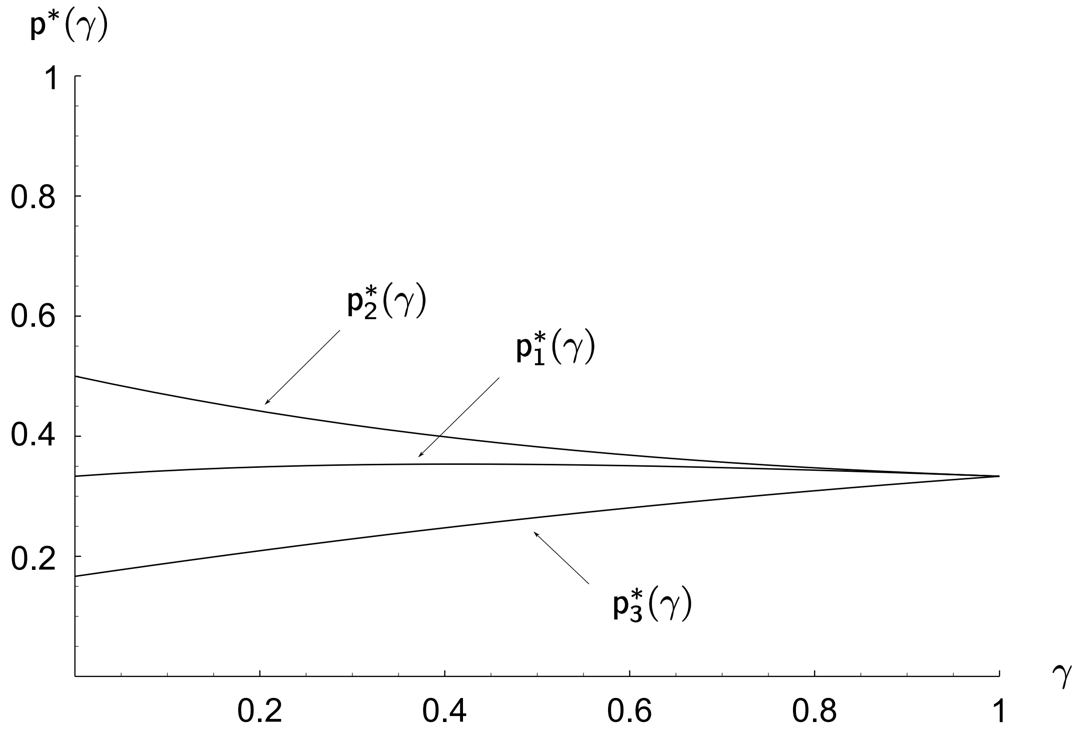
$$\gamma_0 = \sqrt{2}-1 \approx 0.412 \quad \text{and} \quad p_1^*(\gamma_0) = \frac{\sqrt{2}}{4} \approx 0.353\,.$$

The function $f(x)$ determining the equilibrium payoffs is monotonely increasing from $f(0) = 5/12$ to $f(1) = 2/3 = 8/12$. Recall that $\lambda=\gamma=0$ corresponds to the playing for time game and $\lambda=\gamma=1$ to the critical time game.

So far we discussed only the illegal game, i.e., the game in which the operator acts illegally with certainty. We assume that the operator will behave legally if his expected payoff vector in case of legal behavior is larger than in case of illegal behavior in both components. Since the operator´s payoff vector in case of legal behavior is zero in both components, the condition for legal behavior is:

$$Op_3^* < 0 \quad \text{or equivalently} \quad f(\gamma) < \frac{1}{1+b_1/d_1}$$

Figure 5.12 Illustration of $p_j^*(\gamma)$, $j = 1, 2, 3$, as functions of γ.



for all $\gamma \in [0, 1]$. Since $f(x)$ is a monotonely increasing function, this condition can be assured if we postulate

$$f(1) < \frac{1}{1 + b_1/d_1} \quad \text{or explicitly} \quad 2 \cdot d_1 < b_1 \,.$$

Like in the games considered in sections 5.2.1 and 5.2.2 the equilibrium strategy of the inspector is not unique, as the Nash condition for the operator show (for fixed γ):

$$\begin{aligned} 0 &\geq -b_1 \cdot p_1^* + a \cdot p_2^* + d_1 \cdot p_3^* \\ 0 &\geq d_1 \cdot p_1^* - b_1 \cdot p_2^* + a \cdot p_3^* \\ 0 &\geq a \cdot (p_1^* + p_2^*) - b_1 \cdot p_3^* \,, \end{aligned}$$

where a has been defined before.

We see that these conditions for the operator are fulfilled for any $\gamma \in [0, 1]$ if they are fulfilled for $\gamma = 1$. This means that the inspector is on the safe side if he plays the critical time game.

One can understand this simple result if one realizes that, if the payoffs of the critical and the playing for time game are adjusted in the way we did it, the condition for legal behavior in the critical time game implies that condition in the playing for time game.

5.2.6 Concluding Remarks

It is obvious, that the critical time concept is appealing from a theoretical point of view since it describes very well the real intentions of the NPT verification, in particular the aspect of inducing a State to legal behavior. Nevertheless, we preferred the playing for time concept for real applications for two reasons:

(i) As mentioned already before, it is difficult to justify the critical time concept in practice in view of the long delays of a reaction of the international community once an illegal activity of a State has been detected.

(ii) The true value of the critical time concept lies in its evaluation of the inducing to legal behavior principle which in turn requires the definition of payoff parameter. The estimation of numerical values of these parameters, however, has not yet been tackled in practice.

Instead, the safeguards authorities went into a different direction which will be discussed in the next section.

5.3 Global inspection effort

In the central analytical parts of this study, e.g., in the third and fourth chapters, we considered inspections in single on-site interim storage and fuel element fabrication facilities. We assumed that the number of unannounced interim inspections in one facility and one year was either one or two, in one case also three (only for the purpose of illustration), and we considered this number as one of the parameters of the inspection problem.

In this section we discuss in which way this number might be determined in practice. For this reason, we have to consider the total number of unannounced interim inspections in one State and one year. Given that total number, a rule has to be formulated according to which the number of unannounced interim inspections in one facility and one year is determined.

It should be mentioned that this discussion goes beyond the scope of the study, therefore, it will not be carried through here in detail. Instead it is meant to complement the work of the central chapters of this study and also to give an outlook on possible future work concerning unannounced interim inspections in the European Union (EU) as a whole.

In the following we will first present the way in which EURATOM and IAEA have fixed the total number of unannounced interim inspections of the IAEA for on-site interim storage facilities (SFSFs) in each State of the EU and per year. Similar considerations could be performed for fuel element fabrication facilities. Thereafter we will discuss different schemes for allocating the total number of unannounced interim inspections to the SFSFs in a State. This is a non-trivial problem only for Germany since in the other (non-nuclear weapons) States of the EU only one inspection is foreseen. Finally, we present for the purpose of illustration a very simplified game theoretical analysis of the allocation of the global inspection effort within one State of the EU.

Quite generally, i.e., not considering unannounced interim inspections specifically, has the allocation of (IAEA) inspections resources to different States already been investigated some years age, see [1]. The purpose of those analysis was, among others, to analyze the effect of different incentives of States for illegal behavior. Here we assume that these incentives, if at all, are the same for all States of the EU.

5.3.1 Total number of inspections in one State of the EU

According to the IAEA/EURATOM partnership approach the total number of unannounced interim inspections in one facility and one year is specified for each type of nuclear facility. In the IAEA/EURATOM partnership approach under Integrated Safeguards for SFSFs, this is done for this type of facility. Even though it is not quite clear how the procedure presented there has to be implemented, see [4], we propose here a rule which then leads to the numbers given in the IAEA/EURATOM partnership approach.

The rule may be formulated as follows: The number k of unannounced interim inspections in the SFSFs of one State of the EU and one year is the smallest integer which guarantees that each SFSF is inspected with at least 20% probability. Let us mention in passing that no justification for that rule has been given in the IAEA/EURATOM partnership approach. At the end of this section we will indicate what this rule means in decision theoretical terms.

As already stated have all (non-nuclear weapons) States of the EU except for Germany so few SFSFs that there is only one unannounced interim inspection per year. Germany has 16 SFSFs and we will show now that 4 inspections fulfill the rule given above.

In doing so we assume in line with section 2.1.3 – at least in principle, see the remark at the end of this section – that each SFSF can be inspected at most three times. Then the probability that one SFSF is inspected at least once is determined as follows with the help of Laplace's rule:

(i) Number of possible events:

- One SFSF is inspected 3 times: $16 \cdot \binom{15}{1} = 16 \cdot 15$.
- Two SFSFs are inspected twice: $\binom{16}{2} = 16 \cdot 15/2$
- One SFSF is inspected twice: $16 \cdot \binom{15}{2} = 16 \cdot 15 \cdot 14/2$
- Each SFSF is inspected at most once: $\binom{16}{4}$.

(ii) Number of favorable events (for the inspection of one specific SFSF):

- The SFSF is inspected 3 times: 15.
- The SFSF is inspected twice: $15 + \binom{15}{2}$
- The SFSF is inspected once: $15 + 15 \cdot 14 + \binom{15}{3}$.

Thus the probability that one (specific) SFSF will be inspected is

$$\begin{aligned}
&\mathbf{P}(\{\text{ a specific SFSF is inspected }\}) \\
&\qquad = \mathbf{P}(\{\text{ the specific SFSF is inspected exactly 3 times }\}) \\
&\qquad + \mathbf{P}(\{\text{ the specific SFSF is inspected twice }\}) \\
&\qquad + \mathbf{P}(\{\text{ the specific SFSF is inspected once }\}) \\
&\qquad = \frac{\binom{15}{1} + \left(\binom{15}{1} + \binom{15}{2}\right) + \left(\binom{15}{1} + 2\cdot\binom{15}{2} + \binom{15}{3}\right)}{16\cdot\binom{15}{1} + \left(\binom{16}{2} + 16\cdot\binom{15}{2}\right) + \binom{16}{4}} = \frac{163}{772} \approx 0.21\,,
\end{aligned}$$

i.e., slightly above 20%.

Let us finally determine the probability that 1 out of 16 SFSFs is inspected exactly 3 times:

$$\begin{aligned}
&\mathbf{P}(\{\text{ 1 SFSF out of 16 SFSFs is inspected exactly three times }\}) \\
&\qquad = \frac{16\cdot 15}{16\cdot\binom{15}{1} + \left(\binom{16}{2} + 16\cdot\binom{15}{2}\right) + \binom{16}{4}} = \frac{12}{193} \approx 0.062\,.
\end{aligned}$$

In the third and fourth chapter we did not consider explicitly this possibility (with a few exception in the subsections on page 29, 53 and 62); from a common sense point of view it appears not reasonable, anyhow, to inspect 1 out of 16 facilities three times, if only four inspections can be performed in total. Therefore, we made the additional assumption that at most two unannounced interim inspections are performed in one facility and the reference time interval.

5.3.2 Allocation of inspections to facilities

Once the total number of unannounced interim inspections for one category of facilities in one State is fixed, the question arise how these inspections have to be distributed on the single facilities (if there is more than one) and on the possible time points.

There are at least three different procedures which one might imagine. We will take the example of Germany.

First, what one might call a purely statistical approach: There are $16 \cdot 3$=48 possibilities for inspections. Thus one has to allocate randomly the four inspections to these 48 possibilities. Of course, this procedure ignores any kind of incentives of the operator resp. the State.

Second, one may randomly allocate the four inspections to the 16 facilities such that at most two inspections can be performed in one facility. Thereafter, one may proceed as outlined in chapter 3 and 4 of this study. This procedure ignores the fact that the State is party to the NPT and not the operator of the single facilities.

Third, one allocates the inspection to the different facilities and time points with the help of a strategic analysis. Assuming that in one facility at most two inspections can be performed, the inspection authority has

- $\binom{16}{2}$ possibilities to perform two inspections in two facilities,
- $16 \cdot \binom{15}{2} = 3 \cdot \binom{16}{3}$ possibilities to perform two inspections in one and one in inspection in two facilities,
- $\binom{14}{4}$ possibilities to perform one inspection in four facilities.

Since, furthermore, there are three possibilities to distribute both one or two inspections on three time points, we have in the first case 3^2, in the second 3^3 and in the third case 3^4 possibilities. Thus, in total we have

$$3^2 \cdot \binom{16}{2} + 3^3 \cdot 3 \cdot \binom{16}{3} + 3^4 \cdot \binom{16}{4} = 193860$$

pure strategies of the inspection authority. Furthermore, if only one illegal activity is planned in all facilities (which is a worst case assumption from the inspection authority's point of view) the State has

$$16 \cdot 3 = 48$$

pure strategies. Such a huge game is hardly mathematically feasible with standard techniques. Therefore, one has to look for approaches which are feasible.

In order to illustrate the third approach, however, which from a theoretical point of view represents a consequent generalization of the work performed in this study, we will discuss a simplified problem.

5.3.3 Strategic analysis

Let us consider the hypothetical case that there are two interim storage facilities in one State, and that in total two unannounced interim inspections will be performed at the possible three different time points in each facility. As mentioned before, in line with the fact that the State is the party to the NPT, we assume that only one illegal activity will take place. Also, we model this problem like in the third chapter of this study: Both players choose their strategies simultaneously, and the payoff to the the State is the detection which in turns is the negative payoff to the inspector. For the sake of simplicity, we ignore errors of the second kind, i.e., we take $\beta = 0$.

The normal form of this game is represented in Figure 5.13. A subset of the set of all optimal strategies of both players, together with the optimal expected detection time 2, is given in Figure 5.14. Let us look at some interesting properties of these optimal strategies

- No pure strategy, which contains the inspection at time point 3, is ever played in a saddle point determined by Canty's program, see [10].
- The two pure strategies $(1,2)_1$ and $(1,2)_2$, according to which in each facility the time points 1 and 2 are selected for inspection, are always played with positive probability. In fact, one optimal strategy of the inspector is to use only these two strategies.
- The optimal inspection strategies need not be symmetrical with respect to both States: consider the third one in Figure 5.14: Here the expected number of inspections in the first facility is

$$2 \cdot \frac{5}{18} + 1 \cdot \frac{2}{9} + 1 \cdot \frac{1}{6} = \frac{17}{18} < 1\,,$$

  whereas that of the second facility, of course, is greater one.
- The State starts his illegal activity either in the first or in the second facility at time point 2, or uses any mixture of these two pure strategies in an optimal strategy.

Taking the first and third property together, one understands that the State takes time point 2 for the start of his illegal activity since no inspection takes place at time point 3. Vice versa it is not so clear why time point 3 is not used for inspection, perhaps due to the fact that nine pure strategies contain this choice and therefore, the State would be tempted to start the illegal activity earlier if this time point would be taken into account.

It remains to be investigated if these properties of the equilibria hold also for more complicated games which are closer to the reality in Germany, or if, on the contrary with each new model new properties surface, as it is so frequently the case with game theoretical analyses.

One final remark shall be made as regards to the determination of the necessary number of inspections in one State. In the case discussed here, the two inspections lead to the optimal expected detection time 2. Introducing the payoff parameters b and d like in section 5.1, the equilibrium payoff to the State in case of illegal activity is $2 \cdot d - b$.Taking again payoff zero for legal behavior we see that the State will behave legally if $b/d > 2$. Thus, if we would have solved the game for Germany (16 SFSFs and 4 inspections), we would be able to show which relation between b and d has been assumed for Germany.

Finally, from a decision theoretical point of view it would be more satisfying, not to determine the total number of inspections for one type of facility in a State of the EU with the help of the probabilistic rule described in section 5.3.1, but to fix the same ratio b/d for all States of the EU and derive this way the necessary number of inspections in the way described above.

Figure 5.13 The matrix game explained in detail section 5.3.3.

	$(1,2)_1$	$(1,3)_1$	$(2,3)_1$	$(1,2)_2$	$(1,3)_2$	$(2,3)_2$	$1_1,1_2$	$2_1,1_2$	$3_1,1_2$	$1_1,2_1$	$2_1,2_2$	$3_1,2_2$	$1_1,3_2$	$2_1,3_2$	$3_1,3_2$
0_1	1	1	2	4	4	4	1	2	3	1	2	3	1	2	3
1_1	1	2	1	3	3	3	3	1	2	3	1	2	3	1	2
2_1	2	1	1	2	2	2	2	2	1	2	2	1	2	2	1
0_2	4	4	4	1	1	2	1	1	1	2	2	2	3	3	3
1_2	3	3	3	1	2	1	3	3	3	1	1	1	2	2	2
2_2	2	2	2	2	1	1	2	2	2	2	2	2	1	1	1

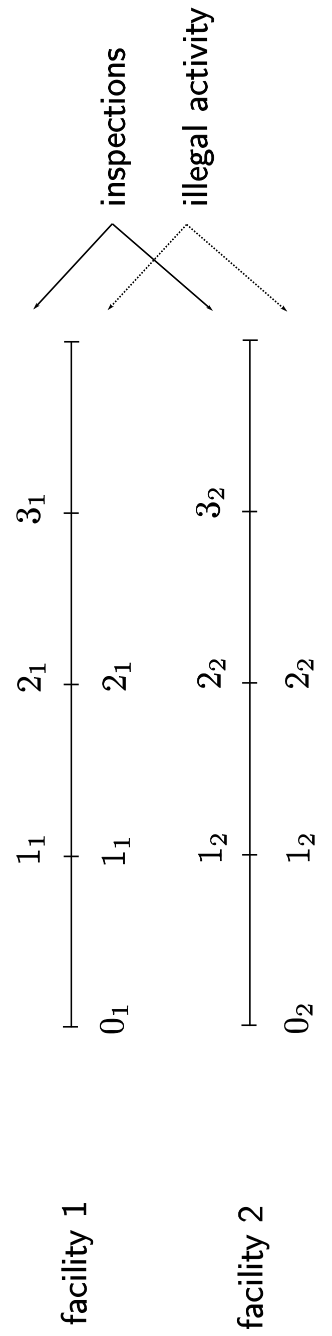


Figure 5.14 Optimal strategies of the matrix game given in Figure 5.13.

$p^*_{(1,2)_1}$	$p^*_{(1,3)_1}$	$p^*_{(2,3)_1}$	$p^*_{(1,2)_2}$	$p^*_{(1,3)_2}$	$p^*_{(2,3)_2}$	$p^*_{1_1,1_2}$	$p^*_{2_1,1_2}$	$p^*_{3_1,1_2}$	$p^*_{1_1,2_2}$	$p^*_{2_1,2_2}$	$p^*_{3_1,2_2}$	$p^*_{1_1,3_2}$	$p^*_{2_1,3_2}$	$p^*_{3_1,3_2}$
$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0
$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	$\frac{1}{4}$	0	0	0	0
$\frac{5}{18}$	0	0	$\frac{1}{3}$	0	0	$\frac{2}{9}$	0	0	$\frac{1}{6}$	0	0	0	0	0
$\frac{1}{3}$	0	0	$\frac{5}{18}$	0	0	$\frac{2}{9}$	$\frac{1}{6}$	0	0	0	0	0	0	0
$\frac{1}{3}$	0	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0	0	0	0	0	0	0	0
$\frac{1}{3}$	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{6}$	0	0	0	0	0	0	0
$\frac{1}{3}$	0	0	$\frac{1}{2}$	0	0	$\frac{1}{6}$	0	0	0	0	0	0	0	0
$\frac{1}{2}$	0	0	$\frac{1}{3}$	0	0	0	0	0	$\frac{1}{6}$	0	0	0	0	0
$\frac{1}{2}$	0	0	$\frac{1}{3}$	0	0	$\frac{1}{6}$	0	0	0	0	0	0	0	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0	0	0	0

$q^*_{0_1}$	$q^*_{1_1}$	$q^*_{1_2}$	$q^*_{0_2}$	$q^*_{1_2}$	$q^*_{2_2}$
0	0	0	0	0	1
0	0	1	0	0	0

Chapter 6

Summary and Recommendations

Having analyzed two models – simultaneous and hybrid-sequential – in two versions – time discrete and time continuous – for the planning of unannounced interim inspections in nuclear facilities in States of the European Union (EU), and having applied the results to on-site interim storage facilities and fuel element fabrication facilities, two question arise. First, what kind of conclusion can be drawn from these results? And second, which recommendations can be given to practitioners, who are expected to be users of these results?

From the theorists' point of view one might respond to the second question that practitioners have to choose those models and versions, whose underlying assumptions fit best to their views of the inspection problem, and then appropriate recommendations can be given. Of course, reality is not that simple and therefore, in this chapter conclusions from the results obtained in previous chapters are formulated which then lead to some recommendations which have to be weighted with arguments additional to those taken into account in this study.

In the following an overview of the models considered in this study is given. Thereafter, some conclusions are drawn, in particular concerning the comparison of the simultaneous and hybrid-sequential models, since comparisons of the time discrete and continuous versions have already been made at the ends of the third and fourth chapters. Finally, with all due care some recommendations are formulated.

6.1 Overview of models and versions

In Table 6.1 an overview of the two models and their two versions analyzed in this study is given, together with the sections in which they are analyzed and where the results are applied to the two types of nuclear facilities.

Table 6.1 Overview of inspection models and versions considered in this study and sections in which they are analyzed.

time (version) → / operator's behavior (model) ↓	discrete	continuous
simultaneous	3.1.1 (Storage), 3.1.2 (Fuel fabrication)	3.2 (Both)
hybrid-sequential	4.1.1 (Storage), 4.1.2 (Fuel fabrication)	4.2 (Both)

The arguments for the selection of these four models and versions have already been given in previous chapters, in particular in the first one. They will not be repeated here; instead it is assumed for the sake of practicability that they are the only practically relevant ones, and that the problem is now to draw conclusions from their results and to formulate recommendations, as stated above.

Just for the sake of clarity, let us repeat the basic assumptions underlying the four variants, i.e., models and versions.

- Discrete time models are characterized by the assumption that IAEA inspectors can perform their unannounced interim inspections only together with EURATOM inspections which take place every three months in on-site interim storage facilities and every two months in fuel element fabrication facilities.
- Continuous time models, on the contrary, permit IAEA inspections at any time point of the reference time interval.
- Simultaneous models are characterized by the assumption that both "players" plan their activities in advance, before the beginning of the reference time interval. This means that they choose their time points for inspection and start of the illegal activity, eventually with the help of a random experiment using a random number generator, in advance.
- Hybrid-sequential models permit the facility operator to only decide before the beginning of the reference time interval whether or not to start immediately an illegal activity or to delay it. The same after the first inspection and so on.

Of course there are many more assumptions which are listed in the third and fourth chapter. They are, however, less important for the discussion to follow.

6.2 Overview of solutions and conclusions

In Figure 6.1 the optimal expected detection times for the models and versions considered in this study and applied to the on-site interim storages for spent nuclear fuel elements are given, and in Figure 6.2 they are given as applied to the fuel element fabrication facility. In the following we discuss only the case of the on-site interim storage facility since the conclusions to be drawn for the fuel element fabrication facility are more or less the same.

We do not present here again the graphical representations of the optimal expected detection times as function of the non-detection probability β. Putting for each type of facility the eight functions into one figure would not improve the understanding of these results, as we think.

Before comparing the contents of the different formulae presented in Figure 6.1 let us remember the different mathematical methods for obtaining the results: Whereas in the cases of the on-site interim storage facility standard techniques for the solution of normal and extensive form games could be used by hand, in case of the fuel element fabrication facility computer programs had to be applied. In the time continuous version advanced analytical techniques had to be used; for the case of two unannounced interim inspections original research had to be performed.

Even more generally formulated, one realizes that any change of assumptions leads to different models and mathematical problems, and results from the solution of one problem cannot be extrapolated to those of the other ones.

On the other hand once the assumptions are made one obtains solutions which are very interesting and cannot be gained with common sense considerations. For example, in the continuous time models, where pure optimal strategies exists, game theory offers a unique possibility to link together the non-detection probability β with the optimal time point(s) of inspection(s).

In section 3.3 and 4.3 we have already compared the results of the time discrete and continuous versions of the simultaneous and hybrid-sequential models separately. Let us shortly repeat the main findings.

For the simultaneous model all optimal expected detection times as function of the non-detection probability β, i.e., the graphical representations of the analytical expressions given in the two upper boxes of Figure 6.1, are given in Figure 3.11. Both for $k = 1$ and $k = 2$ inspections in the continuous time model result in shorter expected detection times for all values of β. However, for $k = 2$ the discrete time model gives a shorter optimal expected detection time than the continuous time model with $k = 1$. Also, as it was demonstrated in Figure 3.8, $k = 1$ inspection with a smaller value of β may lead to the same optimal expected detection time as $k = 2$ inspections with a larger value of β.

Correspondingly, for the hybrid-sequential model all optimal expected detection times as function of β, i.e., the graphical representation of the analytical expressions given in the two lower boxes of Figure 6.1 are given in Figure 4.11. Their properties are similar to those described above, however, for $k = 1$ inspection both the time discrete and continuous models lead to the same optimal expected detection time.

Let us now turn to the comparison of the simultaneous and the hybrid-sequential models for the discrete time version – the two left hand boxes in Figure 6.1 – and for the continuous time version – the two right hand boxes in Figure 6.1.

In all cases we get for $\beta = 1$ the same optimal expected detection time four. This is obvious since in that case the detection of an illegal activity is only possible at the end of the reference time interval and consequently, the operator starts his illegal activity right at the beginning of that interval. Thus, we consider in the following only small values of β.

For the discrete time version, both models lead to very similar results. For $k = 1$ and $\beta = 0$ we get for the optimal expected detection time $11/6$ for the simultaneous, and 2 for the hybrid-sequential model. For $k = 2$ and $\beta = 0$ we get $4/3$ for both models.

For the continuous time version the situation is a little different. For $k = 1$ and $\beta = 0$ we get $4/e \approx 1.47$ for the simultaneous, and 2 for the hybrid-sequential model, that is, relatively large differences. For $k = 2$ we get $4/(e \cdot (e-1)) \approx 0.85$ for the simultaneous, and $4/3 \approx 1.33$ for the hybrid-sequential model.

Let us put these results for $\beta = 0$ together: For $k = 1$ inspection both versions of the hybrid-sequential model lead to an optimal expected detection time two, which at the same time is the longest among the four models and versions. The shortest optimal expected detection can be found in the continuous time version of the simultaneous model with $4/e \approx 1.47$. For $k = 2$ the situation is similar: Here, interesting enough, three of the four models and versions lead to $4/3 \approx 1.33$, the shortest being again the continuous time version of the simultaneous model with $4/(e \cdot (e-1)) \approx 0.85$.

In sum, the hybrid-sequential model leads to longer optimal expected detection times than the simultaneous one, both for the discrete and the continuous time version. This is intuitive, since the operator is able to make use of the information he gains in the course of the game. Of course, $k = 2$ leads to shorter expected detection times than $k = 1$ for all models and versions and given values of the non-detection probability β.

6.3 Recommendations

Let us finally turn to the question of what can be recommended to the two inspection authorities after having analyzed the two models and versions. In the following we consider again only the on-site interim storage facility since similar arguments hold for the fuel element fabrication facility.

The scope of our study was the planning of unannounced interim inspections in one single facility. Thus, let us consider first those States of the EU which have only one on-site interim storage facility. For those States, $k = 1$ unannounced interim inspection is foreseen, and it has to be decided with the help of which model and version this inspection has to be planned.

If the inspection authorities cannot agree on which model and version should be used and therefore, want to be on the safe side, the recommendation is simple. The time continuous version of the hybrid-sequential model leads to the longest optimal expected detection time therefore, this variant should be used. There are two additional advantages, namely that there is a complete freedom of choice in time, and the inspectorate can use a pure strategy, i.e., it can plan and announce it. A disadvantage of this procedure may be that the IAEA inspection may take place at different time points than the EURATOM inspection thus, an additional burden may be posed on the plant operator.

At first sight, it may be surprising that at the end of a detailed mathematical study on *unannounced* interim inspections it is a recommendation to *announce* the inspections. It should be kept in mind, however, that it is also possible not to announce the inspections, but rather plan them with the help of a random experiment using a random number generator as long as it leads to the same optimal expected detection time. Nevertheless, the fact remains that it is not really intuitive that the announcement works as well.

For those States of the EU which have so few on-site interim storage facilities that for them again just one unannounced interim inspection is foreseen, this procedure can also be applied: Since they are all considered to be equal, the inspectorate chooses one of them at random (each with the same probability) and then proceeds as described above.

For Germany the situation is more complicated, and its analysis goes beyond the scope of this study, since four inspections have to be distributed on 16 facilities. It has been sketched in section 5.3 that in principle this problem leads to a rather complicated mathematical model, independently of the variant to be used. Therefore, at this point no recommendation can be formulated except for some pragmatic approach like one of those mentioned in section 5.3, the effectiveness of which still has to be demonstrated.

Figure 6.1 Optimal expected detection times for the variants considered in this study as applied to the on-site interim storage facility.

operator's behavior \ time	discrete	continuous
simultaneous	$k=1: \begin{cases} \frac{11}{6}+\beta & \text{for} \quad 0 \le \beta < \frac{1}{6} \\ \frac{10}{6}+2\cdot\beta & \text{for} \quad \frac{1}{6} \le \beta < \frac{2}{3} \\ \frac{6}{6}+3\cdot\beta & \text{for} \quad \frac{2}{3} \le \beta \le 1 \end{cases}$ $k=2: \begin{cases} \frac{4+6\cdot\beta+5\cdot\beta^2+2\cdot\beta^3}{3+2\cdot\beta+\beta^2} & \text{for} \quad 0 \le \beta < \frac{1}{2} \\ 1+\beta+2\cdot\beta^2 & \text{for} \quad \frac{1}{2} \le \beta \le 1 \end{cases}$	$k=1: \quad \frac{4}{e^{1-\beta}}$ $k=2: \quad \frac{4}{e^{2\cdot(1-\beta)}-(1-\beta)^2\cdot e^{(1-\beta)}}$
hybrid-sequential	$k=1: \quad \frac{4}{2-\beta}$ $k=2: \begin{cases} \frac{4-\beta+\beta^2}{3-3\cdot\beta+2\cdot\beta^2-\beta^3} & \text{for} \quad 0 \le \beta < \frac{1}{2} \\ 1+\beta+2\cdot\beta^2 & \text{for} \quad \frac{1}{2} \le \beta \le 1 \end{cases}$	$k=1: \quad \frac{4}{2-\beta}$ $k=2: \quad \frac{4}{3-2\cdot\beta}$

Figure 6.2 Optimal expected detection times for the variants considered in this study as applied to the fuel element fabrication facility.

time / operator's behavior	discrete	continuous
simultaneous	$k=1:\ \begin{cases} \frac{70}{30}+2\cdot\beta & \text{for} \quad 0\le\beta<\frac{13}{60} \\ \frac{47}{20}+3\cdot\beta & \text{for} \quad \frac{13}{60}\le\beta<\frac{11}{20} \\ \frac{9}{5}+4\cdot\beta & \text{for} \quad \frac{11}{20}\le\beta<\frac{4}{5} \\ 1+5\cdot\beta & \text{for} \quad \frac{4}{5}\le\beta\le 1 \end{cases}$ $k=2:\ \begin{cases} \frac{59+133\cdot\beta+128\cdot\beta^2+62\cdot\beta^3+12\cdot\beta^4}{34+48\cdot\beta+30\cdot\beta^2+8\cdot\beta^3} & \text{for} \quad 0\le\beta<0.172965 \\ \frac{26+48\cdot\beta+23\cdot\beta^2+2\cdot\beta^3-14\cdot\beta^4-12\cdot\beta^5}{16+10\cdot\beta-5\cdot\beta^2-4\cdot\beta^3-2\cdot\beta^4} & \text{for} \quad 0.179265\le\beta<0.249989 \\ \frac{23+77\cdot\beta+102\cdot\beta^2+72\cdot\beta^3}{15+27\cdot\beta+18\cdot\beta^2} & \text{for} \quad 0.249989\le\beta<0.529234 \\ \frac{10+28\cdot\beta+33\cdot\beta^2+36\cdot\beta^3}{7+10\cdot\beta+2\cdot\beta^2} & \text{for} \quad 0.529234\le\beta<0.75 \\ 1+\beta+4\cdot\beta^2 & \text{for} \quad 0.75\le\beta\le 1 \end{cases}$	$k=1:\ \frac{6}{e^{1-\beta}}$ $k=2:\ \frac{6}{e^{2\cdot(1-\beta)}-(1-\beta)^2\cdot e^{(1-\beta)}}$
hybrid-sequential	$k=1:\ \frac{6}{2-\beta}$ $k=2:\ \begin{cases} \frac{6}{3-2\cdot\beta} & \text{for} \quad 0\le\beta<2/3 \\ \frac{8-5\cdot\beta+3\cdot\beta^2}{5-9\cdot\beta+8\cdot\beta^2-3\cdot\beta^3} & \text{for} \quad 2/3\le\beta<3/4 \\ 1+\beta+4\cdot\beta^2 & \text{for} \quad 3/4\le\beta\le 1 \end{cases}$	$k=1:\ \frac{6}{2-\beta}$ $k=2:\ \frac{6}{3-2\cdot\beta}$

Chapter 7

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Chapter 8

Bibliography

[1] R. Avenhaus and M. J. Canty. *Compliance Quantified - An Introduction to Data Verification*. Cambridge University Press, Cambridge, UK, 1996.

[2] R. Avenhaus and M. J. Canty. Playing for time: A sequential inspection game. *European Journal of Operational Research*, 167(2):475–492, December 2005.

[3] R. Avenhaus, M. J. Canty, and K. Sohrweide. Extension of a Model for Timely Inspection. In *Proceedings of the ESARDA 25th Annual Meeting, Symposium on Safeguards and Nuclear Material Management*, Stockholm, 13-15 May, 2003. JRC Ispra, Italy.

[4] R. Avenhaus and Th. Krieger. Comments on the IAEA/EURATOM Partnership Approach under Integrated Safeguards for Spent Fuel Storage Facilities (SFSFs). Technical report, ITIS e.V. an der Universität der Bundeswehr München, 29. 10. 2008.

[5] R. Avenhaus and Th. Krieger. A Critical Evaluation of the Critical Time Concept in Nuclear Material Safeguards. Aix en Provence, 22-24 May, 2007. JRC Ispra, Italy.

[6] R. Avenhaus, B. von Stengel, and S. Zamir. Inspection games. In R. J. Aumann and S. Hart, editors, *Handbook of Game Theory with Economic Applications*, volume 3 of *Handbook of Game Theory with Economic Applications*, chapter 51, pages 1947–1987. Elsevier, 2002.

[7] C. P. Behrens, H. H. Remagen, A. Rezniczek, B. Richter, K. Rudolf, G. Stein, and M. Weis. Safeguards for spent nuclear fuel in transfer from wet storage to dry storage on on-site interim storage facilities. In *Proceedings of an International Safeguards Symposium*, Paper No. IAEA-CN-148/114, IAEA, Vienna, October 2006.

[8] P. Borm, S. Tijs, and J van den Aarssen. Pareto Equilibria in Multiobjective Games. *Mathematical Methods of Operations Research*, 60:302–312, 1988.

[9] M. Braun. *Differential Equations and Their Applications*. Springer-Verlag, New York, 1975.

[10] M. J. Canty. *Resolving Conflicts with Mathematica® - Algorithms for Two-Person Games*. Academic Press, Amsterdam, 2003.

[11] M. J. Canty, D. Rothenstein, and R. Avenhaus. Timely Inspection and Deterence. *European Journal of Operational Research*, 131(1):208–223, 2001.

[12] M. Carter and B. van Brunt. *The Lebesque-Stieltjes Integral*. Springer-Verlag, New York, 2000.

[13] H. Diamond. Minimax Policies for Unobservable Inspections. *Mathematics of Operations Research*, 7(1):139–153, 1982.

[14] M. Dresher. A sampling inspection problem in arms control agreements: a game theoretical analysis. *Memorandum No. RM-2872-ARPA (RAND Corporation, Santa Monica)*, 1962.

[15] EURATOM. Commision Regulation (EURATOM) No. 3227/76. *Official Journal of the European Communities, Brussels*, 19(L363), 1976.

[16] EURATOM. Commission Regulation (EURATOM) No. 302/2005. *Official Journal of the European Union*, (L54), 2005.

[17] U. Gocke, W. Braun, and E. Weber. Nutzung des Brennstoffs aus der Wiederaufarbeitung etabliert: Über 30 Jahre MOX- und ERU-Brennelemente für Leichtwasserreaktoren. *International Journal for Nuclear Power (ATW)*, (51):686–694, November 2006.

[18] H. Grümm. Safeguards Verification - Its Credibility and the Diversion Hypothesis. *IAEA Bulletin*, 25(4):27–28, 1983.

[19] S. Hart. Games in extensive and strategic forms. In R. J. Aumann and S. Hart, editors, *Handbook of Game Theory with Economic Applications*, volume 1 of *Handbook of Game Theory with Economic Applications*, chapter 2, pages 19–40. Elsevier, 1992.

[20] http://www.umwelt.niedersachsen.de/master/C11676795_N11613212_L20_D0_I598.html, July 18, 2008.

[21] IAEA. The Structure and Content of Agreements Between the Agency and States Required in Connection with the Treaty on the Non-Proliferation of Nuclear Weapons. INFCIRC/153 (corrected). Technical report, IAEA, Vienna, 1972.

[22] IAEA. Model Protocol Additional to the Agreement(s) Between State(s) and the International Atomic Energy Agency for the Application of Safeguards. INFCIRC/540. Technical report, IAEA, Vienna, 1997.

[23] IAEA. IAEA Safeguards Glossary, 2001 Edition. Technical report, IAEA International Nuclear Verification Series No. 3, Vienna, 2002.

[24] D. M. Kilgour. Site Selection for On-Site Inspection in Arms Control. *Arms Control*, 13(3):439–462, 1992.

[25] Th. Krieger. Nicht-kooperative extensive n-Personenspiele mit vektorwertigen Auszahlungen (Non-cooperative extensive form games with vector-valued payoffs), PhD-Dissertation, Universität der Bundeswehr München, 2003.

[26] Th. Krieger. On Pareto equilibria in vector-valued extensive form games. *Mathematical Methods of Operations Research*, 58:449–458, 2003.

[27] Th. Krieger. On the asymptotic behavior of a discrete time inspection game: proofs and illustrations. *Technischer Bericht, Fakultät für Informatik*, 5, 2007.

[28] E. Leitner, R. Weh, R. Avenhaus, and M. J. Canty. Sequential Test Procedures for Near-Real-Time Accountancy. *Journal of the Instiute of Nuclear Materials Management*, XV(2):29–37, 1987.

[29] T. Lundqvist. Safeguards Aspects of Recycling Plutonium into MOX Fuel in Light Water Reactors. In *Proceedings of the ESARDA 29th Annual Meeting, Symposium on Safeguards and Nuclear Material Management*, Aix en Provence, 22-24 May 2007. JRC Ispra, Italy.

[30] P. Morris. *Introduction to Game Theory*. Springer Verlag, Berlin, 1994.

[31] R. B. Myerson. *Game Theory - Analysis of Conflict*. Harvard University Press, Boston, MA, 1999.

[32] J. F. Nash. Non-Cooperative Games. *Ann. of Math.*, 54(2):286 – 295, 1951.

[33] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1947.

[34] G. Owen. *Game Theory*. Academic Press, New York, 2nd edition, 1988.

[35] Personal communication from A. Reznizcek, UBA Unternehmensberatung GmbH.

[36] Personal communication from B. Bendick, H. Wessling and W. Kösters, ANF AREVA Lingen, Germany.

[37] Personal communication from K. Rudolf and Dr. A. Jussofie, GNS (Gesellschaft Nuklear-Service mbH).

[38] Personal communication from O. Alique, EURATOM DG TREN, Luxembourg.

[39] A. Rezniczek, B. Richter, and A Jussofie. Assessment of the Performance of Containment and Surveillance Equipment Part II: Trial Application. *ESARDA Bulletin*, 41:51–61, 2009.

[40] V. K. Rohatgi. *An Introduction to Probability Theory and Mathematical Statistics*. John Wiley & Sons, New York, 1976.

[41] K. Rudolf, C. P. Behrens, and B. Richter. Safeguards Techniques at Storage Facilities - Operator´s Thoughts and Experiences. In *Proceedings of the ESARDA 29th Annual Meeting, Symposium on Safeguards and Nuclear Material Management*, Aix en Provence, 22-24 May 2007. JRC Ispra, Italy.

[42] L. S. Shapley. Equilibrium Points in Games with Vector Payoffs. *Naval Research Logistics Quarterly*, 1:57–61, 1959.

[43] www.bfs.de/www/extfs/transport/gv/dezentrale_zl/standort/kle_szl_kb.pdf, June 16, 2008.

[44] P. L. Yu. *Multiple-Criteria Decision Making*. Plenum Press, New York, 1985.

Appendices

Appendix A

Notation of the (optimal) expected detection time Op and Op^*

According to the assumptions, see Figure 1.1, a large number of game theoretical models has been considered; all of them leading to an optimal expected detection time Op^*. Figure 6.1 presents them for the on-site interim storage facility; the same number exists for the fuel element fabrication facility – even though for the time continuous version the factor 4 just has to be replaced by 6. Thus, we have in total 16 different optimal expected detection times.

It is reasonable mathematical practice not to use the same symbol for different analytical forms therefore we have to use different symbols. The easiest way would be to number them from 1 to 16, however, this way one would lose any intuition. What would mean for example Op^*_{13}?

In order to maintain some kind of intuition and simplicity at the same time, we proceed as follows:

- We do not differentiate between the (optimal) expected detection times for on-site interim storage facilities and for fuel element fabrication facilities, since we never compare them. So we use the same symbols for them, even though the analytical forms are different, of course.

- Also we do not differentiate between the (optimal) expected detection times for the simultaneous and for the hybrid-sequential models: They are treated in different chapters, and when they are put together and compared – in the sixth chapter – we can easily avoid to use the same symbols for different analytical forms.

- Thus, the possible time points for inspections n and the number of unannounced interim inspections k remain to be indicated. For the time discrete version we use a pair of indices, $Op_{n,k}$ and $Op^*_{n,k}$, and for the time continuous version just one index, Op_k and Op^*_k. This way, as already mentioned, we use different forms for the time discrete versions for the on-site interim storage facilities and for the fuel element fabrication facilities: In the former ones, we have $n = 3$, whereas in the latter ones $n = 5$.

In addition, we describe functions in the usual form, namely, e.g., $Op^*(\beta)$ or $Op^*(\beta; \mathbf{q}^*, \mathbf{p}^*)$ means the optimal expected detection time as a function of the non-detection probability β.

Appendix B

Zero-sum matrix games

In this appendix we give a short introduction into zero-sum matrix games. There is a large amount of literature on this subject which differs in length, mathematical rigor and applications. We present the most important concepts along the lines of [30].

B.1 Pure strategies and the payoff matrix

Let Φ_{Op} and Φ_{Insp} be the finite sets of pure strategies of the operator and the inspector. If the combination $(i, j) \in \Phi_{Op} \times \Phi_{Insp}$ of pure strategies is played, then the operator receives the payoff $Op(i, j)$. In the games of section 3.1 this payoff is the elapsed time between start and detection of the illegal activity, when the operator starts his illegal activity at time point i and the inspector performs his inspection(s) at time point(s) j.

Since the operator wants to maximize this detection time and the inspector wants to minimize this time, the payoff to the inspector is the negative of the payoff to the operator, i.e., $Insp(i, j) = -Op(i, j)$ for each pair $(i, j) \in \Phi_{Op} \times \Phi_{Insp}$. These games are called zero-sum games. The game is played as follows: Each player chooses an $i \in \Phi_{Op}$ and $j \in \Phi_{Insp}$. These choices are made simultaneously and independently of the other player. Then the operator receives $Op(i, j)$ and the inspector $-Op(i, j)$.

Since we have only two players in the games discussed in this study, the conflict situation can be described with the help of a payoff matrix A. The two players are referred to as *row player* and the *column player*, respectively. The row player has $|\Phi_{Op}|$ strategies[1] which are identified with the rows of A. The column player has $|\Phi_{Insp}|$ strategies which are identified with the columns of A. If the row player plays strategy i and the column players plays strategy j, then the payoff to the row player is $Op(i, j)$ and the payoff to the column player is $-Op(i, j)$. It is important to make it clear form the beginning that larger numbers in A are favored by the row player and smaller ones by the column player. Thus, a negative entry is a loss to the row player but a gain (of the absolute value) to the column player.

[1]For a finite set Φ the number of elements is denoted $|\Phi|$.

B.2 Mixed strategies, expected payoffs and the saddle point condition

We first want to explain the idea of the saddle point concept - the solution concept of zero-sum games - using the above introduced pure strategies: A combination (i^*, j^*) of pure strategies is a *saddle point* of the game, if any unilateral deviation does not improve the deviator's payoff, i.e., mathematically speaking

$$Op(i, j^*) \leq Op(i^*, j^*) \leq Op(i^*, j) \quad \text{for all} \quad i \in \Phi_{Op} \quad \text{and} \quad j \in \Phi_{Insp}\,.$$

The left hand inequality reflects the idea that the operator wants to maximize the detection time, while the right hand inequality shows the minimization of the detection time by the inspector. The pair (i^*, j^*) fulfilling the inequalities above is called saddle point (in pure strategies). It should be noted that the saddle point concept was introduced by *von Neumann and Morgenstern*, see [33], is a special case of the famous and widely used concept of the *Nash equilibrium*, see [32].

Let us consider for the purpose of illustration the zero-sum matrix games in Figure B.1.

Figure B.1 Two zero-sum matrix games with different payoff matrices.

	1	2	3
0	2	1	1
1	-1	0	-1
2	3	1	1
3	0	0	0

	1	2	3
0	1	2	3
1	3	1	2
2	2	2	1
3	1	1	1

In both games the operator has the set of pure strategies $\Phi_{Op} = \{0, 1, 2, 3\}$ and the inspector's set of pure strategies is $\Phi_{Insp} = \{1, 2, 3\}$. If for instance the pair $(0, 3)$ is played in the left matrix game, the operator receives the payoff 1 and the inspector -1. It can be easily checked, that in the left matrix game the pairs $(0, 2)$, $(0, 3)$, $(2, 2)$ and $(2, 3)$ are the only saddle points. Furthermore, it can be seen that the right zero-sum matrix game does not possesses a saddle point in pure strategy combinations!

It is important to note that in game theory it is always assumed that both players know the pure strategy sets (the own and that of the other player) and also the payoff matrix. This condition is called *common knowledge* without which game theory - as used in this study - would not work.

A new idea is needed in order to get closer to a satisfactory concept of a solution of a matrix game if there is no saddle point in pure strategies: Each player should choose a strategy at *random*. In this way, the other player has no way of predicting which strategy will be used. The *probabilities* with which the various pure strategies are chosen will probably be known to both opponents (since they can make the same considerations and because of the common knowledge assumption), the particular strategy chosen for a particular game will not be known. The problem of each player will then be to set these probabilities in an optimal way. Thus, we have to introduce the concept of mixed strategies:

A *mixed strategy* of a player is a probability distribution over his set of pure strategies. Although all of the following concepts can be introduced quite abstract, we restrict ourselves to the case that $|\Phi_{Op}| = 4$ and $|\Phi_{Insp}| = 3$, because these are the interesting cases in this study. We get for the operator's set of mixed strategies

$$Q_{Op} = \left\{ \mathbf{q}^T = (q_0, q_1, q_2, q_3) \in \mathbb{R}^4 : q_i \geq 0 \quad \text{for} \quad i = 0, \ldots, 3 \quad \text{and} \quad \sum_{i=0}^{3} q_i = 1 \right\}$$

and for the inspector's one

$$Q_{Insp} = \left\{ \mathbf{p}^T = (p_1, p_2, p_3)^T \in \mathbb{R}^3 : p_j \geq 0 \quad \text{for} \quad j = 1, \ldots, 3 \quad \text{and} \quad \sum_{j=1}^{3} p_j = 1 \right\} .$$

A pure strategies can be seen as a special case of a mixed strategy. If the players play the mixed strategy combination $(\mathbf{q}, \mathbf{p})$, the operator's expected payoff defined on the set $Q_{Op} \times Q_{Insp}$ is given by

$$Op(\mathbf{q}, \mathbf{p}) = \mathbf{q}^T A\, \mathbf{p} = \sum_{i=0}^{3} \sum_{j=1}^{3} q_i\, p_j\, Op(i, j) \,. \tag{B.1}$$

According to our assumptions the inspector's expected payoff is
$Insp(\mathbf{q}, \mathbf{p}) = -Op(\mathbf{q}, \mathbf{p})$.

Now the idea of the saddle point in pure strategies can be generalized to the saddle point criterion, see, e.g., [31]:

A mixed strategy combination $(\mathbf{q}^*, \mathbf{p}^*) \in Q_{Op} \times Q_{Insp}$ constitutes a saddle point in mixed strategies of the zero-sum game with payoff matrix A if and only if

$$Op(\mathbf{q}, \mathbf{p}^*) \leq Op(\mathbf{q}^*, \mathbf{p}^*) \leq Op(\mathbf{q}^*, \mathbf{p}) \quad \text{for all} \quad \mathbf{q} \in Q_{Op} \quad \text{and all} \quad \mathbf{p} \in Q_{Insp} \,,$$

where $Op(\mathbf{q}, \mathbf{p})$ is given by (B.1). □

$Op^* = Op(\mathbf{q}^*, \mathbf{p}^*)$ is called the *value* of the game. For the matrix games considered in this study we often write for the value of the game $Op_k^*(\beta)$ in order to emphasize the dependence of Op^* from the number k of unannounced interim inspections and the error of the second kind β. $\mathbf{q}^*$ resp. $\mathbf{p}^*$ are also called optimal strategies of the operator resp.

the inspector. It can be shown that every zero-sum game with finite pure strategy sets possesses at least on saddle point in mixed strategies, see [32] or [33], but of course - see the argumentation above - not always a saddle point in pure strategy combinations.

If one only has to prove that a given pair of mixed strategies is a saddle point, the following characterization is very useful: $(\mathbf{q}^*, \mathbf{p}^*)$ is a saddle point of the zero-sum game if only if

$$Op(i, \mathbf{p}^*) \leq Op(\mathbf{q}^*, \mathbf{p}^*) \quad \text{for all} \quad i = 0, 1, 2, 3$$

and

$$Op(\mathbf{q}^*, \mathbf{p}^*) \leq Op(\mathbf{q}^*, j) \quad \text{for all} \quad j = 1, 2, 3\,,$$

see, e.g., [31], i.e., both inequalities have only to be proven for the pure strategies of the players.

If a zero-sum game has the saddle points $(\mathbf{q}^*, \mathbf{p}^*)$ and $(\mathbf{q}_1^*, \mathbf{p}_1^*)$, then $(\mathbf{q}^*, \mathbf{p}_1^*)$ and $(\mathbf{q}_1^*, \mathbf{p}^*)$ are also saddle points of the game with the property

$$Op(\mathbf{q}^*, \mathbf{p}^*) = Op(\mathbf{q}^*, \mathbf{p}_1^*) = Op(\mathbf{q}_1^*, \mathbf{p}^*) = Op(\mathbf{q}_1^*, \mathbf{p}_1^*)\,,$$

i.e., all saddle points are interchangeable and lead to the same value. For this reason finding all saddle points is more a mathematical challenge than necessary for applications.

Appendix C

Extensive form games

Matrix games are deceptively simple. The concept of a strategy comprises many different aspects, for example sequencing, information, chance and others. These aspects which are so important for real life conflicts, are much better expressed in *extensive form games*.

C.1 Definition of extensive form games

A non-cooperative game in extensive form is a graphical representation of the possible moves of all players from the beginning of the game until its end. It has the form of the tree - growing from the top to the bottom - where a set of branches starting at some point indicate a player´s alternative at that point. For the sake of illustration we refer in the following to the extensive form game in Figure 4.3 and its reduced form in Figure 4.4.

A precise mathematical definition of extensive form games has been given, for example, by [19] and [31] and goes as follows. Let us mention in passing that we present the general definition for n-person extensive form games, even though we consider only the case $n = 2$, since it is not more complicated then the special one.

An n-person game non-cooperative extensive form game is a rooted tree – usually growing from the top to the bottom – together with labels at every decision point or node and decision alternative or branch, defined as follows:

- Each nonterminal node has a player label that is taken from the set $\{0, 1, \ldots, n\}$. Nodes that are assigned a player label 0 are called chance nodes. The set $\{1, 2, \ldots, n\}$ represents the set of players in the game, and for each individual player i in this set, the nodes with the player label i are decision nodes that are controlled by that player.
- Every alternative at a chance node has a label that specifies its probability. At each chance node, these chance probabilities of the alternatives are nonnegative numbers that sum to one.

- Every decision point or node that is controlled by a player has a second label that specifies the information state that the player would have if the path of the play reached this node. When the path of the play reaches a node controlled by a player, the player knows only the information state on the current node. Thus, two nodes that belong to the same player should have the same information state only if the player would be unable to distinguish between the situations represented by these nodes when either occurs in the play of the game.

- Each alternative or branch at a node that is controlled by a player has an alternative or move label. Furthermore, for any two nodes x and y that have the same player label and the same information label, there must be one alternative or move at both nodes that has the same move label.

- Each terminal or outcome node has a payoff label for each player, such that for each player i, there is a payoff u_i, measured on some utility scale.

As mentioned above, we consider only two-person games. Chance nodes play a major role in our games since, once the inspector performs an inspection after the beginning of an illegal activity, a chance node describes whether or not the illegal activity will be detected (with probability $1 - \beta$) or not (with probability β).

C.2 Pure strategies, mixed strategies and behavioral strategies

A *pure strategy* in an extensive form game is any rule for determining a move at every possible information state in the game. Mathematically, a strategy is a function that maps information states into moves. For each player i let S_i denote the set of possible information states of player i in the game. For each information state s in S_i let D_s denote the set of moves that would be available to player i when he moved at a node with information state s. Then the set of pure strategies for player i in the extensive form game is the cartesian product $\times_{s \in S_i} D_s$. In other words, a pure strategy of a player is a complete plan for his choices at all his information sets.

A *mixed strategy* means that the player chooses, before the beginning of the game, one such comprehensive plan at random according to a certain probability distribution.

An alternative method of randomization for the player is to make an independent random choice at each one of his information states. That is, rather than selecting for every information set, one definite choice – as in a pure strategy – he specifies instead a probability distribution over the set of choices there; moreover, the choices at different information sets are stochastically independent. These randomization procedures are called *behavior(al) strategies*.

Without going into details of *games with perfect recall* – which we are considering exclusively in our applications – we assert that mixed strategies and behavioral strategies of these games are equivalent to each other in the sense that they lead to the same expected payoffs, see, e.g. Hart [19].

Let us illustrate these concepts with the help of the extensive form game represented graphically in Figure 4.4 and its normal form representation in Figure D.3.

From Figure 4.4 and formula (4.5) we get with slightly rearranging the terms

$$\begin{aligned} Op_{3,2}(\beta;\mathbf{g},\mathbf{p}) &= g_0 \cdot [\, A \cdot p_{(1,2)} + B \cdot p_{(1,3)} + C \cdot p_{(2,3)} \,] \\ &+ p_{(1,2)} \cdot [\, (1-g_0) \cdot g_1 \cdot d + (1-g_0) \cdot (1-g_1) \cdot 2 \,] \\ &+ p_{(1,3)} \cdot [\, (1-g_0) \cdot g_1 \cdot 3 + (1-g_0) \cdot (1-g_1) \cdot 1 \,] \\ &+ p_{(2,3)} \cdot (1-g_0) \cdot f \,. \end{aligned}$$

From Figure D.3 with $\mathbf{q}^T = (q_1, q_2, q_3, q_4)$ being the mixed strategy of the operator we get

$$\begin{aligned} Op_{3,2}(\beta;\mathbf{g},\mathbf{p}) &= (q_1+q_2) \cdot [\, A \cdot p_{(1,2)} + B \cdot p_{(1,3)} + C \cdot p_{(2,3)} \,] \\ &+ q_3 \cdot [\, p_{(1,2)} \cdot d + p_{(1,3)} \cdot e + p_{(2,3)} \cdot f \,] \\ &+ q_4 \cdot [\, p_{(1,2)} \cdot 2 + p_{(1,3)} \cdot 1 + p_{(2,3)} \cdot f \,] \\ &= (q_1+q_2) \cdot [\, A \cdot p_{(1,2)} + B \cdot p_{(1,3)} + C \cdot p_{(2,3)} \,] \\ &+ p_{(1,2)} \cdot [\, q_3 \cdot d + q_4 \cdot 2 \,] \\ &+ p_{(1,3)} \cdot [\, q_3 \cdot e + q_4 \cdot 1 \,] \\ &+ p_{(2,3)} \cdot [\, q_3 + q_4 \cdot f \,] \,. \end{aligned}$$

Thus we see immediately, that

$$\begin{aligned} g_0 = q_1 + q_2 & \\ (1-g_0) \cdot g_1 &= q_3 \\ (1-g_0) \cdot (1-g_1) &= q_4 \end{aligned}$$

lead to the same expected payoff to the operator.

Let us add that because of $q_1 + q_2 = 1 - q_3 - q_4$ we have in both cases two free variables for optimizing payoffs, say q_3 and q_4 on the one hand, and g_0 and g_1 on the other.

A Nash equilibrium, i.e., a saddle point in a zero-sum game, is defined in the same way in extensive form games as in normal form games. It can be determined in different ways, let us just mentioned three of them: In extensive form games with *perfect information*, i.e., in extensive form games where all information states of all players consists of exactly one decision node, a backward induction procedure is used which means that non-optimal moves are eliminated from the bottom to the top (in our application we do not use such games). Or one uses behavioral strategies and tries to find a Nash equilibrium with the help of the Nash conditions. Or one transforms the extensive form game into a normal form game and applies the solution techniques available for this type of games. In our application we use both methods.

It should be mentioned in passing that normal form games, which corresponds to extensive form games, may have more Nash equilibria than the latter ones, but in our applications we do not encounter this difficulty.

Appendix D

Proofs

D.1 Proof of the saddle point solution for the simultaneous discrete time game

For the proofs in this section we use the following property of optimal strategies in zero-sum games: $\mathbf{q}^*$ and $\mathbf{p}^*$ are optimal strategies for the operator resp. the inspector if only if

$$Op(i, \mathbf{p}^*) \leq Op(\mathbf{q}^*, \mathbf{p}^*) \quad \text{for all} \quad i = 0, 1, 2, 3 \tag{D.1}$$

and

$$Op(\mathbf{q}^*, \mathbf{p}^*) \leq Op(\mathbf{q}^*, j) \quad \text{for all} \quad j = 1, 2, 3\,. \tag{D.2}$$

For the sake of simplicity we will write throughout this section $Op_{n,k}(\mathbf{q}, \mathbf{p})$ instead of $Op_{n,k}(\beta; \mathbf{q}, \mathbf{p})$.

D.1.1 The case $n = 3$ and $k = 1$

We now prove that the strategies given on the pages 23 to 25 are indeed optimal strategies. The proof is presented in several steps.

The case $0 \leq \beta < 1/6$: One easily sees that (3.1) and (3.2) fulfill the conditions

$$q_i^* \geq 0 \quad \text{for} \quad i = 0, \ldots, 3 \quad \text{and} \quad \sum_{i=0}^{3} q_i^* = 1 \quad \text{and} \tag{D.3}$$

$$p_j^* \geq 0 \quad \text{for} \quad j = 1, 2, 3 \quad \text{and} \quad \sum_{j=1}^{3} p_j^* = 1 \tag{D.4}$$

and are therefore mixed strategies of the operator resp. the inspector.

With (3.1) and (3.2) we get

$$\begin{aligned} & Op_{3,1}(0,\mathbf{p}^*) = Op_{3,1}(1,\mathbf{p}^*) = Op_{3,1}(2,\mathbf{p}^*) \\ = \;& Op_{3,1}(\mathbf{q}^*,1) = Op_{3,1}(\mathbf{q}^*,2) = Op_{3,1}(\mathbf{q}^*,3) \end{aligned}$$

and therefore

$$Op^*_{3,1}(\beta) = Op_{3,1}(\mathbf{q}^*,1) = \frac{11}{6} + \beta \quad (> 1)\,,$$

i.e., (3.3). For $i = 3$ we obtain $Op_{3,1}(3,\mathbf{p}^*) = 1 < Op^*_{3,1}(\beta)$. Thus, both inequalities (D.1) and (D.2) are fulfilled and therefore $\mathbf{q}^*$ and $\mathbf{p}^*$ are optimal strategies.

The case $1/6 < \beta < 2/3$**:** One easily sees that (3.7) and (3.8) fulfill the conditions (D.3) and (D.4) and are therefore mixed strategies of the operator resp. the inspector.

Furthermore, we obtain with (3.7) and (3.8)

$$Op_{3,1}(0,\mathbf{p}^*) = Op_{3,1}(1,\mathbf{p}^*) = Op_{3,1}(\mathbf{q}^*,1) = Op_{3,1}(\mathbf{q}^*,2)$$

and therefore

$$Op^*_{3,1}(\beta) = Op_{3,1}(\mathbf{q}^*,1) = \frac{10}{6} + 2\cdot\beta \quad (> 2)\,,$$

i.e., (3.9). For $i = 2, 3$ we get

$$\begin{aligned} Op_{3,1}(2,\mathbf{p}^*) &= 2 < Op^*_{3,1}(\beta) \qquad \text{and} \\ Op_{3,1}(3,\mathbf{p}^*) &= 1 < Op^*_{3,1}(\beta) \end{aligned}$$

and for $j = 3$

$$Op_{3,1}(\mathbf{q}^*,3) = \frac{16}{6} + \beta > Op^*_{3,1}(\beta)\,.$$

Thus, both inequalities (D.1) and (D.2) are fulfilled and therefore $\mathbf{q}^*$ and $\mathbf{p}^*$ are optimal strategies.

The case $2/3 < \beta \leq 1$**:** One easily sees that (3.13) and (3.14) fulfill the conditions (D.3) and (D.4) and are therefore mixed strategies of the operator resp. the inspector.

Furthermore, we obtain with (3.13) and (3.14)

$$Op_{3,1}(0,\mathbf{p}^*) = Op_{3,1}(\mathbf{q}^*,1)$$

and therefore

$$Op^*_{3,1}(\beta) = Op_{3,1}(\mathbf{q}^*,1) = \frac{6}{6} + 3\cdot\beta \quad (> 3)\,.$$

For $i = 1, 2, 3$ we get

$$\begin{aligned} Op_{3,1}(1,\mathbf{p}^*) &= 3 < Op^*_{3,1}(\beta) \qquad \text{and} \\ Op_{3,1}(2,\mathbf{p}^*) &= 2 < Op^*_{3,1}(\beta) \qquad \text{and} \\ Op_{3,1}(3,\mathbf{p}^*) &= 1 < Op^*_{3,1}(\beta) \end{aligned}$$

and for $j = 2, 3$

$$\begin{aligned} Op_{3,1}(\mathbf{q}^*, 2) &= \frac{12}{6} + 2 \cdot \beta > Op^*_{3,1}(\beta) \\ Op_{3,1}(\mathbf{q}^*, 3) &= \frac{18}{6} + \beta > Op^*_{3,1}(\beta) \,. \end{aligned}$$

Thus, both inequalities (D.1) and (D.2) are fulfilled and therefore $\mathbf{q}^*$ and $\mathbf{p}^*$ are optimal strategies.

The proof that (3.4) and (3.5) are optimal strategies leading to the optimal expected detection time (3.6) can be shown in the same ways as in case $1/6 < \beta < 2/3$. The same holds for $\beta = 2/3$ and the corresponding optimal strategies (3.10) and (3.11) with the optimal expected detection time (3.12). □

D.1.2 The case $n = 3$ and $k = 2$

We now prove that the strategies given on the pages 27 to 28 are indeed optimal strategies. The proof is again presented in several steps.

The case $0 \le \beta < 1/2$: Again one sees that (3.19) and (3.20) fulfill condition (D.3) and – with some lengthy calculations – that (3.16) - (3.18) fulfill the condition

$$p^*_{(1,2)} \ge 0 \,, \quad p^*_{(1,3)} \ge 0 \,, \quad p^*_{(2,3)} \ge 0 \quad \text{and} \quad p^*_{(1,2)} + p^*_{(1,3)} + p^*_{(2,3)} = 1 \,,$$

and are therefore mixed strategies of the operator resp. the inspector.

With (3.16) - (3.18) and (3.19) - (3.20) we obtain

$$\begin{aligned} & Op_{3,2}(0, \mathbf{p}^*) = Op_{3,2}(1, \mathbf{p}^*) = Op_{3,1}(3, \mathbf{p}^*) \\ = \; & Op_{3,2}(\mathbf{q}^*, 1) = Op_{3,2}(\mathbf{q}^*, 2) = Op_{3,2}(\mathbf{q}^*, 3) \end{aligned}$$

and therefore

$$Op^*_{3,2}(\beta) = Op_{3,2}(\mathbf{q}^*, 1) = \frac{4 + 6 \cdot \beta + 5 \cdot \beta^2 + 2 \cdot \beta^3}{3 + 2 \cdot \beta + \beta^2} \quad (\ge \frac{4}{3} > 1) \,,$$

i.e., (3.21). For $i = 3$ we have $Op_{3,2}(3, \mathbf{p}^*) = 1 < Op^*_{3,2}(\beta)$. Thus, both inequalities (D.1) and (D.2) are fulfilled and therefore $\mathbf{q}^*$ and $\mathbf{p}^*$ are optimal strategies.

The case $1/2 < \beta \le 1$: One easily sees that (3.25) and (3.26) are correctly normalized and are therefore mixed strategies of the operator resp. the inspector.

Furthermore, we get

$$Op_{3,2}(0, \mathbf{p}^*) = Op_{3,2}(\mathbf{q}^*, 1)$$

and therefore

$$Op^*_{3,2}(\beta) = Op_{3,2}(\mathbf{q}^*, 1) = 1 + \beta + 2 \cdot \beta^2 \quad (> 2) \,,$$

i.e., (3.27). For $i = 1, 2, 3$ we have

$$\begin{aligned} Op_{3,2}(1, \mathbf{p}^*) &= 1 + 2 \cdot \beta < Op^*_{3,2}(\beta) \quad \text{and} \\ Op_{3,2}(2, \mathbf{p}^*) &= 2 < Op^*_{3,1}(\beta) \quad \text{and} \\ Op_{3,2}(3, \mathbf{p}^*) &= 1 < Op^*_{3,1}(\beta) \end{aligned}$$

and for $j = 2, 3$

$$\begin{aligned} Op_{3,2}(\mathbf{q}^*, 2) &= 1 + 2 \cdot \beta + \beta^2 \geq Op^*_{3,2}(\beta) \\ Op_{3,2}(\mathbf{q}^*, 3) &= 2 + \beta + \beta^2 \geq Op^*_{3,2}(\beta)\,. \end{aligned}$$

Thus, both inequalities (D.1) and (D.2) are fulfilled and therefore $\mathbf{q}^*$ and $\mathbf{p}^*$ are optimal strategies.

For the limiting case $\beta = 1/2$ the optimal strategies are given by (3.22) and (3.23) with the corresponding optimal expected detection time (3.24). The proof goes along the same line like in case $1/2 < \beta \leq 1$. □

D.1.3 The case $n = 5$ and $k = 1$

We now prove that the strategies given on the pages 33 to 35 are indeed optimal strategies. The proof is again presented in several steps.

The case $0 \leq \beta < 13/60$**:** One easily sees that (3.28) and (3.29) as well as (3.30) are correctly normalized and are therefore mixed strategies of the operator resp. the inspector.

Furthermore we obtain with some lengthy calculations

$$\begin{aligned} & Op_{5,1}(0, \mathbf{p}^*) = Op_{5,1}(1, \mathbf{p}^*) = Op_{5,1}(2, \mathbf{p}^*) = Op_{5,1}(3, \mathbf{p}^*) \\ = \; & Op_{5,1}(\mathbf{q}^*, 1) = Op_{5,1}(\mathbf{q}^*, 2) = Op_{5,1}(\mathbf{q}^*, 3) = Op_{5,1}(\mathbf{q}^*, 4) \end{aligned}$$

and therefore

$$Op^*_{5,1}(\beta) = Op_{5,1}(\mathbf{q}^*, 1) = \frac{77}{30} + 2 \cdot \beta \quad (\geq \frac{77}{30} > 2)\,,$$

i.e., (3.31). For $i = 4, 5$ we obtain $Op_{5,1}(i, \mathbf{p}^*) = 6 - i \leq 2 < Op^*_{5,1}(\beta)$. For $j = 5$ we get

$$Op_{5,1}(\mathbf{q}^*, 5) = \beta + \frac{107}{30} > \frac{77}{30} + 2 \cdot \beta = Op^*_{5,1}(\beta)\,.$$

Thus, both inequalities (D.1) and (D.2) are fulfilled and therefore $\mathbf{q}^*$ and $\mathbf{p}^*$ are optimal strategies.

The case $13/60 < \beta < 11/20$**:** Again, one easily sees that (3.32) and (3.33) as well as (3.34) are correctly normalized and are therefore mixed strategies of the operator resp. the inspector.

Furthermore we get – again with some lengthy calculations –

$$\begin{aligned} & Op_{5,1}(0,\mathbf{p}^*) = Op_{5,1}(1,\mathbf{p}^*) = Op_{5,1}(2,\mathbf{p}^*) \\ = \; & Op_{5,1}(\mathbf{q}^*,1) = Op_{5,1}(\mathbf{q}^*,2) = Op_{5,1}(\mathbf{q}^*,3) \end{aligned}$$

and therefore

$$Op^*_{5,1}(\beta) = Op_{5,1}(\mathbf{q}^*,1) = \frac{47}{20} + 3\cdot\beta \quad (>3)\,,$$

i.e., (3.35). For $i = 3,4,5$ we have

$$\begin{aligned} Op_{5,1}(3,\mathbf{p}^*) &= 3 < Op^*_{5,1}(\beta) && \text{and} \\ Op_{5,1}(4,\mathbf{p}^*) &= 2 < Op^*_{5,1}(\beta) && \text{and} \\ Op_{5,1}(5,\mathbf{p}^*) &= 1 < Op^*_{5,1}(\beta) \end{aligned}$$

and for $j = 4$

$$\begin{aligned} Op_{5,1}(\mathbf{q}^*,4) &= (4+2\cdot\beta)\cdot\frac{12}{20} + (3+2\cdot\beta)\cdot\frac{3}{20} + (2+2\cdot\beta)\cdot\frac{5}{20} \\ &= 2\cdot\beta + 4\cdot\frac{12}{20} + 3\cdot\frac{3}{20} + 2\cdot\frac{5}{20} \\ &= 2\cdot\beta + \frac{67}{20} > \frac{47}{20} + 3\cdot\beta = Op^*_{5,1}(\beta)\,. \end{aligned}$$

For $j = 5$ we obtain in analogy

$$\begin{aligned} Op_{5,1}(\mathbf{q}^*,5) &= (5+\beta)\cdot\frac{12}{20} + (4+\beta)\cdot\frac{3}{20} + (3+\beta)\cdot\frac{5}{20} \\ &= \beta + \frac{87}{20} > \frac{47}{20} + 3\cdot\beta = Op^*_{5,1}(\beta)\,. \end{aligned}$$

Thus, both inequalities (D.1) and (D.2) are fulfilled and therefore $\mathbf{q}^*$ and $\mathbf{p}^*$ are optimal strategies.

The case $11/20 < \beta < 4/5$**:** Again, one easily sees that (3.36) as well as (3.37) are correctly normalized and are therefore mixed strategies of the operator resp. the inspector.

Furthermore we obtain

$$Op_{5,1}(0,\mathbf{p}^*) = Op_{5,1}(1,\mathbf{p}^*) = Op_{5,1}(\mathbf{q}^*,1) = Op_{5,1}(\mathbf{q}^*,2)$$

and therefore

$$Op^*_{5,1}(\beta) = Op_{5,1}(\mathbf{q}^*,1) = \frac{9}{5} + 4\cdot\beta \quad (>4)\,,$$

i.e., (3.38). For $i = 2,\ldots,5$ we have

$$Op_{5,1}(i,\mathbf{p}^*) \;=\; -(i-2)+4 \le 4 < Op^*_{5,1}(\beta)$$

and for $j = 3, 4, 5$

$$\begin{aligned} Op_{5,1}(\mathbf{q}^*, 3) &= 3 \cdot \beta + \frac{14}{5} > \frac{9}{5} + 4 \cdot \beta = Op^*_{5,1}(\beta) \\ Op_{5,1}(\mathbf{q}^*, 4) &= 2 \cdot \beta + \frac{19}{5} > \frac{9}{5} + 4 \cdot \beta = Op^*_{5,1}(\beta) \\ Op_{5,1}(\mathbf{q}^*, 5) &= \beta + \frac{24}{5} > \frac{9}{5} + 4 \cdot \beta = Op^*_{5,1}(\beta) \,. \end{aligned}$$

Thus, both inequalities (D.1) and (D.2) are fulfilled and therefore $\mathbf{q}^*$ and $\mathbf{p}^*$ are optimal strategies.

The case $4/5 < \beta \leq 1$**:** It is clear that (3.39) as well as (3.40) are correctly normalized and are therefore mixed strategies of the operator resp. the inspector.

Furthermore we obtain

$$Op_{5,1}(0, \mathbf{p}^*) = Op_{5,1}(\mathbf{q}^*, 1)$$

and therefore

$$Op^*_{5,1}(\beta) = Op_{5,1}(\mathbf{q}^*, 1) = 1 + 5 \cdot \beta \quad (> 5) \,,$$

i.e., (3.41). For $i = 1, \ldots, 5$ we have

$$Op_{5,1}(i, \mathbf{p}^*) \;=\; -(i-2) + 4 \leq 5 < Op^*_{5,1}(\beta)$$

and for $j = 2 \ldots, 5$

$$Op_{5,1}(\mathbf{q}^*, j) \;=\; j + (6-j) \cdot \beta > 1 + 5 \cdot \beta = Op^*_{5,1}(\beta) \,,$$

since $j - 1 > \beta \cdot (j-1)$ for all $j = 2, \ldots, 5$. Thus, both inequalities (D.1) and (D.2) are fulfilled and therefore $\mathbf{q}^*$ and $\mathbf{p}^*$ are optimal strategies.

The proof of the limiting cases is omitted. □

D.2 Proof of the saddle point solution for the simultaneous continuous time game

In order to solve the game with payoff kernel given on page 41 we transform the game into another one to simplify the analysis: We consider a game with the payoff kernel

$$\widetilde{A}_2(x, (y_1, y_2)) = \begin{cases} -(1-\beta) \cdot y_2 - \beta \cdot (1-\beta) \cdot y_1 + x & 0 \leq y_1 < y_2 < x \leq t_0 \\ -(1-\beta) \cdot y_1 + x & 0 \leq y_1 < x \leq y_2 \leq t_0 \\ x & 0 \leq x \leq y_1 < y_2 \leq t_0 \end{cases} \quad \text{(D.5)}$$

The idea of this transformation is that we consider the remaining time until the game ends. Let

$$Y := \{\, (y_1, y_2) : 0 \leq y_1 < y_2 \leq t_0 \,\} \,.$$

In the following we are going to solve the game with payoff kernel $\widetilde{A}_2(x,(y_1,y_2))$, i.e., we want to determine probability distribution functions $\widetilde{Q}^*(x)$ on $[0,t_0]$ for the operator and $\widetilde{P}^*_{(Y_1,Y_2)}(y_1,y_2)$ on Y for the inspector such that

$$\mathbf{E}(x,\widetilde{P}^*) \le \mathbf{E}(\widetilde{Q}^*,\widetilde{P}^*) \le \mathbf{E}(\widetilde{Q}^*,(y_1,y_2)) \tag{D.6}$$

for all $x \in [0,t_0]$ and all $(y_1,y_2) \in Y$.

A fundamental role for the construction of the optimal strategies, i.e., the probability distribution functions $\widetilde{Q}^*(x)$ and $\widetilde{P}^*_{(Y_1,Y_2)}(y_1,y_2)$ play the functions $h_1(x)$ and $h_2(x)$ fulfilling the following differential equation system

$$h_1'(x) = (1-\beta)\cdot h_1(x) \tag{D.7}$$
$$h_2'(x) = (1-\beta)\cdot h_2(x) - (1-\beta)^2\cdot h_1(x) \tag{D.8}$$

with

$$h_1(0) = 1 \quad \text{and} \quad h_2(0) = h_1(1)\,. \tag{D.9}$$

The solution of this differential equation system was already given (3.45) and (3.46). At this point the concrete structure of the solution is not really needed. It is only important that the functions $h_1(x)$ and $h_2(x)$ fulfill (D.7) and (D.8) with the boundary condition (D.9) and that they are monotone increasing function of x for $i = 1,2$.

Without loss of generality we assume in the following analysis that $t_0 = h_2(1)$, see [13].

We first show that the each pair of strategies of player 2 (the player corresponding to the inspector in the original game) (y_1^*,y_2^*) with $y_1^* := h_1(u)$ and $y_2^* := h_2(u)$ for all $u \in [0,1]$ makes player 1 (the player corresponding to the operator in the original game) indifferent with respect to his pure strategies $x \in [0,t_0]$.

For $x \in (h_1(0),h_1(1))$ we have

$$0 \le U < h_1^{-1}(x) \iff y_1^* < x \le y_2^* \iff -(1-\beta)\cdot h_1(U) + x$$

and

$$h_1^{-1}(x) \le U \le 1 \iff x \le y_1^* < y_2^* \iff x\,.$$

Therefore we get for all $x \in (h_1(0),h_1(1))$

$$\begin{aligned}
\widetilde{Op}_2(x,\widetilde{P}^*) &= \int_{[0,t_0)}\int_{(t_1,t_0]} \widetilde{A}_2(x,(y_1,y_2))\,d\widetilde{P}^*_{(Y_1,Y_2)}(y_1,y_2) \\
&= \int_0^1 \widetilde{A}_2(h_1^{-1}(x),(h_1(u),h_2(u)))\cdot \mathbf{1}\,du \\
&= \int_0^{h_1^{-1}(x)} [-(1-\beta)\cdot h_1(u) + x]\,du + \int_{h_1^{-1}(x)}^1 x\,du \\
&= x - (1-\beta)\cdot \int_0^{h_1^{-1}(x)} h_1(u)\,du\,.
\end{aligned} \tag{D.10}$$

Differentiation with respect to x leads for all $x \in (h_1(0), h_1(1))$ to

$$\begin{aligned}\frac{d}{dx}\,\widetilde{Op}(x,\widetilde{P}^*) &= 1-(1-\beta)\cdot h_1(h_1^{-1}(x))\cdot \frac{d}{ds}\,h_1^{-1}(x)\\ &= 1-(1-\beta)\cdot x\cdot \frac{d}{ds}\,h_1^{-1}(x)\,.\end{aligned}$$

Since

$$\frac{d}{ds}\,h_1^{-1}(x) = \frac{1}{h_1'(h_1^{-1}(x))} \overset{(D.7)}{=} \frac{1}{(1-\beta)\cdot h_1(h_1^{-1}(x))} = \frac{1}{(1-\beta)\cdot x}$$

we get

$$\frac{d}{dx}\,\widetilde{Op}(x,\widetilde{P}^*) = 0 \quad \text{for all} \quad x \in (h_1(0), h_1(1))\,.$$

For $x \in (h_2(0), h_2(1))$ we obtain in analogy to the considerations made above

$$\begin{aligned}\widetilde{Op}(x,\widetilde{P}^*) &= \int_0^{h_2^{-1}(x)} [\,-(1-\beta)\cdot h_2(u) - \beta\cdot(1-\beta)\cdot h_1(u) + x\,]\,du\\ &+ \int_{h_2^{-1}(x)}^{1} [\,-(1-\beta)\cdot h_1(u) + x\,]\,du\,. \end{aligned} \tag{D.11}$$

Differentiation with respect to x leads for all $x \in (h_2(0), h_2(1))$ to

$$\begin{aligned}\frac{d}{dx}\,\widetilde{Op}(x,\widetilde{P}^*) &= 1+\big(-(1-\beta)\cdot x + (1-\beta)^2\cdot h_1(h_2^{-1}(x))\big)\cdot\frac{1}{h_2'(h_2^{-1}(x))}\\ &\overset{(D.8)}{=} 0\,.\end{aligned}$$

From (D.10) and (D.11) we see that $\widetilde{Op}(x,\widetilde{P}^*)$ is a continuous function in x and therefore we obtain $\widetilde{Op}(x,\widetilde{P}^*) = C$ for all $x \in [h_1(0), h_2(1)]$. This leads for $x = h_1(0)$ to the value of the transformed game (remember $h_1(0) = 1$)

$$\widetilde{Op}(h_1(0),\widetilde{P}^*) = h_1(0) = 1\,.$$

For all $x \in [0, h_1(0))$ we have $\widetilde{Op}(x,\widetilde{P}^*) = x < h_1(0)$. Therefore, we have shown

$$\mathbf{E}(x,\widetilde{P}^*) \le \mathbf{E}(\widetilde{Q}^*,\widetilde{P}^*)\,,$$

i.e., the left hand inequality of (D.6). Furthermore we have $\mathbf{E}(\widetilde{Q}^*,\widetilde{P}^*) = h_1(0) = 1$.

We now want to show the right hand inequality of (D.6), i.e.,

$$\mathbf{E}(\widetilde{Q}^*,\widetilde{P}^*) \le \mathbf{E}(\widetilde{Q}^*,(y_1,y_2))$$

for all $(y_1, y_2) \in Y$.

We first get for all $(y_1, y_2) \in Y$ and all every distribution function $\widetilde{Q}(x)$, see , e.g., [12]

$$\begin{aligned}
\mathbf{E}(\widetilde{Q}, (y_1, y_2)) &= \int_0^1 \widetilde{A}_2(x, (y_1, y_2))\, d\widetilde{Q}(x) \\
&= \int_{[0,y_1]} x\, d\widetilde{Q}(x) + \int_{(y_1,y_2]} [\,-(1-\beta)\cdot y_1 + x\,]\, d\widetilde{Q}(x) \\
&+ \int_{(y_2,t_0]} [\,-(1-\beta)\cdot y_2 - \beta\cdot(1-\beta)\cdot y_1 + x\,]\, d\widetilde{Q}(x) \\
&= \int_{[0,t_0]} x\, d\widetilde{Q}(x) - [\,(1-\beta)\cdot y_1\,]\cdot(\widetilde{Q}(y_2^+) - \widetilde{Q}(y_1^+)) \\
&- [\,(1-\beta)\cdot y_2 + \beta\cdot(1-\beta)\cdot y_1\,]\cdot(\widetilde{Q}(t_0^+) - \widetilde{Q}(y_2^+))\,.
\end{aligned}$$

Rearranging leads us finally to

$$\begin{aligned}
\mathbf{E}(\widetilde{Q}, (y_1, y_2)) &= \int_{[0,t_0]} x\, d\widetilde{Q}(x) + [\,(1-\beta)\cdot y_1\,]\cdot\widetilde{Q}(y_1^+) \\
&+ (1-\beta)\cdot[\,y_2 - (1-\beta)\cdot y_1\,]\cdot\widetilde{Q}(y_2^+) \\
&- [\,(1-\beta)\cdot y_2 + \beta\cdot(1-\beta)\cdot y_1\,]\cdot\widetilde{Q}(t_0^+)
\end{aligned} \tag{D.12}$$

We now define the optimal distribution function of player 1:

$$\widetilde{Q}_2^*(x) := \begin{cases} 0 & : \quad 0 \le x < h_1(0) \\ 1 - \dfrac{h_2(1-h_1^{-1}(x))}{h_2(1)} & : \quad h_1(0) \le x < h_1(1) = h_2(0) \\ 1 - \dfrac{h_1(1-h_2^{-1}(x))}{h_2(1)} & : \quad h_2(0) \le x < h_2(1) = t_0 \\ 1 & : \quad x = h_2(1) = t_0 \end{cases} . \tag{D.13}$$

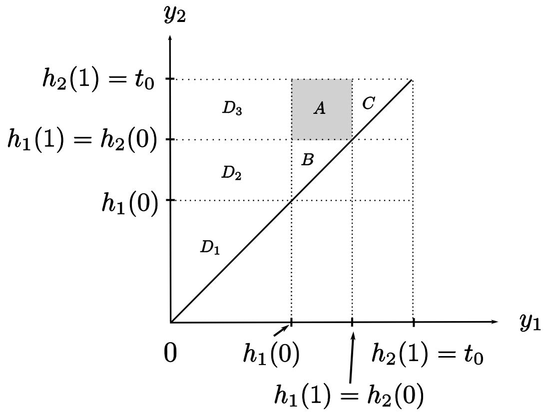
It can be shown that $\widetilde{Q}_2^*(x)$ is indeed a probability distribution function, that is

$$\lim_{x\to-\infty} \widetilde{Q}_2^*(x) = 0\,, \quad \lim_{x\to\infty} \widetilde{Q}_2^*(x) = 1$$

and monotone increasing in x. Furthermore, it can be easily seen that $\widetilde{Q}_2^*(x)$ is a continuous function on $[0, t_0)$, but not differentiable at $x \in \{h_1(0), h_1(1), h_2(1)\}$. At $x = h_2(1) = t_0$ this distribution function has an atom.

We now have to show that $\widetilde{P}^*(y_1, y_2)$ minimizes the function $\mathbf{E}(\widetilde{Q}^*, \widetilde{P}(y_1, y_2))$. This is done in several steps. We use the notation of Figure D.1.

Figure D.1 Different domains used in the proof.



Before starting we note that we get with (D.12) for all $y_1 \in (0, t_0) \setminus \{h_1(0), h_2(0)\}$

$$\frac{1}{1-\beta} \cdot \frac{\partial}{\partial y_1} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) = \widetilde{Q}^*(y_1) + y_1 \cdot \frac{d}{dy_1} \widetilde{Q}^*(y_1) - (1-\beta) \cdot \widetilde{Q}^*(y_2) - \beta \tag{D.14}$$

and for all $y_2 \in (y_1, t_0) \setminus \{h_1(0), h_2(0)\}$

$$\frac{1}{1-\beta} \cdot \frac{\partial}{\partial y_2} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) = \widetilde{Q}^*(y_2) + (y_2 - (1-\beta) \cdot y_1) \cdot \frac{d}{dy_2} \widetilde{Q}^*(y_2) - 1\,. \tag{D.15}$$

The domains D_1, D_2 and D_3: We get for all $y_1 \in (0, h_1(0))$ and for all $y_2 \in (y_1, t_0]$ with (D.14) and definition (D.13)

$$\frac{1}{1-\beta} \cdot \frac{\partial}{\partial y_1} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) = -(1-\beta) \cdot \widetilde{Q}^*(y_2) - \beta\,.$$

We now distinguish two cases:

If $y_2 \in (y_1, h_1(0)]$, then we have $\widetilde{Q}^*(y_2) = 0$ and therefore

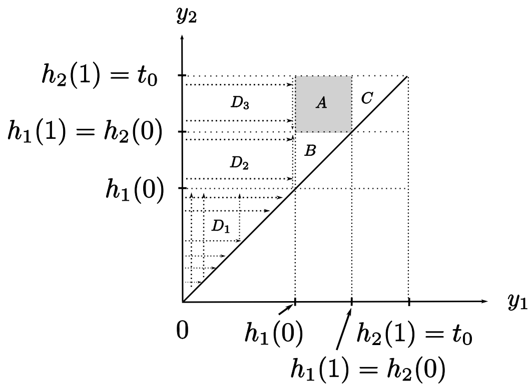
$$\frac{1}{1-\beta} \cdot \frac{\partial}{\partial y_1} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) = -\beta \leq 0\,.$$

In addition we have with (D.15) that

$$\frac{1}{1-\beta} \cdot \frac{\partial}{\partial y_2} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) = -1\,,$$

i.e., the function $\mathbf{E}(\widetilde{Q}^*, (y_1, y_2))$ is monotone increasing in y_2 in the set D_1, see Figure D.2.

Figure D.2 Monotonicity properties of $\mathbf{E}(\widetilde{Q}^*, (y_1, y_2))$.



If $y_2 \in (h_1(0), t_0]$, then $\widetilde{Q}^*(y_2) > 0$ and therefore

$$\frac{1}{1-\beta} \cdot \frac{\partial}{\partial y_1} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) < 0 \,.$$

Because of the continuity of $\mathbf{E}(\widetilde{Q}^*, (y_1, y_2))$ on Y we obtain

$$\inf_{(y_1,y_2)\in[0,h_1(0)]\times(y_1,t_0]} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) = \min_{y_2\in(h_1(0),t_0]} \mathbf{E}(\widetilde{Q}^*, (h_1(0), y_2)) \,.$$

We now want to determine this minimum. Again we consider two cases

If $y_2 \in (h_1(0), h_1(1))$ we have with (D.15)

$$\begin{aligned}
& \frac{1}{1-\beta} \cdot \frac{\partial}{\partial y_2} \mathbf{E}(\widetilde{Q}^*, (h_1(0), y_2)) \\
= \quad & \widetilde{Q}^*(y_2) + (y_2 - (1-\beta) \cdot h_1(0)) \cdot \frac{d}{dy_2} \widetilde{Q}^*(y_2) - 1 \\
= \quad & -\frac{h_2(1 - h_1^{-1}(y_2))}{h_2(1)} + (y_2 - (1-\beta) \cdot h_1(0)) \cdot \frac{h_2'(1 - h_1^{-1}(y_2))}{h_2(1) \cdot h_1'(h_1^{-1}(y_2))} \\
\overset{(D.7),(D.8)}{=} \quad & \frac{h_2(1 - h_1^{-1}(y_2))}{h_2(1)} \cdot \left(-1 + \frac{y_2 - (1-\beta) \cdot h_1(0)}{y_2} \right) \\
- \quad & \frac{(1-\beta) \cdot (y_2 - (1-\beta) \cdot h_1(0)) \cdot h_1(1 - h_1^{-1}(y_2))}{h_2(1) \cdot y_2} \\
< \quad & 0 \,,
\end{aligned}$$

since $(1-\beta) \cdot h_1(0) < y_2$. The result of this analysis is, that $\mathbf{E}(\widetilde{Q}^*, (h_1(0), y_2))$ is a monotone decreasing function on $y_2 \in (h_1(0), h_1(1))$, see Figure D.2.

If $y_2 \in (h_2(0), t_0)$ we have with (D.15)

$$\begin{aligned}
&\frac{1}{1-\beta} \cdot \frac{\partial}{\partial y_2} \mathbf{E}(\widetilde{Q}^*, (h_1(0), y_2)) \\
= \;& \widetilde{Q}^*(y_2) + (y_2 - (1-\beta) \cdot h_1(0)) \cdot \frac{d}{dy_2} \widetilde{Q}^*(y_2) - 1 \\
= \;& -\frac{h_1(1 - h_2^{-1}(y_2))}{h_2(1)} + (y_2 - (1-\beta) \cdot h_1(0)) \cdot \frac{h_1'(1 - h_2^{-1}(y_2))}{h_2(1) \cdot h_2'(h_2^{-1}(y_2))} \\
\overset{(D.7)}{=} \;& \frac{h_1(1 - h_2^{-1}(y_2))}{h_2(1)} \cdot \left(-1 + \frac{(1-\beta) \cdot (y_2 - (1-\beta) \cdot h_1(0))}{h_2'(h_2^{-1}(y_2))} \right) \\
> \;& 0\,,
\end{aligned}$$

whereas the last inequality can be seen as follows: The condition

$$(1-\beta) \cdot (y_2 - (1-\beta) \cdot h_1(0)) > h_2'(h_2^{-1}(y_2))$$

is with (D.8) equivalent to

$$y_2 - (1-\beta) \cdot h_1(0) > y_2 - (1-\beta) \cdot h_1(h_2^{-1}(y_2))$$

and equivalent to (because of the construction of h_1)

$$h_1(0) < h_1(h_2^{-1}(y_2))\,,$$

which is always fulfilled. The result of this analysis is, that $\mathbf{E}(\widetilde{Q}^*, (h_1(0), y_2))$ is a monotone increasing function on $y_2 \in (h_2(0), t_0]$, see again Figure D.2.

Together we have shown, that

$$\begin{aligned}
\inf_{(y_1,y_2) \in [0,h_1(0)] \times (y_1,t_0]} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) &= \min_{y_2 \in (h_1(0),t_0]} \mathbf{E}(\widetilde{Q}^*, (h_1(0), y_2)) \\
&= \mathbf{E}(\widetilde{Q}^*, (h_1(0), h_2(0)))\,,
\end{aligned}$$

i.e. the infimum is reached in the left lower edge of the square A in Figure D.2.

The domain B: In this case we have with (D.14)

$$\begin{aligned}
&\frac{1}{1-\beta} \cdot \frac{\partial}{\partial y_1} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) \\
= \;& \widetilde{Q}^*(y_1) + y_1 \cdot \frac{d}{dy_1} \widetilde{Q}^*(y_1) - (1-\beta) \cdot \widetilde{Q}^*(y_2) - \beta \\
= \;& 1 - \frac{h_2(1 - h_1^{-1}(y_1))}{h_2(1)} + y_1 \cdot \frac{1}{h_2(1)} \cdot \frac{h_2'(1 - h_1^{-1}(y_1))}{h_1'(h_1^{-1}(y_1))} \\
- \;& (1-\beta) \cdot \left(1 - \frac{h_2(1 - h_1^{-1}(y_2))}{h_2(1)} \right) - \beta \\
\overset{(D.7)}{=} \;& -\frac{h_2(1 - h_1^{-1}(y_1))}{h_2(1)} + \frac{h_2'(1 - h_1^{-1}(y_1))}{h_2(1) \cdot (1-\beta)} + (1-\beta) \cdot \frac{h_2(1 - h_1^{-1}(y_2))}{h_2(1)} \\
\overset{(D.8)}{=} \;& \frac{(1-\beta)}{h_2(1)} \cdot \left(h_2(1 - h_1^{-1}(y_2)) - h_1(1 - h_1^{-1}(y_1)) \right) \\
> \;& 0\,,
\end{aligned}$$

because $h_1(x)$ and $h_2(x)$ have disjunct ranges (except the end points of the interval). In the previous paragraph we have shown, that $\mathbf{E}(\widetilde{Q}^*, (h_1(0), y_2))$ is a monotone decreasing function on $y_2 \in (h_1(0), h_2(0))$. Therefore, we finally get

$$\begin{aligned}\inf_{(y_1,y_2)\in(h_1(0),h_2(0))\times(y_1,h_2(0)]} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) &= \min_{y_2\in(h_1(0),h_2(0)]} \mathbf{E}(\widetilde{Q}^*, (h_1(0), y_2)) \\ &= \mathbf{E}(\widetilde{Q}^*, (h_1(0), h_2(0)))\,,\end{aligned}$$

see again for a graphical illustration Figure D.2.

The domain C: In this case we get with (D.15)

$$\begin{aligned}&\frac{1}{1-\beta}\cdot\frac{\partial}{\partial y_2}\mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) \\ =\;& \widetilde{Q}^*(y_2) + (y_2 - (1-\beta)\cdot y_1)\cdot\frac{d}{dy_2}\widetilde{Q}^*(y_2) - 1 \\ =\;& -\frac{h_1(1-h_2^{-1}(y_2))}{h_2(1)} + (y_2 - (1-\beta)\cdot y_1)\cdot\frac{h_1'(1-h_2^{-1}(y_2))}{h_2(1)\cdot h_2'(h_2^{-1}(y_2))} \\ \overset{(D.7)}{=}\;& \frac{h_1(1-h_2^{-1}(y_2))}{h_2(1)}\cdot\left(-1+\frac{(1-\beta)\cdot(y_2-(1-\beta)\cdot y_1)}{h_2'(h_2^{-1}(y_2))}\cdot\right).\end{aligned}$$

Since $y_1 > h_2(0)$ we get from construction of h_1 that

$$1 < h_1(h_2^{-1}(y_2)) < h_1(1) = h_2(0) < y_1\,.$$

Therefore we get

$$\frac{(1-\beta)\cdot(y_2-(1-\beta)\cdot y_1)}{h_2'(h_2^{-1}(y_2))} \overset{(D.8)}{=} \frac{(y_2-(1-\beta)\cdot y_1)}{y_2-(1-\beta)\cdot h_1(h_2^{-1}(y_2))} < 1\,.$$

This leads us finally to

$$\frac{1}{1-\beta}\cdot\frac{\partial}{\partial y_2}\mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) < 0$$

and therefore

$$\inf_{(y_1,y_2)\in[h_2(0),t_0)\times(y_1,t_0]} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) = \min_{y_1\in[h_2(0),t_0)} \mathbf{E}(\widetilde{Q}^*, (y_1, t_0))\,.$$

We now evaluate the right expression in this equation. First we get with (D.14)

$$\begin{aligned}&\frac{1}{1-\beta}\cdot\frac{\partial}{\partial y_1}\mathbf{E}(\widetilde{Q}^*, (y_1, t_0)) = \widetilde{Q}^*(y_1) + y_1\cdot\frac{d}{dy_1}\widetilde{Q}^*(y_1) - 1 \\ =\;& -\frac{h_1(1-h_2^{-1}(y_1))}{h_2(1)} + y_1\cdot\frac{h_1'(1-h_2^{-1}(y_1))}{h_2(1)\cdot h_2'(h_2^{-1}(y_1))} \\ \overset{(D.7)}{=}\;& \frac{h_1(1-h_2^{-1}(y_1))}{h_2(1)}\cdot\left(-1+\frac{(1-\beta)\cdot y_1}{h_2'(h_2^{-1}(y_1))}\right) \\ >\;& 0\,,\end{aligned}$$

where the validity of the last inequality follows from the fact that (for $\beta < 1$)

$$(1-\beta)^2 \cdot h_1(h_2^{-1}(y_1)) > 0$$

and therefore

$$\begin{aligned} (1-\beta)\cdot y_1 &> (1-\beta)\cdot y_1 - (1-\beta)^2 \cdot h_1(h_2^{-1}(y_1)) \\ &= h_2'(h_2^{-1}(y_1)) \quad (>0)\,. \end{aligned}$$

This shows that $\mathbf{E}(\widetilde{Q}^*, (y_1, t_0))$ is a monotone increasing function for $y_1 \in (h_2(0), t_0)$ and leads us finally to

$$\begin{aligned} \inf_{(y_1,y_2)\in[h_2(0),t_0)\times(y_1,t_0]} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) &= \min_{y_1\in[h_2(0),t_0)} \mathbf{E}(\widetilde{Q}^*, (y_1, t_0)) \\ &= \mathbf{E}(\widetilde{Q}^*, (h_2(0), t_0))\,, \end{aligned}$$

the right upper corner of set A, see Figure D.2.

The domain A: The necessary condition for a minimum (y_1^*, y_2^*) is, that

$$\frac{\partial}{\partial y_1} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2))\Big|_{(y_1^*,y_2^*)} = 0 \quad \text{and} \quad \frac{\partial}{\partial y_2} \mathbf{E}(\widetilde{Q}^*, (y_1, y_2))\Big|_{(y_1^*,y_2^*)} = 0\,. \tag{D.16}$$

We obtain

$$\begin{aligned} &\frac{1}{1-\beta}\cdot\frac{\partial}{\partial y_2}\mathbf{E}(\widetilde{Q}^*, (y_1, y_2)) \\ =\ & \widetilde{Q}^*(y_2) + (y_2 - (1-\beta)\cdot y_1)\cdot\frac{d}{dy_2}\widetilde{Q}^*(y_2) - 1 \\ =\ & -\frac{h_1(1-h_2^{-1}(y_2))}{h_2(1)} + (y_2 - (1-\beta)\cdot y_1)\cdot\frac{h_1'(1-h_2^{-1}(y_2))}{h_2(1)\cdot h_2'(h_2^{-1}(y_2))} \\ \overset{(D.7)}{=}\ & \frac{h_1(1-h_2^{-1}(y_2))}{h_2(1)}\cdot\left(-1 + \frac{(1-\beta)\cdot(y_2-(1-\beta)\cdot y_1)}{h_2'(h_2^{-1}(y_2))}\right)\,. \end{aligned}$$

This expression is zero if and only if

$$\begin{aligned} (1-\beta)\cdot(y_2 - (1-\beta)\cdot y_1) &= h_2'(h_2^{-1}(y_2)) \\ &\overset{(D.8)}{=} (1-\beta)\cdot y_2 - (1-\beta)^2\cdot h_1(h_2^{-1}(y_2)) \end{aligned}$$

which is equivalent to

$$h_1^{-1}(y_1) = h_2^{-1}(y_2)\,.$$

If one defines $x := h_1^{-1}(y_1)$, then $x \in (0,1)$ and we have, that all pairs (y_1^*, y_2^*) with

$$y_1^* = h_1(x) \quad \text{and} \quad y_2^* = h_2(x) \tag{D.17}$$

fulfill the optimal condition on the right hand side of (D.16). But do they also fulfill the left hand equality? From (D.14) we obtain

$$\begin{aligned}
&\frac{1}{1-\beta}\cdot\frac{\partial}{\partial y_1}\mathbf{E}(\widetilde{Q}^*,(y_1,y_2)) = \widetilde{Q}^*(y_1) + y_1\cdot\frac{d}{dy_1}\widetilde{Q}^*(y_1) - (1-\beta)\cdot\widetilde{Q}^*(y_2) - \beta \\
= \quad & 1 - \frac{h_2(1-h_1^{-1}(y_1))}{h_2(1)} + y_1\cdot\frac{1}{h_2(1)}\cdot\frac{h_2'(1-h_1^{-1}(y_1))}{h_1'(h_1^{-1}(y_1))} \\
- \quad & (1-\beta)\cdot\left(1-\frac{h_1(1-h_2^{-1}(y_2))}{h_2(1)}\right) - \beta \\
\overset{(D.7)}{=} \quad & -\frac{h_2(1-h_1^{-1}(y_1))}{h_2(1)} + \frac{h_2'(1-h_1^{-1}(y_1))}{h_2(1)} + (1-\beta)\cdot\frac{h_1(1-h_2^{-1}(y_2))}{h_2(1)} \\
\overset{(D.8)}{=} \quad & \frac{(1-\beta)}{h_2(1)}\cdot\left(h_1(1-h_2^{-1}(y_2)) - h_1(1-h_1^{-1}(y_1))\right)\,.
\end{aligned}$$

The pair (y_1^*, y_2^*) with (D.17) makes this equation to zero. Summing up we have shown that $\widetilde{Q}^*(x)$ is an optimal strategy of player 1 and that player 2 has to play the pair (y_1^*, y_2^*) according to (D.17). This completes the proof. □

From the construction of the game with payoff kernel (D.5) we can immediately determine the solution of the original game:

Let $t_0 > 0$ be an arbitrary number, i.e., $t_0 \neq h_2(1)$ is possible. Then the inspector has to perform his inspections at the time points

$$t_1^* = t_0 - y_2^* \quad \text{and} \quad t_2^* = t_0 - y_1^*$$

which leads to

$$t_1^* = t_0\cdot\left(1-\frac{h_2(1-u)}{h_2(1)}\right) \quad \text{and} \quad t_2^* = t_0\cdot\left(1-\frac{h_1(1-u)}{h_2(1)}\right)\,,$$

where u is the realization of a uniformly distributed random variable U on $[0,1]$.

The optimal expected detection time is

$$Op_2^*(\beta) = \frac{t_0}{h_2(1)}\,.$$

This completes the proof. □

D.3 Proof of the saddle point solution for the hybrid-sequential discrete time game

D.3.1 The case $k = 1$ and arbitrary n

We show that the strategies given by (4.1) and (4.2) indeed constitutes a saddle point of the game with value (4.3).

The saddle point criterion is in this case given by

$$Op_{n,1}(i,\mathbf{p}^*;\beta) \leq Op^*_{n,1}(\beta) \leq Op_{n,1}(\mathbf{q}^*,j;\beta) \tag{D.18}$$

for $i = \overline{l_0}, l_0$ and $j = 1,\ldots,n$. In order to verify the left hand inequality of (D.18) we have to show

$$Op_{n,1}(\bar{l}_0,\mathbf{p}^*;\beta) \leq \frac{n+1}{2-\beta} \quad \text{and} \quad Op_{n,1}(l_0,\mathbf{p}^*;\beta) \leq \frac{n+1}{2-\beta}\,.$$

This is equivalent to

$$\sum_{j=1}^{n} [\,(1-\beta)\cdot j + \beta\cdot(n+1)\,]\cdot p_j^* \leq \frac{n+1}{2-\beta} \quad \text{and} \quad \sum_{j=1}^{n} [\,n+1-j\,]\cdot p_j^* \leq \frac{n+1}{2-\beta}\,,$$

which holds as equalities because of (4.2).

The right hand side of (D.18) is equivalent to

$$\frac{n+1}{2-\beta} \leq \frac{1}{2-\beta}\cdot[\,(1-\beta)\cdot j + \beta\cdot(n+1)\,] + \frac{1-\beta}{2-\beta}\cdot(n+1-j)$$

for $j = 1,\ldots,n$, which is fulfilled as equality for all j. This completes the proof. □

D.3.2 The case $n = 3$ and $k = 2$

With the notation introduced in Figure 4.4 the expected detection time, i.e., the payoff to the operator is

$$\begin{aligned} Op_{3,2}(\beta) &= g_0\cdot[\,A\cdot p_{(1,2)} + B\cdot p_{(1,3)} + C\cdot p_{(2,3)}\,] \\ &+ (1-g_0)\cdot[\,g_1\cdot(d\cdot p_{(1,2)} + e\cdot p_{(1,3)}) \\ &\qquad + (1-g_1)\cdot(2\cdot p_{(1,2)} + p_{(1,3)}) + f\cdot p_{(2,3)}\,] \end{aligned}$$

We determine the saddle point of this game in such a way that we choose the optimal strategies of the two players in such a way that the adversaries are rendered indifferent with respect to their own strategies. Thus, the indifference condition for the operator are

$$\begin{aligned} g_1 &: \; d\cdot p^*_{(1,2)} + e\cdot p^*_{(1,3)} - 2\cdot p^*_{(1,2)} - p^*_{(1,3)} = 0\,, \\ g_0 &: \; A\cdot p^*_{(1,2)} + B\cdot p^*_{(1,3)} + C\cdot p^*_{(2,3)} - 2\cdot p^*_{(1,2)} - p^*_{(1,3)} - f\cdot p^*_{(2,3)} = 0\,, \\ & \; p^*_{(1,2)} + p^*_{(1,3)} + p^*_{(2,3)} = 1\,. \end{aligned} \tag{D.19}$$

The indifference condition for the inspector are

$$\begin{aligned} p_{(1,2)} &: \; g_0^*\cdot(A-C) + (1-g_0^*)\cdot[\,g_1^*\cdot d + (1-g_1^*)\cdot 2 - f\,] = 0\,, \\ p_{(1,3)} &: \; g_0^*\cdot(B-C) + (1-g_0^*)\cdot[\,g_1^*\cdot e + (1-g_1^*)\cdot 1 - f\,] = 0\,. \end{aligned} \tag{D.20}$$

Furthermore the value of the game is

$$Op_{3,2}(\beta) = g_0^* \cdot C + (1 - g_0^*) \cdot f \,. \tag{D.21}$$

Solving the systems of equations (D.19) and (D.20) we get (4.6), (4.7) and (4.8). Writing the nominator of $p^*_{(1,3)}$ in the form $(1 - 2 \cdot \beta) \cdot (1 + \beta^2)$ we see immediately that this solution holds only for $\beta < 0.5$. In order to prove the saddle point condition for $\beta \geq 0.5$ we either have to guess and to prove it, or more systematically, transform the extensive form game into a normal form one.

To arrive at the normal form, we have to determine all pure strategies of both players which means all choices of all information sets (even if some of them actually cannot be reached. Thus, for the inspector the pure strategies are again $(1, 2)$, $(1, 3)$ and $(2, 3)$, whereas for the operator they are

$$\bar{l}_0\, \bar{l}_1, \bar{l}_0\, l_1, l_0\, \bar{l}_1 \quad \text{and} \quad l_0\, l_1 \,.$$

The normal form of this game is given in Figure D.3.

Figure D.3 Normal form of the extensive form game given in Figure 4.4. $\mathbf{q} = (q_1, q_2, q_3)$ and $\mathbf{p} = (p_{(1,2)}, p_{(1,3)}, p_{(2,3)})$ are the mixed strategies of both players. The second row can be deleted.

	$(1,2)$	$(1,3)$	$(2,3)$
$\bar{l}_0\, \bar{l}_1$	$1 + \beta + 2 \cdot \beta^2$	$1 + 2 \cdot \beta + \beta^2$	$2 + \beta + \beta^2$
$\bar{l}_0\, l_1$	$1 + \beta + 2 \cdot \beta^2$	$1 + 2 \cdot \beta + \beta^2$	$2 + \beta + \beta^2$
$l_0\, \bar{l}_1$	$1 + 2 \cdot \beta$	$2 + \beta$	$1 + \beta$
$l_0\, l_1$	2	1	$1 + \beta$

This normal form game can still be solved analytically. More easily is it, to use the Mathematica® program adapted by M. Canty, see [10], for the solution of zero-sum games. For $\beta \leq 0.5$ we get the solution

$$\begin{aligned} q_1^* &= \frac{1}{N} \cdot (1 - \beta + \beta^2)\,, \\ q_2^* &= \frac{1}{N} \cdot (1 - \beta + \beta^2 - \beta^3)\,, \\ q_3^* &= \frac{1}{N} \cdot (1 - \beta)\,. \end{aligned} \tag{D.22}$$

$\mathbf{p}^*$ and the value of the game are given by (4.7) and (4.8). In order to complete the analysis we identify the two solutions for $\beta \leq 0.5$. According to Figure 4.4 and D.3 we have

$$prob(\bar{l}_0) = q_1^* = g_0^* = \frac{1}{N} \cdot (1 - \beta + \beta^2)$$

and furthermore

$$prob(l_0\,\bar{l}_1) = q_2^* = (1 - g_0^*) \cdot g_1^* \,,$$

thus,

$$g_1^* = \frac{q_2^*}{1 - g_0^*} = \frac{q_2^*}{1 - q_1^*} \,.$$

With (D.22) we arrive indeed at the form for g_1^* as given by (4.6).

D.4 Proof of the saddle point solution for the hybrid-sequential continuous time game

D.4.1 The case $k = 1$

In order to determine the solution of this game, we apply again the technique of rendering the adversary indifferent. If this works the saddle point conditions are fulfilled as equalities.

We see immediately that the operator is indifferent with respect to his mixed strategy $(g_2, 1 - g_2)$, if the inspector chooses his inspection time point t_1^* for given values of t_2 and t_0 according to

$$(1 - \beta) \cdot (t_1^* - t_2) + \beta \cdot (t_0 - t_2) - (t_0 - t_1^*) = 0 \qquad \text{(D.23)}$$

which gives

$$t_1^* - t_2 = \frac{1 - \beta}{2 - \beta} \cdot (t_0 - t_2) \,,$$

i.e., (4.14). (D.23) gives immediately with (4.13)

$$Op_1^*(\beta) = t_0 - t_1^* \qquad \text{(D.24)}$$

and therefore (4.16). Conversely, the inspector is indifferent with respect to his strategy t_1, if the operator chooses his (behavioral) strategy g_2^* according to

$$(1 - \beta) \cdot g_2^* - (1 - g_2^*) = 0 \,,$$

which gives us (4.15), which completes the proof. □

D.4.2 The case $k = 2$

The indifference conditions for the operator are with (4.17)

$$g_2(t_2) \ : \ (1-\beta)\cdot(t_1^* - t_2^*) + \beta\cdot(t_0 - t_2^*) - (t_0 - t_1^*) = 0\,, \tag{D.25}$$

$$g_3 \ : \ (1-\beta)\cdot(t_2^* - t_3^*) + \beta\cdot(1-\beta)\cdot(t_1^* - t_3) + \beta^2\cdot(t_0 - t_3) - (t_0 - t_1^*) = 0\,. \tag{D.26}$$

The indifference conditions for the inspector are:

$$t_2 \ : \ g_3^*\cdot(1-\beta) + (1-g_3^*)\cdot[\,g_2^*\cdot(-(1-\beta)-\beta)\,] = 0\,, \tag{D.27}$$

$$t_1 \ : \ g_3^*\cdot\beta\cdot(1-\beta) + (1-g_3^*)\cdot[\,g_2^*\cdot(1-\beta) - (1-g_2^*)\,] = 0\,. \tag{D.28}$$

The optimal expected detection time is therefore

$$Op_2^*(\beta) = t_0 - t_1^* \tag{D.29}$$

and has the same form as (D.24).

We show that the solutions (4.18) - (4.21) fulfill equations (D.25) - (D.28). (4.22) then follows immediately from (D.29).

(D.25) is identical to (D.23). Therefore, (4.19) holds as well. Furthermore we get with some lengthy calculation

$$t_1^* - t_3 = 2\cdot\frac{1-\beta}{3-2\cdot\beta}\cdot(t_0 - t_3)$$

and

$$t_0 - t_1^* = \frac{1}{3-2\cdot\beta}\cdot(t_0 - t_3)\,.$$

Inserting these expressions into (D.28) shows the validity of our results.

Finally, it is straightforward to show that (4.20) and (4.21) fulfill equations (D.27) and (D.28), which completes the proof. □

D.4.3 The case $k = 3$

In this case the expected detection time is

$$\begin{aligned} Op_3(\beta;\mathbf{g},\mathbf{t}) \ = \ & g_4\Big[(1-\beta)\,(t_3-t_4) + \beta\,(1-\beta)\,(t_2-t_4) + \beta^2\,(1-\beta)\,(t_1-t_4) + \beta^3\,(t_0-t_4)\Big] \\ & + \ (1-g_4)\Big[\,g_3(t_3)\cdot\Big\{(1-\beta)\,(t_2-t_3) + \beta\,(1-\beta)\,(t_1-t_3) + \beta^2\,(t_0-t_3)\Big\} \\ & \quad + (1-g_3(t_3))\Big\{\,g_2(t_3,t_2)\cdot\Big\{(1-\beta)\,(t_1-t_2) + \beta\,(t_0-t_2)\Big\} \\ & \quad\quad + (1-g_2(t_3,t_2))\,(t_0-t_1)\Big\}\Big]\,. \end{aligned} \tag{D.30}$$

Therefore the indifference condition for the operator are (we delete the stars)

$$g_2(t_3, t_2): \quad (1-\beta)(t_1 - t_2) + \beta(t_0 - t_2) - (t_0 - t_1) = 0 \tag{D.31}$$

$$g_3(t_3): \quad (1-\beta)(t_2 - t_3) + \beta(1-\beta)(t_1 - t_3) + \beta^2(t_0 - t_3) - (t_0 - t_1) = 0 \tag{D.32}$$

$$g_4: \quad (1-\beta)(t_3 - t_4) + \beta(1-\beta)(t_2 - t_4) + \beta^2(1-\beta)(t_1 - t_4) + \beta^3(t_0 - t_4) - (t_0 - t_1) = 0\,. \tag{D.33}$$

The optimal expected detection time is

$$Op^*(\beta) = t_0 - t_1\,. \tag{D.34}$$

Equations (D.31) and (D.32) are formally identical to equations (D.25) and (D.26), therefore, their solutions are identical to (4.18) and (4.19), i.e., (4.24) and (4.25) hold. In order to prove (4.23), we show, again with some lengthly calculations

$$\begin{aligned} t_2 - t_3 &\overset{(4.24)}{=} \frac{1-\beta}{3-2\beta}(t_0 - t_3) = \frac{1-\beta}{3-2\beta}(t_0 - t_4 + t_4 - t_3) \\ &\overset{(4.23)}{=} \frac{1-\beta}{3-2\beta}\left(1 - \frac{1-\beta}{4-3\beta}\right)(t_0 - t_4) = \frac{1-\beta}{4-3\beta}(t_0 - t_4) \end{aligned}$$

and

$$\begin{aligned} t_1 - t_2 &\overset{(4.25)}{=} \frac{1-\beta}{2-\beta}(t_0 - t_2) = \frac{1-\beta}{2-\beta}(t_0 - t_4 + t_4 - t_3 + t_3 - t_2) \\ &= \frac{1-\beta}{2-\beta}\left(1 - \frac{1-\beta}{4-3\beta} - \frac{1-\beta}{4-3\beta}\right)(t_0 - t_4) = \frac{1-\beta}{4-3\beta}(t_0 - t_4) \end{aligned}$$

and

$$\begin{aligned} t_2 - t_4 &= t_2 - t_3 + t_3 - t_4 = 2\,\frac{1-\beta}{4-3\beta}(t_0 - t_4) \\ t_1 - t_4 &= t_1 - t_2 + t_2 - t_4 = 3\,\frac{1-\beta}{4-3\beta}(t_0 - t_4) \\ t_0 - t_1 &= t_0 - t_4 + t_4 - t_1 = \left(1 - 3\,\frac{1-\beta}{4-3\beta}\right)(t_0 - t_4) = \frac{t_0 - t_4}{4-3\beta}\,. \end{aligned} \tag{D.35}$$

Inserting these forms into the left hand side of equation (D.33) and using the identity

$$(1-\beta)^2 + 2\,\beta(1-\beta)^2 + 3\,\beta^2(1-\beta)^2 + \beta^3(4-3\beta) - 1 = 0$$

for all $\beta \in [0,1]$, we see that the left hand side of equation (D.33) is equal to zero. Equation (4.29) follows immediately from (D.34) and (D.35).

The indifference conditions for the inspector are

$$t_3: \quad g_4\,(1-\beta) + (1-g_4)\,g_3 = 0$$

$$t_2: \quad g_4\,\beta\,(1-\beta) + (1-g_4)\,[\,g_3\,(1-\beta) - (1-g_3)\,g_2\,] = 0$$

$$t_1: \quad g_4\,\beta^2\,(1-\beta) + (1-g_4)\,[\,g_3\,\beta\,(1-\beta) + (1-g_3)\,(g_2\,(2-\beta)-1)\,] = 0\,.$$

They are equivalent to

$$\frac{g_4}{1-g_4}\,(1-\beta) - g_3 = 0 \qquad \text{(D.36)}$$

$$\frac{g_4}{1-g_4}\,\beta\,(1-\beta) + g_3\,(1-\beta) - (1-g_3)\,g_2 = 0 \qquad \text{(D.37)}$$

$$\frac{g_4}{1-g_4}\,\beta^2\,(1-\beta) + g_3\,\beta\,(1-\beta) + (1-g_3)\,(g_2\,(2-\beta)-1) = 0\,. \qquad \text{(D.38)}$$

Inserting (D.36) in (D.37) we get

$$\frac{g_3}{1-g_3} = g_2$$

and therefore with (D.36) and (D.38)

$$g_2 = \frac{1}{2}$$

from which g_3 and g_4 follow as given by (4.27) and (4.26).

This completes the proof. □

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Abstract

In this study unannounced interim inspections are analyzed in general, and specifically in view of the recently defined IAEA/EURATOM Partnership Approach. In particular the following aspects are addressed: A general discussion of the problem of unannounced interim inspections which includes an identification and description of all the types of assumptions which are necessary to be made for a quantitative treatment, an outline of the general methods identified and proposed for the solution of problems of that kind, the identification of two examples for a detailed analysis to be carried out with the general method identified in the previous steps, and lessons learned and reflections aiming at identifying also practical recommendations on the matter of unannounced interim inspections.

As a major result, for a specific facility and a given number of inspections per reference time, e.g., one year, optimal inspection strategies are determined. Whereas for the case that inspection are perfect the results can be guessed, this does not hold anymore for imperfect inspections, e.g., limited sample sizes of seals to be checked. Also, together with the optimal inspection strategies, optimal expected detection times are determined as functions of the parameters of the model, namely number of inspections per reference time interval and probability of not detecting an illegal activity during an inspection.

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