Contributions to Discrete Mathematics

Volume 15, Number 3, Pages 117–129
ISSN 1715-0868

# DECOMPOSITION OF COMPLETE TRIPARTITE GRAPHS INTO CYCLES AND PATHS OF LENGTH THREE

SHANMUGASUNDARAM PRIYADARSINI AND APPU MUTHUSAMY

ABSTRACT. Let $C_k$ and $P_k$ denote a cycle and a path on $k$ vertices, respectively. In this paper, we obtain necessary and sufficient conditions for the decomposition of $K_{r,s,t}$ into $p$ copies of $C_3$ and $q$ copies of $P_4$ for all possible values of $p$, $q \geq 0$.

## 1. INTRODUCTION

We consider only finite undirected simple graphs. Let $K_{n_1,n_2,\ldots,n_r}$ denote a complete $r$-partite graph with part sizes $n_1, n_2, \ldots, n_r$, where each $n_i > 0$ is an integer. A partition of a graph $G$ into edge disjoint subgraphs $G_1, G_2, G_3, \ldots, G_n$ such that their union gives $G$ is called a *decomposition* of $G$. Let $C_k$ and $P_k$ respectively denote a cycle and a path on $k$ vertices. They are also called a $k$-cycle and $k$-path, respectively. The problem of finding necessary and sufficient conditions to decompose complete $n$-partite graphs into $k$-cycles has been considered for many values of $n$ and $k$. The case $n$= 2 was completely solved by Sotteau [13]. Smith [12] proved that the necessary conditions for the decomposition of complete equipartite graphs into cycles of length $2p$ (where $p \geq 3$ is a prime) are also sufficient. In the case of complete tripartite graphs, Cavenagh [5] has shown that $K_{m,m,m}$ can be decomposed into $k$-cycles if and only if $k \leq 3m$ and $k$ divides $3m^2$. Billington [2] gave necessary and sufficient conditions for the existence of a decomposition of any complete tripartite graph into specified number of 3-cycles and 4-cycles. Mahmoodian and Mirzakhani [10] proved the existence of a $C_5$-decomposition of $K_{r,s,t}$ whenever the necessary conditions are satisfied and two of the partite sets have equal size, except when $r = s = 0 \pmod 5$ and $t \neq 0 \pmod 5$. The authors of [1, 3, 6, 7] also studied this problem. Billington et al. [4] gave necessary and sufficient conditions for the path and cycle decomposition of complete equipartite graphs with 3 and 5 parts. Priyadharsini and Muthusamy [11] gave necessary and sufficient conditions for the existence of $(G_n, H_n)$-decomposition of $\lambda K_n$ and $\lambda K_{n,n}$, where $G_n$,

Received by the editors October 4, 2018, and in revised form February 9, 2020.
*Key words and phrases.* Cycle, Path, Complete tripartite graph, Decomposition of graphs.

$H_n \in \{C_n, P_n, S_{n-1}\}$. Jeevadoss and Muthusamy [8] gave necessary and sufficient conditions for the existence of $\{P_{k+1}, C_k\}_{p,q}$-decomposition of $K_{m,n}$ and $K_n$, when $m \geq k/2$, $n \geq \lceil (k+1)/2 \rceil$ for $k \equiv 0 \pmod 4$ and when $m$, $n \geq 2k$ for $k \equiv 2 \pmod 4$.

In this paper we give necessary and sufficient conditions for decomposing $K_{r,s,t}$ with $r \leq s \leq t$ into $p$ copies of $C_3$ and $q$ copies of $P_4$ for all possible values of $p$, $q \geq 0$. Definitions and notation not defined here can be referred to in [9].

**Lemma 1.1** ([7]). *Let $r$, $s$, and $t$ be integers such that $r \leq s \leq t$. A Latin rectangle of order $r \times s$ based on $t$ elements is equivalent to the existence of $rs$ edge-disjoint triangles sitting inside the complete tripartite graph $K_{r,s,t}$.*

The triangle $(i, j, k)$ in the 3-partite graph $K_{r,s,t}$ is the subgraph of $K_{r,s,t}$ induced by the $i$th vertex of part 1, $j$th vertex of part 2, and $k$th vertex of part 3.

**Definition 1.2** ([7]). *Consider a rectangular array of order $r \times s$ with entries from the set $T = \{1, 2, \ldots, t\}$. If each element of $T$ appears at most once in each row and at most once in each column, we call such an array a Latin rectangle of order $r \times s$ on $t$ elements.*

**Definition 1.3** ([7]). *Let $r$, $s$, and $t$ be integers such that $r \leq s \leq t$. A Latin representation of the complete tripartite graph $K_{r,s,t}$ is a Latin rectangle of order $r \times s$ on $t$ elements, together with a set of $t - s$ elements at the end of each row and a set of $t - r$ elements at the bottom of every column so that each element from the set $T = \{1, 2, 3, \ldots, t\}$ occurs once in each of the $r$ rows and once in each of the $s$ columns.*

*Remark:* To construct a Latin representation of the complete tripartite graph $K_{r,s,t}$ we first take a Latin rectangle of order $r \times s$ on $t$ elements. We then adjoin to the end of each row a set of remaining elements from the set $\{1, 2, 3, \ldots, t\}$ not already used in that row and to the bottom of each column we adjoin a set of remaining elements from the set $\{1, 2, 3, \ldots, t\}$ not already used in that column as in Figure 1.

Each entry $k$ of the set appended at the end of the $i$th row represents an edge from the $i$th element of the partite set of size $r$ to the element $k$ of the partite set of size $t$. Similarly, each entry $k$ of the set appended at the bottom of the $j$th column represents an edge from the $j$th element of the partite set of size $s$ to the element $k$ of the partite set of size $t$. So a Latin representation of $K_{r,s,t}$ is in fact equivalent to a decomposition of $K_{r,s,t}$ into $rs$ triangles and $rK_{1,t-s} + sK_{1,t-r}$.

Here we define *trade* to be a set of elements in the Latin representation, corresponding to a set of triangles and edges in $K_{r,s,t}$ which are $P_4$-decomposable. We define *relabelling* of the elements of a trade to be a bijection $\phi$ from the set of elements of $T = \{1, 2, \ldots, t\}$ onto itself. Thus every occurrence of $i \in T$ in the trade is replaced by $\phi(i)$. The relabelling of

| | | | | | | |
|---|---|---|---|---|---|---|
| $1$ | $2$ | $\cdots$ | $s$ | $s+1$ | $\cdots$ | $t$ |
| $2$ | $3$ | $\cdots$ | $s+1$ | $s+2$ | $\cdots$ | $1$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ |
| $r$ | $r+1$ | $\cdots$ | $r+s-1$ | $r+s$ | $\cdots$ | $r-1$ |

| | | | |
|---|---|---|---|
| $r+1$ | $r+2$ | $\cdots$ | $r+s$ |
| $\vdots$ | $\vdots$ | | $\vdots$ |
| $t$ | $1$ | $\cdots$ | $s-1$ |

FIGURE 1.

the elements in a trade does not change the structure of the corresponding set of edges in $K_{r,s,t}$.

**Construction 1.5.** *Two copies of $C_3$, with a common vertex, is equivalent to two copies of $P_4$. Let $(a_1, b_1, c_1)$, $(a_1, b_2, c_2)$ be two copies of $C_3$ with a common vertex $a_1$; then it can be written as two copies of $P_4$, $P(c_1, a_1, b_2, c_2)$, $P(c_2, a_1, b_1, c_1)$. In general, $n$ copies of $C_3$ with a common vertex is equivalent to $n$ copies of $P_4$. Let $(a_1, b_1, c_1), (a_1, b_2, c_2), \ldots, (a_1, b_{(n-1)}, c_{(n-1)})$, $(a_1, b_n, c_n)$ be $n$ copies of $C_3$ with a common vertex $a_1$; then it can be written $n$ copies of $P_4$ as $P(c_1, a_1, b_2, c_2), P(c_2, a_1, b_3, c_3), \ldots, P(c_{(n-1)}, a_1, b_n, c_n)$, $P(c_n, a_1, b_1, c_1)$.*

**Construction 1.6.** *Here we define two types of trades, in the first type we use elements from outside the Latin rectangle which are $P_4$-decomposable. The trades of first type are $T_1$, $T_2$, $T_3$, $T_4$, as shown in Figure 2 from the elements outside the Latin rectangle in which each copy of trades in $K_{r,s,t}$ are all edge-disjoint and $P_4$-decomposable.*

*The trade $T_1$ can be obtained from the newly adjoined elements on the right side of the Latin rectangle which can be decomposed into three copies of $P_4$ as follows: $P(c_{(s+1)}, a_i, c_{(s+2)}, a_j)$, $P(a_i, c_{(s+3)}, a_k, c_{(s+5)})$, $P(c_{(s+3)}, a_j, c_{(s+4)}, a_k)$. Similarly, by relabelling we can obtain the trade $T_1$ from the newly adjoined elements on the bottom of the Latin rectangle.*

*The trade $T_2$ can be obtained from the newly adjoined elements on the right side of the Latin rectangle which can be decomposed into two copies of $P_4$ as $P(c_{(s+1)}, a_i, c_{(s+2)}, a_j)$, $P(a_j, c_{(s+3)}, a_k, c_{(s+4)})$. Similarly, by relabelling we can obtain the trade $T_2$ from the newly adjoined elements on the bottom of the Latin rectangle.*

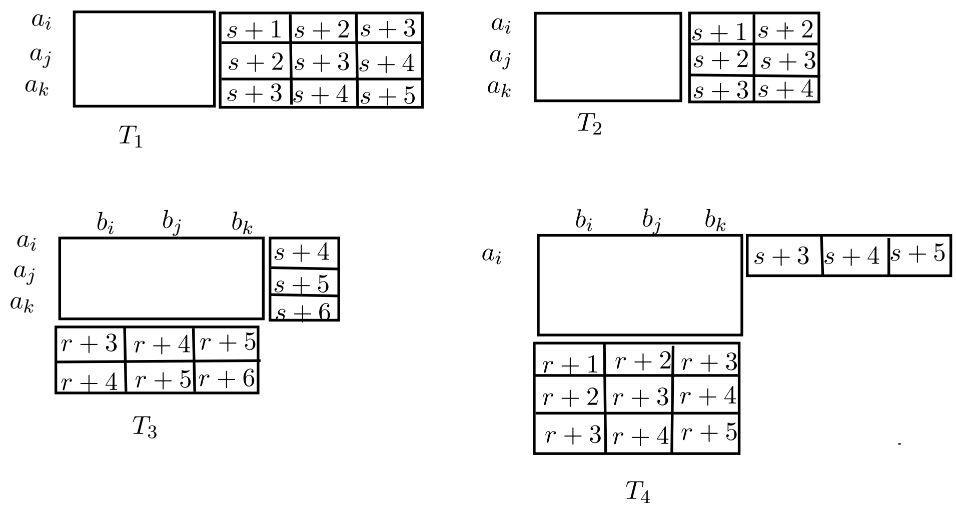


FIGURE 2.

*The trade $T_3$ can be obtained from the newly adjoined elements on the right side and the bottom of the Latin rectangle which can be decomposed into three copies of $P_4$ as $P(c_{(r+3)}, b_i, c_{(r+4)}, a_i)$, $P(c_{(r+4)}, b_j, c_{(s+5)}, a_j)$, $P(c_{(r+5)}, b_k, c_{(r+6)}, a_k)$, where $s+4$, $s+5$, $s+6$ in the right side of the Latin rectangle are equivalent to $r+4$, $r+5$, $r+6$ respectively.*

*The trade $T_4$ can be obtained from the newly adjoined elements on the right side and the bottom of the Latin rectangle which can be decomposed into four copies of $P_4$ as $P(c_{(r+1)}, b_i, c_{(r+2)}, b_j)$, $P(b_i, c_{(s+3)}, b_j, c_{(s+4)})$, $P(c_{(s+3)}, b_k, c_{(s+4)}, a_i)$, $P(b_k, c_{(s+5)}, a_i, c_{(s+3)})$ where $s+3$, $s+4$, $s+5$ in the right side of the Latin rectangle are equivalent to $r+3$, $r+4$, $r+5$ respectively.*

*In the second type, the elements from both inside and outside of the Latin rectangle are used. We use these two types of suitable trades untill all the edges in $K_{r,s,t}$ are used.*

## 2. NECESSARY CONDITIONS

**Theorem 2.1.** *If the complete tripartite graph $K_{r,s,t}$, where $r \le s \le t$, has a decomposition into $p$ copies of $C_3$ and $q$ copies of $P_4$, then the following holds:*

(i) $3|(rs + st + tr)$,
(ii) $q \neq 1$.

*Proof.* By a counting argument, we get the required condition $(i)$. We prove $(ii)$ by a contradiction. Suppose that $q = 1$. Then the end vertices of the only path $P_4$ have odd degree in $(K_{r,s,t} - E(P_4))$. Therefore the resulting graph $(K_{r,s,t} - E(P_4))$ cannot be decomposed into $C_3$, a contradiction. Hence $q \neq 1$. □

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 3 | 1 |
| 3 | 1 | 2 |

FIGURE 3.

**Corollary 2.2.** *If the complete tripartite graph $K_{r,s,t}$ can be decomposed into $pC_3$ and $qP_4$, where $r \le s \le t$, then $r$, $s$, and $t$ must satisfy one of the following:*

(a) *any two of $r$, $s$, $t$ are congruent to $0 \pmod 3$,*
(b) *all of $r$, $s$, $t$ are congruent to $1 \pmod 3$,*
(c) *all of $r$, $s$, $t$ are congruent to $2 \pmod 3$.*

*Proof.* The proof follows from the fact that the number of edges of $K_{r,s,t}$ is divisible by 3. □

## 3. SUFFICIENT CONDITIONS

**Lemma 3.1.** *The graph $K_{3,3,3}$ can be decomposed into $p$ copies of $C_3$ and $q$ copies of $P_4$, where $0 \le p \le 9$ and $0 \le q \le 9$, $q \ne 1$.*

*Proof.* Form a Latin square of order $3\times 3$ on 3 elements as shown in Figure 3. By Lemma 1.1, we have nine edge-disjoint 3-cycles as follows:

$$(a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3),$$
$$(a_2, b_3, c_1), (a_3, b_1, c_3), (a_3, b_2, c_1), (a_3, b_3, c_2).$$

In fact, this gives the required decomposition when $p = 9$, $q = 0$. The required decomposition for the other choices of $p$ and $q$ can be obtained by using Construction 1.5. □

**Lemma 3.2.** *The graph $K_{3,3,4}$ can be decomposed into $pC_3$ and $qP_4$, where $0 \le p \le 7$ and $4 \le q \le 11$.*

*Proof.* We form a Latin rectangle of order $3 \times 3$ on 4 elements. By Lemma 1.1, we have nine copies of $C_3$ as follows:

$$(a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3),$$
$$(a_2, b_3, c_4), (a_3, b_1, c_3), (a_3, b_2, c_4), (a_3, b_3, c_1).$$

The newly added element to the right side of each row in the Latin rectangle represents a single edge which cannot be decomposed into $P_4$. Similarly the newly added element to the bottom of each column of the Latin rectangle represents a single edge which cannot be decomposed into $P_4$. Here we use trades of the second type to get required number of copies of $P_4$. The single

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 1 |
| 3 | **4** | **1** | 2 |
| 4 | 1 | 2 | |

FIGURE 4.

edges outside the Latin rectangle along with the two copies of $C_3$ indicated by bold letters in Figure 4 give four copies of $P_4$:

$$P(a_1, c_4, b_2, a_3), P(b_1, c_4, a_3, b_3), P(a_2, c_1, a_3, c_2), P(b_2, c_1, b_3, c_2).$$

Also, when $p = 6$, $q = 5$, we have six copies of $C_3$:

$$(a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3), (a_2, b_3, c_1)$$

and five copies of $P_4$:

$$P(c_3, b_3, c_1, a_2), P(c_1, a_3, b_1, c_3), P(c_2, a_3, b_2, c_4),$$
$$P(a_1, c_4, b_2, c_1), P(b_1, c_4, a_3, b_3).$$

The other choices of $p$ and $q$ can be obtained by using Construction 1.5. Hence the graph $K_{3,3,4}$ has the desired decomposition. □

**Theorem 3.3.** *If $r \equiv 0 \pmod 3$, $s \equiv 0 \pmod 3$, and for any $t$, then the complete tripartite graph $K_{r,s,t}$, $r \le s \le t$, can be decomposed into $p$ copies of $C_3$ and $q$ copies of $P_4$, where $q \ne 1$.*

*Proof.* The proof is separated into three cases.

CASE 1: $r \equiv 0 \pmod 3, s \equiv 0 \pmod 3, t \equiv 0 \pmod 3$.

The Latin rectangle of order $r \times s$ on $t$ elements give $rs$ triangles. The other choices of $p$ and $q$ can be obtained by Construction 1.5. Now the newly added elements to the right side of the Latin rectangle form $(r/3)[(t-s)/3]$ copies of $3 \times 3$ arrays each representing the trade $T_1$. Similarly the newly added elements to the bottom of the Latin rectangle form $(s/3)[(t-r)/3]$ copies of $3 \times 3$ arrays each representing the trade $T_1$. By Construction 1.6, the copies of trade $T_1$ are all edge-disjoint and $P_4$-decomposable. Hence we have the required decomposition, where $0 \le p \le rs$,

$$r\left(\frac{t-s}{3}\right) + s\left(\frac{t-r}{3}\right) \le q \le rs + r\left(\frac{t-s}{3}\right) + s\left(\frac{t-r}{3}\right).$$

CASE 2: $r \equiv 0 \pmod 3, s \equiv 0 \pmod 3, t \equiv 1 \pmod 3$.

For the graph $K_{r,r,r+1}$, the newly added elements to the right side of Latin rectangle form an $r \times 1$ array which cannot be decomposed into $P_4$. Similarly the newly added elements to the bottom of Latin rectangle form a $1 \times s$ array which cannot be decomposed into $P_4$. Therefore we use trades of the second type to obtain the required number of copies of $P_4$. The single edges on the both side of the Latin rectangle along with $2r/3$ copies of $C_3$ give $4r/3$ copies of $P_4$. These $4r/3$ copies of $P_4$ and the remaining $(r^2 - (2r/3))$ copies of $C_3$ give the maximum number of copies of $P_4$ by Construction 1.5. Hence the graph has the required decomposition, where $0 \leq p \leq (r^2 - \frac{2r}{3})$,

$$\frac{4r}{3} \leq q \leq r^2 + \frac{2r}{3}.$$

For the graph $K_{r,s,s+1}$, the newly added elements to the right side of the Latin rectangle is an $r \times 1$ array. The newly added elements to the bottom of the Latin rectangle form $(s/3)[((t-r)-4)/3]$ copies of $3 \times 3$ arrays which represents the trade $T_1$ and the remaining elements form $2s/3$ copies of $2 \times 3$ arrays which represents the trade $T_2$. The elements of $r/3$ copies of $3 \times 1$ array in the right side of the Latin rectangle along with the elements of $r/3$ copies of $2 \times 3$ array at the bottom of the Latin rectangle which contain the same elements of a $3 \times 1$ array form the trade $T_3$. By Construction 1.6, the edge-disjoint copies of $T_1$, $T_2$, $T_3$ are $P_4$-decomposable. The remaining possible choices of $p$ and $q$ can be obtained by using Construction 1.5. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$s\left[\frac{(t-r)-4}{3}\right] + \frac{4s}{3} - \frac{2r}{3} + r \leq q$$
$$\leq s\left[\frac{(t-r)-4}{3}\right] + \frac{4s}{3} - \frac{2r}{3} + r + rs.$$

For $t > s+1$, the newly added elements to the right side of the Latin rectangle form $(r/3)[((t-s)-4)/3]$ copies of $3 \times 3$ arrays which represents the trade $T_1$ and the remaining elements form $2r/3$ copies of $3 \times 2$ arrays which represents the trade $T_2$. The newly added elements to the bottom of the Latin rectangle form $(s/3)[((t-r)-4)/3]$ copies of $3 \times 3$ arrays which represents the trade $T_1$ and $2s/3$ copies of $2 \times 3$ arrays which represents the trade $T_2$. By Construction 1.6, the edge-disjoint copies of $T_1$, $T_2$ are $P_4$-decomposable. The remaining possible choices of $p$ and $q$ can be obtained by using Construction 1.5. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$s\left[\frac{(t-r)-4}{3}\right] + r\left[\frac{(t-s)-4}{3}\right] + 4\left(\frac{r}{3} + \frac{s}{3}\right) \leq q$$
$$\leq s\left[\frac{(t-r)-4}{3}\right] + r\left[\frac{(t-s)-4}{3}\right] + 4\left(\frac{r}{3} + \frac{s}{3}\right) + rs.$$

Case 3: $r \equiv 0 \pmod 3, s \equiv 0 \pmod 3, t \equiv 2 \pmod 3$.

In this graph, the newly added elements to the right side of Latin rectangle form $(r/3)[((t-s)-2)/3]$ copies of $3 \times 3$ arrays which represents the trade $T_1$ and the remaining elements form $r/3$ copies of $3 \times 2$ arrays which represents the trade $T_2$. Similarly the newly added elements to the bottom of the Latin rectangle form $(s/3)[((t-r)-2)/3]$ copies of $3\times 3$ arrays which represents the trade $T_1$ and the remaining elements form $s/3$ copies of $2\times 3$ arrays which represents the trade $T_2$. Hence we have the required decomposition, where $0\le p\le rs$,

$$s\left[\frac{(t-r)-2}{3}\right]+r\left[\frac{(t-s)-2}{3}\right]+\frac{2(r+s)}{3}\le q$$
$$\le s\left[\frac{(t-r)-2}{3}\right]+r\left[\frac{(t-s)-2}{3}\right]+\frac{2(r+s)}{3}+rs.$$

□

**Theorem 3.4.** *If $r\equiv 0 \pmod 3$, $s\equiv 1 \pmod 3$, and $t\equiv 0 \pmod 3$, then the complete tripartite graph $K_{r,s,t}$, $r<s<t$, can be decomposed into $pC_3$ and $qP_4$, where $q\neq 1$.*

*Proof.* The Latin rectangle of order $r\times s$ on $t$ elements form $rs$ triangles. The other choices of $p$ and $q$ can be obtained by Construction 1.5. The newly added elements to the right side of the Latin rectangle form $(r/3)[((t-s)-2)/3]$ copies of $3\times 3$ arrays each representing the trade $T_1$ and the remaining elements form $r/3$ copies of $3\times 2$ arrays which represents the trade $T_2$. Similarly the newly added elements to the bottom of the Latin rectangle form $[(t-r)/3][(s-4)/3]$ copies of $3\times 3$ arrays each representing the trade $T_1$ and the remaining elements form $[2(t-r)/3]$ copies of $3\times 2$ arrays which represents the trade $T_2$. By Construction 1.6, all the trades are edge-disjoint and $P_4$-decomposable. Hence we have the required decomposition, where $0\le p\le rs$,

$$r\left[\frac{(t-s)-2}{3}\right]+\frac{4(t-r)}{3}+\frac{2r}{3}+\left[\frac{(t-r)(s-4)}{3}\right]\le q$$
$$\le rs+r\left[\frac{(t-s)-2}{3}\right]+\frac{4(t-r)}{3}+\frac{2r}{3}+\frac{(t-r)(s-4)}{3}.$$

□

**Theorem 3.5.** *If $r\equiv 0 \pmod 3$, $s\equiv 2 \pmod 3$, and $t\equiv 0 \pmod 3$, then the complete tripartite graph $K_{r,s,t}$, $r<s<t$, can be decomposed into $pC_3$ and $qP_4$, where $q\neq 1$.*

*Proof.* We consider $s=r+2$, then the newly added elements to the right side of the Latin rectangle is an $r\times 1$ array which cannot be decomposed into $P_4$. Therefore, we use trades of the second type. These $r$ single edges along with $2r/3$ triangles give $5r/3$ copies of $P_4$. Now the newly added elements to the bottom of the Latin rectangle have $(s-2)/3$ copies of $3\times 3$ arrays and one copy of a $3\times 2$ array. Hence we have the required decomposition, where $0\le p\le r(s-(2/3))$,

$$\frac{5r}{3}+s\le q\le\frac{5r}{3}+s+r\left(s-\frac{2}{3}\right).$$

For $t = s+1$, the newly added elements to the right side of the Latin rectangle form $r/3$ copies of $3\times 1$ arrays which cannot be decomposed into copies of $P_4$. Now the elements of $r/3$ copies of $3\times 1$ arrays in the right side of the Latin rectangle along with the elements of $r/3$ copies of $2\times 3$ arrays in the bottom of the Latin rectangle which contain the same elements of a $3\times 1$ array form the trade $T_3$. The newly added elements to the bottom of the Latin rectangle form $([((t-r)-6)/3][(s-2)/3]$ copies of $3\times 3$ arrays, $(t-r)/3$ copies of $3\times 2$ arrays and $3(s-2)/3$ copies of $2\times 3$ arrays. Therefore we get $[((t-r)-6)/3][(s-2)/3]$ copies of the trade $T_1$, $(t-r)/3$, $3(s-2)/3$, and $(3s-6-r)/3$ copies of $T_2$ in which all are edge-disjoint. Hence we have the required decomposition, where $0\le p\le rs$ and $q\ne 1$,

$$(s-2)\left[\frac{(t-r)-6}{3}\right]+r+\frac{2(t-r)}{3}+2(s-2)+\frac{2(3s-6-r)}{3}\le q$$
$$\le (s-2)\left[\frac{(t-r)-6}{3}\right]+r+\frac{2(t-r)}{3}+2(s-2)+\frac{2(3s-6-r)}{3}+rs.$$

Now for $t > s+1$, the newly added elements to the right side of the Latin rectangle form $\frac{r}{3}[\frac{(t-s)-4}{3}]$ copies of $3\times 3$ arrays and $2r/3$ copies of $3\times 2$ arrays which represents the trade $T_1$ and $T_2$ respectively. The newly added elements to the bottom of the Latin rectangle form $[(t-r)/3][(s-2)/3]$ copies of $3\times 3$ arrays and $[(t-r)/3]$ copies of $3\times 2$ arrays which represents the trade $T_1$ and $T_2$ respectively. Hence we have the required decomposition, where $0\le p\le rs$ and $q\ne 1$,

$$\frac{4r}{3}+\frac{2(t-r)}{3}+\frac{(t-r)(s-2)}{3}+r\left[\frac{(t-s)-4}{3}\right]\le q$$
$$\le \frac{4r}{3}+\frac{2(t-r)}{3}+\frac{(t-r)(s-2)}{3}+r\left[\frac{(t-s)-4}{3}\right]+rs.$$

□

**Theorem 3.6.** *If $r\equiv 1 \pmod 3$, $s\equiv 1 \pmod 3$, and $t\equiv 1 \pmod 3$, then the complete tripartite graph $K_{r,s,t}$, $r\le s\le t$, can be decomposed into $pC_3$ and $qP_4$, where $q\ne 1$.*

*Proof.* We consider three cases:

CASE 1: $r > 1$.

In this case the Latin rectangle of order $r\times s$ on $t$ elements give $rs$ triangles. The newly added elements to the right side of the Latin rectangle form $[(t-s)/3][(r-4)/3]$ copies of $3\times 3$ arrays which represents the trade $T_1$ and the remaining elements form $2(t-s)/3$ copies of $2\times 3$ arrays which represents the trade $T_2$. The newly added elements to the bottom of the Latin rectangle form $[(t-r)/3][(s-4)/3]$ copies of $3\times 3$ arrays each representing the trade $T_1$ and the remaining elements form $2(t-r)/3$ copies of $3\times 2$ arrays which represents the trade $T_2$. The other choices of $p$ and $q$ can be obtained by Construction 1.5. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$\frac{(t-s)(r-4)}{3} + \frac{(t-r)(s-4)}{3} + \frac{4(t-s)}{3} + \frac{4(t-r)}{3} \leq q$$
$$\leq rs + \frac{(t-s)(r-4)}{3} + \frac{(t-r)(s-4)}{3} + \frac{4(t-s)}{3} + \frac{4(t-r)}{3}.$$

CASE 2: $r = 1$, $s = 1$.

We have one copy of $C_3$ and $(t-1)/3$ copies of $K_{2,3}$ which can be decomposed into two copies of $P_4$. Therefore in this case we get $p = 1$ and $q = 2(t-1)/3$.

CASE 3: $r = 1$, $s > 1$.

In this case we have $p = s$. The newly added elements to the bottom of the Latin rectangle form $[(t-1)/3][(s-4)/3]$ copies of $3 \times 3$ arrays which represents the trade $T_1$ and the remaining elements form $2(t-1)/3$ copies of $3 \times 2$ arrays which represents the trade $T_2$. The newly added elements to the right side of the Latin rectangle form $(t-s)/3$ copies of $1 \times 3$ arrays which cannot be decomposed into copies of $P_4$. The elements of $(t-s)/3$ copies of $1 \times 3$ arrays in the right side of the Latin rectangle along with the elements of $(t-s)/3$ copies of $3 \times 3$ arrays in the bottom of the Latin rectangle which contain the same elements of $1 \times 3$ arrays form the trade $T_4$. Therefore we get $([(t-1)/3][(s-4)/3] - [(t-s)/3])$ copies of $3 \times 3$ arrays which represents the trade $T_1$. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$\frac{4(t-s)}{3} + \left[\frac{(t-1)(s-4) - 3(t-s)}{3}\right] + \frac{4(t-1)}{3} \leq q$$
$$\leq s + \frac{4(t-s)}{3} + \left[\frac{(t-1)(s-4) - 3(t-s)}{3}\right] + \frac{4(t-1)}{3}.$$

□

**Theorem 3.7.** *If* $r \equiv 1 \pmod 3$, $s \equiv 0 \pmod 3$, *and* $t \equiv 0 \pmod 3$, *then the complete tripartite graph* $K_{r,s,t}$, $r < s \leq t$, *can be decomposed into* $pC_3$ *and*$qP_4$, *where* $q \neq 1$.

*Proof.* We consider two cases:

CASE 1: $r = 1$.

The Latin rectangle of order $1 \times s$ on $t$ elements give $s$ triangles. The newly added elements to the bottom of the Latin rectangle have $(s/3)[(t-3)/3]$ copies of $3 \times 3$ arrays and $s/3$ copies of $2 \times 3$ arrays. The newly added elements to the right side of the Latin rectangle have $(t-s)/3$ copies of $1 \times 3$ arrays. The elements of $(t-s)/3$ copies of $1 \times 3$ arrays on the right side of the Latin rectangle along with the elements of $(t-s)/3$ copies of $3 \times 3$ array in the bottom of the Latin rectangle which contain the same elements of a $1 \times 3$ array form the trade $T_4$. Therefore we have $((s/3)[(t-3)/3] - [(t-s)/3])$ copies of $T_1$, $(t-s)/3$ copies of $T_4$

and $s/3$ copies of $T_2$. Hence we have the required decomposition, where $0 \leq p \leq s$,

$$\left[\frac{s(t-3)-3(t-s)}{3}\right] + \frac{2s}{3} + \frac{4(t-s)}{3} \leq q \leq s + \left[\frac{s(t-3)-3(t-s)}{3}\right] + \frac{2s}{3} + \frac{4(t-s)}{3}.$$

CASE 2: $r > 1$.

The Latin rectangle of order $r \times s$ on $t$ elements give $rs$ triangles. The newly added elements to the right side of the Latin rectangle form $(t-s)(r-4)/9$ copies of $3 \times 3$ arrays and $2(t-s)/3$copies of $2 \times 3$ arrays. The newly added elements to the bottom of the Latin rectangle form $[s(t-r-2)/9]$ copies of $3 \times 3$ arrays and $s/3$ copies of $2 \times 3$ arrays. Therefore each copy of $3 \times 3$ arrays and $2 \times 3$ arrays representing the trades $T_1$ and $T_2$ respectively. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$\frac{s(t-r-2)}{3} + \frac{(t-s)(r-4)}{3} + \frac{4(t-s)}{3} + \frac{2s}{3} \leq q \leq rs + \frac{s(t-r-2)}{3} + \frac{(t-s)(r-4)}{3} + \frac{4(t-s)}{3} + \frac{2s}{3}.$$

□

**Theorem 3.8.** *If* $r \equiv 2 \pmod 3$, $s \equiv 2 \pmod 3$, *and* $t \equiv 2 \pmod 3$, *then the complete tripartite graph* $K_{r,s,t}$, $r \leq s \leq t$, *can be decomposed into* $pC_3$ *and* $qP_4$, $q \neq 1$.

*Proof.* The Latin rectangle of order $r \times s$ on $t$ elements give $rs$ triangles. The other choices of $p$ and $q$ can be obtained by using Construction 1.5. The newly added elements to the right side of the Latin rectangle form $[(t-s)/3][(r-2)/3]$ copies of $3 \times 3$ arrays and $(t-s)/3$ copies of $2 \times 3$ arrays each representing the trades $T_1$ and $T_2$ respectively. Similarly the newly added elements to the bottom of the Latin rectangle form $[(t-r)/3][(s-2)/3]$ copies of $3 \times 3$ arrays and $(t-r)/3$ copies of $3 \times 2$ arrays each representing the trades $T_1$ and $T_2$ respectively. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$\frac{(t-s)(r-2)}{3} + \frac{2(t-s)}{3} + \frac{(t-r)(s-2)}{3} + \frac{2(t-r)}{3} \leq q \leq \frac{(t-s)(r-2)}{3} + \frac{2(t-s)}{3} + \frac{(t-r)(s-2)}{3} + \frac{2(t-r)}{3} + rs.$$

□

**Theorem 3.9.** *If* $r \equiv 2 \pmod 3$, $s \equiv 0 \pmod 3$, *and* $t \equiv 0 \pmod 3$, *then the complete tripartite graph* $K_{r,s,t}$, $r < s \leq t$, *can be decomposed into* $pC_3$ *and* $qP_4$, $q \neq 1$.

*Proof.* We consider two cases:
CASE 1: $s = t$.

The Latin rectangle of order $r \times s$ on $s$ elements give $rs$ triangles. The newly added elements to the bottom of the Latin rectangle is a $1 \times (s/3)$ array which cannot be decomposed into $P_4$. Here we use trades of the second type to decompose $P_4$. The edges in the bottom of the Latin rectangle along with $2s/3$ triangles give $s$ copies of $P_4$. Therefore we get $0 \leq p \leq (rs - (2s/3))$ and $s \leq q \leq (rs + s/3)$.

CASE 2: $s < t$.

The newly added elements to the right side of the Latin rectangle form $[(t-s)/3][(r-2)/3]$ copies of $3 \times 3$ arrays and the remaining elements form $[(t-s)/3]$ copies of $2\times 3$ arrays. Similarly the newly added elements to the bottom of the Latin rectangle form $[(t-r-4)/3][s/3]$ copies of $3\times 3$ arrays and the remaining elements form $2s/3$ copies of $2 \times 3$ arrays. Therefore each copy of $3 \times 3$ arrays and $2 \times 3$ arrays represent the trades $T_1$ and $T_2$, respectively. The other choices of $p$ and $q$ can be obtained by using Construction 1.5. Hence we get the required decomposition, where $0 \leq p \leq rs$,

$$\frac{(t-s)(r-2)}{3} + \frac{2(t-s)}{3} + \frac{s(t-r-4)}{3} + \frac{4s}{3} \leq q \leq \frac{(t-s)(r-2)}{3} + \frac{2(t-s)}{3} + \frac{s(t-r-4)}{3} + \frac{4s}{3} + rs.$$

□

## 4. Conclusion

**Main Theorem.** *Let $p$ and $q$ be nonnegative integers and let $r$, $s$, $t$ be positive integers. There exists a decomposition of $K_{r,s,t}$, $r \leq s \leq t$, into $pC_3$ and $qP_4$ if and only if $3(p+q) = rs + st + tr$, $q \neq 1$, where $r$, $s$, $t$ satisfy the following conditions:*

(a) *any two of $r$, $s$, $t$ are congruent to* $0 \pmod 3$,
(b) *all of $r$, $s$, $t$ are congruent to* $1 \pmod 3$,
(c) *all of $r$, $s$, $t$ are congruent to* $2 \pmod 3$.

*Proof.* This follows from the Theorems 2.1, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, and 3.9. □

## Acknowledgement

The authors thank the UGC New Delhi, India (Grant No. F.510/7/DRS-I/2016(SAP-DRS-I)) and the Department of Science and Technology, New Delhi (Grant No. SR/FIST/MSI-115/2016(Level-I)), for their generous financial support.

## References

1. S. Alipour, E. S. Mahmoodian, and E. Mollaahmadi, *On decomposing complete tripartite graphs into* 5*-cycles*, Australas. J. Combin. **54** (2012), 289–301.
2. E. J. Billington, *Decomposing complete tripartite graphs into cycles of length* 3 *and* 4, Discrete Math. **197–198** (1999), 123–135.
3. E. J. Billington and N. J. Cavenagh, *Decomposing complete tripartite graphs into* 5*-cycles when the partite sets have similar size*, Aequationes Math. **82** (2011), 277–289.
4. E. J. Billington, N. J. Cavenagh, and B. R. Smith, *Path and cycle decompositions of complete equipartite graphs:* 3 *and* 5 *parts*, Discrete Math. **310** (2010), 241–252.
5. N. J. Cavenagh, *Decompositions of complete tripartite graphs into $k$ cycles*, Australas. J. Combin. **18** (1998), 193–200.
6. ______, *Further decompositions of complete tripartite graphs into* 5*-cycles*, Discrete Math. **256** (2002), 55–81.
7. N. J. Cavenagh and E. J. Billington, *On decomposing complete tripartite graphs into* 5*-cycles*, Australas. J. Combin. **22** (2000), 41–62.
8. S. Jeevadoss and A. Muthusamy, *Decomposition of complete bipartite graphs into paths and cycles*, Discrete Math. **331** (2014), 98–108.
9. C. C. Lindner and C. A. Rodger, *Design theory*, 2nd ed., CRC Press, Boca Raton, 2009.
10. E. S. Mahmoodian and M. Mirzakhani, *Combinatorial advances*, Mathematics and Its Applications, vol. 329, ch. Decomposition of complete tripartite graphs into 5-cycles, pp. 235–241, Kluwer Academic Publishers, Dordrechtn, 1995.
11. H. M. Priyadharsini and A. Muthusamy, $(G_m, H_m)$*-multidecomposition of* $K_{m,m}$, Bull. ICA **66** (2012), 42–48.
12. B. R. Smith, *Decomposing complete equipartite graphs into cycles of length* $2p$, J. Combin. Des. **16** (2008), 244–252.
13. D. Sotteau, *Decomposition of* $K_{m,n}\,(K*_{m,n})$ *into cycles(circuits) of length* $2k$, J. Combin. Theory Ser. B **30** (1981), 75–81.

Department of Mathematics, Periyar University, Salem Tamilnadu, India.
*E-mail address*: priyaah.ss@gmail.com

Department of Mathematics, Periyar University, Salem Tamilnadu, India.
*E-mail address*: appumuthusamy@gmail.com