NUMERICAL HOMOGENIZATION OF FRACTAL INTERFACE PROBLEMS

RALF KORNHUBER, JOSCHA PODLESNY, AND HARRY YSERENTANT

Abstract. We consider the numerical homogenization of a class of fractal elliptic interface problems inspired by related mechanical contact problems from the geosciences. A particular feature is that the solution space depends on the actual fractal geometry. Our main results concern the construction of projection operators with suitable stability and approximation properties. The existence of such projections then allows for the application of existing concepts from localized orthogonal decomposition (LOD) and successive subspace correction to construct first multiscale discretizations and iterative algebraic solvers with scale-independent convergence behavior for this class of problems.

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1. Introduction

Classical homogenization aims at deriving computationally feasible, effective mathematical descriptions of multiscale phenomena by capturing the fine scales in terms of local cell problems. Starting from elliptic problems with oscillating coefficients [2, 3] and its random counterparts [26, 51] (stochastic) homogenization has become a flourishing field of research and a well-established, powerful tool in mathematical modelling with multiple scales. An enormous variety of applications include multiscale materials, featuring irregular or even fractal boundaries, transmission conditions across fractal interfaces, or long, thin fibers [18, 30, 32], biological materials like lung tissue [4, 9], or polycrystals giving rise to multiscale interface problems with jump conditions across a fine scale network of interfaces [10, 12, 19]. Corresponding stochastic variants have been studied in [21, 25].

Classical homogenization typically relies on scale separation and periodicity of fine scale behavior. To overcome these limitations in practical computations, numerical homogenization aims at deriving multiscale discretizations and iterative algebraic solution methods that are robust with respect to the inherent lack of smoothness of multiscale problems. A natural approach to multiscale discretization is to build all relevant fine scale features of a given problem directly into the approximating ansatz space. Over more than two decades, this basic idea has led to composite finite elements [20, 39], variational multiscale methods [24], heterogeneous multiscale methods [1, 47], and multiscale finite elements [14, 23]. A certain breakthrough in the mathematical understanding of multiscale discretization methods for elliptic self-adjoint problems with oscillating coefficients came with the seminal paper on localized orthogonal decomposition (LOD) by Målqvist and Peterseim [31]. Starting from a projection $\Pi : \mathcal{H} \to \mathcal{S}_h$

that maps the solution space $\mathcal{H}$ onto some given finite element space $\mathcal{S}_h \subset \mathcal{H} \subset L^2$ with mesh size h and satisfies the following stability and approximation property

$$\|\Pi v\|_{\mathcal{H}} \le c\|v\|_{\mathcal{H}}, \qquad \|v - \Pi v\|_{L^2} \le Ch\|v\|_{\mathcal{H}} \qquad \forall v \in \mathcal{H}, \tag{1}$$

they observed that the a-orthogonal complement $\mathcal{W}$ of the kernel of Π (the orthogonal complement with respect to the underlying energy scalar product) has the same dimension as $\mathcal{S}_h$ and, without any additional assumptions on periodicity or scale separation, provides an approximation with optimal accuracy. Moreover, optimal accuracy is preserved under localization of the a-orthogonalized nodal basis of $\mathcal{W}$. The actual computation of these localized basis functions amounts to an approximate solution of local problems, utilizing a much larger finite element space $\mathcal{S}$ that resolves all fine scale features of the given problem.

An alternative to multiscale discretization methods is to use such a large finite element space $\mathcal{S}$ directly for discretization and derive iterative algebraic solution methods that converge independently both of the discretization parameters and of the regularity of the continuous solution. The construction of such methods has been carried out successfully in the framework of iterative subspace correction [29, 48, 49, 50]. Each iteration step typically requires the solution of a set of fully decoupled local subproblems that capture the different frequencies of the actual error. In particular, subspace correction methods can be applied to localization in LOD [27] and are often merged with multiscale discretization techniques e.g., to enhance convergence of multigrid methods by enrichment of coarse grid spaces [20, 28]. While the LOD approach to the construction of multiscale discretizations makes explicit use of a projection $\Pi : \mathcal{H} \to \mathcal{S}_h$ with stability and approximation property (1), such kind of projections play a crucial role in the convergence analysis of subspace correction methods (see, e.g., [28] and the references cited therein). The explicit construction and analysis of such operators for standard Sobolev and finite element spaces has therefore quite a history with further applications in finite element convergence theory and a posteriori error analysis [7, 8, 11, 15, 35, 46].

In this paper, we consider numerical homogenization of a class of elliptic fractal interface problems without periodicity and scale separation that is motivated by geology. Experimental studies suggest that grains in fractured rock are distributed in a fractal manner [33, 44], an observation which is also reflected by geophysical modelling of fragmentation due to tectonic deformation [41]. All spatial scales ranging from grains and rocks even up to tectonic plates are interacting in geophysical fault networks that play an essential role in the dynamics of earthquake sources (see, e.g., [40] and the literature cited therein). Mathematical modelling of stress accumulation and release in fault networks gives rise to continuum mechanical problems with frictional contact along the interfaces (see, e.g. [37] and the literature cited therein). Linearization of contact conditions leads to elliptic interface problems, where frictional motion along interfaces is replaced by weighted jumps of diplacement.

Scalar versions of such interface problems with fractal interface geometry have recently been suggested and analyzed by Heida et al. [22]. More precisely, the fractal interface Γ is the limit of level-k interface networks $\Gamma^{(k)}$ for $k \to \infty$ and a level-k interface network $\Gamma^{(k)} = \bigcup_{j=1}^{k} \Gamma_j$ consists of single faults Γ_j. Here, the single faults Γ_j are ordered from "strong" to "weak" in the sense that discontinuities of displacements along Γ_j are expected to decrease for increasing k, because "more fractured" media are expected to show higher resistance [17, 34]. For each fixed k, the level-k networks $\Gamma^{(k)}$ divide the computational domain Ω into a finite number of cells representing, e.g., geological grains, rocks, and plates. For each $k \in \mathbb{N}$, we define a Hilbert space $\mathcal{H}_k$ by completion of piecewise smooth functions in $\Omega\backslash\Gamma^{(k)}$ with respect to a scalar product involving the broken H^1-seminorm and weighted L^2-norms of jumps across Γ_j, $j=1,\dots k$. The solution space $\mathcal{H}$ for interface problems on the limiting fractal geometry Γ is finally defined by completion of $\bigcup_{k=1}^{\infty}\mathcal{H}_k$. We consider self-adjoint elliptic variational problems in $\mathcal{H}$. Observe that the multiscale character of such problems goes beyond the usual lack of smoothness, because the solution space $\mathcal{H}$ itself depends on the actual fractal geometry which is not accessible by a fixed classical finite element space. This suggests multiscale modifications of classical finite elements as ansatz spaces allowing for a priori discretization error estimates.

The main results of this paper concern the construction of projection operators $\Pi_k:\mathcal{H}\to\mathcal{S}_k$ with the stability and approximation property (1) for spaces $\mathcal{S}_k$ of piecewise linear finite elements with respect to a triangulation $\mathcal{T}^{(k)}$ resolving the level-k interface network $\Gamma^{(k)}$, $k\in\mathbb{N}$. These results allow for direct access to existing approaches to numerical homogenization, e.g., by LOD or subspace correction. Our construction consists of two steps. We first consider projections $\Pi_{\mathcal{H}_k}:\mathcal{H}\to\mathcal{H}_k$ and then $\Pi_{\mathcal{S}_k}:\mathcal{H}_k\to\mathcal{S}_k$, both with the desired properties (1). As projections $\Pi_{\mathcal{S}_k}$ can be essentially taken from the literature [7, 8, 11, 15, 35, 46], we mainly concentrate on the construction and analysis of $\Pi_{\mathcal{H}_k}$ by extending common concepts based on local Poincaré inequalities [8, 46]. Here, the presence of jump terms creates various technical difficulties. In particular, counterexamples show that it is not possible to bound jumps of local averages by jumps of the original functions. Therefore, stability of $\Pi_{\mathcal{H}_k}$ requires strong assumptions on the locality of Γ that rule out, e.g. the Cantor network [22, 44]. The existence of suitable projections Π_k then opens the door to a variety of existing numerical homogenization methods. We only consider two simple examples to fix ideas (see [38] for more advanced applications). The application of LOD with cell-based localization by subspace correction in the spirit of [27, 31] provides a multiscale discretization with optimal error estimates. Using concepts from [29], we also present continuous and discrete versions of a two-level multigrid method with cell-based block Gauss-Seidel smoother and convergence rates that are independent of mesh and scale parameters. In the concluding numerical experiments with a highly localized fractal geometry, we found the theoretically predicted behavior of this method. Moreover, application to a geologically inspired crystalline structure illustrates the potential of our approach in future applications.

The paper is organized as follows. The first section contains the continuous problem formulation. After a detailed description of the geometry of the multiscale network $\Gamma^{(k)}$, $k\in\mathbb{N}$, together with some assumptions capturing its shape regularity and fractal character, we introduce a fractal interface problem and state existence and uniqueness. In the next section, we discuss convergence of its k-scale approximation associated with the subspaces $\mathcal{H}_k\subset\mathcal{H}$. Then we introduce suitable piecewise linear finite element spaces $\mathcal{S}_k\subset\mathcal{H}_k$ for the approximation of these k-scale problems and state some error estimates. The ensuing Section 4 is the core of the paper. It contains the construction and analysis of projections $\Pi_k=\Pi_{\mathcal{S}_k}\circ\Pi_{\mathcal{H}_k}$ via local Poincaré inequalities, a trace lemma, and quasi-interpolation. The next two sections are devoted to first applications of these projections Π_k to construct and analyze a LOD-type multiscale discretization with optimal error estimates and a mesh- and scale-independent subspace correction method. We finally report on some numerical experiments that illustrate our theoretical findings and open a perspective to future practical applications.

2. Fractal interface problems

2.1. **Interface networks.** Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a bounded domain with Lipschitz boundary $\partial\Omega$ that contains a countable set of mutually disjoint interfaces Γ_j, $j \in \mathbb{N}$. We assume that each interface Γ_j is piecewise affine with finite $(d-1)$-dimensional Hausdorff measure. We consider the k-scale interface networks $\Gamma^{(k)}$ and their fractal limit Γ, given by

$$\Gamma^{(k)} = \bigcup_{j=1}^{k} \Gamma_j, \quad k \in \mathbb{N}, \qquad\qquad \Gamma = \bigcup_{j=1}^{\infty} \Gamma_j,$$

respectively. Since all interfaces Γ_j, $j \in \mathbb{N}$, have Lebesgue measure zero in $\mathbb{R}^d$, their countable union Γ has Lebesgue measure zero as well. However, Γ might have fractal (Hausdorff-) dimension $d-s$ for some $s \in (0,1)$ and infinite $(d-1)$-dimensional measure.

For each fixed $k \in \mathbb{N}$, the set $\Omega\backslash\Gamma^{(k)}$ consists of finitely many, mutually disjoint, open, and simply connected cells $G \in \Omega^{(k)}$, i.e.

$$\Omega\backslash\Gamma^{(k)} = \bigcup_{G\in\Omega^{(k)}} G.$$

We assume that $\partial G = \partial\overline{G}$ (no slits) and that either $G \cap \partial\Omega$ has positive $(d-1)$-dimensional Hausdorff measure or $G \cap \partial\Omega = \varnothing$. We also assume that the cells $G \in \Omega^{(k)}$ are star-shaped in the sense that for each $G \in \Omega^{(k)}$ there is a center $p_G \in G$ of G and a continuous function ρ_G defined on the unit sphere S^{d-1} in $\mathbb{R}^d$ with values in $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ such that

$$G = \left\{p_G + rs \mid s \in S^{d-1},\, 0 \leq r < \rho_G(s)\right\}. \tag{2}$$

Denoting

$$R_G = 2 \max_{s\in S^{d-1}} \rho_G(s), \quad r_G = 2 \min_{s\in S^{d-1}} \rho_G(s) \tag{3}$$

we assume that the cell partitions $\Omega^{(k)}$, $k \in \mathbb{N}$, are shape regular in the sense that

$$\frac{R_G}{r_G} \leq \gamma \qquad \forall G \in \Omega^{(k)} \quad \forall k \in \mathbb{N} \tag{4}$$

holds with some constant $\gamma \geq 1$.

Introducing the subset of invariant cells

$$\Omega^{(k)}_\infty = \left\{G \in \Omega^{(k)} \mid G \in \mathcal{G}^{(j)}\ \forall j > k\right\}$$

we define the maximal size

$$d_k = \max\left\{R_G \mid G \in \Omega^{(k)}\backslash\Omega^{(k)}_\infty\right\} \tag{5}$$

of cells $G \in \Omega^{(k)}$ to be divided on higher levels. Hence, $R_G \leq d_k$ for all $G \in \Omega^{(k)}\backslash\Omega^{(k)}_\infty$. Observe that d_k is monotonically decreasing in $k \in \mathbb{N}$. We assume

$$d_k \to 0 \quad \text{for} \quad k \to \infty. \tag{6}$$

Let $|M| \in \mathbb{N} \cup \{+\infty\}$ stand for the number of elements of some set M. Denoting

$$(x,y) = \{x + s(y-x) \mid s \in (0,1)\},$$

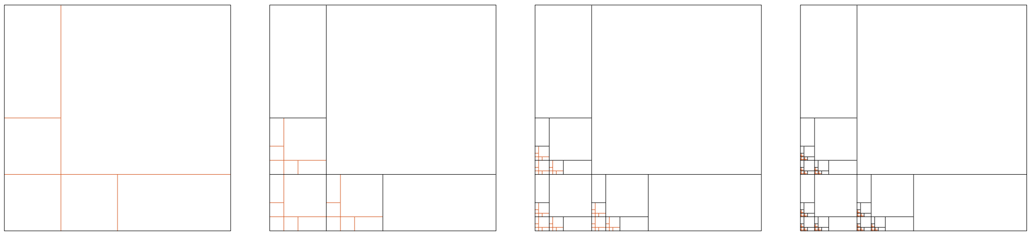

FIGURE 1. Highly localized interface network in $d = 2$ space dimensions: $\Gamma^{(1)} = \Gamma_1$ (red) and $\Gamma^{(k)}$ with Γ_k (red) for $k = 2, 3, 4$.

we also assume that for each fixed $k \in \mathbb{N}$ and all $j \in \mathbb{N}$ with $j > k$, there is a constant $C_{k,j} \geq 0$ such that

$$|(x,y) \cap G \cap \Gamma_j| \leq C_{k,j} \qquad \forall G \in \Omega^{(k)} \tag{7}$$

holds for almost all x, $y \in \Omega$. We set $C_1 = 1$, $C_j = C_{1,j}$, $j = 2, \dots,$ and

$$r_k = \sup_{j>k} \frac{C_{k,j}}{C_{1,j}}, \qquad k \in \mathbb{N}. \tag{8}$$

We finally assume that the interface networks $\Gamma^{(k)}$ are self-similar in the sense that

$$r_k C_k \leq C_0, \qquad \forall k \in \mathbb{N} \tag{9}$$

holds with some constant C_0.

As an example, we consider a highly localized interface network in $d = 2$ space dimensions. Let $\Omega = (0,1)^2$ be the unit square and $\{e_1, e_2\}$ denote the canonical basis in $\mathbb{R}^2$. Then the interface networks $\Gamma^{(k)}$, $k \in \mathbb{N}$, are inductively constructed as follows. Let

$$\Gamma^{(1)} = \Gamma_1 = \{\tfrac{1}{4}e_1 + (0,e_2)\} \cup \{\tfrac{1}{4}e_2 + (0,e_1)\} \cup \{\tfrac{1}{2}e_1 + (0,\tfrac{1}{4}e_2)\} \cup \{\tfrac{1}{2}e_2 + (0,\tfrac{1}{4}e_1)\}.$$

For given $\Gamma^{(k)}$, $k \geq 1$, we define

$$\tilde{\Gamma}_{k+1} = \Gamma^{(k)} \cup \{e_1 + \Gamma^{(k)}\} \cup \{e_2 + \Gamma^{(k)}\}$$

and set $\Gamma_{k+1} = \frac{1}{4}\tilde{\Gamma}_{k+1} \backslash \Gamma^{(k)}$. See Figure 1 for an illustration. The resulting interface network is self-similar by construction which can be directly extended to $d = 3$ space dimensions. We have $d_k = \sqrt{2}\, 4^{-k}$, $C_k = 2^k + 2^{k-1} - 2$ and $C_{k,l} = C_{l-k+1}$, $k = 2, \dots$. Thus $r_k = 2^{1-k}$ and (9) holds with $C_0 = 3$.

2.2. **Fractal function spaces.** For each fixed $k \in \mathbb{N}$, we introduce the space of piecewise smooth functions

$$\mathcal{C}^1_{k,0}(\Omega) = \left\{ v : \overline{\Omega} \backslash \Gamma^{(k)} \to \mathbb{R} \;\middle|\; v|_G \in C^1(\overline{G}) \; \forall G \in \Omega^{(k)} \text{ and } v|_{\partial\Omega} \equiv 0 \right\}$$

on $\Omega \backslash \Gamma^{(k)}$. Let $j = 1, \dots, k$. As Γ_j is piecewise affine, there is a normal ν_ξ to Γ_j at almost all $\xi \in \Gamma_j$ and we fix the orientation of ν_ξ such that $\nu_\xi \cdot e_m > 0$ with $m = \min\{i = 1, \dots, d \mid \nu_\xi \cdot e_i \neq 0\}$, and $\{e_1, \dots, e_d\}$ denotes the canonical basis of $\mathbb{R}^d$. For $\xi \in \Gamma^{(k)}$ such that ν_ξ exists and for $x \neq y \in \mathbb{R}^d$ such that $(x-y) \cdot \nu_\xi \neq 0$, the jump of $v \in C^1_{k,0}(\Omega)$ across Γ_j at ξ in the direction $y - x$ is defined by

$$[\![v]\!]_{x,y}(\xi) = \lim_{s \downarrow 0} \left(v\left(\xi + s(y-x)\right) - v\left(\xi - s(y-x)\right) \right).$$

Up to the sign, $[\![v]\!]_{x,y}(\xi)$ is equal to the normal jump of $v \in C^1_{k,0}(\Omega)$

$$[\![v]\!](\xi) := [\![v]\!]_{\xi-\nu_\xi,\xi+\nu_\xi}(\xi).$$

For some fixed material constant $\mathfrak{c} > 0$, that, e.g., determines the growth of resistance to jumps with increasing fracturing, and the geometrical constant $C_j = C_{1,j}$ taken from (7), we introduce the scalar product

$$\langle v,\, w\rangle_k = \int_{\Omega\setminus\Gamma^{(k)}} \nabla v\cdot\nabla w\; dx + \sum_{j=1}^{k} (1+\mathfrak{c})^j\, C_j \int_{\Gamma_j} [\![v]\!][\![w]\!]\; d\Gamma_j,\quad v, w \in C^1_{k,0}(\Omega), \tag{10}$$

with the associated norm $\|v\|_k = \langle v,\, v\rangle_k^{1/2}$. Observe that $(1+\mathfrak{c})^j$ generates an exponential scaling of the resistance to jumps across Γ_j.

Standard completion of $\mathcal{C}^1_{k,0}(\Omega)$ leads to a hierarchy of k-scale Hilbert spaces

$$\mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_k, \qquad k \in \mathbb{N},$$

with the scalar products $\langle\cdot,\cdot\rangle_k$ and dense subspaces $\mathcal{C}^1_{k,0}(\Omega) \subset \mathcal{H}_k$, $k \in \mathbb{N}$. A limiting fractal Hilbert space $\mathcal{H}$ with scalar product

$$\langle v,\, w\rangle = \int_{\Omega\setminus\Gamma} \nabla v\cdot\nabla w\; dx + \sum_{j=1}^{\infty} (1+\mathfrak{c})^j\, C_j \int_{\Gamma_j} [\![v]\!][\![w]\!]\; d\Gamma_j,\quad v, w \in \mathcal{H}, \tag{11}$$

and associated norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ is obtained by completion of $\bigcup_{k\in\mathbb{N}}\mathcal{H}_k$. We recall the main properties of $\mathcal{H}$ for later use and refer to [22] for details.

The smooth subspaces $(\mathcal{C}^1_{k,0}(\Omega))_{k\in\mathbb{N}}$, and thus the finite-scale spaces $(\mathcal{H}_k)_{k\in\mathcal{H}}$, are dense in $\mathcal{H}$ in the sense that for any $v, w \in \mathcal{H}$ there are sequences $(v_k)_{k\in\mathbb{N}}, (w_k)_{k\in\mathbb{N}} \subset (\mathcal{C}^1_{k,0}(\Omega))_{k\in\mathbb{N}}$, i.e., with $v_k, w_k \in \mathcal{C}^1_{k,0}(\Omega)$ for all $k \in \mathbb{N}$, such that

$$\|v - v_k\| \to 0, \qquad \langle v_k, w_k\rangle_k \to \langle v, w\rangle \quad \text{for } k \to \infty. \tag{12}$$

Observe that

$$\Omega\setminus\Gamma = \Omega \cap \Big(\bigcup_{j=1}^{\infty} \Gamma_j\Big)^{\complement} \subset \Omega\setminus\Gamma^{(k)}$$

is Lebesgue measurable so that the space $L^2(\Omega\setminus\Gamma)$ implicitly appearing in (11) is well-defined. For the definition of generalized jumps $[\![v]\!]$, $v \in \mathcal{H}$, also appearing in (11), we introduce the sequence space $(L^2(\Gamma_j))_{j\in\mathbb{N}}$ equipped with the weighted norm

$$\|z\|_\Gamma = \left(\sum_{j=1}^{\infty} (1+\mathfrak{c})^j C_j\, \|z_j\|^2_{0,\Gamma_j}\right)^{\frac{1}{2}}, \quad z = (z_j)_{j\in\mathbb{N}} \in (L^2(\Gamma_j))_{j\in\mathbb{N}},$$

with $\|\cdot\|_{0,\Gamma_j}$ denoting the usual norm in $L^2(\Gamma_j)$. Then, for each $v \in \mathcal{H}$ and each sequence $(v_k)_{k\in\mathbb{N}}$ with $v_k \in \mathcal{H}_k$, the limits

$$\nabla v = \lim_{k\to\infty} \nabla v_k \quad \text{in } L^2(\Omega\setminus\Gamma) \quad \text{and} \quad [\![v]\!] = \lim_{k\to\infty} [\![v_k]\!] \quad \text{in } (L^2(\Gamma_k))_{k\in\mathbb{N}}$$

exist and are called weak gradient ∇v and generalized jump $[\![v]\!]$ of v, respectively. We have the Green's formula

$$\int_\Omega v\nabla\cdot\varphi\; dx = -\int_{\Omega\setminus\Gamma} \nabla v\cdot\varphi\; dx + \sum_{j=1}^{\infty} \int_{\Gamma_j} [\![v]\!]\varphi\cdot\nu_j\; d\Gamma_j \qquad \forall\varphi \in C_0^\infty(\mathbb{R}^d)^d \tag{13}$$

and the Poincaré-type inequality

$$\|v\|_{0,\Omega} \le C_P \left(|v|^2_{1,\Omega\setminus\Gamma} + \sum_{j=1}^{\infty} (1+\mathfrak{c})^j \, C_j \, \|[\![v]\!]\|^2_{0,\Gamma_j} \right)^{1/2} \tag{14}$$

where $|v|_{1,\Omega\setminus\Gamma} = \|\,|\nabla v|\,\|_{0,\Omega\setminus\Gamma}$ and the constant C is bounded in terms of $(1+\frac{1}{\mathfrak{c}})\mathrm{diam}(\Omega)$. Moreover, the continuous embedding $\mathcal{H} \subset H^s(\Omega)$, $s \in [0, \frac{1}{2})$, into Sobolev-Slobodeckij spaces $H^s(\Omega)$ (see, e.g. [42, 43]) allows to identify $\mathcal{H}$ with a subspace of $\bigcap_{s\in[0,\frac{1}{2})} H^s(\Omega)$.

2.3. **Fractal interface problem.** We consider the fractal interface problem

$$u \in \mathcal{H}: \qquad a(u,v) = (f,v) \qquad \forall v \in \mathcal{H} \tag{15}$$

with $f \in L^2(\Omega)$, the usual scalar product $(\cdot,\cdot)$ in $L^2(\Omega)$, and the bilinear form

$$a(v,w) = \int_{\Omega\setminus\Gamma} A\nabla v \cdot \nabla w \; dx + \sum_{j=1}^{\infty} (1+\mathfrak{c})^j \, C_j \int_{\Gamma_j} B[\![v]\!][\![w]\!] \; d\Gamma_j, \quad v, w \in \mathcal{H}, \tag{16}$$

involving the functions $A : \Omega\setminus\Gamma \to \mathbb{R}^{d\times d}$ and $B : \Gamma = \bigcup_{j=1}^{\infty} \Gamma_j \to \mathbb{R}$. We assume that $A(x) \in \mathbb{R}^{d\times d}$ is symmetric for all $x \in \Omega\setminus\Gamma$ and has the properties

$$\alpha_0|\xi|^2 \le A(x)\xi\cdot\xi, \qquad |A(x)\xi\cdot\eta| \le \alpha_1|\xi||\eta|, \qquad \forall \xi,\eta \in \mathbb{R}^d \qquad \forall x \in \Omega\setminus\Gamma \tag{17}$$

with positive constants $\alpha_0, \alpha_1 \in \mathbb{R}$. We also assume that B satisfies

$$0 < \beta_0 \le B(x) \le \beta_1 \qquad \forall x \in \Gamma \tag{18}$$

with constants $\beta_0, \beta_1 \in \mathbb{R}$. The assumptions (17) and (18) imply that $a(\cdot,\cdot)$ is symmetric and elliptic in the sense that

$$\mathfrak{a}\,\|v\|^2 \le a(v,v), \qquad |a(v,w)| \le \mathfrak{A}\,\|v\|\,\|w\| \qquad \forall v, w \in \mathcal{H} \tag{19}$$

holds with $\mathfrak{a} = \min\{\alpha_0, \beta_0\}$ and $\mathfrak{A} = \min\{\alpha_1, \beta_1\}$. Hence, $a(\cdot,\cdot)$ is a scalar product in $\mathcal{H}$ and the associated energy norm $\|\cdot\|_a = a(\cdot,\cdot)^{1/2}$ is equivalent to $\|\cdot\|$.

Note that we have $(f,\cdot) \in \mathcal{H}^{-1}$ due to the continuous embedding (14) of $\mathcal{H}$ into $L^2(\Omega)$. Hence, well-posedness follows directly from the Lax-Milgram lemma.

Proposition 2.1. *The fractal interface problem* (15) *admits a unique solution* $u \in \mathcal{H}$ *satisfying the stability estimate*

$$\|u\| \le \tfrac{1}{\mathfrak{a}}\, C_P \, \|f\|_{0,\Omega} \, . \tag{20}$$

We now focus on the numerical approximation of the solution u of the fractal interface problem (15).

3. Finite-scale discretization

3.1. **Finite scales.** As $\mathcal{H}$ is characterized by limiting properties of the k-scale spaces $\mathcal{H}_k$, $k \in \mathbb{N}$, it is natural to consider the interface problems

$$u_{\mathcal{H}_k} \in \mathcal{H}_k: \qquad a(u_{\mathcal{H}_k}, v) = (f,v) \qquad \forall v \in \mathcal{H}_k \tag{21}$$

on finite scales $k \in \mathbb{N}$. Note that

$$a(v,w) = a_k(v,w) = \int_{\Omega\setminus\Gamma} A\nabla v \cdot \nabla w \; dx + \sum_{j=1}^{k} (1+\mathfrak{c})^j \, C_j \int_{\Gamma_j} B[\![v]\!][\![w]\!] \; d\Gamma_j, \quad v, w \in \mathcal{H}_k. \tag{22}$$

While the Lax-Milgram lemma implies existence and uniqueness, a straightforward error estimate follows from Céa's lemma.

Proposition 3.1. *For each $k \in \mathbb{N}$ the k-scale interface problem* (21) *admits a unique solution $u_{\mathcal{H}_k} \in \mathcal{H}_k$ satisfying the error estimate*

$$\|u - u_{\mathcal{H}_k}\| \le \frac{\mathfrak{A}}{\mathfrak{a}} \inf_{v \in \mathcal{H}_k} \|u - v\| . \tag{23}$$

In the light of (12), this directly implies convergence

$$\|u - u_{\mathcal{H}_k}\| \to 0 \qquad \text{for } k \to \infty. \tag{24}$$

In the case $A(x) = I$ and (quite restrictive) shape regularity conditions on $G \in \Omega^{(k)}$, $k \in \mathbb{N}$, there are even exponential error estimates of the form

$$\|u - u_{\mathcal{H}_k}\| \le C \|f\|_{0,\Omega} \frac{1}{\mathfrak{c}} (1 + \mathfrak{c})^{-(k-1)} \tag{25}$$

with C depending only on the space dimension d, the Poincaré-type constant in (14), and shape regularity [22, Theorem 4.2].

3.2. **Finite elements on finite scales.** Let $\mathcal{T}^{(0)}$ be a partition of Ω into simplices with maximal diameter $h_0 > 0$, which is regular in the sense that the intersection of two different simplices T, $T' \in \mathcal{T}^{(0)}$ is either a common n-simplex for some $n = 0, \dots, d-1$ or empty. The shape regularity $\sigma > 0$, i.e., the maximal ratio of the radii of the circumscribed and the inscribed ball of $T \in \mathcal{T}^{(0)}$ is preserved under uniform regular refinement [5, 6]. We assume that the sequence of partitions resulting from successive uniform regular refinement of $\mathcal{T}^{(0)}$ resolves the interface network in the sense that for each fixed $k \in \mathbb{N}$ there is a partition $\mathcal{T}^{(k)}$, as obtained by a finite number of refinement steps, such that the interfaces Γ_j, $j = 1, \dots, k$, can be represented by faces of simplices $T \in \mathcal{T}^{(k)}$, i.e.

$$\Gamma^{(k)} = \bigcup_{E \in \mathcal{E}_\Gamma^{(k)} \subset \mathcal{E}^{(k)}} E \tag{26}$$

holds with a suitable subset $\mathcal{E}_\Gamma^{(k)}$ of the set $\mathcal{E}^{(k)}$ of faces of simplices $T \in \mathcal{T}^{(k)}$. In particular, this implies that for all $G \in \Omega^{(k)}$ the set $\mathcal{T}_G^{(k)} = \{T \in \mathcal{T}^{(k)} \mid T \subset \overline{G}\}$ is a local partition of G and that the maximal diameter h_k of $T \in \mathcal{T}^{(k)}$ is bounded by the maximal diameter d_k of $G \in \Omega^{(k)}$. We additionally assume that $\Omega^{(k)}$ is not over-resolved in the sense that d_k can be uniformly bounded by h_k, i.e., that

$$\delta d_k \le h_k \le d_k, \quad k \in \mathbb{N}, \tag{27}$$

holds with a constant $\delta > 0$ independent of $k \in \mathbb{N}$. Let $\mathcal{N}_G^{(k)}$ denote the set of vertices of $T \in \mathcal{T}_G^{(k)}$ that are not located on the boundary $\partial\Omega$. Observe that each vertex located on an interface Γ_j with two (or more) adjacent cells G, $G' \in \Omega^{(k)}$, gives rise to two (or more) different nodes $p \in \mathcal{N}_G^{(k)}$ and $p' \in \mathcal{N}_{G'}^{(k)}$. For each $G \in \Omega^{(k)}$, we introduce the local finite element space $\mathcal{S}_k(G)$ of piecewise affine functions with respect to $\mathcal{T}_G^{(k)}$ that are vanishing on $\partial G \cap \partial\Omega$. The space $\mathcal{S}_k(G)$ is spanned by the standard nodal basis $\lambda_p^{(k)}$, $p \in \mathcal{N}_G^{(k)}$. Extending these functions by zero from $\overline{G}$ to Ω, we define the broken finite element space

$$\mathcal{S}_k = \operatorname{span}\left\{\lambda_p^{(k)} \mid p \in \mathcal{N}^{(k)}\right\}, \qquad \mathcal{N}^{(k)} = \bigcup_{G \in \Omega^{(k)}} \mathcal{N}_G^{(k)}.$$

The discretization of the k-scale interface problem (21) with respect to $\mathcal{S}_k$ is given by

$$u_{\mathcal{S}_k} \in \mathcal{S}_k : \qquad a_k(u_{\mathcal{S}_k}, v) = (f, v) \qquad \forall v \in \mathcal{S}_k \,. \tag{28}$$

with $a_k(\cdot,\cdot)$ taken from (22). Existence and uniqueness of the resulting finite element approximation $u_{\mathcal{S}_k}$ of $u_{\mathcal{H}_k} \in \mathcal{H}_k$ follows from the Lax-Milgram lemma. Convergence is implied by Céa's lemma together with (24).

Proposition 3.2. *The finite element approximations $(u_{\mathcal{S}_k})_{k\in\mathbb{N}}$ converge to the solution u of* (15) *in the sense that for each $\varepsilon > 0$ there is a sufficiently large $k \in \mathbb{N}$ such that*

$$\|u_{\mathcal{H}_k} - u_{\mathcal{S}_k}\| < \varepsilon. \tag{29}$$

For each fixed $k \in \mathbb{N}$, the expected order of convergence is obtained under suitable regularity conditions on $u_{\mathcal{H}_k}$.

Proposition 3.3. *Let $k \in \mathbb{N}$ and assume that $u_{\mathcal{H}_k}|_G \in H^r(G)$ $\forall G \in \Omega^{(k)}$ with $r = 2$, if $d = 1, 2$, and $r = 2 + \varepsilon$, $\varepsilon > 0$, if $d = 3$. Then the a priori error estimate*

$$\|u_{\mathcal{H}_k} - u_{\mathcal{S}_k}\| \le C h_k \sum_{G\in\Omega^{(k)}} \|u_{\mathcal{H}_k}\|_{H^r(G)} \tag{30}$$

holds with a constant C depending only on the shape regularity σ of $\mathcal{T}^{(k)}$.

Proof. The proof follows from well-known interpolation error estimates [13]. □

A priori error estimates for the discretization error $\|u - u_{\mathcal{S}_k}\|$ can be obtained by combining (30) with exponential convergence of $u_{\mathcal{H}_k}$ (see [22, Theorem 4.2]). In section 5 below, we will discuss multiscale modifications of classical finite elements that provide optimal a priori error estimates directly.

4. Projections

This section is devoted to the construction of stable, surjective projections

$$\Pi_k : \quad \mathcal{H} \to \mathcal{S}_k, \qquad k \in \mathbb{N},$$

satisfying an approximation property. To this end, we extend well-known arguments [8, 11, 46] to the present situation.

4.1. Local Poincaré-type inequalities.

This subsection is devoted to local Poincaré-type inequalities on (subsets of) the cells $G \subset \Omega^{(k)} \backslash \Omega^{(k)}_\infty$, which, in contrast to cells from $\Omega^{(k)}_\infty$, have non-empty intersection with Γ_j for $j > k$. We will frequently use the notation

$$B(G, R) = \{p_G + rs \mid s \in S^{d-1}, 0 \le r \le R\}$$

for $G \in \Omega^{(k)}$ and some $R > 0$.

Differences can be expressed in terms of derivatives and intermediate jumps.

Lemma 4.1.
Let $k \in \mathbb{N}$, $G \in \Omega^{(k)} \backslash \Omega^{(k)}_\infty$, $x, y \in G$ with $(x,y) \subset G$ and $\left|(x,y) \cap \Gamma^{(k)}\right| < \infty$, and $K > k$. Then we have

$$|v(x) - v(y)|^2 \le \left(1 + \tfrac{1}{\mathfrak{c}}\right) |x-y|^2 \left(\int_0^1 |\nabla v\,(x + t(y-x))|\; dt \right)^2$$
$$+ \left(1 + \tfrac{1}{\mathfrak{c}}\right) \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k}\, C_{k,j} \sum_{\xi \in (x,y)\cap\Gamma_j} [\![v]\!]^2(\xi) \qquad \forall v \in \mathcal{C}^1_{K,0}(\Omega),$$

where $\nabla v\,(x + t(y-x))$ is understood to be zero, if $x + t(y-x) \in \Gamma^{(K)}$.

Proof. The assertion follows in the same way as [22, Lemma 3.5]. □

The next lemma provides control of intermediate jumps in terms of integrals along interfaces.

Lemma 4.2. *Let $k \in \mathbb{N}$, $B = B(G, r_G) \subset G \in \Omega^{(k)} \backslash \Omega^{(k)}_\infty$, and $K \ge j > k$. Then*

$$\int_B \int_B \sum_{\xi \in (x,y)\cap\Gamma_j} [\![v]\!]^2(\xi)\; dx\, dy \le C|B|\, r_G \int_{\Gamma_j \cap B} [\![v]\!]^2\; d\Gamma_j \qquad \forall v \in \mathcal{C}^1_{K,0}(\Omega) \tag{31}$$

holds with a constant C only depending on the space dimension d.

Proof. By similar arguments as in the proof of [22, Theorem 3.6], the transformation of variables $(x,y) = \Psi(x,\eta) = (x, x+\eta)$ leads to

$$\begin{aligned} \int_B \int_B \sum_{\xi \in (x,y)\cap\Gamma_j} [\![v]\!]^2(\xi)\; dx\, dy \;&=\; \int_{\{|\eta| \le 2r_G\}} \int_{M(\eta)} \sum_{\xi \in (x, x+\eta)} [\![v]\!]^2(\xi)\; dx\, d\eta \\ \le \int_{\{|\eta| \le 2r_G\}} |\eta| \int_{\Gamma_j \cap B} [\![v]\!]^2\; d\Gamma_j\; d\eta \;&\le\; C|B|\, r_G \int_{\Gamma_j \cap B} [\![v]\!]^2\; d\Gamma_j \end{aligned}$$

with $M(\eta) = \{x \in B \mid x + \eta \in B\}$ and a constant C only depending on the space dimension d. □

We are now ready to prove a Poincaré inequality on balls $B = B(G, r_G) \subset G \in \Omega^{(k)} \backslash \Omega^{(k)}_\infty$. We will use the notation

$$\fint_M v\; dx = \frac{1}{|M|} \int_M v\; dx$$

with suitable subsets $M \subset G$.

Proposition 4.3. *Let $k \in \mathbb{N}$ and $B = B(G, r_G) \subset G \in \Omega^{(k)} \backslash \Omega^{(k)}_\infty$. Then*

$$\left\| v - \fint_B v\; dx \right\|^2_{0,B} \le \left(1 + \tfrac{1}{\mathfrak{c}}\right) C\, r_G \left(r_G\, |v|^2_{1,B\backslash\Gamma} + \sum_{j=k+1}^{\infty} (1+\mathfrak{c})^{j-k}\, C_{k,j}\, \|[\![v]\!]\|^2_{0,\Gamma_j \cap B(G,r_G)} \right) \tag{32}$$

holds for all $v \in \mathcal{H}$ with a constant C depending only on the space dimension d.

Proof. As $\{v \in \mathcal{C}^1_{K,0}(\Omega) \mid K \in \mathbb{N}\}$ is dense in $\mathcal{H}$ and the quantities in (32) are depending continuously on v, it is sufficient to prove the assertion for $v \in \mathcal{C}^1_{K,0}(\Omega)$. Let $v \in \mathcal{C}^1_{K,0}(\Omega)$ with arbitrary $K > k$ and note that the triangle inequality and Fubini's theorem imply

$$\left\| v - \fint_B v^2\; dx \right\|^2_{0,B} = \int_B \left| \fint_B v(x) - v(y)\; dy \right|^2\; dx \le \fint_B \int_B |v(x) - v(y)|^2\; dx\, dy. \tag{33}$$

Lemma (4.1) and the Cauchy-Schwarz inequality provide

$$
\begin{aligned}
|v(x)-v(y)|^2 \le\ & \left(1+\tfrac{1}{\mathfrak{c}}\right)|x-y|^2 \int_0^1 |\nabla v\left(x+t(y-x)\right)|^2 \, dt \\
& +\left(1+\tfrac{1}{\mathfrak{c}}\right) \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k} C_{k,j} \sum_{\xi\in(x,y)\cap\Gamma_j} [\![v]\!]^2(\xi).
\end{aligned}
$$

Treating the gradient part in the same way as in well-known proofs of the classical Poincaré inequality on balls (cf, e.g., [16, Lemma 4.1]), we obtain

$$
\int_B \fint_B |x-y|^2 \int_0^1 |\nabla v\left(x+t(y-x)\right)|^2 \, dt \, dy \, dx \le c r_G^2 \, |v|^2_{1,B\backslash\Gamma^{(K)}} \tag{34}
$$

with a positive constant c depending only on the space dimension d. Application of Lemma 4.2 to the jump term provides

$$
\int_B \fint_B \sum_{\xi\in(x,y)\cap\Gamma_j} [\![v]\!]^2(\xi) \, dy \, dx \le c' r_G \int_{\Gamma_j\cap B} [\![v]\!]^2 \, d\Gamma_j \tag{35}
$$

with a constant c' depending only on d. Inserting (34) and (35) into (33) concludes the proof. $\square$

The lines of proof of Proposition 4.3 carry over to the following trace analogue on spheres. We refer to [38] for details.

Lemma 4.4. *Let $k\in\mathbb{N}$, $B=B(G,r_G)\subset G\in\Omega^{(k)}\backslash\Omega^{(k)}_\infty$, and $K>k$. Then*

$$
\left\| v - \fint_B v \, dx \right\|^2_{0,\partial B} \le \left(1+\tfrac{1}{\mathfrak{c}}\right) C \left(r_G \, |v|^2_{1,B\backslash\Gamma^{(K)}} + \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k} C_{k,j} \, \|[\![v]\!]\|^2_{0,\Gamma_j\cap B} \right) \qquad \forall v\in\mathcal{C}^1_{K,0}(\Omega)
$$

holds with a constant C depending only on the space dimension d.

The following lemmata prepare the extension of the Poincaré inequality from balls to cells $G\in\Omega^{(k)}\backslash\Omega^{(k)}_\infty$. We start by controlling intermediate jumps in $G\backslash B(G,r_G)$.

Lemma 4.5. *Let $k\in\mathbb{N}$, $B=B(G,r_G)\subset G\in\Omega^{(k)}\backslash\Omega^{(k)}_\infty$, $M=G\backslash B(G,r_G)\subset G$, and $K\ge j>k$. Then we have*

$$
\int_M \sum_{\xi\in(p_G,y)\cap\Gamma_j\cap M} [\![v]\!]^2(\xi) \, dy \le \tfrac{\gamma^{d-1}}{d} R_G \int_{\Gamma_j\cap M} [\![v]\!]^2 \, d\Gamma_j \qquad \forall v\in\mathcal{C}^1_{K,0}(\Omega).
$$

Proof. Assume $p_G=0$ without loss of generality and let $v\in\mathcal{C}^1_{K,0}(\Omega)$ with arbitrary $K\ge j>k$. As the interfaces are piecewise affine, $\Gamma_j=\bigcup_{i\in I}\Gamma_{j,i}$ can be represented as a countable union of its affine components $\Gamma_{j,i}$, $i\in I\subset\mathbb{N}$. For almost all $y\in M$, the set $(0,y)\cap\Gamma_j\cap M$ is finite and we set

$$
\sum_{\xi\in(0,y)\cap\Gamma_j\cap M} [\![v]\!]^2(\xi) = \sum_{i\in I} \varphi_i(y) \tag{36}
$$

denoting

$$
\varphi_i(y) = [\![v]\!]^2(\xi), \quad \text{if} \quad (0,y)\cap\Gamma_{j,i}\cap M = \xi\in\mathbb{R}^d,
$$

and $\varphi_i(y) = 0$, if there is no intersection of $(0, y)$ with $\Gamma_{j,i}$ in M. We extend φ_i by zero to the ball $B(G, R_G) \supset G \supset M$. This leads to

$$\int_M \varphi_i(y)\, dy = \int_{B(G,R_G)\setminus B(G,r_G)} \varphi_i(y)\, dy = \int_{S^{d-1}} \int_{r_G}^{R_G} \varphi_i(\Psi(r,s))\, r^{d-1}\, dr\, ds, \tag{37}$$

where Ψ stands for the transformation from d-dimensional spherical to Cartesian coordinates. We introduce the section $S_i = \{s \in S^{d-1} \mid (0, R_G s) \cap \Gamma_{j,i} \cap M \neq \varnothing\}$ of directions that contribute to the integral in (37), and $\partial B_i = \{R_G s \mid s \in S_i\}$ is the corresponding subset of the boundary $\partial B(G, R_G)$ of $B(G, R_G)$. If these sets are empty or if $\Gamma_{j,i}$ is normal to ∂B_i, i.e., ∂B_i is a singleton, then the integral in (37) vanishes. Otherwise, there is an explicit parametrization $\xi(s) = \Psi(g_i(s)R_G, s)$ of $\Gamma_{j,i} \cap M$ over ∂B_i with a smooth function $g_i : \partial B_i \to (0, 1]$ and, by definition,

$$0 \le \varphi_i(\Psi(r,s)) \le [\![v]\!]^2\, (\xi(s)), \qquad s \in S_i.$$

Therefore, integration over r and substitution yields

$$\int_{S^{d-1}} \int_{r_G}^{R_G} \varphi_i(\Psi(r,s))\, r^{d-1}\, dr\, ds \le \tfrac{1}{d}\, R_G \int_{S_i} [\![v]\!]^2(\xi(s)) R_G^{d-1}\, ds \le \tfrac{1}{d}\, R_G \int_{\partial B_i} [\![v]\!]^2(\xi(s))\, ds \tag{38}$$

and $g_i(s)R_G \ge r_G$, $s \in S_i$, together with shape regularity $R_G \le \gamma r_G$ implies

$$\int_{\partial B_i} [\![v]\!]^2(\xi(s))\, ds \le \gamma^{d-1} \int_{\partial B_i} [\![v]\!]^2(\xi(s)) g_i^{d-2} \sqrt{g_i^2 + |\nabla g_i|^2 R_G^2}\, ds = \gamma^{d-1} \int_{\Gamma_{j,i}\cap M} [\![v]\!]^2\, d\Gamma_{j,i}. \tag{39}$$

In light of (36), (37), (38), and (39), summation over $i \in I$ finally leads to

$$\begin{aligned}
\int_M \sum_{\xi \in (0,y)\cap\Gamma_j\cap M} [\![v]\!]^2(\xi)\, dy &= \sum_{i\in I} \int_M \varphi_i(y)\, dy \\
&\le \frac{\gamma^{d-1}}{d}\, R_G \sum_{i\in I} \int_{\Gamma_{j,i}\cap M} [\![v]\!]^2\, d\Gamma_{j,i} = \frac{\gamma^{d-1}}{d}\, R_G \int_{\Gamma_j \cap M} [\![v]\!]^2\, d\Gamma_j.
\end{aligned}$$

□

The next lemma is an analogue of Lemma 4.1 in [46].

Lemma 4.6. *Let $k \in \mathbb{N}$, $G \in \Omega^{(k)} \setminus \Omega^{(k)}_\infty$, and $K > k$. Then*

$$\begin{aligned}
\|v\|^2_{0,G} \le \|v\|_{0,B(G,r_G)} &+ C R_G \|v\|^2_{0,\partial B(G,r_G)} \\
&+ \left(1 + \tfrac{1}{\mathfrak{c}}\right) C R_G \left(R_G |v|^2_{1,G\setminus\Gamma^{(K)}} + \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k}\, C_{k,j}\, \|[\![v]\!]\|^2_{0,\Gamma_j \cap (G\setminus B(G,r_G))} \right)
\end{aligned}$$

holds for all $v \in \mathcal{C}^1_{K,0}(\Omega)$ with a constant C depending only on the dimension d and shape regularity γ of $\Omega^{(k)}$.

Proof. Utilizing

$$\|v\|^2_{0,G} = \|v\|^2_{0,B(G,r_G)} + \|v\|^2_{0,G\setminus B(G,r_G)}$$

we have to derive a suitable bound for $\|v\|^2_{0,G\setminus B(G,r_G)}$. We set $M = G\setminus B(G, r_G)$ for notational convenience and assume $p_G = 0$ without loss of generality. Transformation to spherical coordinates then yields the splitting

$$
\begin{aligned}
\|v\|_{0,M}^2 &= \int_{S^{d-1}} \int_{r_G}^{\rho_G(s)} r^{d-1} |v(rs)|^2 \, dr\, ds \\
&= \int_{S^{d-1}} \int_{r_G}^{\rho_G(s)} r^{d-1} |v(rs) - v(r_G s) + v(r_G s)|^2 \, dr\, ds \\
&\le \underbrace{2\int_{S^{d-1}} \int_{r_G}^{\rho_G(s)} r^{d-1} |v(rs) - v(r_G s)|^2 \, dr\, ds}_{=:I_1} + \underbrace{2\int_{S^{d-1}} \int_{r_G}^{\rho_G(s)} r^{d-1} |v(r_G s)|^2 \, dr\, ds}_{=:I_2} .
\end{aligned}
$$

We will provide suitable bounds for these two parts and first consider I_1. Lemma 4.1 leads to

$$
\begin{aligned}
I_1 \le\ & 2\left(1+\tfrac{1}{\mathfrak{c}}\right) \int_{S^{d-1}} \int_{r_G}^{\rho_G(s)} r^{d-1} \left(\int_{r_G}^{r} |\nabla v(zs)| \, dz \right)^2 dr\, ds \\
& + 2\left(1+\tfrac{1}{\mathfrak{c}}\right) \int_{S^{d-1}} \int_{r_G}^{\rho_G(s)} r^{d-1} \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k} C_{k,j} \sum_{\xi \in (r_G s, rs)\cap\Gamma_j} [\![v]\!]^2(\xi)\, dr\, ds.
\end{aligned} \tag{40}
$$

By the Cauchy-Schwarz inequality and straightforward computations, as in the proof of [46, Lemma 4.1], the gradient term in (40) can be bounded according to

$$
\begin{aligned}
& \int_{S^{d-1}} \int_{r_G}^{\rho_G(s)} r^{d-1} \left| \int_{r_G}^{r} \nabla v(zs)\, dz \right|^2 dr\, ds \\
\le\ & \int_{S^{d-1}} \left(\int_{r_G}^{\rho_G(s)} z^{d-1} |\nabla v(zs)|^2 \, dz \right) \left(\int_{r_G}^{\rho_G(s)} r^{d-1} \int_{r_G}^{r} z^{1-d}\, dz\, dr \right) ds \\
\le\ & c R_G^2 |v|_{1,M\setminus\Gamma^{(K)}}^2
\end{aligned} \tag{41}
$$

with a constant c depending only on the dimension d and shape regularity $\gamma \ge \frac{R_G}{r_G}$ of $\Omega^{(k)}$. In order to bound the jump contributions in (40) in terms of integrals along interfaces, we apply Lemma 4.5 to obtain

$$
\begin{aligned}
& \int_{S^{d-1}} \int_{r_G}^{\rho_G(s)} r^{d-1} \sum_{\xi\in(r_G s, rs)\cap\Gamma_j} [\![v]\!]^2(\xi)\, dr\, ds = \int_M \sum_{\xi\in(0,y)\cap\Gamma_j\cap M} [\![v]\!]^2(\xi)\, dy \\
& \le \tfrac{\gamma^{d-1}}{d} R_G \int_{\Gamma_j\cap M} [\![v]\!]^2 \, d\Gamma_j = \tfrac{\gamma^{d-1}}{d} R_G \, \|[\![v]\!]\|_{0,\Gamma_j\cap M}^2 .
\end{aligned} \tag{42}
$$

Inserting $M = G\setminus B(G, r_G) \subset G$, the estimates (41) and (42) provide

$$
I_1 \le 2c\left(1+\tfrac{1}{\mathfrak{c}}\right) \left(R_G^2 |v|_{1,G\setminus\Gamma^{(K)}}^2 + \tfrac{\gamma^{d-1}}{d} R_G \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k} C_{k,j} \, \|[\![v]\!]\|_{0,\Gamma_j\cap(G\setminus B(G,r_G))}^2 \right). \tag{43}
$$

Straightforward calculation leads to

$$
\begin{aligned}
I_2 &= 2\int_{S^{d-1}} \int_{r_G}^{\rho_G(s)} r^{d-1} |v(r_G s)|^2 \, dr\, ds = 2\int_{S^{d-1}} r_G^{d-1} |v(r_G s)|^2 \int_{r_G}^{\rho_G(s)} \left(\tfrac{r}{r_G}\right)^{d-1} dr\, ds \\
&= 2\int_{S^{d-1}} r_G^{d-1} |v(r_G s)|^2 \, \tfrac{r_G}{d} \left(\left(\tfrac{\rho_G(s)}{r_G}\right)^d - 1 \right) ds \le \tfrac{2}{d} \left(\left(\tfrac{R_G}{r_G}\right)^d - 1 \right) r_G \, \|v\|_{0,\partial B(G,r_G)}^2 .
\end{aligned} \tag{44}
$$

Together with (43) this concludes the proof. □

As a direct extension of Lemma 4.3 in [46], we are now ready to state a local Poincaré inequality on cells $G \in \Omega^k \backslash \Omega_\infty^{(k)}$.

Proposition 4.7.
For every $k \in \mathbb{N}$ and every cell $G \in \Omega^{(k)} \backslash \Omega_\infty^{(k)}$, the local Poincaré inequality

$$\left\| v - \fint_G v\, dx \right\|_{0,G}^2 \le C\left(1 + \tfrac{1}{\mathfrak{c}}\right) d_k \left(d_k \, |v|_{1,G\backslash\Gamma}^2 + \sum_{j=k+1}^{\infty} (1+\mathfrak{c})^{j-k} C_{k,j} \, \|[\![v]\!]\|_{0,\Gamma_j \cap G}^2 \right) \tag{45}$$

holds for all $v \in \mathcal{H}$ with a constant c depending only on the dimension d and shape regularity γ of $\Omega^{(k)}$.

Proof. It is sufficient to show (45) for $v \in \mathcal{C}_{K,0}^1(\Omega)$ with arbitrary $K > k$, and then use a density argument. Observe that $\fint_G v\, dx$ minimizes the functional $\|v - \cdot\|_{0,G}^2$. Denoting $B = B(G, r_G)$, we conclude from Lemma 4.6

$$\begin{aligned}
\left\| v - \fint_G v\, dx \right\|_{0,G}^2 &\le \left\| v - \fint_B v\, dx \right\|_{0,G}^2 \\
&\le \left\| v - \fint_B v\, dx \right\|_{0,B}^2 + C R_G \left\| v - \fint_B v\, dx \right\|_{0,\partial B}^2 \\
&\quad + C R_G \left(1 + \tfrac{1}{\mathfrak{c}}\right) \left(R_G \, |v|_{1,G\backslash\Gamma^{(K)}}^2 + \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k} C_{k,j} \, \|[\![v]\!]\|_{0,\Gamma_j \cap (G\backslash B)}^2 \right).
\end{aligned}$$

Now the assertion follows from the Poincaré inequality on balls stated in Proposition 4.3 together with its trace analogue for spheres Lemma 4.4. ☐

4.2. **A trace lemma.** In order to control the jump contributions in the stability estimates below, we provide some estimates of traces on the interfaces Γ_j of functions $v \in \mathcal{C}_{K,0}^1(\Omega)$ with arbitrary $K \in \mathbb{N}$. For this purpose, we follow the approach by Verfürth [46] and utilize the triangulations $\mathcal{T}^{(k)}$ introduced in Subsection 3.2. The following lemma is a direct extension of [46, Lemma 3.2] and can be shown along the same lines of proof. The additionally arising jump contributions are controlled in a similar way as in Lemma 4.2 and [22, Theorem 3.6]. We refer to [38] for details.

Lemma 4.8. *Let $k \in \mathbb{N}$, $T \in \mathcal{T}^{(k)}$, and $E \in \mathcal{E}^{(k)}$ be a face of T. Then*

$$\|v\|_{0,E}^2 \le c\left(1 + \tfrac{1}{\mathfrak{c}}\right) \left(h_k^{-1} \|v\|_{0,T}^2 + h_k \, |v|_{1,T\backslash\Gamma^{(K)}}^2 + \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k} C_{k,j} \, \|[\![v]\!]\|_{0,\Gamma_j \cap T}^2 \right)$$

holds for all $v \in \mathcal{C}_{K,0}^1(\Omega)$ with $K > k$ and a constant c depending only on the space dimension d and shape regularity σ of $\mathcal{T}^{(k)}$.

Now we are ready to state the desired trace lemma.

Lemma 4.9.
Let $k \in \mathbb{N}$ and $G \in \Omega^{(k)} \backslash \Omega_\infty^{(k)}$ and $l = 1, \ldots, k$. Then

$$\|v\|_{0,\Gamma_l \cap \partial G}^2 \le C\left(1 + \tfrac{1}{\mathfrak{c}}\right) \left(d_k^{-1} \|v\|_{0,G}^2 + d_k \, |v|_{1,G\backslash\Gamma^{(K)}}^2 + \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k} C_{k,j} \, \|[\![v]\!]\|_{0,\Gamma_j \cap G}^2 \right)$$

holds for all $v \in \mathcal{H}_K$ with $K > k$ and a constant C depending only on the space dimension d, shape regularity σ of $\mathcal{T}^{(k)}$ and the constant δ in (27).

Proof. By a density argument, it is sufficient to consider $v \in \mathcal{C}^1_{K,0}(\Omega)$. Let $G \in \Omega^{(k)} \backslash \Omega^{(k)}_\infty$ and recall that $\mathcal{T}^{(k)}_G \subset \mathcal{T}^{(k)}$ is a local partition of $\mathcal{E}^{(k)}_G \subset \mathcal{E}^{(k)}$. Denoting the set faces of simplices $T \in \mathcal{T}^{(k)}_G$ by $\mathcal{E}^{(k)}_G$, select the subset of faces $\mathcal{E}^{(k)}_{\partial G} \subset \mathcal{E}^{(k)}_G$ such that

$$\partial G = \bigcup_{E \in \mathcal{E}^{(k)}_{\partial G}} E.$$

Note that for each $E \in \mathcal{E}^{(k)}_{\partial G}$ there is a simplex $T_E \in \mathcal{T}^{(k)}_G$ with face E and a simplex $T \in \mathcal{T}^{(k)}_G$ can contribute at most all of its $d+1$ faces to $\mathcal{E}^{(k)}_{\partial G}$. Utilizing the trace Lemma 4.8 and (27), we get

$$\begin{aligned}
\|v\|^2_{0,\Gamma_l \cap \partial G} &\leq \sum_{E \in \mathcal{E}^{(k)}_{\partial G}} \|v\|^2_{0,E} \\
&\leq c\left(1 + \tfrac{1}{\mathfrak{c}}\right) \sum_{E \in \mathcal{E}^{(k)}_{\partial G}} \left(h_k^{-1} \|v\|^2_{0,T_E} + h_k |v|^2_{1,T_E \backslash \Gamma^{(K)}} + \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k} C_{k,j} \|[\![v]\!]\|^2_{0,\Gamma_j \cap T_E} \right) \\
&\leq c(d+1)\left(1 + \tfrac{1}{\mathfrak{c}}\right) \sum_{T \in \mathcal{T}^{(k)}_G} \left(h_k^{-1} \|v\|^2_{0,T} + h_k |v|^2_{1,T \backslash \Gamma^{(K)}} + \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k} C_{k,j} \|[\![v]\!]\|^2_{0,\Gamma_j \cap T} \right) \\
&\leq C\left(1 + \tfrac{1}{\mathfrak{c}}\right) \left(d_k^{-1} \|v\|^2_{0,G} + d_k |v|^2_{1,G \backslash \Gamma^{(K)}} + \sum_{j=k+1}^{K} (1+\mathfrak{c})^{j-k} C_{k,j} \|[\![v]\!]\|^2_{0,\Gamma_j \cap G} \right)
\end{aligned}$$

with a constant c depending only on the space dimension d, shape regularity σ of $\mathcal{T}^{(k)}$, and the constant δ in (27). □

4.3. Projections on finite-scale spaces $\mathcal{H}_k$.

Definition 4.10. *For every $k \in \mathbb{N}$, we define the linear projection $\Pi_{\mathcal{H}_k} : \mathcal{H} \to \mathcal{H}_k$ by setting*

$$\Pi_{\mathcal{H}_k} v|_G = \begin{cases} \underset{v_k \in H^1(G)}{\arg\min} \{ |v - v_k|_{1,G\backslash\Gamma} \mid \int_G v - v_k \, dx = 0 \}, & G \in \Omega^{(k)} \backslash \Omega^{(k)}_\infty \\ v|_G, & G \in \Omega^{(k)}_\infty \end{cases} \tag{46}$$

for all $G \in \Omega^{(k)}$ and $v \in \mathcal{H}$.

The operator $\Pi_{\mathcal{H}_k}$ is well-defined. Indeed, for every $G \in \Omega^{(k)} \backslash \Omega^{(k)}_\infty$ its local contribution v_k is the unique solution of a quadratic minimization problem on the affine space $⨍_G v \, dx + W$, $W = \{ w \in H^1(G) \mid \int_G w \, dx = 0 \}$, which is characterized by the variational equality

$$(\nabla v_k, \nabla w) = (\nabla v, \nabla w) \qquad \forall w \in W. \tag{47}$$

Lemma 4.11. *For every $k \in \mathbb{N}$ the linear projection $\Pi_{\mathcal{H}_k}$ satisfies*

$$⨍_G v - \Pi_{\mathcal{H}_k} v \, dx = 0 \quad \text{and} \quad |\Pi_{\mathcal{H}_k} v|_{1,G} \leq |v|_{1,G\backslash\Gamma} \qquad \forall v \in \mathcal{H}. \tag{48}$$

Proof. Setting $v_k = \Pi_{\mathcal{H}_k} v|_G$, the first equality follows by definition (46) and after testing with $w = v_k - \fint_G v\, dx$ in (47), the remaining local stability of $\Pi_{\mathcal{H}_k}$ follows from the Cauchy-Schwarz inequality. □

We now state an approximation property of the projections $\Pi_{\mathcal{H}_k} v$, $k \in \mathbb{N}$.

Theorem 4.12. *Assume that the condition*

$$r_k (1 + \mathfrak{c})^{-k} \le d_k \tag{49}$$

on the geometry of the interface network Γ *is satisfied. Then the projections* $\Pi_{\mathcal{H}_k} : \mathcal{H} \to \mathcal{H}_k$, $k \in \mathbb{N}$, *have the approximation property*

$$\|v - \Pi_{\mathcal{H}_k} v\|_0^2 \le c \left(1 + \tfrac{1}{\mathfrak{c}}\right) d_k^2 \, \|v\|^2 \qquad \forall v \in \mathcal{H} \tag{50}$$

with a constant c *depending only on the space dimension* d *and shape regularity* γ *of* $\Omega^{(k)}$.

Proof. Let $G \in \Omega^{(k)} \backslash \Omega^{(k)}_\infty$ and $v \in \mathcal{H}$. As $v - \Pi_{\mathcal{H}_k} v$ has mean-value zero and $\Pi_{\mathcal{H}_k} v$ does not jump across Γ_l for $l \ge k+1$, the local Poincaré inequality stated in Proposition 4.7 yields

$$\|v - \Pi_{\mathcal{H}_k} v\|_{0,G}^2 \le c \left(1 + \tfrac{1}{\mathfrak{c}}\right) d_k \left(d_k \, |v - \Pi_{\mathcal{H}_k} v|_{1,G\backslash\Gamma}^2 + \sum_{j=k+1}^{\infty} (1+\mathfrak{c})^{j-k} C_{k,j} \, \|[\![v]\!]\|_{0,\Gamma_j \cap G}^2 \right) \tag{51}$$

with a constant c depending only on the dimension d and shape regularity γ of $\Omega^{(k)}$. Assumption (49) and the definition (8) of r_k imply

$$(1+\mathfrak{c})^{-k} C_{k,j} \le r_k (1+\mathfrak{c})^{-k} C_j \le d_k C_j. \tag{52}$$

Now we insert these estimates into (51) and make use of the Cauchy-Schwarz inequality and of the local stability (48) to obtain

$$\|v - \Pi_{\mathcal{H}_k} v\|_{0,G}^2 \le c \left(1 + \tfrac{1}{\mathfrak{c}}\right) d_k^2 \left(2\, |v|_{1,G\backslash\Gamma}^2 + \sum_{j=k+1}^{\infty} (1+\mathfrak{c})^{j} C_j \, \|[\![v]\!]\|_{0,\Gamma_j \cap G}^2 \right).$$

As $\|v - \Pi_{\mathcal{H}_k} v\|_{0,G} = 0$ for all $G \in \Omega^{(k)}_\infty$, summation over $G \in \Omega^{(k)} \backslash \Omega^{(k)}_\infty$ completes the proof. □

For each fixed $k \in \mathbb{N}$ boundedness

$$\|\Pi_{\mathcal{H}_k} v\| \le \mu_k \, \|v\| \qquad \forall v \in \mathcal{H} \tag{53}$$

of $\Pi_{\mathcal{H}_k}$ holds with a constant μ_k as a consequence of the closed graph theorem [38, ????]. In order to identify sufficient conditions for uniform stability of $\Pi_{\mathcal{H}_k}$, we want to further clarify the dependence of μ_k on $k \in \mathbb{N}$. To this end, the following lemma provides a bound for the jump contributions to $\|\Pi_{\mathcal{H}_k} v\|$ in terms of $\|v\|$.

Lemma 4.13. *Let* $k \in \mathbb{N}$, $G \in \Omega^{(k)} \backslash \Omega^{(k)}_\infty$ *and assume that conditions* (27) *and* (49) *are satisfied. Then*

$$\sum_{l=1}^{k} (1+\mathfrak{c})^l C_l \, \|[\![v - \Pi_{\mathcal{H}_k} v]\!]\|_{0,\Gamma_l}^2 \le C \left(1 + \tfrac{1}{\mathfrak{c}}\right)^2 d_k \left(\sum_{l=1}^{k} (1+\mathfrak{c})^l C_l \right) \|v\|^2 .$$

holds for all $v \in \mathcal{C}^1_{K,0}(\Omega)$ *with* $K > k$ *and a constant* C *depending only on the space dimension* d, *shape regularity* σ *of* $\mathcal{T}^{(k)}$, *and the constant* δ *in* (27).

Proof. Let $k \in \mathbb{N}$ and $v \in \mathcal{C}^1_{K,0}(\Omega)$ with $K > k$. Note that

$$\begin{aligned} \|[\![v - \Pi_{\mathcal{H}_k} v]\!]\|^2_{0,\Gamma_l} &= \sum_{\substack{G,G' \in \Omega^{(k)} \\ G \neq G'}} \int_{\Gamma_l \cap \partial G \cap \partial G'} \left((v - \Pi_{\mathcal{H}_k} v)|_G - (v - \Pi_{\mathcal{H}_k} v)|_{G'} \right)^2 \, d\Gamma_l \\ &\leq 4 \sum_{G \in \Omega^{(k)}} \|v - \Pi_{\mathcal{H}_k} v\|^2_{0,\Gamma_l \cap \partial G} . \end{aligned}$$

holds for $l = 1, \ldots, k$. Inserting (52) (a consequence of assumption (49)) into the local approximation property (51), we get

$$\|v - \Pi_{\mathcal{H}_k} v\|^2_{0,G} \leq c \left(1 + \tfrac{1}{\mathfrak{c}}\right) d_k^2 \left(|v - \Pi_{\mathcal{H}_k} v|^2_{1,G \setminus \Gamma^{(K)}} + \sum_{j=k+1}^{K} (1 + \mathfrak{c})^j C_j \, \|[\![v]\!]\|^2_{0,\Gamma_j \cap G} \right). \tag{54}$$

As $[\![\Pi_{\mathcal{H}_k} v]\!] = 0$ on Γ_j for $j > k$, application of the trace Lemma 4.9, together with (54), Lemma 4.11, and (52) lead to

$$\begin{aligned} &\|v - \Pi_{\mathcal{H}_k} v\|^2_{0,\Gamma_l \cap \partial G} \\ \leq &c' \left(1 + \tfrac{1}{\mathfrak{c}}\right) \left(d_k^{-1} \|v - \Pi_{\mathcal{H}_k} v\|^2_{0,G} + d_k \, |v - \Pi_{\mathcal{H}_k} v|^2_{1,G \setminus \Gamma^{(K)}} + \sum_{j=k+1}^{K} (1 + \mathfrak{c})^{s-k} \, C_{k,j} \, \|[\![v]\!]\|^2_{0,\Gamma_j \cap G} \right) \\ \leq &C' \left(1 + \tfrac{1}{\mathfrak{c}}\right)^2 d_k \left(|v|^2_{1,G \setminus \Gamma^{(K)}} + \sum_{j=k+1}^{K} (1 + \mathfrak{c})^j \, C_j \, \|[\![v]\!]\|^2_{0,\Gamma_j \cap G} \right) \end{aligned}$$

with constants c', C' depending on the space dimension d, shape regularity γ of $\Omega^{(k)}$, shape regularity σ of $\mathcal{T}^{(k)}$, and the constant δ in (27). Summation over $G \in \Omega^{(k)}$ yields

$$\|v - \Pi_{\mathcal{H}_k} v\|^2_{0,\Gamma_l} \leq C \left(1 + \tfrac{1}{\mathfrak{c}}\right)^2 d_k \, \|v\|^2$$

and the assertion follows. ☐

We are ready to state stability of the projections $\Pi_{\mathcal{H}_k}$, $k \in \mathbb{N}$.

Theorem 4.14. *Assume that conditions* (27) *and* (49) *are satisfied. Then the projections* $\Pi_{\mathcal{H}_k} : \mathcal{H} \to \mathcal{H}_k$, $k \in \mathbb{N}$, *are stable in the sense that*

$$\|\Pi_{\mathcal{H}_k} v\|^2 \leq c \left(1 + \tfrac{1}{\mathfrak{c}}\right)^3 d_k \left(\sum_{l=1}^{k} (1 + \mathfrak{c})^l C_l \right) \|v\|^2 \qquad \forall v \in \mathcal{H} \tag{55}$$

holds for each $k \in \mathbb{N}$ *with a constant* c *depending only on the space dimension* d*, shape regularity* γ *of* $\Omega^{(k)}$*, shape regularity* σ *of* $\mathcal{T}^{(k)}$*, and the constant* δ *in* (27).

Proof. As $\mathcal{C}^1_{K,0}(\Omega)$, $K \in \mathbb{N}$, is dense in $\mathcal{H}$ and $\Pi_{\mathcal{H}_k}$ is continuous for each fixed $k \in \mathbb{N}$, it is sufficient to prove (55) for $v \in \mathcal{C}^1_{K,0}(\Omega)$ with arbitrary $K \geq k$. In light of

$$\|\Pi_{\mathcal{H}_k} v\| \leq \|v - \Pi_{\mathcal{H}_k} v\|_{\mathcal{H}_k} + \|v\|$$

it is sufficient to derive a corresponding bound for $\|v - \Pi_{\mathcal{H}_k} v\|$. Utilizing boundedness of $\Pi_{\mathcal{H}_k}$ with respect to $|\cdot|_{1,\Omega \setminus \Gamma}$, cf. Lemma 4.11, and that, by construction, $\Pi_{\mathcal{H}_k} v$ does not jump across Γ_l, $l > k$, we obtain

$$\begin{aligned} \|v - \Pi_{\mathcal{H}_k} v\|^2 &= |v - \Pi_{\mathcal{H}_k} v|^2_{1,\Omega\setminus\Gamma} + \left(1 + \tfrac{1}{\mathfrak{c}}\right) \sum_{l=1}^{K} (1+\mathfrak{c})^l C_l \, \|[\![v - \Pi_{\mathcal{H}_k} v]\!]\|^2_{0,\Gamma_l} \\ &\le 4\|v\|^2 + \left(1 + \tfrac{1}{\mathfrak{c}}\right) \sum_{l=1}^{k} (1+\mathfrak{c})^l C_l \, \|[\![v - \Pi_{\mathcal{H}_k} v]\!]\|^2_{0,\Gamma_l}. \end{aligned}$$

Now the assertion follows from Lemma 4.13. □

Uniform stability of $\Pi_{\mathcal{H}_k}$ is obtained under an additional condition on the geometry of the interface network Γ.

Corollary 4.15. *Assume that conditions* (27) *and* (49) *are satisfied and that the additional condition*

$$d_k \left(\sum_{l=1}^{k} (1+\mathfrak{c})^l C_l \right) \le C_\Gamma, \qquad k \in \mathbb{N}, \tag{56}$$

holds with a constant C_Γ independent of k. Then the projections $\Pi_{\mathcal{H}_k}$, $k \in \mathbb{N}$, are uniformly stable, i.e.,

$$\|\Pi_{\mathcal{H}_k} v\| \le c\,\|v\| \qquad \forall v \in \mathcal{H} \tag{57}$$

holds for each $k \in \mathbb{N}$ with a constant c depending only on the space dimension d, shape regularity γ of $\Omega^{(k)}$, shape regularity σ of $\mathcal{T}^{(k)}$, the constant δ in (27)*, the constant C_Γ in* (56)*, and the material constant $\mathfrak{c}$.*

The additional condition (56) reflects the fact that the jump contributions to $\|\Pi_{\mathcal{H}_k} v\|$ cannot be bounded by the jump contributions to $\|v\|$ (see [38, ???] for a simple counterexample). Relating the material constant $\mathfrak{c}$ to the geometry of the interface network, it implies that the interfaces $\Gamma^{(k)}$ are highly localized for feasible $\mathfrak{c} > 0$ and thus excludes, e.g., the Cantor network [22, 44, 45]. For example, the highly localized network described in Subsection 2.1 above satisfies condition (56) for $\mathfrak{c} \le 1$.

4.4. **Quasi-interpolation on finite element spaces $\mathcal{S}_k$.** We now construct and analyse suitable projections $\Pi_{\mathcal{S}_k} : \mathcal{H}_k \to \mathcal{S}_k$, utilizing well-known concepts from finite element analyis.

Definition 4.16. *For every $k \in \mathbb{N}$, we define the Clément-type quasi-interpolation $\Pi_{\mathcal{S}_k} : \mathcal{H}_k \to \mathcal{S}_k$ by setting*

$$\Pi_{\mathcal{S}_k} v = \sum_{p \in \mathcal{N}^{(k)}} (\Pi_p v)\ \lambda_p^{(k)} \tag{58}$$

with $\Pi_p : \mathcal{H}_k \to \mathbb{R}$ defined by

$$\Pi_p v = \fint_{\omega_p} v \, dx, \qquad \omega_p = \operatorname{supp} \lambda_p^{(k)}, \quad p \in \mathcal{N}^{(k)}, \tag{59}$$

for $v \in \mathcal{H}_k$.

Proposition 4.17. *Let $k \in \mathbb{N}$ and $G \in \Omega^{(k)}$. Then the projection $\Pi_{\mathcal{S}_k}$ defined in* (58) *has the local approximation property*

$$\|v - \Pi_{\mathcal{S}_k} v\|_{0,G} \le c h_k\, |v|_{1,G} \qquad \forall v \in \mathcal{H}_k \tag{60}$$

with a constant c depending only on the dimension d and shape regularity σ of $\mathcal{T}^{(k)}$.

Proof. Let $v \in \mathcal{H}_k$, $G \in \Omega^{(k)}$, and $T \in \mathcal{T}_G^{(k)} \subset \mathcal{T}^{(k)}$. Then

$$\|v - \Pi_{\mathcal{S}_k} v\|_{0,T}^2 \le C h_k^2 \sum_{p \in T \cap \mathcal{N}_G^{(k)}} |v|_{1,\omega_p}^2$$

holds with a constant C depending only on the dimension d and shape regularity σ of $\mathcal{T}^{(k)}$ [8, 46]. The assertion now follows by summation over $T \in \mathcal{T}_G^{(k)}$. □

Proposition 4.18. *The projections $\Pi_{\mathcal{S}_k}$, $k \in \mathbb{N}$, defined in (58) are stable in the sense that*

$$\|\Pi_{\mathcal{S}_k} v\| \le c d_k \left(\sum_{l=1}^{k} (1 + \mathfrak{c})^l C_l \right) \|v\| \qquad \forall v \in \mathcal{H}_k \tag{61}$$

holds with a constant c depending only on the dimension d and shape regularity σ of $\mathcal{T}^{(k)}$.

Proof. Let $v \in \mathcal{H}_k$ and observe that

$$\|\Pi_{\mathcal{S}_k} v\|^2 \le 2\,\|v\|^2 + |\Pi_{\mathcal{S}_k} v|_{1,\Omega\setminus\Gamma^{(k)}}^2 + 2 \sum_{l=1}^{k} (1 + \mathfrak{c})^l C_l \,\|[\![v - \Pi_{\mathcal{S}_k} v]\!]\|_{0,\Gamma_l}^2 \tag{62}$$

follows from the triangle inequality and the Cauchy-Schwarz inequality. It is well-known, e.g., from [8, Theorem 2.4] that

$$|\Pi_{\mathcal{S}_k} v|_{1,\Omega\setminus\Gamma^{(k)}}^2 = \sum_{G \in \Omega^{(k)}} |\Pi_{\mathcal{S}_k} v|_{1,G}^2 \le c \sum_{G \in \Omega^{(k)}} |v|_{1,G}^2 = c\,|v|_{1,\Omega\setminus\Gamma^{(k)}}^2 \le c\,\|v\|^2 \tag{63}$$

holds with a constant c depending only on shape regularity σ of $\mathcal{T}^{(k)}$ and the space dimension d. We now derive a corresponding bound for the jump terms occurring in (62). As $\mathcal{T}^{(k)}$ resolves the interface network $\Gamma^{(k)}$ according to (26), there are subsets $\mathcal{E}_l^{(k)} \subset \mathcal{E}^{(k)}$ such that

$$\Gamma_l = \bigcup_{E \in \mathcal{E}_l^{(k)}} E, \quad l = 1, \ldots, k.$$

Now let $E \subset \overline{G}_{E,1} \cap \overline{G}_{E,2} \subset \Gamma_l$ with $G_{E,i} \in \Omega^{(k)}$, $i = 1,2$, and we set $v_i = v|_{G_{E,i}}$, $i = 1,2$. Then the Cauchy-Schwarz inequality yields

$$\|[\![v - \Pi_{\mathcal{S}_k} v]\!]\|_{0,\Gamma_l}^2 = \sum_{E \in \mathcal{E}_l^{(k)}} \int_E [\![v - \Pi_{\mathcal{S}_k} v]\!]^2 \, dE \le 2 \sum_{E \in \mathcal{E}_l^{(k)}} \int_E |v_1 - \Pi_{\mathcal{S}_k} v_1|^2 + |v_2 - \Pi_{\mathcal{S}_k} v_2|^2 \, dE. \tag{64}$$

It is well-known [8, 46] that

$$\int_E |v_i - \Pi_{\mathcal{S}_k} v_i|^2 \, dE \le c \sum_{p \in \mathcal{N}_{E,i}} h_k\,|v_i|_{1,\omega_p}^2, \quad i = 1,2,$$

holds with $\omega_p = \operatorname{supp} \lambda_p$, $\mathcal{N}_{E,i} = E \cap \mathcal{N}_{G_{E,i}}^{(k)}$ denoting the vertices of E located in $\overline{G}_{E,i}$, and a constant c depending only on shape regularity σ of $\mathcal{T}^{(k)}$ and the space dimension d. After inserting this bound into (64), summation over $l = 1, \ldots, k$, and shape regularity of $\mathcal{T}^{(k)}$ leads to

$$\sum_{l=1}^{k} (1 + \mathfrak{c})^l C_l \,\|[\![v - \Pi_{\mathcal{S}_k} v]\!]\|_{0,\Gamma_l}^2 \le c h_k \left(\sum_{l=1}^{k} (1 + \mathfrak{c})^l C_l \right) |v|_1^2$$

with c only depending on σ and d and the assertion follows from (27). □

Note that uniform stability of $\Pi_{\mathcal{S}_k}$, $k \in \mathbb{N}$, is obtained under the additional assumption (56).

Definition 4.19. *For every $k \in \mathbb{N}$, we define the projection*

$$\Pi_k = \Pi_{\mathcal{S}_k} \circ \Pi_{\mathcal{H}_k} : \mathcal{H} \to \mathcal{S}_k. \tag{65}$$

Theorem 4.20. *Assume that the conditions* (27), (49), (56) *hold. Then the projections $\Pi_k : \mathcal{H} \to \mathcal{S}_k$, $k \in \mathbb{N}$, defined in* (65) *have the approximation property*

$$\|v - \Pi_k v\|_0 \le c h_k \|v\| \qquad \forall v \in \mathcal{H} \tag{66}$$

with a constant c depending only on the space dimension d, shape regularity γ of $\Omega^{(k)}$, shape regularity σ of $\mathcal{T}^{(k)}$, the constant δ in (27)*, the constant C_Γ in* (56)*, and the material constant $\mathfrak{c}$.*

Proof. The assertion is an immediate consequence of the triangle inequality

$$\|v - \Pi_k v\|_0 \le \|v - \Pi_{\mathcal{H}_k} v\|_0 + \|\Pi_{\mathcal{H}_k} v - \Pi_{\mathcal{S}_k} (\Pi_{\mathcal{H}_k} v)\|_0 ,$$

Theorem 4.12, Proposition 4.17, and Corollary 4.15. □

Uniform stability of the projections Π_k is an immediate consequence of Corollary 4.15 and Proposition 4.18.

Theorem 4.21. *Assume that the conditions* (27), (49), (56) *hold. Then the projections $\Pi_k : \mathcal{H} \to \mathcal{S}_k$, $k \in \mathbb{N}$, defined in* (65) *are uniformly stable in the sense that*

$$\|\Pi_k v\| \le c \|v\| \qquad \forall v \in \mathcal{H} \tag{67}$$

holds with a constant c depending only on the space dimension d, shape regularity γ of $\Omega^{(k)}$, shape regularity σ of $\mathcal{T}^{(k)}$, the constant δ in (27)*, the constant C_Γ in* (56)*, and the material constant $\mathfrak{c}$.*

5. Multiscale finite element discretization

For some fixed $k \in \mathbb{N}$, we now construct novel multiscale finite element spaces with the same dimension as $\mathcal{S}_k$ that provide discretization errors of order h_k. Utilizing the projection $\Pi_k : \mathcal{H} \to \mathcal{S}_k$ defined in (65), we can readily apply local orthogonal decomposition (LOD) as introduced by Målqvist & Peterseim [31] with localization by subspace decomposition as suggested in [27].

Let $\mathcal{V}_k = \ker \Pi_k \subset \mathcal{H}$ denote the kernel of Π_k and $\mathcal{C} : \mathcal{H} \to \mathcal{V}_k$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{V}_k$ with respect to the scalar product $a(\cdot,\cdot)$ in $\mathcal{H}$. Then the multiscale finite element space

$$\mathcal{W}_k = \{v - \mathcal{C}v \mid v \in \mathcal{H}\} = \{v - \mathcal{C}v \mid v \in \mathcal{S}_k\} = \operatorname{span}\{(I - \mathcal{C})\lambda_p^{(k)} \mid p \in \mathcal{N}_k\}$$

is isomorphic to $\mathcal{S}_k$. We consider the multiscale discretization

$$u_k \in \mathcal{W}_k : \qquad a(u_k, v) = (f, v) \qquad \forall v \in \mathcal{W}_k. \tag{68}$$

The following error analysis is due to Peterseim [36] and Målqvist & Peterseim [31] (see also [27]).

Theorem 5.1. *The unique solution u_k of the discrete problem* (68) *is given by*

$$u_k = (I - \mathcal{C})\Pi_k u. \tag{69}$$

The discretization error has the representation $u - u_k = Cu$ and the error estimate

$$\|u - u_k\| \le c h_k \|f\|_0$$

holds with c depending only on the constants appearing in Theorems 4.20, 4.21, and the ellipticity constant $\mathfrak{a}$ from (19).

In spite of these desired properties, the space $\mathcal{W}_k$ is problematic, because its multiscale basis functions $(I - \mathcal{C})\lambda_p^{(k)}$, $p \in \mathcal{N}_k$, in general have global support. We therefore consider (intentionally local) approximations $\mathcal{C}_\nu : \mathcal{H} \to \mathcal{H}$, $\nu \in \mathbb{N}$, of $\mathcal{C}$ giving rise to the approximate subspaces

$$\mathcal{W}_k^{(\nu)} = \operatorname{span}\{(I - \mathcal{C}_\nu)\lambda_p^{(k)} \mid p \in \mathcal{N}_k\}$$

and corresponding Galerkin discretizations

$$u_k^{(\nu)} \in \mathcal{W}_k^{(\nu)} : \qquad a(u_k^{(\nu)}, v) = (f, v) \qquad \forall v \in \mathcal{W}_k^{(\nu)}. \tag{70}$$

The following discretization error estimate is taken from [27].

Theorem 5.2. *Assume that the approximations $\mathcal{C}_\nu : \mathcal{H} \to \mathcal{H}$, $\nu \in \mathbb{N}$, of $\mathcal{C}$ are convergent in the sense that*

$$\|\mathcal{C}v - \mathcal{C}_\nu v\|_a \le q \|\mathcal{C}v\|_a, \qquad \nu \in \mathbb{N}, \tag{71}$$

holds for all $v \in \mathcal{H}$ with some convergence rate $q < 1$. Then we have the discretization error estimate

$$\left\|u - u_k^{(\nu)}\right\| \le (1 + q^\nu)\,\tfrac{\mathfrak{A}}{\mathfrak{a}}\,\|u - u_k\| + q^\nu \tfrac{\mathfrak{A}}{\mathfrak{a}}\,\|u - \Pi_k u\|, \qquad \nu \in \mathbb{N}. \tag{72}$$

Proof. Exploiting $(I - \mathcal{C}_\nu)\Pi_k u \in \mathcal{W}_k^{(\nu)}$ and (69), we obtain

$$\left\|u - u_k^{(\nu)}\right\|_a \le \|u - (I - \mathcal{C}_\nu)\Pi_k u\|_a = \|(u - u_k) - (\mathcal{C}\Pi_k u - \mathcal{C}_\nu \Pi_k u)\|_a.$$

Convergence (71) together with identity (69) provides

$$\|\mathcal{C}\Pi_k u - \mathcal{C}_\nu \Pi_k u\|_a \le q^\nu \|\mathcal{C}\Pi_k u\|_a \le q^\nu (\|u - u_k\|_a + \|u - \Pi_k u\|_a).$$

Now the assertion follows from the triangle inequality and the norm equivalence (19). □

We now concentrate on the construction of convergent local approximations $\mathcal{C}_\nu : \mathcal{H} \to \mathcal{H}$, $\nu \in \mathbb{N}$, by local subspace correction. Here, we make heavy use of the fact that the kernel $\mathcal{V}_k$ of Π_k is high-frequency. Locality (46), (58) of the projection $\Pi_k = \Pi_{\mathcal{S}_k} \circ \Pi_{\mathcal{H}_k}$ motivates the splitting

$$\mathcal{V}_k = \sum_{G \in \Omega^{(k)}} \mathcal{V}_G \tag{73}$$

into the subspaces

$$\mathcal{V}_G = \{(I - \Pi_k)v|_G \mid v \in \mathcal{H}\} \subset \mathcal{V}_k, \qquad G \in \Omega^{(k)}.$$

Here, $v|_G$ is defined by $v|_G(x) = v(x)$ for $x \in G$ and $v|_G(x) = 0$ otherwise. Note that the linear mapping $\mathcal{H} \ni v \to v|_G \in \mathcal{H}$ is uniformly bounded in $\mathcal{H}$ for all $G \in \Omega^{(k)}$ and each fixed $k \in \mathbb{N}$ as a consequence of the trace Lemma 4.9 and the continuous embedding of $\mathcal{H}$ into $L^2(\Omega)$. The subspaces $\mathcal{V}_G$ are closed, because convergence of a sequence $(v_i)_{i \in \mathbb{N}} \subset \mathcal{V}_G \subset \mathcal{V}_k$ to some $v \in \mathcal{H}$

implies $v \in \mathcal{V}_k$, i.e., $\Pi_k v = 0$, as $\mathcal{V}_k$ is closed, $v = v|_G$, as supp $v_i \subset G$ for all $i \in \mathbb{N}$, and therefore $v = (I - \Pi_k)v|_G \in \mathcal{V}_G$. The following lemma is the main result of this section.

Lemma 5.3. *The splitting* (73) *is stable in the sense that for each $v \in \mathcal{V}_k$ there is a decomposition $(v_G)_{G\in\Omega^{(k)}}$ of v with $v_G \in \mathcal{V}_G$, $G \in \Omega^{(k)}$, such that*

$$\sum_{G\in\Omega^{(k)}} \|v_G\|_a^2 \le K_1 \|v\|_a^2 \tag{74}$$

holds with a constant K_1 depending only on the constants appearing in Theorems 4.20, 4.21, the geometric constant C_0 in (9) *and the ellipticity constants $\mathfrak{a}$, $\mathfrak{A}$ from* (19)*.*

Assume that for all $k \in \mathbb{N}$ and each G in $\Omega^{(k)}$ the number of neighboring cells of G from $\Omega^{(k)}$ is uniformly bounded by $c_N \in \mathbb{R}$. Then the splitting (73) *is bounded in the sense that for each $v \in \mathcal{V}_k$ all decompositions $(v_G)_{G\in\Omega^{(k)}}$ of v with $v_G \in \mathcal{V}_G$, $G \in \Omega^{(k)}$, satisfy*

$$\|v\|_a^2 \le K_2 \sum_{G\in\Omega^{(k)}} \|v_G\|_a^2 \tag{75}$$

with a constant K_2 depending only on c_N.

Proof. Boundedness (75) with a constant K_2 depending only on the maximal number of neighbors of each cell G is a direct consequence of the Cauchy-Schwarz inequality.

By a density argument, it is sufficient to show (74) for $v \in \mathcal{V}_k \cap \mathcal{H}_K$. We consider the splitting of v into its local components

$$v_G = (I - \Pi_k)v|_G \in \mathcal{V}_G, \qquad G \in \Omega^{(k)}.$$

Exploiting the locality of Π_k, i.e., $(I - \Pi_k)(v|_G) = ((I - \Pi_k)v)\,|_G$, we have

$$\|v_G\|^2 = |v - \Pi_k v|_{1,G\setminus\Gamma^{(K)}}^2 + \sum_{j=1}^{k}(1+\mathfrak{c})^j C_j \|v - \Pi_k v\|_{0,\Gamma_j\cap\partial G}^2 + \sum_{j=k+1}^{K}(1+\mathfrak{c})^j C_j \|[\![v]\!]\|_{0,\Gamma_j\cap G}^2 . \tag{76}$$

As a consequence of the Cauchy-Schwarz inequality, Lemma 4.11 and the local boundedness (63) of $\Pi_{\mathcal{S}_k}$, we have

$$|v - \Pi_k v|_{1,G\setminus\Gamma^{(K)}}^2 \le C\,|v|_{1,G\setminus\Gamma^{(K)}}^2 \tag{77}$$

with a constant C depending only on the space dimension d and shape regularity σ of $\mathcal{T}^{(k)}$. After utilizing the trace Lemma 4.9, we apply local boundedness (77), and the geometric conditions (9), (49), and (56) to obtain

$$\sum_{j=1}^{k}(1+\mathfrak{c})^j C_j \|v - \Pi_k v\|_{0,\Gamma_j\cap\partial G}^2 \le C' \left(d_k^{-2} \|v - \Pi_k v\|_{0,G}^2 + |v|_{1,G\setminus\Gamma^{(K)}}^2 + \sum_{j=k+1}^{K}(1+\mathfrak{c})^j C_j \|[\![v]\!]\|_{0,\Gamma_j\cap G}^2 \right)$$

with C' additionally depending on the material constant $\mathfrak{c}$, the constant δ in (27) and the constants appearing in (9) and (56). After inserting the above estimates in (76), summation over G and (27) lead to

$$\sum_{G\in\Omega^{(k)}} \|v_G\|^2 \le C'' \left(h_k^{-2} \|v - \Pi_k v\|_0^2 + \|v\|^2 \right).$$

Now the approximation property stated in Proposition 4.20 together with the norm equivalence (19) concludes the proof. $\square$

Let $P_G : \mathcal{H} \to \mathcal{V}_G$, $G \in \Omega^{(k)}$, denote the a-orthogonal Ritz projections defined by

$$P_G w \in \mathcal{V}_G : \qquad a(P_G w, v) = a(w, v) \qquad \forall v \in \mathcal{V}_G \tag{78}$$

for $w \in \mathcal{H}$ and

$$T = \sum_{G \in \Omega^{(k)}} P_G$$

the resulting preconditioner. Lemma 5.3 implies

$$1/K_1 a(v, v) \le a(Tv, v) \le K_2 a(v, v) \qquad \forall v \in \mathcal{V}_k \tag{79}$$

or, equivalently, the bound $\kappa \le K_1 K_2$ of the condition number $\kappa = \|T\|_a \left\|T^{-1}\right\|_a$ of T restricted to $\mathcal{V}_k$. We consider straightforward damped Richardson iteration

$$\mathcal{C}_{\nu+1} = \mathcal{C}_\nu + \omega T(I - \mathcal{C}_\nu), \qquad \mathcal{C}_0 = 0, \tag{80}$$

with a suitable damping factor ω. Note that $\mathcal{C}_\nu v \in \mathcal{V}_k$, $\nu \in \mathbb{N}$, holds for any $v \in \mathcal{H}$. Now convergence of (80) follows by well-known arguments.

Theorem 5.4. *Assume that for all $k \in \mathbb{N}$ and each $G \in \Omega^{(k)}$ the number of neighboring cells of G from $\Omega^{(k)}$ is uniformly bounded by $c_N \in \mathbb{R}$. Then the approximations $\mathcal{C}_\nu$, $\nu \in \mathbb{N}$, of $\mathcal{C}$ defined in* (80) *are convergent for $\omega < 2/K_2$ in the sense of* (71)*, and we have $q = 1 - 1/K_1K_2$ for the optimal damping factor $\omega = 1/K_2$ with K_1, K_2 depending only on the constants appearing in Theorems 4.20, 4.21, the geometric constant C_0 in* (9)*, c_N, and the ellipticity constants $\mathfrak{a}$, $\mathfrak{A}$ from* (19)*.*

More sophisticated iterative schemes with better convergence rates are discussed, e.g., in [27].

Utilizing Theorems 5.1 and 5.2, the desired discretization error estimate

$$\left\| u - u_k^{(\nu)} \right\| = \mathcal{O}(h_k)$$

is obtained by choosing $\nu \in \mathbb{N}$ such that the stopping criterion $q^\nu \frac{\mathfrak{A}}{\mathfrak{a}} \|u - \Pi_k u\| = \mathcal{O}(h_k)$ is fulfilled.

Note that the support of the first iterate $(I - \mathcal{C}_1)\lambda_p^{(k)} = (I - \omega T)\lambda_p^{(k)}$ is contained in $\overline{G}$, if p is located in G and contained in $\overline{G} \cup \overline{G'}$, if $p \in \overline{G} \cap \Gamma_k \cap \overline{G'}$. Similarly, the support of the approximate multiscale basis functions $(I - \mathcal{C}_\nu)\lambda_p^{(k)} = (I - \omega T)^\nu \lambda_p^{(k)})$, $p \in \mathcal{N}_k$, spreads at most by one layer of cells in each iteration step and therefore depends logarithmically on the prescribed accuracy of order h_k.

The construction of $\mathcal{W}_k^{(\nu)}$ requires the successive solution of local problems (78) in the infinite dimensional function spaces $\mathcal{V}_G$. In order to derive a computationally feasible analogue of the multiscale finite element discretization (70), we start from a typically very large, maybe computationally inaccessible finite element space $\mathcal{S}$ associated with a very strong refinement $\mathcal{T}$ of $\mathcal{T}^{(k)}$ that resolves all fine scale features of the multiscale interface problem as necessary to provide the desired accuracy of order h_k. Proceeding literally as above with $\mathcal{H}$ replaced by $\mathcal{S}$, we obtain discrete versions of Theorems 5.1, 5.2, and 5.4, where the iteration (80) takes the form of a damped block Jacobi iteration.

6. Iterative subspace correction

We now consider the construction and convergence analysis of subspace correction methods for the fractal interface problem (15) together with computationally feasible discrete versions for k-scale finite element approximations (28). Their convergence rates neither depend on the scales $k \in \mathbb{N}$ nor on the meshsize h_k.

The starting point is the two-level splitting

$$\mathcal{H} = \mathcal{V}_0 + \sum_{G \in \Omega^{(k)}} \mathcal{V}_G \tag{81}$$

with

$$\mathcal{V}_0 = \mathcal{S}_\ell, \qquad \mathcal{V}_G = \{v|_G \mid v \in \mathcal{H}\}, \quad G \in \Omega^{(k)}$$

and $1 \le \ell < k$. In particular, each $v \in \mathcal{H}$ can be decomposed into its local components

$$v_\ell = \Pi_\ell v \in \mathcal{S}_\ell, \qquad v_G = (v - \Pi_\ell v)|_G \in \mathcal{V}_G, \quad G \in \Omega^{(k)}.$$

Utilizing stability and approximation properties of $\Pi_\ell : \mathcal{H} \to \mathcal{S}_\ell$, stability and boundedness of the splitting (81) with corresponding constants K_1 and K_2 follows by similar arguments as in the proof of Lemma 5.3. Therefore, the corresponding preconditioner

$$T = P_0 + \sum_{G \in \Omega^{(k)}} P_G$$

with Ritz projections $P_0 : \mathcal{H} \to \mathcal{V}_0$ and $P_G : \mathcal{H} \to \mathcal{V}_G$, $G \in \Omega^{(k)}$, respectively, admits the bound $\kappa \le K_1 K_2$ of the condition number κ of $T : \mathcal{H} \to \mathcal{H}$. This property directly entails corresponding bounds for the convergence rates of preconditioned linear and nonlinear iterative schemes like Richardson or conjugate gradient methods.

In order to describe a sequential subspace correction method induced by the splitting (81), we introduce a numbering $\{G_1, \dots, G_m\} = \Omega^{(k)}$ of the cells and of the corresponding subspaces $\mathcal{V}_i = \mathcal{V}_{G_i}$ and Ritz projections $P_i = P_{\mathcal{V}_i}$, $i = 1, \dots, m$. We now consider the linear iteration

$$w_0 = u^{(\nu)}, \quad w_{i+1} = w_i + P_{m-i}(u - w_i),\ i = 0, \dots, m, \quad u^{(\nu+1)} = w_{m+1}, \tag{82}$$

for $\nu = 0, 1, \dots$ with arbitrary given iterate $u^{(0)} \in \mathcal{H}$. Instead of boundedness (75), convergence of (82) relies on the following Cauchy-Schwarz-type inequality.

Lemma 6.1. *Assume that for all $k \in \mathbb{N}$ and each G in $\Omega^{(k)}$ the number of neighboring cells of G from $\Omega^{(k)}$ is uniformly bounded by $c_N \in \mathbb{R}$. Then the Cauchy-Schwarz-type inequality*

$$\sum_{i,j=0}^{m} a(v_i, w_j) \le K_3 \left(\sum_{i=0}^{m} a(v_i, v_i) \right)^{1/2} \left(\sum_{j=0}^{m} a(w_j, w_j) \right)^{1/2}$$

holds for all $v_i \in \mathcal{V}_i$, $w_j \in \mathcal{V}_j$, $i, j = 0, \dots, m$, with a constant K_3 depending only on c_N.

Proof. For some fixed $G \in \Omega^{(k)}$, we introduce the local scalar product

$$a_G(v, w) = \int_{G \setminus \Gamma} A \nabla v \cdot \nabla w \, dx + \tfrac{1}{2} \sum_{j=1}^{k} (1 + \mathfrak{c})^j \, C_j \int_{\Gamma_j \cap \partial G} B[\![v]\!][\![w]\!] \, d\Gamma_j$$
$$+ \sum_{j=k+1}^{\infty} (1 + \mathfrak{c})^j \, C_j \int_{\Gamma_j \cap G} B[\![v]\!][\![w]\!] \, d\Gamma_j, \qquad v, w \in \mathcal{H},$$

with the property

$$\sum_{G\in\Omega^{(k)}} a_G(v,w) = a(v,w), \qquad v,w\in\mathcal{H}. \tag{83}$$

As the common support of $v_i \in \mathcal{V}_i$ and $w_j \in \mathcal{V}_j$ is contained in $\overline{G}_i \cap \overline{G}_j$ for $i,j = 1,\dots,m$, the Cauchy-Schwarz inequality and Gershgorin's theorem lead to

$$\sum_{i,j=0}^{m} a_G(v_i,w_j) \le (c_G+1)\left(\sum_{i=0}^{m} a_G(v_i,v_i)\right)^{1/2}\left(\sum_{j=0}^{m} a_G(w_j,w_j)\right)^{1/2}$$

with c_G denoting the number of neighboring cells of G from $\Omega^{(k)}$. After summation over $G\in\Omega^{(k)}$, the Cauchy-Schwarz inequality in $\mathbb{R}^{m+1}$ together with (83) complete the proof. □

The following convergence result is based on the error propagation

$$u - u^{(\nu+1)} = (I-P_0)\cdots(I-P_m)(u-u^{(\nu)}). \tag{84}$$

Its proof can be taken literally, e.g., from [29, Theorem 5.2].

Theorem 6.2. *Assume that for all $k\in\mathbb{N}$ and each G in $\Omega^{(k)}$ the number of neighboring cells of G from $\Omega^{(k)}$ is uniformly bounded by $c_N\in\mathbb{R}$. Then the iterative scheme* (82) *is convergent with respect to the energy norm, and*

$$\left\|u-u^{(\nu+1)}\right\|_a \le \left(1-\tfrac{1}{K_1K_3^2}\right)\left\|u-u^{(\nu)}\right\|_a$$

holds for any initial iterate $u^{(0)}\in\mathcal{H}$ with K_1, K_3 depending only on the constants appearing in Theorems 4.20, 4.21, the geometric constant C_0 in (9)*, c_N and the ellipticity constants $\mathfrak{a}$, $\mathfrak{A}$ from* (19)*.*

We emphasize that the two-level iteration (82) is just a simple illustrative example for a subspace correction method that can be analyzed using the projection operators suggested in Section 4. More efficient methods can be constructed in a similar way. For example, a symmetric variant of (82) that can be accelerated by conjugate gradients, is obtained by augmenting each iteration step by additional corrections $P_i(u-w_{m+1+i})$ taken in reverse order $i=1,\dots,m$. For detailed investigations, we refer to [38].

The linear iteration (82) takes place in $\mathcal{H}$ and thus requires the successive evaluation of Ritz projections P_i to infinite dimensional subspaces $\mathcal{V}_i\subset\mathcal{H}$, $i=1,\dots,m$. However, replacing $\mathcal{H}$ by a finite element space $\mathcal{S}_K$ with some $K\ge k>\ell$, the above considerations and convergence results literally translate to corresponding subspace correction methods for the finite element discretization (28) with respect to $\mathcal{S}_K$. In particular, the discrete analogue of (82) leads to a two-grid iteration with block Gauß-Seidel smoother on the fine grid $\mathcal{T}^{(k)}$ that is globally converging with convergence rate independent of the level K and corresponding meshsize h_K of the discrete solution space $\mathcal{S}_K$.

7. Numerical Experiments

In our two numerical experiments, we consider the finite element discretization (28) of the fractal interface problem (15) with $\Omega=(0,1)^2\subset\mathbb{R}^2$, $\mathfrak{c}=1$, the identity matrix $A=I\in\mathbb{R}^{d\times d}$, $B=1$ and two different kinds of fractal interface networks.

In order to illustrate the theoretical findings of Section 6, we consider the discrete analogue of the linear iteration (82) in function space, i.e., the two-grid method with block Gauß-Seidel smoother as induced by the two-level splitting (81), with coarse space $\mathcal{S}_\ell = \mathcal{S}_1$. The fine grid level $k = K$ is selected to coincide with the level of the underlying discrete solution space $\mathcal{S}_K$, $K = 1, \ldots, K_{\max}$. We always use the initial iterate $u^{(0)} = u_{\mathcal{S}_1}$, i.e., the finite element approximation on the coarse grid $\mathcal{T}^{(1)}$.

In light of the hierarchical lower bound

$$\|u_{\mathcal{S}_{K+1}} - u_{\mathcal{S}_K}\| \le \|u - u_{\mathcal{S}_K}\|$$

of the discretization error, the algebraic error is reduced up to discretization accuracy once the computationally feasible criterion

$$\left\|u_{\mathcal{S}_K} - u_{\mathcal{S}_K}^{(\nu)}\right\| \le \|u_{\mathcal{S}_{K+1}} - u_{\mathcal{S}_K}\| \tag{85}$$

is fulfilled. We will use (85) to determine the minimal number of iteration steps as required to reduce the algebraic error below discretization accuracy.

7.1. **Highly localized interface network.** In our first numerical experiment, we consider the highly localized fractal interface network as depicted in Figure 1. In this case, we have $d_k = \sqrt{2}\,4^{-k}$, $C_k = 2^k$, and $r_k = 2^{1-k}$. Hence, conditions (6), (9) hold true and the conditions (49), (56) are satisfied for $\mathfrak{c} = 1$.

Starting with the triangulation $\mathcal{T}^{(1)}$ as obtained by two uniform regular refinements of the partition $\mathcal{T}^{(0)}$ consisting of two congruent triangles, the triangulation $\mathcal{T}^{(k)}$ results from two uniform regular refinement steps applied to $\mathcal{T}^{(k-1)}$ for $k = 2, 3, \ldots$. We have $h_k = \sqrt{2}\,4^{-k}$ so that (27) holds with $\delta = 1$. For all $k \in \mathbb{N}$ and each G in $\Omega^{(k)}$, the number of neighboring cells of G from $\Omega^{(k)}$ is uniformly bounded by $c_N = 6$. As a consequence, the conditions for uniform stability and approximation property of the projections Π_k, $k \in \mathbb{N}$, as stated in Theorem 4.20 and Theorem 4.21, respectively, and for the uniform convergence result in Theorem 6.2 are satisfied in this case.

Table 1 displays the error reduction factors

$$\rho_K^{(\nu)} = \frac{\|u_{\mathcal{S}_K} - u_{\mathcal{S}_K}^{(\nu)}\|}{\|u_{\mathcal{S}_K} - u_{\mathcal{S}_K}^{(\nu-1)}\|}, \qquad \nu = 1, \ldots, 9,$$

together with their geometric mean ρ_K for the levels $K = 1, \ldots, K_{\max} = 5$. We observe that the error reduction factors nicely converge to the convergence rates on each level K and appear to saturate at 0.266 with increasing K. According to the criterion (85) the discretization accuracy is already reached after 3 steps.

7.2. **Geologically inspired interface network.** In our second numerical experiment, we consider an interface network mimicking a fractal crystalline structure. The triangulation $\mathcal{T}^{(1)}$ is obtained by four uniform regular refinement steps applied to the partition $\mathcal{T}^{(0)}$ consisting of two congruent triangles, and the triangulation $\mathcal{T}^{(k+1)}$ results from uniform regular refinement of $\mathcal{T}^{(k)}$ for $k = 1, 2, \ldots$. The level-k interfaces are inductively constructed as follows.
Let $G_0 = \Omega$ denote the initial cell with center $c = (0.5, 0.5)^T$ and midpoints $l, t, r, b \in \mathbb{R}^2$ of its left, top, right, and bottom boundary. The level-1 interface Γ_1, as shown in the left picture of Figure 2, then consists of four connected paths of edges in $\mathcal{E}^{(1)}$ starting with l, t, r, b and

ν	$K=2$	$K=3$	$K=4$	$K=5$
1	0.208	0.247	0.252	0.252
2	0.221	0.259	0.263	0.263
3	0.223	0.261	0.265	0.265
4	0.224	0.261	0.266	0.266
5	0.224	0.261	0.266	0.266
6	0.224	0.261	0.266	0.266
7	0.224	0.261	0.266	0.266
8	0.224	0.261	0.266	0.266
9	0.224	0.261	0.266	0.266
ρ_K	0.222	0.259	0.264	0.264

TABLE 1. Highly localized interface network:
Error reduction factors and geometric mean ρ_K of two-level subspace correction method

ending with c. These four paths must not self-intersect and must meet in and only in c. With these constraints, the actual selection of edges is made randomly with strong bias towards the straight line connecting the corresponding start and end points. Once $\Gamma^{(1)} = \Gamma_1$ is constructed, centers $c_i = (c_{i,1}, c_{i,2})^T$ of the four resulting cells $G_i \in \Omega^{(1)}$, $i = 1, \ldots, 4$, are determined in a similar way as described above. Each cell $G_i \in \Omega^{(1)}$ is either refined now or never. The decision about refinement or $G_i \in \Omega^{(1)}_\infty$ is made randomly according to the probability $\mathcal{P}(\min\{c_{i,1}, c_{i,2}\})$ with density $\rho(\xi) = 2(1-\xi)$, $\xi \in (0,1)$, i.e., with a linear bias towards the left and the lower boundary of Ω. In case of refinement, G_i is split into four subcells by four paths of edges in $\mathcal{E}^{(2)}$ starting with midpoints of its left, top, right, and bottom boundary and ending with c_i in analogy to the splitting of the initial cell G_0. The union of all these paths constitutes the level-2 interface Γ_2. This procedure is repeated inductively to construct the interface networks Γ_k, $k = 2, \ldots, 6$ (see Figure 2).

Apparently, the resulting interface network does not satisfy the locality condition (56) and the other conditions stated in Theorems 4.20, 4.21 that are finally sufficient for the convergence result in Theorem 6.2 are also unclear.

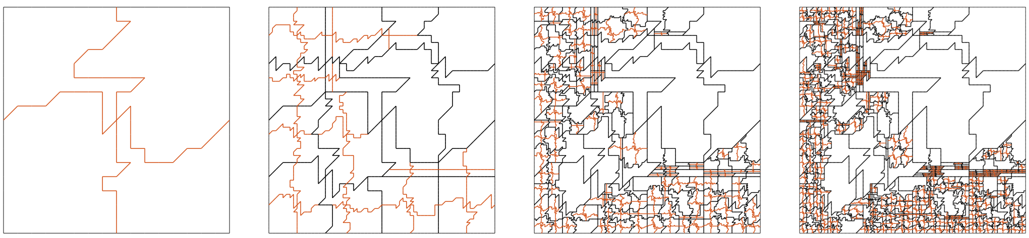

FIGURE 2. Geologically inspired interface network in $d = 2$ space dimensions: $\Gamma^{(1)} = \Gamma_1$ (red) and $\Gamma^{(k)}$ with Γ_k (red) for $k = 3, 5, 6$.

Nevertheless, the error reduction factors as displayed Table 2 only moderately deteriorate in comparison with the highly localized case and even seem to saturate with increasing level K. According to the criterion (85) the discretization accuracy is already reached after 5 steps.

ν	$K=2$	$K=3$	$K=4$	$K=5$	$K=6$
1	0.624	0.696	0.732	0.744	0.748
2	0.675	0.735	0.766	0.775	0.777
3	0.711	0.758	0.781	0.788	0.790
4	0.733	0.773	0.791	0.796	0.798
5	0.746	0.785	0.798	0.803	0.804
6	0.753	0.792	0.804	0.808	0.809
7	0.758	0.798	0.809	0.812	0.813
8	0.761	0.802	0.813	0.816	0.816
9	0.763	0.805	0.816	0.818	0.819
ρ_K	0.723	0.771	0.790	0.795	0.797

Table 2. Geologically inspired interface network:
Error reduction factors and geometric mean ρ_K of two-level subspace correction method

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Ralf Kornhuber, Institut für Mathematik, Freie Universität Berlin, 14195 Berlin, Germany

Email address: `kornhuber@math.fu-berlin.de`

Joscha Podlesny, Institut für Mathematik, Freie Universität Berlin, 14195 Berlin, Germany

Email address: `podlesjo@math.fu-berlin.de`

Harry Yserentant, Institut für Mathematik, Technische Universität Berlin, 10623 Berlin, Germany

Email address: `yserentant@math.tu-berlin.de`