Environment. Technology. Resources. Rezekne, Latvia
Proceedings of the 12th International Scientific and Practical Conference. Volume II, 165-169

Probabilistic Inference for Interval Probabilities in Decision-Making Processes

Oleg Uzhga-Rebrov
Information and Communication Technologies Research Centre
Rezekne Academy of Technologies
Rezekne, Latvia
rebrovs@tvnet.lv

Galina Kuleshova
Institute of Information Technology
Riga Technical University
Riga, Latvia
galina.kulesova@rtu.lv

***Abstract*— The present paper considers one approach to Bayes' formula based probabilistic inference under interval values of relevant probabilities; the necessity of it is caused by the impossibility to obtain reliable deterministic values of the required probabilistic evaluations. The paper shows that the approach proves to be the best from the viewpoint of the required amount of calculations and visual representation of the results. The execution of the algorithm of probabilistic inference is illustrated using a classical task of decision making related to oil mining. For visualisation purposes, the state of initial and target information is modelled using probability trees.**

Keywords*— *Bayes' formula, event indicators, interval probabilistic inference, interval probability, probability tree.

I. Non-formal Introduction to Interval Probabilities

Probabilistic inference procedures are widely used in different scientific, technical and economic areas. To exemplify widespread procedures of this kind, one can mention marginalization of a set of joint probabilities, calculation of conditional probabilities in the set of joint probabilities, and calculation of the posterior probabilities of events on the basis of their prior probabilities and information provided by event indicators.

All original procedures of probabilistic inference have been developed for the cases when the values of relevant probabilities are set in a unique deterministic form. Unfortunately, it is not always possible to obtain such probability values. The main reasons for that are the shortage or complete absence of suitable statistical data and low confidence in the evaluations provided by experts.

What can be done in situations like that? One possible way is to use available deterministic evaluations ignoring their potential unreliability. Another way is to introduce some extent of uncertainty into relevant evaluations assuming that their real values lie within the specified bounds of uncertainty. In situations where for some reason or other it is impossible to obtain reliable evaluations of the necessary probabilities, the second option seems to be preferable. By introducing controlled extents of uncertainty in the evaluations that are of interest to us, we extend the possibility of obtaining uncertain results, but real value of that result lies within the set bounds of uncertainty.

The idea to apply non-point probabilities has a long history. The first formal use dated back at least to the middle of the 19th century, is connected with the name of George Boole who intended to co-ordinate theory of logic (that can express complete ignorance) and probability theory.

Since the 1990s, the theory has received strong impulse initiated by exhaustive foundations of P. Walley [1] who had introduced the term *imprecise probabilities*. To evaluate boundary values of probabilities, P.Walley [1] introduced into consideration both buying and selling price for a hypothetical gamble. Those two prices correspond to the lower and the upper probabilities that form an interval of possible values of relevant probability. That interpretation underlies the theory of uncertain probabilities of P. Walley. The theory can be regarded as a specific extension of the traditional subjective probability theory.

Walley's theory extends the traditional subjective probability theory via buying and selling prices for gambles, whereas Weichselberger's approach generalizes Kolmogorov's axioms without imposing an interpretation. On the other hand, Weichselberger [2]-[4] treats interval values of probabilities as initial data; based on it, he builds his theory of interval probabilities. Strict theory of interval probabilities is also described in [5].

II. Formal Concepts and Definitions of Interval Probabilities

Let there be a set of random events $A=\{a_i, i=1,...,n\}$. Let us assume that probabilities of occurrence of those events are set not in the deterministic form but in the form of intervals of possible values of those probabilities.

$$[l_i, u_i],\ i=1,...,n,$$

where l_i - lower (the least) possible value of

Print ISSN 1691-5402
Online ISSN 2256-070X
http://dx.doi.org/10.17770/etr2019vol2.4112

probability $p_i = p(a_i)$; u_i - upper (the largest) possible value of probability $p_i = p(a_i)$.

It is evident that while choosing one of the values $p_i \in [l_i, u_i]$, $i = 1,...,n$ in a random or systematized way, we get a set of deterministic probabilistic evaluations $\{p_i, i = 1,...,n\}$. In [6], a set of all possible probabilities of this kind is determined as follows:

$$P = \{p_i \in p(A) / l_i \le p_i \le u_i, \forall i\}, \tag{1}$$

where $p(A)$ denotes a set of all possible probabilistic evaluations defined in the set of random events *A*.

To avoid situations when $P = \varnothing$, boundary values of probabilistic intervals have to satisfy these restricting conditions:

$$\sum_{i=1}^{n} l_i \le 1 \le \sum_{i=1}^{n} u_i . \tag{2}$$

Probabilistic intervals satisfying conditions (2) are called *proper* intervals in [6]. It is evident that in the tasks of interval probabilistic inference we should only use proper intervals.

In common case $p_i \ge l_i$ and $p_i \le u_i$, $\forall a_i \in A$. If

$$l_i = \inf_{p_i \in \mathbf{P}} \text{ and } u_i = \sup_{p_i \in \mathbf{P}}, \forall i, \tag{3}$$

it means that deterministic values of probabilities can be selected over the whole interval $[l_i, u_i]$, also including its boundaries. In [5], probabilistic intervals satisfying (3) are called *reachable intervals*; whereas in [2]-[4], probabilities defined in the probabilistic intervals of general type are called R *-probabilities* but probabilities defined in the reachable intervals are called F *-probabilities with M structure.*

In [6], it is proved that for reachable probabilistic intervals these inequalities are valid:

$$\sum_{j \ne i} l_j + u_i \le 1, \forall i; \tag{4}$$

$$\sum_{j \ne i} u_j + l_i \le 1, \forall i. \tag{5}$$

In [2]-[4], [6], algorithms are proposed for determining marginal interval probabilities and conditional interval probabilities in the set of joint interval probabilities. It should be noted that the algorithms described in [6] are simpler and more operable than those presented in [2]-[4].

III. Interval Versions of Bayes' Formula

One specific interval version of Bayes' formula is proposed in [2]-[4]. Here, formula derivation is based on the simultaneous use of two concepts of interval probability: intuitive concept and canonical concept. The algorithm is quite complicated, is of rather artificial character and possesses limited operability; that is why, this version is not examined in the present paper.

A much more attractive version is an interval variant of Bayes' formula, which is based on the concept of generalised probabilistic intervals [7]-[8]. The presentation of the theory of generalised intervals can be found in [9]-[11], as well as in other works.

A classical interval is identified as a set of real numbers, while a generalized one is identified by means of predicates that are filled with real numbers, and its boundaries are not ordered in a conventional sense. The generalized interval $x = [\underline{x}, \overline{x}] \in KR$ is called proper if $\underline{x} \le \overline{x}$, and improper if $\underline{x} \ge \overline{x}$. The set of proper intervals is denoted as $IR = \{[\underline{x}, \overline{x}] / \underline{x} \le \overline{x}\}$, but the set of improper intervals is denoted as $\overline{IR} = \{[\underline{x}, \overline{x}], \underline{x} \ge \overline{x}\}$. Operations on the generalised intervals are determined based on Kaucher's arithmetic. [12].

Two specific mathematical operations are defined in the set of generalized intervals:

$$pro\mathbf{x} = [\min(\underline{x}, \overline{x}), \max(\underline{x}, \overline{x})]. \tag{6}$$

The result of that operation is a proper generalised interval.

$$imp\mathbf{x} = [\max(\underline{x}, \overline{x}), \min(\underline{x}, \overline{x})]. \tag{7}$$

This operation yields an improper generalised interval. The operation that follows transforms a proper generalised interval into an improper generalised interval.

$$dual\mathbf{x} = [\overline{x}, \underline{x}]. \tag{8}$$

Wang [7], [8] proposes this interval version of Bayes' formula:

$$p(E_i / A) = \frac{p(A/E_i)\, p(E_i)}{\sum_{j=1}^{n} dualp(A/E_j)\, dualp(E_j)} \tag{9}$$

where E_i, $i = 1,...,n$, - are mutually disjoint event partitions in Ω, and $\sum_{j=1}^{n} p(E_j) = 1$; $dualp(.)$ is defined in (8).

To simplify the calculations, this expression can be used:

$$\sum_{j=1}^{n} dualp(A/E_j)\, dualp(E_j) = \sum_{j=1}^{n} dual\left(p(A/E_j)\, p(E_j)\right) \tag{10}$$

Let us consider a simple illustrative example of probabilistic inference in a task of decision making based on the interval version of Bayes' formula (9). As an example, a classical task of assessing the chances of oil presence on a specific site is described, provided that the prior evaluations of these chances are set, and conditional deterministic evaluations of probabilities of the results of seismic exploration of the site are assigned. The following data are used as initial: a set of random events ("states of nature") $A = \{a_1, a_2\}$ where event a_1 corresponds to real presence of oil on the site, event a_2 corresponds to real absence of oil on the site. Let us call events a_1 and a_2 *geological events*. Let us assume that based on the expert evaluation, these interval values of probabilities of the events were assigned:

$$p(a_1)=[0.50,0.70], \qquad p(a_2)=[0.30,0.50].$$

Let us assume that a manager of an oil mining company has made a decision to arrange seismic exploration of the site to re-evaluate the prior values of probabilities $p(a_1)$ and $p(a_2)$. Let us denote a set of random events, outcomes of seismic exploration as $B=\{b_1,b_2\}$ where b_1 is an outcome indicating the presence of oil on the site but b_2 is an outcome indicating the absence of oil on the site. Let us call events b_1 and b_2 *seismic events*.

The specifics of a seismic exploration is that it can both confirm real presence or absence of oil on a site, and produce erroneous results, i.e., to show the presence of oil when it is missing in reality or to show the absence of oil when it is really present. Let us introduce this system of denotations:

b_1/a_1 - seismic exploration has confirmed real presence of oil on the site;

b_2/a_1 - seismic exploration has erroneously indicated the lack of oil on the site, though in reality oil is present;

b_1/a_2 - seismic exploration has erroneously indicated the presence of oil on the site, though in reality oil is not present on the site;

b_2/a_2 - seismic exploration has confirmed real absence of oil on the site.

Let there be set these interval values of conditional probabilities:

$$p(b_1/a_1)=[0.70,0.90],\ p(b_1/a_2)=[0.10,0.30]$$

$$p(b_2/a_1)=[0.10,0.30],\ p(b_2/a_2)=0.70,0.90.$$

As can easily be seen, these values are reachable values according to conditions (4) and (5). Actually, here we are not interested in conditional probabilities of the results of seismic study depending on the presence or lack of oil at a site, $p(b_j/a_i)$, $i,j=1,2$; instead, we are interested in conditional probabilities $p(a_i/b_j)$, $i,j=1,2$ of the presence or lack of oil on a specific site depending on the results of a seismic exploration.

The task in this example is to calculate the posterior probabilities $p(a_i/b_j)$, $i,j=1,2$, based on the information available. The initial state of information in the form of a probability tree is shown in Fig. 1.

Probabilities of the outcomes are calculated by multiplying the probabilities related to the tree branches leading to the given outcome. The calculated values of these probabilities are given in the end positions of probability tree in Fig. 1. Now all the necessary and sufficient information is available to calculate the required values of the posterior probabilities.

Event b_1 can occur jointly with event a_1 (outcome (1)), and jointly with event a_2 (outcome (3)). Therefore, the total probability of event b_1 can be calculated as follows:

$$p'(b_1)=p(1)+p(3)=[0.35,0.63]+[0.03,0.15]=[0.38,0.78]$$

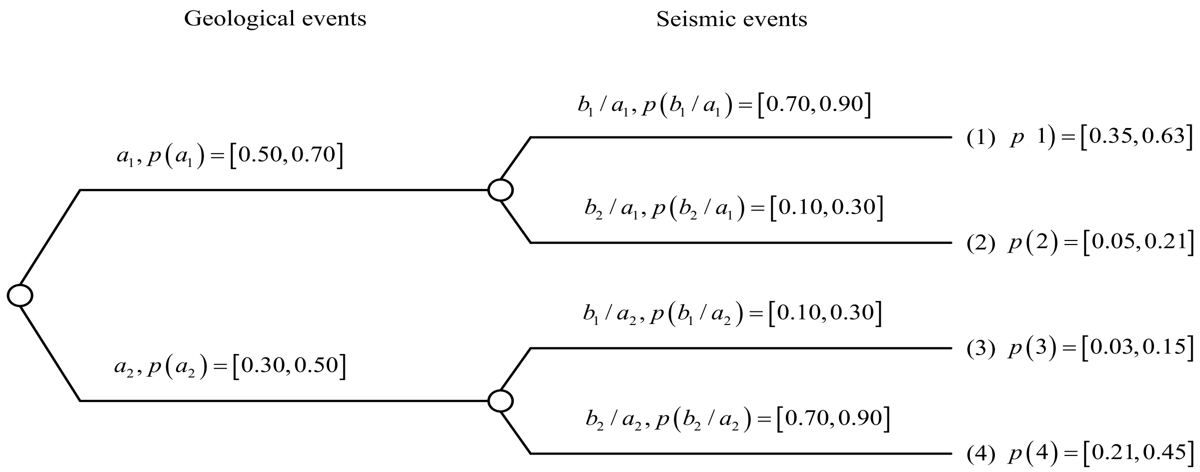

Fig. 1. Initial state of information in the task of oil mining.

Event b_2 can occur jointly with event a_1 (outcome (2)) and with event a_2 (outcome (4)). That is why,

$$p'(b_2)=p(2)+p(4)=[0.05,0.21]+[0.21,0.45]=[0.26,0.66]$$

The values $p'(b_1)$ and $p'(b_2)$ would be the values of denominators in the classic Bayes' formula should all the values in the above example be set in unambiguous deterministic form. However, in reality we deal with interval values of probabilities; due to that, we have to utilize values $dualp'(b_1)$ and $dualp'(b_2)$. Let us determine these values using (10).

$$p''(b_1)=dualp'(b_1)=dual[0.38,0.78]=[0.78,0.38]$$

$$p''(b_2) = dualp'(b_2) = dual[0.26, 0.66] = [0.66, 0.26]$$

Let us calculate the required values of the posterior probabilities using (9) and taking into account (10).

$$p(a_1/b_1) = \frac{p(b_1/a_1)p(b_1)}{dualp\left(p(b_1/a_1)p(b_1) + p(b_1/a_2)p(b_2)\right)} = \frac{p(1)}{p''(b_1)} =$$

$$= \frac{[0.35, 0.63]}{[0.78, 0.38]} = \left[\frac{0.35}{0.38}, \frac{0.63}{0.78}\right] = [0.92, 0.81].$$

The remaining calculations will be done by analogy.

$$p(a_2/b_1) = \frac{p(3)}{p''(b_1)} = \frac{[0.03, 0.15]}{[0.78, 0.38]} = \left[\frac{0.03}{0.38}, \frac{0.15}{0.78}\right] = [0.08, 0.19]$$

$$p(a_1/b_2) = \frac{p(2)}{p''(b_2)} = \frac{[0.05, 0.21]}{[0.66, 0.26]} = \left[\frac{0.05}{0.26}, \frac{0.21}{0.66}\right] = [0.19, 0.32]$$

$$p(a_2/b_2) = \frac{p(4)}{p''(b_2)} = \frac{[0.21, 0.45]}{[0.66, 0.26]} = \left[\frac{0.21}{0.26}, \frac{0.45}{0.66}\right] = [0.81, 0.68]$$

For values $p(a_1/b_1)$ and $p(a_2/b_2)$, we have obtained improper intervals. This result is absolutely correct from the viewpoint of the theory of generalised intervals; however, it is incorrect from the position of common sense. To get the correct result, we simply invert these improper interval using (6). As a result, we have

$$p(a_1/b_1) = [0.81, 0.92], \; p(a_1/b_2) = [0.19, 0.32],$$

$$p(a_2/b_1) = [0.08, 0.19], \; p(a_2/b_2) = [0.68, 0.81].$$

It is easy to see that the resulting intervals are reachable intervals. The check for reachability has to be made for pairs of intervals $p(\cdot/b_1)$ and $p(\cdot/b_2)$).

The target state of information in the form of a probability tree is represented in Fig. 2.

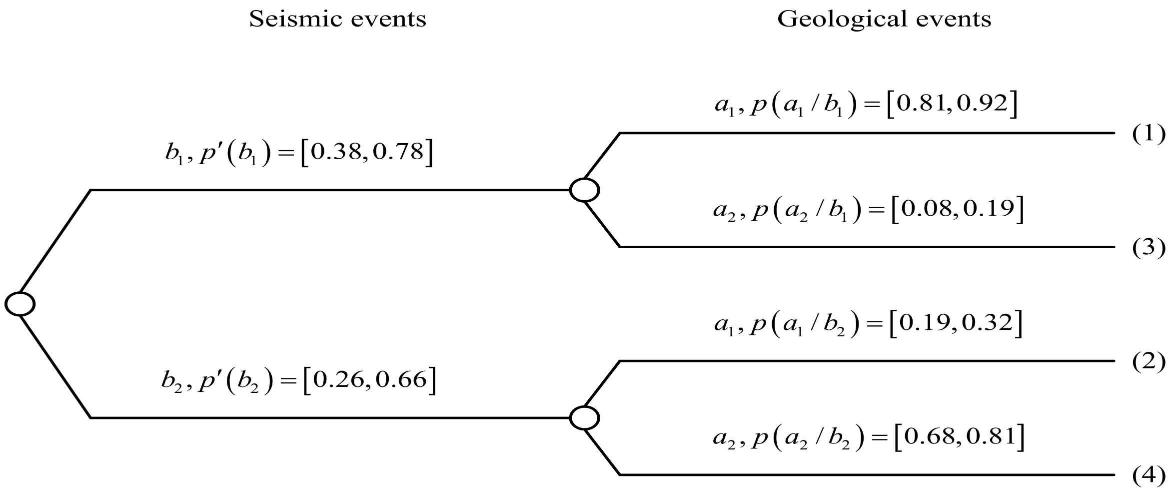

Fig. 2. Target state of information in the task of oil mining.

IV. Conclusions

The use of interval probabilistic values is aimed at modelling uncertainties regarding these values in a specific way. In the past decades, multiple variants of procedures of probabilistic inference at interval values of relevant probabilities have been proposed. The present paper shows that the most appropriate interval version of Bayes' formula is the version proposed in [7]-[8]. Utilization of the concept of generalised probability intervals helps to simplify the necessary calculations. This version is logically validated and does not require application of complicated concepts as it takes place in Weichselberger version [2]-[4]. The shortcoming of the considered technique is that when calculating the posterior values of probabilities, improper intervals of those values might be obtained. This shortcoming, however, can be easily overcome by means of simple inverting of improper probabilistic intervals.

Probabilistic inference under interval values has found wide application in different fields of science and technology. A considerable number of publications on this topic can evidence this. Out of numerous publications, paper [13] should be mentioned which presents a qualitative review on application of imprecise values in engineering. Papers [14]-[16] provide examples of application of interval values of probability in the processes of decision analysis and choice.

The use of procedures of interval probabilistic inference seems to be prospective in tasks of ecological risk assessment and making decisions related to the protection of the environment since a significant degree of uncertainty of initial information is characteristic of this research area.

References

[1] P. Walley, Statistical Reasoning with Imprecise Probabilities, London: Chapman and Hall, 1991.

[2] K. Weichselberger, The Theory of Interval-Probability as a Unifying Concept for Uncertainty: 1st Int. Symposium on Imprecise Probabilities and Their Application, 29 June – 2 July, 1999, Ghent, Belgium. 1999, 9 p.

[3] K. Weichselberger and Th. Augustin. On the Symbiosis of Two Concepts of Conditional Interval Probability. In J. Bernard, T. Seidenfeld, and M. Zaffalon (Editors), ISIPTA'03, Waterloo: Carleton Scientific, 2003, pp. 608 – 630.

[4] K. Weichselberger, Elementare Grundbegriffe einer allgemeinen Wahrscheinlichkeitsrechnung I – Intervallwahrscheinlichkeitn als umfassendes Konzept. In cooperation with T. Augustin and Wallner A., Heidelberg: Physica, 2001.

[5] V. Kuznetsov, Interval Statistical Models. Moscow: Radio i Svyaz Publishers, 1991. (In Russian)

[6] L. M. De Campos, J.F. Huete and S. Moral, "Probability Intervals: A Tool for Uncertain Reasoning, " in Int. Journal of Uncertainty, Fuzziness and Knowledge-Based Systems. Vol. 2, No. 2, 1994. pp. 167 – 196.

[7] Y. Wang, "Imprecise Probabilities with a Generalized Form," Proc. 3rd Int Workshop on Reliability Engineering.

[8] Y. Wang, "Imprecise probabilities based on generalized intervals for system reliability assessment", in Int. Journal Reliability and Safety, Vol. 4, No. 4, 2010.

[9] E.M.A. Gardenes, L. Sainz, R. Jorba et al., "Modal Intervals", Reliable Computing, 7 (2), 2001, pp. 77 – 111.

[10] S.R. Shary, "A new technique in system analysis under interval uncertainty and ambiguity". Reliable Computing, 8 (2), 2002, pp. 321 – 418.

[11] Popova E.D., "Multiplication distributivity of proper and improper intervals". Reliable Computing, 7 (2), 2001, pp. 129 – 140.

[12] E. Kaucher, "Interval analysis in the extended interval space IR". Computing Supplement, 2, 1980, pp. 33 – 49.

[13] M. Beer, S. Ferson, and V.Kreinovich, "Imprecise probabilities in engineering analyses". Mechanical Systems and Signal Processing, Vol. 37, Issue 1 – 2, 2013, pp. 4 – 29.

[14] L.M. Zeinalova, "Decision making with combined states under internal uncertainty". Procedia Computer Science, 102, 2016, pp. 223 – 230.

[15] S. Rinard, "A Decision Theory for Imprecise Probabilities". Philosophers' Imprint, Vol. 15, No. 7, 2015, pp. 1-16.

[16] G. Jeantet and O. Spanjaard, Optimizing the Hurwicz criterion in decision trees with imprecise probabilities: Proceedings of the 1st Int. Conference on Algorithmic Decision Theory. October 20 – 23, 2009, Venice, Italy, pp. 340 – 352.