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The Integer-antimagic Spectra of Graphs with a Chord

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The Integer-antimagic Spectra of Graphs with a Chord

Cover Page Footnote

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Abstract

Let A be a nontrival abelian group. A connected simple graph $G = (V, E)$ is A-antimagic if there exists an edge labeling $f : E(G) \to A \setminus \{0\}$ such that the induced vertex labeling $f^+ : V(G) \to A$, defined by $f^+(v) = \sum_{uv \in E(G)} f(uv)$, is injective. The integer-antimagic spectrum of a graph G is the set $\mathrm{IAM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$. In this paper, we determine the integer-antimagic spectra for cycles with a chord, paths with a chord, and wheels with a chord.

1 Introduction

Labelings form a large and important area of study in graph theory. First formally introduced by Rosa [7] in the 1960s, graph labelings have captivated the interest of many mathematicians in the ensuing decades. In addition to the intrinsic beauty of the subject matter, graph labelings have applications (discussed in papers by Bloom and Golomb [1, 2]) in graph factorization problems, X-ray crystallography, radar pulse code design, and addressing systems in communication networks. The interested reader is directed to Gallian's [4] dynamic survey, which contains thousands of references to research papers and books on the topic of graph labelings.

Let G be a connected simple graph. For any nontrivial abelian group A (written additively), let $A^* = A \setminus \{0\}$, where 0 is the additive identity of A. Let a function $f : E(G) \to A^*$ be an edge labeling of G and $f^+ : V(G) \to A$ be its induced vertex labeling, which is defined by $f^+(v) = \sum\limits_{uv \in E(G)} f(uv)$. If there exists an edge labeling f whose induced vertex labeling f^+ on $V(G)$ is injective, then we say that f is an *A-antimagic labeling* and that G is an *A-antimagic* graph. The *integer-antimagic spectrum* of a graph G is the set $\mathrm{IAM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$.

The concept of the A-antimagicness property for a graph G (introduced independently in [3, 5]) naturally arises as a variation of the A-magic labeling problem (where the induced vertex labeling is a constant map). There is a large body of research on A-magic graphs within the mathematical literature. As for A-antimagic graphs (which is the focus of our paper), cycles, paths, various classes of trees, dumbbells, multi-cyclic graphs, $K_{m,n}$, $K_{m,n} - \{e\}$, tadpoles and lollipop graphs were investigated in [3, 6, 8, 9, 10].

A result of Jones and Zhang [5] finds the minimum element of $\mathrm{IAM}(G)$, for all connected graphs on three or more vertices. In their paper, a $\mathbb{Z}_n$-antimagic labeling of a graph on n vertices is referred to as a *nowhere-zero modular edge-graceful labeling*. They proved the following theorem.

Theorem 1.1 (Jones and Zhang [5]). *If G is a connected simple graph of order $n \geq 3$, then $min\{t : t \in \mathrm{IAM}(G)\} \in \{n, n+1, n+2\}$. Furthermore,*

1. *$min\{t : t \in \mathrm{IAM}(G)\} = n$ if and only if $n \not\equiv 2 \pmod 4$, $G \neq K_3$, and G is not a star of even order,*

2. *$min\{t : t \in \mathrm{IAM}(G)\} = n + 1$ if and only if $G = K_3$ or $n \equiv 2 \pmod 4$ and G is not a star of even order, and*

3. min$\{t : t \in \mathrm{IAM}(G)\} = n + 2$ *if and only if* G *is a star of even order.*

Motivation for our current work is found in the following conjecture [6].

Conjecture 1.1. *Let* G *be a connected simple graph. If* t *is the least positive integer such that* G *is* $\mathbb{Z}_t$*-antimagic, then* $\mathrm{IAM}(G) = \{k : k \geq t\}$.

In [3, 6, 8, 9, 10], Conjecture 1.1 was shown to be true for various classes of graphs. The purpose of this paper is to provide additional evidence for Conjecture 1.1 by verifying it for cycles with a chord, paths with a chord, and wheels with a chord. We use constructive methods to determine the integer-antimagic spectra for these particular classes of graphs.

2 Cycles with a Chord

In this paper, we use the constructed labelings from the proof of the following theorem.

Theorem 2.1 (Chan, Low and Shiu [3]). *If* $r = 0, 1, 3$*, then* C_{4m+r} *is* $\mathbb{Z}_k$*-antimagic, for all* $m \in \mathbb{N}$, $k \geq 4m + r$. C_{4m+2} *is* $\mathbb{Z}_k$*-antimagic, for all* $m \in \mathbb{N}$, $k \geq 4m + 3$.

For the sake of completeness, here are the labelings. Let $e_1, e_2, ..., e_n$ be edges of C_n arranged in counter-clockwise direction. A $\mathbb{Z}_k$-antimagic labeling of C_n can be obtained as follows.

Case 1: $n = 4m$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m; \\ 3 + 2(2m - \lceil \frac{i}{2} \rceil) & \text{if } 2m + 1 \leq i \leq 4m. \end{cases}$$

Case 2: $n = 4m + 1$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m; \\ 3 + 2(2m - \lceil \frac{i}{2} \rceil) & \text{if } 2m + 1 \leq i \leq 4m + 1. \end{cases}$$

Case 3: $n = 4m + 2$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m + 3; \\ 3 + 2(2m - \lceil \frac{i-2}{2} \rceil) & \text{if } 2m + 4 \leq i \leq 4m + 2. \end{cases}$$

Case 4: $n = 4m + 3$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m + 3; \\ 3 + 2(2m - \lceil \frac{i-3}{2} \rceil) & \text{if } 2m + 4 \leq i \leq 4m + 3. \end{cases}$$

Let $(1, 2, ..., n)$ denote the n-cycle with counterclockwise edges $\{i, i+1\}$ for $1 \leq i \leq n-1$ and $\{1, n\}$. Let $[1, 2, ..., n]$ denote the path of length $n-1$ with edges $\{i, i+1\}$ for $1 \leq i \leq n-1$. Suppose that C_n is the cycle $(v_1, v_2, ..., v_n)$, and let $2 \leq l \leq \lfloor \frac{n}{2} \rfloor$. Define $C_n(l)$ to be the graph

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obtained from C_n by adding an edge $c = \{v_i, v_j\}$, where $l = min\{|i-j|, n-|i-j|\}$. We call $C_n(l)$ an *n-cycle* with a *chord* c of *perimeter* l.

Note that $C_n(l)$ is the union of two cycles which share a common edge, namely chord c of $C_n(l)$. The *minor subcycle of* $C_n(l)$ is the shorter of the two cycles, denoted by $C_n^-(l)$. The *major subcycle of* $C_n(l)$ is the longer of the two cycles, denoted by $C_n^+(l)$.

The *alternating cycle labeling* of $C_n = (1, 2, ..., n)$, starting with the edge $\{1, 2\}$, is the function $g : E(C_n) \to \{1, -1\}$, such that $g(\{1, 2\}) = 1$ and g alternates between 1 and -1.

The *alternating path labeling* of the path $P_n = [1, 2, ..., n]$ starting with edge $\{1, 2\}$ is the function $t : E(P_n) \to \{1, -1\}$, such that t alternates between 1 and -1, and the labeling on the first edge must be specified.

Lemma 2.2. *Let $n \geq 4$ be an integer and let l be a positive odd integer with $2 \leq l \leq \lfloor \frac{n}{2} \rfloor$. Then,* $\mathrm{IAM}(C_n(l)) = \{k : k \geq n\}$ *if $n \equiv 0, 1, 3,$ $(mod\ 4)$ and* $\mathrm{IAM}(C_n(l)) = \{k : k \geq n+1\}$ *if $n \equiv 2,$ $(mod\ 4)$.*

Proof. The main idea of the proof is to find an even cycle within $C_n(l)$ containing the chord and to use the alternating cycle labeling on it. One can choose the endpoints of the chord to be v_1 and v_{l+1}. Now let f be the $\mathbb{Z}_k$-antimagic labeling of C_n defined in Theorem 2.1. Since l is odd, $C_n^-(l)$ has even length. This follows from the fact that $C_n^-(l)$ contains an odd number of edges from C_n, along with the chord. Let g be the alternating cycle labeling on the edges of $C_n^-(l)$ starting with edge $\{v_1, v_2\}$.

Now define the labeling $h(e) = f(e) + g(e)$ where $g(e)$ is defined to be 0 for all edges not included in $C_n^-(l)$. Note that since $3 \leq l+1 \leq \lfloor \frac{n}{2} \rfloor + 1$, we have that for each $1 \leq i \leq l+1$, $f(e_i) = 1$ if and only if $i = 1$. The edge labeling on the chord is -1. Therefore, $h(e) \neq 0$ for all $e \in E(C_n(l))$. Furthermore, $h^+(e) = f^+(e)$ for all $e \in C_n(l)$. Since f is a $\mathbb{Z}_k$-antimagic labeling, so is h. □

Figure 1 illustrates the proof of Lemma 2.2.

Theorem 2.3. *Let n and l be postive integers with $2 \leq l \leq \lfloor \frac{n}{2} \rfloor$. Then* $\mathrm{IAM}(C_n(l)) = \{k : k \geq n\}$ *if $n \equiv 0, 1, 3,$ $(mod\ 4)$ and* $\mathrm{IAM}(C_n(l)) = \{k : k \geq n+1\}$ *if $n \equiv 2,$ $(mod\ 4)$.*

Proof. It suffices to consider only the case where <u>l is even</u>, since the case where l is odd is addressed in Lemma 2.2. First, let f be the $\mathbb{Z}_k$-antimagic labeling of C_n defined in Theorem 2.1.

If n is odd, then $C_n^+(l)$ has even length. This follows from the fact that $C_n^+(l)$ contains an odd number of edges from C_n, along with the chord. We will define the labeling $h(e) = f(e) + g(e)$ where g is the alternating cycle labeling on the edges of $C_n^+(l)$ starting with the edge $\{v_1, v_n\}$, and $g(e)$ is defined to be 0 for all edges not in $C_n^+(l)$. To ensure that $h(e) \neq 0$ for all $e \in V(C_n(l))$, we must show that $f(e) \neq -g(e)$. We observe the following about the minimum and maximum values of f.

In the case where $n = 4m+1$, the maximum value of f is given by $f(e_{2m+1}) = 2m+1$. The minimum value of f is 1. Furthermore, $f(e_i) = 1$ if and only if $i \in \{1, 4m+1\}$.

In the case where $n = 4m+3$, the maximum value of f is given by $f(e_{2m+3}) = 2m+3$. The minimum value of f is 1. Furthermore $f(e_i) = 1$ if and only if $i = 1$.

In either case, $f(e) \neq -1$ for all edges e; therefore $f(e) \neq -g(e)$ for all edges where $g(e) = 1$. We now have to check the edge labels on e_1 and e_n. Since e_1 is not in $C_n^+(l)$,

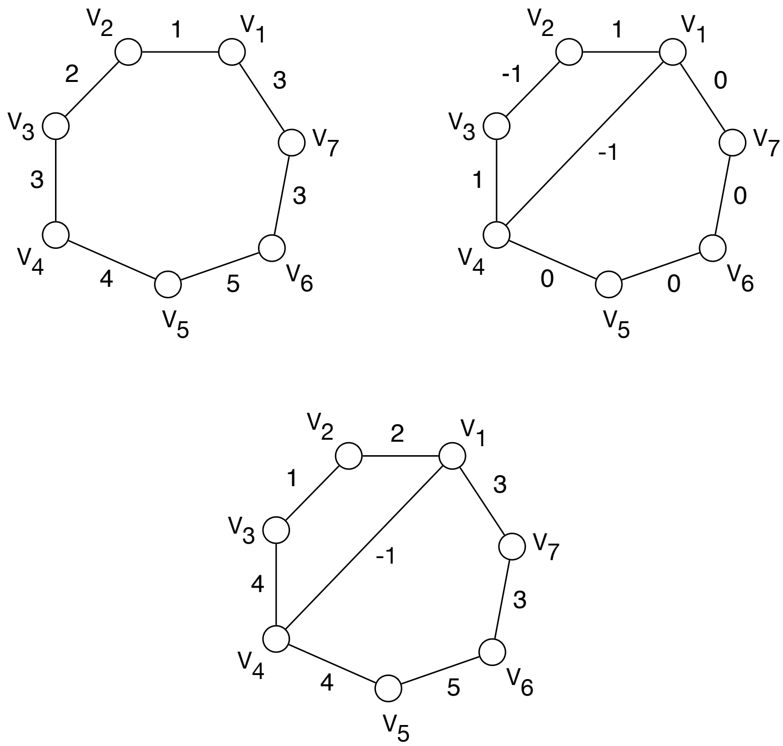

Figure 1: $\text{IAM}(C_7(3)) = \{7, 8, 9, \dots\}$.

$g(e_1) = 0$ by definition; thus, $f(e_1) \neq -g(e_1)$. By the definition of g, we have that $g(e_n) = 1$; thus, $f(e_n) \neq -g(e_n)$. Thus when n is odd, we have that h is the desired $\mathbb{Z}_k$-antimagic labeling.

In the case where n is even, we consider the following two subcases.

Subcase 1: $n = 4m + 2$. We assume chord c has endpoints v_{2m+3} and v_{2m+3+l}. Note that $2 \leq l \leq 2m$. So in the case where $l = 2m$, the endpoints of c are v_{2m+3} and v_1. Define P to be the path

$$[v_{2m+3}, v_{2m+3+l}, v_{2m+2+l}, v_{2m+1+l}, v_{2m+l}, \dots, v_{2m+4}].$$

Now, define $h : E(C_n(l)) \to \mathbb{Z}_k^*$ by

$$h(e) = f(e) + z(e),$$

where addition is in $\mathbb{Z}_k$. Here, f is the $\mathbb{Z}_k$-antimagic labeling for C_n (given in Theorem 2.1, Case 3) and

$$z(e) = \begin{cases} t(e) & \text{if } e \in P; \\ 0 & \text{otherwise.} \end{cases}$$

Here, t is the alternating path labeling of P starting with the chord, which is labeled $t(c) = -1$.

First, we will show that no edge is labeled 0. Since $n \equiv 2 \pmod 4$, $f(e_i) = 1$ if and only if i =1. By the definition of P, we have that $e_1 \notin P$. Therefore, $h(e) - 1 \not\equiv 0 \pmod k$ for all edges $e \in E(C_n(l))$. By the definition of f, we see that $\max\{f(e_i) : 1 \leq i \leq 4m+2\} = f(e_{2m+3}) = 2m+3$. Therefore, $h(e) + 1 \not\equiv 0 \pmod k$. Thus, $h(e) \not\equiv 0 \pmod k$ for all edges $e \in E(C_n(l))$.

Next, we will show that all induced vertex labels are distinct. Since t is the alternating path labeling, we have that $h^+(v) = f^+(v)$ for all vertices v (besides the two endpoints of P, which are v_{2m+3} and v_{2m+4}). We claim that $h^+(v_{2m+3}) = f^+(v_{2m+4})$ and $h^+(v_{2m+4}) = f^+(v_{2m+3})$. To see this, observe that by the definition of f, we have that

$$\begin{aligned} f^+(v_{2m+3}) &= f(e_{2m+2}) + f(e_{2m+3}) \\ &= (2m+2) + (2m+3) \\ &= 4m+5 \end{aligned}$$

and

$$\begin{aligned} f^+(v_{2m+4}) &= f(e_{2m+3}) + f(e_{2m+4}) \\ &= (2m+3) + 3 + 2\Big(2m - \Big\lceil \frac{2m+4-2}{2} \Big\rceil\Big) \\ &= 4m+4. \end{aligned}$$

Then, we also have

$$h^+(v_{2m+3}) = f^+(v_{2m+3}) + t(c) = 4m+5-1 = 4m+4$$

and

$$h^+(v_{2m+4}) = f^+(v_{2m+4}) + t(e_{2m+4}) = 4m+4+1 = 4m+5.$$

Thus, the net result of combining the alternating path labeling of P with the $\mathbb{Z}_k$-antimagic labeling of C_{4m+2} is that the induced vertex labels of v_{2m+3} and v_{2m+4} are transposed. Therefore, h is the desired $\mathbb{Z}_k$-antimagic labeling.

<u>Subcase 2</u>: $n = 4m$. We assume chord c has endpoints v_{2m+1} and v_{2m+1+l}. Define P to be the path

$$[v_{2m+1}, v_{2m+1+l}, v_{2m+l}, v_{2m+l-1}, v_{2m+l-2}, \ldots, v_{2m+2}].$$

We define the same labeling h as in the proof of Subcase 1, but with the alternating path labeling t of the newly defined path P starting with the chord $t(c) = 1$. The argument follows the same structure. The only differences are the calculations of the induced vertex labels, which are as follows.

By the definition of f, we have that

$$\begin{aligned} f^+(v_{2m+1}) &= f(e_{2m}) + f(e_{2m+1}) \\ &= 2m+3+2\Big(2m - \Big\lceil \frac{2m+1}{2} \Big\rceil\Big) \\ &= 4m+1 \end{aligned}$$

and

$$\begin{aligned} f^+(v_{2m+2}) &= f(e_{2m+1}) + f(e_{2m+2}) \\ &= 2m+1+3+2\Big(2m - \Big\lceil \frac{2m+2}{2} \Big\rceil\Big) \\ &= 4m+2. \end{aligned}$$

Then,

$$h^+(v_{2m+1}) = f^+(v_{2m+1}) + t(c) = 4m+1+1 = 4m+2$$

and

$$h^+(v_{2m+2}) = f^+(v_{2m+2}) + t(e_{2m+2}) = 4m+2-1 = 4m+1.$$

Thus, h is the desired $\mathbb{Z}_k$-antimagic labeling. □

Figures 2 and 3 illustrate the proof of Theorem 2.3.

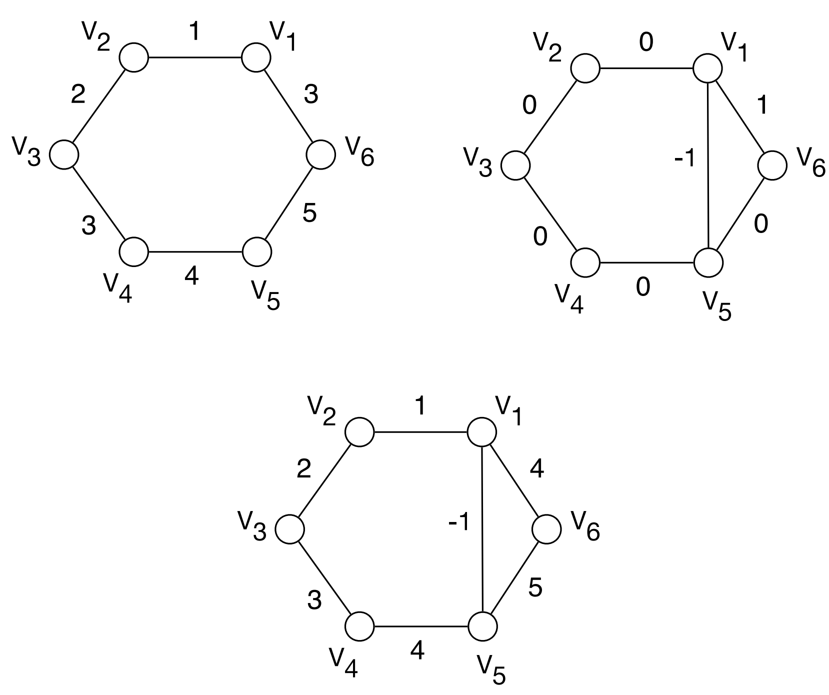

Figure 2: $\mathrm{IAM}(C_6(2)) = \{7, 8, 9, \ldots\}$.

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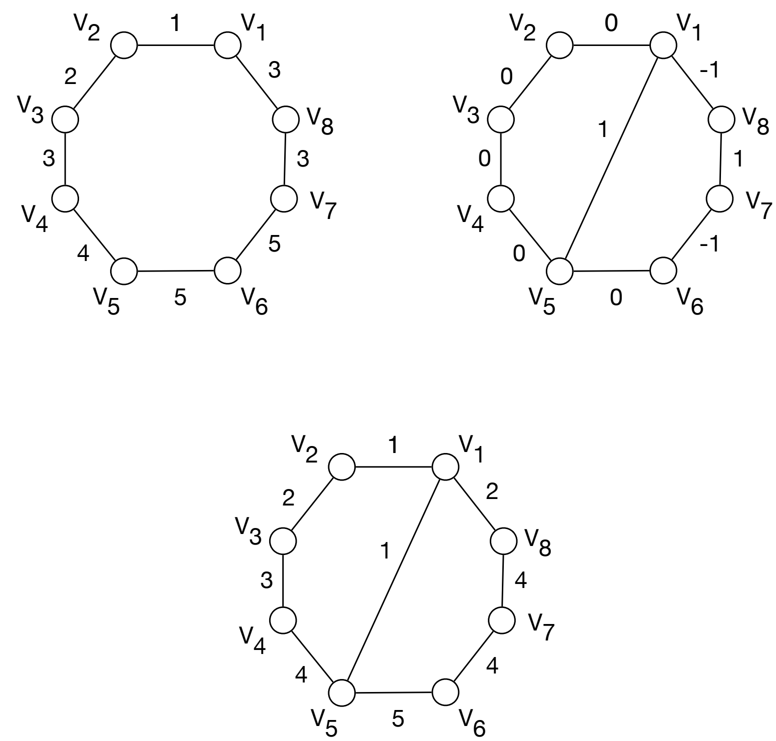

Figure 3: $\mathrm{IAM}(C_8(4)) = \{8, 9, 10, \dots\}$.

3 Paths with a Chord

A *path with a chord c of perimeter l* (denoted $P_n(l)$), is defined similarly to a cycle with a chord of perimeter l. More precisely, $P_n(l)$ denotes the graph obtained by adding an edge, called the *chord*, to the path P_n. The *perimeter* of the chord is the length of the path from one end-vertex to the other which does not consist of the chord itself. We assume the two endpoints of the chord are not the end vertices of the original path, since this would simply be a cycle. The following theorem gives constructions of labelings which characterize the $\mathrm{IAM}(P_n)$. Again in this paper, we use these particular labelings in characterizing the $\mathrm{IAM}(P_n(l))$.

Theorem 3.1 (Chan, Low and Shiu [3]). *If $r = 0, 1, 3$, then P_{4m+r} is $\mathbb{Z}_k$-antimagic, for all $m \in \mathbb{N}$, $k \geq 4m + r$. P_{4m+2} is $\mathbb{Z}_k$-antimagic, for all $m \in \mathbb{N}$, $k \geq 4m + 3$.*

For the sake of completeness, here are the labelings. Let $e_1, e_2, ..., e_{n-1}$ be edges of P_n, from left to right. A $\mathbb{Z}_k$-antimagic labeling of P_n can be obtained as follows.

<u>Case 1</u>: $n = 4m$:

$$f(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd;} \\ \frac{i}{2} & \text{if } i \text{ is even and } 2 \leq i \leq 2m-2; \\ \frac{i+2}{2} & \text{if } i \text{ is even and } 2m \leq i \leq 4m-2. \end{cases}$$

Case 2: $n = 4m + 1$:

$$f(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even;} \\ \frac{i+3}{2} & \text{if } i \text{ is odd and } 1 \le i \le 2m-3; \\ \frac{i+5}{2} & \text{if } i \text{ is odd and } 2m-1 \le i \le 4m-1. \end{cases}$$

Case 3: $n = 4m + 2$:

$$f(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd;} \\ \frac{i+2}{2} & \text{if } i \text{ is even and } 2 \le i \le 2m-2; \\ \frac{i+4}{2} & \text{if } i \text{ is even and } 2m \le i \le 4m. \end{cases}$$

Case 4: $n = 4m + 3$:

$$f(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even;} \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } 1 \le i \le 2m-1; \\ \frac{i+3}{2} & \text{if } i \text{ is odd and } 2m+1 \le i \le 4m+1. \end{cases}$$

A *tadpole* graph $T(r, s)$ is obtained by joining a cycle C_r and an end-vertex of a path P_s by a bridge, where $r \ge 3$ and $s \ge 1$. The following technical lemma will be used in the proof of Theorem 3.3.

Lemma 3.2 (Shiu, Sun and Low [10])**.** *For $r \ge 3$ and $s \ge 1$,*

$$\mathrm{IAM}(T(r,s)) = \begin{cases} r+s, r+s+1, \ldots & \text{if } r+s \not\equiv 2 \pmod 4; \\ r+s+1, r+s+2, \ldots & \text{if } r+s \equiv 2 \pmod 4. \end{cases}$$

We are now ready to establish the integer-antimagic spectrum of a path with a chord.

Theorem 3.3. *Let n and l be postive integers with $2 \le l \le \lfloor \frac{n}{2} \rfloor$. Then,* $\mathrm{IAM}(P_n(l)) = \{k : k \ge n\}$ *if* $n \equiv 0, 1, 3 \pmod 4$, *and* $\mathrm{IAM}(P_n(l)) = \{k : k \ge n+1\}$ *if* $n \equiv 2 \pmod 4$.

Proof. First, let f be the $\mathbb{Z}_k$-antimagic labeling of P_n defined in Theorem 3.1. We make several observations about the nature of f and f^+. From the definition of f, we see that if $f(e_i) = 1$, then $i \in \{1, 2\}$. Thus, at most two edges are labeled with 1. We can also see that for every edge $e \in E(P_n)$, $f(e) \le \frac{n}{2}+2$; in particular, $f(e) \ne k-1$, since $k \ge n$. The last (and most nuanced) observation is that there are at most two edges, with the exception of the first and last edges, which do not have consecutive induced vertex labels on their endpoints. Moreover, the two edges whose induced vertex labels are not consecutive are never adjacent to each other.

The graph $P_n(l)$ contains exactly one cycle which will be denoted C. Without loss of generality, we assume that $l \ge 2$ and that l is even. In the case that l is odd, C would have even length. Here, our claim is established by overlaying the alternating cycle labeling, much like how it was used in the proof of Theorem 2.3.

Case 1: $2 \le l \le n-3$. By symmetry, we may assume that $e_1, e_2 \notin E(C)$. Thus for all $e \in E(C)$, $f(e) \notin \{1, k-1\}$. Due to the observations above, there exists at least one edge (say, $x \in E(C)$) whose endpoints (say, α and β) have consecutive induced vertex labels. Thus, $C - x$ is a path of even length and $f^+(\alpha) = a-1$ and $f^+(\beta) = a$ for some $a \in \mathbb{Z}_n$. We let t be the alternating path labeling of $C - x$ for which the first edge is labeled with 1 and that edge is chosen to be the one whose endpoint is α.

Now, define $h : E(P_n(l)) \to \mathbb{Z}_k^*$ by

$$h(e) = f(e) + z(e),$$

where addition is in $\mathbb{Z}_k$, and

$$z(e) = \begin{cases} t(e) & \text{if } e \in C - x; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $h^+(v) = (f+z)^+(v) = (f+t)^+(v)$ for all $v \notin \{\alpha, \beta\}$. Furthermore, $(f+t)^+(\alpha) = f^+(\beta)$ and $(f+t)^+(\beta) = f^+(\alpha)$. In other words, by overlaying the labeling t on top of f we transpose the induced vertex labels of α and β, and we leave all other induced vertex labels fixed. Thus, h is the desired $\mathbb{Z}_k$-antimagic labeling of $P_n(l)$.

Case 2: $l = n-2$. Here, we have a cycle with a pendant path (i.e., a tadpole graph). Hence, the claim is true for this case, by Lemma 3.2. □

Figures 4 and 5 illustrate the proof of Theorem 3.3.

4 Wheels with a Chord

Let W_n denote the *wheel on n spokes*, which is the graph containing a cycle of length n with another special vertex not on the cycle, called the *central vertex*, that is adjacent to every vertex on the cycle. The integer-antimagic spectra of wheels were determined in [6], and is stated in the following theorem.

Theorem 4.1 (Roberts and Low [6]). *For each integer $m \ge 1$, $\mathrm{IAM}(W_{4m+r}) = \{k : k \ge 4m+r+1\}$ if $r = 0, 2, 3$ and $\mathrm{IAM}(W_{4m+1}) = \{k : k \ge 4m+3\}$.*

Figure 6 illustrates Theorem 4.1.

A *wheel on n spokes with a chord* (denoted W_n^+) is a graph obtained by adding an edge to W_n. Since the central vertex of W_n is adjacent to all other vertices, a chord added to W_n must have both endpoints on the outer cycle. Note that W_3^+ is a multigraph and is not considered in this paper.

Theorem 4.2. *For each integer $n \ge 4$, $\mathrm{IAM}(W_n^+) = \{k : k \ge n+1\}$ if $n \equiv 0, 2, 3 \pmod 4$, and $\mathrm{IAM}(W_n^+) = \{k : k \ge n+2\}$ if $n \equiv 1 \pmod 4$.*

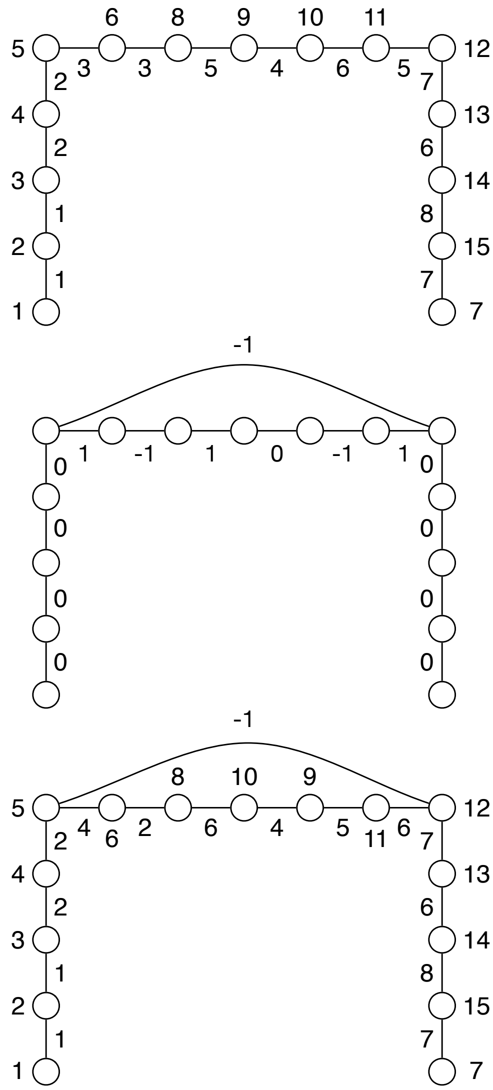

Figure 4: $\text{IAM}(P_{15}(6)) = \{15, 16, 17, \dots\}$.

Proof. For $n \in \{4, 5, 6, 7, 8, 9\}$, the $\mathbb{Z}_k$-antimagic labelings of W_n^+ are given in Figures 7−12.

Suppose $n \geq 10$. Let f be the $\mathbb{Z}_k$-antimagic labeling of W_n defined in Theorem [6]. Let the vertices w and z be the endpoints of the chord in W_n^+, and note that neither w nor z are the central vertex. Denote the central vertex by y. There also must exist a vertex, say x, on the outer cycle which is different from z and is adjacent to w.

Consider the 4-cycle C, with edges $\{w, x\}$, $\{x, y\}$, $\{y, z\}$, and $\{z, w\}$. Since $n \geq 10$, we have that $k \geq 11$. So by the Pigeonhole Principle, there exists some $a \in \mathbb{Z}_k^*$ such that $\pm a \notin \{\pm f(\{w, x\}), \pm f(\{x, y\}), \pm f(\{y, z\}), \pm f(\{z, w\})\}$. We will overlay a variation of an

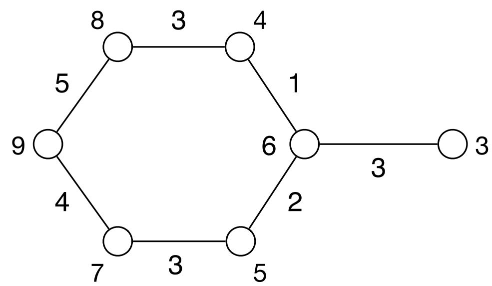

Figure 5: $\mathrm{IAM}(T(6,1)) = \{7, 8, 9, \ldots\}$.

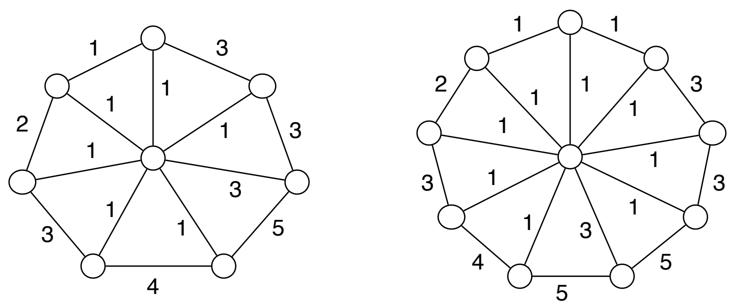

Figure 6: $\mathrm{IAM}(W_7) = \{8, 9, 10, \ldots\}$ and $\mathrm{IAM}(W_9) = \{11, 12, 13, \ldots\}$.

alternating cycle labeling on C by defining the edge labeling $t : E(W_n^+) \to \mathbb{Z}_k^*$ as follows.

$$t(e) = \begin{cases} a & \text{if } e \in \{\{w,x\},\{y,z\}\}; \\ -a & \text{if } e \in \{\{x,y\},\{w,z\}\}; \\ 0 & \text{otherwise.} \end{cases}$$

Now, define $h : E(W_n^+) \to \mathbb{Z}_k^*$ by $h(e) = f(e) + t(e)$, where addition is in $\mathbb{Z}_k$. Note that $h(e) \neq 0$ for all $e \in E(W_n^+)$, since a was chosen appropriately. Furthermore, $h^+(v) = f^+(v)$ for all $v \in V(W_n^+)$. Thus, h is the desired $\mathbb{Z}_k$-antimagic labeling. □

References

[1] G.S. Bloom and S.W. Golomb, Numbered complete graphs, unusual rulers and assorted applications, *Theory and Applications of Graphs: Lecture Notes in Math.*, Vol. 642, Springer-Verlag, New York (1978), pp. 53-65.

[2] G.S. Bloom and S.W. Golomb, Applications of numbered undirected graphs, *Proc. of the IEEE*, 65(4):562–570, (1977).

[3] W.H. Chan, R.M. Low and W.C. Shiu, Group-antimagic labelings of graphs, *Congr. Numer.*, 217:21–31, (2013).

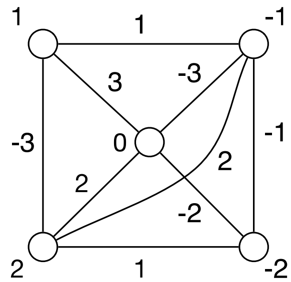

Figure 7: $\mathrm{IAM}(W_4^+) = \{5, 6, 7, \dots\}$.

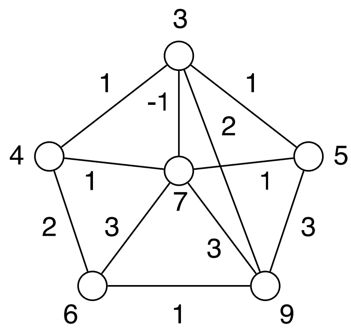

Figure 8: $\mathrm{IAM}(W_5^+) = \{7, 8, 9, \dots\}$.

[4] J.A. Gallian, A dynamic survey of graph labeling, *Electronic Journal of Combinatorics*, Dynamic Survey DS6, (2018), `http://www.combinatorics.org`.

[5] R. Jones and P. Zhang, Nowhere-zero modular edge-graceful graphs, *Discussiones Mathematicae Graph Theory*, 32:487–505, (2012).

[6] D. Roberts and R.M. Low, Group-antimagic labelings of multi-cyclic graphs, *Theory and Applications of Graphs.*, 3(1) Article 6, (2016), electronic.

[7] A. Rosa, On certain valuations of the vertices of a graph, in: Théorie des graphes, journées internationales d'études, Rome 1966 (Dunod, Paris, 1967) 349–355.

[8] W.C. Shiu and R.M. Low, The integer-antimagic spectra of dumbbell graphs, *Bull. Inst. Combin. Appl.*, 77:89–110, (2016).

[9] W.C. Shiu and R.M. Low, Integer-antimagic spectra of complete bipartite graphs and complete bipartite graphs with a deleted edge, *Bull. Inst. Combin. Appl.*, 76:54–68, (2016).

[10] W.C. Shiu, P.K. Sun and R.M. Low, Integer-antimagic spectra of tadpole and lollipop graphs, *Congr. Numer.*, 225:5–22, (2015).

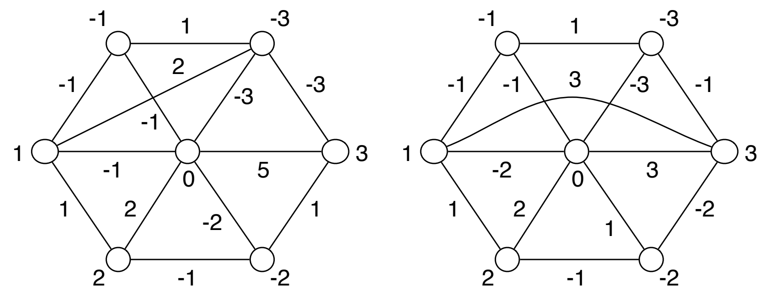

Figure 9: $\mathrm{IAM}(W_6^+) = \{7, 8, 9, \ldots\}$.

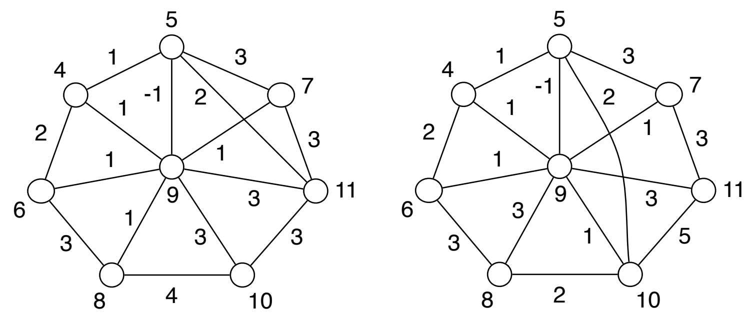

Figure 10: $\mathrm{IAM}(W_7^+) = \{8, 9, 10, \ldots\}$.

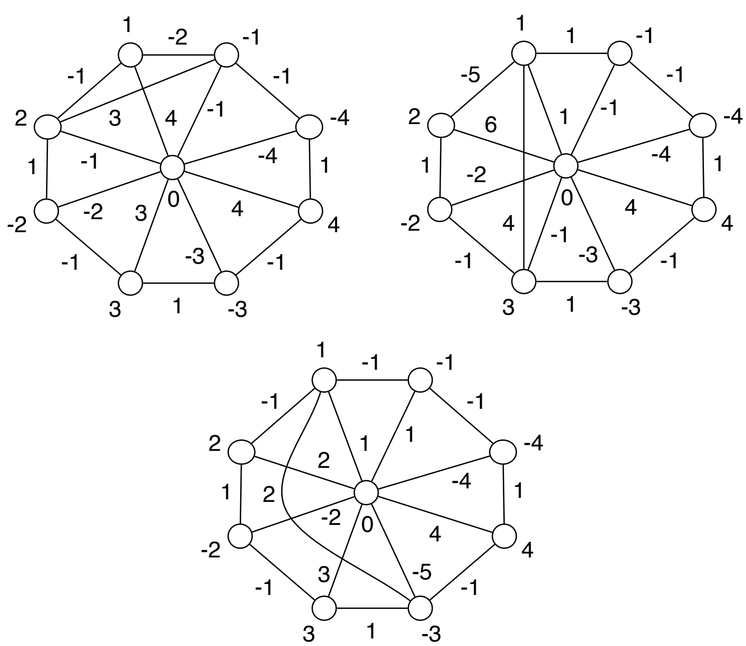

Figure 11: $\mathrm{IAM}(W_8^+) = \{9, 10, 11, \ldots\}$.

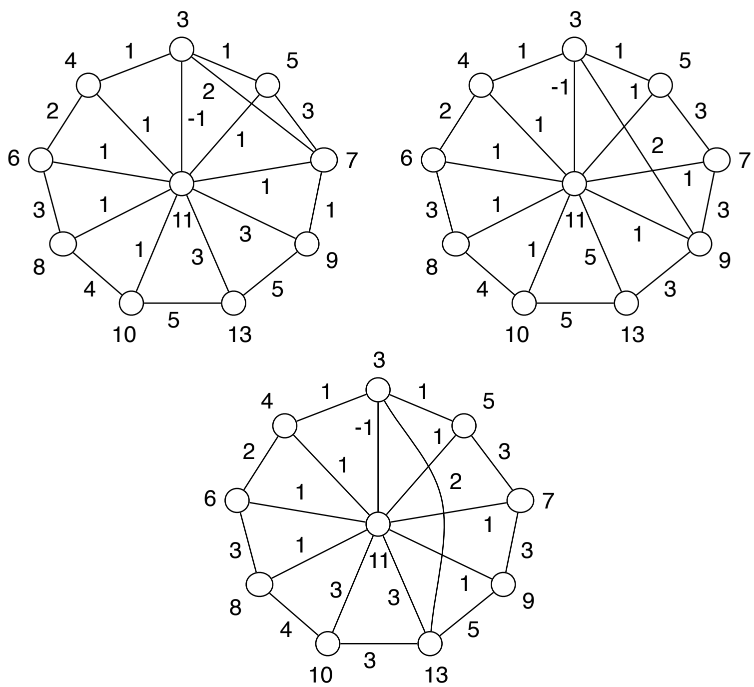

Figure 12: $\mathrm{IAM}(W_9^+) = \{11, 12, 13, \dots\}$.