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POWERS OF A MATRIX AND COMBINATORIAL IDENTITIES

J. MC LAUGHLIN AND B. SURY

ABSTRACT. In this article we obtain a general polynomial identity in k variables, where $k \geq 2$ is an arbitrary positive integer.

We use this identity to give a closed-form expression for the entries of the powers of a $k \times k$ matrix.

Finally, we use these results to derive various combinatorial identities.

1. INTRODUCTION

In [4], the second author had observed that the following ‘curious’ polynomial identity holds:

$$\sum (-1)^i \binom{n-i}{i} (x+y)^{n-2i} (xy)^i = x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n.$$

The proof was simply observing that both sides satisfied the same recursion. He had also observed (but not published the result) that this recursion defines in a closed form the entries of the powers of a 2×2 matrix in terms of its trace and determinant and the entries of the original matrix. The first author had independently discovered this fact and derived several combinatorial identities as consequences [2].

In this article, for a general k , we obtain a polynomial identity and show how it gives a closed-form expression for the entries of the powers of a $k \times k$ matrix. From these, we derive some combinatorial identities as consequences.

2. MAIN RESULTS

Throughout the paper, let K be any fixed field of characteristic zero. We also fix a positive integer k . The main results are the following two theorems:

Theorem 1. *Let x_1, \dots, x_k be independent variables and let s_1, \dots, s_k denote the various symmetric polynomials in the x_i 's of degrees $1, 2, \dots, k$ respectively. Then, in the polynomial ring $K[x_1, \dots, x_k]$, for each positive*

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integer n , one has the identity

$$\sum_{r_1+\dots+r_k=n} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k} = \sum_{2i_2+3i_3+\dots+ki_k \leq n} c(i_2, \dots, i_k, n) s_1^{n-2i_2-3i_3-\dots-ki_k} (-s_2)^{i_2} s_3^{i_3} \dots ((-1)^{k-1} s_k)^{i_k},$$

where

$$c(i_2, \dots, i_k, n) = \frac{(n - i_2 - 2i_3 - \dots - (k-1)i_k)!}{i_2! \dots i_k! (n - 2i_2 - 3i_3 - \dots - (ki_k))!}.$$

Theorem 2. Suppose $A \in M_k(K)$ and let

$$T^k - s_1 T^{k-1} + s_2 T^{k-2} + \dots + (-1)^k s_k I$$

denote its characteristic polynomial. Then, for all $n \geq k$, one has

$$A^n = b_{k-1} A^{k-1} + b_{k-2} A^{k-2} + \dots + b_0 I,$$

where

$$\begin{aligned} b_{k-1} &= a(n - k + 1), \\ b_{k-2} &= a(n - k + 2) - s_1 a(n - k + 1), \\ &\vdots \\ b_1 &= a(n - 1) - s_1 a(n - 2) + \dots + (-1)^{k-2} s_{k-2} a(n - k + 1), \\ b_0 &= a(n) - s_1 a(n - 1) + \dots + (-1)^{k-1} s_{k-1} a(n - k + 1) \\ &= (-1)^{k-1} s_k a(n - k). \end{aligned}$$

and

$$a(n) = c(i_2, \dots, i_k, n) s_1^{n-i_2-2i_3-\dots-(k-1)i_k} (-s_2)^{i_2} s_3^{i_3} \dots ((-1)^{k-1} s_k)^{i_k},$$

with

$$c(i_2, \dots, i_k, n) = \frac{(n - i_2 - 2i_3 - \dots - (k-1)i_k)!}{i_2! \dots i_k! (n - 2i_2 - 3i_3 - \dots - (ki_k))!}.$$

as in Theorem 1.

Proof of Theorems 1 and 2. In Theorem 1, if $a(n)$ denotes either side, it is straightforward to verify that

$$a(n) = s_1 a(n - 1) - s_2 a(n - 2) + \dots + (-1)^{k-1} s_k a(n - k).$$

Theorem 2 is a consequence of Theorem 1 on using induction on n . □

The special cases $k = 2$ and $k = 3$ are worth noting for it is easier to derive various combinatorial identities from them.

Corollary 1. (i) Let $A \in M_3(K)$ and let $X^3 = tX^2 - sX + d$ denote the characteristic polynomial of A . Then, for all $n \geq 3$,

$$(2.1) \quad A^n = a_{n-1}A + a_{n-2}Adj(A) + (a_n - ta_{n-1})I,$$

where

$$a_n = \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} t^{n-2i-3j} s^i d^j$$

for $n > 0$ and $a_0 = 1$.

(ii) Let $B \in M_2(K)$ and let $X^2 = tX - d$ denote the characteristic polynomial of B . Then, for all $n \geq 2$,

$$B^n = b_n I + b_{n-1} Adj(B)$$

for all $n \geq 2$, where

$$b_n = \sum \binom{n-i}{i} (-1)^i t^{n-2i} d^i.$$

Corollary 2. Let $\theta \in K$, $B \in M_2(K)$ and t denote the trace and d the determinant of B . We have the following identity in $M_2(K)$:

$$\begin{aligned} (a_{n-1} - \theta a_{n-2})B + (a_n - (\theta + t)a_{n-1} + \theta a_{n-2}t)I \\ = y_{n-1}B + (y_n - t y_{n-1})I, \end{aligned}$$

where

$$a_n = \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (\theta + t)^{n-2i-3j} (\theta t + d)^i (\theta d)^j$$

and

$$y_n = \sum \binom{n-i}{i} (-1)^i t^{n-2i} d^i.$$

In particular, for any $\theta \in K$, one has

$$b_n - (\theta + 1)b_{n-1} + \theta b_{n-2} = 1,$$

where

$$b_n = \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (\theta + 2)^{n-2i-3j} (1 + 2\theta)^i \theta^j.$$

Corollary 3. The numbers $c_n = \sum_{2i+3j=n} (-1)^i \binom{i+j}{j} 2^i 3^j$ satisfy

$$c_n + c_{n-1} - 2c_{n-2} = 1.$$

Proof. This is the special case of Corollary 2 where we take $\theta = -2$. Note that the sum defining c_n is over only those i, j for which $2i + 3j = n$. \square

Note than when $k = 3$, Theorem 1 can be rewritten as follows:

Theorem 3. *Let n be a positive integer and x, y, z be indeterminates. Then*

$$(2.2) \quad \sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (x+y+z)^{n-2i-3j} (xy+yz+zx)^i (xyz)^j \\ = \frac{xy(x^{n+1}-y^{n+1}) - xz(x^{n+1}-z^{n+1}) + yz(y^{n+1}-z^{n+1})}{(x-y)(x-z)(y-z)}.$$

Proof. In Corollary 1, let

$$A = \begin{pmatrix} x+y+z & 1 & 0 \\ -xy-xz-yz & 0 & 1 \\ xyz & 0 & 0 \end{pmatrix}.$$

Then $t = x + y + z$, $s = xy + xz + yz$ and $d = xyz$. It is easy to show (by first diagonalizing A) that the $(1, 2)$ entry of A^n equals the right side of (2.2), with $n + 1$ replaced by n , and the $(1, 2)$ entry on the right side of (2.1) is a_{n-1} . \square

Corollary 4. *Let x and z be indeterminates and n a positive integer. Then*

$$\sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (2x+z)^{n-2i-3j} (x^2+2xz)^i (x^2z)^j \\ = \frac{x^{2+n} + nx^{1+n}(x-z) - 2x^{1+n}z + z^{2+n}}{(x-z)^2}.$$

Proof. Let $y \rightarrow x$ in Theorem 3. \square

Some interesting identities can be derived by specialising the variables in Theorem 1. For instance, in [5], it was noted that Binet's formula for the Fibonacci numbers is a consequence of Theorem 1 for $k = 2$. Here is a generalization.

Corollary 5. *(Generalization of Binet's formula)*

Let the numbers $F_k(n)$ be defined by the recursion

$$F_k(0) = 1, F_k(r) = 0 \forall r < 0,$$

$$F_k(n) = F_k(n-1) + F_k(n-2) + \cdots + F_k(n-k).$$

Then, we have

$$F_k(n) = \sum_{2i_2 + \cdots + ki_k \leq n} \frac{(n-i_2-2i_3-\cdots-(k-1)i_k)!}{i_1!i_2!\cdots i_k!(n-2i_2-3i_3-\cdots-ki_k)!}.$$

Further, this equals $\sum_{r_1+\cdots+r_k=n} \lambda_1^{r_1} \cdots \lambda_k^{r_k}$ where $\lambda_i, 1 \leq i \leq k$ are the roots of the equation $T^k - T^{k-1} - T^{k-2} - \cdots - 1 = 0$.

Proof. The recursion defining $F_k(n)$'s corresponds to the case $s_1 = -s_2 = \cdots = (-1)^{k-1}s_k = 1$ of the theorem. \square

Corollary 6.

$$\sum c(i_2, \dots, i_k, n) k^n \prod_{j=2}^k \left((-1)^{j-1} k^{-j} \binom{k}{j} \right)^{i_j} = \binom{n+k-1}{k}.$$

where

$$c(i_2, \dots, i_k, n) = \frac{(n - i_2 - 2i_3 - \dots - (k-1)i_k)!}{i_2! \dots i_k! (n - 2i_2 - 3i_3 - \dots - ki_k)!}.$$

Proof. Take $x_i = 1$ for all i in Theorem 1. The left side of Theorem 1 is simply the sum $\sum_{r_1+\dots+r_k=n} 1$. \square

From Theorem 3 we have the following binomial identities as special cases.

Proposition 1. (i) Let λ be the unique positive real number satisfying $\lambda^3 = \lambda + 1$. Let x, y denote the complex conjugates such that $xy = \lambda, x + y = \lambda^2$, and let $z = -\frac{1}{\lambda}$. Then,

$$\begin{aligned} \sum_{2i+3j \leq n} (-1)^j \binom{n-2j}{j} &= \sum_{r+s+t=n} x^r y^s z^t \\ &= \frac{xy(x^{n+1} - y^{n+1}) - xz(x^{n+1} - z^{n+1}) + yz(y^{n+1} - z^{n+1})}{(x-y)(x-z)(y-z)}. \end{aligned}$$

(ii)

$$\sum_{2i+3j \leq n} (-1)^j \binom{i+j}{j} \binom{n-i-2j}{i+j} = [(n+2)/2].$$

(iii)

$$\sum \binom{n-2j}{j} (-4)^j 3^{n-3j} = \frac{(3n+4)2^{n+1} + (-1)^n}{9}.$$

(iv)

$$\begin{aligned} \sum \binom{n-2j}{j} 3^{n-3j} (-2)^j \\ = \frac{(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}}{2\sqrt{3}} + \frac{(1+\sqrt{3})^{n+1} + (1-\sqrt{3})^{n+1}}{6} - \frac{1}{3}. \end{aligned}$$

3. COMMUTATING MATRICES

In this section we derive various combinatorial identities by writing a general 3×3 Matrix A as a product of commuting matrices.

Proposition 2. Let A be an arbitrary 3×3 matrix with characteristic equation $x^3 - tx^2 + sx - d = 0$, $d \neq 0$. Suppose p is arbitrary, with

$p^3 + p^2t + ps + d \neq 0$, $p \neq 0$, $-t$. If n is a positive integer, then

$$(3.1) \quad A^n = \left(\frac{pd}{p^3 + p^2t + sp + d} \right)^n \sum_{r=0}^{3n} \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} \\ \times \left(\frac{-p(p+t)^2}{d} \right)^j \left(\frac{-(p+t)}{p} \right)^k \left(\frac{-A}{p+t} \right)^r.$$

Proof. This follows from the identity

$$A = \frac{-1}{p^3 + p^2t + sp + d} (pA^2 - Ap(p+t) - dI) (A + pI),$$

after raising both sides to the n -th power and collecting powers of A . Note that the two matrices $pA^2 - Ap(p+t) - dI$ and $A + pI$ commute. \square

Corollary 7. *Let p , x , y and z be indeterminates and let n be a positive integer. Then*

$$\sum_{r=0}^{3n} \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} \left(\frac{p(p+x+y+z)^2}{xyz} \right)^j \\ \times \left(\frac{p+x+y+z}{p} \right)^k \frac{xy(x^r - y^r) - xz(x^r - z^r) + yz(y^r - z^r)}{(p+x+y+z)^r} \\ = (xy(x^n - y^n) - xz(x^n - z^n) + yz(y^n - z^n)) \\ \times \left(\frac{p^3 + p^2(x+y+z) + p(xy+xz+yz) + xyz}{pxyz} \right)^n.$$

Proof. Let A be the matrix from Theorem 3 and compare (1,1) entries on both sides of (3.1). \square

Corollary 8. *Let p , x and z be indeterminates and let n be a positive integer. Then*

$$\sum_{r=0}^{3n} \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} \left(\frac{p(p+2x+z)^2}{x^2z} \right)^j \\ \times \left(\frac{p+2x+z}{p} \right)^k \frac{r x^{1+r} - x^r z - r x^r z + z^{1+r}}{(p+2x+z)^r} \\ = (n x^{1+n} - x^n z - n x^n z + z^{1+n}) \\ \times \left(\frac{p^3 + p^2(2x+z) + p(x^2 + 2xz) + x^2 z}{p x^2 z} \right)^n.$$

Proof. Divide both sides in the corollary above by $x - y$ and let $y \rightarrow x$. \square

Corollary 9. *Let p and x be indeterminates and let n be a positive integer. Then*

$$\begin{aligned} \sum_{r=0}^{3n} \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} \left(\frac{p(p+3x)^2}{x^3} \right)^j \\ \times \left(\frac{p+3x}{p} \right)^k \frac{r(1+r)x^{-1+r}}{2(p+3x)^r} \\ = \frac{n(1+n)x^{-1+n}}{2} \left(\frac{(p+x)^3}{px^3} \right)^n. \end{aligned}$$

Proof. Divide both sides in the corollary above by $(x-z)^2$ and let $z \rightarrow x$. \square

Corollary 10. *Let p be an indeterminate and let n be a positive integer. Then*

$$\begin{aligned} \sum_{r=0}^{3n} \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} p^{j-k} (p+3)^{2j+k-r} \frac{r(1+r)}{2} \\ = \frac{n(1+n)(p+1)^{3n}}{2p^n}. \end{aligned}$$

Proof. Replace p by px in the corollary above and simplify. \square

Various combinatorial identities can be derived from Theorem 3 by considering matrices A such that particular entries in A^n have a simple closed form. We give four examples.

Corollary 11. *Let n be a positive integer.*

(i) *If $p \neq 0, -1$, then*

$$\sum_{r=0}^{3n} \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} p^{j-k} (p+3)^{2j+k-r} r = n \frac{(1+p)^{3n}}{p^n}.$$

(ii) *Let F_n denote the n -th Fibonacci number. If $p \neq 0, -1, \phi$ or $1/\phi$ (where ϕ is the golden ratio), then*

$$\begin{aligned} \sum_{r=0}^{3n} \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{k+r} p^{j-k} (p+2)^{2j+k-r} F_r \\ = F_n \frac{(1+p)^n (-1+p+p^2)^n}{(-p)^n}. \end{aligned}$$

(iii) If $p \neq 0, -1$ or -2 , then

$$\begin{aligned} \sum_{r=0}^{3n} \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} p^{j-k} (p+4)^{2j+k-r} 2^{-j} (2^r - 1) \\ = (2^n - 1) \left(\frac{(1+p)^2(p+2)}{2p} \right)^n. \end{aligned}$$

(iv) If $p \neq 0, -1, -g$ or $-h$ and $gh \neq 0$, then

$$\begin{aligned} \sum_{r=0}^{3n} \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \binom{j}{r-j-k} (-1)^{j+k+r} p^{j-k} (p+1+g+h)^{2j+k-r} \\ \times \frac{g^r + h^r}{(gh)^j} = (g^n + h^n) \left(\frac{(1+p)(g+p)(h+p)}{ghp} \right)^n. \end{aligned}$$

Proof. The results follow from considering the $(1, 2)$ entries on both sides in Theorem 3 for the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{g+h}{2} & \frac{(g-h)^2}{4} & 0 \\ 1 & \frac{g+h}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. □

4. A RESULT OF BERNSTEIN

In [1] Bernstein showed that the only zeros of the integer function

$$f(n) := \sum_{j \geq 0} (-1)^j \binom{n-2j}{j}$$

are at $n = 3$ and $n = 12$. We use Corollary 1 to relate the zeros of this function to solutions of a certain cubic Thue equation and hence to derive Bernstein's result.

Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

With the notation of Corollary 1, $t = 1$, $s = 0$, $d = -1$, so that

$$a_n = \sum_{3j \leq n} (-1)^j \binom{n-2j}{j} = f(n),$$

and, for $n \geq 4$,

$$\begin{aligned} A^n &= f(n-2)A^2 + (f(n) - f(n-2))A + (f(n) - f(n-1))I \\ &= \begin{pmatrix} f(n) & f(n-1) & f(n-2) \\ -f(n-2) & -f(n-3) & -f(n-4) \\ -f(n-1) & -f(n-2) & -f(n-3) \end{pmatrix}. \end{aligned}$$

The last equality follows from the fact that $f(k+1) = f(k) - f(k-2)$, for $k \geq 2$.

Now suppose $f(n-2) = 0$. Since the recurrence relation above gives that $f(n-4) = -f(n-1)$ and $f(n) = f(n-1) - f(n-3)$, it follows that

$$\begin{aligned} (-1)^n = \det(A^n) &= \begin{vmatrix} f(n-1) - f(n-3) & f(n-1) & 0 \\ 0 & -f(n-3) & f(n-1) \\ -f(n-1) & 0 & -f(n-3) \end{vmatrix} \\ &= -f(n-1)^3 - f(n-3)^3 + f(n-1)f(n-3)^2. \end{aligned}$$

Thus $(x, y) = \pm(f(n-1), f(n-3))$ is a solution of the Thue equation

$$x^3 + y^3 - xy^2 = 1.$$

One could solve this equation in the usual manner of finding bounds on powers of fundamental units in the cubic number field defined by the equation $x^3 - x + 1 = 0$. Alternatively, the Thue equation solver in PARI/GP [3] gives unconditionally (in less than a second) that the only solutions to this equation are

$$(x, y) \in \{(4, -3), (-1, 1), (1, 0), (0, 1), (1, 1)\},$$

leading to Bernstein's result once again.

REFERENCES

- [1] Bernstein, Leon *Zeros of the functions $f(n) = \sum_{i=0}^n (-1)^i \binom{n-2i}{i}$* . J. Number Theory **6** (1974), 264–270.
- [2] James Mc Laughlin, *Combinatorial Identities Deriving from the n -th Power of a 2×2 Matrix*. Integers **4** (2004), A19, 14 pp.
- [3] PARI/GP: <http://pari.math.u-bordeaux.fr/>
- [4] B. Sury, *A curious polynomial identity*. Nieuw Arch. Wisk. (4) **11** (1993), no. 2, 93–96.
- [5] —, *A parent of Binet's formula?* Math. Magazine **77** (2004), no. 4, 308–310.

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