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Gunaratnam, Trishen

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Trishen Satyendran Gunaratnam

University of Bath
Department of Mathematical Sciences

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Declaration of any previous submission of the work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

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Declaration of authorship

I am the author of this thesis, and the work described therein was carried out by myself personally, with the exception of two articles where the work was fairly and equally distributed amongst collaborators.

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Abstract

In this thesis we are interested in the statistical mechanics of Euclidean field theories in 3D. We solve two problems: the first concerns the relationship between Gaussian measures and nonlinear wave equations; the second concerns phase transitions for ϕ^4_3. The common theme between our contributions is the development of the variational approach of Barashkov and Gubinelli [BG19] to ultraviolet stability, which allows one to control the singular short-distance behaviour of Euclidean field theories in 3D, in the context of statistical mechanics arguments.

Our first contribution is to establish the quasi-invariance of Gaussian measures supported on Sobolev spaces under the dynamics of the cubic defocusing wave equation. This extends previous work in the two-dimensional case [OT20]. Two new ingredients in the three-dimensional case are (i) the construction of certain weighted Gaussian measures based on the variational approach to ultraviolet stability, and (ii) an improved argument in controlling the growth of the truncated weighted Gaussian measures, where we combine a deterministic growth bound of solutions with stochastic estimates on random distributions. This is joint work with Tadahiro Oh, Nikolay Tzvetkov, and Hendrik Weber [GOTW18].

Our second contribution is to quantify the phase transition for ϕ^4_3. In particular, we establish a surface order large deviation estimate for the magnetisation of low temperature ϕ^4_3. As a byproduct, we obtain a decay of spectral gap for its Glauber dynamics given by the ϕ^4_3 singular stochastic PDE. Our main technical results are contour bounds for ϕ^4_3, which extends 2D results by Glimm, Jaffe, and Spencer [GJS75]. We adapt an argument by Bodineau, Velenik, and Ioffe [BIV00] to use these contour bounds to study phase segregation. The main challenge to obtain the contour bounds is to handle the ultraviolet divergences of ϕ^4_3 whilst preserving the structure of the low temperature potential. To do this, we build on the variational approach to ultraviolet stability for ϕ^4_3. This is joint work with Ajay Chandra and Hendrik Weber [CGW20].

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I. Introduction

1 Euclidean field theories

Euclidean field theories in d-dimensions are special types of Borel probability measures on the space of Schwartz distributions $S'(\mathbb{R}^d)$. They can be thought of as Gibbs measures on continuum fields. Indeed, from the viewpoint of statistical mechanics, they exhibit a rich variety of phenomena: they arise as continuum and scaling limits of discrete spin models, undergo phase transitions, and are invariant measures for Hamiltonian and singular stochastic PDEs. Their origins, however, are in quantum field theory, where they arise from evaluating quantum fields at imaginary times. What makes them special is that they allow one to rigorously undo the passage from real time to imaginary time and thereby reconstruct quantum fields from classical/Euclidean fields.

More precisely, Euclidean field theories are probability measures whose correlation functions satisfy (a variant of) the Osterwalder-Schrader axioms [OS73, OS75], which consist of: an appropriate analyticity condition, Euclidean invariance, permutation symmetry, and reflection positivity. The Osterwalder-Schrader reconstruction theorem [OS75, Theorem E $\leftrightarrow$ R] then states that the analytic continuation of the Euclidean fields to (minus) imaginary time yields operator-valued distributions that are densely defined on a Hilbert space $\mathfrak{H}$ and satisfy the Wightman axioms of quantum field theory [Wig56]. Moreover, one can reverse the analytic continuation and obtain a Euclidean quantum field theory from a set of operator-valued distributions satisfying the Wightman axioms.

The most intriguing and least self-explanatory axiom is reflection positivity, which we now state on the level of the measure as opposed to the correlation functions. Let ν be a Euclidean field theory. We distinguish the first coordinate of $x = (x_1, \ldots, x_d) = (x_1, \underline{x}) \in \mathbb{R}^d$ and denote by $\mathbb{H}$ the associated upper half plane. Let θ be the reflection map across $\mathbb{H}$. Define $\mathscr{A}^+ \subset L^2(\nu)$ to be the set of random variables generated by $\phi \in S'(\mathbb{R}^d)$ with support (suitably interpreted) in $\mathbb{H}$. We say ν is reflection positive if, for any $A \in \mathscr{A}^+$,

$$\int_{S'(\mathbb{R}^d)} A(\phi) \cdot \theta A(\phi) d\nu(\phi) \geqslant 0$$

where $\theta A(\phi) = A(\theta\phi)$.

Reflection positivity is significant in both quantum theory and statistical physics, and underlies a deep connection between the two. On the one hand, it allows the construction of the Hilbert space of quantum states: define $\langle A, B \rangle_{\mathfrak{H}} = \int_{S'(\mathbb{R}^d)} A(\phi) \cdot \theta B(\phi) d\nu(\phi)$ for $A, B \in \mathscr{A}^+$ and let $\mathscr{N}$ be the set of null vectors under this bilinear form. Then, reflection positivity implies that the completion of $\mathscr{A}^+/\mathscr{N}$ under $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$

is a Hilbert space $\mathfrak{H}$. See [OS75] or [GJ87, Chapter 6.1]. On the other hand, many classical and quantum spin systems are reflection positive and, as we briefly touch upon later on, this property is fundamental to their theory of phase transitions in $d \geqslant 3$. See [Bis09] for a review. In this thesis it plays a small but essential role in Part III.

2 Examples: the Gaussian free field and the ϕ^4 model

We are interested in Euclidean field theories with formal densities proportional to

$$e^{-\mathscr{H}(\phi)}d\phi. \tag{2.1}$$

Here, $d\phi$ is the (non-existent) Lebesgue measure over $S'(\mathbb{R}^d)$ and $\mathscr{H}$ is the Hamiltonian

$$\mathscr{H}(\phi) = \int_{\mathbb{R}^d} \mathscr{V}(\phi(x)) + \frac{1}{2}|\nabla\phi(x)|^2 dx$$

where $\mathscr{V} : \mathbb{R} \to \mathbb{R}$ is a potential and ∇ is the gradient. Choices of potentials include:

- $\mathscr{V}(\phi(x)) = \frac{1}{2}m^2\phi(x)^2$, corresponding to the d-dimensional Gaussian free field of mass $m \geqslant 0$, or free field for short;
- $\mathscr{V}(\phi(x)) = \lambda\phi(x)^4$, corresponding to the ϕ^4 model in d-dimensions with coupling constant $\lambda > 0$, or ϕ^4_d for short.

We are also interested in generalisations of the free field where ∇ is replaced by a higher order derivative (although strictly speaking these may not satisfy reflection positivity).

The free field is realised as the centred Gaussian measure with covariance $(-\Delta + m^2)^{-1}$, where Δ is the Laplacian. It was first shown to be a Euclidean field theory by Nelson [Nel73] and is considered trivial since it is associated to a quantum field theory without interaction. However, due to the non-existence of Lebesgue measure in infinite dimensions, it is a starting point to rigorously construct non-Gaussian/nontrivial Euclidean field theories. The latter measures are more interesting than their trivial counterparts and exhibit a richer variety of phenomena. They are also of greater physical importance from the quantum field theory point of view since they are associated to quantum field theories with interaction. Candidates for nontrivial Euclidean field theories are given by measures with higher order nonlinearities in the potential $\mathscr{V}$, e.g. the ϕ^4 model.

The construction of nontrivial Euclidean field theories is a notoriously difficult problem for $d \geqslant 2$. This is true even for the easier problem of showing the construction of finite volume approximations to such measures, e.g. replacing $\mathbb{R}^d$

by $\mathbb{T}_N^d = (\mathbb{R}/N\mathbb{Z})^d$, the d-dimensional torus of sidelength $N \in \mathbb{N}$, in (2.1). As alluded to above, it is natural to define these objects using a density with respect to the centred Gaussian measure μ_N with covariance $(-\Delta_N)^{-1}$, where Δ is the Laplacian on $\mathbb{T}_N^d$ (we ignore the problem of constant fields/zeroeth Fourier mode in this discussion). Note that μ_N is the massless free field on $\mathbb{T}_N^d$. However, for $d \geqslant 2$, μ_N is not supported on a space of functions and samples need to be interpreted as Schwartz distributions. This is a serious problem because there is no canonical interpretation of products of distributions, meaning that the nonlinearity $\int_{\mathbb{T}_N^d} \mathcal{V}(\phi(x))dx$ is in general not well-defined on the support of μ_N.

If one introduces an ultraviolet (small-scale) cutoff $K > 0$ on the field to regularise it, then one sees that the nonlinearities of the regularised field $\mathcal{V}(\phi_K)$ fail to converge as the cutoff is removed - there are divergences. The strength of these divergences grow with dimension: they are only logarithmic in the cutoff for $d = 2$, whereas they are polynomial for $d \geqslant 3$. Renormalisation is required to kill these divergences. This is done by looking at the measures defined with respect to the cutoff potential and subtracting appropriate counter-terms from the Hamiltonian. Obtaining a nontrivial limiting measure as the cutoff is removed, which is often called showing ultraviolet stability, is not always possible and depends heavily on the choice of $\mathcal{V}$ and the dimension.

One of the big successes of the constructive field theory programme, initiated by Glimm and Jaffe in the '60s, was the construction of finite volume approximations to ϕ_2^4 and later ϕ_3^4. Renormalisation of the ϕ^4 Hamiltonian is done by subtracting the counter-term $\int_{\mathbb{T}_N^d} \delta m^2(K)\phi_K^2$, where the renormalisation constant $\delta m^2(K)$ is given by $C_1\lambda \log K$ in $d = 2$ and $C_2\lambda K - C_3\lambda^2 \log K$ in $d = 3$, for some $C_1, C_2, C_3 > 0$. If these constants are appropriately chosen (i.e. by perturbation theory), then a nontrivial limiting measure is obtained as $K \to \infty$. Nelson was the first to show ultraviolet stability for ϕ_2^4 [Nel66]. In the significantly harder case of ϕ_3^4, Glimm and Jaffe made the first breakthrough [GJ73] and many results followed [Fel74, MS77, BCG+80, BFS83, BDH95, MW17b, GH18, BG19]. We particularly highlight the recent approach of Barashkov and Gubinelli [BG19] based on the Boué-Dupuis variational formula for Gaussian expectations, which plays a central role in this thesis.

Extensions to infinite volume and (partial) verification of the Osterwalder-Schrader axioms have been achieved through use of cluster expansions [GJS74, FO76], correlation inequalities [SG73, GRS75], random walk expansions [BFS83], PDE techniques [GH18], and other methods. See [GJ87, Parts II and III] for an in-depth treatment of ϕ_2^4 using these methods, and see [GH18] for a review of the state-of-the-art for ϕ_3^4.

In higher dimensions there are triviality results for ϕ^4: in $d \geqslant 5$ these are due to Aizenman and Fröhlich [Aiz82, Frö82], whereas the $d = 4$ case was only recently done by Aizenman and Duminil-Copin [ADC20]. These results imply that if one takes a lattice cutoff as short-scale cutoff for (renormalised) ϕ^4, then any continuum limit whose covariance between points $\phi(x)$ and $\phi(y)$ decays as $|x-y| \to \infty$ is necessarily Gaussian. In other words, the strong ultraviolet divergences of dimensions $d \geqslant 4$ results in the destruction of the ϕ^4 model.

We restrict our attention to Euclidean field theories in $d = 2$ and 3, and often work with these objects in (sometimes large) finite volumes $\mathbb{T}_N^d$. We are particularly concerned with the significantly harder case of $d = 3$, which is the physically relevant dimension in statistical physics.

3 The statistical mechanics of Euclidean field theories

In this thesis we address two areas of research concerning the statistical mechanics of Euclidean field theories. First, these objects arise naturally as Gibbs measures for Hamiltonian PDEs, such as wave and Schrödinger equations. We are interested in exploring this connection further for the specific case of nonlinear wave equations. Second, the ϕ^4 model bears many similarities to the Ising model. Indeed, it is well-known that both models undergo phase transition. However, whilst the phase coexistence regime of the Ising model has been studied extensively, there are comparatively few results for ϕ^4. We are interested in exploring the finer properties of the coexistence regime for ϕ^4: in particular, looking at the phenomenon of phase segregation and implications for relaxation times of its natural Glauber dynamics.

3.1 Nonlinear wave equations and (quasi-)invariant measures

Wave and Schrödinger equations are of great importance in physics since they are known to model a wide variety of phenomena. Wave equations are PDEs of the form

$$\partial_t^2 u = \Delta u \pm u^p \tag{3.1}$$

where $u : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}$. Schrödinger equations are PDEs of the form

$$-i\partial_t \varphi = \Delta\varphi \pm |\varphi|^{p-1}\varphi$$

where $\phi : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$. Above, p is taken to be a positive odd integer; u^p and $|\varphi|^{p-1}\varphi$ are called nonlinearities (i.e. the linear wave and Schrödinger equations correspond to the PDEs above without these terms); the sign of the nonlinearity corresponds to the equation being defocusing (minus) or focusing (plus). We often just consider the cubic ($p = 3$) defocusing case. We do not consider other types of nonlinearities or the equations posed on the full space $\mathbb{R}^d$.

These equations are examples of Hamiltonian PDEs. For the wave equation, this can be seen by rewriting the PDE as the following system:

$$\begin{aligned} \partial_t u &= \partial_v \mathscr{H}^{\mathrm{NLW}}(u, v) \\ \partial_t v &= -\partial_u \mathscr{H}^{\mathrm{NLW}}(u, v) \end{aligned} \tag{3.2}$$

where $(u, \partial_t u) = (u, v)$ and the Hamiltonian given by

$$\mathscr{H}^{\mathrm{NLW}}(u,v) = \int_{\mathbb{T}^d} \frac{1}{p+1} u^{p+1} + \frac{1}{2}|\nabla u|^2 + \frac{1}{2} v^2 dx.$$

is conserved. For the Schrödinger equation, the Hamiltonian structure can be seen by writing it in terms of real and imaginary parts (which we do not do) and then showing that it conserves the Hamiltonian

$$\mathscr{H}^{\mathrm{NLS}}(\varphi) = \int_{\mathbb{T}^d} \frac{1}{p+1} |\varphi|^{p+1} + \frac{1}{2}|\nabla \varphi|^2 dx.$$

Invariant measures of Hamiltonian dynamics are interesting to study because, for example, they are important to the study of long-time behaviour of solutions (i.e. global existence of solutions and ergodicity). In finite dimensional systems, there is a well-known link between conserved quantities, such as the Hamiltonian, and invariant measures. This correspondence is a consequence of Liouville's theorem, which states that Lebesgue measure on phase space (position space $\times$ momentum space) is conserved under the dynamics. The punchline is that the Gibbs measure (i.e. the measure with density given by the exponential of minus the Hamiltonian), or the analogous measure associated to any conserved quantity, is invariant. It is natural to ask whether this correspondence passes on to infinite dimensional Hamiltonian systems. Note that the same argument in finite dimensions does not carry over to this case (for one, Lebesgue measure does not exist in this situation). However, it is relatively straightforward to establish invariance of measures associated to conserved quantities for linear wave and Schrödinger equations because they are Gaussian.

For nonlinear Hamiltonian PDEs, in particular nonlinear Schrödinger equations, the study of invariant measures is significantly harder. For one, their Gibbs measures are finite volume approximations of nontrivial Euclidean field theories (over vector-valued or complex fields). Moreover, the well-posedness theory for nonlinear equations is more difficult than for linear equations. The first breakthrough was in $d = 1$ by Lebowitz, Rose, and Speer [LRS88] and Bourgain [Bou94], where the invariance of Gibbs measures was established for nonlinear defocusing Schrödinger equations with polynomial nonlinearity of order $p \leqslant 5$ (and also the focusing case with an energy cutoff in the Gibbs measure). The next big breakthrough was by Bourgain [Bou96b], who famously established invariance of a complex-valued version of ϕ^4_2 for the two-dimensional renormalised cubic defocusing Schrödinger equation. The analogous problem in $d = 3$ remains open.

A related but more tractable question is to ask how certain Gaussian measures, which arise as invariant measures of linear equations, are transported under the flow of nonlinear equations. We are specifically interested in the case of the cubic nonlinear wave equation, which is easier to analyse than Schrödinger equations. Given $s \in \mathbb{R}$, let $H^s(\mathbb{T}^d)$ denote the classical L^2-based Sobolev space of order σ and define $\vec{H}^s(\mathbb{T}^d) = H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d)$. Let $\vec{\mu}_s$ denote the Gaussian measure with formal density:

$$d\vec{\mu}_s = Z_s^{-1} e^{-\frac{1}{2}\|\vec{u}\|^2_{\vec{H}^{s+1}}} d\vec{u}$$

where $\vec{u} = (u, v)$. The norms of $\vec{H}^{s+1}(\mathbb{T}^d)$ are conserved under the dynamics of the linear wave equation and one can show that their associated measures are invariant. However, the cubic nonlinearity destroys the conservation of these norms and one does not expect these measures to be invariant.

Nevertheless, there have been a series of recent results initiated by Tzvetkov [Tzv15] that has made significant progress in better understanding the relation of Gaussian measures analogous to $\vec{\mu}_s$ and nonlinear Hamiltonian PDEs. See, for example, [OT17, OST18, OT20, OTT19] and references therein. In the case of wave equations, under certain restrictions on s and the dimension, the measure $\vec{\mu}_s$ (or analogously measures) can be shown to be quasi-invariant under these dynamics: this means is that the law of the solution at any time is equivalent to the law of the (random) initial data sampled from μ_s. Whilst not as strong as invariance, this is still very useful in infinite dimensions because many interesting properties concerning small-scale behaviour under a Gaussian measure hold true with probability 0 or 1 (this is an implication of Fernique's theorem [DPZ14, Theorem 2.7]). Indeed, one can show that samples under $\vec{\mu}_s$ almost surely belong to L^p-based Sobolev spaces of appropriate regularity. Then, quasi-invariance implies an almost sure preservation of this L^p-based regularity for nonlinear Hamiltonian PDEs. Such a phenomenon is not in general true in the deterministic setting, even for linear equations. See [Lit63, Per80, Sog93].

These results can also be viewed from the perspective of the study of transport for Gaussian measures. Indeed, it is well-known that Gaussian measures in infinite dimensions are either equivalent or mutually singular. It is interesting to then ask under which transformations is equivalence preserved, i.e. transformations under which the Gaussian measure is quasi-invariant. This has been well-studied in the case of deterministic shifts by Cameron and Martin [CM49], and there are general abstract criterion for nonlinear transformations due to Ramer [Ram74] and Cruzeiro [Cru83b, Cru83a]. The recent works mentioned in the preceding paragraph can be seen as giving concrete and nontrivial examples of nonlinear transformations under which a large class of Gaussian measures are quasi-invariant.

3.2 Phase transitions, Ising, and ϕ^4

Phase transitions are rich and complex phenomena that are ubiquitous in statistical mechanics. An example of central importance to us is the ferromagnet-paramagnet transition where iron, beyond a certain critical temperature, loses its ability to retain a non-zero magnetisation in the presence of no external field.

The Ising (or Lenz-Ising) model was introduced by Lenz [Len20] to capture this phenomenon. It is given by a Gibbs probability measure defined on spin configurations $\{\pm 1\}^{\mathbb{Z}^d}$ such that the probability of a given spin configuration σ is formally proportional to $e^{-\mathscr{H}^{\text{Ising}}_{\beta,h}(\sigma)}$, where $\beta > 0$ is the inverse temperature, and $h \in \mathbb{R}$ is the external field, and

$$\mathscr{H}^{\text{Ising}}_{\beta,h}(\sigma) = -\beta \sum_{i,j \in \mathbb{Z}^d, i \sim j} \sigma_i \sigma_j - h \sum_{i \in \mathbb{Z}^d} \sigma_i$$

where $i \sim j$ means i and j are nearest-neighbours. We write $\langle \cdot \rangle^{\text{Ising}}_{\beta,h}$ to denote expectations with respect to this measure, which is interpreted as the weak limit of Ising models on growing discrete tori $\mathbb{T}^d_N \cap \mathbb{Z}^d$.

Phase transition in the Ising model for $d \geqslant 2$ was famously established by Peierls [Pei36] and later made rigorous by Griffiths [Gri64] and Dobrushin [Dob65]. One can show the existence of long range order when β is sufficiently large and $h = 0$: namely, the quantity

$$|\langle \sigma_0 \sigma_i \rangle^{\text{Ising}}_{\beta,0} - \langle \sigma_0 \rangle^{\text{Ising}}_{\beta,0} \langle \sigma_i \rangle^{\text{Ising}}_{\beta,0}|$$

does not decay as $|i| \to \infty$. Equivalently, one can show the existence of spontaneous magnetisation:

$$\lim_{h \downarrow 0} \langle \sigma_0 \rangle^{\text{Ising}}_{\beta,h} > 0 = \langle \sigma_0 \rangle^{\text{Ising}}_{\beta,0}.$$

These results rely on the development of contour bounds for the Ising model. This is most easily explained in $d = 2$. Under a deformation convention to avoid ambiguities, each spin configuration σ is in bijection with a configuration of simple curves, called contours, that form interfaces between regions of $+$ spins and $-$ spins. The set of contours is called the phase boundary $\partial\sigma$. One can rewrite the Ising measure in terms of contours and show that, for any closed bounded simple curve Γ formed by lattice lines,

$$\langle \mathbf{1}_{\Gamma \in \partial\sigma} \rangle^{\text{Ising}}_{\beta,0} \leqslant e^{-2\beta|\Gamma|}. \tag{3.3}$$

The significance of this in the context of phase transitions is that, by the $\sigma \mapsto -\sigma$ symmetry, one can rewrite

$$\langle \sigma_0 \sigma_i \rangle^{\text{Ising}}_{\beta,0} - \langle \sigma_0 \rangle^{\text{Ising}}_{\beta,0} \langle \sigma_i \rangle^{\text{Ising}}_{\beta,0} = 1 - 4 \langle \mathbf{1}_{\sigma_0 = 1} \mathbf{1}_{\sigma_i = -1} \rangle^{\text{Ising}}_{\beta,0}. \tag{3.4}$$

On the event $\{\sigma_0 = 1\} \cap \{\sigma_i = -1\}$, there must be a contour separating 0 and i (i.e. it encloses either 0 or i). Summing over all possible contours and using the contour bound (3.3), one can show that, uniformly over $i \in \mathbb{Z}^d$, the righthand side

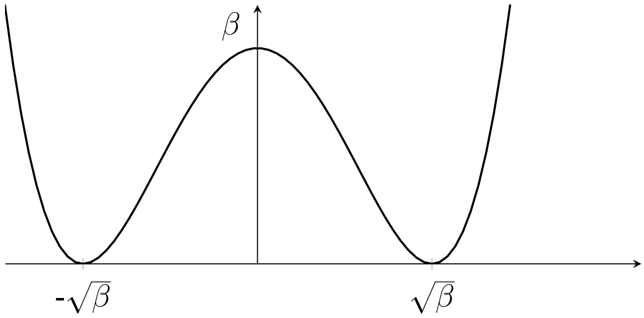

Figure 1: Plot of $\mathscr{V}_\beta$

of (3.4) converges to 1 as $\beta \to \infty$. In particular, there is long range order provided β is sufficiently large.

It turns out that phase transitions also occur in ϕ^4 models in $d = 2$ and 3, and the underlying reason for this is that ϕ^4 and Ising models are very similar. To explain this, first note that due to renormalisation, the ϕ^4 potential for fields with ultraviolet cutoff $K > 0$ becomes infinitely non-convex as $K \to \infty$. The leading order divergence is proportional to λ and this governs the rate at which the potential is becoming more non-convex. Thus, one can formally reparametrise the ϕ^4 potential as a quartic double well of the form $\mathscr{V}(\phi(x)) = \lambda(\phi(x)^2 - 1)^2$. A scaling argument then yields that there exists $\beta = \beta(\lambda) \to \infty$ as $\lambda \to \infty$ such that the above theory is equivalent to a ϕ^4 theory defined formally by the measure ν_β with density

$$d\nu_\beta(\phi) \propto \exp\Big(- \int_{\mathbb{R}^d} \mathscr{V}_\beta(\phi(x)) + \frac{1}{2}|\nabla\phi(x)|^2 dx\Big) d\phi \tag{3.5}$$

where $\mathscr{V}_\beta(\phi(x)) = \frac{1}{\beta}(\phi(x)^2 - \beta)^2$. See [GJS76c] for full details in $d = 2$. We write $\langle \cdot \rangle_\beta$ to denote the corresponding expectation operator. $\mathscr{V}_\beta(\phi(x))$ has minima at $\phi(x) = \pm\sqrt{\beta}$ with a potential barrier at $\phi(x) = 0$ of height β, so the minima become widely separated by a steep barrier as $\beta \to \infty$. See Figure 1. Consequently, ν_β resembles an Ising model with spins at $\pm\sqrt{\beta}$ (i.e. at inverse temperature $\beta > 0$) for large β.

Glimm, Jaffe, and Spencer [GJS75] exploited this similarity with low temperature Ising and proved the existence of long range order and symmetry breaking for ν_β in $d = 2$ using a sophisticated modification of Peierls' argument. In [GJS76a, GJS76b] they further develop the Peierls' expansion for ν_β into a full low temperature expansion and establish spontaneous magnetisation. Moreover, they construct two distinct measures that correspond to ν_β. These two measures satisfy all the Osterwalder-Schrader axioms and exhibit exponential decay of correlations.

The Peierls' argument of [GJS75] relies on contour bounds for ν_β. Discretise $\mathbb{R}^2$ into unit blocks and, for each block $\square \subset \mathbb{R}^2$ and $\phi \sim \nu_\beta$, let $\phi(\square)$ be the block averaged field. The configuration of block averages retains the large-scale information of the field, meaning that it is appropriate to study phase transitions, but it does not contain small-scale divergences. Due to the structure of the potential $\mathcal{V}_\beta$ when β is large, the configuration of block averages resembles an Ising model (however, it is still a continuous spin configuration). The set of blocks are decomposed pathwise into positive and negative blocks depending on the sign of the block averages, i.e. $\square$ is positive if $\phi(\square) > 0$. The phase boundary of a configuration consists of the connected components of the boundary between positive and negative blocks, i.e. contours. Conditional on certain (strong) moment bounds, one can then show that, for any fixed contour Γ, there exists $C > 0$ such that for β sufficiently large,

$$\nu_\beta(\Gamma \text{ is in the phase boundary}) \leqslant e^{-C\sqrt{\beta}|\Gamma|}.$$

The existence of phase transition for ϕ^4 then follows by using this contour bound and arguing as in the case of low temperature Ising. The key difficulty, therefore, is to show the moment bounds. The techniques of [GJS75, GJS76a, GJS76b] fail to establish these moment bounds in the significantly harder case of $d = 3$.

However, phase transition for ν_β in $d = 3$ was established by Fröhlich, Simon, and Spencer [FSS76] using a different argument based fundamentally on reflection positivity. This argument is much more general than Peierls' argument and plays a central role in the theory of phase transitions in $d \geqslant 3$: it applies to models with continuous symmetry [FSS76], quantum spin systems [DLS78], and can be combined with Peierls estimates to yield a very systematic theory of phase transition [FILS78, FILS80]. However, the techniques of [FSS76] alone are less quantitative than the Peierls' theory of [GJS75]. For example, it is not clear how to extend the results of [GJS76a, GJS76b] to $d = 3$. Moreover, these techniques are less natural for the ϕ^4 model, since intuitively the mechanisms which govern phase transition in this case are the same as for the Ising model.

The similarities between Ising and ϕ^4 in the context of phase transition are in fact manifestations of a deeper connection between these models. On the one hand, ϕ^4 arises as the continuum limit of Ising-type models near their critical points [SG73, CMP95, HI18]. On the other hand, one formally obtains Ising as the limit of ϕ^4 models as the coupling constant $\lambda \to \infty$ [GJ85]. It is, moreover, conjectured that the scaling limits of these models at their critical points yield the same limit [GJ85], i.e. Ising and ϕ^4 are in the same universality class, and that this limiting object is a special type of field theory that exhibits conformal symmetries [BPZ84]. The rigorous study of these phenomena is extremely difficult and there are many open problems. Instead, still drawing on the analogy between Ising and ϕ^4, we address much more tractable but still interesting finer properties of the phase transition.

3.2.1 Phase segregation

Although phase coexistence for ν_β has been established, little is known of this regime in comparison to the low temperature Ising model. In the latter model, the study of phase segregation at low temperatures in large but finite volumes was initiated by Minlos and Sinai [MS67, MS68], culminating in the famous Wulff constructions: due to Dobrushin, Kotecký, and Shlosman in $d = 2$ [DKS89, DKS92], with simplifications due to Pfister [Pfi91] and results up to the critical point by Ioffe and Schonmann [IS98]; and Bodineau [Bod99] in $d = 3$, see also results up to the critical point by Cerf and Pisztora [CP00] and the bibliographical review in [BIV00, Section 1.3.4].

An easier point of entry to study phase segregation phenomena for ϕ^4 models is given by surface order large deviation estimates for the average magnetisation of finite volume approximations. For the Ising model, these type of estimates were first established in $d = 2$ by Schonmann [Sch87] and later extended up to the critical point by Chayes, Chayes, and Schonmann [CCS87]; in $d = 3$ they were first established by Pisztora [Pis96]. They are related to the Wulff constructions, which actually allow one to characterise the large deviations for the average magnetisation. See [BIV00]. Moreover, they should be contrasted with the volume order large deviations established for the finite volume average magnetisation in the high temperature regime where there is no phase coexistence [CF86, Ell85, FO88, Oll88].

3.2.2 The Glauber dynamics of ϕ^4

The Glauber dynamics of $\nu_{\beta,N}$, the finite volume approximations of ν_β, is given by the singular stochastic PDE

$$\begin{aligned} (\partial_t - \Delta)\Phi &= -\frac{4}{\beta}\Phi^3 + (4+\infty)\Phi + \sqrt{2}\xi \\ \Phi(0,\cdot) &= \phi_0 \end{aligned} \tag{3.6}$$

where $\Phi \in S'(\mathbb{R}_+ \times \mathbb{T}_N^d)$ is a space-time Schwartz distribution, ϕ_0 is a suitable initial condition, the infinite constant indicates renormalisation, and ξ is space-time white noise. This equation is (a version of) the dynamical ϕ^4 model and has its origins in the theory of stochastic quantisation [PW81]. It also arises naturally as the continuum limit of Glauber dynamics of Ising-type models: this has been established for $d = 2$ in [MW17a] and is conjectured to hold for $d = 3$.

There is now a fairly complete well-posedness theory of (3.6) for $d = 2$ and 3. The local well-posedness for $d = 2$ is classical [DPD03] and global well-posedness on $\mathbb{R}^2$ has also been established [MW17c]. The local well-posedness for $d = 3$ was a major breakthrough in stochastic analysis during the last decade and there are now approaches using regularity structures [Hai14, Hai16], paracontrolled distributions

[GIP15, CC18], and renormalisation group [Kup16]. Global well-posedness on finite volumes was established in [MW17b] and then later extended to infinite volume [GH19, MW18].

By contrast, the long-time/large-scale behaviour of this equation is less understood. On the one hand, in finite volumes one can show that solutions are Markov processes that are reversible with respect to $\nu_{\beta,N}$ and admit a spectral gap $\lambda_{\beta,N} > 0$ - a quantity whose inverse, which is called the relaxation time, governs the rate at which variances converge to equilibrium. See [TW18, HM18a, HS19, ZZ18a]. However, these results are not quantitative and very little is known about the dependency of $\lambda_{\beta,N}$ on β and N. Indeed, due to phase transition one expects that the dynamics in infinite volume does not admit a unique invariant measure when β is sufficiently large. Thus, one expects the limiting behaviour of $\lambda_{\beta,N}$ as $N \to \infty$ to be very sensitive to the choice of β.

This phenomenon has been well-studied for the Glauber dynamics of the 2D Ising model, where a relatively complete picture has been established (in higher dimensions it is less complete). The relaxation times for the Ising dynamics on the 2D torus of sidelength N undergo the following trichotomy as $N \to \infty$: in the high temperature regime, they are uniformly bounded in N [AH87, MO94]; in the low temperature regime, they are exponential in N [Sch87, CCS87, Tho89, MO94, CGMS96]; and at criticality, they are polynomial in N [Hol91, LS12]. It would be interesting to see whether such a trichotomy holds for the relaxation times of dynamical ϕ^4.

4 Main contributions

In this thesis, we solve two problems concerning the statistical mechanics of Euclidean field theories in the physically relevant dimension $d = 3$. One of the reasons why both of these problems had remained open is the difficulties in handling ultraviolet divergences in $d = 3$: previous methods were either too difficult or too delicate to be incorporated successfully with statistical mechanics arguments. The key advancement that has enabled us to attack these problems is the new variational approach to ultraviolet stability for ϕ^4_3 developed by Barashkov and Gubinelli [BG19], which in turn was inspired by methods developed in the context of singular stochastic PDEs in the last decade [Hai14, GIP15]. The common theme underlying our contributions is the development of this variational approach in the context of understanding the statistical mechanics of Euclidean field theories in $d = 3$.

The first contribution of this thesis is to establish the quasi-invariance of Gaussian measures supported on Sobolev spaces under the dynamics of the cubic nonlinear wave equation in three dimensions.

Contribution 1. *Let $s \geqslant 4$ be an even integer. Then, $\vec{\mu}_s$ is quasi-invariant under the dynamics of the defocusing cubic nonlinear wave equation on $\mathbb{T}^3$.*

This is based on the article "*Quasi-invariant Gaussian measures for the nonlinear wave equation in three dimensions*", which is joint work with Tadahiro Oh, Nikolay Tzvetkov, and Hendrik Weber [GOTW18].

We adopt the general strategy of [Tzv15] and study quasi-invariance of the Gaussian measures $\vec{\mu}_s$ indirectly by studying non-Gaussian measures that arise naturally due to the presence of the nonlinearity. The two key steps in this strategy are (i) the construction of the non-Gaussian measure and (ii) an energy estimate on the time derivative of the modified Hamiltonian (that is, the Hamiltonian of the Gaussian measure plus a correction term induced by the presence of the nonlinearity).

In [OT20], this strategy was used to prove the analogue of this result for $d = 2$. This was done by introducing a simultaneous renormalisation on the modified Hamiltonian and its time derivative (this, in particular, allows one to make sense of the nonlinear correction term), and then performing a delicate analysis centered on a quadrilinear Littlewood-Paley expansion. Their analysis does not extend to $d = 3$ because of difficulties in both steps (i) and (ii) of the above strategy.

To prove our result, we combine use of the variational formula, deterministic growth bound on solutions, and stochastic estimates on random distributions to both a) construct the relevant non-Gaussian measures and b) establish softer energy estimates that are sufficient to prove quasi-invariance. This results in a significantly simpler proof of quasi-invariance in the harder, physically relevant three-dimensional case as compared with the two-dimensional case.

The second contribution of this thesis is the development of quantitative methods (in the spirit of [GJS75]) to establish phase transition for ϕ^4_3, and subsequent use of these methods to initiate the study of phase segregation for this model and quantify the decay of the spectral gap for its Glauber dynamics. This is based on the article "*Phase transitions for* ϕ^4_3", which is joint work with Ajay Chandra and Hendrik Weber [CGW20].

We study the behaviour of the average magnetisation

$$\mathfrak{m}_N(\phi) = \frac{1}{N^3} \int_{\mathbb{T}_N} \phi(x) dx$$

for fields ϕ distributed according to $\nu_{\beta,N}$. Our main result is to establish a surface order upper bound on large deviations for $\mathfrak{m}_N$. We state it for $d = 3$ below, but an analogue also holds for $d = 2$.

Contribution 2. *For any* $\zeta \in (0,1)$*, there exists* $C = C(\zeta) > 0$ *such that, for* β *and* N *sufficiently large,*

$$\nu_{\beta,N}\Big(\mathfrak{m}_N \in (-\zeta\sqrt{\beta}, \zeta\sqrt{\beta})\Big) \leqslant e^{-C\sqrt{\beta}N^2}. \tag{4.1}$$

The main difficulty in establishing this result is to handle the ultraviolet divergences of $\nu_{\beta,N}$ whilst preserving the structure of the low temperature potential. We do this by building on the variational approach to ultraviolet stability. Our insight is to separate scales within the corresponding stochastic control problem through a coarse-graining into an effective Hamiltonian and remainder. The effective Hamiltonian captures the macroscopic description of the system and is treated using low temperature expansion techniques adapted from [GJS76b]. The remainder contains the ultraviolet divergences and these are killed using the renormalisation techniques of [BG19].

Morover, we adapt arguments which were used by Bodineau, Velenik, and Ioffe [BIV00] in the context of equilibrium crystal shapes of discrete spin models, to study phase segregation for ϕ^4_3. In particular, we adapt them to handle a block-averaged model with unbounded spins. Technically, this requires control over large fields.

A direct implication of our result is the exponential explosion of relaxation times in the infinite volume limit provided β is sufficiently large. This is a step towards establishing phase transition for the relaxation times of dynamical ϕ^4.

5 The Boué-Dupuis formula in the simplest setting

To close the main body of this introduction, we discuss our main tool - the Boué-Dupuis variational formula for expectations of functionals of Brownian motion - in its simplest setting. The use of this formula in the context of Euclidean field theory in the spirit of [BG19] is explored at depth in the next two parts of this thesis.

We equip $\Omega = C([0,1];\mathbb{R})$ with its Borel σ-algebra and let $\mathbb{P}$ be the probability measure such that the coordinate process $B_\bullet$ is a Brownian motion. We write $\mathbb{E}$ to denote expectation with respect to $\mathbb{P}$. We work on the filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{0\leqslant t\leqslant 1}, \mathbb{P})$, where $\mathcal{A}$ is the $\mathbb{P}$-completion of the Borel σ-alebra on Ω and $(\mathcal{F}_t)_{0\leqslant t\leqslant 1}$ is the natural filtration induced by B augmented with $\mathbb{P}$-null sets of $\mathcal{A}$.

We now define the space of drifts for our control problem. Let $\mathbb{H}$ be the space of processes $v_\bullet$ that are $\mathbb{P}$-almost surely in $L^2([0,1];\mathbb{R})$ and progressively measurable with respect to $(\mathcal{A}_t)_{0\leqslant t\leqslant 1}$. It is convenient in applications, including to show the ultraviolet stability of ϕ^4_3, to also work with bounded drifts $\mathbb{H}_b \subset \mathbb{H}$. These are defined as follows: for every $M \in \mathbb{N}$, let $\mathbb{H}_{b,M} \subset \mathbb{H}$ be drifts such that $\mathbb{P}$-almost surely we have $\int_0^1 v_s^2 ds \leqslant M$. Then, let $\mathbb{H}_b = \bigcup_{M\in\mathbb{N}} \mathbb{H}_{b,M}$. Finally, in the proof of the Boué-Dupuis formula, it is convenient to work with simple drifts $\mathbb{H}_s \subset \mathbb{H}_b$. These are the drifts v of the form

$$v_s = \sum_{j=1}^{k} F_j \mathbf{1}_{(t_j,t_{j+1}]}(s)$$

where $k \in \mathbb{N}$, $0 = t_1 \leqslant \cdots \leqslant t_{k+1} = T$, and $N \in \mathbb{N}$, $F_j : \mathbb{R} \to \mathbb{R}$ is $\mathcal{F}_{t_j}$-measurable, and $|F_j| \leqslant N$ $\mathbb{P}$-almost surely.

The following theorem is the Boué-Dupuis formula.

Theorem 5.1. *Let $\mathscr{H} : \Omega \to \mathbb{R}$ be bounded and measurable. Then,*

$$-\log \mathbb{E}e^{-\mathscr{H}(B)} = \inf \mathbb{E}\left[\mathscr{H}\left(B + \int_0^{\cdot} v_t dt\right) + \frac{1}{2}\int_0^1 v_t^2 dt\right]$$

where the infimum is over $v \in \mathbb{H}$ or $\mathbb{H}_b$.

Proof. Theorem 5.1 was first established in [BD98] but we follow the proof in [BD19, Chapters 8.1.3-8.1.4]. The upper and lower bounds are established in Sections 5.1 and 5.2, respectively. □

Remark 5.2. *Various improvements/extensions of Theorem 5.1 exist. For example, see [Üst14] for a version with $\mathscr{H}$ measurable and satisfying certain integrability conditions. See also [Leh13] for a simplified version of Theorem 5.1 that is sufficient to analyse functional inequalities, i.e. logarithmic Sobolev and Brascamp-Lieb inequalities.*

5.1 Proof of Theorem 5.1: upper bound

We are going to show that for any $v \in \mathbb{H}_s$,

$$-\log \mathbb{E}e^{-\mathscr{H}(B)} \leqslant \mathbb{E}\left[\mathscr{H}\left(B + \int_0^{\cdot} v_t d_t\right) + \frac{1}{2}\int_0^1 v_t^2 dt\right]. \tag{5.1}$$

Showing that this bound extends to all $v \in \mathbb{H}$ (and, hence, all $v \in \mathbb{H}_b$) follows by approximation arguments. See [BD19, Chapter 8.1.3].

Our starting point is a representation of the classical Gibbs variational principle.

Lemma 5.3. *Let $\mathscr{M}_1(\Omega)$ be the space of probability measures on $(\Omega, \mathscr{A})$. Then,*

$$-\log \mathbb{E}e^{-\mathscr{H}(B)} = \inf_{\mathbb{Q}\in\mathscr{M}_1(\Omega)} \left[\mathbb{E}_{\mathbb{Q}}\mathscr{H}(B) + R(\mathbb{Q}\|\mathbb{P})\right]$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes expectation with respect to $\mathbb{Q}$ and $R(\mathbb{Q}\|\mathbb{P}) = \mathbb{E}_{\mathbb{Q}}[\log \frac{d\mathbb{Q}}{d\mathbb{P}}]$ is the relative entropy (with the convention that $R(\mathbb{Q}\|\mathbb{P}) = \infty$ if $\mathbb{Q}$ is not absolutely continuous with respect to $\mathbb{P}$).

Moreover, the infimum is obtained at the measure $\mathbb{Q}^{\text{opt}}$ with density

$$\frac{d\mathbb{Q}^{\text{opt}}}{d\mathbb{P}} = \frac{e^{-\mathscr{H}(B)}}{\mathbb{E}e^{-\mathscr{H}(B)}}.$$

Proof. It suffices to consider $\mathbb{Q} \in \mathscr{M}_1(\Omega)$ absolutely continuous with respect to $\mathbb{P}$. Then, by using the definition of $\mathbb{Q}^{\text{opt}}$,

$$\mathbb{E}_{\mathbb{Q}}\mathscr{H}(B) + R(\mathbb{Q}\|\mathbb{P}) = \mathbb{E}_{\mathbb{Q}}\left[\mathscr{H}(B) + \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}}\Big[\mathscr{H}(B) + \log \frac{d\mathbb{Q}^{\text{opt}}}{d\mathbb{P}} + \log \frac{d\mathbb{Q}}{d\mathbb{Q}^{\text{opt}}}\Big] \\
&= \mathbb{E}_{\mathbb{Q}}\Big[\mathscr{H}(B) - \mathscr{H}(B) - \log \mathbb{E}e^{-\mathscr{H}(B)} + \log \frac{d\mathbb{Q}}{d\mathbb{Q}^{\text{opt}}}\Big] \\
&= -\log \mathbb{E}e^{-\mathscr{H}(B)} + R(\mathbb{Q}\|\mathbb{Q}^{\text{opt}}).
\end{aligned}
$$

We are done with the observation that $R(\mathbb{Q}\|\mathbb{Q}^{\text{opt}}) \geqslant 0$ with equality if and only if $\mathbb{Q} = \mathbb{Q}^{\text{opt}}$. □

For any $\tilde{v} \in \mathbb{H}_s$, denote by $\mathbb{Q}^{\tilde{v}}$ the measure with density

$$
\frac{d\mathbb{Q}^{\tilde{v}}}{d\mathbb{P}} = e^{\int_0^1 \tilde{v}_t dB_t - \frac{1}{2}\int_0^1 \tilde{v}_t^2 dt}. \tag{5.2}
$$

Note that the stochastic exponential in (5.2) has expectation 1 and hence $Q^{\tilde{v}}$ is a probability measure. See e.g. [RY13, Proposition 1.15, Chapter VIII]. By Girsanov's theorem [RY13, Theorem 1.4, Chapter VIII], the process $B^{\tilde{v}} = B - \int_0^{\cdot} \tilde{v}_t dt$ is a Brownian motion under the measure $\mathbb{Q}^{\tilde{v}}$.

Now fix $v \in \mathbb{H}_s$. By direct calculation one can show that there exists $\tilde{v} \in \mathbb{H}_s$ such that the distribution of $(B^{\tilde{v}}, \tilde{v})$ under $\mathbb{Q}^{\tilde{v}}$ is equal to the distribution of (B, v) under $\mathbb{P}$. See [BD19, Lemma 8.7]. Applying the variational principle in Lemma 5.3 with the choice $\mathbb{Q} = \mathbb{Q}^{\tilde{v}}$ then yields

$$
-\log \mathbb{E}e^{-\mathscr{H}(B)} \leqslant \mathbb{E}_{\mathbb{Q}^{\tilde{v}}}\Big[\mathscr{H}(B) + R(\mathbb{Q}^{\tilde{v}}\|\mathbb{P})\Big].
$$

First, note that by Girsanov's theorem and the definition of $\tilde{v}$,

$$
\mathbb{E}_{\mathbb{Q}^{\tilde{v}}}\mathscr{H}(B) = \mathbb{E}_{\mathbb{Q}^{\tilde{v}}}\Big[\mathscr{H}\Big(B^{\tilde{v}} + \int_0^{\cdot} \tilde{v}_t dt\Big)\Big] = \mathbb{E}\Big[\mathscr{H}\Big(B + \int_0^{\cdot} v_t dt\Big)\Big] \tag{5.3}
$$

Second,

$$
\begin{aligned}
R(\mathbb{Q}^{\tilde{v}}\|\mathbb{P}) &= \mathbb{E}_{\mathbb{Q}^{\tilde{v}}}\Big[\int_0^1 \tilde{v}_t dB_t - \frac{1}{2}\int_0^1 \tilde{v}_t^2 dt\Big] = \mathbb{E}_{\mathbb{Q}^{\tilde{v}}}\Big[\int_0^1 \tilde{v}_t dB_t^{\tilde{v}} + \frac{1}{2}\int_0^1 \tilde{v}_t^2 dt\Big] \\
&= \mathbb{E}_{\mathbb{Q}^{\tilde{v}}}\Big[\frac{1}{2}\int_0^1 \tilde{v}_t^2 dt\Big] = \mathbb{E}\Big[\frac{1}{2}\int_0^1 v_t^2 dt\Big]
\end{aligned} \tag{5.4}
$$

where in the first equality we have used the definition of $\mathbb{Q}^{\tilde{v}}$, in the second equality we have used Girsanov's theorem, in the third equality we have used that $\int_0^{\cdot} \tilde{v}_t dB_t^{\tilde{v}}$ is a martingale, and in the last equality we have used the definition of $\tilde{v}_s$.

Combining (5.3) and (5.4) establishes (5.1).

5.2 Proof of Theorem 5.1: lower bound

We restrict to the case where $\mathscr{H}(B)$ is of the form:

$$\mathscr{H}(B) = h\big(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}\big)$$

where $k \in \mathbb{N}$, $0 = t_1 < \cdots < t_k = T$, and $h : \mathbb{R}^k \to \mathbb{R}$ is smooth and compactly supported.

The advantage of this regularisation is that we are able to construct an explicit minimiser for the corresponding variational problem: i.e., we show that there exists $u \in \mathbb{H}_b \subset \mathbb{H}$ such that

$$-\log \mathbb{E}e^{-\mathscr{H}(B)} = \mathbb{E}\Big[\mathscr{H}\Big(B + \int_0^\cdot u_t dt\Big) + \frac{1}{2}\int_0^1 u_t^2 dt\Big].$$

The extension to measurable and bounded $\mathscr{H}$ then requires tedious but straightforward approximation arguments, so we omit them. Note that the infimum in the stochastic control problem may not be attained for general $\mathscr{H}$. See [BD19, Chapter 8.1.4] for more details on this approximation procedure.

The key tool to construct minimisers is given in the following lemma.

Lemma 5.4. *Fix $T > 0$ and $m \in \mathbb{N}$. Let $g : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be smooth and compactly supported, and define $V : [0,T] \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ by*

$$V(t,z,x) = -\log \mathbb{E}e^{-g(z,x+B_{T-t})}.$$

Then,

- *$z \mapsto V(t,z,x)$ is smooth and compactly supported for all $(t,x) \in [0,T] \times \mathbb{R}$;*
- *$x \mapsto V(t,z,x)$ is smooth with bounded derivatives of all order for all $(t,z) \in [0,T] \times \mathbb{R}^m$.*

Moreover, for $z \in \mathbb{R}^m$, let $t \mapsto U(z,t)$ be the unique solution to the equation

$$U(z,t) = -\int_0^t \partial_x V(s,z,U(z,s))ds + B_t$$

and define $u(t) = -\partial_x V(t,z,U(z,t))$.

Then,

$$-\log \mathbb{E}e^{-g(z,B_T)} = \mathbb{E}\Big[g\Big(z, B + \int_0^\cdot u_t dt\Big) + \frac{1}{2}\int_0^T u_t^2 dt\Big]. \tag{5.5}$$

Proof. This lemma is classical in stochastic control theory. Indeed, by the Feynman-Kac formula, $Z = \mathbb{E}e^{g(z,x+B_{T-t})}$ solves a linear PDE. From this, one gets that $V = -\log Z$ solves a nonlinear PDE called a Hamilton-Jacobi-Bellman equation. As such, V can be interpreted as a cost function for a stochastic control problem. Since V is smooth, standard arguments can be used to construct a minimiser. See [FS06, Chapter VI]. □

We define a sequence of potentials $V_j : \mathbb{R}^j \to \mathbb{R}$ for $j \in \{1, \ldots, k\}$ as follows: let $V_k = h$. For $j \in \{1, \ldots, k-1\}$ and $z_j \in \mathbb{R}^j$, let

$$V_j(z_j) = -\log \mathbb{E}e^{V_{j+1}(z_j, B_{t_{j+1}} - B_{t_j})}.$$

Using the independence of Brownian increments, we can rewrite this as a conditional expectation

$$e^{-V_j(z_j)} = \mathbb{E}e^{-V_{j+1}(z_j, B_{t_{j+1}} - B_{t_j})} = \mathbb{E}\Big[e^{-V_{j+1}(z_j, B_{t_{j+1}} - B_{t_j})} | \mathscr{A}_{t_j}\Big].$$

Then,

$$V_0 = -\log \mathbb{E}e^{-V_1(B_{t_1})} = -\log \mathbb{E}e^{-\mathscr{H}(B)}$$

where the first equality is by definition and the second is by successive conditioning. Thus, we can interpret the sequence of potentials $(V_j)_{1 \leqslant j \leqslant k}$ as renormalisations of the potential $V_0 = -\log \mathbb{E}e^{-\mathscr{H}(B)}$.

We construct a minimiser to the stochastic control problem associated to V_0, which is what we are interested in, by analysing the stochastic control problems associated to the renormalised potentials starting with $j = k$ and then running backwards. In particular, we apply Lemma 5.4 to construct minimisers of the control problem associated to V_j for times $t \in [t_j, t_{j+1})$ and then glue these minimisers together.

Let $(U(z_j, t))_{t_j \leqslant t < t_{j+1}}$ be the solution of the equation

$$U(z_j, t) = -\int_{t_j}^{t_{j+1}} \partial_x V_{j+1}(s, z_j, U(z_j, s)) ds + B(t) - B(t_j).$$

Define the process $u \in \mathbb{H}_b$ by $u(t) = -\partial_x V_{j+1}(t, Z_j, U(Z_j, t))$ for $t \in [t_j, t_{j+1})$, where $Z_j = (B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}})$. Then, by recursively applying (5.5) starting from $j = k$ and running backwards, we obtain:

$$\begin{aligned} -\log \mathbb{E}e^{\mathscr{H}(B)} &= \mathbb{E}\Big[h\Big(B_{t_1} + \int_0^{t_1} u_t dt, B_{t_2} - B_{t_1} + \int_{t_1}^{t_2} u_t dt, \\ &\qquad \ldots, B_{t_k} - B_{t_{k-1}} + \int_{t_{k-1}}^{t_k} u_t dt\Big) + \frac{1}{2}\int_0^1 u_t^2 dt\Big] \\ &= \mathbb{E}\Big[\mathscr{H}\Big(B + \int_0^{\cdot} u_t dt\Big) + \frac{1}{2}\int_0^1 u_t^2 dt\Big] \end{aligned}$$

which completes the proof with this specific choice of $\mathscr{H}$.

6 Thesis organisation

The remainder of this thesis is organised into three parts. Part II concerns quasi-invariant Gaussian measures of nonlinear wave equations. In the prologue, we begin by proving the classical correspondence between conserved quantities and invariant measures in finite dimensional Hamiltonian dynamics. Then, we discuss extending this to the linear wave equation. The main body of this part consists of the work [GOTW18], where we establish Contribution 1. In the epilogue, we discuss an extension of our results that has since appeared in the literature.

Part III concerns phase transitions for the ϕ^4 model. In the prologue, as a warm-up for the arguments to come, we recall the classical contour bounds for the low temperature Ising model. The main body of this part consists of the work [CGW20], where we establish Contribution 2. In the epilogue, we sketch how our techniques can be used with the Peierls' argument of [GJS75] to establish phase transition for ϕ^4_3.

In Part IV we conclude this thesis with a discussion of future directions. In particular, we discuss two interesting problems that seem within reach.

II. Nonlinear wave equations

Prologue

We begin this part by deriving the classical fact that Gibbs measures are invariant under finite dimensional Hamiltonian dynamics.

Hamiltonian dynamics in finite dimensions are systems of ODEs of the form

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= -\nabla_{\mathbf{q}}\mathscr{H}(\mathbf{p},\mathbf{q}) \\ \frac{d\mathbf{q}}{dt} &= \nabla_{\mathbf{p}}\mathscr{H}(\mathbf{p},\mathbf{q})\end{aligned} \tag{0.1}$$

where $\mathbf{p},\mathbf{q} \in \mathbb{R}^3$ are generalised momentum and position; $\nabla_{\mathbf{p}}, \nabla_{\mathbf{q}}$ are gradients in momentum and position space, respectively, and $\mathscr{H} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a (e.g. smooth) Hamiltonian.

Note that, for solutions $(\mathbf{p}(t), \mathbf{q}(t))$ of (0.1), we have

$$\frac{d}{dt}\mathscr{H}(\mathbf{p},\mathbf{q}) = \nabla_{\mathbf{p}}\mathscr{H}(\mathbf{p},\mathbf{q}) \cdot \frac{d\mathbf{p}}{dt} + \nabla_{\mathbf{q}}\mathscr{H}(\mathbf{p},\mathbf{q}) \cdot \frac{d\mathbf{q}}{dt} = 0.$$

Thus, the Hamiltonian $\mathscr{H}$ is conserved.

We write $\Phi : \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}^6$ to denote the flow of this ODE, i.e. $\Phi(t)(\mathbf{p_0}, \mathbf{q_0})$ is the solution to (0.1) with initial data $(\mathbf{p_0}, \mathbf{q_0})$. Note that it is reversible, i.e. $\Phi(t)^{-1} = \Phi(-t)$.

Let m be the Gibbs measure, i.e. the measure with density proportional to

$$e^{-\mathscr{H}(\mathbf{p},\mathbf{q})} d\mathbf{p} d\mathbf{q}.$$

Consider $\Phi(t)_* m(A)$ for any measurable set $A \subset \mathbb{R}^6$ and $t \in \mathbb{R}$, where $\Phi(t)_* m$ is the pushforward of m under $\Phi(t)$. Note that by reversibility, $\Phi(t)_* m(A) = m(\Phi(-t)A)$. Then,

$$\begin{aligned}\partial_t \Phi(t)_* m(A) &= \partial_t \int_{\Phi(-t)A} e^{-\mathscr{H}(\mathbf{p},\mathbf{q})} d\mathbf{p} d\mathbf{q} \\ &= \partial_t \int_A e^{-\mathscr{H}(\Phi(t)(\mathbf{p},\mathbf{q}))} \det(\nabla_{\mathbf{p}}\Phi(t)(\mathbf{p},\mathbf{q}), \nabla_{\mathbf{q}}\Phi(t)(\mathbf{p},\mathbf{q})) d\mathbf{p} d\mathbf{q} \\ &= \int_A -\partial_t \mathscr{H}(\Phi(t)(\mathbf{p},\mathbf{q})) \cdot e^{-\mathscr{H}(\Phi(t)(\mathbf{p},\mathbf{q})}\det\big(\nabla_{\mathbf{p}}\Phi(t)(\mathbf{p},\mathbf{q}), \nabla_{\mathbf{q}}\Phi(t)(\mathbf{p},\mathbf{q})\big) \\ &+ e^{-\mathscr{H}(\Phi(t)(\mathbf{p},\mathbf{q}))} \nabla_{\mathbf{p},\mathbf{q}} \cdot \big(- \nabla_{\mathbf{q}}\mathscr{H}(\Phi(t)(\mathbf{p},\mathbf{q})), \nabla_{\mathbf{p}}\mathscr{H}(\Phi(t)(\mathbf{p},\mathbf{q}))\big) d\mathbf{p} d\mathbf{q} \\ &= 0\end{aligned}$$

where the first equality is by a standard change of variables, the second equality is by direct calculation, and the third equality is by the conservation of $\mathscr{H}$ and fact that the vector field $\frac{d}{dt}\Phi(t)(\mathbf{p},\mathbf{q}) = \big(-\nabla_{\mathbf{q}}\mathscr{H}(\Phi(t)(\mathbf{p},\mathbf{q})), \nabla_{\mathbf{p}}\mathscr{H}(\Phi(t)(\mathbf{p},\mathbf{q}))\big)$ is divergence free with respect to $\nabla_{\mathbf{p},\mathbf{q}} = (\nabla_{\mathbf{p}}, \nabla_{\mathbf{q}})$. Hence, m is invariant under Φ. Note that this argument is not special to the Gibbs measure; one can replace m by any measure with density

$$e^{-\tilde{\mathscr{H}}(\mathbf{p},\mathbf{q})}d\mathbf{p}d\mathbf{q}$$

where $\tilde{\mathscr{H}}$ is conserved under (0.1).

The approach above, which is fundamentally Liouville's theorem in statistical physics, is still useful in the infinite dimensional context. Recall that the linear wave equation is given by the system of PDEs

$$\begin{aligned}\partial_t u &= v\\ \partial_t v &= \Delta u.\end{aligned} \tag{0.2}$$

Moreover, recall that $\vec{\mu}_s$ is the Gaussian measure with formal density

$$d\vec{\mu}_s = Z_s^{-1}e^{-\frac{1}{2}\|\vec{u}\|^2_{\vec{H}^{s+1}}}d\vec{u}$$

where $\vec{H}^{s+1}(\mathbb{T}^d) = H^{s+1}(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$ and $\vec{u} = (u, v)$. Note that, for a solution $\vec{u}(t)$ of (0.2),

$$\begin{aligned}\frac{d}{dt}\|\vec{u}\|^2_{\vec{H}^{s+1}} &= \frac{d}{dt}\int_{\mathbb{T}^d} |(-\Delta)^{\frac{s+1}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2 dx\\ &= 2\int_{\mathbb{T}^d} (-\Delta)^{\frac{s+1}{2}}u \cdot (-\Delta)^{\frac{s+1}{2}}v + (-\Delta)^{\frac{s}{2}}v \cdot (-\Delta)^{\frac{s}{2}}(\Delta u)dx = 0\end{aligned}$$

where the third equality is by (fractional) integration by parts. Hence, $\|\vec{u}\|^2_{\vec{H}^{s+1}}$ is conserved under (0.2). A truncation argument in Fourier space along with Liouville's theorem for finite dimensional systems then establishes invariance. Details are given in [Tzv15], where this approach is the first step to establishing quasi-invariance of Gaussian measures under (nonlinear) Hamiltonian PDEs.

Although the approach of the preceding paragraph is more in the spirit of what we do in the upcoming sections, the invariance of the measures $\vec{\mu}_s$ under the linear wave equation can be seen more elegantly by using rotation invariance of Gaussian measures. We give the essential ideas but omit details. Samples from $\vec{\mu}_s$ can be constructed as random Fourier series (so-called Karhunen-Loève expansions) that converge in $\vec{H}^\sigma$ for every $\sigma < s - \frac{d-2}{2}$. The solution map of the linear wave equation acts as a rotation map on frequencies of functions belonging to $\vec{H}^\sigma$. As a result, the solution of the linear wave equation with initial data sampled from $\vec{\mu}_s$ admits a random Fourier representation where each frequency is a rotation from the frequencies of the initial data, hence it is distributed according to $\vec{\mu}_s$. This establishes invariance.

Statement of authorship

Appendix 6B: Statement of Authorship

This declaration concerns the article entitled:

Quasi-invariant Gaussian measures for the nonlinear wave equation in three dimensions

Publication status (tick one)

Draft manuscript	Submitted	In review	Accepted	Published
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Candidate's contribution to the paper (provide details, and also indicate as a percentage)	The candidate contributed to / considerably contributed to / predominantly executed the… Formulation of ideas: All ideas were a result of fair and equal collaboration between all four coauthors. So 25%. Design of methodology: N/A Experimental work: N/A Presentation of data in journal format: N/A
Statement from Candidate	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature.

Signed		**Date**	01/10/20

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1 Introduction

1.1 Main result

We consider the following defocusing cubic nonlinear wave equation (NLW) on the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$:

$$\partial_t^2 u - \Delta u + u^3 = 0, \tag{1.1}$$

where $u : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}$ is the unknown function. With $v = \partial_t u$, we rewrite (1.1) in the following vectorial form:

$$\begin{cases} \partial_t u = v \\ \partial_t v = \Delta u - u^3. \end{cases} \tag{1.2}$$

Given $\sigma \in \mathbb{R}$, let $H^\sigma(\mathbb{T}^3)$ denote the classical L^2-based Sobolev space of order σ defined by the norm:

$$\|u\|_{H^\sigma} = \|\langle n\rangle^\sigma \widehat{u}(n)\|_{\ell^2(\mathbb{Z}^3)},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\widehat{u}$ denotes the Fourier transform of u. A classical argument yields global well-posedness of the Cauchy problem (1.2) in the Sobolev spaces:

$$\vec{H}^\sigma(\mathbb{T}^3) \overset{\text{def}}{=} H^\sigma(\mathbb{T}^3) \times H^{\sigma-1}(\mathbb{T}^3)$$

for $\sigma \geqslant 1$ and, consequently, admits a global flow Φ_{NLW} (see Lemma 2.4 below) on these spaces.

Given $s \in \mathbb{R}$, let $\vec{\mu}_s$ denote the Gaussian measure with Cameron-Martin space $\vec{H}^{s+1}(\mathbb{T}^3)$. Denoting $\vec{u} = (u, v)$, the Gaussian measure $\vec{\mu}_s$ has a formal density:

$$\begin{aligned} d\vec{\mu}_s &= Z_s^{-1} e^{-\frac{1}{2}\|\vec{u}\|^2_{\vec{H}^{s+1}}} d\vec{u} \\ &= \prod_{n \in \mathbb{Z}^3} Z_{s,n}^{-1} e^{-\frac{1}{2}\langle n\rangle^{2(s+1)}|\widehat{u}(n)|^2} e^{-\frac{1}{2}\langle n\rangle^{2s}|\widehat{v}(n)|^2} d\widehat{u}(n) d\widehat{v}(n). \end{aligned}$$

Samples $\vec{u}^\omega = (u^\omega, v^\omega)$ from $\vec{\mu}_s$ can be constructed via the following Karhunen-Loève expansions:[1]

$$u^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^{s+1}} e^{in\cdot x} \qquad \text{and} \qquad v^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{s}} e^{in\cdot x}, \tag{1.3}$$

where $\{g_n\}_{n\in\mathbb{Z}^3}$ and $\{h_n\}_{n\in\mathbb{Z}^3}$ are collections of standard complex-valued Gaussian variables which are independent modulo the condition[2] $g_n = \overline{g_{-n}}$ and $h_n = \overline{h_{-n}}$. It is easy to see that the series (1.3) converge in $L^2(\Omega; \vec{H}^\sigma(\mathbb{T}^3))$ for

$$\sigma < s - \frac{1}{2} \tag{1.4}$$

[1]Henceforth, we drop the harmless factor 2π.

[2]In particular, we impose that g_0 and h_0 are real-valued.

and therefore the map

$$\omega \in \Omega \longmapsto (u^\omega, v^\omega)$$

induces the Gaussian measure $\vec{\mu}_s$ as a probability measure on $\vec{H}^\sigma(\mathbb{T}^3)$ for the same range of σ. Our main goal in this paper is to study the transport property of the Gaussian measure $\vec{\mu}_s$ under the dynamics of (1.2). We state our main result.

Theorem 1.1. *Let $s \geqslant 4$ be an even integer. Then, $\vec{\mu}_s$ is quasi-invariant under the dynamics of the defocusing cubic NLW* (1.2) *on $\mathbb{T}^3$. More precisely, for any $t \in \mathbb{R}$, the Gaussian measure $\vec{\mu}_s$ and its pushforward under $\Phi_{\mathrm{NLW}}(t)$ are mutually absolutely continuous.*

Theorem 1.1 ensures the propagation of almost sure properties of $\vec{\mu}_s$ along the flow. This is important because, in infinite dimensions, many interesting properties concerning small-scale behavior under a Gaussian measure hold true with probability 0 or 1. This is an implication of Fernique's theorem (Theorem 2.7 in [DPZ14]); under a Gaussian measure, any given norm is finite with probability 0 or 1. For example, samples $\vec{u}$ of the Gaussian measure $\vec{\mu}_s$ almost surely belong to the L^p-based Sobolev spaces $\vec{W}^{\sigma,p}(\mathbb{T}^3)$ for any $p \geqslant 1$ and more generally to the Besov spaces, $\vec{B}^\sigma_{p,q}(\mathbb{T}^3)$ for any $p, q \geqslant 1$, including the case $p = q = \infty$ (Hölder-Besov space), provided that σ satisfies (1.4). Theorem 1.1 then implies that these L^p-based regularities are transported along the nonlinear flow. An analogous statement for deterministic initial data is expected to fail in general. See [Lit63, Per80, Sog93].

Theorem 1.1 is an addition to a series of recent results [Tzv15, OT17, OST18, OT20, OTT19] that has made significant progress in the study of transport properties of Gaussian measures under nonlinear Hamiltonian PDEs. The general strategy, as introduced by the third author in [Tzv15], is to study quasi-invariance of the Gaussian measures $\vec{\mu}_s$ indirectly by studying weighted Gaussian measures, where the weight corresponds to a correction term that arises due to the presence of the nonlinearity. See Subsection 3.2. The two key steps in this strategy are (i) the construction of the weighted Gaussian measure and (ii) an energy estimate on the time derivative of the modified energy (that is, the energy of the Gaussian measure plus the correction term). In [OT20], the second and third authors employed this strategy and proved the analogue of Theorem 1.1 in the two-dimensional case. This was done by introducing a simultaneous renormalization on the modified energy functional and its time derivative and then performing a delicate analysis centered on a quadrilinear Littlewood-Paley expansion.

As pointed out in [OT20], the argument in the two-dimensional case does not extend to the current three-dimensional setting. The proof of Theorem 1.1 uses two new key ingredients. The first is the use of a variational formula in constructing weighted Gaussian measures, inspired by Barashkov and Gubinelli [BG19]. The second new ingredient appears in studying the growth of the truncated weighted Gaussian measures, where we combine a deterministic growth bound on solutions (as in a recent paper by Planchon, Visciglia, and the third author [PTV19]) with stochastic estimates on random distributions (as in the two-dimensional case [OT20]). This hybrid argument allows us to use a softer energy estimate to prove quasi-invariance. Our simplification also comes from the use of Besov spaces in the spirit of [MWX17]. This results in a significantly simpler proof of quasi-invariance in the harder, physically relevant three-dimensional case as compared with the two-dimensional case.

1.2 Remarks and comments

(i) A slight modification of the proof of Theorem 1.1 shows that the Gaussian measures $\vec{\mu}_s$ are also quasi-invariant under the nonlinear Klein-Gordon equation:

$$\begin{cases} \partial_t u = v \\ \partial_t v = (\Delta - 1)u - u^3. \end{cases} \tag{1.5}$$

It is easy to see that $\vec{\mu}_s$ is invariant under the linear Klein-Gordon equation, i.e. removing u^3 in (1.5), which trivially implies that almost sure properties of $\vec{\mu}_s$ are transported along the flow of the linear dynamics. The addition of a defocusing cubic nonlinearity into the equation destroys invariance but the quasi-invariance of $\vec{\mu}_s$ for (1.5) can be interpreted as saying that the nonlinear flow retains the small-scale properties of the linear flow.

In order to obtain invariance of $\vec{\mu}_s$ under the linear wave equation, one would need to replace $\langle \cdot \rangle$ with $|\cdot|$ in (1.3), which would raise an issue at the zeroth Fourier mode (see Remark 3.6). Nevertheless, in the study of small-scale properties of solutions, this issue is irrelevant and one can easily show that $\vec{\mu}_s$ is quasi-invariant under the linear wave equation. Theorem 1.1 then implies that the NLW dynamics also retains the small-scale properties of the linear wave dynamics.

(ii) The restriction that s is an even integer in Theorem 1.1 comes from an application of the classical Leibniz rule in order to derive the right correction term for the modified energy and the weighted Gaussian measure. In terms of regularity restrictions, the construction of the weighted Gaussian measure works for any real $s > \frac{3}{2}$ (Proposition 3.7). Our argument for the energy estimate (Proposition 3.8) only requires $s > \frac{5}{2}$ but, in our derivation of a modified energy, we also use the classical Leibniz rule for $(-\Delta)^{\frac{s}{2}}$ which only works if s is an even integer. It may be possible to relax this second condition using a fractional Leibniz rule to go below $s = 4$. At present, however, we do not know how to do this.

(iii) Our new hybrid argument in proving Theorem 1.1 requires a softer energy estimate than that in [OT20] and is also applicable to the two-dimensional case. We point out, however, that the argument in [OT20], involving heavier multilinear analysis, provides better quantitative information on the growth of the truncated weighted Gaussian measures. See Remark 3.12. For example, the argument in [OT20] allows us to prove higher L^p-integrability of the Radon-Nikodym derivative of the weighted Gaussian measures (with an energy cutoff), while our proof of Theorem 1.1 does not provide such extra information.

(iv) It would be of interest to investigate the quasi-invariance property of $\vec{\mu}_s$ for NLW with a higher order nonlinearity or in higher dimensions. Our techniques appear to carry over to higher order nonlinearities. This might even permit to analyze energy-supercritical equations (such as the three-dimensional septic NLW), where global well-posedness is not known. Consequently, one might aim to prove "local-in-time" quasi-invariance (as stated in [Bou96a]). See also [PTV19] for an example of a local-in-time quasi-invariance result. See also Remark 3.4 below.

(v) Quasi-invariance results such as Theorem 1.1 are complimentary to the study of low regularity well-posedness with random initial data. Starting with the seminal work of Bourgain [Bou94, Bou96b], there has been intensive study on the random data Cauchy theory for nonlinear dispersive PDEs. There are two related directions in this study. The first one is the study of invariant measures associated with conservation laws such as Gibbs measures, in particular, the construction of almost sure global-in-time dynamics via the so-called Bourgain's invariant measure argument; see [OT17, BOP19] for the references therein. The other is the study of almost sure well-posedness with respect to random initial data. Here, one can often exploit the higher L^p_x-based regularity made accessible by randomization of initial data to establish well-posedness below critical thresholds, where equations are ill-posed in L^2-based Sobolev spaces. In the context of NLW, see the work [BT08, BT11] by Burq and the third author for almost sure local well-posedness. There are also globalization arguments in this probabilistic setting; see [BT11, Poc, OP16, OP17]. See also a general review [BOP19] on the subject.

As for the defocusing cubic NLW (1.2) on $\mathbb{T}^3$, the scaling symmetry induces the critical regularity $\sigma_{\text{crit}} = \frac{1}{2}$. It is known that (1.2) is locally well-posed in $\vec{H}^\sigma(\mathbb{T}^3)$ for $\sigma \geqslant \frac{1}{2}$, while it is ill-posed for $\sigma < \frac{1}{2}$; see [LS95, CCT03, BT08, OOT]. In [BT08, BT11], Burq and the third author proved almost sure global well-posedness of (1.2) with respect to the random initial data in (1.3) for $s > \frac{1}{2}$, namely for $\sigma > 0$. In this regime, the flow Φ_{NLW} exists almost surely globally in time. Then, it is natural to ask the following question.

Problem. *Study the transport property of the Gaussian measures $\vec{\mu}_s$ for low values of $s > \frac{1}{2}$, in particular in the regime where the global-in-time dynamics is constructed only probabilistically.*

1.3 Organization

In Section 2, we introduce basic tools in our proof: Besov spaces, the Wiener chaos estimate, the classical well-posedness theory of (1.2), and also deterministic growth bounds. In Section 3, we present the proof of Theorem 1.1 assuming (i) the construction of the weighted Gaussian measures (Proposition 3.7) and (ii) the energy estimate (Proposition 3.8). Section 4 is devoted to the construction of the weighted Gaussian measures and, finally, Section 5 deals with the energy estimate.

2 Analytic and stochastic toolbox

2.1 On the phase space

Given $N \in \mathbb{N}$, we denote by π_N the frequency projector on the (spatial) frequencies $\{|n| \leqslant N\}$:

$$(\pi_N u)(x) = \sum_{|n| \leqslant N} \widehat{u}_n \, e^{in\cdot x},$$

We then set

$$\mathscr{E}_N = \pi_N L^2(\mathbb{T}^3).$$

Namely, $\mathscr{E}_N$ is the finite-dimensional vector space of real-valued trigonometric polynomials of degree $\leqslant N$ endowed with the restriction of the $L^2(\mathbb{T}^3)$ scalar product. The product space $\mathscr{E}_N \times \mathscr{E}_N$ is a finite dimensional real inner-product space and thus there is a canonical Lebesgue measure on this space, which we denote by L_N. We also use $(\mathscr{E}_N \times \mathscr{E}_N)^\perp$ to denote the orthogonal complement of $\mathscr{E}_N \times \mathscr{E}_N$ in $\vec{H}^\sigma(\mathbb{T}^3)$, $\sigma < s - \frac{1}{2}$.

2.2 Besov spaces

Let $B(\xi, r)$ denote the ball in $\mathbb{R}^3$ of radius $r > 0$ centered at $\xi \in \mathbb{R}^3$ and let $\mathscr{A}$ denote the annulus $B(0, \frac{4}{3}) \backslash B(0, \frac{3}{8})$. Letting $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define a sequence $\{\chi_j\}_{j\in\mathbb{N}_0}$ by setting

$$\chi_0 = \widetilde{\chi}, \qquad \chi_j(\,\cdot\,) = \chi(2^{-j}\,\cdot\,), \qquad \text{and} \qquad \sum_{j=0}^{\infty} \chi_j \equiv 1$$

for some suitable $\widetilde{\chi}, \chi \in C_c^\infty(\mathbb{R}^3; [0,1])$ such that $\mathrm{supp}(\widetilde{\chi}) \subset B(0, \frac{4}{3})$ and $\mathrm{supp}(\chi) \subset \mathscr{A}$. We then define the Littlewood-Paley projector $\mathbf{P}_j$, $j \in \mathbb{N}_0$, by setting

$$\mathbf{P}_j u(x) = \sum_{n\in\mathbb{Z}^3} \chi_j(n) \widehat{u}(n) e^{in\cdot x}$$

for $u \in \mathscr{D}'(\mathbb{T}^3)$.

Given $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the Besov space $B^s_{p,q}(\mathbb{T}^3)$ is the set of distributions $u \in \mathscr{D}'(\mathbb{T}^3)$ such that

$$\|u\|_{B^s_{p,q}} = \Big\| \big\{ 2^{sj} \|\mathbf{P}_j u\|_{L^p_x} \big\}_{j\in\mathbb{N}_0} \Big\|_{\ell^q_j} < \infty. \tag{2.1}$$

We use the conventions $\vec{B}^s_{p,q}(\mathbb{T}^3) = B^s_{p,q}(\mathbb{T}^3) \times B^{s-1}_{p,q}(\mathbb{T}^3)$ and $\vec{\mathscr{C}}^s(\mathbb{T}^3) = \mathscr{C}^s(\mathbb{T}^3) \times \mathscr{C}^{s-1}(\mathbb{T}^3)$, where $\mathscr{C}^s(\mathbb{T}^3) = B^s_{\infty,\infty}(\mathbb{T}^3)$ denotes the Hölder-Besov space. Note that (i) the parameter s measures differentiability and p measures integrability, (ii) $H^s(\mathbb{T}^3) = B^s_{2,2}(\mathbb{T}^3)$, and (iii) for $s > 0$ and not an integer, $\mathscr{C}^s(\mathbb{T}^3)$ coincides with the classical Hölder spaces; see [Gra09].

Lemma 2.1. *The following estimates hold.*
(i) (interpolation) *For* $0 < s_1 < s_2$, *we have*[3]

$$\|u\|_{H^{s_1}} \lesssim \|u\|_{H^{s_2}}^{\frac{s_1}{s_2}} \|u\|_{L^2}^{\frac{s_2-s_1}{s_2}}. \tag{2.2}$$

(ii) (immediate embeddings) *Let* $s_1, s_2 \in \mathbb{R}$ *and* $p_1, p_2, q_1, q_2 \in [1, \infty]$. *Then, we have*

$$\begin{aligned} &\|u\|_{B^{s_1}_{p_1,q_1}} \lesssim \|u\|_{B^{s_2}_{p_2,q_2}} && \text{for } s_1 \leqslant s_2,\ p_1 \leqslant p_2, \text{ and } q_1 \geqslant q_2, \\ &\|u\|_{B^{s_1}_{p_1,q_1}} \lesssim \|u\|_{B^{s_2}_{p_1,\infty}} && \text{for } s_1 < s_2, \\ &\|u\|_{B^0_{p_1,\infty}} \lesssim \|u\|_{L^{p_1}} \lesssim \|u\|_{B^0_{p_1,1}}. \end{aligned} \tag{2.3}$$

(iii) (algebra property) *Let* $s > 0$. *Then, we have*

$$\|uv\|_{\mathscr{C}^s} \lesssim \|u\|_{\mathscr{C}^s} \|v\|_{\mathscr{C}^s}. \tag{2.4}$$

(iv) (Besov embedding) *Let* $1 \leqslant p_2 \leqslant p_1 \leqslant \infty$, $q \in [1, \infty]$, *and* $s_2 = s_1 + 3\big(\frac{1}{p_2} - \frac{1}{p_1}\big)$. *Then, we have*

$$\|u\|_{B^{s_1}_{p_1,q}} \lesssim \|u\|_{B^{s_2}_{p_2,q}}. \tag{2.5}$$

(v) (duality) *Let* $s \in \mathbb{R}$ *and* $p, p', q, q' \in [1, \infty]$ *such that* $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. *Then, we have*

$$\left| \int_{\mathbb{T}^3} uv\, dx \right| \leqslant \|u\|_{B^s_{p,q}} \|v\|_{B^{-s}_{p',q'}}, \tag{2.6}$$

where $\int_{\mathbb{T}^3} uv\, dx$ *denotes the duality pairing between* $B^s_{p,q}(\mathbb{T}^3)$ *and* $B^{-s}_{p',q'}(\mathbb{T}^3)$.
(vi) (fractional Leibniz rule) *Let* $p, p_1, p_2, p_3, p_4 \in [1, \infty]$ *such that* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}$. *Then, for every* $s > 0$, *we have*

$$\|uv\|_{B^s_{p,q}} \lesssim \|u\|_{B^s_{p_1,q}} \|v\|_{L^{p_2}} + \|u\|_{L^{p_3}} \|v\|_{B^s_{p_4,q}}. \tag{2.7}$$

(vi) (product estimate) *Let* $s_1 < 0 < s_2$ *such that* $s_1 + s_2 > 0$. *Then, we have*

$$\|uv\|_{\mathscr{C}^{s_1}} \lesssim \|u\|_{\mathscr{C}^{s_1}} \|v\|_{\mathscr{C}^{s_2}}. \tag{2.8}$$

[3]We use the convention that the symbol $\lesssim$ indicates that inessential constants are suppressed in the inequality.

Proof. While these estimates are standard, we briefly discuss their proofs for readers' convenience. See also [BCD11] for details of the proofs in the non-periodic case. The log convexity inequality (2.2) and the duality (2.6) follow from Hölder's inequality. The first estimate in (2.3) is immediate from the definition (2.1), while the second one in (2.3) follows from the ℓ^{q_1}-summability of $\{2^{(s_1-s_2)j}\}_{j\in\mathbb{N}_0}$ for $s_1 < s_2$. The last estimate in (2.3) follows from the boundedness of the Littlewood-Paley projector $\mathbf{P}_j$ and Minkowski's inequality. The Besov embedding (2.5) is a direct consequence of Bernstein's inequality:

$$\|\mathbf{P}_j u\|_{L^{p_1}} \lesssim 2^{3j(\frac{1}{p_2}-\frac{1}{p_1})}\|\mathbf{P}_j u\|_{L^{p_2}}.$$

The algebra property (2.4) is immediate from the following paraproduct decomposition due to Bony [Bon81]:

$$uv = \sum_{j\in\mathbb{N}_0} \mathbf{P}_j u \cdot S_j v + \sum_{j\in\mathbb{N}_0}\sum_{|j-k|\leqslant 1} \mathbf{P}_j u \cdot \mathbf{P}_k v + \sum_{k\in\mathbb{N}_0} S_k u \cdot \mathbf{P}_k v \tag{2.9}$$

with Hölder's inequality. Here, S_j is given by

$$S_j u = \sum_{k\leqslant j-2} \mathbf{P}_k u.$$

The fractional Leibniz rule (2.7) also follows from the paraproduct decomposition (2.9). In proving (2.7) for the resonant product, i.e. the second term on the right-hand side of (2.9), one needs to proceed slightly more carefully:

$$\begin{aligned}
&\bigg\| 2^{sm}\Big\|\mathbf{P}_m\Big(\sum_{j\in\mathbb{N}_0}\sum_{|j-k|\leqslant 1}\mathbf{P}_j u\cdot\mathbf{P}_k v\Big)\Big\|_{L^p}\bigg\|_{\ell^q_m}\\
&\quad\lesssim \bigg\|\sum_{j\geqslant m-10} 2^{s(m-j)}2^{sj}\|\mathbf{P}_j u\|_{L^{p_1}}\|\mathbf{P}_j v\|_{L^{p_2}}\bigg\|_{\ell^q_m}\\
&\quad\lesssim \|u\|_{B^s_{p_1,q}}\|v\|_{L^{p_2}},
\end{aligned}$$

where we used Young's and Hölder's inequalities together with the embedding: $L^{p_2}(\mathbb{T}^3) \hookrightarrow B^0_{p_2,\infty}(\mathbb{T}^3)$ in the last step. See also Lemma 2.84 in [BCD11]. Lastly, the product estimate (2.8) follows from a similar consideration. □

2.3 Wiener chaos estimate

Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variables defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathscr{F}$ is the σ-algebra generated by this sequence. Given $k \in \mathbb{N}_0$, we define the homogeneous Wiener chaoses $\mathscr{H}_k$ to be the closure (under $L^2(\Omega)$) of the span of Fourier-Hermite polynomials $\prod_{n=1}^{\infty} H_{k_n}(g_n)$, where H_j is the Hermite polynomial of degree j and $k = \sum_{n=1}^{\infty} k_n$.[4] Then, we have the following Ito-Wiener decomposition:

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

See Theorem 1.1.1 in [Nua06]. We have the following classical Wiener chaos estimate.

Lemma 2.2. *Let $k \in \mathbb{N}_0$. Then, we have*

$$\Big(\mathbb{E}\big[|X|^p\big]\Big)^{\frac{1}{p}} \leqslant (p-1)^{\frac{k}{2}} \Big(\mathbb{E}\big[|X|^2\big]\Big)^{\frac{1}{2}} \tag{2.10}$$

for any random variable $X \in \mathcal{H}_k$ and any $2 \leqslant p < \infty$.

The estimate (2.10) is a direct corollary to the hypercontractivity of the Ornstein-Uhlenbeck semigroup due to Nelson [Nel66] and the fact that any element $X \in \mathcal{H}_k$ is an eigenfunction for the Ornstein-Uhlenbeck operator with eigenvalue $-k$.

For our purpose, we need the following three facts: (i) If Z is a linear combination of $\{g_n\}$, then $Z \in \mathcal{H}_1$. (ii) For $Z \in \mathcal{H}_1$, the random variable $Z^2 - \mathbb{E}[Z^2] \in \mathcal{H}_2$. (iii) If $Y, Z \in \mathcal{H}_1$ are independent, then $YZ \in \mathcal{H}_2$.

The next lemma gives a regularity criterion for stationary random distributions. Recall that a random distribution u on $\mathbb{T}^d$ is said to be stationary if $u(\,\cdot\,)$ and $u(x_0 + \cdot\,)$ have the same law for any $x_0 \in \mathbb{T}^d$. Moreover, we say that $u \in \mathcal{H}_k$ if $u(\varphi) \in \mathcal{H}^k$ for any test function $\varphi \in C^\infty(\mathbb{T}^d)$.

Lemma 2.3. (i) *Let u be a stationary random distribution on $\mathbb{T}^d$, belonging to $\mathcal{H}_k$ for some $k \in \mathbb{N}_0$. Suppose that there exists $s_0 \in \mathbb{R}$ such that*

$$\mathbb{E}\big[|\widehat{u}(n)|^2\big] \lesssim \langle n \rangle^{-d-2s_0} \tag{2.11}$$

for any $n \in \mathbb{Z}^d$. Then, for any $s < s_0$ and finite $p \geqslant 2$, we have $u \in L^p(\Omega; \mathcal{C}^s(\mathbb{T}^d))$.
(ii) *Let $\{u_N\}_{N \in \mathbb{N}}$ be a sequence of stationary random distributions on $\mathbb{T}^d$, belonging to $\mathcal{H}_k$ for some $k \in \mathbb{N}_0$. Suppose that there exists $s_0 \in \mathbb{R}$ such that u_N satisfies* (2.11) *for each $N \in \mathbb{N}$. Moreover, suppose that there exists $\theta > 0$ such that*

$$\mathbb{E}\big[|\widehat{u}_N(n) - \widehat{u}_M(n)|^2\big] \lesssim N^{-2\theta} \langle n \rangle^{-d-2s_0}$$

for any $n \in \mathbb{Z}^d$ and any $M \geqslant N \geqslant 1$. Then, for any $s < s_0$ and finite $p \geqslant 2$, u_N converges to some u in $L^p(\Omega; \mathcal{C}^s(\mathbb{T}^d))$.

The proof is a straightforward computation with the Wiener chaos estimate (Lemma 2.2). See [MWX17, Proposition 3.6] for details of the proof of Part (i). Part (ii) follows from similar considerations.

[4]This implies that $k_n = 0$ except for finitely many n's.

2.4 Truncated NLW dynamics: well-posedness and approximation

In the following, we often work at the level of the truncated dynamics in order to rigorously justify calculations. As such, in this subsection, we briefly go over the well-posedness theory and approximation results of the following Cauchy problem for the truncated NLW on $\mathbb{T}^3$:

$$\begin{cases} \partial_t u = v \\ \partial_t v = \Delta u - \pi_N\big((\pi_N u)^3\big) \\ (u, v)|_{t=0} = (u_0, v_0), \end{cases} \tag{2.12}$$

where $N \geqslant 1$ and π_N denotes the projector onto spatial frequencies $\{|n| \leqslant N\}$. We also use the following shorthand notations:

$$u_N = \pi_N u \qquad \text{and} \qquad v_N = \pi_N v.$$

We allow $N = \infty$ with the convention $\pi_\infty = \mathrm{Id}$, which reduces (2.12) to (1.2).

For the (untruncated) NLW (1.2), the conserved energy is given by

$$E(\vec{u}) = \frac{1}{2}\int_{\mathbb{T}^3} \big(|\nabla u|^2 + v^2\big) + \frac{1}{4}\int_{\mathbb{T}^3} u^4.$$

The truncated system (2.12) also has the following conserved energy:

$$E_N(\vec{u}) = \frac{1}{2}\int_{\mathbb{T}^3} \big(|\nabla u|^2 + v^2\big) + \frac{1}{4}\int_{\mathbb{T}^3} (\pi_N u)^4. \tag{2.13}$$

In the following two lemmas, we state the classical well-posedness theory for (2.12) and the relevant dynamical properties.

Lemma 2.4. *Let $\sigma \geqslant 1$ and $N \in \mathbb{N} \cup \{\infty\}$. Then, the truncated NLW* (2.12) *is globally well-posed in $\vec{H}^\sigma(\mathbb{T}^3)$. Namely, given any $(u_0, v_0) \in \vec{H}^\sigma(\mathbb{T}^3)$, there exists a unique global solution to* (2.12) *in $C(\mathbb{R}; \vec{H}^\sigma(\mathbb{T}^3))$, where the dependence on initial data is continuous. Moreover, if we denote by $\Phi_N(t)$ the data-to-solution map at time t, then $\Phi_N(t)$ is a continuous bijection on $\vec{H}^\sigma(\mathbb{T}^3)$ for every $t \in \mathbb{R}$, satisfying the semigroup property:*

$$\Phi_N(t + \tau) = \Phi_N(t) \circ \Phi_N(\tau)$$

for any $t, \tau \in \mathbb{R}$.

The global well-posedness result stated in Lemma 2.4 follows from a standard local well-posedness theory along with the conservation of the truncated energy $E_N(\vec{u})$. See [OT20, Lemma 2.1] for the proof in the two-dimensional case.[5] The same proof applies to the three-dimensional case in view of the Sobolev embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$ (with a small modification at the zeroth frequency).

[5]This is in the context of the nonlinear Klein-Gordon equation but the proof can be easily adapted.

Lemma 2.5. (i) (Growth bound) *Given $\sigma \geqslant 1$, we denote by B_R the ball of radius $R > 0$ in $\vec{H}^\sigma(\mathbb{T}^3)$ centered at the origin. Then, for any given $T > 0$, there exists $C(R, T) > 0$ such that*

$$\Phi_N(t)(B_R) \subset B_{C(R,T)} \tag{2.14}$$

for any $t \in [0, T]$ and $N \in \mathbb{N} \cup \{\infty\}$.
(ii) (Approximation) *Let $\sigma \geqslant 1$, $T > 0$, and K be a compact set in $\vec{H}^\sigma(\mathbb{T}^3)$. Then, for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that*

$$\|\Phi(t)(\vec{u}) - \Phi_N(t)(\vec{u})\|_{\vec{H}^\sigma(\mathbb{T}^3)} < \varepsilon$$

for any $t \in [0, T]$, $\vec{u} \in K$, and $N \geqslant N_0$. Hence, we have

$$\Phi(t)(K) \subset \Phi_N(t)(K + B_\varepsilon).$$

for any $t \in [0, T]$ and $N \geqslant N_0$. Here, $\Phi(t)$ denotes the solution map $\Phi_\infty(t) = \Phi_{\mathrm{NLW}}(t)$ for the (untruncated) NLW (1.2).

Proof. The solution $\vec{u} = (u, v)$ to (2.12) satisfies the following Duhamel formulation:

$$\begin{aligned} u(t) &= S(t)(u_0, v_0) - \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} \pi_N\big((\pi_N u)^3\big)(t')dt', \\ v(t) &= \partial_t S(t)(u_0, v_0) - \int_0^t \cos((t-t')|\nabla|)\pi_N\big((\pi_N u)^3\big)(t')dt', \end{aligned} \tag{2.15}$$

where $S(t)$ denotes the linear wave propagator given by

$$S(t)(u_0, v_0) = \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|} v_0.$$

From the fractional Leibniz rule (2.7) and (2.5), we have

$$\|u^3\|_{H^{\sigma-1}} \lesssim \|u\|_{B^{\sigma-1}_{6,2}} \|u\|_{L^6}^2 \lesssim \|u\|_{H^\sigma}\|u\|_{H^1}^2 \tag{2.16}$$

for $\sigma \geqslant 1$. Then, from (2.15) and (2.16) with the conservation of the truncated energy E_N in (2.13), we have[6]

$$\begin{aligned} \|\vec{u}(t)\|_{\vec{H}^\sigma} &\leqslant \|(u_0, v_0)\|_{\vec{H}^\sigma} + C(1+|t|) \int_0^t \|u(t')\|_{H^\sigma} \|u(t')\|_{H^1}^2 dt' \\ &\leqslant \|(u_0, v_0)\|_{\vec{H}^\sigma} + C(1+|t|) \cdot E_N(u_0, v_0) \int_0^t \|(u, v)(t')\|_{\vec{H}^\sigma} dt'. \end{aligned}$$

Hence, the growth bound (2.14) follows from Gronwall's inequality.

The approximation property (ii) follows from a modification of the local well-posedness argument. Since the argument is standard, we omit details. See, for example, our previous works: Proposition 2.7 in [Tzv15] and Lemma 6.20/B.2 in [OT17]. □

[6]The factor $1 + |t|$ appears in controlling the zeroth frequency: $\frac{\sin((t-t')|\nabla|)}{|\nabla|} = t - t'$.

3 Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. We first present a general framework of the strategy. We then introduce a renormalized energy and discuss further refinements required for our problem. In Subsection 3.4, we prove Theorem 1.1 by assuming the construction of the weighted Gaussian measure (Proposition 3.7) and the renormalized energy estimate (Proposition 3.8). We present the proofs of Propositions 3.7 and 3.8 in Sections 4 and 5.

3.1 General framework

In [Tzv15], the third author introduced a general strategy, combining PDE techniques and stochastic analysis to prove quasi-invariance of Gaussian measures under nonlinear Hamiltonian PDE dynamics. In the following, we briefly describe a rough idea behind this method [Tzv15, OT20], using NLW on $\mathbb{T}^d$ as an example. See also [OT15] for a survey on this subject. Note that we keep our discussion at a formal level and that some steps need to be justified by working at the level of the truncated dynamics (2.12).

Let $\Phi = \Phi_{\text{NLW}}$ as in the previous section. In order to prove quasi-invariance of $\vec{\mu}_s$ under Φ, we would like to show $\vec{\mu}_s(\Phi(t)(A)) = 0$ for any $t \in \mathbb{R}$ and any measurable set $A \subset \vec{H}^\sigma(\mathbb{T}^d)$ with $\vec{\mu}_s(A) = 0$. Here, $\sigma < s + 1 - \frac{d}{2}$ denotes the regularity of samples on $\mathbb{T}^d$ under $\vec{\mu}_s$. The main idea is to study the evolution of

$$\vec{\mu}_s(\Phi(t)(A)) = Z_s^{-1} \int_{\Phi(t)(A)} e^{-\frac{1}{2}\|\vec{u}\|^2_{\vec{H}^{s+1}}} d\vec{u}$$

for a general measurable set $A \subset \vec{H}^\sigma(\mathbb{T}^d)$ and to control the growth of $\vec{\mu}_s(\Phi(t)(A))$ in time. Here, the main goal is show a differential inequality of the form:

$$\frac{d}{dt}\vec{\mu}_s(\Phi(t)(A)) \leqslant Cp^\beta \big\{\vec{\mu}_s(\Phi(t)(A))\big\}^{1-\frac{1}{p}} \tag{3.1}$$

for some $0 \leqslant \beta \leqslant 1$ and for $p > 1$ sufficiently large. Once (3.1) could be established, Yudovich's argument [Yud63] or its refinement [OT20] when $\beta = 1$ would then yield quasi-invariance for short times. Iterating the argument and using time-reversibility of the equation yields quasi-invariance for all $t \in \mathbb{R}$. In this argument, the linear power of p in the prefactor of the right-hand side of (3.1) is crucial.

By applying a change-of-variable formula, we have

$$\vec{\mu}_s(\Phi(t)(A)) \text{``} = \text{''} Z_s^{-1} \int_A e^{-\frac{1}{2}\|\Phi(t)(\vec{u})\|^2_{\vec{H}^{s+1}}} d\vec{u}. \tag{3.2}$$

For the truncated dynamics (2.12), the formula (3.2) can be justified via invariance of the Lebesgue measure and bijectivity of the flow Φ_N. See Lemma 3.9 below. Fix

$t_0 \in \mathbb{R}$. Then, by taking a time derivative, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\vec{\mu}_s(\Phi(t)(A))\bigg|_{t=t_0} \\
&\quad = -\frac{1}{2}Z_s^{-1}\int_{\Phi(t_0)(A)} \frac{d}{dt}\Big(\|\Phi(t)(\vec{u})\|^2_{\vec{H}^{s+1}}\Big) e^{-\frac{1}{2}\|\Phi(t)(\vec{u})\|^2_{\vec{H}^{s+1}}} d\vec{u}\bigg|_{t=0} \\
&\quad = -\frac{1}{2}\int_{\Phi(t_0)(A)} \frac{d}{dt}\Big(\|\Phi(t)(\vec{u})\|^2_{\vec{H}^{s+1}}\Big)\bigg|_{t=0} d\vec{\mu}_s.
\end{aligned} \tag{3.3}$$

This reduction of the analysis to that at $t = 0$, exploiting the group property $\Phi(t_0 + t) = \Phi(t)\Phi(t_0)$ was inspired from the work [TV14]. Suppose that we had an effective energy estimate (with smoothing) of the form:

$$\frac{d}{dt}\|\Phi(t)(\vec{u})\|^2_{\vec{H}^{s+1}}\bigg|_{t=0} \text{“}\leqslant\text{”}\, C(\|\vec{u}\|_{\vec{H}^1})\|\vec{u}\|^\theta_{\vec{\mathscr{C}}^\sigma} \tag{3.4}$$

for some $\theta \leqslant 2$. Then, the desired estimate (3.1) would follow from (3.2), (3.3), and (3.4) along with the Wiener chaos estimate (Lemma 2.2). Note that, in the energy estimate (3.4), we can afford to place two factors of $\vec{u}$ in the stronger Hölder-Besov $\vec{\mathscr{C}}^\sigma$-norm, while we need to place all the other factors in the (weaker) $\vec{H}^1$-norm, which is controlled by the conserved energy $E(\vec{u})$ in (2.13).

In [Tzv15], the third author established an energy estimate of the form (3.4) for the BBM equation by consideration in the spirit of quasilinear hyperbolic PDEs (namely, integration by parts in x). Unfortunately, an energy estimate of the form (3.4) does not hold in general for nonlinear Hamiltonian PDEs. In [OT17, OT20], the second and third authors circumvented this problem by introducing a modified energy:

$$E_s(\vec{u}) = \frac{1}{2}\|\vec{u}\|^2_{\vec{H}^{s+1}} + R_s(\vec{u})$$

with a suitable correction term $R_s(\vec{u})$ such that the desired energy estimate of the form (3.4) holds for this modified energy. By following the strategy described above, they first established quasi-invariance of the weighted Gaussian measure associated with this modified energy:

$$d\vec{\rho}_s = Z_s^{-1}e^{-E_s(\vec{u})}d\vec{u} = Z_s^{-1}e^{-R_s(\vec{u})}d\vec{\mu}_s$$

(with a cutoff on a conserved quantity). Then, quasi-invariance of $\vec{\mu}_s$ followed from the mutual absolute continuity of $\vec{\mu}_s$ and $\vec{\rho}_s$.

For Schrödinger-type equations, modified energies were introduced by the normal form method (namely, integration by parts in time); see [OT17, OST18, FT19].

In [OT20], the second and third authors derived a modified energy for NLW on $\mathbb{T}^2$ based on integration by parts in x but a certain renormalization was needed to control singularity. We will describe the details of this derivation in the next subsection.

Summary: The study of quasi-invariance has therefore been reduced to two steps: (i) the construction of the weighted Gaussian measure $\vec{\rho}_s$ and (ii) establishing an effective energy estimate on $\partial_t E_s(\vec{u})\big|_{t=0}$.

3.2 Renormalized energy for NLW

In this subsection, we present a discussion on a modified energy for our problem. See (3.18) below for the full modified energy. In the following, we fix $\sigma = s+1-\frac{d}{2}-\varepsilon \geqslant 1$ for some small $\varepsilon > 0$ and let B_R denotes the ball of radius $R > 0$ in $\vec{H}^\sigma(\mathbb{T}^d)$ centered at the origin. Fix a frequency cutoff size N and, instead of using (a suitable truncated version of) the energy of $\vec{\mu}_s$, let us consider the following natural energy to work with for the wave equation (see Remark 3.6):

$$\frac{1}{2}\int_{\mathbb{T}^d}(D^s v_N)^2 + \frac{1}{2}\int_{\mathbb{T}^d}(D^{s+1}u_N)^2,$$

where $D^s = (-\Delta)^{\frac{s}{2}}$ denotes the Riesz potential of order s. Fix an even integer $s \geqslant 4$ and let $\vec{u} = (u, v)$ be a solution to the truncated NLW (2.12). Then, the Leibniz rule yields

$$\begin{aligned}
\partial_t\bigg[\frac{1}{2}\int_{\mathbb{T}^d}(D^s v_N)^2 + \frac{1}{2}\int_{\mathbb{T}^d}(D^{s+1}u_N)^2\bigg] &= \int_{\mathbb{T}^d}(D^{2s}v_N)(-u_N^3)\\
&= -3\int_{\mathbb{T}^d} D^s v_N D^s u_N\, u_N^2\\
&\quad + \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s\\ |\alpha|,|\beta|,|\gamma|<s}} c_{\alpha,\beta,\gamma}\int_{\mathbb{T}^d} D^s v_N \cdot Q_{s,N}(u_N)^\alpha u_N\\
&\qquad \times Q_{s,N}(u_N)^\beta u_N \cdot Q_{s,N}(u_N)^\gamma u_N
\end{aligned} \tag{3.5}$$

for some combinatorial constants $c_{\alpha,\beta,\gamma}$ that depend only on s, where $Q_{s,N}(u_N)^\alpha$ denotes $Q_{s,N}(u_N)_{x_1}^{\alpha_1}\cdots Q_{s,N}(u_N)_{x_d}^{\alpha_d}$ for a multi-index $\alpha = (\alpha_1,\dots,\alpha_d)$. Samples $\vec{u}$ under the Gaussian measure $\vec{\mu}_s$ belong almost surely to $\vec{\mathscr{C}}^\sigma(\mathbb{T}^d)\backslash\vec{\mathscr{C}}^{s+1-\frac{d}{2}}(\mathbb{T}^d)$ for $\sigma < s+1-\frac{d}{2}$. The main issue is how to treat $D^s v_N$ on the right-hand side of (3.5) due to its low regularity $\sigma - 1$. It turns out that all but the first term on the right-hand side of (3.5) can be treated by integration by parts. See Remark 3.3. As for the first term, recalling from (2.12) that $v_N = \partial_t u_N$, we have

$$-3\int_{\mathbb{T}^d} D^s v_N D^s u_N\, u_N^2$$

$$= -\frac{3}{2}\partial_t\bigg[\int_{\mathbb{T}^d}(D^s u_N)^2 u_N^2\bigg] + 3\int_{\mathbb{T}^d}(D^s u_N)^2\, v_N u_N. \tag{3.6}$$

The terms on the right-hand side of (3.6) are better behaved than that on the left-hand side since D^s no longer falls on the less regular term v. This motivates us to define a modified energy with a correction term of the form:

$$R_s(\vec{u}) = \frac{3}{2}\int_{\mathbb{T}^d}(D^s u_N)^2 u_N^2.$$

When $d = 1$, this choice of the correction term allows us to define a suitable modified energy and to construct the weighted Gaussian measure associated with this modified energy (modulo an issue at the zeroth frequency). When $d = 2$ or 3, however, we have $u \notin \mathscr{C}^s(\mathbb{T}^d)$ almost surely and thus the limiting expression $(D^s u)^2$ is ill defined since it is the square of a distribution of negative regularity. Moreover, the singular term $(D^s u)^2$ appears in both terms on the right-hand side of (3.6). As such, we have issues at the level of both the energy and its time derivative, which propagate to both the construction of the weighted Gaussian measure and the energy estimate.

Motivated by Euclidean quantum field theory, we introduce a renormalization. This amounts to replacing $(D^s u)^2$ by $(D^s u)^2 - \infty$, suitably interpreted; given $N \in \mathbb{N}$, we replace $(D^s u_N)^2$ in (3.6) by $Q_{s,N}(u_N)$, where

$$Q_{s,N}(f) \stackrel{\text{def}}{=} (D^s f)^2 - \sigma_N \tag{3.7}$$

and σ_N is given by

$$\sigma_N \stackrel{\text{def}}{=} \mathbb{E}_{\vec{\mu}_s}\Big[(D^s \pi_N u)^2\Big] \sim \sum_{\substack{n\in\mathbb{Z}^d\\ 1\leqslant|n|\leqslant N}} \frac{1}{|n|^2} \sim \begin{cases} \log N & \text{for } d = 2,\\ N & \text{for } d = 3,\end{cases} \tag{3.8}$$

as $N \to \infty$. The crucial observation in [OT20] is that the effect of the renormalization for the two terms on the right-hand side in (3.6) precisely cancels each other, since

$$-\frac{3}{2}\sigma_N\partial_t\bigg[\int_{\mathbb{T}^d} u_N^2\bigg] + 3\sigma_N\int_{\mathbb{T}^d} v_N u_N = 0,$$

where we used the equation (2.12). As a result, we obtain

$$\begin{aligned}
&-3\int_{\mathbb{T}^d} D^s v_N D^s u_N\, u_N^2\\
&\qquad = -\frac{3}{2}\partial_t\bigg[\int_{\mathbb{T}^d} Q_{s,N}(u_N)u_N^2\bigg] + 3\int_{\mathbb{T}^d} Q_{s,N}(u_N)v_N u_N.
\end{aligned} \tag{3.9}$$

In view of (3.5) and (3.9), we define the renormalized energy $\mathscr{E}_{s,N}(\vec{u})$ by

$$\mathscr{E}_{s,N}(\vec{u}) = \frac{1}{2}\int_{\mathbb{T}^d}(D^{s+1}u)^2 + \frac{1}{2}\int_{\mathbb{T}^d}(D^s v)^2 + \frac{3}{2}\int_{\mathbb{T}^d} Q_{s,N}(u_N)u_N^2. \tag{3.10}$$

Then, we have

$$\partial_t\mathscr{E}_{s,N}(\vec{u}) = 3\int_{\mathbb{T}^d} Q_{s,N}(u_N)v_N u_N + \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s\\ |\alpha|,|\beta|,|\gamma|<s}} c_{\alpha,\beta,\gamma}\int_{\mathbb{T}^d} D^s v_N \cdot Q_{s,N}(u_N)^\alpha u_N \times Q_{s,N}(u_N)^\beta u_N \cdot Q_{s,N}(u_N)^\gamma u_N. \tag{3.11}$$

Note that we have renormalized both the energy and its time derivative at the same time. The considerations above motivate the definition of the renormalized weighted Gaussian measure:

$$d\vec{\rho}_{s,r,N} = Z_{s,N,r}^{-1}\mathbf{1}_{\{E_N(\vec{u})\leqslant r\}}e^{-\mathscr{E}_{s,N}(\vec{u})}d\vec{u}, \tag{3.12}$$

where $E_N(\vec{u})$ is as in (2.13). The energy cutoff in (3.12) is necessary to construct this measure due to an issue with the zeroth frequency (see Remark 3.6).

Remark 3.1. If $\vec{u}$ is distributed according to the Gaussian measure $\vec{\mu}_s$, then we can apply Wick renormalization to $(D^s u_N)^2$ and obtain the Wick power $:(D^s u_N)^2:$. Here, Wick renormalization corresponds the orthogonal projection onto a (second) homogeneous Wiener chaos under $L^2(\vec{\mu}_s)$. In this case, we have

$$:(D^s u_N)^2: = Q_{s,N}(u_N).$$

This renormalization allows us to take a limit $:(D^s u)^2: = \lim_{N\to\infty} :(D^s u_N)^2:$ in a suitable space (see Lemmas 4.1 and 4.6 below). In the discussion above for deriving the renormalized energy $\mathscr{E}_{s,N}$, however, $\vec{u}$ denotes a solution to (2.12) and a notation such as $:(D^s u_N)^2:$ is not well defined. This is the reason we needed to introduce $Q_{s,N}$ in (3.7).

Remark 3.2. This simultaneous renormalization of the energy and its time derivative does not introduce any modification to the original truncated equation (2.12) since its Hamiltonian $E_N(\vec{u})$ remains unchanged. We also point out two (related) interesting observations: (i) renormalization is usually applied in the handling of rough functions, whereas we use renormalization in the context of high regularity solutions, and (ii) the simultaneous renormalization is introduced only as a tool to prove Theorem 1.1.

Remark 3.3. In view of the regularity of $\vec{u}$ under $\vec{\mu}_s$, it may seem that some of the lower order terms under the sum on the right-hand side of (3.11) are divergent as $N \to \infty$: for example, when $|\alpha| = s - 1$, $|\beta| = 1$, and $\gamma = 0$. However, by integration by parts (in x) and the independence of u and v, they turn out to be convergent without any renormalization. See the proof of Proposition 3.8.

• **Problem (i): Construction of the weighted Gaussian measure.** The problem of constructing the limiting weighted Gaussian measure measure $\vec{\widetilde{\rho}}_{s,r} = \lim_{N\to\infty} \vec{\widetilde{\rho}}_{s,r,N}$ bears some similarity with the problem of constructing the Φ^4-measures. First of all, the need for renormalization in (3.10) means that the positivity of the random variable $\int (D^s u)^2 u^2$ is destroyed. Moreover, there is a similarity between the measures themselves; despite not having the simple algebraic structure of the Φ^4-measure, the term $\int (D^s u)^2 u^2$ is quartic in u. In [OT20], the second and third authors exploited these similarities and modified Nelson's construction of the Φ^4_2-measure to construct the desired weighted Gaussian measure $\vec{\widetilde{\rho}}_{s,r}$ in the two-dimensional case. The construction in [OT20] heavily uses the logarithmic divergence rate (3.8) of the renormalization constants and uses the energy cutoff $\mathbf{1}_{\{E_N(u,v)\leqslant r\}}$, while they did not make use of the positive quartic potential energy term $\frac{1}{4}\int u^4$.

The analogy between $\vec{\widetilde{\rho}}_{s,r}$ and the Φ^4-measures starts to break down in the three-dimensional case. On the one hand, Nelson's construction fails for both. For the measure $\vec{\widetilde{\rho}}_{s,r}$, this is due to the algebraic divergence rate (3.8) of the renormalization constants σ_N; see Remark 3.6 in [OT20]. For the Φ^4_3-measure, the issue is more subtle and further renormalization beyond Wick renormalization is required. As a consequence, the resulting Φ^4_3-measure is expected to be singular with respect to its underlying Gaussian measure. We point out that one expects a priori that the renormalizations necessary for $\vec{\widetilde{\rho}}_{s,r}$ are different from the Φ^4_3-measure since the singular term in $\int (D^s u)^2 u^2$ is quadratic, not quartic, in u.

In order to construct $\vec{\widetilde{\rho}}_{s,r}$, we use the techniques introduced in a recent paper [BG19] by Barashkov and Gubinelli, where the partition functions of the Φ^4_2- and Φ^4_3-measures were analyzed by way of variational formulas. In particular, we show that the measures $\vec{\widetilde{\rho}}_{s,r}$ are still absolutely continuous with respect to the underlying Gaussian measure.[7] One technical issue with the construction of $\vec{\widetilde{\rho}}_{s,r}$ is that it is not clear whether the term $\int (D^s u)^2 u^2$ is good enough to control the large-scale behavior (= low frequency part) of u. In the following, we circumvent this problem by introducing a new renormalized energy $E_{s,N}(\vec{u})$ in (3.18) by adding the energy $E_N(\vec{u})$ in (2.13) (plus an extra term controlling the zeroth Fourier coefficient of u) to the renormalized energy $\mathscr{E}_{s,N}(\vec{u})$ in (3.10). This allows us to use the potential

[7]In order to avoid an issue at the zeroth frequency, we need to make a modification to the renormalized energy $\mathscr{E}_{s,N}(\vec{u})$. This leads to a slightly different weighted Gaussian measure. See (3.18), (3.20), and (3.21) below.

energy term $\frac{1}{4}\int u_N^4$ in (2.13) to get rid of the need of the energy cutoff $\mathbf{1}_{\{E_N(\vec{u})\leqslant r\}}$. The effect is to change the underlying Gaussian measure $\vec{\mu}_s$ to a different Gaussian measure $\vec{\nu}_s$, which will be shown to be equivalent to $\vec{\mu}_s$ by Kakutani's theorem. See Lemma 3.5 below. The measures that we construct are simple yet interesting examples of measures that require only Wick renormalization but for which Nelson's construction fails.

• **Problem (ii): Energy estimate.** In the two-dimensional case [OT20], it was not possible to establish an energy estimate of the form (3.4). Instead, it was shown that

$$\left|\partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))|_{t=0}\right| \lesssim C(\|\vec{u}\|_{\vec{H}^1})F(\vec{u}). \tag{3.13}$$

for a suitable renormalized energy. Here, $F(\vec{u})$ denotes complicated expressions that contain high regularity information on $\vec{u}$ such as the $\vec{W}^{\sigma,\infty}$-norm as well as the renormalized second power $\int_{\mathbb{T}^2} Q_{s,N}(u_N)$. As mentioned above, all but two factors need to be placed in the weaker H^1-norm so that $F(\vec{u})$ is at most quadratic in $\vec{u}$, which implies that $F(\vec{u}) \in \mathcal{H}_2$. This allows us to obtain the right growth bound of the form (3.1) after applying the Wiener chaos estimate (Lemma 2.2). Here, it is crucial to study the energy estimate (3.13) at time $t = 0$ to exploit the Gaussian initial data in in (1.3). In [OT20], the energy estimate (3.13) involved a delicate quadrilinear Littlewood-Paley expansion balancing the interplay between the energy conservation and the higher order regularity. As pointed out in [OT20], the estimate of the form (3.13) fails for the three-dimensional case.

In a recent paper [PTV19], Planchon, Visciglia, and the third author proved quasi-invariance of the Gaussian measures under the dynamics of the (super-)quintic nonlinear Schrödinger equations (NLS) on $\mathbb{T}$ by establishing a novel energy estimate. The idea is to exploit a deterministic growth bound (2.14) on solutions. Then, the required energy estimate takes the following form:[8]

$$\left|\partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\right| \leqslant C\big(1 + \|\Phi_N(t)(\vec{u})\|_{\vec{H}^\sigma}^k\big). \tag{3.14}$$

Here, $k > 0$ can be *any* positive number. The main point is that if we start dynamics with a measurable set $A \subset B_R$, then (3.14) with the growth bound (2.14) yields

$$\left|\mathbf{1}_A(\vec{u}) \cdot \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\right| \leqslant C\left|\mathbf{1}_{B_{C(R,T)}}(\vec{u}) \cdot \big(1 + \|\vec{u}\|_{\vec{H}^\sigma}^k\big)\right| \leqslant C(R)^k$$

for any $t \in [0,T]$ and $N \in \mathbb{N} \cup \{\infty\}$. This control allows us to prove quasi-invariance for each measurable set $A \subset B_R$ (in the sense of (3.24) below). Then, by a soft argument, we can conclude quasi-invariance of the Gaussian measure $\vec{\mu}_s$. The main

[8]In the case of NLS, we have u instead of $\vec{u} = (u, v)$. For the sake of presentation, we keep the notation adapted to the NLW context.

advantage of this argument is that we are allowed to place any power k in the stronger $\vec{H}^\sigma$-norm. Note that the energy estimate (3.14) is entirely deterministic and hence there is no need to reduce the analysis to time $t = 0$.

In this paper, we combine these two approaches described above and establish an energy estimate of the form:

$$\Big|\mathbf{1}_{B_R}(\vec{u}) \cdot \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))|_{t=0}\Big| \leqslant C(\|\vec{u}\|_{\vec{H}^\sigma}) F(\vec{u}),$$

where we use the deterministic growth bound (2.14) to control $C(\|\vec{u}\|_{\vec{H}^\sigma})$, while we use the Wiener chaos estimate (Lemma 2.2) to control $F(\vec{u})$. The fact that we have access to the stronger $\vec{H}^\sigma$-norm (rather than $\vec{H}^1$-norm as in (3.13)) allows us to get by with a softer energy estimate. Moreover, in our case, $F(\vec{u})$ is given in an explicit manner (see Proposition 5.1). It contains products of derivatives of u_N and v_N as well as the $\mathscr{C}^{-1-\varepsilon}$-norm of the Wick power $Q_{s,N}(u_N) = (D^s u_N)^2 - \sigma_N$. By proceeding as in [MWX17], we establish regularity properties of these random distributions in Proposition 4.3. These two points lead to a significantly simpler proof of quasi-invariance than the two-dimensional case [OT20].

Remark 3.4. Following the discussion of Remark (iv) in Subsection 1.2, one might attempt to implement an analogous construction of weighted Gaussian measure in the case of NLW with a higher order nonlinearity or in higher dimensions. Higher order nonlinearities would result in a higher power of the regular part of the renormalized energy, while the singular part would remain quadratic, i.e. $(D^s u)^2$. Thus, the construction of these measures seems tractable. This is in sharp contrast with the construction of the Φ^{2n}_3 measures, where higher order nonlinearities result in higher powers of distributions which makes the construction of such measures impossible (for $n \geqslant 3$). Higher dimensions would result in a more singular quadratic part.

3.3 Statements of key results

In the remaining part of this paper, we fix $d = 3$. In this subsection, we introduce a new renormalized energy and then state the key propositions in proving Theorem 1.1.

We first introduce a new Gaussian measure, whose energy is more suitable for analysis on NLW (but still controls the zeroth frequency). Define a Gaussian measure $\vec{\nu}_s$ via the following Karhunen-Loève expansions:

$$\begin{aligned} u^\omega(x) &= g_0(\omega) + \sum_{n\in\mathbb{Z}^3\setminus\{0\}} \frac{g_n(\omega)}{(|n|^2 + |n|^{2s+2})^{\frac{1}{2}}} e^{in\cdot x}, \\ v^\omega(x) &= \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{(1 + |n|^{2s})^{\frac{1}{2}}} e^{in\cdot x}, \end{aligned} \tag{3.15}$$

where $\{g_n\}_{n\in\mathbb{Z}^3}$ and $\{h_n\}_{n\in\mathbb{Z}^3}$ are as in (1.3). Then, the formal density of $\vec{\nu}_s$ is given by

$$d\vec{\nu}_s = Z_s^{-1} e^{-H_s(\vec{u})} d\vec{u},$$

where

$$\begin{aligned} H_s(\vec{u}) = \frac{1}{2}\bigg(\int_{\mathbb{T}^3} u\bigg)^2 + \frac{1}{2}\int_{\mathbb{T}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{T}^3} (D^{s+1}u)^2 \\ + \frac{1}{2}\int_{\mathbb{T}^3} v^2 + \frac{1}{2}\int_{\mathbb{T}^3} (D^s v)^2. \end{aligned} \tag{3.16}$$

Lemma 3.5. *Let $s > \frac{3}{4}$. Then, the Gaussian measures $\vec{\mu}_s$ and $\vec{\nu}_s$ are equivalent.*

The proof of this lemma is based on a simple application of Kakutani's theorem [Kak48]; see the proof of Lemma 6.1 in [OT20] for details in the two-dimensional case.

Remark 3.6. The linear wave equation conserves the homogeneous Sobolev norm:

$$\|\vec{u}\|^2_{\dot{\vec{H}}^{s+1}} = \int_{\mathbb{T}^3} (D^{s+1}u)^2 + \int_{\mathbb{T}^3} (D^s v)^2.$$

Hence, we would like to work with Gaussian measures with formal density $e^{-\frac{1}{2}\|u\|^2_{\dot{\vec{H}}^{s+1}}}$. These measures do not exist as probability measures since the zeroth frequency is not controlled. This is the reason we chose to include $g_0(\omega)$ in (3.15), giving rise to the first term in $H_s(\vec{u})$ defined in (3.16).

As we see below, we add the truncated energy $E_N(\vec{u})$ in (2.13) to construct the full renormalized energy, which explains the appearance of the terms with $|\nabla u|^2$ and v^2 in (3.16). This addition of the truncated energy $E_N(\vec{u})$ allows us to include the quartic potential energy $\frac{1}{4}\int u_N^4$ without changing the time derivative of the renormalized energy; see (3.19). We point out that this quartic homogeneity plays an important role in the construction of the weighted Gaussian measure.

Given $N \in \mathbb{N}$, we *redefine* the parameter σ_N, adapted to the new Gaussian measure $\vec{\nu}_s$, by

$$\sigma_N \stackrel{\text{def}}{=} \mathbb{E}_{\vec{\nu}_s}\Big[(D^s u_N)^2\Big] = \sum_{\substack{n\in\mathbb{Z}^3 \\ 1\leqslant |n|\leqslant N}} \frac{|n|^{2s}}{|n|^2 + |n|^{2s+2}} \sim N \longrightarrow \infty \tag{3.17}$$

as $N \to \infty$. We also redefine the operator $Q_{s,N}$ in (3.7) with this new definition of σ_N. In the remaining part of this paper, we will use these new definitions for σ_N and $Q_{s,N}$.

We now define the full renormalized energy $E_{s,N}(\vec{u})$ by

$$E_{s,N}(\vec{u}) = \mathscr{E}_{s,N}(\vec{u}) + E_N(\vec{u}) + \frac{1}{2}\bigg(\int_{\mathbb{T}^3} u_N\bigg)^2, \tag{3.18}$$

where $\mathscr{E}_{s,N}$ is as in (3.10) and E_N is the truncated energy in (2.13). Then, it follows from (3.11) and the conservation of the truncated energy that

$$\begin{aligned}\partial_t E_{s,N}(\vec{u}) = 3\int_{\mathbb{T}^3} & Q_{s,N}(u_N)v_N u_N \\ &+ \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s \\ |\alpha|,|\beta|,|\gamma|<s}} c_{\alpha,\beta,\gamma}\int_{\mathbb{T}^3} D^s v_N \cdot Q_{s,N}(u_N)^\alpha u_N \\ &\qquad \times Q_{s,N}(u_N)^\beta u_N \cdot Q_{s,N}(u_N)^\gamma u_N \\ &+ \bigg(\int_{\mathbb{T}^3} u_N\bigg)\bigg(\int_{\mathbb{T}^3} v_N\bigg)\end{aligned} \tag{3.19}$$

for any solution $\vec{u}$ to the truncated NLW (2.12). Moreover, from (3.16), we have

$$E_{s,N}(\vec{u}) = H_s(\vec{u}) + R_{s,N}(u),$$

where

$$\begin{aligned}R_{s,N}(u) &= \frac{3}{2}\int_{\mathbb{T}^3} Q_{s,N}(u_N)u_N^2 + \frac{1}{4}\int_{\mathbb{T}^3} u_N^4 \\ &= \frac{3}{2}\int_{\mathbb{T}^3}\Big((D^s u_N)^2 - \sigma_N\Big)u_N^2 + \frac{1}{4}\int_{\mathbb{T}^3} u_N^4.\end{aligned} \tag{3.20}$$

We are now ready to state the two key ingredients for proving Theorem 1.1: (i) the construction of the weighted Gaussian measures and (ii) the renormalized energy estimate.

Define the weighted Gaussian measure $\vec{\rho}_{s,N}$ by

$$d\vec{\rho}_{s,N}(\vec{u}) = \mathscr{Z}_{s,N}^{-1} e^{-R_{s,N}(u)} d\vec{\nu}_s(\vec{u}), \tag{3.21}$$

where $\mathscr{Z}_{s,N}$ is the normalization constant. The following proposition establishes uniform integrability of the density $e^{-R_{s,N}(u)}$ in (3.21), which allows us to construct the limiting weighted Gaussian measure $\vec{\rho}_s$ by

$$d\vec{\rho}_s(\vec{u}) = \mathscr{Z}_s^{-1} e^{-R_s(u)} d\vec{\nu}_s(\vec{u}),$$

where $R_s(u)$ is a limit of $R_{s,N}(u)$; see Lemma 4.1.

Proposition 3.7 (Construction of the weighted Gaussian measure)**.** *Let $s > \frac{3}{2}$. Then, the weighted Gaussian measures $\vec{\rho}_{s,N}$ converges strongly to $\vec{\rho}_s$. Namely, we have*

$$\lim_{N\to\infty} \vec{\rho}_{s,N}(A) = \vec{\rho}_s(A)$$

for any measurable set $A \subset \vec{H}^\sigma(\mathbb{T}^3)$, $\sigma < s - \frac{1}{2}$. Moreover, given any finite $p \geqslant 1$, the sequence $\{e^{-R_{s,N}(u)}\}_{N\in\mathbb{N}}$ and $e^{-R_s(u)}$ are uniformly bounded in $L^p(\vec{\nu}_s)$. As a consequence, $\vec{\rho}_s$ is equivalent to $\vec{\nu}_s$.

Next, we state the key renormalized energy estimate. Recall that B_R denotes the ball of radius $R > 0$ in $\vec{H}^\sigma(\mathbb{T}^3)$ centered at the origin. We denote by $\Phi_N(t)$ the flow of the truncated NLW dynamics (2.12).

Proposition 3.8 (Renormalized energy estimate)**.** *Let $s \geqslant 4$ be an even integer. Then, given $R > 0$, there is a constant $C = C(R) > 0$ such that*

$$\left\{ \int \mathbf{1}_{B_R}(\vec{u}) \cdot \left|\partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))|_{t=0}\right|^p d\vec{\nu}_s(\vec{u}) \right\}^{\frac{1}{p}} \leqslant Cp$$

for any finite $p \geqslant 1$ and any $N \in \mathbb{N}$.

Before we state the main proposition on the evolution of the truncated measures $\vec{\rho}_{s,N}$, let us state the following change-of-variable formula. Given $N \in \mathbb{N}$, let $\mathscr{E}_N = \pi_N L^2(\mathbb{T}^3)$ and we endow $\mathscr{E}_N \times \mathscr{E}_N$ with the Lebesgue measure L_N as in Section 2. Then, by viewing the Gaussian measure $\vec{\nu}_s$ as a product measure on $(\mathscr{E}_N \times \mathscr{E}_N) \times (\mathscr{E}_N \times \mathscr{E}_N)^\perp$, we can write the truncated weighted Gaussian measure $\vec{\rho}_{s,N}$ defined in (3.21) as

$$\begin{aligned} d\vec{\rho}_{s,N}(\vec{u}) &= \mathscr{Z}_{s,N}^{-1}\, e^{-R_{s,N}(\pi_N u)} d\vec{\nu}_s(\vec{u}), \\ &= \hat{Z}_{s,N}^{-1}\, e^{-E_{s,N}(\pi_N \vec{u})}\, dL_N \otimes d\vec{\nu}_{s;N}^{\perp}(\vec{u}), \end{aligned} \tag{3.22}$$

where $\hat{Z}_{s,N}$ denotes the normalization constant and $\vec{\nu}_{s;N}^{\perp}$ denotes the marginal Gaussian measure of $\vec{\nu}_s$ on $(\mathscr{E}_N \times \mathscr{E}_N)^\perp$. Then, we have the following change-of-variable formula.

Lemma 3.9. *Let $s > \frac{3}{2}$ and $N \in \mathbb{N}$. Then, we have*

$$\vec{\rho}_{s,N}(\Phi_N(t)(A)) = \hat{Z}_{s,N}^{-1} \int_A e^{-E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))}\, dL_N \otimes d\vec{\nu}_{s;N}^{\perp}(\vec{u})$$

for any $t \in \mathbb{R}$ and any measurable set $A \subset \vec{H}^\sigma(\mathbb{T}^3)$ with $\sigma < s - \frac{1}{2}$.

The proof of Lemma 3.9 is based on (i) the invariance of the Lebesgue measure L_N under (the low frequency part of) the truncated NLW dynamics $\pi_N \Phi_N(t)$, (ii) the conservation of the truncated energy $E_N(\vec{u})$ under $\Phi_N(t)$ and (iii) the bijectivity of the solution map $\Phi_N(t)$. As it follows from similar considerations presented in [Tzv15, OT17], we omit details of the proof.

We now state and prove the main proposition, essentially establishing the differential inequality (3.1). This proposition allows us to control the growth of the pushforward measure $\vec{\rho}_{s,N}(\Phi_N(t)(A))$ of a given measurable set $A \subset \vec{H}^\sigma(\mathbb{T}^3)$ uniformly in $N \in \mathbb{N}$, provided that the set A lies in the ball $B_R \subset \vec{H}^\sigma(\mathbb{T}^3)$ of radius $R > 0$. Namely, it only provides a *set-dependent* control. This dependence on $R > 0$, however, does not cause any trouble in establishing quasi-invariance of the Gaussian measure $\vec{\nu}_s$ (and hence of $\vec{\mu}_s$).

Proposition 3.10. *Let $s \geqslant 4$ be an even integer and $\sigma \in \big(1, s - \frac{1}{2}\big)$. Then, given $R > 0$ and $T > 0$, there exists $C_{R,T} > 0$ such that*

$$\frac{d}{dt}\vec{\rho}_{s,N}(\Phi_N(t)(A)) \leqslant C_{R,T} \cdot p \left\{\vec{\rho}_{s,N}(\Phi_N(t)(A))\right\}^{1-\frac{1}{p}}$$

for any $p \geqslant 2$, any $N \in \mathbb{N}$, any $t \in [0,T]$, and any measurable set $A \subset B_R \subset \vec{H}^\sigma(\mathbb{T}^3)$.

In [OT20], there is an analogous statement, controlling the evolution of the truncated measures (without the restriction on B_R); see [OT20, Lemma 5.2]. The main idea of the proof of Lemma 5.2 in [OT20] is to reduce the analysis to that at $t = 0$, which provides access to the random distributions in (3.15). On the other hand, the main idea in [PTV19] at this step is to use the *deterministic* control (2.14) on the growth of solutions. In the following, we combine both of these ideas, thus introducing a hybrid argument which works more effectively than each of the two methods.

Proof. Fix $R, T > 0$ and $t_0 \in [0,T]$. Let $A \subset B_R$ be a measurable set in $\vec{H}^\sigma(\mathbb{T}^3)$. Using the flow property of $\Phi_N(t)$, we have

$$\begin{aligned}\frac{d}{dt}\vec{\rho}_{s,N}(\Phi_N(t)(A))\bigg|_{t=t_0} &= \mathscr{Z}_{s,N}^{-1}\frac{d}{dt}\int_{\Phi_N(t)(A)} e^{-R_{s,N}(\pi_N u)} d\vec{\nu}_s(\vec{u})\bigg|_{t=t_0} \\ &= \mathscr{Z}_{s,N}^{-1}\frac{d}{dt}\int_{\Phi_N(t)(\Phi_N(t_0)(A))} e^{-R_{s,N}(\pi_N u)} d\vec{\nu}_s(\vec{u})\bigg|_{t=0}.\end{aligned}$$

The change-of-variable argument (Lemma 3.9), (3.22), and the growth bound (2.14) in Lemma 2.5 yield

$$\frac{d}{dt}\vec{\rho}_{s,N}(\Phi_N(t)(A))\bigg|_{t=t_0}$$

$$
\begin{aligned}
&= \hat{Z}_{s,N}^{-1} \frac{d}{dt} \int_{\Phi_N(t_0)(A)} e^{-E_{s,N}(\pi_N \Phi_N(t)(u,v))} dL_N \otimes d\vec{\nu}_{s;N}^{\perp} \bigg|_{t=0} \\
&= -\mathscr{Z}_{s,N}^{-1} \int_{\Phi_N(t_0)(A)} \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\big|_{t=0} e^{-R_{s,N}(\pi_N u)} d\vec{\nu}_s(\vec{u}) \\
&\leqslant \mathscr{Z}_{s,N}^{-1} \int_{B_{C(R,T)}} \Big| \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\big|_{t=0} \Big| e^{-R_{s,N}(\pi_N u)} d\vec{\nu}_s(\vec{u}).
\end{aligned}
$$

Then, from Hölder's inequality, we obtain

$$
\begin{aligned}
\frac{d}{dt} \vec{\rho}_{s,N}(\Phi_N(t)(A)) \bigg|_{t=t_0} &\leqslant \Big\| \mathbf{1}_{B_{C(R,T)}}(\vec{u}) \cdot \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\big|_{t=0} \Big\|_{L^p(\vec{\rho}_{s,N})} \\
&\qquad \times \big\{ \vec{\rho}_{s,N}(\Phi_N(t_0)(A)) \big\}^{1-\frac{1}{p}}.
\end{aligned}
$$

Finally, by Cauchy-Schwarz inequality together with the uniform exponential moment bound on $R_{s,N}(u)$ in Proposition 3.7 and Proposition 3.8, we obtain

$$
\begin{aligned}
&\Big\| \mathbf{1}_{B_{C(R,T)}}(u,v) \cdot \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\big|_{t=0} \Big\|_{L^p(\vec{\rho}_{s,N})} \\
&\leqslant \mathscr{Z}_{s,N}^{-\frac{1}{p}} \Big\| \mathbf{1}_{B_{C(R,T)}}(\vec{u}) \cdot \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\big|_{t=0} \Big\|_{L^{2p}(\vec{\nu}_{s,N})} \\
&\qquad \times \Big\| e^{-R_{s,N}(u)} \Big\|_{L^2(\vec{\nu}_s)}^{\frac{1}{p}} \\
&\leqslant C_{R,T} \cdot p.
\end{aligned} \tag{3.23}
$$

Here, we used the boundedness of $\mathscr{Z}_{s,N}^{-1}$, uniformly in $N \in \mathbb{N}$ (recall that $\mathscr{Z}_{s,N} \to \mathscr{Z}_s > 0$ as $N \to \infty$). This completes the proof of Proposition 3.10. □

3.4 Proof of Theorem 1.1

We conclude this section by presenting the proof of Theorem 1.1. Our aim is to show that for each *fixed* $R > 0$, we have

$$
\vec{\nu}_s(A) = 0 \qquad \text{implies} \qquad \vec{\nu}_s\big(\Phi(t)(A)\big) = 0 \tag{3.24}
$$

for any measurable set $A \subset B_R \subset \vec{H}^\sigma(\mathbb{T}^3)$, $\sigma \in (1, s - \frac{1}{2})$ and any $t > 0$.[9] Since the choice of $R > 0$ is arbitrary, this yields quasi-invariance of $\vec{\nu}_s$ under the NLW dynamics. Then, we invoke Lemma 3.5 to conclude quasi-invariance of $\vec{\mu}_s$ (Theorem 1.1).

Arguing as in [OT20], Proposition 3.10 allows us to establish quasi-invariance of the truncated weighted Gaussian measures $\vec{\rho}_{s,N}$ with the uniform control in

[9]In view of the time reversibility of the equation (1.2), it suffices to consider positive times.

$N \in \mathbb{N}$ (but with dependence on $R > 0$). See Proposition 5.3 in [OT20]. By the approximation property of the truncated NLW dynamics (Lemma 2.5 (ii)) and the strong convergence of $\vec{\rho}_{s,N}$ to $\vec{\rho}_s$ (Proposition 3.7), we can upgrade this to the $N = \infty$ case, thus establishing quasi-invariance of the untruncated weighted Gaussian measure $\vec{\rho}_s$ under the NLW dynamics. See Lemma 5.5 in [OT20] for the proof.

Lemma 3.11. *Given any $R > 0$, there exists $t_* = t_*(R) \in [0, 1]$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ with the following property; if a measurable set $A \subset B_R \subset \vec{H}^\sigma(\mathbb{T}^3)$, $\sigma \in \left(1, s - \frac{1}{2}\right)$ satisfies*

$$\vec{\rho}_s(A) < \delta,$$

then we have

$$\vec{\rho}_s(\Phi(t)(A)) < \varepsilon$$

for any $t \in [0, t_*]$.

Finally, we establish (3.24) by exploiting the mutual absolute continuity between $\vec{\rho}_s$ and $\vec{\nu}_s$ for each fixed $R > 0$. Let $A \subset B_R$ be such that $\vec{\nu}_s(A) = 0$. By the mutual absolute continuity of $\vec{\nu}_s$ and $\vec{\rho}_s$, we have

$$\vec{\rho}_s(A) = 0.$$

Now, fix a target time $T > 0$ and let $C(R, T)$ be as in Lemma 2.5 (i). Namely, we have

$$\Phi(t)(A) \subset B_{C(R,T)} \tag{3.25}$$

for all $t \in [0, T]$. Then, by applying Lemma 3.11 with R replaced by $C(R, T)$, we obtain

$$\vec{\rho}_s(\Phi(t)(A)) = 0 \tag{3.26}$$

for $t \in [0, t_*]$, where $t_* = t_*(C(R, T))$. In view of (3.25), we can iterate this argument and conclude that (3.26) holds for any $t \in [0, T]$. Since the choice of $T > 0$ was arbitrary, we obtain (3.26) for any $t > 0$. Finally, by invoking the mutual absolute continuity of $\vec{\nu}_s$ and $\vec{\rho}_s$ once again, we have

$$\vec{\nu}_s(\Phi(t)(A)) = 0$$

for any $t > 0$. This proves (3.24) and hence Theorem 1.1.

Remark 3.12. While this new hybrid argument allows us to establish quasi-invariance of the Gaussian measure $\vec{\nu}_s$ (and hence $\vec{\mu}_s$) under the NLW dynamics even in the three-dimensional case, it does not provide as good of a quantitative bound as the two-dimensional argument. For example, in the two-dimensional case, the argument in [OT20] yielded

$$\vec{\rho}_s(\Phi(t)(A)) \lesssim \big(\vec{\rho}_s(A)\big)^{\frac{1}{c^{1+|t|}}} \tag{3.27}$$

for a weighted Gaussian measure $\vec{\rho}_{s,r}$ with an energy cutoff $\mathbf{1}_{\{E(u,v)\leqslant r\}}$, where $c = c(r) > 0$; see Remark 5.6 in [OT20]. Our present understanding does not provide an analogous bound to (3.27) in three dimensions.

4 Construction of the weighted Gaussian measure

In this section, we prove Proposition 3.7 by establishing uniform integrability of the densities $R_{s,N}(u)$ of the weighted Gaussian measures $\vec{\rho}_{s,N}$ in (3.21). In Subsection 4.1, we first prove some regularity properties of random distributions (Proposition 4.3) and then the L^p-convergence of $R_{s,N}(u)$ in (3.20). We split the proof of the main result (Proposition 4.2) into two parts. In Subsection 4.2, we follow the argument by Barashkov and Gubinelli [BG19] and express the partition function $\mathscr{Z}_{s,N}$ in terms of a minimization problem involving a stochastic control problem (Proposition 4.4). In Subsection 4.3, we then study the minimization problem and establish boundedness of the partition function $\mathscr{Z}_{s,N}$, uniformly in $N \in \mathbb{N}$.

Let $N \geqslant 1$. Recall that $\vec{\rho}_{s,N}$ has density $e^{-R_{s,N}(u)}$ with respect to $\vec{\nu}_s$. In particular, note that the non-Gaussian part of $\vec{\rho}_{s,N}$ depends only on u. This motivates the following reduction; define $H_s^{(1)}(u)$ and $H_s^{(2)}(v)$ by

$$\begin{aligned} H_s^{(1)}(u) &= \frac{1}{2}\bigg(\int_{\mathbb{T}^3} u\bigg)^2 + \frac{1}{2}\int_{\mathbb{T}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{T}^3} (D^{s+1}u)^2, \\ H_s^{(2)}(v) &= \frac{1}{2}\int_{\mathbb{T}^3} v^2 + \frac{1}{2}\int_{\mathbb{T}^3} (D^s v)^2. \end{aligned}$$

Then, define Gaussian measures $\nu_s^{(j)}$, $j = 1, 2$, with formal densities:

$$d\nu_s^{(1)} = Z_{1,s}^{-1} e^{-H_s^{(1)}(u)} du \qquad \text{and} \qquad d\nu_s^{(2)} = Z_{2,s}^{-1} e^{-H_s^{(2)}(v)} dv.$$

Since $H_s(\vec{u}) = H_s(u, v)$ in (3.16) is now written as

$$H_s(\vec{u}) = H_s^{(1)}(u) + H_s^{(2)}(v),$$

the Gaussian measure $\vec{\nu}_s$ can be rewritten as

$$d\vec{\nu}_s(\vec{u}) = d\nu_s^{(1)}(u) \otimes d\nu_s^{(2)}(v). \tag{4.1}$$

From decomposition (4.1), we have

$$d\vec{\rho}_{s,N}(\vec{u}) = d\rho_{s,N}(u) \otimes d\nu_s^{(2)}(v),$$

where $\rho_{s,N}$ is given by

$$d\rho_{s,N}(u) = \mathscr{Z}_{s,N}^{-1} e^{-R_{s,N}(u)} d\nu_s^{(1)}(u).$$

The partition function $\mathscr{Z}_{s,N}$ is now expressed as

$$\mathscr{Z}_{s,N} = \int e^{-R_{s,N}(u)} d\nu_s^{(1)}(u). \tag{4.2}$$

In the following, we denote $\nu_s^{(1)}$ by ν_s and prove various statements in terms of ν_s but they can be trivially upgraded to the corresponding statement for $\vec{\nu}_s$.

Lemma 4.1. *Let $s > \frac{3}{2}$. Then, given any finite $p < \infty$, $R_{s,N}$ defined in (3.20) converges to some R_s in $L^p(\nu_s)$ as $N \to \infty$.*

The goal of this section is to prove the following proposition on uniform (in $N \in \mathbb{N}$) integrability of the density $e^{-R_{s,N}(u)}$ for $\vec{\rho}_{s,N}$, which allows us to construct the limiting measure $\vec{\rho}_s$. As a consequence of our construction, the weighted Gaussian measure $\vec{\rho}_s$ is equivalent to $\vec{\nu}_s$ (and hence to $\vec{\mu}_s$ in view of Lemma 3.5).

Proposition 4.2. *Let $s > \frac{3}{2}$. Then, given any finite $p < \infty$, there exists $C_p > 0$ such that*

$$\sup_{N\in\mathbb{N}} \left\| e^{-R_{s,N}(u)} \right\|_{L^p(\nu_s)} \leqslant C_p < \infty. \tag{4.3}$$

Moreover, we have

$$\lim_{N\to\infty} e^{-R_{s,N}(u)} = e^{-R_s(u)} \qquad \text{in } L^p(\nu_s). \tag{4.4}$$

While the first part of Proposition 3.7 follows from Proposition 4.2 with $p = 1$, we need to have the uniform bound (4.3) for some $p > 1$ for the proof of Proposition 3.10. See (3.23). Note that this requirement on a higher integrability for some $p > 1$ is analogous to the situation in Bourgain's construction on invariant Gibbs measures for Hamiltonian PDEs [Bou94], where, as in (3.23), the analysis of the weighted Gaussian measure needs to be reduced to that of the underlying Gaussian measure by Cauchy-Schwarz inequality. Since the argument is identical for any $p \geqslant 1$, we only present details for the case $p = 1$. We point out that the L^p-convergence (4.4) is a consequence of the uniform exponential moment bound (4.3) and the softer convergence in measure (as a consequence of Lemma 4.1). See Remark 3.8 in [Tzv08]. Therefore, we focus on proving the uniform bound (4.3).

In the next subsection, we prove Lemma 4.1. The subsequent subsections are devoted to the proof of Proposition 4.2.

4.1 Regularity of random distributions

Let u be distributed according to ν_s and $Q_{s,N}$ be as in (3.7) with σ_N in (3.17). In this case, we have

$$:(D^s u_N)^2: = Q_{s,N}(u_N),$$

where the left-hand side is the standard notation for the Wick renormalization.

We first state and prove the regularity properties of (products of) certain random distributions. The proof of Lemma 4.1 is presented at the end of this subsection.

Proposition 4.3. *Let $s \geqslant 1$ and $\varepsilon > 0$. Then, there exists $C = C(s,\varepsilon) > 0$ such that for any $N \in \mathbb{N}$ and any $2 \leqslant p < \infty$, we have*

$$\| :(D^s u_N)^2: \|_{L^p(\nu_s, \mathscr{C}^{-1-\varepsilon})} \leqslant Cp, \tag{4.5}$$

$$\|Q_{s,N}(u_N)^\kappa v_N\, \partial^\alpha u_N\|_{L^p(\vec{\nu}_s, \mathscr{C}^{-1-\varepsilon})} \leqslant Cp \qquad |\kappa| = s-1,\ |\alpha| = s, \tag{4.6}$$

$$\|Q_{s,N}(u_N)^\kappa v_N\, \partial^\alpha u_N\|_{L^p(\vec{\nu}_s, \mathscr{C}^{-\frac{1}{2}-\varepsilon})} \leqslant Cp \qquad |\kappa| = s-1,\ |\alpha| \leqslant s-1, \tag{4.7}$$

where $u_N = \pi_N u$ and $v_N = \pi_N v$. Moreover, as $N \to \infty$, the sequences above converge to limits denoted by $:(D^s u)^2:$ and $Q_{s,N}(u_N)^\kappa v\, Q_{s,N}(u_N)^\alpha u$ with respect to the same topologies.

We will also use this proposition in proving the renormalized energy estimate in Section 5.

Proof. We only prove (4.5) in the following. The other estimates (4.6) and (4.7) follow in a similar manner, with the simplification that no renormalization is needed due to the independence of u and v under $\vec{\nu}_s$. The regularity $-1-\varepsilon$ in (4.6) is naturally expected in view of the regularities $< -\frac{1}{2}$ for each of $Q_{s,N}(u_N)^\kappa v_N$ and $\partial^\alpha u_N$. A similar comment applies to (4.7), where the regularity of $Q_{s,N}(u_N)^\kappa v$ is less than $-\frac{1}{2}$.

Noting that

$$\frac{|n|^s}{(|n|^2 + |n|^{2s+2})^{\frac{1}{2}}} \lesssim \frac{1}{\langle n \rangle}$$

for any $n \in \mathbb{Z}^3 \backslash \{0\}$, it follows from the Karhunen-Loève expansion (3.15) that

$$\begin{aligned}\mathbb{E}_{\nu_s}\Big[\big|\mathscr{F}\{ :(D^s u_N)^2: \}(n)\big|^2\Big] &\lesssim \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ |n_j| \leqslant N}} \frac{\big|\mathbb{E}[g_{n_1} g_{n-n_1} g_{-n_2} g_{-n+n_2}]\big|}{\langle n_1 \rangle \langle n-n_1 \rangle \langle n_2 \rangle \langle n-n_2 \rangle} \mathbf{1}_{\{n \neq 0\}} \\ &\quad + \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ |n_j| \leqslant N}} \frac{\big|\mathbb{E}\big[(|g_{n_1}|^2 - 1)(|g_{n_2}|^2 - 1)\big]\big|}{\langle n_1 \rangle^2 \langle n_2 \rangle^2} \mathbf{1}_{\{n=0\}}\end{aligned} \tag{4.8}$$

for any $n \in \mathbb{Z}^3$, where $\mathscr{F}$ denotes Fourier transform. In the first sum on the right-hand side of (4.8), we note that due to the independence (modulo the conjugates) of the g_n's and by Wick's theorem, all non-vanishing terms must satisfy $n_1 = n_2$ or $n_1 = n - n_2$. Thus, we obtain

$$\sum_{\substack{n_1,n_2\in\mathbb{Z}^3\\ |n_j|\leqslant N}} \frac{\big|\mathbb{E}[g_{n_1}g_{n-n_1}g_{-n_2}g_{-n+n_2}]\big|}{\langle n_1\rangle\langle n-n_1\rangle\langle n_2\rangle\langle n-n_2\rangle}\mathbf{1}_{\{n\neq 0\}} \lesssim \sum_{n_1\in\mathbb{Z}^3}\frac{1}{\langle n_1\rangle^2\langle n-n_1\rangle^2} \lesssim \frac{1}{\langle n\rangle} \tag{4.9}$$

uniformly in $N \in \mathbb{N}$, where in the last inequality we used a standard result on discrete convolutions (see Lemma 4.2 in [MWX17]). In the second sum on the right-hand side of (4.8), we note that, by Wick's theorem, the contribution from $|n_1| \neq |n_2|$ vanishes. Thus, we obtain

$$\sum_{\substack{n_1,n_2\in\mathbb{Z}^3\\ |n_j|\leqslant N}} \frac{\big|\mathbb{E}\big[(|g_{n_1}|^2-1)(|g_{n_2}|^2-1)\big]\big|}{\langle n_1\rangle^2\langle n_2\rangle^2}\mathbf{1}_{\{n=0\}} \lesssim 1, \tag{4.10}$$

uniformly in $N \in \mathbb{N}$. Putting (4.9) and (4.10) together, we obtain

$$\mathbb{E}\Big[\big|\mathscr{F}\big\{ :(D^s u_N)^2: \big\}(n)\big|^2\Big] \lesssim \frac{1}{\langle n\rangle}$$

for any $n \in \mathbb{Z}^3$ and $N \in \mathbb{N}$.

By a similar computation, we have

$$\mathbb{E}\Big[\big|\mathscr{F}\big\{ :(D^s u_N)^2: - :(D^s u_M)^2: \big\}(n)\big|^2\Big] \lesssim \frac{1}{N^\theta\langle n\rangle^{1-\theta}}$$

for any $n \in \mathbb{Z}^3$, any $M \geqslant N \geqslant 1$, and $\theta \in [0,1]$. Note that $:(D^s u_N)^2:$ lies in the second homogeneous Wiener chaos $\mathscr{H}_2$. Hence, by Lemma 2.3 with $\theta > 0$ sufficiently small, we conclude that $:(D^s u_N)^2:$ converges to some $:(D^s u)^2:$ in $L^p(\nu_s; \mathscr{C}^{-1-\varepsilon}(\mathbb{T}^3))$ for any finite $p \geqslant 2$. □

We now present the proof of Lemma 4.1.

Proof of Lemma 4.1. For $s > \frac{3}{2}$, Lemma 2.3 implies u_N converges to u in $L^p(\nu_s; \mathscr{C}^\sigma)$ for any finite $p \geqslant 2$ and any $\sigma < s - \frac{1}{2}$. In the following, we choose $\sigma > 0$ sufficiently close to $s - \frac{1}{2}$. Then, by the algebra property (2.4), we see that u_N^2 (and u_N^4, respectively) converges to u^2 (and u^4, respectively) in $L^p(\nu_s; \mathscr{C}^\sigma)$ for any finite $p \geqslant 2$.

Proposition 4.3 asserts that $:(D^s u_N)^2:$ converges to $:(D^s u)^2: \in L^p(\nu_s, \mathscr{C}^{-1-\varepsilon}(\mathbb{T}^3))$ for any $\varepsilon > 0$. Recall from (2.8) that the bilinear multiplication map from $\mathscr{C}^{s_1} \times \mathscr{C}^{s_2}$ to $\mathscr{C}^{s_1}$ is a continuous operation for $s_1 < 0 < s_2$ such that $s_1 + s_2 > 0$. Therefore, by choosing $\sigma > 1 + \varepsilon$ (which is possible since $s > \frac{3}{2}$), we conclude that

$$:(D^s u)^2: u^2 = \lim_{N\to\infty} :(D^s u_N)^2: u_N^2$$

exists as an element in $L^p(\nu_s; \mathscr{C}^{-1-\varepsilon}(\mathbb{T}^3))$ for all finite $p \geqslant 2$. This means that

$$\frac{3}{2} :(D^s u)^2: u^2 + \frac{1}{4} u^4 \in L^p(\nu_s, \mathscr{C}^{-1-\varepsilon}(\mathbb{T}^3)). \tag{4.11}$$

Lemma 4.1 then follows from (4.11). □

4.2 Variational formulation

In this subsection, we follow the argument in [BG19] and derive a variational formula for the normalization constant $\mathscr{Z}_{s,N}$ in (4.2). Given small $\varepsilon > 0$, let $\Omega_\varepsilon = C(\mathbb{R}_+, \mathscr{C}^{-\frac{3}{2}-\varepsilon}(\mathbb{T}^3))$ equipped with its Borel σ-algebra. Denote by[10] $\{X_t\}$ the coordinate process on Ω_ε and consider the probability measure $\mathbb{P}$ that makes $\{X_t\}$ a cylindrical Brownian motion in $L^2(\mathbb{T}^3)$. Namely, we have

$$X_t = \sum_{n\in\mathbb{Z}^3} B_t^n e^{in\cdot x},$$

where $\{B_t^n\}_{n\in\mathbb{Z}^3}$ is a sequence of independent complex-valued[11] Brownian motions such that $\overline{B_t^n} = B_t^{-n}$, $n \in \mathbb{Z}^3$. Then, define a centered Gaussian process $\{Y_t\}$ by

$$Y_t = \mathcal{J}^{-s-1} X_t \stackrel{\text{def}}{=} B_t^0 + \sum_{n\in\mathbb{Z}^3\setminus\{0\}} \frac{B_t^n}{(|n|^2 + |n|^{2s+2})^{\frac{1}{2}}} e^{in\cdot x}. \tag{4.12}$$

Then, in view of (3.15), we have $\mathrm{Law}_{\mathbb{P}}(Y_1) = \nu_s$. By truncating the sum in (4.12), we also define the truncated process $Y_t^N = \pi_N Y_t$ with the property $\mathrm{Law}_{\mathbb{P}}(Y_1^N) = \mathrm{Law}_{\nu_s}(\pi_N u)$. Note that we have $\mathbb{E}[(D^s Y_1^N)^2] = \sigma_N$, where σ_N is as in (3.17). For simplicity of notations, we suppress dependence on $N \in \mathbb{N}$ when it is clear from the context.

Let $\mathbb{H}_a$ denote the space of progressively measurable processes that belong to $L^2([0,1]; L^2(\mathbb{T}^3))$, $\mathbb{P}$-almost surely. We say that an element θ of $\mathbb{H}_a$ is a *drift*. Given a drift $\theta \in \mathbb{H}_a$, we define the measure $\mathbb{Q}^\theta$ whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is given by the following stochastic exponential:

$$\frac{d\mathbb{Q}^\theta}{d\mathbb{P}} = e^{\int_0^1 \langle \theta_t, dX_t\rangle - \frac{1}{2}\int_0^1 \|\theta_t\|_{L_x^2}^2 dt}. \tag{4.13}$$

[10] In the remaining part of this section, we use the standard notation in stochastic analysis where subscripts denote parameters for stochastic processes.

[11] We normalize B_t^n so that $\mathrm{Var}(B_t^n) = t$. Moreover, we impose that B_t^0 is real-valued.

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\mathbb{T}^3)$. Then, by letting $\mathbb{H}_c$ denote the space of drifts such that $\mathbb{Q}^\theta(\Omega_\varepsilon) = 1$, it follows from Girsanov's theorem ([DPZ14, Theorem 10.14] and [RY13, Theorems 1.4 and 1.7 in Chapter VIII]) that the process X_t is a semimartingale under $\mathbb{Q}^\theta$ with a decomposition:

$$X_t = X_t^\theta + \int_0^t \theta_{t'} dt', \tag{4.14}$$

where X_t^θ is now a cylindrical Brownian motion in $L^2(\mathbb{T}^3)$ under the new measure $\mathbb{Q}^\theta$. From (4.14), we also obtain the decomposition:

$$Y_t = Y_t^\theta + I_t(\theta), \tag{4.15}$$

where $Y_t^\theta = \mathcal{J}^{-s-1} X_t^\theta$ and $I_t(\theta) = \int_0^t \mathcal{J}^{-s-1} \theta_{t'} dt'$. In the following, we use $\mathbb{E}$ to denote an expectation with respect to $\mathbb{P}$, while we use $\mathbb{E}_\mathbb{Q}$ for an expectation with respect to some other probability measure $\mathbb{Q}$.

Before proceeding further, let us recall the following estimate ([Föl85, Lemma 2.6]):

$$\int_0^1 \|\theta_t\|_{L^2_x}^2 dt \leqslant 2H(\mathbb{Q}^\theta | \mathbb{P}), \tag{4.16}$$

where $H(\mathbb{Q}^\theta|\mathbb{P})$ denotes the relative entropy of $\mathbb{Q}^\theta$ with respect to $\mathbb{P}$ defined by

$$H(\mathbb{Q}^\theta|\mathbb{P}) = \mathbb{E}_{\mathbb{Q}^\theta}\bigg[\log \frac{d\mathbb{Q}^\theta}{d\mathbb{P}} \bigg] = \mathbb{E}\bigg[\frac{d\mathbb{Q}^\theta}{d\mathbb{P}} \log \frac{d\mathbb{Q}^\theta}{d\mathbb{P}} \bigg].$$

With the notations introduced above, we have the following variational characterization of the partition function $\mathscr{Z}_{s,N}$ defined in (4.2).

Proposition 4.4. *For any $N \in \mathbb{N}$, we have*

$$-\log \mathscr{Z}_{s,N} = \inf_{\theta \in \mathbb{H}_c} \mathbb{E}_{\mathbb{Q}^\theta}\bigg[R_{s,N}(Y_1^\theta + I_1(\theta)) + \frac{1}{2}\int_0^1 \|\theta_t\|_{L^2_x}^2 dt \bigg]. \tag{4.17}$$

Proof. As a preliminary step, we first derive bounds on $\mathscr{Z}_{s,N}$ and

$$\mathbb{E}\bigg[\frac{e^{-R_{s,N}(Y_1)}}{\mathscr{Z}_{s,N}} \log\bigg(\frac{e^{-R_{s,N}(Y_1)}}{\mathscr{Z}_{s,N}} \bigg) \bigg].$$

Note that these bounds imply that the measure $\frac{e^{-R_{s,N}(Y_1)}}{\mathscr{Z}_{s,N}} d\mathbb{P}$ has a finite relative entropy with respect to $\mathbb{P}$.

From (4.2), Jensen's inequality, and (3.20), there exists finite $C(N) > 0$ such that

$$\mathscr{Z}_{s,N} \geqslant e^{-\mathbb{E}[R_{s,N}(Y_1)]} \geqslant e^{-\mathbb{E}\left[\frac{3}{2}\int (D^s Y_1^N)^2 (Y_1^N)^2 dx + \frac{1}{4}\int (Y_1^N)^4 dx\right]} \geqslant C(N). \tag{4.18}$$

In view of the following pointwise lower bound:

$$\begin{aligned}\frac{3}{2}(D^s Y_1^N)^2(Y_1^N)^2 - \frac{3}{2}\sigma_N (Y_1^N)^2 + \frac{1}{4}(Y_1^N)^4 &\geqslant -\frac{3}{2}\sigma_N (Y_1^N)^2 + \frac{1}{4}(Y_1^N)^4 \\ &\geqslant -\frac{9}{2}\sigma_N^2 + \frac{1}{8}(Y_1^N)^4 \geqslant -C(N) > -\infty,\end{aligned} \tag{4.19}$$

it follows from (4.18), Cauchy's inequality, and Lemma 4.1 that there exists finite $C(N) > 0$ such that

$$\begin{aligned}\mathbb{E}\bigg[\frac{e^{-R_{s,N}(Y_1)}}{\mathscr{Z}_{s,N}} \log\bigg(\frac{e^{-R_{s,N}(Y_1)}}{\mathscr{Z}_{s,N}}\bigg)\bigg] &\leqslant C(N)\mathbb{E}\Big[e^{-R_{s,N}(Y_1)}\big(1 + \log e^{-R_{s,N}(Y_1)}\big)\Big] \\ &\leqslant C(N)\mathbb{E}\Big[e^{-2R_{s,N}(Y_1)} + |R_{s,N}(Y_1)|^2 + 1\Big] \\ &\leqslant C(N) < \infty.\end{aligned} \tag{4.20}$$

Now, fix $\theta \in \mathbb{H}_c$. We show that

$$-\log \mathscr{Z}_{s,N} \leqslant \mathbb{E}_{\mathbb{Q}^\theta}\bigg[R_{s,N}(Y_1^\theta + I_1(\theta)) + \frac{1}{2}\int_0^1 \|\theta_t\|_{L_x^2}^2 dt\bigg]. \tag{4.21}$$

Suppose that $\mathbb{E}_{\mathbb{Q}^\theta}\Big[\int_0^1 \|\theta_t\|_{L_x^2}^2 dt\Big] = \infty$. Then, (4.21) holds trivially since it follows from the decomposition (4.15) of Y_t under $\mathbb{Q}^\theta$ and Cauchy's inequality with Lemma 4.1, (4.18), and (4.19) that

$$\mathbb{E}_{\mathbb{Q}^\theta}\Big[|R_{s,N}(Y_1^\theta + I_1(\theta))|\Big] = \mathbb{E}\bigg[|R_{s,N}(Y_1)|\frac{e^{-R_{s,N}(Y_1)}}{\mathscr{Z}_{s,N}}\bigg] < \infty.$$

Next, suppose that

$$\mathbb{E}_{\mathbb{Q}^\theta}\bigg[\int_0^1 \|\theta_t\|_{L_x^2}^2 dt\bigg] < \infty. \tag{4.22}$$

Note that $\mathscr{Z}_{s,N} = \mathbb{E}[e^{-R_{s,N}(Y_1)}]$. Then, by changing the measure with (4.13), Jensen's inequality, and applying the decompositions (4.14) and (4.15) of X_t and Y_t under $\mathbb{Q}^\theta$, we obtain

$$\begin{aligned}-\log \mathscr{Z}_{s,N} &\leqslant \mathbb{E}_{\mathbb{Q}^\theta}\bigg[R_{s,N}(Y_1) + \int_0^1 \langle\theta_t, dX_t\rangle - \frac{1}{2}\int_0^1 \|\theta_t\|_{L_x^2}^2 dt\bigg] \\ &= \mathbb{E}_{\mathbb{Q}^\theta}\bigg[R_{s,N}(Y_1^\theta + I_1(\theta)) + \int_0^1 \langle\theta_t, dX_t^\theta\rangle + \frac{1}{2}\int_0^1 \|\theta_t\|_{L_x^2}^2 dt\bigg].\end{aligned} \tag{4.23}$$

From (4.22), we see that the process $\int_0^t \langle \theta_{t'}, dX_{t'}^\theta \rangle$ is a $\mathbb{Q}^\theta$-martingale and hence we conclude that

$$\mathbb{E}_{\mathbb{Q}^\theta}\bigg[\int_0^1 \langle \theta_t, dX_t^\theta \rangle \bigg] = 0. \tag{4.24}$$

Therefore, from (4.23) and (4.24), we obtain (4.21).

Next, we show that the infimum in (4.17) is indeed achieved for a special choice of drift. Given $N \in \mathbb{N}$, define $\mathbb{Q}^N$ by the density

$$\frac{d\mathbb{Q}^N}{d\mathbb{P}} = \frac{e^{-R_{s,N}(Y_1)}}{\mathscr{Z}_{s,N}}. \tag{4.25}$$

By the Brownian martingale representation theorem ([RY13, Proposition 1.6 in Chapter VIII]), there exists a drift $\widetilde{\theta}^N \in \mathbb{H}_c$ such that

$$\frac{d\mathbb{Q}^N}{d\mathbb{P}} = e^{\int_0^1 \widetilde{\theta}_t^N dX_t - \frac{1}{2}\int_0^1 \|\widetilde{\theta}_t^N\|_{L_x^2}^2 dt}. \tag{4.26}$$

Then, from (4.25) and(4.26), we obtain

$$-\log \mathscr{Z}_{s,N} = R_{s,N}(Y_1) + \int_0^1 \langle \widetilde{\theta}_t^N, dX_t \rangle - \frac{1}{2}\int_0^1 \|\widetilde{\theta}_t^N\|_{L_x^2}^2 dt. \tag{4.27}$$

Taking expectations of (4.27) with respect to $\mathbb{Q}^N$ and using the decompositions (4.14) and (4.15) of X_t and Y_t under $\mathbb{Q}^N$, we obtain

$$\begin{aligned} -\log \mathscr{Z}_{s,N} = \mathbb{E}_{\mathbb{Q}^N}\Bigg[& R_{s,N}\big(Y_1^{\widetilde{\theta}^N} + I_1(\widetilde{\theta}^N)\big) + \int_0^1 \langle \widetilde{\theta}_t^N, dX_t^{\widetilde{\theta}^N} \rangle \\ & + \frac{1}{2}\int_0^1 \|\widetilde{\theta}_t^N\|_{L_x^2}^2 dt \Bigg]. \end{aligned} \tag{4.28}$$

On the other hand, from (4.25) and (4.20), we have

$$\mathbb{E}_{\mathbb{Q}^N}\bigg[\log \frac{d\mathbb{Q}^N}{d\mathbb{P}} \bigg] = \mathbb{E}\bigg[\frac{e^{-R_{s,N}(Y_1)}}{\mathscr{Z}_{s,N}} \log\bigg(\frac{e^{-R_{s,N}(Y_1)}}{\mathscr{Z}_{s,N}} \bigg) \bigg] < \infty. \tag{4.29}$$

In particular, it follows from (4.29) and (4.16) that

$$\mathbb{E}_{\mathbb{Q}^N}\bigg[\int_0^1 \|\widetilde{\theta}_t^N\|_{L_x^2}^2 dt \bigg] < \infty.$$

This implies that the stochastic integral $\int_0^t \langle \widetilde{\theta}_{t'}^N, dX_{t'}^{\widetilde{\theta}^N} \rangle$ is a $\mathbb{Q}^N$-martingale. Therefore, from (4.28), we obtain

$$-\log \mathscr{Z}_{s,N} = \mathbb{E}_{\mathbb{Q}^N}\bigg[R_{s,N}\big(Y_1^{\widetilde{\theta}^N} + I_1(\widetilde{\theta}^N)\big) + \frac{1}{2}\int_0^1 \|\widetilde{\theta}_t^N\|_{L_x^2}^2 dt \bigg].$$

This completes the proof of Proposition 4.4. □

Remark 4.5. The material presented above differs from [BG19] in the following ways: (i) we do not need to introduce a time-dependent cutoff in the definition of $\{Y_t\}$ and (ii) we do not need to use the stronger Boué-Dupuis formula [BD98]:

$$- \log \mathscr{Z}_{s,N} = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}\left[R_{s,N}(Y_1 + I_1(\theta)) + \frac{1}{2}\int_0^1 \|\theta_t\|_{L^2}^2 dt \right].$$

See [Üst14] or Theorem 2 in [BG19] for further discussion.

4.3 Exponential integrability

In this subsection, we present the proof of Proposition 4.2 by studying the minimization problem (4.17) in Proposition 4.4. In particular, we show that the infimum in (4.17) is bounded away from $-\infty$, uniformly in $N \in \mathbb{N}$. Our strategy is to use pathwise stochastic bounds on Y_1^θ, uniform in the drift θ and use pathwise deterministic bounds on $I_1(\theta)$ independently of the drift (see Lemmas 4.6 and 4.7).

We first state two lemmas on the pathwise regularity estimates on Y_1^θ and $I_1(\theta)$.

Lemma 4.6. *Let $2 \leqslant p < \infty$. Then, we have*

$$\sup_{\theta \in \mathbb{H}_c} \mathbb{E}_{\mathbb{Q}^\theta}\Big[\|D^s Y_1^\theta\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}}^p + \| :(D^s Y_1^\theta)^2: \|_{\mathscr{C}^{-1-\varepsilon}}^p \Big] < \infty \tag{4.30}$$

for any $\varepsilon > 0$. Here, colons denote Wick renormalization.

Proof. Recall that $\{X_t^\theta\}$ under $\mathbb{Q}^\theta$ is a cylindrical Brownian motion in $L^2(\mathbb{T}^3)$ for any $\theta \in \mathbb{H}_c$. Thus, the supremum in (4.30) is superfluous since the law of $Y_1^\theta = \mathcal{J}^{-s-1} X_1^\theta$ under $\mathbb{Q}^\theta$ is invariant under a change of drifts. In particular, we have $\mathrm{Law}_{\mathbb{Q}^\theta}(Y_1^\theta) = \nu_s$. Then, (4.30) follows from the Hölder-Besov regularity of samples under ν_s and (4.5) in Proposition 4.3. □

Lemma 4.7 (Cameron-Martin drift regularity)**.** *The drift term $\theta \in \mathbb{H}_c$ has the regularity of the Cameron-Martin space $H^{s+1}(\mathbb{T}^3)$:*

$$\|I_1(\theta)\|_{H^{s+1}}^2 \leqslant \int_0^1 \|\theta_t\|_{L^2}^2 dt. \tag{4.31}$$

Proof. This is immediate from Minkowski's integral inequality followed by Cauchy-Schwarz inequality:

$$\|I_1(\theta)\|_{H^{s+1}} = \left\| \int_0^1 \theta_t dt \right\|_{L^2} \leqslant \int_0^1 \|\theta_t\|_{L^2} dt \leqslant \left(\int_0^1 \|\theta_t\|_{L^2}^2 dt \right)^{\frac{1}{2}},$$

yielding (4.31). □

We now present the proof of Proposition 4.2, using Proposition 4.4. Fixing an arbitrary drift $\theta \in \mathbb{H}_c$, the quantity that we wish to bound from below is

$$\mathscr{W}_N(\theta) = \mathbb{E}_{\mathbb{Q}^\theta}\bigg[R_{s,N}(Y_1^\theta + I_1(\theta)) + \frac{1}{2}\int_0^1 \|\theta_t\|_{L_x^2}^2 dt \bigg]. \tag{4.32}$$

Since the drift $\theta \in \mathbb{H}_c$ is fixed, we suppress the dependence on the drift θ henceforth and denote $Y = Y_1^\theta$ and $\Theta = I_1(\theta)$. From the definition (3.20) of $R_{s,N}$, we have

$$\begin{aligned} R_{s,N}(Y+\Theta) = \frac{3}{2}\int_{\mathbb{T}^3} &:(D^sY)^2\!: (Y+\Theta)^2 + 2D^sY D^s\Theta(Y+\Theta)^2 + (D^s\Theta)^2(Y+\Theta)^2 \\ &+ \frac{1}{4}\int_{\mathbb{T}^3}(Y+\Theta)^4. \end{aligned} \tag{4.33}$$

The main strategy is to bound $\mathscr{W}_N(\theta)$ from below pathwise and independently of the drift by utilizing the positive terms:

$$\mathscr{U}_N(\theta) = \frac{3}{2}\int (D^s\Theta)^2\Theta^2 + \frac{1}{4}\int \Theta^4 + \frac{1}{2}\int_0^1 \|\theta_t\|_{L_x^2}^2 dt. \tag{4.34}$$

In the following, we state three lemmas, controlling the other terms appearing in (4.33). The proofs of these lemmas follow from lengthy but straightforward computations and are presented at the end of this section. The first lemma handles the terms quadratic in D^sY.

Lemma 4.8 (Terms quadratic in D^sY)**.** *Let $s > \frac{3}{2}$. Then, given $\delta > 0$ sufficiently small, there exist small $\varepsilon > 0$ and $c(\delta) > 0$ such that*

$$\int_{\mathbb{T}^3} :(D^sY)^2\!: Y^2 \lesssim \| :(D^sY)^2\!: \|_{\mathscr{C}^{-1-\varepsilon}}^2 + \|D^sY\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}}^4, \tag{4.35}$$

$$\int_{\mathbb{T}^3} :(D^sY)^2\!: Y\Theta \leqslant c(\delta)\Big(\| :(D^sY)^2\!: \|_{\mathscr{C}^{-1-\varepsilon}}^4 + \|D^sY\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}}^4\Big) + \delta\|\Theta\|_{H^{s+1}}^2, \tag{4.36}$$

$$\int_{\mathbb{T}^3} :(D^sY)^2\!: \Theta^2 \leqslant c(\delta)\| :(D^sY)^2\!: \|_{\mathscr{C}^{-1-\varepsilon}}^4 + \delta\Big(\|\Theta\|_{H^{s+1}}^2 + \|\Theta\|_{L^4}^4\Big). \tag{4.37}$$

The next lemma handles the terms linear in D^sY.

Lemma 4.9 (Terms linear in D^sY)**.** *Let $s > 1$. Then, given $\delta > 0$ sufficiently small, there exist small $\varepsilon > 0$, $c(\delta) > 0$, and $p_j = p_j(\varepsilon, s) > 1$, $j = 1, 2$, such that*

$$\int_{\mathbb{T}^3} D^sY D^s\Theta Y^2 \leqslant c(\delta)\|D^sY\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}}^6 + \delta\|\Theta\|_{H^{s+1}}^2, \tag{4.38}$$

$$\int_{\mathbb{T}^3} D^sY D^s\Theta Y\Theta \leqslant c(\delta)\Big(1 + \|D^sY\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}}\Big)^{p_1} + \delta\Big(\|\Theta\|_{H^{s+1}}^2 + \|\Theta\|_{L^4}^4\Big), \tag{4.39}$$

$$\begin{aligned} \int_{\mathbb{T}^3} D^sY D^s\Theta\Theta^2 &\leqslant c(\delta)\Big(1 + \|D^sY\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}}\Big)^{p_2} \\ &\quad + \delta\Big(\|\Theta\|_{H^{s+1}}^2 + \|\Theta\|_{L^4}^4 + \|D^s\Theta\Theta\|_{L^2}^2\Big). \end{aligned} \tag{4.40}$$

Lastly, the third lemma controls the term quadratic in $D^s\Theta$.

Lemma 4.10 (Term quadratic in $D^s\Theta$)**.** *Let $s > 1$. Then, given $\delta > 0$, there exist small $\varepsilon > 0$, $c(\delta) > 0$, and $p = p(s, \varepsilon) > 1$ such that*

$$\int_{\mathbb{T}^3} (D^s\Theta)^2 Y\Theta \leqslant c(\delta)\|D^sY\|^p_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} + \delta\Big(\|\Theta\|^2_{H^{s+1}} + \|\Theta\|^4_{L^4} + \|D^s\Theta\Theta\|^2_{L^2}\Big) \tag{4.41}$$

The regularity restriction $s > \frac{3}{2}$ appears in controlling the terms quadratic in D^sY. We now prove Proposition 4.2, assuming Lemmas 4.8, 4.9, and 4.10.

First, note that the remaining terms left to treat in (4.33) are harmless. The terms $\int_{\mathbb{T}^3}(D^s\Theta)^2Y^2$, $\int_{\mathbb{T}^3} Y^4$, and $\int_{\mathbb{T}^3} Y^2\Theta^2$ are positive and thus can be discarded. The remaining two terms can be controlled by Young's inequality:

$$\int_{\mathbb{T}^3} Y^3\Theta + \int_{\mathbb{T}^3} Y\Theta^3 \leqslant c(\delta)\|Y\|^4_{L^4} + \delta\|\Theta\|^4_{L^4}$$

for any $\delta > 0$. We now apply the regularity estimates of Lemmas 4.6 and 4.7 to the bounds obtained in Lemmas 4.8, 4.9, and 4.10, and the bounds on the harmless terms. Then, from (4.32), (4.33), and (4.34), we conclude that, by choosing $\delta > 0$ sufficiently small, there exists finite $C = C(\delta) > 0$ such that

$$\sup_{N\in\mathbb{N}} \sup_{\theta\in\mathbb{H}_c} \mathscr{W}_N(\theta) \geqslant \sup_{N\in\mathbb{N}} \sup_{\theta\in\mathbb{H}_c} \Big\{ -C(\delta) + \frac{1}{4}\mathscr{U}_N(\theta)\Big\} \geqslant -C(\delta) > -\infty.$$

Therefore, by Proposition 4.4, this proves Proposition 4.2 (when $p = 1$).

In the remaining part of this section, we present the proofs of Lemmas 4.8, 4.9, and 4.10.

Proof of Lemma 4.8. By duality (2.6) and the algebra property (2.4), we have

$$\text{LHS of (4.35)} \leqslant \|:(D^sY)^2:\|_{B^{-1-2\varepsilon}_{1,1}} \|Y\|^2_{\mathscr{C}^{1+2\varepsilon}}.$$

Then, by choosing $\varepsilon > 0$ sufficiently small, (4.35) follows from the trivial embeddings (2.3) and Cauchy's inequality, provided that $s > \frac{3}{2}$.

By duality (2.6) and the fractional Leibniz rule (2.7), we have

$$\begin{aligned}\text{LHS of (4.36)} &\lesssim \|:(D^sY)^2:\|_{B^{-1-2\varepsilon}_{\infty,2}} \|Y\Theta\|_{B^{1+2\varepsilon}_{1,2}} \\ &\lesssim \|:(D^sY)^2:\|_{\mathscr{C}^{-1-\varepsilon}} \Big(\|Y\|_{B^{1+2\varepsilon}_{2,2}}\|\Theta\|_{L^2} + \|Y\|_{L^2}\|\Theta\|_{B^{1+2\varepsilon}_{2,2}}\Big).\end{aligned}$$

Then, by choosing $\varepsilon > 0$ sufficiently small, (4.36) follows from (2.3) and Young's inequality, provided that $s > \frac{3}{2}$.

Lastly, proceeding as above with (2.6) and (2.7), we have

$$\text{LHS of (4.37)} \lesssim \|:(D^sY)^2:\|_{B^{-1-2\varepsilon}_{\infty,2}} \|\Theta\|_{B^{1+2\varepsilon}_{2,2}} \|\Theta\|_{L^2}.$$

Then, (4.37) follows from (2.3), $L^4(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3)$, and Young's inequality. □

Next, we present the proof of Lemma 4.9. The main idea is to use (i) $\|\Theta\|_{H^{s+1}}$ for controlling derivatives on Θ and (ii) $\|\Theta\|_{L^4}$ and $\|D^s\Theta\Theta\|_{L^2}$ for controlling homogeneity of Θ.

Proof of Lemma 4.9. By duality (2.6) and the fractional Leibniz rule (2.7) with (2.3), we have

$$
\begin{aligned}
\text{LHS of (4.38)} &\lesssim \|D^sY\|_{B^{-\frac{1}{2}-2\varepsilon}_{\infty,2}} \|D^s\Theta Y^2\|_{B^{\frac{1}{2}+2\varepsilon}_{1,2}} \\
&\lesssim \|D^sY\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \Big(\|Y^2\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} \|D^s\Theta\|_{L^2} + \|Y^2\|_{L^2} \|D^s\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} \Big) \\
&\leqslant \|D^sY\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \|Y\|^2_{\mathscr{C}^{\frac{1}{2}+3\varepsilon}} \|\Theta\|_{H^{s+1}}.
\end{aligned}
$$

Then, by choosing $\varepsilon > 0$ sufficiently small, (4.38) follows from Cauchy's inequality, provided that $s > 1$.

By duality (2.6) and the fractional Leibniz rule (2.7) with (2.3) and (2.4), we have

$$
\begin{aligned}
\text{LHS of (4.39)} &\lesssim \|D^sY\|_{B^{-\frac{1}{2}-2\varepsilon}_{\infty,2}} \|D^s\Theta Y\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{1,2}} \\
&\lesssim \|D^sY\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \Big(\|Y\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} \|D^s\Theta\|_{L^2} + \|Y\Theta\|_{L^2} \|D^s\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} \Big) \\
&=: T_1 + T_2.
\end{aligned}
$$

By Hölder's inequality and (2.3), we have

$$
\begin{aligned}
T_2 &\lesssim \|D^sY\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \|Y\|_{L^4} \|\Theta\|_{H^{s+1}} \|\Theta\|_{L^4} \\
&\lesssim \|D^sY\|^2_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \|\Theta\|_{H^{s+1}} \|\Theta\|_{L^4}
\end{aligned}
\tag{4.42}
$$

for $s > \frac{1}{2}$ and small $\varepsilon > 0$.

By (2.7), (2.3), and the interpolation (2.2), we have

$$
\begin{aligned}
\|Y\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} &\lesssim \|Y\|_{B^{\frac{1}{2}+2\varepsilon}_{\infty,2}} \|\Theta\|_{L^2} + \|Y\|_{L^\infty} \|\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} \\
&\lesssim \|Y\|_{\mathscr{C}^{\frac{1}{2}+3\varepsilon}} \|\Theta\|_{H^{\frac{1}{2}+2\varepsilon}} \\
&\lesssim \|Y\|_{\mathscr{C}^{\frac{1}{2}+3\varepsilon}} \|\Theta\|^{\gamma}_{H^{s+1}} \|\Theta\|^{1-\gamma}_{L^2}
\end{aligned}
$$

for some $\gamma = \gamma(s, \varepsilon) \in (0, 1)$. Thus, we have

$$
T_1 \lesssim \|D^sY\|^2_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \|\Theta\|^{1+\gamma}_{H^{s+1}} \|\Theta\|^{1-\gamma}_{L^4} \tag{4.43}
$$

for $s > 1$ and small $\varepsilon > 0$. Hence, noting that $\frac{1}{2} + \frac{1}{4} < 1$ and $\frac{1+\gamma}{2} + \frac{1-\gamma}{4} < 1$ for $\gamma \in (0, 1)$, the desired estimate (4.39) follows from applying Young's inequality to (4.42) and (4.43).

Finally, we consider (4.40). By (2.6) and (2.7) with (2.3), we have

$$\begin{aligned}
\text{LHS of (4.40)} &\lesssim \|D^s Y\|_{B^{-\frac{1}{2}-2\varepsilon}_{\infty,1}} \|D^s\Theta\Theta^2\|_{B^{\frac{1}{2}+2\varepsilon}_{1,\infty}} \\
&\lesssim \|D^s Y\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \Big(\|D^s\Theta\Theta\|_{L^2} \|\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,\infty}} + \|D^s\Theta\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,\infty}} \|\Theta\|_{L^2} \Big) \\
&=: T_3 + T_4.
\end{aligned}$$

By the interpolation (2.2) with $L^4(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3)$, there exists $\gamma_1 = \gamma_1(s,\varepsilon) \in (0,1)$ such that

$$T_3 \lesssim \|D^s Y\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \|D^s\Theta\Theta\|_{L^2} \|\Theta\|^{\gamma_1}_{H^{s+1}} \|\Theta\|^{1-\gamma_1}_{L^4}.$$

Noting that $\frac{1}{2} + \frac{\gamma_1}{2} + \frac{1-\gamma_1}{4} < 1$, we can apply Young's inequality to bound the contribution from T_3 by the right-hand side of (4.40).

It remains to estimate T_4. By the interpolation (2.2) and (2.7), we have

$$\begin{aligned}
\|D^s\Theta\Theta\|_{H^{\frac{1}{2}+2\varepsilon}} \|\Theta\|_{L^2} &\lesssim \|D^s\Theta\Theta\|^{\gamma_2}_{H^1} \|D^s\Theta\Theta\|^{1-\gamma_2}_{L^2} \|\Theta\|_{L^2} \\
&\lesssim \Big(\|D^s\Theta\|_{B^1_{2,2}} \|\Theta\|_{L^\infty} + \|D^s\Theta\|_{L^6} \|\Theta\|_{B^1_{3,2}} \Big)^{\gamma_2} \\
&\quad \times \|D^s\Theta\Theta\|^{1-\gamma_2}_{L^2} \|\Theta\|_{L^4},
\end{aligned} \tag{4.44}$$

where $\gamma_2 = \gamma_2(\varepsilon) \in (0,1)$ is given by

$$\gamma_2 = \frac{1}{2} + 2\varepsilon. \tag{4.45}$$

By Sobolev's inequality and the interpolation (2.2) (with $s > \frac{1}{2}$), we have

$$\begin{aligned}
\|D^s\Theta\|_{B^1_{2,2}} \|\Theta\|_{L^\infty} + \|D^s\Theta\|_{L^6} \|\Theta\|_{B^1_{3,2}} &\lesssim \|\Theta\|_{H^{s+1}} \|\Theta\|_{H^{\frac{3}{2}+\varepsilon}} \\
&\lesssim \|\Theta\|^{1+\gamma_3}_{H^{s+1}} \|\Theta\|^{1-\gamma_3}_{L^4},
\end{aligned} \tag{4.46}$$

where $\gamma_3 = \gamma_3(s,\varepsilon) \in (0,1)$ is given by

$$\gamma_3 = \frac{3+2\varepsilon}{2(s+1)}. \tag{4.47}$$

Combining (4.44) and (4.46), we obtain

$$T_4 \lesssim \|D^s Y\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \|\Theta\|^{\gamma_2(1+\gamma_3)}_{H^{s+1}} \|D^s\Theta\Theta\|^{1-\gamma_2}_{L^2} \|\Theta\|^{1+\gamma_2(1-\gamma_3)}_{L^4}.$$

From (4.45) and (4.47), we observe that

$$\frac{\gamma_2(1+\gamma_3)}{2} + \frac{1-\gamma_2}{2} + \frac{1+\gamma_2(1-\gamma_3)}{4} < 1,$$

provided that $s > \frac{1}{2}$ and $\varepsilon > 0$ is sufficiently small. Therefore, we can apply Young's inequality to bound the contribution from T_4 by the right-hand side of (4.40). This completes the proof of Lemma 4.9. □

We conclude this section by presenting the proof of Lemma 4.10.

Proof of Lemma 4.10. By Cauchy's inequality, we have

$$\int_{\mathbb{T}^3} (D^s\Theta)^2 Y\Theta \leqslant c(\delta)\int_{\mathbb{T}^3} (D^s\Theta)^2 Y^2 + \delta\|D^s\Theta\Theta\|_{L^2}^2. \tag{4.48}$$

By Hölder's and Sobolev's inequalities followed by the interpolation (2.2) with (2.3) and (2.4), we have

$$\begin{aligned}\int_{\mathbb{T}^3} (D^s\Theta)^2 Y^2 &\lesssim \|D^s\Theta\|_{L^3}^2 \|Y^2\|_{L^3} \lesssim \|\Theta\|_{H^{s+\frac{1}{2}}}^2 \|Y^2\|_{H^{\frac{1}{2}}} \\ &\lesssim \|\Theta\|_{H^{s+1}}^{2\gamma}\|\Theta\|_{L^2}^{2(1-\gamma)}\|Y^2\|_{\mathscr{C}^{\frac{1}{2}+\varepsilon}} \\ &\lesssim \|\Theta\|_{H^{s+1}}^{2\gamma}\|\Theta\|_{L^4}^{2(1-\gamma)}\|D^sY\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}}^2\end{aligned} \tag{4.49}$$

for some $\gamma = \gamma(s) \in (0,1)$, provided that $s > 1$ and $\varepsilon > 0$ is sufficiently small. Noting that $\frac{2\gamma}{2} + \frac{2(1-\gamma)}{4} < 1$, (4.41) follows from (4.48), (4.49), and Young's inequality. □

5 Renormalized energy estimate

Recall from (3.19) that

$$\partial_t E_{s,N}(\pi_N\Phi_N(t)(\vec{u}))\Big|_{t=0} = F_1(\vec{u}_N) + F_2(\vec{u}_N) + F_3(\vec{u}_N),$$

where $\vec{u}_N = (u_N, v_N)$ and

$$\begin{aligned}F_1(\vec{u}_N) &= 3\int_{\mathbb{T}^3} Q_{s,N}(u_N)v_N u_N, \\ F_2(\vec{u}_N) &= \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s \\ |\alpha|,|\beta|,|\gamma|<s}} c_{\alpha,\beta,\gamma}\int_{\mathbb{T}^3} D^s v_N \cdot Q_{s,N}(u_N)^\alpha u_N \cdot Q_{s,N}(u_N)^\beta u_N \cdot Q_{s,N}(u_N)^\gamma u_N, \\ F_3(\vec{u}_N) &= \bigg(\int_{\mathbb{T}^3} u_N\bigg)\bigg(\int_{\mathbb{T}^3} v_N\bigg).\end{aligned}$$

Proposition 5.1. *Let $s \geqslant 4$ be an even integer. Then, there exist $\sigma < s-\frac{1}{2}$ sufficiently close to $s-\frac{1}{2}$ and small $\varepsilon > 0$ such that*

$$\bigg|\partial_t E_{s,N}(\pi_N\Phi_N(t)(\vec{u}))\Big|_{t=0}\bigg| \leqslant \big(1 + \|\vec{u}_N\|_{\vec{H}^\sigma}^2\big)F(\vec{u}_N), \tag{5.1}$$

where

$$F(\vec{u}_N) = 1 + \|Q_{s,N}(u_N)\|_{\mathscr{C}^{-1-\varepsilon}} + \sup_{\substack{|k|=s-1\\|\alpha|=s}} \|\partial^\kappa v_N\, \partial^\alpha u_N\|_{\mathscr{C}^{-1-\varepsilon}} + \sup_{\substack{|k|=s-1\\|\alpha|\leqslant s-1}} \|\partial^\kappa v_N\, \partial^\alpha u_N\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}}.$$

Proposition 3.8 follows from Proposition 5.1, the cutoff in the $\vec{H}^\sigma$-norm, and the Wiener chaos estimate (Lemma 2.2).

Proof. In the following, we prove (5.1) uniformly in $N \in \mathbb{N}$. Thus, we drop the N-dependence and write $Q_s(u)$ for $Q_{s,N}(u_N)$.

First, note that the estimate for F_3 follows trivially from Cauchy-Schwarz inequality. Next, we treat F_1. By duality (2.6) and the fractional Leibniz rule (2.7), we have

$$\begin{aligned} \int_{\mathbb{T}^3} Q_s(u)uv &\lesssim \|Q_s(u)\|_{\mathscr{C}^{-1-\varepsilon}} \|uv\|_{\mathscr{B}^{1+\varepsilon}_{1,1}} \\ &\lesssim \|Q_s(u)\|_{\mathscr{C}^{-1-\varepsilon}} \|u\|_{H^\sigma} \|v\|_{H^{\sigma-1}}, \end{aligned} \tag{5.2}$$

provided that $\sigma > 2+\varepsilon$. This is guaranteed by choosing σ sufficiently close to $s - \frac{1}{2}$, when $s > \frac{5}{2}$.

It remains to consider F_2. By integration by parts, it suffices to consider terms of the form:

$$\int_{\mathbb{T}^3} \partial^\kappa v\, \partial^\alpha u\, \partial^\beta u\, \partial^\gamma u,$$

where $|\kappa| = s-1$, $\max(\alpha, \beta, \gamma) \leqslant s$, and $|\alpha| + |\beta| + |\gamma| = s+1$. Without loss of generality, we assume that $|\alpha| \geqslant |\beta| \geqslant |\gamma|$. The idea is to group the low regularity terms ($\partial^\kappa v$ and $\partial^\alpha u$) and treat them as one piece.

First, let us assume that $|\alpha| = s$. In this case, we have $|\beta| = 1$ and $|\gamma| = 0$. By duality (2.6) and the fractional Leibniz rule (2.7), we have

$$\left| \int_{\mathbb{T}^3} \partial^\kappa v \partial^\alpha u\, \partial u\, u \right| \lesssim \|\partial^\kappa v\, \partial^\alpha u\|_{\mathscr{C}^{-1-\varepsilon}} \|\partial u\, u\|_{\mathscr{B}^{1+\varepsilon}_{1,1}} \lesssim \|\partial^\kappa v\, \partial^\alpha u\|_{\mathscr{C}^{-1-\varepsilon}} \|u\|^2_{H^\sigma}, \tag{5.3}$$

provided that $\sigma > 2+\varepsilon$. By choosing $\varepsilon > 0$ sufficiently small, we can guarantee this condition if $s > \frac{5}{2}$.

This leaves the case $|\alpha| \leqslant s-1$. Noting that $|\beta| \leqslant \frac{s+1}{2}$ and $|\gamma| \leqslant \frac{s+1}{3}$ (under $|\alpha| \geqslant |\beta| \geqslant |\gamma|$), we see that $\partial^\beta u, \partial^\gamma u \in H^{\frac{1}{2}+\varepsilon}(\mathbb{T}^3)$ for $s > 3$. Thus, by duality (2.6) and the fractional Leibniz rule (2.7), we have:

$$\left| \int_{\mathbb{T}^3} \partial^\kappa v\, \partial^\alpha u\, \partial^\beta u Q_{s,N}(u_N)^\gamma u \right| \lesssim \|\partial^\kappa v\, \partial^\alpha u\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \|\partial^\beta u\, u\|_{\mathscr{B}^{\frac{1}{2}+\varepsilon}_{1,1}} \lesssim \|\partial^\kappa v\, \partial^\alpha u\|_{\mathscr{C}^{-\frac{1}{2}-\varepsilon}} \|u\|^2_{H^\sigma}. \tag{5.4}$$

This completes the proof of Proposition 5.1. □

Remark 5.2. The restriction $s > 3$ in the last case appears only when $|\beta| = \frac{s+1}{2}$. In fact, when $|\beta| \leqslant \frac{s}{2}$, the estimate (5.4) holds true for $s > 2$. On the other hand, when $|\beta| = \frac{s+1}{2}$, we must have $|\alpha| = |\beta| = \frac{s+1}{2}$. In this case, by applying dyadic decompositions and working with the Littlewood-Paley pieces $\mathbf{P}_{j_2} Q_{s,N}(u_N)^\alpha u \, \mathbf{P}_{j_3} Q_{s,N}(u_N)^\beta u$, we can move half a derivative from the third factor to the second factor, thus showing that a slight variant of (5.4) holds for $s > 2$. Therefore, the estimates (5.2) and (5.3) on F_1 and F_2 impose the regularity restriction $s > \frac{5}{2}$.

Epilogue

Since the initial upload of [GOTW18], there has been an improvement of our results by [STX20]. Quasi-invariance is established for all $s > \frac{5}{2}$ (i.e. not only even integers) and the analysis is extended to the case of the quintic defocusing nonlinear wave equation. The extension from cubic to quintic is straightforward, but the extension to allow for fractional s is interesting.

In order to extend to fractional s, the key idea is to use commutator estimates rather than integration by parts in the energy estimate to isolate the leading order divergence. Thus, from our perspective the main innovation of [STX20] as compared to [GOTW18] is the much cleverer treatment of the lower order terms in the energy estimates.

III. Phase transitions

Prologue

In this part we focus on the ϕ^4 model and explore its phase transition in depth. The intuition behind our results comes from the classical Peierls' argument for the low temperature Ising model [Pei36], some aspects of which we recall in this prologue. In particular, we are going to derive contour bounds; we have already seen how these bounds can be used to establish long range order in Part I.

We establish the contour bounds in finite volumes and they extend to infinite volume with some care. Let $\Lambda_N = \{1, \ldots, N\}^3 \subset \mathbb{Z}^3$ be the box of sidelength $N \in \mathbb{N}$ and let $\Omega_N = \{\pm 1\}^{\Lambda_N}$ the space of spin configurations; note that we work on boxes rather than tori to avoid handling some topological issues. The Ising model on Λ_N at inverse temperature $\beta > 0$ is given by the measure $\mu^{\text{Ising}}_{\beta,N}$ defined for $\sigma \in \Omega_N$ by

$$\mu^{\text{Ising}}_{\beta,N}(\sigma) = \frac{1}{Z^{\text{Ising}}_{\beta,N}} e^{-\mathscr{H}^{\text{Ising}}_{\beta,N}(\sigma)}$$

where $Z^{\text{Ising}}_{\beta,N}$ is the partition function and

$$\mathscr{H}^{\text{Ising}}_{\beta,N}(\sigma) = -\beta \sum_{i \sim j} \sigma_i \sigma_j.$$

where $i \sim j$ means nearest-neighbours in Λ_N.

Recall that each configuration in Ω_N is in bijection with a configuration of contours; we have already explained this for $d = 2$, but it carries over to $d = 3$. Indeed, consider the partition of $\mathbb{R}^3$ by unit blocks centred on points in $\mathbb{Z}^3$ and restrict to boxes with centres in Λ_N (some care is needed near the boundary points of Λ_N, but we ignore this). Then, each configuration $\sigma \in \Omega_N$ is in bijection with a configuration of connected faces of blocks that, under some deformation convention to avoid ambiguities/self-intersections, form the boundary between $+$ and $-$ spins. We call connected components contours and the phase boundary $\partial\sigma$ the set of contours.

Lemma. *Let $\beta > 0$ and Γ a fixed contour that encloses a volume (i.e. has a well-defined interior). Then,*

$$\mu^{\text{Ising}}_{\beta,N}(\Gamma \in \partial\sigma) \leqslant e^{-2\beta|\Gamma|}$$

where $|\Gamma|$ is the number of faces in Γ.

Proof. By writing the Ising Hamiltonian in terms of agreements and disagreement of spins, we can represent the Ising measure as a gas of contours:

$$\mu_{\beta,N}^{\text{Ising}}(\sigma) = \frac{\prod_{\gamma\in\partial\sigma} e^{-2\beta|\gamma|}}{\sum_{\sigma'\in\Omega_N}\prod_{\gamma\in\partial\sigma'} e^{-2\beta|\gamma|}}.$$

Thus,

$$\mu_{\beta,N}^{\text{Ising}}(\Gamma\in\partial\sigma) = \sum_{\sigma\in\Omega_N:\Gamma\in\partial\sigma} \mu_{\beta,N}^{\text{Ising}}(\sigma) \leqslant e^{-2\beta|\Gamma|}\frac{\prod_{\gamma\in\partial\sigma\setminus\{\Gamma\}} e^{-2\beta|\gamma|}}{\sum_{\sigma\in\Omega_N}\prod_{\gamma\in\partial\sigma} e^{-2\beta|\gamma|}}. \quad (0.1)$$

For each $\sigma \in \Omega_N$ such that $\Gamma \in \partial\sigma$, let σ^Γ be the unique spin configuration obtained by flipping the value of spins in the interior of Γ (which erases this contour). Denote by Ω_N^Γ the set of configurations σ^Γ obtained in this way. Note that $\Omega_N^\Gamma \subset \Omega_N$.

Then,

$$(0.1) \leqslant e^{-2\beta|\Gamma|}\frac{\sum_{\sigma^\Gamma\in\Omega_N^\Gamma}\prod_{\gamma\in\partial\sigma^\Gamma} e^{-2\beta|\gamma|}}{\sum_{\sigma\in\Omega_N}\prod_{\gamma\in\partial\sigma} e^{-2\beta|\gamma|}} \leqslant e^{-2\beta|\Gamma|}$$

which finishes the proof. □

Statement of authorship

Appendix 6B: Statement of Authorship

This declaration concerns the article entitled:	
Phase transitions for ϕ^4_3	
Publication status (tick one)	**Draft manuscript** ☐ **Submitted** ✓ **In review** ☐ **Accepted** ☐ **Published** ☐
Publication details (reference)	Submitted to *Communications in Mathematical Physics*
Copyright status (tick the appropriate statement)	I hold the copyright for this material ✓ Copyright is retained by the publisher, but I have been given permission to replicate the material here ☐
Candidate's contribution to the paper (provide details, and also indicate as a percentage)	The candidate contributed to / considerably contributed to / predominantly executed the… Formulation of ideas: All ideas were a result of fair and equal collaboration between all three coauthors. So 33.33%. Design of methodology: N/A Experimental work: N/A Presentation of data in journal format: N/A
Statement from Candidate	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature.
Signed	[signature] **Date** 23/08/20

Last update: Feb 2019

1 Introduction

We study the behaviour of the average magnetisation

$$\mathfrak{m}_N(\phi) = \frac{1}{N^3} \int_{\mathbb{T}_N} \phi(x) dx$$

for fields ϕ distributed according to the measure $\nu_{\beta,N}$ with formal density

$$d\nu_{\beta,N}(\phi) \propto \exp\Big(- \int_{\mathbb{T}_N} \mathcal{V}_\beta(\phi(x)) + \frac{1}{2}|\nabla\phi(x)|^2 dx \Big) \prod_{x \in \mathbb{T}_N} d\phi(x) \tag{1.1}$$

in the infinite volume limit $N \to \infty$. Above, $\mathbb{T}_N = (\mathbb{R}/N\mathbb{Z})^3$ is the 3D torus of sidelength $N \in \mathbb{N}$, $\prod_{x \in \mathbb{T}_N} d\phi(x)$ is the (non-existent) Lebesgue measure on fields $\phi : \mathbb{T}_N \to \mathbb{R}$, $\beta > 0$ is the inverse temperature, and $\mathcal{V}_\beta : \mathbb{R} \to \mathbb{R}$ is the symmetric double-well potential given by $\mathcal{V}_\beta(a) = \frac{1}{\beta}(a^2 - \beta)^2$ for $a \in \mathbb{R}$.

$\nu_{\beta,N}$ is a finite volume approximation of a ϕ^4_3 Euclidean quantum field theory [Gli68, GJ73, FO76]. Its construction, first in finite volumes and later in infinite volume, was a major achievement of the constructive field theory programme in the '60s-'70s: Glimm and Jaffe made the first breakthrough in [GJ73] and many results followed [Fel74, MS77, BCG+80, BFS83, BDH95, MW17b, GH18, BG19]. The model in 2D was constructed earlier by Nelson [Nel66]. In higher dimensions there are triviality results: in dimensions $\geqslant 5$ these are due to Aizenman and Fröhlich [Aiz82, Frö82], whereas the 4D case was only recently done by Aizenman and Duminil-Copin [ADC20]. By now it is also well-known that the ϕ^4_3 model has significance in statistical mechanics since it arises as a continuum limit of Ising-type models near criticality [SG73, CMP95, HI18].

It is natural to define $\nu_{\beta,N}$ using a density with respect to the centred Gaussian measure μ_N with covariance $(-\Delta)^{-1}$, where Δ is the Laplacian on $\mathbb{T}_N$ (see Remark 1.1 for how we deal with the issue of constant fields/the zeroeth Fourier mode). However, in 2D and higher μ_N is not supported on a space of functions and samples need to be interpreted as Schwartz distributions. This is a serious problem because there is no canonical interpretation of products of distributions, meaning that the nonlinearity $\int_{\mathbb{T}_N} \mathcal{V}_\beta(\phi(x))dx$ is not well-defined on the support of μ_N. If one introduces an ultraviolet (small-scale) cutoff $K > 0$ on the field to regularise it, then one sees that the nonlinearities $\mathcal{V}_\beta(\phi_K)$ fail to converge as the cutoff is removed - there are divergences. The strength of these divergences grow as the dimension grows: they are only logarithmic in the cutoff in 2D, whereas they are polynomial in the cutoff in 3D. In addition, $\nu_{\beta,N}$ and μ_N are mutually singular [BG20] in 3D, which produces technical difficulties that are not present in 2D.

Renormalisation is required in order to kill these divergences. This is done by looking at the cutoff measures and subtracting the corresponding counter-term

$\int_{\mathbb{T}_N} \delta m^2(K)\phi_K^2$ where ϕ_K is the field cutoff at spatial scales less than $\frac{1}{K}$ and the renormalisation constant $\delta m^2(K) = \frac{C_1}{\beta}K - \frac{C_2}{\beta^2}\log K$ for specific constants $C_1, C_2 > 0$ (see Section 2). If these constants are appropriately chosen (i.e. by perturbation theory), then a non-Gaussian limiting measure is obtained as $K \to \infty$. This construction yields a one-parameter family of measures $\nu_{\beta,N} = \nu_{\beta,N}(\delta m^2)$ corresponding to bounded shifts of $\delta m^2(K)$.

Remark 1.1. *For technical reasons, we work with a massive Gaussian free field as our reference measure. We do this by introducing a mass $\eta > 0$ into the covariance. This resolves the issue of the constant fields/zeroeth Fourier mode degeneracy. In order to stay consistent with* (1.1)*, we subtract $\int_{\mathbb{T}_N} \frac{\eta}{2}\phi^2 dx$ from $\mathscr{V}_\beta(\phi)$.*

Once we have chosen η, it is convenient to fix δm^2 by writing the renormalisation constants in terms of expectations with respect to $\mu_N(\eta)$. The particular choice of η is inessential since one can show that changing η corresponds to a bounded shift of δm^2 that is $O\left(\frac{1}{\beta}\right)$ as $\beta \to \infty$.

The large-scale behaviour of $\nu_{\beta,N}$ depends heavily on β as $N \to \infty$. To see why, note that $a \mapsto \mathscr{V}_\beta(a)$ has minima at $a = \pm\sqrt{\beta}$ with a potential barrier at $a = 0$ of height β, so the minima become widely separated by a steep barrier as $\beta \to \infty$. Consequently, $\nu_{\beta,N}$ resembles an Ising model on $\mathbb{T}_N$ with spins at $\pm\sqrt{\beta}$ (i.e. at inverse temperature $\beta > 0$) for large β. Glimm, Jaffe, and Spencer [GJS75] exploited this similarity and proved phase transition for ν_β, the infinite volume analogue of $\nu_{\beta,N}$, in 2D using a sophisticated modification of the classical Peierls' argument for the low temperature Ising model [Pei36, Gri64, Dob65]. See also [GJS76a, GJS76b]. Their proof relies on contour bounds for $\nu_{\beta,N}$ in 2D that hold in the limit $N \to \infty$. Their techniques fail in the significantly harder case of 3D. However, phase transition for ν_β in 3D was established by Fröhlich, Simon, and Spencer [FSS76] using a different argument based heavily on reflection positivity. Whilst this argument is more general (it applies, for example, to some models with continuous symmetry), it is less quantitative than the Peierls' theory of [GJS75]. Specifically, it is not clear how to use it to control large deviations of the (finite volume) average magnetisation $\mathfrak{m}_N$.

Although phase coexistence for ν_β has been established, little is known of this regime in comparison to the low temperature Ising model. In the latter model, the study of *phase segregation* at low temperatures in large but finite volumes was initiated by Minlos and Sinai [MS67, MS68], culminating in the famous Wulff constructions: due to Dobrushin, Kotecký, and Shlosman in 2D [DKS89, DKS92], with simplifications due to Pfister [Pfi91] and results up to the critical point by Ioffe and Schonmann [IS98]; and Bodineau [Bod99] in 3D, see also results up to the critical point by Cerf and Pisztora [CP00] and the bibliographical review in [BIV00, Section 1.3.4]. We are interested in a weaker form of phase segregation: *surface* order large deviation estimates for the average magnetisation $\mathfrak{m}_N$. For the Ising model, this was first established in 2D by Schonmann [Sch87] and later extended up to the critical point by Chayes, Chayes, and Schonmann [CCS87]; in 3D this was first established by Pisztora [Pis96]. These results should be contrasted with the volume order large deviations established for $\mathfrak{m}_N$ in the high temperature regime where there is no phase coexistence [CF86, Ell85, FO88, Oll88].

Our main result is a surface order upper bound on large deviations for the average magnetisation under $\nu_{\beta,N}$.

Theorem 1.2. *Let $\eta > 0$ and $\nu_{\beta,N} = \nu_{\beta,N}(\eta)$ as in Remark 1.1. For any $\zeta \in (0,1)$, there exists $\beta_0 = \beta_0(\zeta,\eta) > 0$, $C = C(\zeta,\eta) > 0$, and $N_0 = N_0(\zeta) \geqslant 4$ such that the following estimate holds: for any $\beta > \beta_0$ and any $N > N_0$ dyadic,*

$$\frac{1}{N^2}\log \nu_{\beta,N}\Big(\mathfrak{m}_N \in (-\zeta\sqrt{\beta}, \zeta\sqrt{\beta})\Big) \leqslant -C\sqrt{\beta}. \tag{1.2}$$

Proof. See Section 3.5. □

The condition that N is a sufficiently large dyadic in Theorem 1.2 comes from Proposition 3.8 (we also need that N is divisible by 4 to apply the chessboard estimates of Proposition 6.5). Our analysis can be simplified to prove Theorem 1.2 in 2D with N^2 replaced by N in (1.2).

Our main technical contributions are contour bounds for $\nu_{\beta,N}$. As a result, the Peierls' argument of [GJS75] is extended to 3D, thereby giving a second proof of phase transition for ϕ^4_3. The main difficulty is to handle the ultraviolet divergences of $\nu_{\beta,N}$ whilst preserving the structure of the low temperature potential. We do this by building on the variational approach to showing ultraviolet stability for ϕ^4_3 recently developed by Barashkov and Gubinelli [BG19]. Our insight is to separate scales within the corresponding stochastic control problem through a coarse-graining into an effective Hamiltonian and remainder. The effective Hamiltonian captures the macroscopic description of the system and is treated using techniques adapted from [GJS76b]. The remainder contains the ultraviolet divergences and these are killed using the renormalisation techniques of [BG19].

Our next contribution is to adapt arguments used by Bodineau, Velenik, and Ioffe [BIV00], in the context of equilibrium crystal shapes of discrete spin models, to study phase segregation for ϕ^4_3. In particular, we adapt them to handle a block-averaged model with unbounded spins. Technically, this requires control over *large fields*.

1.1 Application to the dynamical ϕ^4_3 model

The Glauber dynamics of $\nu_{\beta,N}$ is given by the singular stochastic PDE

$$\begin{aligned}(\partial_t - \Delta + \eta)\Phi &= -\frac{4}{\beta}\Phi^3 + (4+\eta+\infty)\Phi + \sqrt{2}\xi \\ \Phi(0,\cdot) &= \phi_0\end{aligned} \tag{1.3}$$

where $\Phi \in S'(\mathbb{R}_+ \times \mathbb{T}_N)$ is a space-time Schwartz distribution, $\phi_0 \in \mathscr{C}^{-\frac{1}{2}-\kappa}(\mathbb{T}_N)$, the infinite constant indicates renormalisation (see Remark 6.16), and ξ is space-time white noise on $\mathbb{T}_N$. The well-posedness of this equation, known as the dynamical ϕ^4_3 model, has been a major breakthrough in stochastic analysis in recent years [Hai14, Hai16, GIP15, CC18, Kup16, MW17b, GH19, MW18].

In finite volumes the solution is a Markov process and its associated semigroup $(\mathscr{P}^{\beta,N}_t)_{t\geqslant 0}$ is reversible and exponentially ergodic with respect to its unique invariant measure $\nu_{\beta,N}$ [HM18a, HS19, ZZ18a]. As a consequence, there exists a spectral gap $\lambda_{\beta,N} > 0$ given by the optimal constant in the inequality:

$$\Big\langle \Big(\mathscr{P}^{\beta,N}_t F\Big)^2 \Big\rangle_{\beta,N} - \Big(\Big\langle \mathscr{P}^{\beta,N}_t F \Big\rangle_{\beta,N}\Big)^2 \leqslant e^{-\lambda_{\beta,N} t}\Big(\langle F^2\rangle_{\beta,N} - \langle F\rangle^2_{\beta,N}\Big)$$

for suitable $F \in L^2(\nu_{\beta,N})$. $\lambda^{-1}_{\beta,N}$ is called the relaxation time and measures the rate of convergence of variances to equilibrium. An implication of Theorem 1.2 is the exponential explosion of relaxation times in the infinite volume limit provided β is sufficiently large.

Corollary 1.3. *Let $\eta > 0$ and $\nu_{\beta,N} = \nu_{\beta,N}(\eta)$ as in Remark 1.1. Then, there exists $\beta_0 = \beta_0(\eta) > 0$, $C = C(\beta_0, \eta)$, and $N_0 \geqslant 4$ such that, for any $\beta > \beta_0$ and $N > N_0$ dyadic,*

$$\frac{1}{N^2}\log \lambda_{\beta,N} \leqslant -C\sqrt{\beta}. \tag{1.4}$$

Proof. See Section 7. □

Corollary 1.3 is the first step towards establishing phase transition for the relaxation times of the Glauber dynamics of ϕ^4 in 2D and 3D. This phenomenon has been well-studied for the Glauber dynamics of the 2D Ising model, where a relatively complete picture has been established (in higher dimensions it is less complete). The relaxation times for the Ising dynamics on the 2D torus of sidelength N undergo the following trichotomy as $N \to \infty$: in the high temperature regime, they are uniformly bounded in N [AH87, MO94]; in the low temperature regime, they are exponential in N [Sch87, CCS87, Tho89, MO94, CGMS96]; at criticality, they are polynomial in N [Hol91, LS12]. It would be interesting to see whether the relaxation times for the dynamical ϕ^4 model undergo such a trichotomy.

1.2 Paper organisation

In Section 2 we introduce the renormalised, ultraviolet cutoff measures $\nu_{\beta,N,K}$ that converge weakly to $\nu_{\beta,N}$ as the cutoff is removed. In Section 3 we carry out the statistical mechanics part of the proof of Theorem 1.2. In particular, conditional on the moment bounds in Proposition 3.6, we develop contour bounds for $\nu_{\beta,N}$. These contour bounds allow us to adapt techniques in [BIV00], which were developed in the context of discrete spin systems, to deal with $\nu_{\beta,N}$.

In Section 4 we lay the foundation to proving Proposition 3.6 by introducing the Boué-Dupuis formalism for analysing the free energy of $\nu_{\beta,N}$ as in [BG19]. We then use a low temperature expansion and coarse-graining argument within the Boué-Dupuis formalism in Section 5 to establish Proposition 5.1 which contains the key analytic input to proving Proposition 3.6.

In Section 6, we use the chessboard estimates of Proposition 6.5 to upgrade the bounds of Proposition 5.1 to those of Proposition 3.6. Chessboard estimates follow from the well-known fact that $\nu_{\beta,N}$ is reflection positive. We give an independent proof of this fact by using stability results for the dynamics (1.3) to show that lattice and Fourier regularisations of $\nu_{\beta,N}$ converge to the same limit. Then, in Section 7, we prove Corollary 1.3 showing that the spectral gaps for the dynamics decay in the infinite volume limit provided β is sufficiently large.

We collect basic notations and analytic tools that we use throughout the paper in Appendix A.

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2 The model

In the following, we use notation and standard tools introduced in Appendix A.1.

Let $\eta > 0$. Denote by $\mu_N = \mu_N(\eta)$ the centred Gaussian measure with covariance $(-\Delta + \eta)^{-1}$ and expectation $\mathbb{E}_N$. Above, Δ is the Laplacian on $\mathbb{T}_N$. As pointed out in Remark 1.1, the choice of η is inessential. We consider it fixed unless stated otherwise and we do not make η-dependence explicit in the notation.

Fix $\beta > 0$. Let $\mathscr{V}_\beta \colon \mathbb{R} \to \mathbb{R}$ be given by

$$\mathscr{V}_\beta(a) = \frac{1}{\beta}(a^2 - \beta)^2 = \frac{1}{\beta}a^4 - 2a^2 + \beta.$$

$\mathscr{V}_\beta$ is a symmetric double well potential with minima at $a = \pm\sqrt{\beta}$ and a potential barrier at $a = 0$ of height β.

Fix $\rho \in C_c^\infty(\mathbb{R}^3; [0,1])$ rotationally symmetric; decreasing; and satisfying $\rho(x) = 1$ for $|x| \in [0, c_\rho)$, where $c_\rho > 0$. See Lemma 4.6 for why the last condition is important. Note that many of our estimates rely on the choice of ρ, but we omit explicit reference to this.

For every $K > 0$, let ρ_K be the Fourier multiplier on $\mathbb{T}_N$ with symbol $\rho_K(\cdot) = \rho(\frac{\cdot}{K})$. For $\phi \sim \mu_N$, we denote $\phi_K = \rho_K \phi$. Note that ϕ_K is smooth. Let

$$\text{Q}_K = \mathbb{E}_N[\phi_K^2(0)] = \frac{1}{N^3} \sum_{n \in (N^{-1}\mathbb{Z})^3} \frac{\rho_K^2(n)}{\langle n \rangle^2} \tag{2.1}$$

where $\langle \cdot \rangle = \sqrt{\eta + 4\pi^2 |\cdot|^2}$. Note that $\text{Q}_K = O(K)$ as $K \to \infty$. The first four Wick powers of ϕ_K are given by the generalised Hermite polynomials:

$$\begin{aligned}
:\phi_K(x): &= \phi_K(x)\\
:\phi_K^2(x): &= \phi_K^2(x) - \text{Q}_K\\
:\phi_K^3(x): &= \phi_K^3(x) - 3\text{Q}_K\phi_K(x)\\
:\phi_K^4(x): &= \phi_K^4(x) - 6\text{Q}_K\phi_K^2(x) + 3\text{Q}_K^2.
\end{aligned}$$

We define the Wick renormalised potential by linearity:

$$:\mathscr{V}_\beta(\phi_K) := \frac{1}{\beta} :\phi_K^4: - 2 :\phi_K^2: + \beta.$$

Let $\nu_{\beta,N,K}$ be the probability measure with density

$$d\nu_{\beta,N,K}(\phi) = \frac{e^{-\mathscr{H}_{\beta,N,K}(\phi_K)}}{\mathscr{Z}_{\beta,N,K}} d\mu_N(\phi). \tag{2.2}$$

Above, $\mathscr{H}_{\beta,N,K}$ is the renormalised Hamiltonian

$$\mathscr{H}_{\beta,N,K}(\phi_K) = \int_{\mathbb{T}_N} :\mathscr{V}_\beta(\phi_K): - \frac{\gamma_K}{\beta^2} :\phi_K^2: - \delta_K - \frac{\eta}{2} :\phi_K^2: dx \tag{2.3}$$

where γ_K and δ_K are additional renormalisation constants given by (5.25) and (5.26), respectively, and $\mathscr{Z}_{\beta,N,K} = \mathbb{E}_N e^{-\mathscr{H}_{\beta,N,K}(\phi_K)}$ is the partition function.

Proposition 2.1. *For every $\beta > 0$ and $N \in \mathbb{N}$, the measures $\nu_{\beta,N,K}$ converge weakly to a non-Gaussian measure $\nu_{\beta,N}$ on $S'(\mathbb{T}_N)$ as $K \to \infty$. In addition, $\mathscr{Z}_{\beta,N,K} \to \mathscr{Z}_{\beta,N}$ as $K \to \infty$ and satisfies the following estimate: there exists $C = C(\beta,\eta) > 0$ such that*

$$-CN^3 \leqslant -\log \mathscr{Z}_{\beta,N} \leqslant CN^3.$$

Proof. Proposition 2.1 is a variant of the classical ultraviolet stability for ϕ^4_3 first established in [GJ73]. Our precise formulation, i.e. the choice of $\gamma_\bullet$ and $\delta_\bullet$, is taken from [BG19, Theorem 1]. □

We write $\langle\cdot\rangle_{\beta,N}$ and $\langle\cdot\rangle_{\beta,N,K}$ for expectations with respect to $\nu_{\beta,N}$ and $\nu_{\beta,N,K}$, respectively.

Remark 2.2. *The constants Q_K, γ_K, δ_K are, respectively, Wick renormalisation, (second order) mass renormalisation, and energy renormalisation constants. They all depend on η and N. δ_K additionally depends on β and is needed for the convergence of $\mathscr{Z}_{\beta,N,K}$ as $K \to \infty$, but drops out of the definition of the cutoff measures* (2.2)*.*

Remark 2.3. *In 2D a scaling argument [GJS76c] allows one to work with the measure with density proportional to*

$$\exp\Big(- \int_{\mathbb{T}_N} : \mathscr{V}_\beta(\phi_K) : dx \Big) d\tilde{\mu}_N(\phi)$$

where $\tilde{\mu}_N$ is the Gaussian measure with covariance $(-\Delta + \sqrt{\beta}^{-1})^{-1}$, i.e. a β-dependent mass. This measure is significantly easier to work with due to the degenerate mass when β is large. In particular, it is easier to obtain contour bounds which, although suboptimal from the point of view of β-dependence, are sufficient for the Peierls' argument in [GJS75] and for the analogue of our argument in Section 3 carried out in 2D. In 3D one cannot work with such a measure.

3 Surface order large deviation estimate

In this section we carry out the statistical mechanics part of the proof of Theorem 1.2. Recall that for large β, the the minima of potential $\mathscr{V}_\beta$ at $\pm\sqrt{\beta}$ are widely separated by a steep potential barrier of height β, so formally $\nu_{\beta,N}$ resembles an Ising model at inverse temperature β. We use this intuition to prove contour bounds for $\nu_{\beta,N}$ (see Proposition 3.2) conditional on certain moment bounds (see Proposition 3.6). The contour bounds are then used to adapt arguments from [BIV00] to prove Theorem 1.2.

3.1 Block averaging

Let e_1, e_2, e_3 be the standard basis for $\mathbb{R}^3$. We identify $\mathbb{T}_N$ with the set

$$\{a_1e_1 + a_2e_2 + a_3e_3 : a_1, a_2, a_3 \in [0, N)\}.$$

Define

$$\mathbb{B}_N = \Big\{ \prod_{i=1}^{3} [a_i, a_i + 1) \subset \mathbb{T}_N : a_1, a_2, a_3 \in \{0, \ldots, N-1\} \Big\}.$$

We call elements of $\mathbb{B}_N$ blocks. For any $B \subset \mathbb{B}_N$, we overload notation and write $B = \bigcup_{\square \in B} \square \subset \mathbb{T}_N$. Hence, $|B| = \int_B 1dx$ is the number of blocks in B. In addition, we identify any $\vec{f} \in \mathbb{R}^{\mathbb{B}_N}$ with the piecewise continuous function on $\mathbb{T}_N$ given by $\vec{f}(x) = \vec{f}(\square)$ for $x \in \square$.

Let $\phi \sim \nu_{\beta,N}$. For any $\square \in \mathbb{B}_N$, let $\phi(\square) = \int_\square \phi dx$. Here, the integral is interpreted as the duality pairing between ϕ (a distribution) and the indicator function $\mathbf{1}_\square$ (a test function); we use this convention throughout. We let $\vec{\phi} = (\phi(\square))_{\square \in \mathbb{B}_N} \in \mathbb{R}^{\mathbb{B}_N}$ denote the block averaged field obtained from ϕ.

Remark 3.1. *Testing ϕ against $\mathbf{1}_\square$, which is not smooth, yields a well-defined random variable on the support of $\nu_{\beta,N}$. Indeed, ϕ belongs almost surely to L^∞-based Besov spaces of regularity s for every $s < -\frac{1}{2}$ (see Appendix A.2 for a review of Besov spaces and see Section 4 for the almost sure regularity of ϕ). On the other hand, indicator functions of blocks belong to L^1-based Besov spaces of regularity s for every $s < 1$ or, more generally, L^p-based Besov spaces of regularity s for every $s < \frac{1}{p}$ (see, for example, Lemma 1.1 in [FR12]). This is sufficient to test ϕ against indicator functions of blocks (using e.g. Proposition A.1). We also give an alternative proof using a type of Itô isometry in Proposition 5.22.*

3.2 Phase labels

We define a map $\vec{\phi} \in \mathbb{R}^{\mathbb{B}_N} \mapsto \sigma \in \{-\sqrt{\beta}, 0, \sqrt{\beta}\}^{\mathbb{B}_N}$ called a phase label. A basic function of σ is to identify whether the averages $\phi(\square)$ take values around the well at $+\sqrt{\beta}$, the well at $-\sqrt{\beta}$, or neither. We quantify this to a given precision $\delta \in (0, 1)$, which is taken to be fixed in what follows.

- We say that $\square \in \mathbb{B}_N$ is plus (resp. minus) valued if

$$|\phi(\square) \mp \sqrt{\beta}| < \sqrt{\beta}\delta.$$

  The set of plus (resp. minus) valued blocks is denoted $\mathcal{P}$ (resp. $\mathcal{M}$).

- The set of neutral blocks is defined as $\mathcal{N} = \mathbb{B}_N \backslash (\mathcal{P} \cup \mathcal{M})$.

Each block in $\mathbb{B}_N$ contains a midpoint. Given two distinct blocks in $\mathbb{B}_N$, we say that they are nearest-neighbours if their midpoints are of distance 1. They are $*$-neighbours if their midpoints are of distance at most $\sqrt{3}$. For any $\square \in \mathbb{B}_N$, the $*$-connected ball centred at $\square$ is the set $\mathrm{B}^*(\square) \subset \mathbb{B}_N$ consisting of $\square$ and its $*$-neighbours. It contains exactly 27 blocks.

- We say that $\square \in \mathbb{B}_N$ is *plus good* if every $\square' \in \mathrm{B}^*(\square)$ is plus valued. The set of plus good blocks is denoted $\mathscr{P}_G$.
- We say that $\square \in \mathbb{B}_N$ is *minus good* if every $\square' \in \mathrm{B}^*(\square)$ is minus valued. The set of minus good blocks is denoted $\mathscr{M}_G$.
- The set of *bad* blocks is defined as $\mathscr{B} = \mathbb{B}_N \backslash (\mathscr{P}_G \cup \mathscr{M}_G)$.

Define the phase label σ associated to $\vec{\phi}$ of precision $\delta > 0$ by

$$\sigma(\square) = \begin{cases} +\sqrt{\beta}, & \square \in \mathscr{P}_G, \\ -\sqrt{\beta}, & \square \in \mathscr{M}_G, \\ 0, & \square \in \mathscr{B}. \end{cases}$$

The following proposition can be thought of as an extension of the contour bounds developed for ϕ^4 in 2D [GJS75, Theorem 1.2] to 3D.

Proposition 3.2. *Let σ be a phase label of precision $\delta \in (0,1)$. Then, there exists $\beta_0 = \beta_0(\delta, \eta) > 0$ and $C_P = C_P(\delta, \eta) > 0$ such that, for $\beta > \beta_0$, the following holds for any $N \in 4\mathbb{N}$: for any set of blocks $B \subset \mathbb{B}_N$,*

$$\nu_{\beta,N}(\sigma(\square) = 0 \textit{ for all } \square \in B) \leqslant e^{-C_P\sqrt{\beta}|B|}. \tag{3.1}$$

Proof. See Section 3.3.1. The main estimates required in the proof are given in Proposition 3.6, which extends [GJS75, Theorem 1.3] to 3D and improves the β-dependence. Assuming this, we then prove Proposition 3.2 in the spirit of the proof of [GJS75, Theorem 1.2]. $\square$

3.3 Penalising bad blocks

Given a phase label, we partition the set of bad blocks $\mathscr{B}$ into two types.

- Frustrated blocks are blocks $\square \in \mathbb{B}_N$ such that $\mathrm{B}^*(\square)$ contains a neutral block. We denote the set of frustrated blocks $\mathscr{B}_F$.
- Interface block are blocks $\square \in \mathbb{B}_N$ such that $\mathrm{B}^*(\square)$ contains no neutral blocks, but there exists at least one pair of *nearest-neighbours* $\{\square', \square''\} \subset \mathrm{B}^*(\square)$ such that $\square' \in \mathscr{P}$ but $\square'' \in \mathscr{M}$. We denote the set of interface blocks $\mathscr{B}_I$.

For any $\square \in \mathbb{B}_N$ and any nearest-neighbours $\square', \square'' \in \mathbb{B}_N$, define:

$$\begin{aligned} Q_1(\square) &= \frac{1}{\sqrt{\beta}} \int_\square (\beta - :\phi^2(x):) dx \\ Q_2(\square) &= \frac{1}{\sqrt{\beta}} \int_\square (:\phi^2(x): - \phi(\square)^2) dx \\ Q_3(\square', \square'') &= \phi(\square') - \phi(\square''). \end{aligned} \tag{3.2}$$

Remark 3.3. *Note that testing* $:\phi^2:$ *against* $\mathbf{1}_\square$ *yields a well-defined random variable on the support of* $\nu_{\beta,N}$. *We give a proof of this fact in Proposition 5.23.*

We write $\mathrm{B}^*_{\mathrm{nn}}(\square)$ for the set of unordered pairs of nearest-neighbour blocks $\{\square', \square''\}$ in $\mathbb{B}_N$ such that $\square', \square'' \in \mathrm{B}^*(\square)$. There are 54 elements in this set.

Lemma 3.4. *Let* $N \in \mathbb{N}$ *and fix a phase label of precision* $\delta \in (0,1)$. *Then, for every* $\square \in \mathbb{B}_N$,

$$\mathbf{1}_{\square \in \mathscr{B}_F} \leqslant 2e^{-C_\delta \sqrt{\beta}} \sum_{\square' \in \mathrm{B}^*(\square)} \Big(\cosh Q_1(\square') + \cosh Q_2(\square') \Big) \tag{3.3}$$

$$\mathbf{1}_{\square \in \mathscr{B}_I} \leqslant 2e^{-C_\delta \sqrt{\beta}} \sum_{\{\square', \square''\} \in \mathrm{B}^*_{\mathrm{nn}}(\square)} \cosh Q_3(\square', \square'') \tag{3.4}$$

where $C_\delta = \min\left(\frac{\delta}{2}, 2 - 2\delta\right) > 0$.

Frustrated blocks are penalised by the potential $\mathscr{V}_\beta$ whereas interface blocks are penalised by the gradient term in the Gaussian measure. Lemma 3.4 formalises this through use of the random variables Q_1, Q_2 and Q_3, which (up to trivial modifications) were introduced in [GJS75]. Q_1 penalises frustrated blocks. Q_2 is an error term coming from the fact that the potential is written in terms of ϕ rather than $\vec{\phi}$. Q_3 penalises interface blocks.

Proof of Lemma 3.4. For any $\square \in \mathbb{B}_N$,

$$\begin{aligned}
\mathbf{1}_{\square\in\mathcal{N}} &= \mathbf{1}_{|\phi(\square)|<(1-\delta)\sqrt{\beta}} + \mathbf{1}_{|\phi(\square)|>(1+\delta)\sqrt{\beta}} \\
&= \mathbf{1}_{\frac{1}{\sqrt{\beta}}(\beta-\phi(\square)^2)>(2\delta-\delta^2)\sqrt{\beta}} + \mathbf{1}_{\frac{1}{\sqrt{\beta}}(\phi(\square)^2-\beta)>(2\delta+\delta^2)\sqrt{\beta}} \\
&= \mathbf{1}_{\frac{1}{\sqrt{\beta}}\int_\square \beta-:\phi^2(x):dx+\frac{1}{\sqrt{\beta}}\int_\square :\phi^2(x):-\phi(\square)^2 dx>(2\delta-\delta^2)\sqrt{\beta}} \\
&\quad + \mathbf{1}_{\frac{1}{\sqrt{\beta}}\int_\square :\phi^2(x):-\beta dx+\frac{1}{\sqrt{\beta}}\int_\square \phi(\square)^2-:\phi^2(x):dx>(2\delta+\delta^2)\sqrt{\beta}} \\
&\leqslant \mathbf{1}_{\frac{1}{\sqrt{\beta}}\int_\square \beta-:\phi^2(x):dx>\frac{2\delta-\delta^2}{2}\sqrt{\beta}} + \mathbf{1}_{\frac{1}{\sqrt{\beta}}\int_\square :\phi^2(x):-\phi(\square)^2 dx>\frac{2\delta-\delta^2}{2}\sqrt{\beta}} \qquad (3.5)\\
&\quad + \mathbf{1}_{\frac{1}{\sqrt{\beta}}\int_\square :\phi^2(x):-\beta dx>\frac{2\delta+\delta^2}{2}\sqrt{\beta}} + \mathbf{1}_{\frac{1}{\sqrt{\beta}}\int_\square \phi(\square)^2-:\phi^2(x):dx>\frac{2\delta+\delta^2}{2}\sqrt{\beta}} \\
&\leqslant e^{-\frac{\delta}{2}\sqrt{\beta}}\Big(e^{Q_1(\square)} + e^{Q_2(\square)} + e^{-Q_1(\square)} + e^{-Q_2(\square)}\Big) \\
&= 2e^{-\frac{\delta}{2}\sqrt{\beta}}\Big(\cosh Q_1(\square) + \cosh Q_2(\square)\Big)
\end{aligned}$$

where in the penultimate line we have used that $\delta^2 \leqslant \delta$.

By the definition of $\mathcal{B}_F$,

$$\mathbf{1}_{\square\in\mathcal{B}_F} \leqslant \sum_{\square'\in \mathrm{B}^*(\square)} \mathbf{1}_{\square'\in\mathcal{N}}. \qquad (3.6)$$

Using (3.5) applied to $\mathbf{1}_{\square'\in\mathcal{N}}$ in (3.6) yields (3.3).

(3.4) is established by the following estimates: by the definition of $\mathcal{B}_I$,

$$\begin{aligned}
\mathbf{1}_{\square\in\mathcal{B}_I} &\leqslant \sum_{\{\square',\square''\}\in \mathrm{B}^*_{\mathrm{nn}}(\square)} (\mathbf{1}_{\square'\in\mathcal{P}}\mathbf{1}_{\square''\in\mathcal{M}} + \mathbf{1}_{\square'\in\mathcal{M}}\mathbf{1}_{\square''\in\mathcal{P}}) \\
&\leqslant \sum_{\{\square',\square''\}\in \mathrm{B}^*_{\mathrm{nn}}(\square)} (\mathbf{1}_{\phi(\square')-\phi(\square'')>(2-2\delta)\sqrt{\beta}} + \mathbf{1}_{\phi(\square'')-\phi(\square')>(2-2\delta)\sqrt{\beta}}) \\
&\leqslant \sum_{\{\square',\square''\}\in \mathrm{B}^*_{\mathrm{nn}}(\square)} e^{-(2-2\delta)\sqrt{\beta}}\Big(e^{Q_3(\square',\square'')} + e^{-Q_3(\square',\square'')}\Big) \\
&= \sum_{\{\square',\square''\}\in \mathrm{B}^*_{\mathrm{nn}}(\square)} 2e^{-(2-2\delta)\sqrt{\beta}}\cosh Q_3(\square',\square'').
\end{aligned}$$

□

In order to use Lemma 3.4 to prove Proposition 3.2, we want to control expectations of $\cosh Q_1, \cosh Q_2$ and $\cosh Q_3$ by the exponentially small (in $\sqrt{\beta}$) prefactor in (3.3) and (3.4). Moreover, we want to control these expectations over a set of blocks as opposed to just single blocks.

Let $B_1, B_2 \subset \mathbb{B}_N$ and let B_3 be any set of unordered pairs of nearest-neighbours in $\mathbb{B}_N$. Define

$$\begin{aligned}
\cosh Q_1(B_1) &= \prod_{\square \in B_1} \cosh Q_1(\square) \\
\cosh Q_2(B_2) &= \prod_{\square \in B_2} \cosh Q_2(\square) \\
\cosh Q_3(B_3) &= \prod_{\{\square,\square'\} \in B_3} \cosh Q_3(\square,\square').
\end{aligned} \tag{3.7}$$

Remark 3.5. *Although the random variable $Q_3(\square,\square')$ does depend on the ordering of $\square$ and $\square'$, $\cosh Q_3(\square,\square')$ does not.*

Proposition 3.6. *For every $a_0 > 0$, there exist $\beta_0 = \beta_0(a_0,\eta) > 0$ and $C_Q = C_Q(a_0,\beta_0,\eta) > 0$ such that the following holds uniformly for all $\beta > \beta_0$, $a_1, a_2, a_3 \in \mathbb{R}$ such that $|a_i| \leqslant a_0$, and $N \in 4\mathbb{N}$: let $B_1, B_2 \subset \mathbb{B}_N$ and B_3 a set of unordered pairs of nearest-neighbour blocks in $\mathbb{B}_N$. Then,*

$$\Big\langle \prod_{i=1}^{3} \cosh\big(a_i Q_i(B_i)\big) \Big\rangle_{\beta,N} \leqslant e^{C_Q(|B_1|+|B_2|+|B_3|)} \tag{3.8}$$

where $|B_3|$ is given by the number of pairs in B_3.

Proof. Proposition 3.6 is established in Section 6.3, but its proof takes up most of this article. The overall strategy is as follows: the crucial first step is to obtain upper and lower bounds on the free energy $-\log \mathscr{Z}_{\beta,N}$ that are uniform in β and extensive in the *volume*, N^3. We then build on this analysis to obtain upper bounds on expectations of the form $\langle \exp Q \rangle_{\beta,N}$ that are uniform in β and extensive in N^3. Here, Q is a placeholder for random variables that are derived from the Q_i's, but that are supported on the whole of $\mathbb{T}_N$ rather than arbitrary unions of blocks. This is all done in Section 5, where the key results are Propositions 5.3 and 5.1, within the framework developed in Section 4.

The next step in the proof is to use the chessboard estimates of Proposition 6.5 (which requires $N \in 4\mathbb{N}$) to bound the lefthand side of (3.8) in terms of $|B_1| + |B_2| + |B_3|$ products of expectations of the form $\langle \exp Q \rangle_{\beta,N}^{\frac{1}{N^3}}$. Applying the results of Section 5 then completes the proof. □

Key features of the estimate (3.8) used in the proof of Proposition 3.2 are that it is *uniform in β* and *extensive in the support of the Q_i's.*

3.3.1 Proof of the Proposition 3.2 assuming Proposition 3.6

We first show that we can reduce to the case where B contains no $*$-neighbours, which simplifies the combinatorics later on. Identify $\mathbb{B}_N$ with a subset of $\mathbb{Z}^3$. For every $e_l \in \{-1, 0, 1\}^3$, let $\mathbb{Z}^3_l = e_l + (3\mathbb{Z})^3$. There are 27 such sub-lattices which we order according to $l \in \{1, \ldots, 27\}$. Note that $\mathbb{Z}^3 = \bigcup_{l=1}^{27} \mathbb{Z}^3_l$. Let $\mathbb{B}^l_N = \mathbb{B}_N \cap \mathbb{Z}^3_l$. Each $*$-connected ball in $\mathbb{B}_N$ contains at most one block from each of these $\mathbb{B}^l_N$.

Assume that (3.1) has been established for sets with no $*$-neighbours with constant C'_P. Then, by Hölder's inequality,

$$\begin{aligned}\nu_{\beta,N}(\sigma(\square) = 0 \text{ for all } \square \in B) &= \Big\langle \prod_{\square \in B} \mathbf{1}_{\square \in \mathscr{B}} \Big\rangle_{\beta,N} \\ &\leqslant \prod_{l=1}^{27} \Big\langle \prod_{\square \in B \cap \mathbb{B}^l_N} \mathbf{1}_{\square \in \mathscr{B}} \Big\rangle_{\beta,N}^{\frac{1}{27}} \\ &\leqslant e^{-\frac{C'_P}{27}|B|}\end{aligned} \tag{3.9}$$

thereby establishing (3.1) with $C_P = \frac{C'_P}{27}$.

Now assume that B contains no $*$-neighbours. Fix any $A \subset B$. Let $\mathrm{B}^*(A) = \bigcup_{\square \in A} \mathrm{B}^*(\square)$ and let $\mathrm{B}^*_{\mathrm{nn}}(A) = \bigcup_{\square \in A} \mathrm{B}^*_{\mathrm{nn}}(\square)$. By our assumption, A contains no $*$-neighbours. Hence, for any $\square' \in \mathrm{B}^*(A)$ there exists a unique $\square \in A$ such that $\square' \in \mathrm{B}^*(\square)$; we define the root of $\square'$ to be $\square$. Similarly, for any $\{\square', \square''\} \in \mathrm{B}^*_{\mathrm{nn}}(A)$ there exists a unique $\square \in A$ such that $\{\square', \square''\} \in \mathrm{B}^*_{\mathrm{nn}}(\square)$; we define the root of $\{\square', \square''\}$ to be $\square$. Note that the definition of root is A-dependent in both cases.

By Lemma 3.4, there exists C_δ such that

$$\begin{aligned}\prod_{\square \in B} \mathbf{1}_{\square \in \mathscr{B}} &= \sum_{A \subset B} \Big(\prod_{\square \in A} \mathbf{1}_{\square \in \mathscr{B}_F}\Big)\Big(\prod_{\square \in B \setminus A} \mathbf{1}_{\square \in \mathscr{B}_I}\Big) \\ &\leqslant 2^{|B|} e^{-C_\delta \sqrt{\beta}|B|} \sum_{A \subset B} \Big(\prod_{\square \in A} \sum_{\square' \in \mathrm{B}^*(\square)} \big(\cosh Q_1(\square') + \cosh Q_2(\square')\big)\Big) \\ &\quad \times \Big(\prod_{\square \in B \setminus A} \sum_{\{\square', \square''\} \in \mathrm{B}^*_{\mathrm{nn}}(\square)} \cosh Q_3(\square', \square'')\Big) \\ &= 2^{|B|} e^{-C_\delta \sqrt{\beta}|B|} \sum_{A \subset B} \sum_{A_1, A_2, A_3} \cosh Q_1(A_1) \cosh Q_2(A_2) \cosh Q_3(A_3)\end{aligned} \tag{3.10}$$

where the last sum is over all $A_1, A_2 \subset \mathrm{B}^*(A)$ and $A_3 \subset \mathrm{B}^*_{\mathrm{nn}}(B \setminus A)$ such that: no two blocks in $A_1 \cup A_2$ share a root, and no two pairs of blocks in A_3 share a root; and, $|A_1| + |A_2| = |A|$ and $|A_3| = |B \setminus A|$. We note that there are $(2 \cdot 27)^{|A|} = 54^{|A|}$ possible A_1 and A_2, and $54^{|B \setminus A|}$ possible A_3.

By Proposition 3.6, there exists C_Q such that, after taking expectations in (3.10) and using that $|A| + |B\backslash A| = |B|$, we obtain

$$\nu_{\beta,N}(\sigma(\square) = 0 \text{ for all } \square \in B) \leqslant 2^{|B|} e^{-C_\delta \sqrt{\beta}|B|} 2^{|B|} 54^{|B|} e^{C_Q|B|}.$$

Thus, choosing

$$\sqrt{\beta} > \frac{4\log 2 + 2\log 54 + 2C_Q}{C_\delta}$$

yields (3.1) with $C_P = \frac{C_\delta}{2}$. This completes the proof.

3.4 Exchanging the block averaged field for the phase label

We now show that Propositions 3.2 and 3.6 allow one to reduce the problem of analysing the block averaged field to that of analysing the phase label. The main difficulty here is dealing with *large fields*, i.e. those $\vec{\phi}$ for which $\int_{\mathcal{B}} |\vec{\phi}|$ is large.

Proposition 3.7. *Let $\delta, \delta' \in (0,1)$ satisfy $\delta' \leqslant \frac{\delta}{2}$. Then, there exists $\beta_0 = \beta_0(\delta, \eta) > 0$, $C = C(\delta, \beta_0, \eta) > 0$ and $N_0 = N_0(\delta) > 0$ such that, for all $\beta > \beta_0$ and $N \in 4\mathbb{N}$ with $N > N_0$,*

$$\frac{1}{N^3} \log \nu_{\beta,N}\Big(\int_{\mathbb{T}_N} \Big|\sigma - \vec{\phi}\Big| dx > \delta\sqrt{\beta}N^3 \Big) \leqslant -C\sqrt{\beta} \tag{3.11}$$

where σ is the phase label of precision $\delta' \leqslant \frac{\delta}{2}$.

Proof. Observe that

$$\begin{aligned} &\nu_{\beta,N}\left(\int_{\mathbb{T}_N} |\sigma - \vec{\phi}| dx > \delta\sqrt{\beta}N^3 \right) \\ &\qquad \leqslant \nu_{\beta,N}\left(\int_{\mathbb{T}_N} |\sigma - \vec{\phi}| dx > \delta\sqrt{\beta}N^3, |\mathcal{B}| < \frac{\delta}{8}N^3 \right) \\ &\qquad\quad + \nu_{\beta,N}\left(|\mathcal{B}| \geqslant \frac{\delta}{8}N^3 \right). \end{aligned} \tag{3.12}$$

By Proposition 3.2, there exists $\beta_0 > 0$ and $C_P > 0$ such that, for $\sqrt{\beta} >$

$\max\left(\sqrt{\beta_0}, \frac{16\log 2}{C_P\delta}\right)$,

$$\begin{aligned}
\nu_{\beta,N}\left(|\mathscr{B}| \geqslant \frac{\delta}{8}N^3\right) &\leqslant \sum_{m=\lceil \frac{\delta}{8}N^3\rceil}^{N^3} \nu_{\beta,N}(|\mathscr{B}| = m) \\
&\leqslant \sum_{m=\lceil \frac{\delta}{8}N^3\rceil}^{N^3} \binom{N^3}{m} e^{-C_P\sqrt{\beta}m} \\
&\leqslant 2^{N^3} e^{-\frac{C_P\delta}{8}\sqrt{\beta}N^3} \\
&\leqslant e^{-\frac{C_P\delta}{16}\sqrt{\beta}N^3}.
\end{aligned} \tag{3.13}$$

Now consider the first term on the right hand side of (3.12). We decompose one step further:

$$\nu_{\beta,N}\left(\int_{\mathbb{T}_N} |\sigma - \vec{\phi}|dx > \delta\sqrt{\beta}N^3, |\mathscr{B}| < \frac{\delta}{8}N^3\right) \leqslant \nu_{\beta,N}(T_1) + \nu_{\beta,N}(T_2)$$

where

$$\begin{aligned}
T_1 &= \left\{\int_{\mathbb{T}_N} |\sigma - \vec{\phi}|dx > \delta\sqrt{\beta}N^3, \int_{\mathscr{B}} |\vec{\phi}|dx \leqslant \frac{\delta}{2}\sqrt{\beta}N^3\right\} \\
T_2 &= \left\{|\mathscr{B}| < \frac{\delta}{8}N^3, \int_{\mathscr{B}} |\vec{\phi}|dx > \frac{\delta}{2}\sqrt{\beta}N^3\right\}.
\end{aligned}$$

We show that $T_1 = \varnothing$ and that

$$\nu_{\beta,N}(T_2) \leqslant e^{-C\sqrt{\beta}N^3} \tag{3.14}$$

for some constant $C = C(\delta) > 0$ and for β sufficiently large. Combining these estimates with (3.13) completes the proof.

First, we treat T_1. On good blocks $|\phi(\square) - \sigma|$ is bounded by the $\sqrt{\beta}$ multiplied by the precision of the phase label ($\delta' \leqslant \frac{\delta}{2}$ in this instance) and $\sigma = 0$ on bad blocks. Therefore, on the set $\left\{\int_{\mathscr{B}} |\vec{\phi}|dx \leqslant \frac{\delta}{2}\sqrt{\beta}N^3\right\}$, we have:

$$\begin{aligned}
\int_{\mathbb{T}_N} |\sigma - \vec{\phi}|dx &= \int_{\mathscr{P}_G\cup\mathscr{M}_G} |\sigma - \vec{\phi}|dx + \int_{\mathscr{B}} |\sigma - \vec{\phi}|dx \\
&\leqslant \frac{\delta}{2}\sqrt{\beta}(|\mathscr{P}_G| + |\mathscr{M}_G|) + \int_{\mathscr{B}} |\vec{\phi}|dx \\
&\leqslant \delta\sqrt{\beta}N^3
\end{aligned}$$

which shows that the first condition in T_1 is inconsistent with the second, so $T_1 = \emptyset$.

We turn our attention to T_2. Fix $B \subset \mathbb{B}_N$. By Chebyschev's inequality, Young's inequality, and Proposition 3.6, there exists $\beta_0 > 0$ and $C_Q > 0$ such that, for $\beta > \beta_0$,

$$\begin{aligned}
\nu_{\beta,N}\left(\int_B |\vec{\phi}| > \frac{\delta}{2}\sqrt{\beta}N^3\right) &\leqslant e^{-\frac{\delta}{2}\sqrt{\beta}N^3}\langle e^{\sum_{\square\in B}|\phi(\square)|}\rangle_{\beta,N} \\
&\leqslant e^{-\frac{\delta}{2}\sqrt{\beta}N^3} e^{\frac{\sqrt{\beta}}{2}|B|}\langle e^{\frac{1}{2\sqrt{\beta}}\sum_{\square\in B}\phi(\square)^2}\rangle_{\beta,N} \\
&\leqslant e^{-\frac{\delta}{2}\sqrt{\beta}N^3} e^{\sqrt{\beta}|B|}\langle e^{\frac{1}{2\sqrt{\beta}}\sum_{\square\in B}(\phi(\square)^2-\beta)}\rangle_{\beta,N} \\
&\leqslant e^{-\frac{\delta}{2}\sqrt{\beta}N^3} e^{\sqrt{\beta}|B|}\langle \prod_{\square\in B} e^{-\frac{1}{2}Q_1(\square)} e^{-\frac{1}{2}Q_2(\square)}\rangle_{\beta,N} \\
&\leqslant e^{-\frac{\delta}{2}\sqrt{\beta}N^3} e^{\sqrt{\beta}|B|} 2^{|B|}\left\langle \cosh\left(\frac{1}{2}Q_1(B)\right)\cosh\left(\frac{1}{2}Q_2(B)\right)\right\rangle_{\beta,N} \\
&\leqslant e^{-\frac{\delta}{2}\sqrt{\beta}N^3} e^{\sqrt{\beta}|B|} 2^{|B|} e^{C_Q|B|}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nu_{\beta,N}(T_2) &\leqslant \sum_{m=1}^{\lfloor\frac{\delta}{8}N^3\rfloor}\sum_{B:|B|=m}\nu_{\beta,N}\left(\int_B|\vec{\phi}|dx > \frac{\delta}{2}\sqrt{\beta}N^3\right) \\
&\leqslant e^{-\frac{\delta}{2}\sqrt{\beta}N^3}\sum_{m=1}^{\lfloor\frac{\delta}{8}N^3\rfloor}\binom{N^3}{m} e^{\sqrt{\beta}m} e^{(C_Q+\log 2)m} \\
&\leqslant e^{-\frac{\delta}{2}\sqrt{\beta}N^3} 2^{N^3} e^{\frac{\delta}{8}\sqrt{\beta}N^3} e^{\frac{(C_Q+\log 2)\delta}{8}N^3} \\
&= e^{\left(-\frac{3\delta}{8}\sqrt{\beta}+\log 2+\frac{(C_Q+\log 2)\delta}{8}\right)N^3}.
\end{aligned} \tag{3.15}$$

Taking

$$\sqrt{\beta} > \frac{16\log 2}{3\delta} + \frac{2}{3}(C_Q + \log 2)$$

yields (3.14) with $C = \frac{3\delta}{16}$. □

3.5 Proof of the main result

Adapting an argument from [Bod02], we reduce the proof of Theorem 1.2 to bounding the probability that $\vec{\phi}$ is far from $\pm\sqrt{\beta}$-valued functions on $\mathbb{B}_N$ whose boundary (between regions of opposite spins) is of certain fixed area. Proposition 3.7 then allows us to go from analysing $\vec{\phi}$ to the phase label, for which we use existing results from [BIV00].

For any $B \subset \mathbb{B}_N$, let ∂B denotes its boundary, which is given by the union of faces of blocks in B. Let $|\partial B| = \int_{\partial B} 1 ds(x)$, where $ds(x)$ is the 2D Hausdorff measure (normalised so that faces have unit area). Thus, $|\partial B|$ is the number of faces in ∂B.

For any $a > 0$, let C_a be the set of functions $\vec{f} \in \{\pm 1\}^{\mathbb{B}_N}$ such that $|\partial\{\vec{f} = +1\}| \leqslant aN^2$. For any $\delta > 0$, let $\mathfrak{B}(C_a, \delta)$ be the set of integrable functions g on $\mathbb{T}_N$ such that there exists $\vec{f} \in C_a$ that satisfies $\int_{\mathbb{T}_N} |g - \vec{f}| dx \leqslant \delta N^3$.

Proposition 3.8. *Let $\delta, \delta' \in (0,1)$ satisfy $\delta' \leqslant \delta$. Then, there exists $\beta_0 = \beta_0(\delta, \eta) > 0$ and $C = C(\delta, \beta_0, \eta) > 0$ such that, for all $\beta > \beta_0$, the following estimate holds: for all $a > 0$, there exists $N_0 = N_0(a, \delta) \geqslant 4$ such that, for all $N > N_0$ dyadic,*

$$\frac{1}{N^2} \log \nu_{\beta,N}\Big(\frac{1}{\sqrt{\beta}}\sigma \notin \mathfrak{B}(C_a, \delta)\Big) \leqslant -C\sqrt{\beta} a$$

where σ is the phase label of precision δ'.

Proof. See [BIV00, Theorem 2.2.1] where Proposition 3.8 is proven for a more general class of phase labels that satisfy a Peierls' type estimate such as the one in Proposition 3.2. We give a self-contained proof for our setting in Section 3.6. □

The following lemma is our main geometric tool. It is a weak form of the isoperimetric inequality on $\mathbb{T}_N$, although it can be reformulated in arbitrary dimension. Its proof is a standard application of Sobolev's inequality and we include it for the reader's convenience.

Lemma 3.9. *There exists $C_I > 0$ such that the following estimate holds for every $N \in \mathbb{N}$:*

$$\min(|\{\vec{f} = 1\}|, |\{\vec{f} = -1\}|) \leqslant C_I |\partial\{\vec{f} = 1\}|^{\frac{3}{2}}$$

for every $\vec{f} \in \{\pm 1\}^{\mathbb{B}_N}$.

Proof. Let $\theta \in C_c^\infty(\mathbb{R}^3)$ be rotationally symmetric with $\int_{\mathbb{R}^3} \theta dx = 1$. By Sobolev's inequality, there exists C such that, for every ε,

$$\int_{\mathbb{T}_N} |f_\varepsilon - c_\varepsilon|^{\frac{3}{2}} dx \leqslant C\Big(\int_{\mathbb{T}_N} |\nabla f_\varepsilon| dx\Big)^{\frac{3}{2}} \tag{3.16}$$

where $\vec{f_\varepsilon} = \vec{f} * \varepsilon^{-3}\theta(\varepsilon^{-1}\cdot)$ and $c_\varepsilon = \frac{1}{N^3}\int_{\mathbb{T}_N} \vec{f_\varepsilon} dx$. Note that C is independent of N by scaling.

Letting $\varepsilon \to 0$ in the left hand side of (3.16), we obtain

$$\int_{\mathbb{T}_N} |\vec{f_\varepsilon} - c_\varepsilon|^{\frac{3}{2}} dx \to \int_{\mathbb{T}_N} |\vec{f} - c|^{\frac{3}{2}} dx \tag{3.17}$$

where $c = \frac{|\{\vec{f}=1\}|-|\{\vec{f}=-1\}|}{N^3}$. Note that $c \in [-1,1]$.

Without loss of generality, assume $c \geqslant 0$. This implies that $|\{\vec{f} = 1\}| \geqslant \{\vec{f} = -1\}|$. Then, evaluating the integral on the righthand side of (3.17), we find that

$$\begin{aligned}\int_{\mathbb{T}_N} |\vec{f} - c|^{\frac{3}{2}} dx &= (1-c)^{\frac{3}{2}} |\{\vec{f} = 1\}| + (1+c)^{\frac{3}{2}} |\{\vec{f} = -1\}| \\ &= (1-c)^{\frac{3}{2}} cN^3 + \Big((1-c)^{\frac{3}{2}} + (1+c)^{\frac{3}{2}}\Big) |\{\vec{f} = -1\}| \\ &\geqslant 2|\{\vec{f} = -1\}|\end{aligned} \tag{3.18}$$

where we have used that the function

$$c \mapsto (1-c)^{\frac{3}{2}} + (1+c)^{\frac{3}{2}}$$

has minimum at $c = 0$ on the interval $[0,1]$.

For the term on the right hand side of (3.16), using duality we obtain

$$\int_{\mathbb{T}_N} |\nabla \vec{f}_\varepsilon| dx = \sup_{\mathbf{g} \in C^\infty(\mathbb{T}_N, \mathbb{R}^3) : |\mathbf{g}|_\infty \leqslant 1} \Big| \int_{\mathbb{T}_N} \nabla \vec{f}_\varepsilon \cdot \mathbf{g} dx \Big| \tag{3.19}$$

where $| \cdot |_\infty$ denotes the supremum norm on $C^\infty(\mathbb{T}_N, \mathbb{R}^3)$.

For any such $\mathbf{g}$, using integration by parts and commuting the convolution with differentiation,

$$\Big| \int_{\mathbb{T}_N} \nabla \vec{f}_\varepsilon \mathbf{g} dx \Big| = \Big| \int_{\mathbb{T}_N} \vec{f}_\varepsilon \nabla \cdot \mathbf{g} dx \Big| = \Big| \int_{\mathbb{T}_N} \vec{f} \nabla \cdot \mathbf{g}_\varepsilon dx \Big| \tag{3.20}$$

where the $\mathbf{g}_\varepsilon$ is interpreted as convolving each component of $\mathbf{g}$ with $\varepsilon^{-3}\theta(\varepsilon^{-1}\cdot)$ separately.

Hence, by the divergence theorem, Young's inequality for convolutions, and using the supremum norm bound on $\mathbf{g}$,

$$(3.20) = 2 \Big| \int_{\partial\{\vec{f}=1\}} \mathbf{g}_\varepsilon \cdot \hat{n} ds(x) \Big| \leqslant 2|\partial\{\vec{f} = 1\}| \tag{3.21}$$

where $\hat{n}$ denotes the unit normal to $\partial\{\vec{f} = 1\}$ pointing into $\{\vec{f} = -1\}$.

Inserting (3.21) in (3.19) implies that, for any ε,

$$\int_{\mathbb{T}_N} |\nabla \vec{f}_\varepsilon| dx \leqslant 2|\partial\{\vec{f} = 1\}|. \tag{3.22}$$

Thus, by inserting (3.22), (3.17) and (3.18) into (3.16), we obtain

$$|\{\vec{f} = -1\}| \leqslant \sqrt{2} C |\partial\{\vec{f} = 1\}|^{\frac{3}{2}}.$$

□

Proof of Theorem 1.2. Let $\zeta \in (0,1)$. Choose $a > 0$ and $\delta \in (0,1)$ such that

$$1 - 2C_I a^{\frac{3}{2}} - \delta = \zeta \tag{3.23}$$

where C_I is the same constant as in Lemma 3.9. We first show that

$$\{\mathfrak{m}_N(\phi) \in (-\zeta\sqrt{\beta}, \zeta\sqrt{\beta})\} \subset \Big\{\frac{1}{\sqrt{\beta}}\vec{\phi} \notin \mathfrak{B}(C_a, \delta)\Big\}. \tag{3.24}$$

Assume $\frac{1}{\sqrt{\beta}}\vec{\phi} \in \mathfrak{B}(C_a, \delta)$. Then, there exists $\vec{f} \in C_a$ such that

$$\int_{\mathbb{T}_N} \Big|\frac{1}{\sqrt{\beta}}\vec{\phi} - \vec{f}\Big| dx \leqslant \delta N^3.$$

This implies

$$\Bigg|\Big|\int_{\mathbb{T}_N} \frac{1}{\sqrt{\beta}}\vec{\phi} dx\Big| - \Big|\int_{\mathbb{T}_N} \vec{f} dx\Big|\Bigg| \leqslant \delta N^3$$

from which we deduce, together with Lemma 3.9,

$$\begin{aligned}\Big|\frac{1}{\sqrt{\beta}}\mathfrak{m}_N(\phi)\Big| &\geqslant 1 - \frac{2\min\Big(|\{\vec{f} = +1\}|, |\{\vec{f} = -1\}|\Big)}{N^3} - \delta. \\ &\geqslant 1 - \frac{2C_I|\partial\{\vec{f} = +1\}|^{\frac{3}{2}}}{N^3} - \delta.\end{aligned}$$

Since $\vec{f} \in C_a$, we obtain

$$|\mathfrak{m}_N(\phi)| \geqslant \sqrt{\beta}(1 - 2C_I a^{\frac{3}{2}} - \delta) = \zeta\sqrt{\beta}$$

by (3.23).

Hence,

$$\Big\{\frac{1}{\sqrt{\beta}}\vec{\phi} \in \mathfrak{B}(C_a, \delta)\Big\} \subset \{|\mathfrak{m}_N(\phi)| \geqslant \zeta\sqrt{\beta}\}.$$

Taking complements establishes (3.24).

Now let σ be the phase label of precision $\frac{\delta}{4}$. Note that

$$\Big\{\frac{1}{\sqrt{\beta}}\vec{\phi} \notin \mathfrak{B}(C_a, \delta)\Big\} \subset \Big\{\frac{1}{\sqrt{\beta}}\sigma \notin \mathfrak{B}\Big(C_a, \frac{\delta}{2}\Big)\Big\} \bigcup \Big\{\int_{\mathbb{T}_N} |\vec{\phi} - \sigma| dx > \frac{\delta}{2}\sqrt{\beta}N^3\Big\}.$$

Applying Proposition 3.7, Proposition 3.8, and using (3.24) finishes the proof. □

3.6 Proof of Proposition 3.8

For any $B \subset \mathbb{B}_N$, let $\partial^* B$ be the set of blocks in B with $*$-neighbours in $\mathbb{T}_N \backslash B$. Note that this is not the same as ∂B, which was defined earlier. Let $\mathcal{D}$ be the set of $*$-connected components of $\partial^*(\mathbb{T}_N \backslash \mathcal{M}_G)$. We call this the set of *defects*. Necessarily, any $\Gamma \in \mathcal{D}$ satisfies $\Gamma \subset \mathcal{B}$.

Fix $\gamma \in (0,1)$. Let $\mathcal{D}^\gamma \subset \mathcal{D}$ be the set of $\Gamma \in \mathcal{D}$ such that $|\Gamma| \leqslant 6N^\gamma$. The elements of $\mathcal{D}^\gamma$ are called γ-small defects and the elements of $\mathcal{D} \backslash \mathcal{D}^\gamma$ are called γ-large defects.

Take any $\Gamma \in \mathcal{D}^\gamma$. Recall that we identify Γ with the subset of $\mathbb{T}_N$ given by the union of blocks in Γ. Write Cl(Γ) for its closure in $\mathbb{T}_N$. The condition $\gamma < 1$ ensures that, provided N is taken sufficiently large depending on γ, any $\Gamma \in \mathcal{D}^\gamma$ is contained in a (translate of a) sphere of radius $\frac{N}{4}$ in $\mathbb{T}_N$. Let Ext(Γ) be the unique connected component of $\mathbb{T}_N \backslash \text{Cl}(\Gamma)$ that intersects with the complement of this sphere. Let Int$(\Gamma) = \mathbb{T}_N \backslash \text{Ext}(\Gamma)$. We identify Ext$(\Gamma)$ and Int(Γ) with their representations as subsets of $\mathbb{B}_N$. Note that $\Gamma \subset \text{Int}(\Gamma)$ and generically the inclusion strict, e.g. when Γ encloses a region.

Let $\mathcal{D}^{\gamma,\max}$ be the set of $\Gamma \in \mathcal{D}^\gamma$ such that $\Gamma \bigcap \text{Int}(\tilde{\Gamma}) = \varnothing$ for any $\tilde{\Gamma} \in \mathcal{D}^\gamma \backslash \{\Gamma\}$. In other words, $\mathcal{D}^{\gamma,\max}$ is the set of γ-small defects that are not contained in the interior of any other γ-small defects, and we call these maximal γ-small defects.

We define two events, one corresponds to the total surface area of γ-large defects being small and the other corresponding to the total volume contained within maximal γ-small defects being small. Let

$$S_1 = \left\{ \sum_{\Gamma \in \mathcal{D} \backslash \mathcal{D}^\gamma} |\Gamma| \leqslant \frac{a}{6} N^2 \right\}$$

$$S_2 = \left\{ \sum_{\Gamma \in \mathcal{D}^{\gamma,\max}} |\text{Int}(\Gamma)| \leqslant \frac{\delta}{4} N^3 \right\}.$$

We now show that for $\phi \in S_1 \cap S_2 \cap \{|\mathcal{B}| < \frac{\delta}{2} N^3\}$, we have $\frac{1}{\sqrt{\beta}}\sigma \in \mathfrak{B}(C_a, \delta)$.

We obtain a $\pm\sqrt{\beta}$-valued spin configuration from σ by erasing all γ-small defects in two steps: First, we reset the values on bad blocks to $\sqrt{\beta}$. Define $\sigma_1 \in \{\pm\sqrt{\beta}\}^{\mathbb{B}_N}$ by $\sigma_1(\square) = \sqrt{\beta}$ if $\square \in \mathcal{B}$, otherwise $\sigma_1(\square) = \sigma(\square)$. Second, define $\sigma_2 \in \{\pm\sqrt{\beta}\}^{\mathbb{B}_N}$ as follows: Given $\square \in \text{Int}(\Gamma)$ for some $\Gamma \in \mathcal{D}^{\gamma,\max}$, let $\sigma_2(\square) = \sigma_1(\tilde{\square})$, where $\tilde{\square}$ is any block in Ext(Γ) that is $*$-neighbours with a block in Γ. Note that the second step is well-defined since the first step ensures that every block in Ext(Γ) that is $*$-neighbours with Γ has the same value. See Figure 2 for an example of this procedure.

From the definition of S_1 and using that the factor 6 in the definition of γ-small defects accounts for the discrepancy between $|\partial \cdot|$ and $|\partial^* \cdot|$,

$$|\partial\{\sigma_2 = +\sqrt{\beta}\}| \leqslant aN^2$$

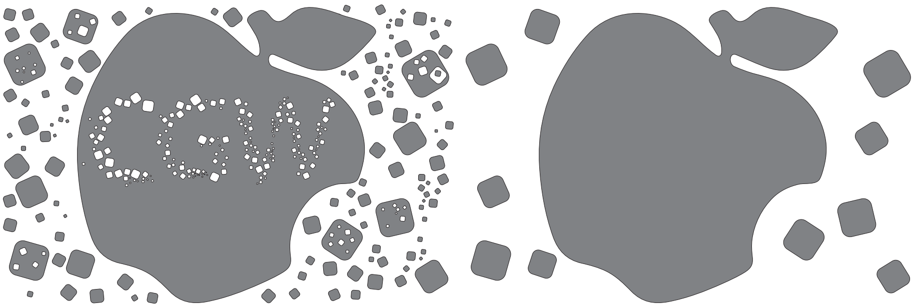


Figure 2: An example of the σ to σ_2 procedure (left to right). Image courtesy of J. N. Gunaratnam

yielding $\frac{1}{\sqrt{\beta}}\sigma_2 \in C_a$. Then, from the definition of S_2 and using the smallness assumption on the number of bad blocks,

$$\int_{\mathbb{T}_N} \frac{1}{\sqrt{\beta}}\Big|\sigma - \sigma_2\Big| dx \leqslant 2 \sum_{\Gamma \in \mathfrak{D}^{\gamma,\max}} |\text{Int}(\Gamma)| + |\mathscr{B}| < 2\frac{\delta}{4}N^3 + \frac{\delta}{2}N^3 < \delta N^3$$

which establishes that $\frac{1}{\sqrt{\beta}}\sigma \in \mathfrak{B}(C_a, \delta)$.

We deduce that the event $\left\{\frac{1}{\sqrt{\beta}}\sigma \notin \mathfrak{B}(C_a,\delta)\right\}$ necessarily implies one of three things: either there are many bad blocks; or, the total surface area of γ-large defects is large; or, the density of γ-small defects is high. That is,

$$\begin{aligned} &\nu_{\beta,N}\Big(\frac{1}{\sqrt{\beta}}\sigma \notin \mathfrak{B}(C_a,\delta)\Big) \\ &\qquad\qquad \leqslant \nu_{\beta,N}\Big(|\mathscr{B}| \geqslant \frac{\delta}{2}N^3\Big) + \nu_{\beta,N}(S_1^c) + \nu_{\beta,N}(S_2^c). \end{aligned} \tag{3.25}$$

Proposition 3.2 gives control on the first event. The other two are controlled by the following lemmas.

Lemma 3.10. *Let* $\gamma, \delta \in (0,1)$. *Then, there exists* $\beta_0 = \beta_0(\gamma,\delta,\eta) > 0$ *and* $C = C(\gamma,\delta,\beta_0,\eta) > 0$ *such that, for all* $\beta > \beta_0$, *the following holds: for any* $a > 0$, *there exists* $N_0 = N_0(\gamma, a) > 0$ *such that, for any* $N \in 4\mathbb{N}$ *with* $N > N_0$,

$$\frac{1}{N^2}\log \nu_{\beta,N}\left(\sum_{\Gamma \in \mathfrak{D}\setminus\mathfrak{D}^\gamma} |\Gamma| > aN^2\right) \leqslant -C\sqrt{\beta}\Big(a + \frac{N^\gamma}{N^2}\Big)$$

where the underlying phase label is of precision δ.

Proof. We give a proof based on arguments from [DKS92, Theorem 6.1] in Section 3.6.1. □

Lemma 3.11. *Let $\gamma, \delta, \delta' \in (0,1)$. Then, there exists $\beta_0 = \beta_0(\gamma, \delta, \delta', \eta) > 0$, $C = C(\gamma, \delta, \delta', \beta_0, \eta) > 0$ and $N_0 = N_0(\gamma, \delta) \geqslant 4$ such that, for all $\beta > \beta_0$ and $N > N_0$ dyadic,*

$$\frac{1}{N^2} \log \nu_{\beta,N}\Bigg(\sum_{\Gamma \in \mathscr{D}^{\gamma,\max}} |\mathrm{Int}(\Gamma)| > \delta N^3 \Bigg) \leqslant -C\sqrt{\beta}\frac{N}{N^{3\gamma}}$$

where the underlying phase label is of precision δ'.

Proof. See [BIV00, Section 5.1.3] for a proof in a more general setting. We give an alternative proof in Section 3.6.2 that avoids the use of techniques from percolation theory. □

As in (3.13), by Proposition 3.2 there exists $C_P > 0$ such that

$$\nu_{\beta,N}(|\mathscr{B}| \geqslant \delta N^3) \leqslant e^{-\frac{C_P\delta}{4}\sqrt{\beta}N^3} \tag{3.26}$$

provided $\sqrt{\beta} > \frac{4\log 2}{\delta C_P}$.

Therefore, from (3.25), (3.26), Lemma 3.10 and Lemma 3.11, there exists $C > 0$ such that

$$\frac{1}{N^2} \log \nu_{\beta,N}\Big(\sigma \notin \mathfrak{B}(C_a, \delta)\Big) \leqslant -C\sqrt{\beta}\min\Big(N, a + \frac{N^\gamma}{N^2}, \frac{N}{N^{3\gamma}}\Big).$$

Taking $\gamma < \frac{1}{3}$ and N sufficiently large completes the proof. All that remains is to show Lemmas 3.10 and 3.11.

3.6.1 Proof of Lemma 3.10

By a union bound

$$\begin{aligned}\nu_{\beta,N}\Bigg(\sum_{\Gamma \in \mathscr{D}\setminus\mathscr{D}^\gamma} |\Gamma| > aN^2 \Bigg) &= \sum_{\substack{\{\Gamma_i\}:|\Gamma_i|>N^\gamma \\ \sum_i |\Gamma_i| > aN^2}} \nu_{\beta,N}\Big(\mathscr{D}\setminus\mathscr{D}^\gamma = \{\Gamma_i\}\Big) \\ &\leqslant \sum_{\substack{\{\Gamma_i\}:|\Gamma_i|>N^\gamma \\ \sum_i |\Gamma_i| > aN^2}} \nu_{\beta,N}\Big(\Gamma_i \subset \mathscr{B} \text{ for all } \Gamma_i \in \{\Gamma_i\}\Big),\end{aligned} \tag{3.27}$$

where$\{\Gamma_i\}$ refers to a non-empty set of distinct $*$-connected subsets of $\mathbb{B}_N$.

By Proposition 3.2 there exists C_P such that, for any $\{\Gamma_i\}$,

$$\nu_{\beta,N}\Big(\Gamma_i \subset \mathscr{B} \text{ for all } \Gamma_i \in \{\Gamma_i\}\Big) = \Big\langle \prod_{\Gamma_i \in \{\Gamma_i\}} \prod_{\square \in \Gamma_i} \mathbf{1}_{\square \in \mathscr{B}} \Big\rangle_{\beta,N}$$

$$\leqslant e^{-C_P\sqrt{\beta}\sum|\Gamma_i|}.$$

Inserting this into (3.27) and using the trivial estimate $\sum|\Gamma_i| \geqslant \frac{1}{2}aN^2 + \frac{1}{2}\sum|\Gamma_i|$,

$$\begin{aligned}
\nu_{\beta,N}\left(\sum_{\Gamma\in\mathcal{D}\setminus\mathcal{D}^\gamma}|\Gamma| > aN^2\right) &\leqslant \sum_{\substack{\{\Gamma_i\}:|\Gamma_i|>N^\gamma\\ \sum_i|\Gamma_i|>aN^2}} e^{-C_P\sqrt{\beta}\sum|\Gamma_i|}\\
&\leqslant e^{-\frac{C_P}{2}\sqrt{\beta}aN^2}\sum_{\{\Gamma_i\}:|\Gamma_i|>N^\gamma} e^{-\frac{C_P}{2}\sqrt{\beta}\sum|\Gamma_i|} \qquad (3.28)\\
&= e^{-\frac{C_P}{2}\sqrt{\beta}aN^2}\sum_{\{\Gamma_i\}:|\Gamma_i|>N^\gamma}\prod_{\Gamma_i\in\{\Gamma_i\}} e^{-\frac{C_P}{2}\sqrt{\beta}|\Gamma_i|}.
\end{aligned}$$

Summing first over the number of elements in $\{\Gamma_i\}$ and then the number of $*$-connected regions containing a fixed number of blocks,

$$\begin{aligned}
\sum_{\substack{\{\Gamma_i\}\\|\Gamma_i|>N^\gamma}}\prod_{\Gamma_i\in\{\Gamma_i\}} e^{-\frac{C_P}{2}\sqrt{\beta}|\Gamma_i|} &= \sum_{m=1}^{\infty}\sum_{\{\Gamma_i\}_{i=1}^m:|\Gamma_i|>N^\gamma}\prod_{i=1}^{m} e^{-\frac{C_P}{2}\sqrt{\beta}|\Gamma_i|}\\
&\leqslant \sum_{m=1}^{\infty}\Bigg(\sum_{\Gamma\ *\text{-connected}\ :|\Gamma|\geqslant N^\gamma} e^{-\frac{C_P}{2}\sqrt{\beta}|\Gamma|}\Bigg)^m\\
&\leqslant \sum_{m=1}^{\infty}\Bigg(\sum_{n\geqslant N^\gamma} N^3 27\cdot 26^{n-1}e^{-\frac{C_P}{2}\sqrt{\beta}n}\Bigg)^m \qquad (3.29)\\
&\leqslant \sum_{m=1}^{\infty} e^{3m\log N-\frac{C_P}{4}\sqrt{\beta}mN^\gamma}\Bigg(\sum_{n\geqslant 1} e^{-\frac{C_P}{4}\sqrt{\beta}n}\Bigg)^m\\
&\leqslant \sum_{m=1}^{\infty} e^{\left(3\log N-\frac{C_P}{4}\sqrt{\beta}N^\gamma\right)m}\\
&\leqslant e^{-\frac{C_P}{8}\sqrt{\beta}N^\gamma}\sum_{m=1}^{\infty} e^{3m\log N-\frac{C_P}{8}\sqrt{\beta}mN^\gamma}
\end{aligned}$$

provided $\sqrt{\beta} > \max\left(\frac{4\log 27}{C_P}, \frac{4\log 2}{C_P}\right) = \frac{4\log 27}{C_P}$ (note that the condition arises so that $e^{-\frac{C_P}{4}\sqrt{\beta}} < \frac{1}{2}$, so that the geometric series with this rate is bounded by 1).

For any $\gamma > 0$, the final series in (3.29) is summable provided $N^\gamma > \log N$ and $\sqrt{\beta} > \frac{24}{C_P}$, thereby finishing the proof.

3.6.2 Proof of Lemma 3.11

Choose $2N^\gamma \leqslant K \leqslant 4N^\gamma$ such that K divides N. Such a choice is possible since we take N to be a sufficiently large dyadic. Let

$$\mathbb{B}_N^K = \Big\{ \blacksquare = \prod_{i=1}^{3} [n_i, n_i + K) \subset \mathbb{T}_N : n_1, n_2, n_3 \in \{0, K, \dots, N-K\} \Big\}.$$

Elements of $\mathbb{B}_N^K$ are called K-blocks.

We say that two distinct K-blocks are $*_K$-neighbours if their corresponding midpoints are of distance at most $K\sqrt{3}$. We define the $*_K$-connected ball around $\blacksquare \in \mathbb{B}_N^K$ to be the set containing itself and its $*_K$-neighbours. As in the proof of Proposition 3.2, we can decompose $\mathbb{B}_N^K = \bigcup_{l=1}^{27} \mathbb{B}_N^{K,l}$ such that any $*_K$-connected ball in $\mathbb{B}_N^K$ contains exactly one K-block from each element of the decomposition.

For each $\blacksquare = [n_1, n_1 + K) \times [n_2, n_2 + K) \times [n_3, n_3 + K)$, distinguish the unit block $\blacksquare = [n_1, n_1 + 1) \times [n_2, n_2 + 1) \times [n_3, n_3 + 1)$. For every $h \in \{0, \dots, K-1\}^3$, let τ_h be the translation map on $\mathbb{B}_N$ induced from the translation map on $\mathbb{T}_N$. We identify $\blacksquare = \bigcup_{h \in \{0,\dots,K-1\}^3} \tau_h \blacksquare$. Denote the set of distinguished unit blocks in $\mathbb{B}_N^K$ (respectively, $\mathbb{B}_N^{K,l}$) as $\mathbb{UB}_N^K$ (respectively, $\mathbb{UB}_N^{K,l}$).

By our choice of K, Int(Γ) is entirely contained in a translation of a K-block for any $\Gamma \in \mathscr{D}^\gamma$. As a result, Int$(\Gamma)$ intersects at most one K-block in $\mathbb{B}_N^{K,l}$ for any fixed l.

Using the correspondence between K-blocks and unit blocks described above, we have

$$\begin{aligned}
\sum_{\Gamma \in \mathscr{D}^{\gamma,\max}} |\mathrm{Int}(\Gamma)| &= \sum_{\square \in \mathbb{B}_N} \sum_{\Gamma \in \mathscr{D}^{\gamma,\max}} \mathbf{1}_{\square \in \mathrm{Int}(\Gamma)} \\
&= \sum_{\blacksquare \in \mathbb{UB}_N^K} \sum_{h \in \{0,\dots,K-1\}^3} \sum_{\Gamma \in \mathscr{D}^{\gamma,\max}} \mathbf{1}_{\tau_h \blacksquare \in \mathrm{Int}(\Gamma)} \\
&= \sum_{l=1}^{27} \sum_{\blacksquare \in \mathbb{UB}_N^{K,l}} \sum_{h \in \{0,\dots,K-1\}^3} \sum_{\Gamma \in \mathscr{D}^{\gamma,\max}} \mathbf{1}_{\tau_h \blacksquare \in \mathrm{Int}(\Gamma)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\nu_{\beta,N}\Bigg(\sum_{\Gamma \in \mathscr{D}^{\gamma,\max}} |\mathrm{Int}(\Gamma)| > \delta N^3 \Bigg) \\
&\qquad \leqslant 27K^3 \max_{h,l} \nu_{\beta,N}\Bigg(\sum_{\blacksquare \in \mathbb{UB}_N^{K,l}} \sum_{\Gamma \in \mathscr{D}^{\gamma,\max}} \mathbf{1}_{\tau_h \blacksquare \in \mathrm{Int}(\Gamma)} > \frac{\delta}{27}\Big(\frac{N}{K}\Big)^3 \Bigg).
\end{aligned} \tag{3.30}$$

where the maximum is over $h \in \{0, \dots, K-1\}^3$ and $1 \leqslant l \leqslant 27$.

Let E_k be the event that precisely k indicator functions appearing on the right hand side of (3.30) are nonzero. In other words, E_k is the event that there are k distinct defects of size at most N^γ such that the k distinct $\tau_h \blacksquare$, where $\blacksquare \in \mathbb{UB}_N^{K,l}$, are contained in their interiors.

Given a block there are $27 \cdot 26^{n-1}$ possible defects of size n that contain this block. Thus, by Proposition 3.2, there exists C_P such that

$$\begin{aligned} \nu_{\beta,N}(E_k) &\leqslant \binom{\frac{N^3}{27K^3}}{k} \sum_{1\leqslant n_1,\dots,n_k\leqslant N^\gamma} \prod_{j=1}^k n_j \cdot 26 \cdot 27^{n_j-1} e^{-C_P\sqrt{\beta} n_j} \\ &\leqslant \binom{\frac{N^3}{27K^3}}{k} e^{-\frac{C_P}{2}\sqrt{\beta}k} \Big(\sum_{n=1}^{N^\gamma} n \cdot 26 \cdot 27^{n-1} e^{-\frac{C_P}{2}\sqrt{\beta} n}\Big)^k \\ &\leqslant \binom{\frac{N^3}{27K^3}}{k} e^{-\frac{C_P}{2}\sqrt{\beta}k} \end{aligned} \tag{3.31}$$

provided e.g. $\sqrt{\beta} > \max\left(\frac{4\log 27}{C_P}, \frac{2\log 2}{C_P}\right) = \frac{4\log 27}{C_P}$. This estimate is uniform over the choice of h and l.

By a union bound on (3.30), using (3.31), and that $2N^\gamma \leqslant K \leqslant 4N^\gamma$,

$$\begin{aligned} \nu_{\beta,N}\Big(\sum_{\Gamma\in\mathfrak{D}^{\gamma,\max}} |\mathrm{Int}(\Gamma)| > \delta N^3\Big) &\leqslant 27K^3 \sum_{k=\lfloor \frac{\delta N^3}{27K^3}\rfloor+1}^{\frac{N^3}{27K^3}} \binom{\frac{N^3}{27K^3}}{k} e^{-\frac{C_P}{2}\sqrt{\beta}k} \\ &\leqslant 27K^3 \cdot 2^{\frac{N^3}{27K^3}} e^{-\frac{\delta C_P}{2\cdot 27}\sqrt{\beta}\frac{N^3}{K^3}} \\ &\leqslant 27\cdot 64 e^{3\gamma\log N + \frac{\log 2}{27\cdot 8}\frac{N^3}{N^{3\gamma}} - \frac{\delta C_P}{27\cdot 16}\sqrt{\beta}\frac{N^3}{N^{3\gamma}}} \\ &\leqslant 27\cdot 64 e^{-\frac{\delta C_P}{27\cdot 32}\sqrt{\beta}\frac{N^3}{N^{3\gamma}}} \end{aligned}$$

provided $\gamma \log N < N^{3-3\gamma}$ and $\sqrt{\beta} > \frac{81\cdot 32 + 4\log 2}{\delta C_P}$. Taking logarithms and dividing by N^2 completes the proof.

4 Boué-Dupuis formalism for ϕ^4_3

In this section we introduce the underlying framework that we build on to analyse expectations of certain random variables under $\nu_{\beta,N}$, as required in the proof of Proposition 3.6. This framework was originally developed in [BG19] to show ultraviolet stability for ϕ^4_3 and identify its Laplace transform.

In particular, we want to obtain estimates on expectations of the form $\langle e^{Q_K}\rangle_{\beta,N,K}$, where Q_K are random variables that converge (in an appropriate sense) to some random variable Q of interest. We always work with a fixed ultraviolet cutoff K and establish estimates on $\langle e^{Q_K}\rangle_{\beta,N,K}$ that are uniform in K: this requires handling of ultraviolet divergences. The first observation is that we can represent such expectations as a ratio of Gaussian expectations:

$$\langle e^{Q_K} \rangle_{\beta,N,K} = \frac{\mathbb{E}_N e^{-\mathscr{H}_{\beta,N,K}(\phi_K)+Q_K(\phi_K)}}{\mathscr{Z}_{\beta,N,K}} \tag{4.1}$$

where we recall $\mathbb{E}_N$ denotes expectation with respect to μ_N and $\mathscr{Z}_{\beta,N,K} = \mathbb{E}_N e^{-\mathscr{H}_{\beta,N,K}(\phi_K)}$ is the partition function.

We then introduce an auxiliary time variable that continuously varies the ultraviolet cutoff between 0 and K, and use it to represent these Gaussian expectations in terms of expectations of functionals of finite dimensional Brownian motions. This allows us to use the Boué-Dupuis variational formula given in Proposition 4.7 to write these expectations in terms of a stochastic control problem. Hence, the problem of obtaining bounds is translated into choosing appropriate controls. An insight made in [BG19] is that one can use methods developed in the context of singular stochastic PDEs, specifically the paracontrolled calculus approach of [GIP15], within the control problem to kill ultraviolet divergences.

Remark 4.1. *In the following, we make use of tools in Appendices A.2 and A.3 concerning Besov spaces and paracontrolled calculus. In addition, for the rest of Sections 4 and 5, we consider $N \in \mathbb{N}$ fixed and drop it from notation when clear.*

4.1 Construction of the stochastic objects

Fix $\kappa_0 > 0$ sufficiently small. We equip $\Omega = C(\mathbb{R}_+; \mathscr{C}^{-\frac{3}{2}-\kappa_0})$ with its Borel σ-algebra. Denote by $\mathbb{P}$ the probability measure on Ω under which the coordinate process $X_\bullet = (X_k)_{k\geqslant 0}$ is an L^2 cylindrical Brownian motion. We write $\mathbb{E}$ to denote expectation with respect to $\mathbb{P}$. We consider the filtered probability space $(\Omega, \mathscr{A}, (\mathscr{A}_k)_{k\geqslant 0}, \mathbb{P})$, where $\mathscr{A}$ is the $\mathbb{P}$-completion of the Borel σ-algebra on Ω, and $(\mathscr{A}_k)_{k\geqslant 0}$ is the natural filtration induced by X and augmented with $\mathbb{P}$-null sets of $\mathscr{A}$.

Given $n \in (N^{-1}\mathbb{Z})^3$, define the process $B^n_\bullet$ by $B^n_k = \frac{1}{N^{\frac{3}{2}}} \int_{\mathbb{T}_N} X_k e_{-n} dx$, where $e_n(x) = e^{2\pi i n\cdot x}$ and we recall that the integral denotes duality pairing between distributions and test functions. Then, $\{B^n_\bullet : n \in (N^{-1}\mathbb{Z})^3\}$ is a set of complex Brownian motions defined on $(\Omega, \mathscr{A}, (\mathscr{A}_k)_{k\geqslant 0}, \mathbb{P})$, independent except for the constraint $\overline{B^n_k} = B^{-n}_k$. Moreover,

$$X_k = \frac{1}{N^3} \sum_{n\in(N^{-1}\mathbb{Z})^3} B^n_k N^{\frac{3}{2}} e_n$$

where $\mathbb{P}$-almost surely the sum converges in $\mathscr{C}^{-\frac{3}{2}-\kappa_0}$.

Let $\mathcal{J}_k$ be the Fourier multiplier with symbol

$$\mathcal{J}_k(\cdot) = \frac{\sqrt{\partial_k \rho_k^2(\cdot)}}{\langle\cdot\rangle}$$

where ρ_k is the ultraviolet cutoff defined in Section 2 and we recall $\langle \cdot \rangle = \sqrt{\eta + 4\pi^2 |\cdot|^2}$. $\mathcal{J}_k$ arises from a continuous decomposition of the covariance of the pushforward measure μ_N under ρ_k:

$$\int_0^k \mathcal{J}_{k'}^2(\cdot) dk' = \frac{\rho_k^2(\cdot)}{\langle \cdot \rangle^2} = \mathcal{F}\Big\{ \mathcal{F}^{-1}(\rho_k) * (-\Delta + \eta)^{-1} * \mathcal{F}^{-1}(\rho_k) \Big\}(\cdot)$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ denotes its inverse (see Appendix A.1). Note that the function $\partial_k \rho_k^2$ has decay of order $\langle k \rangle^{-\frac{1}{2}}$ and the corresponding multiplier is supported frequencies satisfying $|n| \in (c_\rho k, C_\rho k)$ for some $c_\rho < C_\rho$. Thus, we may think of $\mathcal{J}_k$ as having the same regularising properties as the multiplier $\frac{\mathcal{F}\{(-\Delta+\eta)^{-\frac{1}{2}}\}}{\langle k \rangle^{\frac{1}{2}}} \mathbf{1}_{c_\rho k \leqslant |\cdot| \leqslant C_\rho k}$; precise statements are given in Proposition A.9.

Define the process $\<1>_\bullet$ by

$$\<1>_k = \int_0^k \mathcal{J}_{k'} dX_{k'} = \frac{1}{N^{\frac{3}{2}}} \sum_{n \in (N^{-1}\mathbb{Z})^3} \left(\int_0^k \frac{\sqrt{\partial_{k'} \rho_{k'}^2(n)}}{\langle n \rangle} dB_{k'}^n \right) e_n. \tag{4.2}$$

$\<1>_\bullet$ is a centred Gaussian process with covariance:

$$\mathbb{E}\Big[\int_{\mathbb{T}_N} \<1>_k f dx \int_{\mathbb{T}_N} \<1>_{k'} g dx \Big] = \frac{1}{N^3} \sum_{n \in (N^{-1}\mathbb{Z})^3} \frac{\rho^2_{\min(k,k')}}{\langle n \rangle^2} \mathcal{F} f(n) \mathcal{F} g(n)$$

for any $f, g \in L^2$. Thus, the law of $\<1>_k$ is the law of $\rho_k \phi$ where $\phi \sim \mu_N$. As with other processes in the following, we simply write $\<1> = \<1>_\bullet$.

4.1.1 Renormalised multilinear functions of the free field

The second, third, and fourth Wick powers of $\<1>$ are the space-stationary stochastic processes $\<2>, \<3>, \<4>$ defined by:

$$\begin{aligned}
\<2>_k &= \<1>_k^2 - \<tadpole>_k \\
\<3>_k &= \<1>_k^3 - 3\<tadpole>_k \\
\<4>_k &= \<1>_k^4 - 6\<tadpole>_k \<1>_k^2 + 3\<tadpole>_k^2
\end{aligned}$$

where we recall from Section 2 that $\<tadpole>_k = \mathbb{E}_N[\phi_k^2(0)] = \mathbb{E}[\<1>_k^2(0)]$. Note that $\<2>_k, \<3>_k$, and $\<4>_k$ are equal in law to $:\phi_k^2:$, $:\phi_k^3:$, and $:\phi_k^4:$, respectively.

The Wick powers of $\<1>$ can be expressed as iterated integrals using Itô's formula (see [Nua06, Section 1.1.2]). We only need the iterated integral representation $\<3>$:

$$\<3>_k = \frac{3!}{N^{\frac{9}{2}}} \sum_{n_1, n_2, n_3} \int_0^k \int_0^{k_1} \int_0^{k_2} \prod_{i=1}^3 \frac{\sqrt{\partial_{k_i} \rho_{k_i}^2(n_i)}}{\langle n_i \rangle} dB_{k_3}^{n_3} dB_{k_2}^{n_2} dB_{k_1}^{n_1} \tag{4.3}$$

where we have used the convention that sums over frequencies n_i range over $(N^{-1}\mathbb{Z})^3$.

We define additional space-stationary stochastic processes Υ, $\Upsilon^{\bullet}$, $\Upsilon^{\mathbf{V}}$, $\mathbf{VV}$ by

$$\begin{aligned}
\Upsilon_k &= \int_0^k \mathcal{J}_{k'}^2 \Psi_{k'}\, dk' \\
\Upsilon^{\bullet}_k &= \uparrow_k \circeq \Upsilon_k \\
\Upsilon^{\mathbf{V}}_k &= \mathbf{V}_k \circeq \Upsilon_k - \frac{12}{N^6}\uparrow_k \sum_{n_1+n_2+n_3} \int_0^k \frac{\rho_{k'}^2(n_1)\rho_{k'}^2(n_2)\partial_{k'}\rho_{k'}^2(n_3)}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2} dk' \\
\mathbf{VV}_k &= \mathcal{J}_k\mathbf{V}_k \circeq \mathcal{J}_k\mathbf{V}_k - \frac{4}{N^6}\sum_{n_1+n_2+n_3=0} \frac{\rho_k^2(n_1)\rho_k^2(n_2)\partial_k\rho_k^2(n_3)}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2}.
\end{aligned}$$

We make two observations: first, a straightforward calculation shows that Ψ_k diverges in variance as $k \to \infty$. However, due to the presence of $\mathcal{J}_k$, Υ_k can be made sense of as $k \to \infty$. See Lemma 4.6.

Second, $\Upsilon^{\bullet}_k$, $\Upsilon^{\mathbf{V}}_k$, and $\mathbf{VV}_k$ are renormalised resonant products of $\uparrow_k\Upsilon_k$, $\mathbf{V}_k\Upsilon_k$, and $(\mathcal{J}_k\mathbf{V}_k)^2$, respectively. The latter products are classically divergent in the limit $k \to \infty$. We refer to Remark 4.2 for an explanation of why the resonant product is used.

Remark 4.2. *Let $f \in \mathscr{C}^{s_1}$ and $g \in \mathscr{C}^{s_2}$ for $s_1 < 0 < s_2$. Bony's decomposition states that, if the product exists, $fg = f \olessthan g + f \circeq g + f \ogreaterthan g$ and is of regularity s_1 (see Appendix A.3). Since paraproducts are always well-defined (see Proposition A.5), the resonant product contains all of the difficulty in defining the product. However, the resonant product gives regularity information of order $s_1 + s_2$ (see Proposition A.6), which is strictly stronger than the regularity information of the product: i.e. the bound on $\|f \circeq g\|_{\mathscr{C}^{s_1+s_2}}$ is strictly stronger than the bound on $\|fg\|_{\mathscr{C}^{s_1}}$. This is the key property that makes paracontrolled calculus useful in this context [GIP15].*

The required renormalisations of $\Upsilon^{\mathbf{V}}_K$ and $\mathbf{VV}_K$ are related to the usual "sunset" diagram appearing in the perturbation theory for ϕ^4_3,

$$\Theta_k = \frac{1}{N^6}\sum_{n_1+n_2+n_3=0} \frac{\rho_k^2(n_1)\rho_k^2(n_2)\rho_k^2(n_3)}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2}. \tag{4.4}$$

See [Fel74, Theorem 1]. We emphasise that Θ_k depends on η, N and k.

By the fundamental theorem of calculus, the Leibniz rule, and symmetry,

$$\Theta_k = \frac{1}{N^6}\sum_{n_1+n_2+n_3=0} \int_0^k \frac{\partial_{k'}\left(\rho_{k'}^2(n_1)\rho_{k'}^2(n_2)\rho_{k'}^2(n_3)\right)}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2} dk'$$

$$= \frac{3}{N^6} \sum_{n_1+n_2+n_3=0} \frac{\int_0^k \rho_{k'}^2(n_1)\rho_{k'}^2(n_2)\partial_{k'}\rho_k^2(n_3)}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2}.$$

Thus, the renormalisations of $\Upsilon^{\vee}_K$ and $\vee\!\vee_k$ are given by $4\ominus_k \uparrow_k$ and $\frac{4}{3}\partial_k \ominus_k$, respectively.

Remark 4.3. *It is straightforward to verify that there exists $C = C(\eta) > 0$ such that*

$$\ominus_k \leqslant \frac{C(\eta)}{N^6}\log\langle k\rangle \quad \text{and} \quad \partial_k \ominus_k \leqslant \frac{C(\eta)}{N^6}\frac{\log\langle k\rangle}{\langle k\rangle}.$$

Let $\Xi = (\uparrow, \vee, \Upsilon, \Upsilon^{\bullet}, \Upsilon^{\vee}, \vee\!\vee)$. We refer to the coordinates of Ξ as diagrams. The following proposition gives control over arbitrarily high moments of diagrams in Besov spaces.

Proposition 4.4. *For any $p, p' \in [1, \infty)$, $q \in [1, \infty]$, and $\kappa > 0$ sufficiently small, there exists $C = C(p, p', q, \kappa, \eta) > 0$ such that*

$$\begin{aligned} \sup_{k>0} \mathbb{E}\Big[&\|\uparrow_k\|^p_{B^{-\frac{1}{2}-\kappa}_{p',q}} + \|\vee_k\|^p_{B^{-1-\kappa}_{p',q}} + \|\Upsilon_k\|^p_{B^{\frac{1}{2}-\kappa}_{p',q}} \\ &+ \|\Upsilon^{\bullet}_k\|^p_{B^{-\kappa}_{p',q}} + \|\Upsilon^{\vee}_k\|^p_{B^{-\frac{1}{2}-\kappa}_{p',q}} + \Big(\int_0^k \|\vee\!\vee_{k'}\|_{B^{-\kappa}_{p',q}}\Big)^p dk' \Big] \leqslant C. \end{aligned} \tag{4.5}$$

Proof. See [BG19, Lemma 24]. □

Remark 4.5. *The constant on the righthand side of (4.5) is independent of N because our Besov spaces are defined with respect to normalised Lebesgue measure $đx = \frac{dx}{N^3}$ (see Appendix A.2). For $p = \infty$, bounds that are uniform in N do not hold. Indeed, for L^∞-based norms, there is in general no chance of controlling space-stationary processes uniformly in the volume. Thus, we cannot work in Besov-Hölder spaces.*

We prove the bound in (4.5) for Υ_k since it illustrates the role of $\mathcal{J}_k$, is used later in the proof of Proposition 5.22, and gives the reader a flavour of how to prove the bounds on the other diagrams.

Lemma 4.6. *There exists $C = C(\eta) > 0$ such that, for any $n \in (N^{-1}\mathbb{Z})^3$,*

$$\sup_{k>0} \mathbb{E}\Big|\mathscr{F}\Upsilon_k(n)\Big|^2 \leqslant \frac{CN^3}{\langle n\rangle^4}. \tag{4.6}$$

As a consequence, for every $p \in [1, \infty)$ and $s < \frac{1}{2}$, there exists $C = C(p, s, \eta) > 0$ such that

$$\sup_{k>0} \mathbb{E}\Big[\|\Upsilon_K\|^p_{B^s_{p,p}}\Big] \leqslant C.$$

Proof. Inserting (4.3) in the definition of Ψ_k and switching the order of integration,

$$\begin{aligned}\mathscr{F}\Psi_k(n) &= \frac{6}{N^{\frac{3}{2}}}\sum_{n_1+n_2+n_3=n}\int_0^k \frac{\partial_{k'}\rho_{k'}^2(n)}{\langle n\rangle^2}\int_0^{k'}\int_0^{k_1}\int_0^{k_2} \\ &\qquad\qquad \times\left(\prod_{i=1}^3 \frac{\sqrt{\partial_{k_i}\rho_{k_i}^2(n_i)}}{\langle n_i\rangle}\right) dB_{k_3}^{n_3}dB_{k_2}^{n_2}dB_{k_1}^{n_1}dk' \\ &= \frac{6}{N^{\frac{3}{2}}}\sum_{n_1+n_2+n_3=n}\int_0^k\int_0^{k_1}\int_0^{k_2}\left(\int_{k_1}^k \frac{\partial_{k'}\rho_{k'}^2(n)}{\langle n\rangle^2}dk'\right) \\ &\qquad\qquad \times\left(\prod_{i=1}^3 \frac{\sqrt{\partial_{k_i}\rho_{k_i}^2(n_i)}}{\langle n_i\rangle}\right) dB_{k_3}^{n_3}dB_{k_2}^{n_2}dB_{k_1}^{n_1}.\end{aligned}$$

Therefore, by Itô's formula,

$$\begin{aligned}&\mathbb{E}\left|\mathscr{F}\Psi_K(n)\right|^2 \\ &\leqslant \frac{36}{N^3}\sum_{n_1+n_2+n_3=n}\int_0^k\int_0^{k_1}\int_0^{k_2}\left(\int_{k_1}^k \frac{\partial_{k'}\rho_{k'}^2(n)}{\langle n\rangle^2}dk'\right)^2 \\ &\qquad\qquad \times\left(\prod_{i=1}^3 \frac{\partial_{k_i}\rho_{k_i}^2(n_i)}{\langle n_i\rangle^2}\right) dk_3dk_2dk_1 \\ &\leqslant \frac{36}{N^3}\sum_{n_1+n_2+n_3=n}\frac{1}{\langle n_2\rangle^2\langle n_3\rangle^2}\int_0^k\left(\int_{k_1}^k \frac{\partial_{k'}\rho_{k'}^2(n)}{\langle n\rangle^2}dk'\right)^2 \frac{\partial_{k_1}\rho_{k_1}^2(n_1)}{\langle n_1\rangle^2}dk_1\end{aligned} \tag{4.7}$$

where we have performed the k_2 and k_3 integrations, and used that $|\rho_k| \leqslant 1$.

Recall that $\partial_{k'}\rho_{k'}^2$ is supported on frequencies $|n| \in (c_\rho k', C_\rho k')$. Hence, for any $\kappa > 0$,

$$\begin{aligned}(4.7) &\lesssim \frac{1}{N^3}\sum_{n_1+n_2+n_3=n}\frac{1}{\langle n_2\rangle^2\langle n_3\rangle^2}\int_0^K\left(\int_{k_1}^k \frac{\partial_{k'}\rho_{k'}^2(n)}{\langle n\rangle^{2-\frac{\kappa}{2}}}dk'\right)^2 \frac{\partial_{k_1}\rho_{k_1}^2(n_1)}{\langle n_1\rangle^2\langle k_1\rangle^\kappa}dk_1 \\ &\leqslant \frac{1}{N^3}\sum_{n_1+n_2+n_3=n}\frac{1}{\langle n\rangle^{4-\kappa}\langle n_2\rangle^2\langle n_3\rangle^2}\int_0^k \frac{\partial_{k_1}\rho_{k_1}^2(n_1)}{\langle n_1\rangle^{2+\kappa}}dk_1 \qquad (4.8) \\ &\lesssim \frac{1}{N^3}\sum_{n_1+n_2+n_3=n}\frac{1}{\langle n\rangle^{4-\kappa}\langle n_1\rangle^{2+\kappa}\langle n_2\rangle^2\langle n_3\rangle^2} \lesssim \frac{N^3}{\langle n\rangle^4},\end{aligned}$$

where $\lesssim$ means $\leqslant$ up to a constant depending only on η, c_ρ and C_ρ; the last inequality uses standard bounds on discrete convolutions contained in Lemma A.12; and we have used that the double convolution produces a volume factor of N^6. Note that, as said in Section 2, we omit the dependence on c_ρ and C_ρ in the final bound.

By Fubini's theorem, Nelson's hypercontractivity estimate [Nel73] (or the related Burkholder-Davis-Gundy inequality [RY13, Theorem 4.1]), and space-stationarity

$$\begin{aligned}
\mathbb{E}\|\Upsilon_k\|_{B^s_{p,p}}^p &= \sum_{j\geqslant -1} 2^{jps}\mathbb{E}\|\Delta_j \Upsilon_k\|_{L^p}^p \\
&= \sum_{j\geqslant -1} 2^{jps}\int_{\mathbb{T}_N} \mathbb{E}|\Delta_j \Upsilon_k(x)|^p đx \\
&\lesssim \sum_{j\geqslant -1} 2^{jps}\int_{\mathbb{T}_N} \left(\mathbb{E}|\Delta_j \Upsilon_k(x)|^2\right)^{\frac{p}{2}} đx \\
&= \sum_{j\geqslant -1} 2^{jps}\left(\mathbb{E}|\Delta_j \Upsilon_k(0)|^2\right)^{\frac{p}{2}}
\end{aligned} \tag{4.9}$$

where Δ_j is the j-th Littlewood-Paley block defined in Appendix A and we recall $đx = \frac{dx}{N^3}$.

We overload notation and also write Δ_j to mean its corresponding Fourier multiplier. Then, by space-stationarity, for any $j \geqslant -1$,

$$\begin{aligned}
\mathbb{E}|\Delta_j \Upsilon_k(0)|^2 &= \int_{\mathbb{T}_N} \mathbb{E}|\Delta_j \Upsilon_k(x)|^2 đx \\
&= \frac{1}{N^6}\sum_n |\Delta_j(n)|^2 \mathbb{E}|\mathscr{F}\Upsilon_k(n)|^2 \\
&\lesssim \frac{1}{N^3}\sum_n \frac{\Delta_j(n)^2}{\langle n\rangle^4} \lesssim \frac{2^{3j}}{2^{4j}} = \frac{1}{2^j}.
\end{aligned} \tag{4.10}$$

Inserting (4.10) into (4.9) we obtain

$$\mathbb{E}\|\Upsilon_K\|_{B^s_{p,p}}^p \lesssim \sum_{j\geqslant -1} 2^{jps}2^{-\frac{p}{2}j}$$

which converges provided $s < \frac{1}{2}$, thus finishing the proof. □

4.2 The Boué-Dupuis formula

Fix an ultraviolet cutoff K. Recall that we are interested in Gaussian expectations of the form

$$\mathbb{E}_N e^{-\mathscr{H}(\phi_K)}$$

where $\mathscr{H}(\phi_K) = \mathscr{H}_{\beta,N,K}(\phi_K) + Q_K(\phi_K)$.

We may represent such expectations on $(\Omega, \mathscr{A}, (\mathscr{A}_k)_{k\geqslant 0}, \mathbb{P})$:

$$\mathbb{E}_N e^{-\mathscr{H}(\phi_K)} = \mathbb{E} e^{-\mathscr{H}(\Phi_K)}. \tag{4.11}$$

The key point is that the righthand side of (4.11) is written in terms of a measurable functional of Brownian motions. This allows us to exploit continuous time martingale techniques, crucially Girsanov's theorem [RY13, Theorems 1.4 and 1.7], to reformulate (4.11) as a stochastic control problem.

Let $\mathbb{H}$ be the set of processes $v_\bullet$ that are $\mathbb{P}$-almost surely in $L^2(\mathbb{R}_+; L^2(\mathbb{T}_N))$ and progressively measurable with respect to $(\mathscr{A}_k)_{k\geqslant 0}$. We call this the space of drifts. For any $v \in \mathbb{H}$, let $V_\bullet$ be the process defined by

$$V_k = \int_0^k \mathcal{J}_{k'} v_{k'} dk'.$$

For our purposes, it is sufficient to consider the subspace of drifts $\mathbb{H}_K \subset \mathbb{H}$ consisting of $v \in \mathbb{H}$ such that $v_k = 0$ for $k > K$.

We also work with the subset of bounded drifts $\mathbb{H}_{b,K} \subset \mathbb{H}_K$, defined as follows: for every $M \in \mathbb{N}$, let $\mathbb{H}_{b,M,K}$ be the set of $v \in \mathbb{H}_K$ such that

$$\int_0^K \int_{\mathbb{T}_N} v_k^2 dx dk \leqslant M \tag{4.12}$$

$\mathbb{P}$-almost surely. Set $\mathbb{H}_{b,K} = \bigcup_{M\in\mathbb{N}} \mathbb{H}_{b,M,K}$.

The following proposition is the main tool of this section.

Proposition 4.7. *Let $N \in \mathbb{N}$ and $\mathscr{H} : C^\infty(\mathbb{T}_N) \to \mathbb{R}$ be measurable and bounded. Then, for any $K > 0$,*

$$-\log \mathbb{E}\Big[e^{-\mathscr{H}(\Phi_K)}\Big] = \inf_v \mathbb{E}\Big[\mathscr{H}(\Phi_K + V_K) + \frac{1}{2}\int_0^K \int_{\mathbb{T}_N} v_k^2 dx dk\Big] \tag{4.13}$$

where the infimum can be taken over v in $\mathbb{H}_K$ or $\mathbb{H}_{b,K}$.

Proof. (4.13) was first established by Boué and Dupuis [BD98], but we use the version in [BD19, Theorem 8.3], adapted to our setting. □

We cannot directly apply Proposition 4.7 for the case $\mathscr{H} = \mathscr{H}_{\beta,N,K} + Q_K$ because it is not bounded. To circumvent this technicality, we introduce a *total energy cutoff* $E \in \mathbb{N}$. Since K is taken *fixed*, $\mathscr{H}_{\beta,N,K} + Q_K$ is bounded from below. Hence, by dominated convergence

$$\lim_{E\to\infty} \mathbb{E}_N e^{-\big(\mathscr{H}_{\beta,N,K}(\phi_K) + Q_K(\phi_K)\big)\wedge E} = \mathbb{E}_N e^{-\mathscr{H}_{\beta,N,K}(\phi_K) + Q_K(\phi_K)}. \tag{4.14}$$

We apply Proposition 4.7 to $\mathcal{H} = \big(\mathcal{H}_{\beta,N,K} + Q_K\big) \wedge E$. For the lower bound on the corresponding variational problem, we establish estimates that are uniform over $v \in \mathbb{H}_{b,K}$. For the upper bound, we establish estimates for a specific choice of $v \in \mathbb{H}_K$ which is constructed via a fixed point argument. All estimates that we establish are independent of E. Hence, using (4.14) and the representation (4.11), they carry over to $\mathbb{E}_N e^{-\mathcal{H}_{\beta,N,K}(\phi_K)+Q_K(\phi_K)}$. We suppress mention of E unless absolutely necessary.

Remark 4.8. *The assumption that $\mathcal{H}$ is bounded allows the infimum in* (4.13) *to be interchanged between $\mathbb{H}_K$ and $\mathbb{H}_{b,K}$. The use of $\mathbb{H}_{b,K}$ allows one to overcome subtle stochastic analysis issues that arise later on: specifically, justifying certain stochastic integrals appearing in Lemmas 5.14 and 5.16 are martingales and not just local martingales. See Lemma 5.13. The additional boundedness condition is important in the lower bound on the variational problem as the only other a priori information that we have on v there is that $\mathbb{E}\int_0^K \int_{\mathbb{T}_N} v_k^2 dx dk < \infty$, which alone is insufficient. On the other hand, the candidate optimiser for the upper bound is constructed in $\mathbb{H}_K$, but it has sufficient moments to guarantee the aforementioned stochastic integrals in Lemma 5.13 are martingales. See Lemma 5.21.*

Remark 4.9. *A version of the Boué-Dupuis formula for $\mathcal{H}$ measurable and satisfying certain integrability conditions is given in [Üst14, Theorem 7]. These integrability conditions are broad enough to cover the cases that we are interested in, and it is required in [BG19] to identify the Laplace transform of ϕ^4_3. However, it is not clear to us that the infimum in the corresponding variational formula can be taken over $\mathbb{H}_{b,K}$. Therefore, it seems that the stochastic analysis issues discussed in Remark 4.8 cannot be resolved directly using this version without requiring some post-processing (e.g. via a dominated convergence argument with a total energy cutoff as above).*

4.2.1 Relationship with the Gibbs variational principle

Given a drift $v \in \mathbb{H}_K$, we define the measure $\mathbb{Q}$ whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is given by the following stochastic exponential:

$$d\mathbb{Q} = e^{\int_0^K v_k dX_k - \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K v_k^2 dk dx} d\mathbb{P}. \tag{4.15}$$

Let $\mathbb{H}_{c,K}$ be the set of $v \in \mathbb{H}_K$ such that its associated measure defined in (4.15) is a probability measure, i.e. the expectation of the stochastic integral is 1. Then, by Girsanov's theorem [RY13, Theorems 1.4 and 1.7 in Chapter VIII] it follows that the process $X_\bullet$ is a semi-martingale under $\mathbb{Q}$ with decomposition:

$$X_K = X_K^v + \int_0^K v_k dx$$

where $X_\bullet^v$ is an L^2 cylindrical Brownian motion with respect to $\mathbb{Q}$. This induces the decomposition

$$\uparrow_K = \uparrow_K^v + V_K \tag{4.16}$$

where $\uparrow_K^v = \int_0^K \mathcal{J}_k dX_k^v$.

Lemma 4.10. *Let $N \in \mathbb{N}$ and $\mathcal{H} : C^\infty(\mathbb{T}_N) \to \mathbb{R}$ be measurable and bounded from below. Then, for any $K > 0$,*

$$-\log \mathbb{E} e^{-\mathcal{H}(\uparrow_K)} = \min_{v \in \mathbb{H}_{c,K}} \mathbb{E}_{\mathbb{Q}}\Big[\mathcal{H}(\uparrow_K^v + V_K) + \frac{1}{2}\int_0^\infty \int_{\mathbb{T}_N} v_k^2 dx dk\Big] \tag{4.17}$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes expectation with respect to $\mathbb{Q}$.

Proof. (4.17) is a well-known representation of the classical Gibbs variational principle [DE11, Proposition 4.5.1]. Indeed, one can verify that $R(\mathbb{Q}\|\mathbb{P}) = \mathbb{E}_{\mathbb{Q}}\Big[\int_0^\infty \int_{\mathbb{T}_N} v_k^2 dx dk\Big]$, where $R(\mathbb{Q}\|\mathbb{P}) = \mathbb{E}_{\mathbb{Q}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}$ is the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$. A full proof in our setting is given in [GOTW18, Proposition 4.4]. $\square$

Proposition 4.7 has several upshots over Lemma 4.10. The most important for us is that drifts can be taken over a Banach space, thus allowing candidate optimisers to be constructed using fixed point arguments via contraction mapping. In addition, the underlying probability space is fixed (i.e. with respect to the canonical measure $\mathbb{P}$), although this is a purely aesthetic advantage in our case. The cost of these upshots is that the minimum in (4.17) is replaced by an infimum in (4.13), and more rigid conditions on $\mathcal{H}$ are required. We refer to [BD19, Section 8.1.1] or [BG19, Remark 1] for further discussion.

With the connection with the Gibbs variational principle in mind, we call $\mathcal{H}(V_K)$ the drift (potential) energy and we call $\int_0^K \int_{\mathbb{T}_N} v_k^2 dx dk$ the drift entropy.

4.2.2 Regularity of the drift

In our analysis we use intermediate scales between 0 and K. As we explain in Section 5.1, this means that we require control over the process $V_\bullet$ in terms of the drift energy and drift entropy terms in (4.13).

The drift entropy allows a control of $V_\bullet$ in L^2-based topologies.

Lemma 4.11. *For every $v \in L^2(\mathbb{R}_+; L^2(\mathbb{T}_N))$ and $K > 0$,*

$$\sup_{0 \leqslant k \leqslant K} \|V_k\|_{H^1}^2 \leqslant \int_0^K \int_{\mathbb{T}_N} v_k^2 dx dk. \tag{4.18}$$

Proof. (4.18) is a straightforward consequence definition of $\mathcal{J}_k$, see [BG19, Lemma 2]. $\square$

To control the homogeneity in our estimates, we also require bounds on $\|V_\bullet\|_{L^4}^4$. This is a problem: for our specific choices of $\mathscr{H}$, the drift energy allows a control in L^4-based topologies at the *endpoint* V_K. It is in general impossible to control the history of the path by the endpoint (for example, consider an oscillating process $V_\bullet$ with $V_K = 0$). We follow [BG19] to sidestep this issue.

Let $\tilde{\rho} \in C_c^\infty(\mathbb{R}_+; \mathbb{R}_+)$ be non-increasing such that

$$\tilde{\rho}(x) = \begin{cases} 1 & |x| \in \left[0, \frac{c_\rho}{2}\right] \\ 0 & |x| \in \left[c_\rho, \infty\right) \end{cases}$$

and let $\tilde{\rho}_k(\cdot) = \tilde{\rho}(\frac{\cdot}{k})$ for every $k > 0$.

Define the process $V_\bullet^\flat$ by

$$V_k^\flat = \frac{1}{N^3} \sum_n \tilde{\rho}_k(n) \left(\int_0^k \mathcal{J}_{k'}(n) \mathscr{F} v_{k'}(n) dk' \right) e^n.$$

Note that $\mathscr{F}(V_k^\flat)(n) = \mathscr{F}(V_k)(n)$ if $|n| \leqslant \frac{c_\rho}{2}$. Thus, $V_\bullet^\flat$ and $V_\bullet$ have the same low frequency/large-scale behaviour (hence the notation).

The two processes differ on higher frequencies/small-scales. Indeed, as a Fourier multiplier, $\tilde{\rho}_k \mathcal{J}_k = 0$ for $k' > k$. Hence, for any $k \leqslant K$,

$$V_k^\flat = \frac{1}{N^3} \sum_n \tilde{\rho}_k(n) \left(\int_0^K \mathcal{J}_{k'}(n) \mathscr{F} v_{k'}(n) dk' \right) e^n = \tilde{\rho}_k V_K.$$

This is sufficient for our purposes because $\tilde{\rho}_k$ is an L^p multiplier for $p \in (1, \infty)$, and hence the associated operator is L^p bounded for $p \in (1, \infty)$.

Lemma 4.12. *For any* $p \in (1, \infty)$, *there exists* $C = C(p, \eta) > 0$ *such that, for every* $v \in L^2(\mathbb{R}_+; L^2(\mathbb{T}_N))$,

$$\sup_{0 \leqslant k \leqslant K} \|V_k^\flat\|_{L^p} \leqslant C \|V_K\|_{L^p}. \tag{4.19}$$

Moreover, for any $s, s' \in \mathbb{R}$, $p \in (1, \infty)$, $q \in [1, \infty]$, *there exists* $C = C(s, s', p, q, \eta)$ *such that, for every* $v \in L^2(\mathbb{R}_+; L^2(\mathbb{T}_N))$,

$$\sup_{0 \leqslant k \leqslant K} \|\partial_k V_k^\flat\|_{B^{s'}_{p,q}} \leqslant C \frac{\|V_K\|_{B^s_{p,q}}}{\langle k \rangle^{1+s-s'}}. \tag{4.20}$$

Proof. (4.19) and (4.20) are a consequence of the preceding discussion together with the observation that $\partial_k V_k^\flat$ is supported on an annulus in Fourier space and, subsequently, applying Bernstein's inequality (1.6). See [BG19, Lemma 20]. □

5 Estimates on Q-random variables

The main results of this section are upper bounds on expectations of certain random variables, derived from Q_1, Q_2, and Q_3 defined in (3.2), that are uniform in β and extensive in N^3.

Proposition 5.1. *For every $a_0 > 0$, there exist $\beta_0 = \beta_0(a_0, \eta) \geqslant 1$ and $C_Q = C_Q(a_0, \beta_0, \eta) > 0$ such that the following estimates hold: for all $\beta > \beta_0$ and $a \in \mathbb{R}$ satisfying $|a| \leqslant a_0$,*

$$-\frac{1}{N^3} \log \Big\langle \prod_{\square \in \mathbb{B}_N} \exp(aQ_1(\square)) \Big\rangle_{\beta,N} \geqslant -C_Q$$
$$-\frac{1}{N^3} \log \Big\langle \prod_{\square \in \mathbb{B}_N} \exp(aQ_2(\square)) \Big\rangle_{\beta,N} \geqslant -C_Q.$$

In addition,

$$-\frac{1}{N^3} \log \Big\langle \prod_{\{\square,\square'\} \in B} \exp(|aQ_3(\square,\square')|) \Big\rangle_{\beta,N} \geqslant -C_Q$$

where B is any set of unordered pairs of nearest-neighbour blocks that partitions $\mathbb{B}_N$.

Proof. See Section 5.9. □

Proposition 5.1 is used in Section 6.3, together with the chessboard estimates of Proposition 6.5, to prove Proposition 3.6. Indeed, chessboard estimates allow us to obtain estimates on expectations of random variables, derived from the Q_i, that are extensive in their support *from* estimates that are extensive in N^3. Note that the latter are significantly easier to obtain than the former since these random variables may be supported on arbitrary unions of blocks.

Remark 5.2. *For the remainder of this section, we assume $\eta < \frac{1}{392 C_P}$ where C_P is the Poincaré constant on unit blocks (see Proposition A.11). This is for convenience in the analysis of Sections 5.8.1 and 5.9 (see also Lemma 5.20). Whilst this may appear to fix the specific choice of renormalisation constants δm^2, we can always shift into this regime by absorbing a finite part of δm^2 into $\mathcal{V}_\beta$.*

Most of the difficulties in the proof of Proposition 5.1 are contained in obtaining the following upper and lower bounds on the free energy $-\log \mathscr{Z}_{\beta,N}$ that are uniform in β.

Proposition 5.3. *There exists $C = C(\eta) > 0$ such that, for all $\beta \geqslant 1$,*

$$\liminf_{K\to\infty} -\frac{1}{N^3}\log \mathscr{Z}_{\beta,N,K} \geqslant -C \tag{5.1}$$

and

$$\limsup_{K\to\infty} -\frac{1}{N^3}\log \mathscr{Z}_{\beta,N,K} \leqslant C. \tag{5.2}$$

Proof. See Sections 5.8.1 and 5.8.2 for a proof of (5.1) and (5.2), respectively. These proofs rely on Sections 5.2 - 5.7, and the overall strategy is sketched in Section 5.1. □

Remark 5.4. *In [BG19] estimates on $-\log \mathscr{Z}_{\beta,N,K}$ are obtained that are uniform in $K > 0$ and extensive in N^3. However, one can show that these estimates are $O(\beta)$ as $\beta \to \infty$. This is insufficient for our purposes (compare with the uniform in β estimates required to prove Proposition 3.2).*

5.1 Strategy to prove Proposition 5.3

The lower bound on $-\log \mathscr{Z}_{\beta,N,K}$, given by (5.1), is the harder bound to establish in Proposition 5.3. Our approach builds on the analysis of [BG19] by incorporating a low temperature expansion inspired by [GJS76a, GJS76b]. This is explained in more detail in Section 5.1.1.

On the other hand, we establish the upper bound on $-\log \mathscr{Z}_{\beta,N,K}$, given by (5.2), by a more straightforward modification of the analysis in [BG19]. See Section 5.8.2.

We now motivate our approach to establishing (5.1) by first isolating the the difficulty in obtaining β-independent bounds when using [BG19] straight out of the box. The starting point is to apply Proposition 4.7 with $\mathscr{H} = \mathscr{H}_{\beta,N,K}$, together with a total energy cutoff that we refrain from making explicit (see Remark 4.8 and the discussion that precedes it), to represent $-\log \mathscr{Z}_{\beta,N,K}$ as a stochastic control problem.

For every $v \in \mathbb{H}_{b,K}$, define

$$\Psi_K(v) = \mathscr{H}_{\beta,N,K}(\uparrow_K + V_K) + \frac{1}{2}\int_0^K \int_{\mathbb{T}_N} v_k^2 dx dk.$$

Ultraviolet divergences occur in the expansion of $\mathscr{H}_{\beta,N,K}(\uparrow_K + V_K)$ since the integrals $\int_{\mathbb{T}_N} \Psi_K V_K dx$ and $\int_{\mathbb{T}_N} \mathsf{V}_K V_K^2 dx$ appear and cannot be bounded uniformly in K:

- For the first integral, there are difficulties in even interpreting Ψ_K as a random distribution in the limit $K \to \infty$. Indeed, the variance of Ψ_K tested against a smooth function diverges as the cutoff is removed.

- On the other hand, one can show that $\vee_K$ does converge as $K \to \infty$ to a random distribution of Besov-Hölder regularity $-1-\kappa$ for any $\kappa > 0$ (see Proposition 4.4). However, this regularity is insufficient to obtain bounds on the second integral uniform on K . Indeed, V_K can be bounded in at most H^1 uniformly in K (see Lemma 4.11), and hence we cannot test $\vee_K$ against V_K (or V_K^2) in the limit $K \to \infty$.

This is where the need for renormalisation beyond Wick ordering appears.

To implement this, we follow [BG19] and postulate that the small-scale behaviour of the drift v is governed by explicit renormalised polynomials of $\uparrow$ through the change of variables:

$$v_k = -\frac{4}{\beta}\mathcal{J}_k \Psi_k - \frac{12}{\beta}\mathcal{J}_k(\vee_k \otimes V_k^\flat) + r_k \tag{5.3}$$

where the remainder term $r = r(v)$ is defined by (5.3). Since $v \in \mathbb{H}_K \supset \mathbb{H}_{b,K}$, we have that $r \in \mathbb{H}_K$ and, hence, has finite drift entropy; however, note that $r \notin \mathbb{H}_{b,K}$. The optimisation problem is then changed from optimising over $v \in \mathbb{H}_{b,K}$ to optimising over $r(v) \in \mathbb{H}_K$.

The change of variables (5.3) means that the drift entropy of any v now contains terms that are divergent as $K \to \infty$. One uses Itô's formula to decompose the divergent integrals identified above into intermediate scales, and then uses these divergent terms in the drift entropy to mostly cancel them. Using the renormalisation counterterms beyond Wick ordering (i.e. the terms involving γ_K and δ_K), the remaining divergences can be written in terms of well-defined integrals involving the diagrams $\Xi = (\uparrow, \vee, \Psi, \Psi^{\bullet}, \Psi\!\Psi^{\bullet}, \vee\!\vee)$ defined in Section 4.1.1.

One can then establish that, for every $\varepsilon > 0$, there exists $C = C(\varepsilon, \beta, \eta) > 0$ such that, for every $v \in \mathbb{H}_{b,K}$,

$$\mathbb{E}\Psi_K(v) \geqslant -CN^3 + (1-\varepsilon)\mathbb{E}[G_K(v)] \tag{5.4}$$

where

$$G_K(v) = \frac{1}{\beta}\int_{\mathbb{T}_N} V_K^4 dx + \frac{1}{2}\int_0^K \int_{\mathbb{T}_N} r_k^2 dk dx \geqslant 0.$$

The quadratic term in $G_K(v)$ allows one to control the $H^{\frac{1}{2}-\kappa}$ norm of V_K for any $\kappa > 0$, uniformly in K (see Proposition 5.9). These derivatives on V_K appear when analysing terms in $\Psi_K(v)$ involving Wick powers of $\uparrow_K$ tested against (powers of) V_K. However, some of these integrals have quadratic or cubic dependence on the drift, thus the quadratic term in $G_K(v)$ is insufficient to control the homogeneity in these estimates; instead, this achieved by using the quartic term in $G_K(v)$. Note that the good sign of the quartic term in the $\mathcal{H}_{\beta,N,K}$ ensures that $G_K(v)$ is indeed non-negative.

Using the representation (4.11) on $\mathbb{E}_N e^{-\mathscr{H}_{\beta,N,K}(\phi_K)}$ and applying Proposition 4.7, one obtains $-\log \mathscr{Z}_{\beta,N,K} \geqslant -CN^3$ from (5.4) and the positivity of $G_K(v)$.

As pointed out in Remark 5.4, this argument gives $C = O(\beta)$ for β large and this is insufficient for our purposes. The suboptimality in β-dependence comes from the treatment of the integral

$$\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K) - \frac{\eta}{2} V_K^2 dx \tag{5.5}$$

in $\mathscr{H}_{\beta,N,K}(\mathring{\vert}_K + V_K)$. The choice of $G_K(v)$ in the preceding discussion is not appropriate in light of (5.5) since the term $\int_{\mathbb{T}_N} V_K^4 dx$ destroys the structure of the non-convex potential $\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K) dx$. On the other hand, replacing $\frac{1}{\beta}\int_{\mathbb{T}_N} V_K^4 dx$ with the whole integral (5.5) in $G_K(v)$ does not work. This is because (5.5) does not admit a β-independent lower bound.

5.1.1 Fixing β dependence via a low temperature expansion

We expand (5.5) as two terms

$$(5.5) = \int_{\mathbb{T}_N} \frac{1}{2}\mathscr{V}_\beta(V_K) dx + \int_{\mathbb{T}_N} \frac{1}{2}\mathscr{V}_\beta(V_K) - \frac{\eta}{2} V_K^2 dx. \tag{5.6}$$

The first integral in (5.6) is non-negative so we use it as a stability/good term for the deterministic analysis, i.e. replacing $G_K(v)$ by

$$\int_{\mathbb{T}_N} \frac{1}{2}\mathscr{V}_\beta(V_K) dx + \frac{1}{2}\int_0^K \int_{\mathbb{T}_N} r_k^2 dx dk. \tag{5.7}$$

This requires a comparison of L^p norms of V_K for $p \leqslant 4$ on the one hand, and $\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K) dx$ on the other. Due to the non-convexity of $\mathscr{V}_\beta$, this produces factors of β; these have to be beaten by the good (i.e. negative) powers of β appearing in $\mathscr{H}_{\beta,N,K}(\mathring{\vert}_K + V_K)$. We state the required bounds in the following lemma.

Lemma 5.5. *For any $p \in [1,4]$, there exists $C = C(p) > 0$ such that, for all $a \in \mathbb{R}$,*

$$|a|^p \leqslant C(\sqrt{\beta})^{\frac{p}{2}} \mathscr{V}_\beta(a)^{\frac{p}{4}} + C(\sqrt{\beta})^p. \tag{5.8}$$

Hence, for any $f \in C^\infty(\mathbb{T}_N)$,

$$\|f\|_{L^p} \leqslant C(\sqrt{\beta})^{\frac{1}{2}} \Big(\int_{\mathbb{T}_N} \mathscr{V}_\beta(f) đx \Big)^{\frac{1}{4}} + C\sqrt{\beta} \tag{5.9}$$

where we recall $đx = \frac{dx}{N^3}$.

Proof. (5.8) follows from a straightforward computation. (5.9) follows from using (5.8) and Jensen's inequality. □

The difficulty lies in bounding the second integral in (5.6) uniformly in β. In 2D an analogous problem was overcome in [GJS76a, GJS76b] in the context of a low temperature expansion for Φ^4_2. Those techniques rely crucially on the logarithmic ultraviolet divergences in 2D, and the mutual absolute continuity between Φ^4_2 and its underlying Gaussian measure. Thus, they do not extend to 3D. However, we use the underlying strategy of that low temperature expansion in our approach.

We write $\mathscr{Z}_{\beta,N,K}$ as a sum of 2^{N^3} terms, where each term is a modified partition function that enforces the block averaged field to be either positive or negative on blocks. For each term in the expansion, we change variables and shift the field on blocks to $\pm\sqrt{\beta}$ so that the new mean of the field is small. We then apply Proposition 4.7 to each of these 2^{N^3} terms.

We separate the scales in the variational problem by coarse-graining the resulting Hamiltonian. Large scales are captured by an effective Hamiltonian, which is of a similar form to the second integral in (5.6). We treat this using methods inspired by [GJS76b, Theorem 3.1.1]: the expansion and translation allow us to obtain a β-independent bound on the effective Hamiltonian with an error term that depends only on the difference between the field and its block averages (the *fluctuation field*). The fluctuation field can be treated using the massless part of the underlying Gaussian measure (compare with [GJS76b, Proposition 2.3.2]).

The remainder term contains all the small-scale/ultraviolet divergences and we renormalise them using the pathwise approach of [BG19] explained above. Patching the scales together requires uniform in β estimates on the error terms from the renormalisation procedure using an analogue of the stability term (5.7) that incorporates the translation, and Lemma 5.5.

5.2 Expansion and translation by macroscopic phase profiles

Let $\chi_+, \chi_- : \mathbb{R} \to \mathbb{R}$ be defined as

$$\chi_+(a) = \frac{1}{\sqrt{\pi}} \int_{-a}^{\infty} e^{-c^2} dc, \quad \chi_-(a) = \chi_+(-a).$$

They satisfy

$$\chi_+(a) + \chi_-(a) = 1$$

and hence

$$\sum_{\sigma \in \{\pm 1\}^{\mathbb{B}_N}} \prod_{\square \in \mathbb{B}_N} \chi_{\sigma(\square)}\big(\phi(\square)\big) = 1$$

for any $\vec{\phi} = (\phi(\square))_{\square \in \mathbb{B}_N} \in \mathbb{R}^{\mathbb{B}_N}$.

For any $K > 0$, we expand

$$\begin{aligned} \mathscr{Z}_{\beta,N,K} &= \sum_{\sigma \in \{\pm 1\}^{\mathbb{B}_N}} \mathbb{E}_N \Big[e^{-\mathscr{H}_{\beta,N,K}} \prod_{\square \in \mathbb{B}_N} \chi_{\sigma(\square)}\big(\phi_K(\square)\big) \Big] \\ &= \sum_{\sigma \in \{\pm 1\}^{\mathbb{B}_N}} \mathscr{Z}^{\sigma}_{\beta,N,K} \end{aligned} \tag{5.10}$$

where we recall $\phi_K(\square) = \int_{\mathbb{T}_N} \phi_K \mathbf{1}_\square dx$.

We fix σ in what follows and sometimes suppress it from notation. Let $h = \sqrt{\beta}\sigma$. We then have

$$\begin{aligned} \mathscr{Z}^{\sigma}_{\beta,N,K} = \mathbb{E}_N \exp \Bigg(&- \int_{\mathbb{T}_N} :\mathscr{V}_\beta(\phi_K): - \frac{\gamma_K}{\beta^2} :\phi_K^2: - \delta_K \\ &- \frac{\eta}{2} :(\phi_K - h)^2: - \eta \phi_K h + \frac{\eta}{2} h^2 dx \\ &+ \sum_{\square \in \mathbb{B}_N} \log \Big(\chi_{\sigma(\square)}\big(\phi_K(\square)\big)\Big)\Bigg). \end{aligned}$$

We translate the Gaussian fields so that their new mean is approximately h. The translation we use is related to the *classical magnetism*, or response to the external field ηh, used in the 2D setting [GJS76a] and given by $\eta(-\Delta + \eta)^{-1} h$.

Lemma 5.6. *For every $K > 0$, let $h_K = \rho_K h$. Define $\tilde{g}_K = \eta(-\Delta + \eta)^{-1} h_K$ and $g_K = \rho_K \tilde{g}_K$. Then, there exists $C = C(\eta)$ such that*

$$|g_K|_\infty, |\nabla g_K|_\infty \leqslant C\sqrt{\beta} \tag{5.11}$$

where $|\cdot|_\infty$ denotes the supremum norm. Moreover,

$$\int_{\mathbb{T}_N} |\nabla g_K|^2 dx \leqslant \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx. \tag{5.12}$$

Finally, let

$$g^\flat_k = \sum_{n \in (N^{-1}\mathbb{Z})^3} \frac{1}{N^3} \tilde{\rho}_k \int_0^k \mathcal{J}_{k'}(n) \mathcal{F} g(n) dk$$

where $\tilde{\rho}_k$ is as in Section 4.2.2. Then, for any $s, s' \in \mathbb{R}$, $p \in (1,\infty)$ and $q \in [1,\infty]$, there exists $C_1 = C_1(\eta, s, p, q)$ and $C_2 = C_2(\eta, s, s', p, q)$ such that

$$\|g^\flat_k\|_{B^s_{p,q}} \leqslant C_1 \|g_K\|_{B^s_{p,q}} \tag{5.13}$$

and

$$\|\partial_k g^\flat_k\|_{B^{s'}_{p,q}} \leqslant C_2 \frac{1}{\langle k \rangle^{1+s-s'}} \|g_K\|_{B^s_{p,q}}. \tag{5.14}$$

Proof. The estimate (5.11) follows from the fact that $\eta(-\Delta+\eta)^{-1}$ and $\nabla\eta(-\Delta+\eta)^{-1}$ are L^∞ bounded operators. This is because the (η-dependent) Bessel potential and its first derivatives are absolutely integrable on $\mathbb{R}^3$. Hence, by applying Young's inequality for convolutions one obtains the L^∞ boundedness. The uniformity of the estimate over σ follows from $\|\sigma\|_{L^\infty}=1$ for every $\sigma\in\mathbb{B}_N$. The other estimates follow from standard results about smooth multipliers, the observation that $g_k^\flat=\tilde{\rho}_k g_K$ for any $K\geqslant k$, and Lemma 4.12. □

Remark 5.7. *Note that g_K is given by the covariance operator of μ_N applied to ηh. Moreover, note that $g_K\neq\tilde{g}_K$ since $\rho_K^2\neq\rho_K$, i.e. the Fourier cutoff is not sharp.*

By the Cameron-Martin theorem the density of μ_N under the translation $\phi=\psi+\tilde{g}_K$ transforms as

$$d\mu_N(\psi+\tilde{g}_K)=\exp\Big(-\int_{\mathbb{T}_N}\frac{1}{2}\tilde{g}_K(-\Delta+\eta)\tilde{g}_K+\psi(-\Delta+\eta)\tilde{g}_K dx\Big)d\mu_N(\psi).$$

Hence,

$$\mathscr{Z}^\sigma_{\beta,N,K}=\mathbb{E}_N e^{-\mathscr{H}^\sigma_{\beta,N,K}(\psi_K)-F^\sigma_{\beta,N,K}}(\psi)$$

where

$$\begin{aligned}\mathscr{H}^\sigma_{\beta,N,K}(\psi_K)=\int_{\mathbb{T}_N}&:\mathscr{V}_\beta(\psi_K+g_K):-\frac{\gamma_K}{\beta^2}:(\psi_K+g_K)^2:-\delta_K\\&-\frac{\eta}{2}:(\psi_K+g_K-h)^2:dx-\sum_{\square\in\mathbb{B}_N}\log\Big(\chi_{\sigma(\square)}\big((\psi_K+g_K)(\square)\big)\Big)\end{aligned}$$

and

$$F^\sigma_{\beta,N,K}(\psi)=\int_{\mathbb{T}_N}-\eta(\psi_K+g_K)h+\frac{\eta}{2}h^2+\frac{1}{2}\tilde{g}_K(-\Delta+\eta)\tilde{g}_K+\psi(-\Delta+\eta)\tilde{g}_K dx.$$

By integration by parts, the self-adjointness of ρ_K, and the definition of $\tilde{g}_K$

$$\begin{aligned}F^\sigma_{\beta,N,K}(\psi)&=\int_{\mathbb{T}_N}-\eta(\psi+\tilde{g}_K)h_K+\frac{\eta}{2}h^2+\frac{1}{2}|\nabla\tilde{g}_K|^2+\frac{\eta}{2}(\tilde{g}_K)^2+\eta\psi h_K dx\\&=\int_{\mathbb{T}_N}\frac{\eta}{2}(\tilde{g}_K-h_K)^2+\frac{\eta}{2}(1-\rho_K^2)h^2+\frac{1}{2}|\nabla\tilde{g}_K|^2dx.\end{aligned}\tag{5.15}$$

Thus, $F^\sigma_{\beta,N,K}(\psi)$ is independent of ψ and non-negative.

Remark 5.8. *Let $g=\eta(-\Delta+\eta)^{-1}h$. Then,*

$$\lim_{K\to\infty}F^\sigma_{\beta,N,K}=\int_{\mathbb{T}_N}\frac{\eta}{2}(g-h)^2+\frac{1}{2}|\nabla g|^2dx.\tag{5.16}$$

The second integrand on the righthand side of (5.16) *penalises the discontinuities of* σ. *Indeed,* $e^{-\int_{\mathbb{T}_N}\frac{1}{2}|\nabla g|^2dx}$ *is approximately equal to* $e^{-C\sqrt{\beta}|\partial\sigma|}$, *where* $\partial\sigma$ *denotes the surfaces of discontinuity of* σ, $|\partial\sigma|$ *denotes the area of these surfaces, and* $C > 0$ *is an inessential constant. Thus, for* β *sufficiently large,* $\mathscr{Z}_{\beta,N}$ *is approximately equal to*

$$e^{-C\sqrt{\beta}|\partial\sigma|} \times O(1) = \prod_{\Gamma_i\in\sigma} e^{-C\sqrt{\beta}|\Gamma_i|} \times O(1)$$

where Γ_i *are the connected components of* $\partial\sigma$ *(called contours). It would be interesting to further develop this contour representation for* $\nu_{\beta,N}$ *(compare with the 2D expansions of [GJS76a, GJS76b]).*

5.3 Coarse-graining of the Hamiltonian

We apply Proposition 4.7 to $-\log\mathbb{E}_N e^{-\mathscr{H}^{\sigma}_{\beta,N,K}(\uparrow_K)}$. For every $v\in\mathbb{H}_{b,K}$, define

$$\Psi^{\sigma}_K(v) = \mathscr{H}^{\sigma}_{\beta,N,K}(\uparrow_K + V_K) + \frac{1}{2}\int_0^K\int_{\mathbb{T}_N} v_k^2\, dxdk. \tag{5.17}$$

Let $Z_K = \vec{\uparrow}_K + V_K + g_K$, where $\vec{\uparrow}_K = (\uparrow_K(\square))_{\square\in\mathbb{B}_N}$. We split the Hamiltonian as

$$\mathscr{H}^{\sigma}_{\beta,N,K}(\uparrow_K + V_K) = \mathscr{H}^{\mathrm{eff}}_K(Z_K) + \mathscr{R}_K + \frac{1}{2}\int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K+g_K)dx \tag{5.18}$$

where

$$\mathscr{H}^{\mathrm{eff}}_K(Z_K) = \int_{\mathbb{T}_N}\frac{1}{2}\mathscr{V}_{\beta,N,K}(Z_K) - \frac{\eta}{2}(Z_K-h)^2dx - \sum_{\square\in\mathbb{B}_N}\log\Big(\chi_{\sigma(\square)}\big(Z_K(\square)\big)\Big)$$

is an effective Hamiltonian introduced to capture macroscopic scales of the system. The quantity $\mathscr{R}_K$ is then determined by (5.18) and is explicitly given by

$$\begin{aligned}\mathscr{R}_K = \int_{\mathbb{T}_N} &: \mathscr{V}_\beta(\uparrow_K + V_K + g_K) : -\frac{\gamma_K}{\beta^2} : (\uparrow_K + V_K + g_K)^2 : -\delta_K \\ &- \frac{1}{2}\mathscr{V}_\beta(\vec{\uparrow}_K + V_K + g_K) - \frac{1}{2}\mathscr{V}_\beta(V_K+g_K) \\ &- \frac{\eta}{2} : (\uparrow_K + V_K + g_K - h)^2 : + \frac{\eta}{2}(\vec{\uparrow}_K + V_K + g_K - h)^2 dx.\end{aligned}$$

All analysis/cancellation of ultraviolet divergences occurs within the sum of $\mathscr{R}_K$ and the drift entropy, see (5.27). Finally, the last term in (5.18) is a stability term which is key for our non-perturbative analysis, namely it allows us to obtain estimates that are uniform in the drift.

The key point is that we coarse-grain the field by block averaging $\uparrow_K$, the most singular term. This allows us to preserve the structure of the low temperature potential $\mathscr{V}_\beta$ on macroscopic scales (captured in $\mathscr{H}^{\mathrm{eff}}_K(Z_K)$), which is crucial to obtaining estimates independent of β on the free energy.

5.4 Killing divergences

5.4.1 Changing drift variables

For any $v \in \mathbb{H}_{b,K}$, define $r = r(v) \in \mathbb{H}_K$ by

$$r_k = v_k + \frac{4}{\beta}\mathcal{J}_k \Psi_k + \frac{12}{\beta}\mathcal{J}_k(\vee_k \circledcirc (V_k^\flat + g_k^\flat)). \tag{5.19}$$

In our analysis it is convenient to use an intermediate change of variables for the drift. Define $u = u(v) \in \mathbb{H}_K$ by

$$u_k = v_k + \frac{4}{\beta}\mathcal{J}_k \Psi_k. \tag{5.20}$$

Inserting (5.19) and (5.20) into the definition of the integrated drift, $V_k = \int_0^k \mathcal{J}_{k'} v_{k'} dk'$, we obtain

$$\begin{aligned} V_k &= -\frac{4}{\beta}\Upsilon_k - \frac{12}{\beta}\int_0^k \mathcal{J}_{k'}^2(\vee_{k'} \circledcirc (V_{k'}^\flat + g_{k'}^\flat))dk' + R_k \\ &= -\frac{4}{\beta}\Upsilon_k + U_k \end{aligned} \tag{5.21}$$

where $R_k = \int_0^k \mathcal{J}_{k'} r_{k'} dk'$ and $U_k = \int_0^k \mathcal{J}_{k'} u_{k'} dk'$.

The following proposition contains useful estimates estimates on U_K and V_K.

Proposition 5.9. *For any $\varepsilon > 0$ and $\kappa > 0$ sufficiently small, there exists $C = C(\varepsilon, \kappa, \eta) > 0$ such that, for all $\beta > 1$,*

$$\sup_{0 \leqslant k \leqslant K} \|U_k\|^2_{H^{1-\kappa}} \leqslant \frac{CN_K^\Xi}{N^3} + \frac{\varepsilon}{\beta^3}\int_{\mathbb{T}_N} \mathcal{V}_\beta(V_K + g_K)đx + C\int_0^K \int_{\mathbb{T}_N} r_k^2 đx dk \tag{5.22}$$

$$\sup_{0 \leqslant k \leqslant K} \|V_k\|^2_{H^{\frac{1}{2}-\kappa}} \leqslant \frac{CN_K^\Xi}{N^3} + \frac{\varepsilon}{\beta^3}\int_{\mathbb{T}_N} \mathcal{V}_\beta(V_K + g_K)đx + C\int_0^K \int_{\mathbb{T}_N} r_k^2 đx dk \tag{5.23}$$

where we recall $đx = \frac{dx}{N^3}$; and N_K^Ξ is a positive random variable on Ω that is $\mathbb{P}$-almost surely given by a finite linear combination of powers of (finite integrability) Besov and Lebesgue norms of the diagrams $\Xi = \{\mathsf{I}, \vee, \Upsilon, \Upsilon^{\vee}, \Psi^{\vee}, \vee^{\vee}\}$ on the interval $[0, K]$.

Proof. See Section 5.6.1. □

Remark 5.10. *As a consequence of Proposition 4.4, the random variable N_K^Ξ satisfies the following estimate: there exists $C = C(\eta) > 0$ such that*

$$\mathbb{E} N_K^\Xi \leqslant C N^3. \tag{5.24}$$

In the following we denote by N_K^Ξ any positive random variable on Ω that satisfies (5.24). In practice it is always $\mathbb{P}$-almost surely given by a finite linear combination of powers of (finite integrability) Besov norms of the diagrams in Ξ on $[0,K]$. Note that N_K^Ξ includes constants of the form $C = C(\eta) > 0$.

5.4.2 The main small-scale estimates

In the following we write $\approx$ to mean equal up to a term with expectation 0 under $\mathbb{P}$.

Proposition 5.11. *Let $\beta > 0$. For every $K > 0$, define*

$$\gamma_K = -4^2 \cdot 3 \cdot \ominus_K \tag{5.25}$$

where $\ominus_K$ is defined in (4.4), and

$$\delta_K = \mathbb{E}\Bigg[\int_{\mathbb{T}_N} \int_0^K -\frac{8}{\beta^2}(\mathcal{J}_k \Psi_k)^2 dk - \frac{256}{\beta^4} \uparrow_K \Big(\Psi_K \Big)^3 + \frac{96}{\beta^3} \Big(\Psi_K \Big)^2 \vee_K dx \Bigg]. \tag{5.26}$$

Then, for every $v \in \mathbb{H}_{b,K}$,

$$\mathcal{R}_K + \frac{1}{2}\int_0^K \int_{\mathbb{T}_N} v_k^2 dk dx \approx \sum_{i=1}^4 \mathcal{R}_K^i + \frac{1}{2}\int_0^K \int_{\mathbb{T}_N} r_k^2 dx dk \tag{5.27}$$

where

$$\begin{aligned}
\mathcal{R}_K^1 = \int_{\mathbb{T}_N} & -\frac{1}{2\beta}(\vec{\uparrow}_K)^4 - \frac{2}{\beta}(\vec{\uparrow}_K)^3 (V_K + g_K) - \frac{3}{\beta}(\vec{\uparrow}_K)^2 (V_K + g_K)^2 \\
& - \frac{2}{\beta}\vec{\uparrow}_K V_K^3 - \frac{6}{\beta}\vec{\uparrow}_K V_K^2 g_K - \frac{6}{\beta}\vec{\uparrow}_K V_K g_K^2 \\
& + \frac{\eta+2}{2}(\vec{\uparrow}_K)^2 + (\eta+2)\vec{\uparrow}_K V_K dx \\
\mathcal{R}_K^2 = \int_{\mathbb{T}_N} & \frac{192}{\beta^3} \uparrow_K \Psi_K^2 U_K - \frac{48}{\beta^2} \uparrow_K \Psi_K U_K^2 - \frac{96}{\beta^2} \uparrow_K \Psi_K g_K U_K
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{4}{\beta}\uparrow_K U_K^3 + \frac{12}{\beta}\uparrow_K g_K U_K^2 + \frac{12}{\beta}\uparrow_K g_K^2 U_K - (4+\eta)\uparrow_K U_K dx \\
\mathscr{R}_K^3 &= \int_{\mathbb{T}_N} \frac{12}{\beta}(\curlyvee_K \circeq g_K)U_K + \frac{6}{\beta}(\curlyvee_K \circeq U_K - \curlyvee_K \ogreaterthan U_K)U_K \\
&\quad - \frac{48}{\beta^2}(\curlyvee_K \olessthan \Upsilon_K)U_K + \frac{12}{\beta}(\curlyvee_K \olessthan g_K)U_K + \frac{6}{\beta}(\curlyvee_K \olessthan U_K)U_K dx \\
&+ \int_{\mathbb{T}_N}\int_0^K \frac{12}{\beta}\Big(\curlyvee_k \ogreaterthan (\partial_k V_k^\flat + \partial_k g_k^\flat)\Big)U_k \\
&\quad + \frac{12}{\beta}\Big(\curlyvee_K \ogreaterthan (V_K + g_K - V_K^\flat - g_K^\flat)\Big)U_K \\
&\quad - \frac{72}{\beta^2}\left(\Big(\mathscr{J}_k(\curlyvee_k \ogreaterthan (V_k^\flat + g_k^\flat))\Big)^2 - \Big(\mathscr{J}_k\curlyvee_k \circeq \mathscr{J}_k\curlyvee_k\Big)(V_k^\flat + g_k^\flat)^2\right)dkdx \\
\mathscr{R}_K^4 &= -\int_{\mathbb{T}_N} \frac{48}{\beta^2}\Upsilon^{\bullet}_K U_K + \frac{2\gamma_K}{\beta^2}(V_K^\flat + g_K)(V_K + g_K - V_K^\flat - g_K^\flat) \\
&\quad + \frac{\gamma_K}{\beta^2}(V_K + g_K - V_K^\flat - g_K^\flat)^2 dx \\
&+ \int_{\mathbb{T}_N}\int_0^K \frac{2\gamma_k}{\beta^2}(\partial_k V_k^\flat + \partial_k g_k^\flat)(V_k^\flat + g_k^\flat) + \frac{72}{\beta^2}\curlyvee\!\curlyvee_k (V_k^\flat + g_k^\flat)^2 dkdx.
\end{aligned}
$$

Moreover, the following estimate holds: for any $\varepsilon > 0$, there exists $C = C(\varepsilon, \eta) > 0$ such that, for all $\beta > 1$,

$$
\max_{i=1,\dots,4}\left|\mathscr{R}_K^i\right| \leqslant CN_K^{\Xi} + \varepsilon\Big(\int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K + g_K)dx + \frac{1}{2}\int_0^K\int_{\mathbb{T}_N} r_k^2 dxdk\Big) \tag{5.28}
$$

where N_K^{Ξ} is as in Remark 5.10.

Proof. We establish (5.27) in Section 5.5 by arguing as in [BG19, Lemma 5]. The remainder estimates (5.28) are then established in Section 5.6. □

Remark 5.12. *The products $\uparrow_K \Upsilon_K$ and $\uparrow_K \Upsilon_K^2$ appearing above are classically ill-defined in the limit $K \to \infty$. However, (probabilistic) estimates on the resonant product $\Upsilon^{\bullet}_K$ uniform in K are obtained in Proposition 4.4. Hence, the first product can be analysed using a paraproduct decompositions* (1.7)*. The second product is less straightforward and requires a double paraproduct decomposition (see [BG19, Lemma 21 and Proposition 6] and [CC18, Proposition 2.22]).*

5.5 Proof of (5.27): Isolating and cancelling divergences

Using that $\uparrow_K, \vec{\uparrow}_K, \curlyvee_K, \Psi_K$ and $\mathbf{\Psi}_K$ all have expectation zero,

$$
\mathscr{R}_K = \int_{\mathbb{T}_N} \frac{1}{\beta}\mathbf{\Psi}_K + \frac{4}{\beta}\Psi_K(V_K + g_K) + \frac{6}{\beta}\curlyvee_K(V_K + g_K)^2 + \frac{4}{\beta}\uparrow_K(V_K + g_K)^3
$$

$$
\begin{aligned}
&- 2\vee_K - 4\uparrow_K(V_K + g_K) + \mathscr{V}_\beta(V_K + g_K) \\
&- \frac{\gamma_K}{\beta^2}\Big(\vee_K + 2\uparrow_K(V_K + g_K) + (V_K + g_K)^2\Big) - \delta_K \\
&- \frac{1}{2\beta}(\vec{\uparrow}_K)^4 - \frac{2}{\beta}(\vec{\uparrow}_K)^3(V_K + g_K) - \frac{3}{\beta}(\vec{\uparrow}_K)^2(V_K + g_K)^2 \\
&- \frac{2}{\beta}\vec{\uparrow}_K(V_K + g_K)^3 + (\vec{\uparrow}_K)^2 + 2\vec{\uparrow}_K(V_K + g_K) - \frac{1}{2}\mathscr{V}_\beta(V_K + g_K) \\
&- \frac{1}{2}\mathscr{V}_\beta(V_K + g_K) \\
&- \frac{\eta}{2}\vee_K - \eta\uparrow_K(V_K + g_K - h) - \frac{\eta}{2}(V_K + g_K - h)^2 \\
&+ \frac{\eta}{2}(\vec{\uparrow}_K)^2 + \eta\vec{\uparrow}_K(V_K + g_K - h) + \frac{\eta}{2}(V_K + g_K - h)^2 dx \\
\approx \int_{\mathbb{T}_N} &\frac{4}{\beta}\Psi_K V_K + \frac{6}{\beta}\vee_K(V_K + g_K)^2 + \frac{4}{\beta}\uparrow_K(V_K + g_K)^3 - 4\uparrow_K V_K \\
&- \frac{2\gamma_K}{\beta^2}\uparrow_K V_K - \frac{\gamma_K}{\beta^2}(V_K + g_K)^2 - \delta_K \\
&- \frac{1}{2\beta}(\vec{\uparrow}_K)^4 - \frac{2}{\beta}(\vec{\uparrow}_K)^3(V_K + g_K) - \frac{3}{\beta}(\vec{\uparrow}_K)^2(V_K + g_K)^2 \\
&- \frac{2}{\beta}\vec{\uparrow}_K V_K^3 - \frac{6}{\beta}\vec{\uparrow}_K V_K^2 g_K - \frac{6}{\beta}\vec{\uparrow}_K V_K g_K^2 + (\vec{\uparrow}_K)^2 + 2\vec{\uparrow}_K V_K \\
&- \eta\uparrow_K V_K + \frac{\eta}{2}(\vec{\uparrow}_K)^2 + \eta\vec{\uparrow}_K V_K dx
\end{aligned}
$$

Hence, by reordering terms,

$$
\begin{aligned}
\mathscr{R}_K \approx \mathscr{R}_K^1 + \int_{\mathbb{T}_N} &\frac{4}{\beta}\Psi_K V_K + \frac{6}{\beta}\vee_K(V_K + g_K)^2 + \frac{4}{\beta}\uparrow_K(V_K + g_K)^3 \\
&- (4 + \eta)\uparrow_K V_K - \frac{2\gamma_K}{\beta^2}\uparrow_K V_K - \frac{\gamma_K}{\beta^2}(V_K + g_K)^2 - \delta_K dx.
\end{aligned}
\tag{5.29}
$$

Ignoring the renormalisation counterterms (i.e. those involving γ_K and δ_K), the divergences in (5.29) are contained in the integrals $\int_{\mathbb{T}_N} \frac{4}{\beta}\Psi_K V_K dx$ and $\int_{\mathbb{T}_N} \frac{6}{\beta}(V_K + g_K)^2$. In order to kill these divergences, we use changes of variables in the drift entropy to mostly cancel them; the remaining divergences are killed by the renormalisation counterterms. We renormalise the leading order divergences, ie. those polynomial in K, in Section 5.5.1. The divergences that are logarithmic in K are renormalised in Section 5.5.2.

In order to use the drift entropy to cancel divergences, we decompose certain (spatial) integrals across ultraviolet scales $k \in [0, K]$ using Itô's formula. Error terms are produced that are stochastic integrals with respect to martingales (specifically, with respect to $d\Psi_k$ and $d\vee_k$). The following lemma allows us to argue that these stochastic integrals are ≈ 0.

Lemma 5.13. *For any $v \in \mathbb{H}_{b,K}$, the stochastic integrals*

$$\int_{\mathbb{T}_N} \int_0^K V_k d\<3>_k dx \tag{5.30}$$

and

$$\sum_{i<j-1} \int_{\mathbb{T}_N} \int_0^K U_k \Delta_i (V_k^\flat + g_k^\flat) d(\Delta_j \<2>_k) dx \tag{5.31}$$

are martingales. We recall that, above, Δ_i denotes the i-th Littlewood-Paley block.

Proof. In this proof, for any continuous local martingale $Z_\bullet$, we write $\langle\!\langle Z, Z \rangle\!\rangle_\bullet$ for the corresponding quadratic variation process. Moreover, for any Z-adapted process $Y_\bullet$, we write $\int_0^K Y_k \cdot dZ_k$ to denote the stochastic integral $\int_{\mathbb{T}_N} \int_0^K Y_k dZ_k dx$.

We begin with two observations: first, let $v \in \mathbb{H}_{b,M,K}$ for some $M > 0$, i.e. those $v \in \mathbb{H}_{b,K}$ satisfying (4.18). Then, by Sobolev embedding, there exists $C = C(M, N, K, \eta) > 0$ such that

$$\sup_{0 \leqslant k \leqslant K} \|V_k\|_{L^6}^6 \leqslant C$$

$\mathbb{P}$-almost surely.

Second, recalling the iterated integral representation of the Wick powers $\<3>_k$ and $\<2>_k$ (see e.g. (4.3)), one can show $d\<3>_k = 3\<2>_k d\<1>_k$ and $d(\Delta_j \<2>_k) = \Delta_j d\<2>_k = 2\Delta_j \<1>_k d\<1>_k$. Thus, we can write the stochastic integrals (5.30) and (5.31) in terms of stochastic integrals with respect to $d\<1>_k$. It suffices to show that their quadratic variations are finite in expectation.

Using that $d\langle\!\langle \<1>, \<1> \rangle\!\rangle_k = \mathcal{J}_k^2(1) dk = \mathcal{J}_k^2 dk$ and by Young's inequality,

$$\begin{aligned} \mathbb{E}\Big[\langle\!\langle \int_0^\bullet V_k \cdot d\<3>_k \rangle\!\rangle_K \Big] &= 3^2 \mathbb{E}\Big[\int_0^K \int_{\mathbb{T}_N} V_k^2 \<2>_k^2 \mathcal{J}_k^2 dx dk \Big] \\ &\leqslant 3^2 \mathbb{E}\Big[\frac{1}{2} \int_0^K \int_{\mathbb{T}_N} V_k^4 + \<2>_k^4 \mathcal{J}_k^4 dx dk \Big] < \infty. \end{aligned}$$

Hence, (5.30) is a martingale.

Now consider (5.31). By (5.21),

$$\sum_{i<j-1} \int_0^K \int_{\mathbb{T}_N} U_k \Delta_i (V_k^\flat + g_k^\flat) \cdot d(\Delta_j \<2>_k) = Z_K^a + Z_K^b$$

where

$$Z_K^a = 2 \sum_{i<j-1} \int_0^K \int_{\mathbb{T}_N} V_k \Delta_i V_k^\flat \Delta_j \<1>_k d\<1>_k$$

$$Z_K^b = 2\sum_{i<j-1}\int_0^K\int_{\mathbb{T}_N}\Big(\frac{4}{\beta}\Upsilon_k\Delta_i V_k^\flat + U_k\Delta_i g_k^\flat\Big)\Delta_j \dagger_k d\dagger_k.$$

Arguing as for (5.30), one can show $\mathbb{E}\langle\!\langle Z_\bullet^b\rangle\!\rangle_K < \infty$.

By Young's inequality and using that Littlewood-Paley blocks and the $\flat$ operator are L^p multipliers, we have

$$\begin{aligned}\mathbb{E}\langle\!\langle Z_\bullet^a\rangle\!\rangle_K &= 2^2\mathbb{E}\Bigg[\sum_{i<j-1}\int_0^K\int_{\mathbb{T}_N} V_k^2(\Delta_i V_k^\flat)^2(\Delta_j\dagger_k)^2\mathcal{J}_k^2 dk\Bigg]\\ &\leqslant 2^2\mathbb{E}\Bigg[\frac{2}{3}\int_0^K\int_{\mathbb{T}_N} V_k^6 + \frac{1}{3}(\Delta_k\dagger_k)^6\mathcal{J}_k^6 dx dk\Bigg] < \infty\end{aligned}$$

thus establishing that (5.31) is a martingale. □

5.5.1 Energy renormalisation

In the next lemma, we cancel the leading order divergence using the change of variables (5.20) in the drift entropy. The error term does not depend on the drift and is divergent in expectation (as $K \to \infty$); it is cancelled by one part of the energy renormalisation δ_K (see (5.26)).

Lemma 5.14.

$$\int_{\mathbb{T}_N}\frac{4}{\beta}\Psi_K V_K dx + \frac{1}{2}\int_0^K\int_{\mathbb{T}_N} v_k^2 dx dk \approx \int_0^K\int_{\mathbb{T}_N} -\frac{8}{\beta^2}(\mathcal{J}_k\Psi_k)^2 + \frac{1}{2}u_k^2 dx dk.$$

Proof. By Itô's formula, Lemma 5.13, and the self-adjointness of $\mathcal{J}_k$,

$$\int_{\mathbb{T}_N}\frac{4}{\beta}\Psi_K V_K dx = \int_{\mathbb{T}_N}\Big(\int_0^K\frac{4}{\beta}\Psi_k\partial_k V_k dk + \frac{4}{\beta}V_k d\Psi_k\Big)dx \approx \int_{\mathbb{T}_N}\int_0^K\frac{4}{\beta}\mathcal{J}_k\Psi_k v_k dk dx.$$

Hence, by (5.20),

$$\begin{aligned}&\int_{\mathbb{T}_N}\frac{4}{\beta}\Psi_K V_K dx + \frac{1}{2}\int_0^K\int_{\mathbb{T}_N} v_k^2 dx dk\\ &\qquad\approx \int_{\mathbb{T}_N}\int_0^K\frac{4}{\beta}\mathcal{J}_k\Psi_k\big(-\frac{4}{\beta}\Psi_k + u_k\big) + \frac{1}{2}\big(-\frac{4}{\beta}\Psi_k + u_k\big)^2 dk dx\\ &\qquad= \int_{\mathbb{T}_N}\int_0^K -\frac{8}{\beta^2}(\mathcal{J}_k\Psi_k)^2 + \frac{1}{2}u_k^2 dk dx.\end{aligned}$$

□

As a consequence of (5.20), the remaining (non-counterterm) integrals in (5.29) acquire additional divergences that are independent of the drift. We isolate them in the next lemma; they are also renormalised by parts of the energy renormalisation (see (5.26)).

Lemma 5.15.

$$\int_{\mathbb{T}_N} \frac{4}{\beta} \uparrow_K (V_K + g_K)^3 - (4+\eta) \uparrow_K V_K dx \approx \mathscr{R}_K^2 - \int_{\mathbb{T}_N} \frac{256}{\beta^4} \uparrow_K \Psi_K^3 dx \tag{5.32}$$

and

$$\int_{\mathbb{T}_N} \frac{6}{\beta} \vee_K (V_K + g_K)^2 dx \approx \int_{\mathbb{T}_N} \frac{96}{\beta^3} \vee_K \Psi_K^2 - \frac{48}{\beta^2} \vee_K \Psi_K U_K + \frac{6}{\beta} \vee_K U_K^2 + \frac{12}{\beta} \vee_K g_K U_K dx. \tag{5.33}$$

Proof. By (5.21),

$$\begin{aligned}
&\int_{\mathbb{T}_N} \frac{4}{\beta} \uparrow_K (V_K + g_K)^3 dx \\
&= \int_{\mathbb{T}_N} \frac{4}{\beta} \uparrow_K \Big(-\frac{4}{\beta}\Psi_K + U_K + g_K\Big)^3 dx \\
&= \int_{\mathbb{T}_N} \frac{4}{\beta} \uparrow_K \Bigg(-\frac{64}{\beta^3}\Psi_K^3 + \frac{48}{\beta^2}\Psi_K^2 (U_K + g_K) \\
&\qquad - \frac{12}{\beta}\Psi_K U_K^2 - \frac{24}{\beta}\Psi_K U_K g_K - \frac{12}{\beta}\Psi_K g_K^2 \\
&\qquad + U_K^3 + 3U_K^2 g_K + 3U_K g_K^2 + g_K^3 \Bigg) dx \\
&\approx \int_{\mathbb{T}_N} -\frac{256}{\beta^4} \uparrow_K \Psi_K^3 + \frac{192}{\beta^3} \uparrow_K \Psi_K^2 U_K \\
&\qquad - \frac{48}{\beta^2} \uparrow_K \Psi_K U_K^2 - \frac{96}{\beta^2} \uparrow_K \Psi_K g_K U_K \\
&\qquad + \frac{4}{\beta} \uparrow_K U_K^3 + \frac{12}{\beta} \uparrow_K g_K U_K^2 + \frac{12}{\beta} \uparrow_K g_K^2 U_K dx.
\end{aligned} \tag{5.34}$$

Above we have used Wick's theorem and the fact that Ψ_K is Wick ordered to conclude $\mathbb{E}\big[\uparrow_K \Psi_K^2 g_K\big] = \mathbb{E}\big[\uparrow_K \Psi_K g_K^2\big] = 0$.

Similarly, $\mathbb{E} \uparrow_K \Psi_K = 0 \; \Psi_K$. Hence, by (5.21)

$$\int_{\mathbb{T}_N} (4+\eta) \uparrow_K V_K \approx \int_{\mathbb{T}_N} (4+\eta) \uparrow_K U_K dx. \tag{5.35}$$

Combining (5.34) and (5.35) establishes (5.32).

By (5.21),

$$\begin{aligned}
\int_{\mathbb{T}_N} \frac{6}{\beta}\mathsf{V}_K(V_K+g_K)^2dx &= \int_{\mathbb{T}_N} \frac{6}{\beta}\mathsf{V}_K\Big(-\frac{4}{\beta}\Psi_K + U_K + g_K\Big)^2 dx \\
&= \int_{\mathbb{T}_N} \frac{6}{\beta}\mathsf{V}_K\Big(\frac{16}{\beta^2}\Psi_K^2 - \frac{8}{\beta}(U_K+g_K)\Psi_K \\
&\qquad U_K^2 + 2U_Kg_K + g_K^2\Big)dx \\
&\approx \int_{\mathbb{T}_N} \frac{96}{\beta^3}\mathsf{V}_K\Psi_K^2 + \frac{12}{\beta}\mathsf{V}_K\Big(-\frac{4}{\beta}\Psi_K\Big)U_K \\
&\qquad + \frac{6}{\beta}\mathsf{V}_KU_K^2 + \frac{12}{\beta}\mathsf{V}_Kg_KU_Kdx
\end{aligned}$$

where we have used that $\mathbb{E}[\mathsf{V}_Kg_K]=0$ and, by Wick's theorem, $\mathbb{E}\big[\mathsf{V}_K\Psi_K\big]=0$. This establishes (5.33). □

The divergences encountered in Lemmas 5.14 and 5.15 that are independent of the drift are killed by the energy renormalisation δ_K since, by definition,

$$\delta_K \approx \int_{\mathbb{T}_N} -\int_0^K \frac{8}{\beta^2}(\mathcal{J}_k\mathsf{W}_k)^2dk - \frac{256}{\beta^4}\mathsf{I}_K\Big(\Psi_K\Big)^3 + \frac{96}{\beta^3}\Big(\Psi_K\Big)^2\mathsf{V}_Kdx. \quad (5.36)$$

5.5.2 Mass renormalisation

The integrals on the righthand side of (5.33) that involve the drift cannot be bounded uniformly as $K\to\infty$. We isolate divergences using a paraproduct decomposition and expand the drift entropy using (5.19) to mostly cancel them. This is done in Lemma 5.16. The remaining divergences are then killed in Lemma 5.17 using the mass renormalisation.

Lemma 5.16.

$$\begin{aligned}
\int_{\mathbb{T}_N} -\frac{48}{\beta^2}\mathsf{V}_K\Psi_KU_K + \frac{6}{\beta}\mathsf{V}_KU_K^2 + \frac{12}{\beta}\mathsf{V}_Kg_KU_Kdx + \frac{1}{2}\int_0^K u_k^2dkdx \\
\approx \mathcal{R}_K^3 + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K r_k^2dkdx \\
+ \int_{\mathbb{T}_N} \frac{96}{\beta^3}\mathsf{V}_K\Psi_K^2 - \frac{48}{\beta^2}\mathsf{V}_K \circleddash \Psi_KU_Kdx \\
- \int_{\mathbb{T}_N}\int_0^K \frac{72}{\beta^2}(\mathcal{J}_k\mathsf{V}_k \circleddash \mathcal{J}_k\mathsf{V}_k)(V_k^\flat + g_k)^2dkdx.
\end{aligned} \quad (5.37)$$

Proof. We write

$$-\frac{48}{\beta^2}\<2>_K\<30>_K U_K + \frac{12}{\beta}\<2>_K U_K g_K = \frac{12}{\beta}\<2>_K\Big(-\frac{4}{\beta}\<30>_K + g_K\Big)U_K.$$

Thus, using (5.21) and a paraproduct decomposition on the most singular products,

$$\begin{aligned}
&\int_{\mathbb{T}_N} \frac{12}{\beta}\<2>_K\Big(-\frac{4}{\beta}\<30>_K + g_K\Big)U_K + \frac{6}{\beta}\<2>_K U_K^2 dx \\
&\quad= \int_{\mathbb{T}_N} \frac{12}{\beta}\left(\<2>_K \varogreaterthan \Big(-\frac{4}{\beta}\<30>_K + g_K\Big)\right)U_K + \frac{6}{\beta}(\<2>_K \varogreaterthan U_K)U_K \\
&\qquad\quad + \frac{12}{\beta}\left(\<2>_K \circeq \Big(-\frac{4}{\beta}\<30>_K + g_K\Big)\right)U_K + \frac{6}{\beta}(\<2>_K \circeq U_K)U_K \\
&\qquad\quad + \frac{12}{\beta}\left(\<2>_K \varolessthan \Big(-\frac{4}{\beta}\<30>_K + g_K\Big)\right)U_K + \frac{6}{\beta}(\<2>_K \varolessthan U_K)U_K \\
&\quad= \int_{\mathbb{T}_N} \frac{12}{\beta}\Big(\<2>_K \varogreaterthan (V_K + g_K)\Big)U_K - \frac{48}{\beta^2}\Big(\<2>_K \circeq \<30>_K\Big)U_K \\
&\qquad\quad + \frac{12}{\beta}(\<2>_K \circeq g_K)U_K + \frac{6}{\beta}(\<2>_K \circeq U_K - \<2>_K \varogreaterthan U_K)U_K \\
&\qquad\quad - \frac{48}{\beta^2}\Big(\<2>_K \varolessthan \<30>_K\Big)U_K + \frac{12}{\beta}(\<2>_K \varolessthan g_K)U_K + \frac{6}{\beta}(\<2>_K \varolessthan U_K)U_K dx.
\end{aligned} \tag{5.38}$$

All except the first two integrals are absorbed into $\mathscr{R}_K^3$.

For the first integral, we use the (drift-dependent) change of variables (5.19) in the drift entropy of u to mostly cancel the divergence. Due to the paraproduct term, using Itô's formula to decompose into scales requires us to control $V_k + g_k$ for $k < K$. In order to be able to do this, we replace $V_K + g_K$ by $V_K^\flat + g_K^\flat$ first. Then, applying Itô's formula, Lemma 5.13, and using the self-adjointness of $\mathcal{J}_k$,

$$\begin{aligned}
&\int_{\mathbb{T}_N} \frac{12}{\beta}\Big(\<2>_K \varogreaterthan (V_K + g_K)\Big)U_K dx \\
&= \int_{\mathbb{T}_N} \frac{12}{\beta}\Big(\<2>_K \varogreaterthan (V_K^\flat + g_K^\flat)\Big)U_K + \frac{12}{\beta}\Big(\<2>_K \varogreaterthan (V_K + g_K - V_K^\flat - g_K^\flat)\Big)U_K dx \\
&\approx \int_{\mathbb{T}_N}\int_0^K \frac{12}{\beta}\mathcal{J}_k\Big(\<2>_k \varogreaterthan (V_k^\flat + g_k^\flat)\Big)u_k + \frac{12}{\beta}\Big(\<2>_k \varogreaterthan (\partial_k V_k^\flat + \partial_k g_k^\flat)\Big)U_k dk dx \\
&\qquad + \int_{\mathbb{T}_N} \frac{12}{\beta}\Big(\<2>_K \varogreaterthan (V_K + g_K - V_K^\flat - g_K^\flat)\Big)U_K \Bigg] dx.
\end{aligned} \tag{5.39}$$

From (5.19) and (5.20)

$$\begin{aligned}
\int_{\mathbb{T}_N}\int_0^K \frac{12}{\beta}\mathcal{J}_k(\curlyvee_k \obslash (V_k^\flat + g_k^\flat))u_k + \frac{1}{2}u_k^2 dk dx \\
&= \int_{\mathbb{T}_N}\int_0^K -\frac{72}{\beta^2}\Big(\mathcal{J}_k(\curlyvee_k \obslash (V_k^\flat + g_k^\flat))\Big)^2 + \frac{1}{2}r_k^2 dk dx \\
&= \int_{\mathbb{T}_N}\int_0^K -\frac{72}{\beta^2}\Big(\mathcal{J}_k\curlyvee_k \circledequal \mathcal{J}_k\curlyvee_k\Big)(V_k^\flat + g_k^\flat)^2 + \frac{1}{2}r_k^2 \\
&\quad - \frac{72}{\beta^2}\Bigg(\Big(\mathcal{J}_k(\curlyvee_k \obslash (V_k^\flat + g_k^\flat))\Big)^2 - \Big(\mathcal{J}_k\curlyvee_k \circledequal \mathcal{J}_k\curlyvee_k\Big)(V_k^\flat + g_k^\flat)^2\Bigg) dk dx.
\end{aligned} \tag{5.40}$$

Combining (5.38), (5.39), and (5.40) yields (5.37). □

We now cancel the divergences in the last two terms of (5.37) using the mass renormalisation.

Lemma 5.17.

$$\begin{aligned}
\mathscr{R}_K^4 \approx &\int_{\mathbb{T}_N} -\frac{48}{\beta^2}\curlyvee_K \circledequal \Psi_K U_K dx \\
&- \int_{\mathbb{T}_N}\int_0^K \frac{72}{\beta^2}\Big(\mathcal{J}_k\curlyvee_k \circledequal \mathcal{J}_k\curlyvee_k\Big)(V_k^\flat + g_k^\flat)^2 dk dx \\
&- \int_{\mathbb{T}_N} \frac{2\gamma_K}{\beta^2}\uparrow_K V_K - \frac{\gamma_K}{\beta^2}(V_K + g_K)^2 dx
\end{aligned} \tag{5.41}$$

Proof. By the definition of $\clubsuit_K$ (see Section 4.1.1),

$$\begin{aligned}
-\int_{\mathbb{T}_N} \frac{48}{\beta^2}\curlyvee_K \circledequal \Psi_K U_K - \frac{2\gamma_K}{\beta^2}\uparrow_K V_K dx \\
&= -\int_{\mathbb{T}_N} \frac{48}{\beta^2}\clubsuit_K U_K + \frac{8\gamma_K}{\beta^3}\uparrow_K \Psi_K dx \\
&\approx -\int_{\mathbb{T}_N} \frac{48}{\beta^2}\clubsuit_K U_K dx
\end{aligned} \tag{5.42}$$

where we have used that, by Wick's theorem, $\mathbb{E}\big[\uparrow_K \Psi_K\big] = 0$.

To renormalise the second integral in (5.41), we need to rewrite the remaining counterterm in terms of $V_K^\flat$:

$$
\begin{aligned}
&-\int_{\mathbb{T}_N} \frac{\gamma_K}{\beta^2}(V_K + g_K)^2 dx \\
&\quad = -\int_{\mathbb{T}_N} \frac{\gamma_K}{\beta^2}(V_K^\flat + g_K^\flat)^2 + 2(V_K^\flat + g_K^\flat)(V_K + g_K - V_K^\flat - g_K^\flat) \\
&\qquad\qquad + (V_K + g_K - V_K^\flat - g_K^\flat)^2 dx.
\end{aligned}
\tag{5.43}
$$

Using Itô's formula on the first integral of the right hand side of (5.43),

$$
\begin{aligned}
&-\int_{\mathbb{T}_N} \frac{\gamma_K}{\beta^2}(V_K^\flat + g_K^\flat)^2 dx \\
&\quad = -\int_{\mathbb{T}_N}\int_0^K \frac{\partial_k \gamma_k}{\beta^2}(V_k^\flat + g_k^\flat)^2 + \frac{2\gamma_k}{\beta^2}(\partial_k V_k^\flat + \partial_k g_k^\flat)(V_k^\flat + g_k^\flat) dk dx.
\end{aligned}
$$

By the definition of W_k (see Section 4.1.1),

$$
\begin{aligned}
&\int_{\mathbb{T}_N}\int_0^K \Big(-\frac{72}{\beta^2}\mathcal{J}_k \mathsf{V}_k \circledequal \mathcal{J}_k \mathsf{V}_k - \frac{\partial_k \gamma_k}{\beta^2}\Big)(V_k^\flat + g_k^\flat)^2 dk dx \\
&\qquad\qquad = \int_{\mathbb{T}_N}\int_0^K \frac{72}{\beta^2}\mathsf{W}_k (V_k^\flat + g_k^\flat)^2 dk dx.
\end{aligned}
\tag{5.44}
$$

Hence, combining (5.42), (5.43), and (5.44) establishes (5.41). □

Proof of (5.27). Lemmas 5.14, 5.15, 5.16, and 5.17, together with (5.36), establish (5.27). □

5.6 Proof of (5.28): Estimates on remainder terms

Define

$$
\mathcal{R}_K^a = \mathcal{R}_K^{a,1} + \mathcal{R}_K^{a,2} + \mathcal{R}_K^{a,3}
$$

where

$$
\begin{aligned}
\mathcal{R}_K^{a,1} &= \int_{\mathbb{T}_N} -\frac{4}{\beta}\vec{\mathsf{l}}_K V_K^3 + \frac{4}{\beta}\mathsf{l}_K U_K^3 dx \\
\mathcal{R}_K^{a,2} &= \int_{\mathbb{T}_N}\int_0^K \frac{12}{\beta}\Big(\mathsf{V}_k \varogreaterthan (\partial_k V_k^\flat + \partial_k g_k^\flat)\Big) U_k + \frac{2\gamma_k}{\beta^2}(\partial_k V_k^\flat + \partial_k g_k^\flat)(V_k^\flat + g_k^\flat) dk dx \\
\mathcal{R}_K^{a,3} &= \int_{\mathbb{T}_N}\int_0^K \frac{72}{\beta^2}\Bigg(\Big(\mathcal{J}_k(\mathsf{V}_k \varogreaterthan (V_k^\flat + g_k^\flat))\Big)^2 - \Big(\mathcal{J}_k \mathsf{V}_k \circledequal \mathcal{J}_k \mathsf{V}_k\Big)(V_k^\flat + g_k^\flat)^2\Bigg) dk dx
\end{aligned}
$$

and let $\mathscr{R}_K^b = \sum_{i=1}^4 \mathscr{R}_K^i - \mathscr{R}_K^a$.

$\mathscr{R}_K^a$ contains the most difficult terms to bound, either due to analytic considerations or β-dependence; $\mathscr{R}_K^b$ contains the terms that follow almost immediately from [BG19, Lemmas 18-23].

Proposition 5.18. *For any* $\varepsilon > 0$*, there exists* $C = C(\varepsilon, \eta) > 0$ *such that, for all* $\beta > 1$*,*

$$|\mathscr{R}_K^{a,1}| \leqslant CN_K^{\Xi} + \varepsilon\Big(\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K)dx + \frac{1}{2}\int_0^K \int_{\mathbb{T}_N} r_k^2 dxdk\Big) \tag{5.45}$$

$$|\mathscr{R}_K^{a,2}| \leqslant CN_K^{\Xi} + \varepsilon\Big(\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K)dx + \frac{1}{2}\int_0^K \int_{\mathbb{T}_N} r_k^2 dxdk\Big) \tag{5.46}$$

$$|\mathscr{R}_K^{a,3}| \leqslant CN_K^{\Xi} + \varepsilon\Big(\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K)dx + \frac{1}{2}\int_0^K \int_{\mathbb{T}_N} r_k^2 dxdk\Big). \tag{5.47}$$

Proof. The estimates (5.45), (5.46), and (5.47) are established in Sections 5.6.2, 5.6.3, and 5.6.4 respectively. (5.46) and (5.47) are established by a relatively straightforward combination of techniques in [BG19, Lemmas 18-23] together with Lemmas 5.5 and 5.6. On the other hand, the terms with cubic dependence in the drift (5.45) require a slightly more involved analysis.

Note that, since our norms on functions/distributions were defined using $đx = \frac{dx}{N^3}$ instead of dx to track N dependence, in the proof we rewrite the integrals above in terms of $đx$ by dividing both sides by N^3. □

Proposition 5.19. *For any* $\varepsilon > 0$*, there exists* $C = C(\varepsilon, \eta) > 0$ *such that, for all* $\beta > 1$*,*

$$|\mathscr{R}_K^b| \leqslant CN_K^{\Xi} + \varepsilon\Big(\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K)dx + \frac{1}{2}\int_0^K \int_{\mathbb{T}_N} r_k^2 dxdk\Big).$$

Proof. Follows from a direct combination of arguments in [BG19, Lemmas 18-23] with Lemmas 5.5 and 5.6. We omit it. □

Proof of (5.28). Since $\sum_{i=1}^4 \mathscr{R}_K^i = \mathscr{R}_K^a + \mathscr{R}_K^b$, Propositions 5.18 and 5.19 establish (5.28). □

The proofs of Propositions 5.18 and 5.19 rely heavily on bounds on the drift established in Proposition 5.9, so we prove this first in the next subsection. Throughout the remainder of this section, we use the notation $a \lesssim b$ to mean $a \leqslant Cb$ for some $C = C(\varepsilon, \eta)$, and we also allow for this constant to depend on other inessential parameters (i.e. not β, N, or K).

5.6.1 Proof of Proposition 5.9

First, note that (5.23) is a direct consequence of (5.22) along with (5.21) and bounds contained in Proposition 4.4.

We now prove (5.22). Fix any $k' \in [0, K]$. As a consequence of (5.21),

$$\|U_{k'}\|_{H^{1-\kappa}}^2 \leqslant \frac{288}{\beta^2}\Big\| \int_0^{k'} \mathcal{J}_k^2(\mathbf{\Psi}_k \circledcirc (V_k^\flat + g_k^\flat))dk\Big\|_{H^{1-\kappa}}^2 + 2\|R_{k'}\|_{H^{1-\kappa}}^2. \quad (5.48)$$

By Minkowski's integral inequality, Bernstein's inequality (1.6), the multiplier estimate on $\mathcal{J}_k$ (1.13), the paraproduct estimate (1.8), and the $\flat$-estimates (4.19),

$$\begin{aligned}\Big\| \int_0^{k'} \mathcal{J}_k^2(\mathbf{\Psi}_k \circledcirc (V_k^\flat + g_k^\flat))dk\Big\|_{H^{1-\kappa}} &\lesssim \int_0^{k'} \frac{\|\mathcal{J}_k^2(\mathbf{\Psi}_k \circledcirc (V_k^\flat + g_k^\flat))\|_{H^{-1-2\kappa}}}{\langle k\rangle^\kappa}dk \\ &\lesssim \int_0^{k'} \frac{\|\mathbf{\Psi}_k \circledcirc (V_k^\flat + g_k^\flat)\|_{H^{-1-2\kappa}}}{\langle k\rangle^{1+\kappa}}dk \\ &\lesssim \bigg(\int_0^{k'} \frac{\|\mathbf{\Psi}_k\|_{B_{4,\infty}^{-1-2\kappa}}}{\langle k\rangle^{1+\kappa}}dk\bigg)\|V_K + g_K\|_{L^4}.\end{aligned}$$

Hence, by Cauchy-Schwarz with respect to the finite measure $\frac{dk}{\langle k\rangle^{1+\kappa}}$, the potential bound (5.9), and Young's inequality,

$$\begin{aligned}&\frac{1}{\beta^2}\Big\| \int_0^{k'} \mathcal{J}_k^2(\mathbf{\Psi}_k \circledcirc (V_k^\flat + g_k^\flat))dk\Big\|_{H^{1-\kappa}}^2 \\ &\lesssim \frac{1}{\beta^2}\bigg(\int_0^{k'} \frac{\|\mathbf{\Psi}_k\|_{B_{4,\infty}^{-1-2\kappa}}}{\langle k\rangle^{1+\kappa}}dk\bigg)^2\|V_K + g_K\|_{L^4}^2 \\ &\lesssim \frac{1}{\beta^2}\bigg(\int_0^{k'} \frac{\|\mathbf{\Psi}_k\|_{B_{4,\infty}^{-1-2\kappa}}^2}{\langle k\rangle^{1+\kappa}}dk\bigg)\|V_K + g_K\|_{L^4}^2 \\ &\lesssim \int_0^{k'} \frac{\|\mathbf{\Psi}_k\|_{B_{4,\infty}^{-1-2\kappa}}^2}{\langle k\rangle^{1+\kappa}}dk\Bigg(\frac{\Big(\int_{\mathbb{T}_N} \mathcal{V}_\beta(V_K + g_K)\,dx\Big)^{\frac12}}{\beta^{\frac32}} + \frac{1}{\beta}\Bigg) \\ &\leqslant \frac{1}{\beta}\int_0^{k'} \frac{\|\mathbf{\Psi}_k\|_{B_{4,\infty}^{-1-2\kappa}}^2}{\langle k\rangle^{1+\kappa}}dk + \frac{1}{4\varepsilon}\bigg(\int_0^{k'} \frac{\|\mathbf{\Psi}_k\|_{B_{4,\infty}^{-1-2\kappa}}^2}{\langle k\rangle^{1+\kappa}}dk\bigg)^2 \\ &\quad + \frac{\varepsilon}{\beta^3}\int_{\mathbb{T}_N} \mathcal{V}_\beta(V_K + g_K)\,dx.\end{aligned} \quad (5.49)$$

For the remaining term in (5.48), note that by the trivial embedding $H^1 \hookrightarrow H^{1-\kappa}$ and the bound (4.18) applied to $R_{k'}$,

$$\|R_{k'}\|^2_{H^{1-\kappa}} \lesssim \int_0^K \int_{\mathbb{T}_N} r_k^2 đx dk. \tag{5.50}$$

Inserting (5.49) and (5.50) into (5.48) establishes (5.22).

5.6.2 Proof of (5.45)

We start with the first integral in $\mathscr{R}_K^{a,1}$. Fix $\kappa > 0$ and let q be such that $(1+\kappa)^{-1} + q^{-1} = 1$. Then, by Young's inequality (and remembering $\beta > 1$),

$$\Big| \int_{\mathbb{T}_N} \frac{2}{\beta} \vec{\mathsf{r}}_K V_K^3 đx \Big| \leqslant C_\varepsilon \int_{\mathbb{T}_N} |\vec{\mathsf{r}}_K|^q đx + \varepsilon \int_{\mathbb{T}_N} \Big(\frac{V_K}{\sqrt{\beta}} \Big)^{2+2\kappa} |V_K|^{1+\kappa} đx. \tag{5.51}$$

Adding and subtracting g_K into the second term on the righthand side and using the pointwise potential bound (5.8),

$$\begin{aligned}
&\int_{\mathbb{T}_N} \Big(\frac{|V_K|}{\sqrt{\beta}} \Big)^{2+2\kappa} |V_K|^{1+\kappa} đx \\
&\quad \lesssim \int_{\mathbb{T}_N} \Big(\frac{|V_K + g_K|^4}{\beta^2}^{(\frac{1+\kappa}{2})} + \Big|\frac{g_K}{\sqrt{\beta}}\Big|^{2+2\kappa} \Big) |V_K|^{1+\kappa} đx \\
&\quad \lesssim \int_{\mathbb{T}_N} \bigg(\Big(\frac{\mathscr{V}_\beta(V_K + g_K)}{\beta} \Big)^{\frac{1+\kappa}{2}} + 1 + \Big|\frac{g_K}{\sqrt{\beta}}\Big|_\infty^{2+2\kappa} \bigg) |V_K|^{1+\kappa} đx
\end{aligned} \tag{5.52}$$

where we recall that $|\cdot|_\infty$ is the supremum norm.

By the bounds on g_K (5.11) and V_K (5.23), taking $\kappa < 1$ yields

$$\begin{aligned}
&\int_{\mathbb{T}_N} \Big(1 + \Big|\frac{g_K}{\sqrt{\beta}}\Big|_\infty^{2+2\kappa} \Big) |V_K|^{1+\kappa} đx \\
&\qquad \leqslant C(\varepsilon, \kappa, \eta) + \varepsilon \|V_K\|^2_{L^2} \\
&\qquad \leqslant C \frac{N_K^\Xi}{N^3} + \varepsilon \Big(\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K) đx + \frac{1}{2} \int_0^K \int_{\mathbb{T}_N} r_k^2 đx dk \Big).
\end{aligned} \tag{5.53}$$

Above, we recall that N_K^Ξ can contain constants $C = C(\eta) > 0$.

For the remaining term on the righthand side of (5.52), we reorganise terms and iterate the preceding argument:

$$
\begin{aligned}
&\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K+g_K)^{\frac{1+\kappa}{2}}\Big(\frac{|V_K|}{\sqrt{\beta}}\Big)^{1+\kappa}\text{đ}x \\
&\quad\lesssim \int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K+g_K)^{\frac{1+\kappa}{2}}\left(|\frac{g_K}{\sqrt{\beta}}|_\infty^{1+\kappa}+1+\frac{\mathscr{V}_\beta(V_K+g_K)^{\frac{1+\kappa}{4}}}{\beta^{\frac{1+\kappa}{4}}}\right)\text{đ}x \qquad (5.54)\\
&\quad\leqslant C(\varepsilon,\kappa,\eta)+\varepsilon\int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K+g_K)\text{đ}x
\end{aligned}
$$

provided that $\kappa<\frac{1}{4}$.

We now estimate the second integral in $\mathscr{R}_K^{a,1}$. Let $\tilde{\kappa}>0$ be sufficiently small. Let q be such that $\frac{3-\tilde{\kappa}}{4(1+\tilde{\kappa})(1-\tilde{\kappa})}+\frac{1}{q}=1$. Moreover, let $\theta=\frac{2\tilde{\kappa}(1-\tilde{\kappa})}{(1+\tilde{\kappa})(1-2\tilde{\kappa})}$. By duality (1.1), the fractional Leibniz rule (1.2) and interpolation (1.4),

$$
\begin{aligned}
\Big|\int_{\mathbb{T}_N}\frac{4}{\beta}\text{⫯}_K U_K^3\text{đ}x\Big| &\lesssim \frac{1}{\beta}\|\text{⫯}_K\|_{B^{-\frac{1}{2}-\kappa}_{q,\infty}}\|U_K^3\|_{B^{\frac{1}{2}+\kappa}_{\frac{4(1+\tilde{\kappa})(1-\tilde{\kappa})}{3-\tilde{\kappa}},1}}\\
&\lesssim \frac{1}{\beta}\|\text{⫯}_K\|_{B^{-\frac{1}{2}-\kappa}_{q,\infty}}\|U_K\|_{B^{\frac{1}{2}+\kappa}_{2+\tilde{\kappa},1}}\|U_K\|^2_{L^{4-2\tilde{\kappa}}} \qquad (5.55)\\
&\lesssim \frac{1}{\beta}\|\text{⫯}_K\|_{B^{-\frac{1}{2}-\kappa}_{q,\infty}}\|U_K\|^{1-\theta}_{H^{1-\kappa}}\|U_K\|^{2+\theta}_{L^{4-2\tilde{\kappa}}}.
\end{aligned}
$$

By the change of variables (5.21) in reverse, reorganising terms, Young's inequality, the bound on U_K (5.22), and using $\varepsilon<1$,

$$
\begin{aligned}
(5.55) &\leqslant C\frac{N_K^\Xi}{N^3}+\varepsilon\|U_K\|^2_{H^{1-\kappa}}\\
&\quad+\|\text{⫯}_K\|_{B^{-\frac{1}{2}-\kappa}_{q,\infty}}\|U_K\|^{1-\theta}_{H^{1-\kappa}}\Big(\frac{1}{\beta^{\frac{1}{2+\theta}}}\|V_K\|_{L^{4-2\tilde{\kappa}}}\Big)^{2+\theta}\\
&\leqslant C\frac{N_K^\Xi}{N^3}+\varepsilon\|U_K\|^2_{H^{1-\kappa}}+\frac{1}{\sqrt{\beta}^{\frac{8}{2+\theta}}}\int_{\mathbb{T}_N}V_K^4\text{đ}x \qquad (5.56)\\
&\leqslant C\frac{N_K^\Xi}{N^3}+\varepsilon\Big(\int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K+g_K)\text{đ}x+\frac{1}{2}\int_0^K\int_{\mathbb{T}_N}r_k^2\text{đ}x dk\Big)\\
&\qquad+\frac{1}{\sqrt{\beta}^{\frac{8}{2+\theta}}}\int_{\mathbb{T}_N}V_K^4\text{đ}x.
\end{aligned}
$$

For the last term on the righthand side of (5.56), we iterate the potential bound

(5.8) and bound on g_K (5.11) as in the estimate of (5.52):

$$\begin{aligned}
\frac{1}{\sqrt{\beta}^{\frac{8}{2+\theta}}}\int_{\mathbb{T}_N} V_K^4 đx &= \int_{\mathbb{T}_N}\Big(\frac{|V_K|}{\sqrt{\beta}}\Big)^{\frac{4}{1+\frac{\theta}{2}}}|V_K|^{\frac{2\theta}{2+\frac{\theta}{2}}}đx \\
&\lesssim \int_{\mathbb{T}_N}\Big(|\frac{g_K}{\sqrt{\beta}}|^{\frac{4}{1+\frac{\theta}{2}}}+1\Big)|V_K|^{\frac{2\theta}{2+\frac{\theta}{2}}}đx \\
&\quad + \int_{\mathbb{T}_N}\frac{\mathscr{V}_\beta(V_K+g_K)^{\frac{1}{1+\frac{\theta}{2}}}}{\beta^{\frac{2}{1+\frac{\theta}{2}}}}|V_K|^{\frac{2\theta}{2+\frac{\theta}{2}}}đx \\
&\lesssim C(\varepsilon,\eta)+\frac{\varepsilon\eta}{2}\int_{\mathbb{T}_N}|V_K|^2 đx \\
&\quad + \int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K+g_K)^{\frac{1}{1+\frac{\theta}{2}}}\frac{|V_K|^{\frac{2\theta}{2+\frac{\theta}{2}}}}{\beta^{\frac{2}{1+\frac{\theta}{2}}}}đx \\
&\lesssim C(\varepsilon,\eta)+\varepsilon\|V_K\|^2_{H^{\frac{1}{2}-\kappa}} \\
&\quad + \int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K+g_K)^{\frac{1}{1+\frac{\theta}{2}}}\Big(1+|\frac{g_K}{\sqrt{\beta}}|^{\frac{2\theta}{2+\frac{\theta}{2}}}\Big)đx \\
&\quad + \int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K+g_K)^{\frac{1}{1+\frac{\theta}{2}}+\frac{\theta}{4+\theta}}đx \\
&\leqslant C(\varepsilon,\eta)+\varepsilon\Big(\|V_K\|^2_{H^{\frac{1}{2}-\kappa}}+\int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K+g_K)đx\Big) \\
&\leqslant C\frac{N_K^{\Xi}}{N^3}+2\varepsilon\Big(\int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K+g_K)đx+\frac{1}{2}\int_0^K\int_{\mathbb{T}_N}r_k^2 đx dk\Big)
\end{aligned} \tag{5.57}$$

where in the penultimate line we used Young's inequality and in the last line we have used (5.23).

Combining (5.51), (5.53), (5.54), (5.56), and (5.57) establishes (5.45).

5.6.3 Proof of (5.46)

For any $\theta \in (0,1)$ let $\frac{1}{p} = \frac{\theta}{4} + \frac{1-\theta}{2}$ and let $\frac{1}{p'} = 1 - \frac{1}{p}$. Then, by duality (1.1), the paraproduct estimate (1.8), the Bernstein-type bounds on the derivatives of the drift

(4.20), and bounds on the $\partial_k g_k^\flat$ (5.14),

$$\begin{aligned}
\Big|\int_{\mathbb{T}_N}\int_0^K \frac{12}{\beta}\mathsf{V}_k \otimes (\partial_k V_k^\flat + \partial_k g_k^\flat) U_k dk đx\Big| \\
&\lesssim \frac{1}{\beta}\int_0^K \|\mathsf{V}_k \otimes (\partial_k V_k^\flat + \partial_k g_k^\flat)\|_{H^{-1+\kappa}} \|U_k\|_{H^{1-\kappa}} dk \\
&\lesssim \frac{1}{\beta}\int_0^K \|\mathsf{V}_k\|_{B^{-1+\kappa}_{p',2}} \|\partial_k V_k^\flat + \partial_k g_k^\flat\|_{L^p} \|U_k\|_{H^{1-\kappa}} dk \\
&\lesssim \sup_{0\leqslant k\leqslant K} \|U_k\|_{H^{1-\kappa}} \frac{1}{\beta}\|V_K + g_K\|_{B^{3\kappa}_{p,1}} \int_0^K \|\mathsf{V}_k\|_{B^{-1+\kappa}_{p',2}} \frac{dk}{\langle k\rangle^{1+3\kappa}}
\end{aligned} \tag{5.58}$$

where in the last inequality we have reordered terms.

Then,

$$\begin{aligned}
(5.58) &\lesssim \sup_{0\leqslant k\leqslant K} \|U_k\|_{H^{1-\kappa}} \frac{1}{\beta} \|V_K + g_K\|^{\theta}_{B^0_{4,\infty}} \|V_K + g_K\|^{1-\theta}_{B^{6\kappa}_{2,1}} \\
&\quad \times \int_0^K \|\mathsf{V}_k\|_{B^{-1-\kappa}_{p',2}} \frac{dk}{\langle k\rangle^{1+\kappa}} \\
&\lesssim \sup_{0\leqslant k\leqslant K} \|U_k\|_{H^{1-\kappa}} \frac{\|V_K + g_K\|^{\theta}_{L^4}}{\beta^{\theta}} \Big(\frac{\|V_K\|^{1-\theta}_{H^{\frac{1}{2}-\kappa}}}{\beta^{1-\theta}} + 1\Big) \\
&\quad \times \int_0^K \|\mathsf{V}_k\|_{B^{-1-\kappa}_{p',2}} \frac{dk}{\langle k\rangle^{1+\kappa}} \\
&\leqslant C(\varepsilon)\left(1 + \Big(\int_0^K \|\mathsf{V}_k\|_{B^{-1-\kappa}_{p',2}} \frac{dk}{\langle k\rangle^{1+\kappa}}\Big)^{\frac{4}{4-\theta}}\right) \\
&\quad + \frac{\varepsilon}{2}\left(\|V_K\|^2_{H^{\frac{1}{2}-\kappa}} + \sup_{0\leqslant k\leqslant K}\|U_k\|^2_{H^{1-\kappa}} + \frac{1}{\beta^4}\|V_K + g_K\|^4_{L^4}\right) \\
&\leqslant C(\varepsilon,\eta)\frac{N_K^{\Xi}}{N^3} + \varepsilon\Big(\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K) đx + \frac{1}{2}\int_0^K\int_{\mathbb{T}_N} r_k^2 đx dk\Big)
\end{aligned} \tag{5.59}$$

where in the first line we have used Bernstein's inequality (1.5); in the second line we have used interpolation (1.4); in the penultimate line we used Young's inequality; and in the last line we have used the bounds on V_K (5.23), U_k (5.22), together with the potential bound (5.9).

In order to bound the second integrand in $\mathscr{R}_K^{a,2}$, we use Fubini's theorem, the Cauchy-Schwarz inequality, the bounds on $V_k^\flat$ (4.19) and $\partial_k V_K^\flat$ (4.20), and the bounds on g_K (5.11) to obtain

$$
\begin{aligned}
\Big| \int_{\mathbb{T}_N} \int_0^K & \frac{2\gamma_k}{\beta^2} (\partial_k V_k^\flat + \partial_k g_k^\flat)(V_k^\flat + g_k^\flat) dk đx \Big| \\
&\lesssim \frac{1}{\beta^2} \int_0^K \gamma_k \|\partial_k V_k^\flat + \partial_k g_k^\flat\|_{L^2} \|V_k^\flat + g_k^\flat\|_{L^2} dk \\
&\lesssim \frac{1}{\beta^2} \|V_K + g_K\|_{H^{2\kappa}} \|V_K + g_K\|_{L^2} \int_0^K \frac{\gamma_k}{\langle k \rangle^\kappa} \frac{dk}{\langle k \rangle^{1+\kappa}} \\
&\lesssim \Big(\frac{\|V_K\|_{H^{\frac{1}{2}-\kappa}}}{\beta^2} + \frac{1}{\beta^{\frac{3}{2}}} \Big) \|V_K + g_K\|_{L^4}
\end{aligned}
\tag{5.60}
$$

where in the last inequality we have used the observation made in Remark 4.3 that $|\gamma_k| \lesssim \log\langle k \rangle$.

Thus, by Young's inequality (applied to each term after expanding the sum), the potential bound (5.9), and the bound on V_K (5.23),

$$
\begin{aligned}
(5.60) &\leqslant C(\varepsilon, \eta) + \varepsilon \Bigg(\|V_K\|^2_{H^{\frac{1}{2}-\kappa}} + \Big(\frac{1}{\beta^8} + \frac{1}{\beta^6} \Big) \|V_K + g_K\|^4_{L^4} \Bigg) \\
&\leqslant C(\varepsilon, \eta) \frac{N_K^\Xi}{N^3} + \varepsilon \Big(\int_{\mathbb{T}_N} \mathcal{V}_\beta (V_K + g_K) đx + \frac{1}{2} \int_0^K \int_{\mathbb{T}_N} r_k^2 đx dk \Big).
\end{aligned}
\tag{5.61}
$$

Combining (5.59) and (5.61) yields (5.46).

5.6.4 Proof of (5.47)

We write $\mathcal{R}_K^{a,3} = I_1 + I_2 + I_3$, where

$$
\begin{aligned}
I_1 &= \int_{\mathbb{T}_N} \int_0^K \frac{72}{\beta^2} \Big(\mathcal{J}_k (\mathbf{V}_k \otimes (V_k^\flat + g_k^\flat)) \Big)^2 - \Big(\mathcal{J}_k \mathbf{V}_k \otimes (V_k^\flat + g_k^\flat) \Big)^2 dk dx \\
I_2 &= \int_{\mathbb{T}_N} \int_0^K \frac{72}{\beta^2} \Big(\mathcal{J}_k \mathbf{V}_k \otimes (V_k^\flat + g_k^\flat) \Big)^2 \\
&\qquad - \Big(\big(\mathcal{J}_k \mathbf{V}_k \otimes (V_k^\flat + g_k^\flat) \big) \ominus \mathcal{J}_k \mathbf{V}_k \Big) (V_k^\flat + g_k^\flat) dk dx \\
I_3 &= \int_{\mathbb{T}_N} \int_0^K \frac{72}{\beta^2} \Big(\big(\mathcal{J}_k \mathbf{V}_k \otimes (V_k^\flat + g_k^\flat) \big) \ominus \mathcal{J}_k \mathbf{V}_k \\
&\qquad - (\mathcal{J}_k \mathbf{V}_k \ominus \mathcal{J}_k \mathbf{V}_k)(V_k^\flat + g_k^\flat) \Big) \Big(V_k^\flat + g_k^\flat \Big) dk đx.
\end{aligned}
$$

Let $\theta \in (0,1)$ be sufficiently small and let $\frac{1}{p} = \frac{\theta}{4} + \frac{1-\theta}{2}$, $\frac{1}{q} = \frac{1-\theta}{2}$ and $\frac{1}{p'} = \frac{1}{2} - \frac{1}{p}$, $\frac{1}{q'} = \frac{1}{2} - \frac{1}{q}$. Then,

$$
\begin{aligned}
|I_1| &\lesssim \frac{1}{\beta^2} \int_0^K \|\mathcal{J}_k\big(\mathbf{v}_k \circledcirc (V_k^\flat + g_k^\flat)\big) - \mathcal{J}_k \mathbf{v}_k \circledcirc (V_k^\flat + g_k^\flat)\|_{H^{2\kappa}} \\
&\qquad\qquad \times \|\mathcal{J}_k\big(\mathbf{v}_k \circledcirc (V_k^\flat + g_k^\flat)\big) + \mathcal{J}_k \mathbf{v}_k \circledcirc (V_k^\flat + g_k^\flat)\|_{H^{-2\kappa}} dk \\
&\lesssim \frac{1}{\beta^2} \int_0^K \|\mathbf{v}_k\|_{B^{-1-\kappa}_{p',q'}} \|V_k^\flat + g_k^\flat\|_{B^{4\kappa}_{p,q}} \\
&\qquad\qquad \times \Big(\|\mathcal{J}_k\big(\mathbf{v}_k \circledcirc (V_k^\flat + g_k^\flat)\big)\|_{H^{-2\kappa}} + \|\mathcal{J}_k \mathbf{v}_k \circledcirc (V_k^\flat + g_k^\flat)\|_{H^{-2\kappa}} \Big) dk \\
&\lesssim \frac{1}{\beta^2} \int_0^K \|\mathbf{v}_k\|_{B^{-1-\kappa}_{p',q'}} \|V_k^\flat + g_k^\flat\|_{B^{4\kappa}_{p,q}} \|\mathbf{v}_k\|_{B^{-1-2\kappa}_{4,2}} \|V_k^\flat + g_k^\flat\|_{L^4} \frac{dk}{\langle k \rangle}
\end{aligned}
\tag{5.62}
$$

where the first inequality is by duality (1.1); the second inequality is by the commutator estimate (1.14) and the triangle inequality; and the third inequality is by the multiplier estimate (1.13) and the paraproduct estimate (1.8).

Thus,

$$
\begin{aligned}
(5.62) &\lesssim \frac{1}{\beta^2} \int_0^K \|\mathbf{v}_k\|_{B^{-1-\kappa}_{p',q'}} \|V_k^\flat + g_k^\flat\|_{B^{4\kappa}_{p,q}} \|\mathbf{v}_k\|_{B^{-1-\kappa}_{4,2}} \|V_k^\flat + g_k^\flat\|_{L^4} \frac{dk}{\langle k \rangle^{1+\kappa}} \\
&\lesssim \frac{1}{\beta^2} \|V_K + g_K\|^{1-\theta}_{H^{\frac{4\kappa}{1-\theta}}} \|V_K + g_K\|^{1+\theta}_{L^4} \int_0^K \|\mathbf{v}_k\|_{B^{-1-\kappa}_{p',q'}} \|\mathbf{v}_k\|_{B^{-1-\kappa}_{4,2}} \frac{dk}{\langle k \rangle^{1+\kappa}}
\end{aligned}
\tag{5.63}
$$

where the first inequality is by Bernstein's inequality (1.5); and the second inequality is by the $\flat$-bounds applied to $V_k^\flat + g_k^\flat$ (4.19), interpolation (1.4), and the trivial bound $\|V_K + g_K\|_{B^{4\kappa\theta}_{4,\infty}} \lesssim \|V_K + g_K\|_{L^4}$.

By applying Young's inequality, the potential bound (5.9), and the bound on V_K (5.23), we have

$$
\begin{aligned}
(5.63) &\leqslant C \frac{N_K^\Xi}{N^3} + \varepsilon \left(\|V_K + g_K\|^2_{H^{\frac{4\kappa}{1-\theta}}} + \frac{1}{\beta^{\frac{8}{1+\theta}}} \|V_K + g_K\|^4_{L^4} \right) \\
&\leqslant C \frac{N_K^\Xi}{N^3} + \varepsilon \Big(\int_{\mathbb{T}_N} \mathcal{V}_\beta(V_K + g_K) \text{đ}x + \frac{1}{2} \int_0^K \int_{\mathbb{T}_N} r_k^2 \text{đ}x dk \Big).
\end{aligned}
\tag{5.64}
$$

Now consider I_2. Using the commutator estimate (1.10) with $f = \mathcal{J}_k \mathbf{v}_k$, $g = V_k^\flat + g_k^\flat$ and $h = \mathcal{J}_k \mathbf{v}_k \circledcirc (V_k^\flat + g_k^\flat)$, followed by the paraproduct estimate (1.8), we obtain

$$
\begin{aligned}
I_2 &\lesssim \frac{1}{\beta^2} \int_0^K \|\mathcal{J}_k \mathbf{v}_k\|_{B^{-2\kappa}_{6,\infty}} \|V_k^\flat + g_k^\flat\|_{H^{4\kappa}} \|\mathcal{J}_k \mathbf{v}_k \circledcirc (V_k^\flat + g_k^\flat)\|_{B^{-2\kappa}_{3,2}} dk \\
&\lesssim \frac{\|V_K + g_K\|_{H^{4\kappa}} \|V_K + g_K\|_{L^4}}{\beta^2} \int_0^K \|\mathbf{v}_k\|^2_{B^{-\kappa}_{12,2}} \frac{dk}{\langle k \rangle^{1+2\kappa}}.
\end{aligned}
\tag{5.65}
$$

By applying Young's inequality, the potential bound (5.9), and the a priori bound on V_K (5.23),

$$\begin{aligned}(5.65) &\leqslant C(\varepsilon,\eta)\frac{N_K^{\Xi}}{N^3}+\varepsilon\left(\|V_K+g_K\|_{H^{4\kappa}}^2+\frac{1}{\beta^8}\|V_K+g_K\|_{L^4}^4\right)\\ &\leqslant C(\varepsilon,\eta)\frac{N_K^{\Xi}}{N^3}+\varepsilon\Big(\int_{\mathbb{T}_N}\mathcal{V}_\beta(V_K+g_K)dx+\frac{1}{2}\int_0^K\int_{\mathbb{T}_N}r_k^2 dx dk\Big).\end{aligned} \tag{5.66}$$

where the final inequality uses the multiplier estimate (1.13), the $\flat$-bounds applied to V_K+g_K (4.19), and Bernstein's inequality (1.6).

For I_3, we apply duality (1.1), the commutator estimate (1.11) with $f=h=\mathcal{J}_k\mathbf{V}_k$ and $g=V_k^\flat+g_k^\flat$, followed by the $\flat$-bounds applied to V_K+g_K (4.19), to obtain

$$\begin{aligned}I_3 &\lesssim \frac{1}{\beta^2}\int_0^K \|\big(\mathcal{J}_k\mathbf{V}_k\circledcirc(V_k^\flat+g_k^\flat)\big)\circleddash\mathcal{J}_k\mathbf{V}_k-(\mathcal{J}_k\mathbf{V}_k\circleddash\mathcal{J}_k\mathbf{V}_k)(V_k^\flat+g_k^\flat)\|_{B^{\kappa}_{\frac{4}{3},\infty}}\\ &\qquad\times\|V_k^\flat+g_k^\flat\|_{B^{-\kappa}_{4,1}}dk\\ &\lesssim\frac{1}{\beta^2}\int_0^K\|\mathcal{J}_k\mathbf{V}_k\|^2_{B^{-2\kappa}_{8,\infty}}\|V_k^\flat+g_k^\flat\|_{B^{5\kappa}_{2,\infty}}\|V_k^\flat+g_k^\flat\|_{L^4}dk\\ &\lesssim\frac{1}{\beta^2}\|V_K+g_K\|_{H^{5\kappa}}\|V_K+g_K\|_{L^4}\int_0^K\|\mathbf{V}_k\|^2_{B^{-\kappa}_{8,\infty}}\frac{dk}{\langle k\rangle^{1+2\kappa}}\\ &\leqslant C(\varepsilon,\eta)\frac{N_K^{\Xi}}{N^3}+\varepsilon\Big(\int_{\mathbb{T}_N}\mathcal{V}_\beta(V_K+g_K)dx+\frac{1}{2}\int_0^K\int_{\mathbb{T}_N}r_k^2dxdk\Big)\end{aligned} \tag{5.67}$$

where in the last line we have used Young's inequality, the potential bound (5.9), and the bound on V_K (5.23) as in (5.66).

Using that $\mathcal{R}_K^{a,3}=I_1+I_2+I_3$, the estimates (5.64), (5.66), and (5.67) establish (5.47).

5.7 A lower bound on the effective Hamiltonian

The following lemma, based on [GJS76b, Theorem 3.1.1], gives a β-independent lower bound on $\mathcal{H}_K^{\text{eff}}(Z_K)$ in terms of the L^2-norm of the fluctuation field $Z_K^\perp=Z_K-\vec{Z}_K$, where we recall $Z_K=\vec{\mathbf{1}}_K+V_K+g_K$ and $\vec{Z}_K(x)=Z_K(\square)$ for $x\in\square\in\mathbb{B}_N$. This is useful for us because the latter can be bounded in a β-independent way (see Section 5.8.1).

Lemma 5.20. *There exists $C>0$ such that, for any $\zeta>0$ and $K\in(0,\infty)$,*

$$\mathcal{H}_K^{\text{eff}}(Z_K)\geqslant -CN^3-\zeta\int_{\mathbb{T}_N}\big(Z_K^\perp\big)^2dx \tag{5.68}$$

provided $\eta<\min\left(\frac{1}{32},\frac{2\zeta}{49}\right)$.

Proof. First, we write

$$\mathcal{H}_K^{\text{eff}}(Z_K) = \sum_{\square\in\mathbb{B}_N} \int_\square \frac{1}{2}\mathcal{V}_{\beta,N,K}(Z_K) - \frac{\eta}{2}(Z_K - h)^2 - \log\Big(\chi_{\sigma(\square)}\big(Z_K(\square)\big)\Big)dx.$$

Fix $x \in \square \in \mathbb{B}_N$. Without loss of generality, assume $\sigma(x) = 1$ and, hence, $h(x) = \sqrt{\beta}$. Define

$$I(x) = \frac{1}{2}\mathcal{V}_\beta(Z_K(x)) - \frac{\eta}{2}(Z_K(x) - \sqrt{\beta})^2 - \log\chi_+(\vec{Z}_K(x)).$$

In order to show (5.68), it suffices to show that, for some $C > 0$,

$$I(x) + \zeta Z_K^\perp(x)^2 \geqslant -C.$$

The fundamental observation is that $Z_K(x) \mapsto \frac{1}{2}\mathcal{V}_\beta(Z_K(x))$ can be approximated from below near the minimum at $Z_K(x) = \sqrt{\beta}$ by the quadratic $Z_K(x) \mapsto \frac{\eta}{2}(Z_K(x) - \sqrt{\beta})^2$ provided η is taken sufficiently small. Indeed, we have

$$\frac{1}{2}\mathcal{V}_\beta(Z_K(x)) - \frac{\eta}{2}(Z_K(x) - \sqrt{\beta})^2 = \frac{1}{2\beta}(Z_K(x) - \sqrt{\beta})^2\Big((Z_K(x) + \sqrt{\beta})^2 - \eta\beta\Big)$$

which is non-negative provided $|Z_K(x) + \sqrt{\beta}| \geqslant \sqrt{\eta\beta}$. Thus, this approximation is valid except for the region near the opposite potential well satisfying $(-1-\sqrt{\eta})\sqrt{\beta} < Z_K(x) < (-1+\sqrt{\eta})\sqrt{\beta}$ (see Figure 3). When $Z_K(x)$ sits in this region, we split $Z_K(x) = \vec{Z}_K(x) + Z_K^\perp(x)$ and observe that:

- either the deviation to the opposite well is caused by $\vec{Z}_K(x)$, which is penalised by the logarithm in $I(x)$;
- or, the deviation is caused by $Z_K^\perp(x)$, which produces the integral involving $Z_K^\perp$ in (5.68).

Motivated by these observations, we split the analysis of $I(x)$ into two cases. First we treat the case $Z_K(x) \in \mathbb{R}\backslash\Big(-\frac{4\sqrt{\beta}}{3}, -\frac{2\sqrt{\beta}}{3}\Big)$. Under this condition, we have

$$\frac{1}{2}\mathcal{V}_\beta(Z_K(x)) \geqslant \eta(Z_K(x) - \sqrt{\beta})^2$$

provided that $\eta \leqslant \frac{1}{9}$. Since $\chi_+(\cdot) \leqslant 1$, $-\log\chi_+(\cdot) \geqslant 0$. It follows that $I(x) \geqslant 0$.

Now let $Z_K(x) \in \Big(-\frac{4\sqrt{\beta}}{3}, -\frac{2\sqrt{\beta}}{3}\Big)$. Necessarily, either $\vec{Z}_K(x) \leqslant -\frac{\sqrt{\beta}}{3}$ or $Z_K^\perp(x) \leqslant -\frac{\sqrt{\beta}}{3}$.

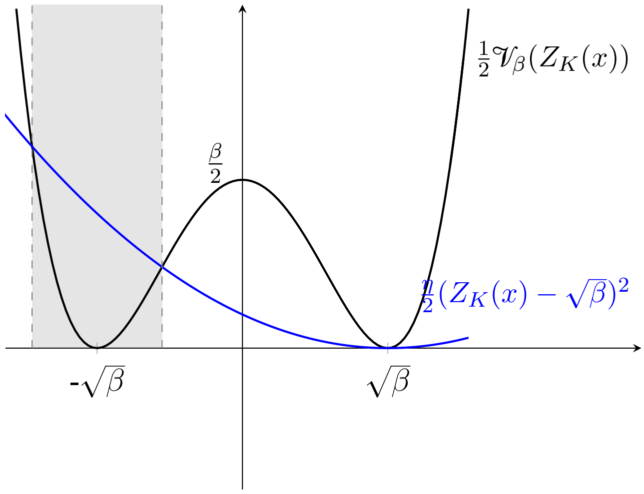


Figure 3: Plot of $\mathscr{V}_\beta(Z_K(x))$ and $\frac{\eta}{2}(Z_K(x)-\sqrt{\beta})^2$.

We first assume that $\vec{Z}_K(x) \leqslant -\frac{\sqrt{\beta}}{3}$. By standard bounds on the Gaussian error function (see e.g. [GJS76b, Lemma 2.6.1]), for any $\theta \in (0,1)$ there exists $C = C(\theta) > 0$ such that

$$-\log \chi_+\big(Z_K(\square)\big) \geqslant -\theta(\vec{Z}_K(x))^2 + C.$$

Applying this with $\theta \in (\frac{1}{2}, 1)$ and that, by our assumption, $\vec{Z}_K(x) - \sqrt{\beta} > 4\vec{Z}_K(x)$,

$$\begin{aligned} I(x) + \zeta(Z_K^\perp(x))^2 &\geqslant -\frac{\eta}{2}(Z_K^\perp(x) + \vec{Z}_K(x) - \sqrt{\beta})^2 - \log \chi_+\big(Z_K(\square)\big) + \zeta(Z_K^\perp(x))^2 \\ &\geqslant (\zeta - \eta)(Z_K^\perp(x))^2 - 16\eta(\vec{Z}_K(x))^2 - \tilde{\theta}(\vec{Z}_K(x)^2) - C \\ &\geqslant -C \end{aligned}$$

provided $\eta < \min\Big(\zeta, \frac{1}{32}\Big)$.

Finally, assume that $Z_K^\perp(x) < -\frac{\sqrt{\beta}}{3}$. Since $Z_K(x) - \sqrt{\beta} \in \Big(-\frac{7\sqrt{\beta}}{3}, -\frac{-5\sqrt{\beta}}{3}\Big)$, we have

$$I(x) + \zeta(Z_K^\perp(x))^2 \geqslant -\frac{49\eta}{18}\beta + \zeta(Z_K^\perp(x))^2 \geqslant 0 \tag{5.69}$$

provided that $\eta \leqslant \frac{2\zeta}{49}$.

□

5.8 Proof of Proposition 5.3

5.8.1 Proof of the lower bound on the free energy (5.1)

We derive bounds uniform in σ for each term in the expansion (5.10). Since there are 2^{N^3} terms, this is sufficient to establish (5.1). Fix $\sigma \in \{\pm 1\}^{\mathbb{B}_N}$.

Recall

$$-\log \mathscr{Z}^\sigma_{\beta,N,K} = -\log \mathbb{E}_N e^{-\mathscr{H}^\sigma_{\beta,N,K}} + F^\sigma_{\beta,N,K}. \tag{5.70}$$

Let $C_P > 0$ be the sharpest constant in the Poincaré inequality (1.15) on unit boxes. Note that C_P is independent of N. Fix $\zeta < \frac{1}{8C_P}$ and let $\varepsilon = 1 - 8C_P\zeta > 0$. By Proposition 5.11 and Lemma 5.20 there exists $C = C(\zeta, \eta) > 0$ such that, for every $v \in \mathbb{H}_{b,K}$,

$$\begin{aligned}
\Psi_K(v) &= \mathscr{H}^\sigma_{\beta,N,K}(\text{\textbullet}_K + V_K) + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K v_k^2 dk dx \\
&\approx \sum_{i=1}^4 \mathscr{R}^i_K + \mathscr{H}^{\text{eff}}_K(Z_K) + \frac{1}{2}\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K)dx + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K r_k^2 dk dx \\
&\geqslant -C(\varepsilon)N^\Xi_K + \mathscr{H}^{\text{eff}}_K + \frac{1-\varepsilon}{2}\Big(\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K)dx + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K r_k^2 dk dx\Big). \\
&\geqslant -C(\zeta)N^\Xi_K - \zeta\int_{\mathbb{T}_N}(Z^\perp_K)^2 dx \\
&\qquad + 4\zeta C_P\Big(\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K)dx + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K r_k^2 dk dx\Big)
\end{aligned}$$

provided $\eta < \frac{2\zeta}{49} < \frac{1}{196C_P}$.

Note that for any $f \in L^2$, $\int_{\mathbb{T}_N}(f^\perp)^2 dx \leqslant \int_{\mathbb{T}_N} f^2 dx$. Therefore, using the inequality $(a_1+a_2+a_3+a_4)^2 \leqslant 4(a_1^2+a_2^2+a_3^2+a_4^2)$ and that $Z^\perp_K(x) = (V_K+g_K)^\perp(x)$, we have

$$\begin{aligned}
\int_{\mathbb{T}_N}(Z^\perp_K)^2 dx \leqslant 4\int_{\mathbb{T}_N} \frac{16}{\beta^2}\Big(\text{\Psi}_K\Big)^2 + \frac{144}{\beta^2}\left(\int_0^K \mathcal{J}_k^2 \text{\textbullet\textbullet}_k \circledcirc (V_k^\flat + g_k))dk\right)^2 \\
+ (R^\perp_K)^2 + (g^\perp_K)^2 dx.
\end{aligned}$$

Arguing as in (5.49),

$$\begin{aligned}
4\int_{\mathbb{T}_N} \frac{16}{\beta^2}\Big(\text{\Psi}_K\Big)^2 + \frac{144}{\beta^2}\left(\int_0^K \mathcal{J}_k^2 \text{\textbullet\textbullet}_k \circledcirc (V_k^\flat + g_k))dk\right)^2 dx \\
\leqslant C(\zeta, C_P)N^\Xi_K + \frac{4\zeta C_P}{\beta^3}\int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K)dx.
\end{aligned}$$

By the Poincaré inequality (1.15) on unit boxes,

$$\int_{\mathbb{T}_N}(R^\perp_K)^2 dx = \sum_{\square \in \mathbb{B}_N}\int_\square\Big(R_K - \int_\square R_K dx\Big)^2 dx$$

$$\leqslant C_P \sum_{\square \in \mathbb{B}_N} \int_{\square} |\nabla R_K|^2 dx$$

$$\leqslant C_P \int_{\mathbb{T}_N} \int_0^K r_k^2 dk dx$$

where in the last inequality we used that $\int_{\mathbb{T}_N} |\nabla R_K|^2 dx \leqslant \|R_K\|_{H^1}^2$ and Lemma 4.11 (applied to R_K).

Similarly, by the Poincaré inequality (1.15) and the (trivial) bound $\|\nabla g_K\|_{L^2}^2 \leqslant \|\nabla \tilde{g}_K\|_{L^2}^2$ (5.12),

$$\int_{\mathbb{T}_N} (g_K^{\perp})^2 dx \leqslant C_P \int_{\mathbb{T}_N} |\nabla g_K|^2 dx \leqslant C_P \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx.$$

Then, recalling that $\beta > 1$,

$$\begin{aligned} \mathbb{E}\Psi_K(v) \geqslant \mathbb{E}\Bigg[& - CN_K^{\Xi} + 4\zeta C_P\Big(1 - \frac{1}{\beta^3}\Big) \int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K) dx \\ & + \Big(4\zeta C_P - 4\zeta C_P\Big) \int_{\mathbb{T}_N} \int_0^K r_k^2 dk dx - 4\zeta C_P \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx \Bigg] \\ \geqslant \mathbb{E}\Bigg[& - CN_K^{\Xi} - 4\zeta C_P \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx \Bigg] \end{aligned}$$

from which, by Proposition 4.7, we obtain

$$- \log \mathbb{E}_N e^{-\mathscr{H}^{\sigma}_{\beta,N,K}} \geqslant -CN^3 - 4\zeta C_P \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx.$$

Inserting this into (5.70) and using that $F^{\sigma}_{\beta,N,K} \geqslant \frac{1}{2} \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx$ (see (5.15)) yields:

$$- \log \mathscr{Z}^{\sigma}_{\beta,N,K} \geqslant -CN^3 + \Big(\frac{1}{2} - 4\zeta C_P\Big) \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx \geqslant -CN^3$$

which establishes (5.1).

5.8.2 Proof of the upper bound on the free energy (5.2)

We (globally) translate the field to one of the minima of $\mathscr{V}_\beta$: this kills the constant β term. Thus, under the translation $\phi = \psi + \sqrt{\beta}$,

$$\mathscr{Z}_{\beta,N,K} = \mathbb{E}_N e^{-\mathscr{H}^{+}_{\beta,N,K}(\psi_K)}$$

where

$$\mathscr{H}^{+}_{\beta,N,K}(\psi_K) = \int_{\mathbb{T}_N} \mathscr{V}^{+}_{\beta}(\psi_K) - \frac{\gamma_K}{\beta^2} : (\psi_K + \sqrt{\beta})^2 : -\delta_K - \frac{\eta}{2} : \psi_K^2 : dx$$

and

$$\mathscr{V}^{+}_{\beta}(a) = \frac{1}{\beta} a^2 (a + 2\sqrt{\beta})^2 = \frac{1}{\beta} a^4 + \frac{4}{\sqrt{\beta}} a^3 + 4a^2.$$

We apply the Proposition 4.7 to $\mathscr{Z}_{\beta,N,K}$ with the infimum taken over $\mathbb{H}_K$. In order to obtain an upper bound, we choose a particular drift in the corresponding stochastic control problem (4.13). Following [BG19], we seek a drift that satisfies sufficient moment/integrability conditions with estimates that are extensive in N^3, as formalised in Lemma 5.21 below. Such a drift is constructed using a fixed point argument, hence the need to work in the Banach space $\mathbb{H}_K$ as opposed to $\mathbb{H}_{b,K}$.

Lemma 5.21. *There exist processes $\mathscr{U}_{\leqslant}\curlyvee_{\bullet}$ and $\mathscr{U}_{>}\curlyvee_{\bullet}$ satisfying $\mathscr{U}_{\leqslant>}\curlyvee_{\bullet} + \mathscr{U}_{\geqslant}\curlyvee_{\bullet} = \curlyvee_{\bullet}$ and a unique fixed point $\check{v} \in \mathbb{H}_K$ of the equation*

$$\check{v}_k = -\frac{4}{\beta} \mathscr{J}_k \Psi_k - \frac{12}{\sqrt{\beta}} \mathscr{J}_k \curlyvee_k - \frac{12}{\beta} \mathscr{J}_k (\mathscr{U}_{>} \curlyvee_k \varogreaterthan \check{V}^{\flat}_k) \tag{5.71}$$

where $\check{V}_K = \int_0^K \mathscr{J}_k \check{v}_k dk$, such that the following estimate holds: for all $p \in [1, \infty)$, there exists $C = C(p, \eta) > 0$ such that, for all $\beta > 1$,

$$\mathbb{E}\left[\int_{\mathbb{T}_N} |\check{V}_K|^p dx + \frac{1}{2} \int_{\mathbb{T}_N} \int_0^K \check{r}_k^2 dk dx \right] \leqslant C N^3 \tag{5.72}$$

where $\check{r}_k = -\frac{12}{\beta} \mathscr{J}_k (\mathscr{U}_{\leqslant} \curlyvee_k \varogreaterthan \check{V}^{\flat}_k)$.

Proof. See [BG19, Lemma 6]. Note that the key difficulty lies in obtaining the right N dependence in (5.72). Due to the paraproduct in the definition of (5.71), one can show that this requires finding a decomposition of $\curlyvee_k$ such that $\mathscr{U}_{>}\curlyvee_k$ has Besov-Hölder norm that is uniformly bounded in N^3 (see Proposition A.5). Such a bound is not true for $\curlyvee_k$ (see Remark 4.5). This is overcome by defining $\mathscr{U}_{\leqslant}\curlyvee_k$ to be a random truncation of the Fourier series of $\curlyvee_k$, where the location of the truncation is chosen to depend on the Besov-Hölder norm of $\curlyvee_k$. □

For $v \in \mathbb{H}_K$, let

$$\Psi^{+}_{K}(v) = \mathscr{H}^{+}_{\beta,N,K}(\uparrow_K + V_K) + \frac{1}{2} \int_{\mathbb{T}_N} \int_0^K v_k^2 dk dx$$

and define $\mathscr{R}_K^+$ by

$$\Psi_K^+(v) = \mathscr{R}_K^+ - \frac{\eta}{2}\int_{\mathbb{T}_N} V_K^2 dx + \int_{\mathbb{T}_N} \mathscr{V}_\beta^+(V_K)dx + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K v_k^2\, dk dx.$$

We observe

$$\Psi_K^+(v) \leqslant \mathscr{R}_K^+ + \int_{\mathbb{T}_N} \mathscr{V}_\beta^+(V_K)dx + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K v_k^2 dk dx. \tag{5.73}$$

Thus, unlike the lower bound, the negative mass $-\frac{\eta}{2}\int_{\mathbb{T}_N} V_K^2 dx$ can be ignored in bounding the upper bound on the free energy.

Now fix $\check{v}$ as in (5.71). Arguing as in Proposition 5.11, there exists $\tilde{\mathscr{R}}_K^+$ such that

$$\mathscr{R}_K^+ + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K \check{v}_k^2 dk dx \approx \tilde{\mathscr{R}}_K^+ + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K \check{r}_k^2 dk dx \tag{5.74}$$

and $\tilde{\mathscr{R}}_K^+$ satisfies the following estimate: for every $\varepsilon > 0$, there exists $C = C(\varepsilon, \eta) > 0$ such that, for all $\beta > 1$,

$$|\tilde{\mathscr{R}}_K^+| \leqslant C N_K^\Xi + \varepsilon\Big(\int_{\mathbb{T}_N} \mathscr{V}_\beta^+(\check{V}_K)dx + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K \check{r}_k^2 dk dx\Big). \tag{5.75}$$

Above, we have used that the moment conditions (5.72) are sufficient for conclusions of Lemma 5.13 to apply to $\check{v}$.

Thus, by (5.73), (5.74), and (5.75),

$$\mathbb{E}[\Psi_K^+(\check{v})] \leqslant CN^3 + (1+\varepsilon)\mathbb{E}\Big[\int_{\mathbb{T}_N} \mathscr{V}_\beta^+(\check{V}_K) + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K \check{r}_k^2 dk dx\Big]. \tag{5.76}$$

By Young's inequality, $\frac{1}{\beta}a^4 + \frac{4}{\sqrt{\beta}}a^3 + 4a^2 \leqslant 3a^4 + 6a^2 \leqslant 9a^4 + 9$ for all $\beta > 1$ and $a \in \mathbb{R}$. Thus,

$$\int_{\mathbb{T}_N} \mathscr{V}_\beta^+(\check{V}_K)dx \leqslant 9\int_{\mathbb{T}_N} \check{V}_K^4 dx + 9N^3.$$

Inserting this into (5.76) and using the moment estimates on the drift (5.72) yields

$$\mathbb{E}[\Psi_K^+(\check{v})] \leqslant CN^3 + (1+\varepsilon)\mathbb{E}\Big[9\int_{\mathbb{T}_N} \check{V}_K^4 dx + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K \check{r}_k^2 dk dx\Big] \leqslant CN^3.$$

Hence, by Proposition 4.7,

$$-\log \mathscr{Z}_{\beta,N,K} = \inf_{v\in\mathbb{H}_K} \mathbb{E}\Psi_K^+(v) \leqslant \mathbb{E}\Psi_K^+(\check{v}) \leqslant CN^3$$

thereby establishing (5.2).

5.9 Proof of Proposition 5.1

We begin with two propositions, the first of which is a type of Itô isometry for fields under $\nu_{\beta,N}$ and the second of characterises functions against which the Wick square field can be tested against. Together, they imply that the random variables in Proposition 5.1 are integrable and that these expectations can be approximated using the cutoff measures $\nu_{\beta,N,K}$. Recall also Remarks 3.1 and 3.3.

Proposition 5.22. *Let $f \in H^{-1+\delta}$ for some $\delta > 0$. For every $K \in (0,\infty)$, let $\phi^{(K)} \sim \nu_{\beta,N,K}$ and $\phi \sim \nu_{\beta,N}$.*

The random variables $\{\int_{\mathbb{T}_N} \phi^{(K)} f dx\}_{K>0}$ converge weakly as $K \to \infty$ to a random variable

$$\phi(f) = \int_{\mathbb{T}_N} \phi f dx \in L^2(\nu_{\beta,N}).$$

Moreover, for every $c > 0$,

$$\left\langle \exp\left(c\phi(f)^2\right)\right\rangle_{\beta,N} < \infty.$$

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subset C^\infty(\mathbb{T}_N)$ such that $f_n \to f$ in $H^{-1+\delta}$. We first show that $\{\phi(f_n)\}$ is Cauchy in $L^2(\nu_{\beta,N})$.

Let $\varepsilon > 0$. Choose n_0 such that, for all $n, m > n_0$, $\|f_n - f_m\|_{H^{-1+\delta}} < \frac{\varepsilon}{N^3}$.

Fix $n, m > n_0$ and let $\delta f = f_n - f_m$. Then,

$$|\phi(f_n) - \phi(f_m)|^2 = \varepsilon \cdot \frac{1}{\varepsilon}\phi(\delta f)^2 \leqslant \varepsilon e^{\frac{1}{\varepsilon}\phi(\delta f)^2}. \tag{5.77}$$

By Proposition 5.3, there exists $C = C(\eta) > 0$ such that

$$\begin{aligned}\left\langle e^{\frac{1}{\varepsilon}\phi(\delta f)^2}\right\rangle_{\beta,N} &= \lim_{K\to\infty} \frac{1}{\mathscr{Z}_{\beta,N,K}} \mathbb{E}_N e^{-\mathscr{H}_{\beta,N,K}(\phi_K) + \frac{1}{\varepsilon}\phi_K(\delta f)^2} \\ &\leqslant e^{CN^3} \limsup_{K\to\infty} \mathbb{E}_N e^{-\mathscr{H}_{\beta,N,K}(\phi_K) + \frac{1}{\varepsilon}\phi_K(\delta f)^2}.\end{aligned}$$

We apply Proposition 4.7 to the expectation on the righthand side (with total energy cutoff suppressed, see Remark 4.8 and the paragraph that precedes it).

For $v \in \mathbb{H}_{b,K}$, define

$$\Psi_K^{\delta f}(v) = \mathscr{H}_{\beta,N,K}(\uparrow_K + V_K) - \frac{1}{\varepsilon}\Big(\int_{\mathbb{T}_N}(\uparrow_K + V_K)\delta f dx\Big)^2 + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K v_k^2 dk dx.$$

Expanding out the second term (and ignoring the prefactor $\frac{1}{\varepsilon}$ for the moment), we obtain:

$$\mathbb{E}\left[\Big(\int_{\mathbb{T}_N} \uparrow_K \delta f dx\Big)^2 + \Big(\int_{\mathbb{T}_N} \curlyvee_K \delta f dx\Big)^2 + \Big(\int_{\mathbb{T}_N} U_K \delta f dx\Big)^2\right]. \tag{5.78}$$

Consider the first integral in (5.78). By Parseval's theorem, the Fourier coefficients of $\uparrow_K$ (see (4.2)), and Itô's isometry,

$$\mathbb{E}\Big[\int_{\mathbb{T}_N} \uparrow_K \delta f dx\Big]^2 = \frac{1}{N^6}\sum_{n,m}\mathbb{E}[\mathcal{F}\uparrow_K(n)\mathcal{F}\uparrow_K(m)]\mathcal{F}\delta f(m)\mathcal{F}\delta f(n) \\ \lesssim \frac{1}{N^3}\sum_n \frac{|\mathcal{F}\delta f(n)|^2}{\langle n\rangle^2} \lesssim N^3\|\delta f\|^2_{H^{-1+\delta}} \tag{5.79}$$

where sums are taken over frequencies $n_i \in (N^{-1}\mathbb{Z})^3$. Above, the N dependency in the last inequality is due to our Sobolev spaces being defined with respect to normalised Lebesgue measure $\text{đ}x$.

For the second term in (5.78), by Parseval's theorem, Itô's isometry, and the Fourier coefficients of Υ_K (see (4.6)), we obtain

$$\mathbb{E}\Big(\int_{\mathbb{T}_N} \Upsilon_K \delta f dx\Big)^2 = \frac{1}{N^6}\mathbb{E}\Big(\sum_n \mathcal{F}\Upsilon_K(n)\mathcal{F}\delta f(n)\Big)^2 \\ = \frac{1}{N^6}\sum_n |\mathcal{F}\delta f(n)|^2\mathbb{E}\Big|\mathcal{F}\Upsilon_K(n)\Big|^2 \\ \lesssim \sum_n \frac{|\mathcal{F}\delta f(n)|^2}{\langle n\rangle^4} \lesssim N^6\|\delta f\|^2_{H^{-1+\delta}}. \tag{5.80}$$

For the final term in (5.78), by duality (1.1)

$$\Big(\int_{\mathbb{T}_N} U_K \delta f dx\Big)^2 \leqslant N^6\|\delta f\|^2_{H^{-1+\delta}}\|U_K\|^2_{H^{1-\delta}}. \tag{5.81}$$

Therefore, using that $\|\delta f\|^2_{H^{1-\delta}} \leqslant \frac{\varepsilon^2}{N^6}$, the estimates (5.79), (5.80), and (5.81) yield:

$$\mathbb{E}\Bigg[\frac{1}{\varepsilon}\Big(\int_{\mathbb{T}_N}(\uparrow_K + V_K)\delta f dx\Big)^2\Bigg] \\ \leqslant C(\eta)N^6(N^{-3}+1)\frac{\|\delta f\|^2_{H^{-1+\delta}}}{\varepsilon} \\ + C(\eta)N^6\frac{\|\delta f\|^2_{H^{-1+\delta}}}{\varepsilon}\mathbb{E}\Big[\|U_K\|^2_{H^{1-\delta}}\Big] \\ \leqslant C(\eta)\varepsilon(N^{-3}+1+\mathbb{E}\|U_K\|^2_{H^{1-\delta}}). \tag{5.82}$$

Using arguments in Section 5.8.1, it is straightforward to show that there exists $C = C(\eta,\beta) > 0$ such that, for ε sufficiently small,

$$\mathbb{E}\Psi^{\delta f}_K(v) \geqslant -CN^3$$

for every $v \in \mathbb{H}_{b,K}$ (note that β dependence is not important here).

Inserting this into Proposition 4.7 gives

$$\limsup_{K\to\infty} \langle e^{-\mathscr{H}_{\beta,N,K}(\phi_K)+\frac{1}{\varepsilon}\phi_K(\delta f)^2}\rangle_{\beta,N,K} \leqslant e^{CN^3}. \tag{5.83}$$

Taking expectations in (5.77) and using (5.83) finishes the proof that $\{\phi(f_n)\}$ is Cauchy in $L^2(\nu_{\beta,N})$.

Similar arguments can be used to show exponential integrability of the limiting random variable, $\phi(f)$ and that,

$$\sup_{K>0} |\langle|\phi^{(K)}(f_n) - \phi^{(K)}(f)|\rangle_{\beta,N,K} \to 0 \quad \text{as } n \to \infty.$$

We now show that $\phi^{(K)}(f)$ converges weakly to $\phi(f)$ as $K \to \infty$. Let $G : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz with Lipschitz constant $|G|_{\text{Lip}}$, and let $\varepsilon > 0$. Choose n sufficiently large so that

$$\sup_{K>0} |\langle|\phi^{(K)}(f_n) - \phi^{(K)}(f)|\rangle_{\beta,N,K} < \frac{\varepsilon}{2|G|_{\text{Lip}}}$$

and

$$\langle|\phi(f_n)) - \phi(f)|\rangle_{\beta,N} < \frac{\varepsilon}{2|G|_{\text{Lip}}}.$$

Then,

$$\begin{aligned} |\langle G(\phi^{(K)}(f))\rangle_{\beta,N,K} - \langle G(\phi(f))\rangle_{\beta,N} &\leqslant \sup_{K>0} |\langle G(\phi^{(K)}(f_n)) - G(\phi^{(K)}(f))\rangle_{\beta,N,K}| \\ &\quad + |\langle G(\phi^{(K)}(f_n))\rangle_{\beta,N,K} - \langle G(\phi(f_n))\rangle_{\beta,N}| \\ &\quad + |\langle G(\phi(f_n)) - G(\phi(f))\rangle_{\beta,N}| \\ &\leqslant |\langle G(\phi^{(K)}(f_n))\rangle_{\beta,N,K} - \langle G(\phi(f_n))\rangle_{\beta,N}| + \varepsilon. \end{aligned}$$

The first term on the righthand side goes to zero as $K \to \infty$ since $f_n \in C^\infty$. Thus,

$$\lim_{K\to\infty} |\langle G(\phi(f))\rangle_{\beta,N,K} - \langle G(\phi(f))\rangle_{\beta,N} \leqslant \varepsilon.$$

Since ε is arbitrary, we have shown that $\phi^{(k)}(f)$ converges weakly to $\phi(f)$. $\square$

Proposition 5.23. *Let $f \in B^s_{\frac{4}{3},1} \cap L^2$ for some $s > \frac{1}{2}$. For every $K \in (0,\infty)$, let $\phi^{(K)} \sim \nu_{\beta,N,K}$ and $\phi \sim \nu_{\beta,N}$.*

The random variables $\{\int_{\mathbb{T}_N} : (\phi^{(K)})^2 : f dx\}_{K>0}$ converge weakly as $K \to \infty$ to a random variable

$$: \phi^2 : (f) = \int_{\mathbb{T}_N} : \phi^2 : f dx \in L^2(\nu_{\beta,N}).$$

Moreover, for $c > 0$,

$$\Big\langle \exp\big(c : \phi^2 : (f)\big)\Big\rangle_{\beta,N} < \infty.$$

Proof. The proof of Proposition 5.23 follows the same strategy as the proof of Proposition 5.22, so we do not give all the details. The only real key difference is the analytic bounds required in the stochastic control problem. Indeed, these require one to tune the integrability assumptions on f in order to get the required estimates.

It is not too difficult to see that the term we need to control is the integral

$$\int_{\mathbb{T}_N} \curlyvee_K f + 2\Psi_K V_K f + V_K^2 f dx. \tag{5.84}$$

Strictly speaking, we need to control the above integral with f replaced by $\delta f = f_n - f_m$, where $\{f_n\}_{n\in\mathbb{N}} \subset C^\infty(\mathbb{T}_N)$ such that $f_n \to f$ in $B^s_{\frac{3}{4},1} \cap L^2$, but the analytic bounds are the same.

Note that $\mathbb{E}\int_{\mathbb{T}_N} \curlyvee_K f dx = \int_{\mathbb{T}_N} \mathbb{E}\curlyvee_K f dx = 0$. Moreover by Young's inequality and the additional integrability assumption $f \in L^2$, for any $\varepsilon > 0$ we have

$$\int_{\mathbb{T}_N} V_K^2 f dx \lesssim \frac{1}{\varepsilon}\int_{\mathbb{T}_N} f^2 dx + \varepsilon \int_{\mathbb{T}_N} V_K^4 dx$$

which can be estimated as in the proof of Proposition 5.22. Thus, we only need to estimate the second integral in (5.84). Note that the product $\Psi_K f$ is a well-defined distribution from a regularity perspective as $K \to \infty$ since $f \in B^s_{\frac{4}{3},1}$ for $s > \frac{1}{2}$. The difficulty in obtaining the required estimates comes from integrability issues.

We split the integral into three terms by using the paraproduct decomposition $\Psi f = \Psi \varolessthan f + \Psi \varodot f + \Psi \varogreaterthan f$. The integral associated to $\Psi \varogreaterthan f$ is straightforward to estimate, so we focus on the first two terms. Since $f \in L^2$ and $\Psi_K \in \mathscr{C}^{-\frac{1}{2}-\kappa}$, by the paraproduct estimate 1.8 we have $\Psi \varolessthan f \in H^{-\frac{1}{2}-\kappa}$. Thus, the integral $\int_{\mathbb{T}_N}(\Psi_K \varolessthan f) V_K dx$ can be treated similarly as in the proof of Proposition 5.22. Note that, in this proposition the use of Hölder-Besov norms is fine because we are not concerned with issues of N dependency. Moreover, note that if we just used that $f \in B^s_{\frac{4}{3},1}$ the resulting integrability of $\Psi_K \varolessthan f$ is not sufficient to justify testing against V_K, which can be bounded in L^2-based Sobolev spaces. For the final integral, by the resonant product estimate (1.9) we have $\Psi_K \varodot f \in L^{\frac{4}{3}}$. Hence, we can use Young's inequality to estimate $\int_{\mathbb{T}_N}(\Psi_K \varodot f) V_K dx$ and then argue similarly as in the proof of Proposition 5.22. □

Without loss of generality, we assume $a_0 = a = 1$ in Proposition 5.1 and we split its proof into Lemmas 5.24, 5.25, and 5.26.

Lemma 5.24. *There exists $\beta_0 > 1$ and $C_Q > 0$ such that, for any $\beta > \beta_0$,*

$$-\frac{1}{N^3}\log\Big\langle \prod_{\square\in\mathbb{B}_N} \exp Q_1(\square)\Big\rangle_{\beta,N} \geqslant -C_Q.$$

Proof. For any $K \in (0,\infty)$, define

$$\mathscr{H}^{Q_1}_{\beta,N,K}(\phi_K) = \int_{\mathbb{T}_N} :\mathscr{V}^{Q_1}_\beta(\phi_K): -\frac{\gamma_K}{\beta^2} :\phi_K^2: -\delta_K - \frac{\eta}{2} :\phi_K^2: dx$$

where

$$\mathscr{V}^{Q_1}_\beta(a) = \mathscr{V}_\beta(a) - \frac{1}{\sqrt{\beta}}(\beta - a^2) - \frac{1}{4} = \frac{1}{\beta}\bigg(a^2 - \Big(\beta + \frac{\sqrt{\beta}}{2}\Big)\bigg)^2.$$

Then, by Propositions 5.23 and 5.3, there exists $C = C(\eta) > 0$ such that

$$\begin{aligned}\Big\langle \prod_{\square\in\mathbb{B}_N} \exp Q_1(\square)\Big\rangle_{\beta,N} &= \lim_{K\to\infty} \Big\langle \exp\Big(\frac{1}{\sqrt{\beta}}\int_{\mathbb{T}_N} \beta - :\phi_k^2: dx\Big)\Big\rangle_{\beta,N,K} \\ &\leqslant e^{\frac{1}{4}N^3} \lim_{K\to\infty} \frac{1}{\mathscr{Z}_{\beta,N,K}} \mathbb{E}_N e^{-\mathscr{H}^{Q_1}_{\beta,N,K}(\phi_K)} \\ &\leqslant e^{\left(C+\frac{1}{4}\right)N^3} \limsup_{K\to\infty} \mathbb{E}_N e^{-\mathscr{H}^{Q_1}_{\beta,N,K}(\phi_K)}\end{aligned}$$

where

Therefore, we have reduced the problem to proving Proposition 5.3 for the potential $\mathscr{V}^{Q_1}_\beta$ instead of $\mathscr{V}_\beta$. The proof follows essentially word for word after two observations: first, the same γ_K and δ_K works for both $\mathscr{V}_\beta$ and $\mathscr{V}^{Q_1}_\beta$ since the quartic term is unchanged. Second, since $\sqrt{\beta + \frac{\sqrt{\beta}}{2}} = \sqrt{\beta} + o(\sqrt{\beta})$ as $\beta \to \infty$, the treatment of β-dependence of the estimates in Section 5.6 is exactly the same. □

Lemma 5.25. *There exists* $\beta_0 > 1$ *and* $C_Q > 0$ *such that, for any* $\beta > \beta_0$,

$$-\frac{1}{N^3}\log\Big\langle \prod_{\square\in\mathbb{B}_N} \exp Q_2(\square)\Big\rangle_{\beta,N} \geqslant -C_Q. \tag{5.85}$$

Proof. By Propositions 5.22, 5.23, and 5.3, there exists $C = C(\eta) > 0$ such that, for β sufficiently large,

$$\begin{aligned}\Big\langle \prod_{\square\in\mathbb{B}_N} \exp Q_2(\square)\Big\rangle_{\beta,N} &= \lim_{K\to\infty} \Big\langle \exp\Big(\frac{1}{\sqrt{\beta}}\sum_{\square\in\mathbb{B}_N} \phi_K(\square)^2 - \frac{1}{\sqrt{\beta}}\int_{\mathbb{T}_N} :\phi_K^2: dx\Big\rangle_{\beta,N,K} \\ &\leqslant e^{CN^3} \limsup_{K\to\infty} \mathbb{E}_N e^{-\mathscr{H}^{Q_2}_{\beta,N,K}(\phi_K)}\end{aligned}$$

where

$$\mathscr{H}^{Q_2}_{\beta,N,K}(\phi_K) = \mathscr{H}_{\beta,N,K}(\phi_K) + \frac{1}{\sqrt{\beta}}\sum_{\square\in\mathbb{B}_N} \phi_K(\square)^2 - \frac{1}{\sqrt{\beta}}\int_{\mathbb{T}_N} :\phi_K^2: dx.$$

As in Section 5.2, we perform the expansion

$$-\log \mathbb{E}_N e^{-\mathscr{H}^{Q_2}_{\beta,N,K}(\phi_K)} = \sum_{\sigma\in\{\pm1\}^{\mathbb{B}_N}} e^{-F^{\sigma}_{\beta,N,K}} \mathbb{E}_N e^{-\mathscr{H}^{Q_2,\sigma}_{\beta,N,K}(\phi_K)} \tag{5.86}$$

where $F^{\sigma}_{\beta,N,K}$ is defined in (5.15) and

$$\mathscr{H}^{Q_2,\sigma}_{\beta,N,K}(\phi_K) = \mathscr{H}^{Q_2}_{\beta,N,K}(\phi_K + g_K) - \sum_{\square\in\mathbb{B}_N} \log\Big(\chi_{\sigma(\square)}\big((\phi_K+g_K)(\square)\big)\Big)$$

Fix $\sigma \in \{\pm1\}^{\mathbb{B}_N}$. For $v \in \mathbb{H}_{b,K}$, define

$$\begin{aligned}\Psi^{Q_2}_K(v) = \Psi_K(v) &+ \frac{1}{\sqrt{\beta}} \sum_{\square\in\mathbb{B}_N} \Big(\int_\square \text{¡}_K + V_K + g_K dx\Big)^2 \\ &- \frac{1}{\sqrt{\beta}} \int_{\mathbb{T}_N} :(\text{¡}_K + V_K + g_K)^2: dx\end{aligned} \tag{5.87}$$

where $\Psi_K = \Psi^{\sigma}_K$ is defined in (5.17).

We estimate second term in (5.87). First, note that

$$\begin{aligned}&\frac{1}{\sqrt{\beta}} \sum_{\square\in\mathbb{B}_N} \Big(\int_\square \text{¡}_K + V_K + g_K dx\Big)^2 \\ &\qquad\leqslant \sum_{\square\in\mathbb{B}_N} \frac{2}{\sqrt{\beta}}\Big(\int_\square \text{¡}_K dx\Big)^2 + \frac{2}{\sqrt{\beta}}\Big(\int_\square V_K + g_K dx\Big)^2.\end{aligned}$$

By a standard Gaussian covariance calculation, there exists $C = C(\eta) > 0$ such that

$$\sum_{\square\in\mathbb{B}_N} \mathbb{E}\Big(\int_\square \text{¡}_K dx\Big)^2 = \sum_{\square\in\mathbb{B}_N} \int_\square\int_\square \mathbb{E}[\text{¡}_K(x)\text{¡}_K(x')]dxdx' \leqslant CN^3.$$

For the other term, by the Cauchy-Schwarz inequality followed by bounds on the potential (5.8) and g_K (5.11), the following estimate holds: for any $\zeta > 0$,

$$\begin{aligned}\frac{2}{\sqrt{\beta}} \sum_{\square\in\mathbb{B}_N} \Big(\int_\square V_K + g_K dx\Big)^2 &\leqslant \int_{\mathbb{T}_N} \frac{2}{\sqrt{\beta}}(V_K+g_K)^2 dx \\ &\leqslant C(\zeta, C_P)N^3 + \zeta C_P \int_{\mathbb{T}_N} \mathscr{V}_\beta(V_K + g_K)dx\end{aligned}$$

where $C_P > 0$ is the Poincaré constant on unit boxes (1.15).

We now estimate the third term in (5.87). Since $\mathbb{E}\curlyvee_K = \mathbb{E}[\text{¡}_K g_K] = 0$,

$$\frac{1}{\sqrt{\beta}} \int_{\mathbb{T}_N} :(\text{¡}_K + V_K + g_K)^2: dx \approx \frac{1}{\sqrt{\beta}} \int_{\mathbb{T}_N} 2\text{¡}_K V_K + (V_K+g_K)^2 dx. \tag{5.88}$$

For the first integral on the righthand side of (5.88), by change of variables (5.21), and the paraproduct decomposition (1.7), we have

$$\frac{1}{\sqrt{\beta}}\int_{\mathbb{T}_N} 2\<1>_K V_K dx = \int_{\mathbb{T}_N} -\frac{8}{\beta^{\frac{5}{2}}}(\<1>_K \varolessthan \<30>_K + \<31>_K + \<1>_K \varogreaterthan \<30>_K) + \frac{2}{\sqrt{\beta}}\<1>_K U_K dx.$$

Hence, by (5.88), Proposition 4.4, duality (1.1), the potential bounds (5.8), and the bounds on U_K (5.22), for any $\varepsilon > 0$ there exists $C = C(\varepsilon, \eta) > 0$ such that

$$\Big|\frac{1}{\sqrt{\beta}}\int_{\mathbb{T}_N} 2\<1>_K V_K dx\Big| \leqslant CN_K^{\Xi} + \varepsilon\Big(\int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K + g_K)dx + \frac{1}{2}\int_{\mathbb{T}_N}\int_0^K r_k^2 dk dx\Big).$$

For the second integral on the righthand side of (5.88), again by (5.8) and (5.11), there exists an inessential constant $C > 0$ such that

$$\int_{\mathbb{T}_N}\frac{1}{\sqrt{\beta}}(V_K + g_K)^2 dx \leqslant CN^3 + \zeta C_P \int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K + g_K)dx.$$

Arguing as in Section 5.8.1 and taking into account the calculations above, the following estimate holds: let $\zeta < \frac{1}{8C_P}$ and $\varepsilon = 1 - 8C_P\zeta > 0$ as in Section 5.8.1. Then, provided $\eta < \frac{1}{196C_P}$ and $\beta > 1$,

$$\begin{aligned}
\mathbb{E}\Psi_K^{Q_2}(v) \geqslant \mathbb{E}\Bigg[&- C(\varepsilon,\zeta,\eta)N_K^{\Xi} + \Big(\frac{1-\varepsilon}{2} - \frac{4C_P\zeta}{2\beta^3} - 2C_P\zeta\Big)\int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K + g_K)dx \\
&+ \Big(\frac{1-\varepsilon}{2} - 4\zeta C_P\Big)\int_{\mathbb{T}_N}\int_0^K r_k^2 dk dx - 4\zeta C_P\int_{\mathbb{T}_N}|\nabla\tilde{g}_K|^2 dx\Bigg] \\
\geqslant{}& -CN^3 - 4\zeta C_P\int_{\mathbb{T}_N}|\nabla\tilde{g}_K|^2 dx.
\end{aligned}$$

Hence, by Proposition 4.7 applied with the Hamiltonian $\mathscr{H}^{Q_2,\sigma}_{\beta,N,K}(\phi_K)$ with total energy cutoff suppressed (see Remark 4.8),

$$F^\sigma_{\beta,N,K} - \log\mathbb{E}_N e^{-\mathscr{H}^{Q_2,\sigma}_{\beta,N,K}} \geqslant -CN^3 + \Big(\frac{1}{2} - 4\zeta C_P\Big)\int_{\mathbb{T}_N}|\nabla\tilde{g}_K|^2 dx \geqslant -CN^3$$

This estimate is uniform in σ, thus summing over the 2^{N^3} terms in the expansion (5.86) yields (5.85). □

Lemma 5.26. *There exists $\beta_0 > 1$ and $C_Q > 0$ such that, for any $\beta > \beta_0$,*

$$-\frac{1}{N^3}\log\Big\langle\prod_{\{\square,\square'\}\in B}\exp|Q_3(\square,\square')|\Big\rangle_{\beta,N} \geqslant -C_Q \tag{5.89}$$

where B is a set of unordered pairs of nearest-neighbour blocks that partitions $\mathbb{B}_N$.

Proof. By Propositions 5.22 and 5.3 there exists $C = C(\eta) > 0$ such that, for β sufficiently large,

$$\Big\langle \prod_{\{\square,\square'\}\in B} \exp|Q_3(\square,\square')|\Big\rangle_{\beta,N} = \lim_{K\to\infty}\Big\langle \exp\Big(\sum_{\{\square,\square'\}\in B}\Big|\int_\square \phi_K dx - \int_{\square'}\phi_K dx\Big|\Big\rangle_{\beta,N,K}$$
$$\leqslant e^{CN^3}\limsup_{K\to\infty}\mathbb{E}_N e^{-\mathscr{H}^{Q_3}_{\beta,N,K}(\phi_K)}$$

where

$$\mathscr{H}^{Q_3}_{\beta,N,K}(\phi_K) = \mathscr{H}^{Q_3}_{\beta,N,K}(\phi_K) - \sum_{\{\square,\square'\}\in B}\Big|\int_\square \phi_K dx - \int_{\square'}\phi_K dx\Big|.$$

We expand

$$-\log\mathbb{E}_N e^{-\mathscr{H}^{Q_3}_{\beta,N,K}(\phi_K)} = \sum_{\sigma\in\{\pm1\}^{\mathbb{B}_N}} e^{-F^\sigma_{\beta,N,K}}\mathbb{E}_N e^{-\mathscr{H}^{Q_3,\sigma}_{\beta,N,K}}$$

where $F^\sigma_{\beta,N,K}$ is defined in (5.15) and

$$\mathscr{H}^{Q_3,\sigma}_{\beta,N,K}(\phi_K) = \mathscr{H}^{Q_3}_{\beta,N,K}(\phi_K + g_K) - \sum_{\square\in\mathbb{B}_N}\log\Big(\chi_{\sigma(\square)}\big((\phi_K+g_K)(\square)\big)\Big).$$

Fix $\sigma\in\{\pm1\}^{\mathbb{B}_N}$. For $v\in\mathbb{H}_{b,K}$, define

$$\Psi^{Q_3}_K(v) = \Psi_K(v) - \sum_{\{\square,\square'\}\in B}\Big|\int_\square \text{\textbf{!}}_K + V_K + g_K dx - \int_{\square'} \text{\textbf{!}}_K + V_K + g_K dx\Big|$$

where $\Psi_K(v) = \Psi^\sigma_K(v)$ is defined in (5.17).

A standard Gaussian calculation yields $\mathbb{E}|\text{\textbf{!}}_K| \leqslant CN^3$ for some constant $C = C(\eta) > 0$. Hence, by the triangle inequality, Proposition 4.4 and the Cauchy-Schwarz inequality,

$$\sum_{\{\square,\square'\}\in B}\Big|\int_\square \text{\textbf{!}}_K + V_K + g_K dx - \int_{\square'} \text{\textbf{!}}_K + V_K + g_K dx\Big|$$
$$\lesssim CN^\Xi_K + \frac{1}{\beta^2}\Big|\int_{\mathbb{T}_N}\int_0^K \mathscr{J}_k\big(\text{\textbf{V}}_k \otimes (V_k^\flat + g_k^\flat)\big)dkdx$$
$$+ \sum_{\{\square,\square'\}\in B}\Big|\int_\square (R_K+g_K)dx - \int_{\square'}(R_K+g_K)dx\Big|.$$

The integral with the paraproduct can be estimated as in (5.49) to establish: for any $\zeta > 0$,

$$\frac{1}{\beta^2}\Big|\int_{\mathbb{T}_N}\int_0^K \mathscr{J}_k\big(\text{\textbf{V}}_k \otimes (V_k^\flat + g_k^\flat)\big)dkdx \leqslant C(\zeta, C_P)N^3 + \frac{2\zeta C_P}{\beta^3}\int_{\mathbb{T}_N}\mathscr{V}_\beta(V_K+g_K)dx$$

where $C_P > 0$ is the Poincaré constant on unit blocks (1.15).

We now estimate the remaining integral. Assume without loss of generality that $\square' = \square + e_1$. Then, by the triangle inequality and the fundamental theorem of calculus,

$$\begin{aligned}\Big| \int_\square (R_K + g_K) dx - \int_{\square'} (R_K + g_K) dx \Big| \\ &= \int_\square \Big(R_K(x) - R_K(x+e_1) + g_K(x) - g_k(x+e_1) \Big) dx \\ &\leqslant \int_0^1 \int_\square |\nabla R_K(x+te_1)| + |\nabla g_K(x+te_1)| dx dt \\ &\leqslant \int_{\square \cup \square'} |\nabla R_K| + |\nabla g_K| dx.\end{aligned}$$

Hence, by the Cauchy-Schwarz inequality, the bound on the drift (4.18) and the bound on ∇g_K (5.12), we have the following estimate: for any $\zeta > 0$,

$$\begin{aligned}\sum_{\{\square,\square'\}\in B} \Big| \int_\square (R_K + g_K) dx - \int_{\square'} (R_K + g_K) dx \Big| \\ &\leqslant C(\zeta, C_P) N^3 + 4\zeta C_P \Big(\int_{\mathbb{T}_N} |\nabla R_K|^2 dx + \int_{\mathbb{T}_N} |\nabla g_K|^2 dx \Big) \\ &\leqslant C(\zeta, C_P) N^3 + 4\zeta C_P \Big(\int_{\mathbb{T}_N} \int_0^K r_k^2 dk dx + \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx \Big).\end{aligned}$$

Thus, by arguing as in Section 5.8.1, one can show the following estimate: let $\zeta < \frac{1}{16 C_P}$ and $\varepsilon = 1 - 8\zeta C_P > 0$. Then, provided $\eta < \frac{1}{392 C_P}$ and $\beta > 1$,

$$\begin{aligned}\mathbb{E}\Psi_K^{Q_3}(v) \geqslant \mathbb{E}\Bigg[&- C N_K^{\Xi} + \Big(\frac{1-\varepsilon}{2} - \frac{2\zeta C_P}{\beta^3} - \frac{2\zeta C_P}{\beta^3} \Big) \int_{\mathbb{T}_N} \mathscr{V}_\beta (V_K + g_K) dx \\ &+ \Big(\frac{1-\varepsilon}{2} - 4\zeta C_P - 4\zeta C_P \Big) \int_{\mathbb{T}_N} \int_0^K r_k^2 dk dx \\ &- (4\zeta C_P + 4\zeta C_P) \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx \Bigg] \\ \geqslant -CN^3 - 8\zeta C_P \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx.\end{aligned}$$

Applying Proposition 4.7 with Hamiltonian $\mathscr{H}_{\beta,N,K}^{Q_3,\sigma}(\phi_K)$, with total energy cutoff suppressed (see Remark 4.8), yields

$$F_{\beta,N,K}^\sigma - \log \mathbb{E}_N e^{-\mathscr{H}_{\beta,N,K}^{Q_3}(\phi_K)} \geqslant -CN^3 + \Big(\frac{1}{2} - 8\zeta C_P \Big) \int_{\mathbb{T}_N} |\nabla \tilde{g}_K|^2 dx \geqslant -CN^3.$$

This estimate is uniform over all 2^{N^3} choices of σ, hence establishing (5.89). $\square$

6 Chessboard estimates

In this section we prove Proposition 3.6 using the chessboard estimates of Proposition 6.5 and the estimates obtained in Section 5. In addition, we establish that $\nu_{\beta,N}$ is reflection positive.

6.1 Reflection positivity of $\nu_{\beta,N}$

We begin by defining reflection positivity for general measures on spaces of distributions following [Shl86] and [GJ87].

For any $a \in \{0, \dots, N-1\}$ and $\{i,j,k\} = \{1,2,3\}$, let

$$\mathscr{R}_{\Pi_{a,i}}(x) = (2a - x_i)e_i + e_j + e_k$$

where $x = x_i e_i + x_j e_j + x_k e_j \in \mathbb{T}_N$ and addition is understood modulo N. Define

$$\Pi_{a,i} = \{x \in \mathbb{T}_N : \mathscr{R}_{\Pi_{a,i}}(x) = x\}. \tag{6.1}$$

Note that for any $x \in \Pi_{a,i}$, $x_i = a$ or $a + \frac{N}{2}$. We say that $\mathscr{R}_{\Pi_{a,i}}$ is the reflection map across the hyperplane $\Pi_{a,i}$.

Fix such a hyperplane Π. It separates $\mathbb{T}_N = \mathbb{T}_N^+ \sqcup \Pi \sqcup \mathbb{T}_N^-$ such that $\mathbb{T}_N^+ = \mathscr{R}_\Pi \mathbb{T}_N^-$. For any $f \in C^\infty(\mathbb{T}_N)$, we say f is $\mathbb{T}_N^+$-measurable if $\mathrm{supp} f \subset \mathbb{T}_N^+$. The reflection of f in Π is defined pointwise by $\mathscr{R}_\Pi f(x) = f(\mathscr{R}_\Pi x)$. For any $\phi \in S'(\mathbb{T}_N)$, we say that ϕ is $\mathbb{T}_N^+$-measurable if $\phi(f) = 0$ unless f is $\mathbb{T}_N^+$ measurable, where $\phi(f)$ denotes the duality pairing between $S'(\mathbb{T}_N)$ and $C^\infty(\mathbb{T}_N)$. For any such ϕ, we define $\mathscr{R}_\Pi \phi$ pointwise by $\mathscr{R}_\Pi \phi(f) = \phi(\mathscr{R}_\Pi f)$.

Let ν be a probability measure on $S'(\mathbb{T}_N)$. We say that $F \in L^2(\nu)$ is $\mathbb{T}_N^+$-measurable if it is measurable with respect to the σ-algebra generated by the set of $\phi \in S'(\mathbb{T}_N)$ that are $\mathbb{T}_N^+$-measurable. For any such F, we define $\mathscr{R}_\Pi F$ pointwise by $\mathscr{R}_\Pi F(\phi) = F(\mathscr{R}_\Pi \phi)$.

The measure ν on $S'(\mathbb{T}_N)$ is called *reflection positive* if, for any hyperplane Π of the form (6.1),

$$\int_{S'(\mathbb{T}_N)} F(\phi) \cdot \mathscr{R}_\Pi F(\phi) d\nu(\phi) \geqslant 0$$

for all $F \in L^2(\nu)$ that are $\mathbb{T}_N^+$-measurable.

Proposition 6.1. *The measure $\nu_{\beta,N}$ is reflection positive.*

6.1.1 Proof of Proposition 6.1

In general, Fourier approximations to $\nu_{\beta,N}$ (such as $\nu_{\beta,N,K}$) are not reflection positive. Instead, we prove Proposition 6.1 by considering lattice approximations to $\nu_{\beta,N}$ for which reflection positivity is straightforward to show.

Let $\mathbb{T}_N^\varepsilon = (\varepsilon\mathbb{Z}/N\mathbb{Z})^3$ be the discrete torus of sidelength N and lattice spacing $\varepsilon > 0$. In order to use discrete Fourier analysis, we assume that $\varepsilon^{-1} \in \mathbb{N}$. Note that any hyperplane Π of the form (6.1) is a subset of $\mathbb{T}_N^\varepsilon$.

For any $\varphi \in (\mathbb{R})^{\mathbb{T}_N^\varepsilon}$, define the lattice Laplacian

$$\Delta^\varepsilon \varphi(x) = \frac{1}{\varepsilon^2} \sum_{\substack{y \in \mathbb{T}_N^\varepsilon \\ |x-y|=\varepsilon}} (\varphi(y) - \varphi(x)).$$

Let $\tilde{\mu}_{N,\varepsilon}$ be the Gaussian measure on $\mathbb{R}^{\mathbb{T}_N^\varepsilon}$ with density

$$d\tilde{\mu}_{N,\varepsilon}(\varphi) \propto \exp\Big(-\frac{\varepsilon^3}{2} \sum_{x \in \mathbb{T}_N^\varepsilon} \varphi(x) \cdot (-\Delta^\varepsilon + \eta)\varphi(x)\Big) \prod_{x \in \mathbb{T}_N^\varepsilon} d\varphi(x)$$

where $d\vec{\phi}(x)$ is Lebesgue measure.

A natural lattice approximation to $\nu_{\beta,N}$ is given by the probability measure $\tilde{\nu}_{\beta,N,\varepsilon}$ with density proportional to

$$d\tilde{\nu}_{\beta,N,\varepsilon}(\varphi) \propto e^{-\tilde{\mathscr{H}}_{\beta,N,\varepsilon}(\varphi)} d\tilde{\mu}_{N,\varepsilon}(\varphi)$$

where

$$\tilde{\mathscr{H}}_{\beta,N,\varepsilon}(\varphi) = \varepsilon^3 \sum_{x \in \mathbb{T}_n^\varepsilon} \mathscr{V}_\beta(\varphi(x)) - \Big(\frac{\eta}{2} + \frac{1}{2}\delta m^2(\varepsilon,\eta)\Big)\varphi(x)^2$$

where $\frac{1}{2}\delta m^2(\varepsilon,\eta)$ is a renormalisation constant that diverges as $\varepsilon \to 0$ (see Proposition 6.19). Note two things: first, the renormalisation constant is chosen dependent on η for technical convenience. Second, no energy renormalisation is included since we are only interested in convergence of measures.

Remark 6.2. *By embedding $\mathbb{R}^{\mathbb{T}_N^\varepsilon}$ into $S'(\mathbb{T}_N)$, we can define reflection positivity for lattice measures. We choose this embedding so that the pushforward of $\tilde{\nu}_{\beta,N,\varepsilon}$ is automatically reflection positive, but other choices are possible.*

For any $\varphi \in \mathbb{R}^{\mathbb{T}_N^\varepsilon}$, we write $\text{ext}^\varepsilon\varphi$ for its unique extension to a trigonometric polynomial on $\mathbb{T}_N$ of degree less than ε^{-1} that coincides with φ on lattice points (i.e. in $\mathbb{T}_N^\varepsilon$). Precisely,

$$\text{ext}^\varepsilon(\varphi)(x) = \frac{\varepsilon^3}{N^3} \sum_n \sum_{y \in \mathbb{T}_N^\varepsilon} e_n(y-x)\varphi(y)$$

where the sum ranges over all $n = (a_1, a_2, a_3) \in (N^{-1}\mathbb{Z})^3$ such that $|a_i| \leqslant \varepsilon^{-1}$, and we recall $e_n(x) = e^{2\pi i n \cdot x}$.

Lemma 6.3. *Let $\varepsilon > 0$ such that $\varepsilon^{-1} \in \mathbb{N}$. Denote by $\text{ext}^\varepsilon_* \tilde{\nu}_{\beta,N,\varepsilon}$ the pushforward of $\tilde{\nu}_{\beta,N,\varepsilon}$ by the map ext^ε. Then, the measure $\text{ext}^\varepsilon_* \tilde{\nu}_{\beta,N,\varepsilon}$ is reflection positive.*

Proof. Fix a hyperplane Π of the form (6.1) and recall that Π separates $\mathbb{T}_N = \mathbb{T}_N^+ \sqcup \Pi \sqcup \mathbb{T}_N^-$. Write $\mathbb{T}^+_{N,\varepsilon} = \mathbb{T}^+_N \cap \mathbb{T}^\varepsilon_N$.

Since the measure $\tilde{\nu}_{\beta,N,\varepsilon}$ is reflection positive on the lattice by [Shl86, Theorem 2.1], the following estimate holds: let $F^\varepsilon \in L^2(\tilde{\nu}_{\beta,N,\varepsilon})$ be $\mathbb{T}^+_{N,\varepsilon}$-measurable - i.e. $F^\varepsilon(\varphi)$ depends only on $\varphi(x)$ for $x \in \mathbb{T}^+_{N,\varepsilon}$. Then,

$$\int F^\varepsilon(\varphi) \cdot \mathscr{R}_\Pi F^\varepsilon(\varphi) d\tilde{\nu}_{\beta,N,\varepsilon}(\varphi) \geqslant 0. \tag{6.2}$$

Let $F \in L^2(\text{ext}^\varepsilon_* \tilde{\nu}_{\beta,N,\varepsilon})$ be $\mathbb{T}^+_N$-measurable. Then, $F \circ \text{ext}^\varepsilon \in L^2(\tilde{\nu}_{\beta,N,\varepsilon})$ is $\mathbb{T}^+_{N,\varepsilon}$-measurable. Using that ext^ε and $\mathscr{R}_\Pi$ (the reflection across Π) commute,

$$\begin{aligned}\int F(\phi) \cdot \mathscr{R}_\Pi F(\phi) d\text{ext}^\varepsilon_* \tilde{\nu}_{\beta,N,\varepsilon}(\phi) &= \int (F \circ \text{ext}^\varepsilon)(\varphi) \cdot (F \circ \mathscr{R}_\Pi \circ \text{ext}^\varepsilon)(\varphi) d\tilde{\nu}_{\beta,N,\varepsilon}(\varphi) \\ &= \int (F \circ \text{ext}^\varepsilon)(\varphi) \cdot (F \circ \text{ext}^\varepsilon)(\mathscr{R}_\Pi \varphi) d\tilde{\nu}_{\beta,N,\varepsilon}(\varphi) \\ &\geqslant 0\end{aligned}$$

where the last inequality is by (6.2). Hence, $\text{ext}^\varepsilon_* \tilde{\nu}_{\beta,N,\varepsilon}$ is reflection positive. □

Proposition 6.4. *There exist constants $\frac{1}{2}\delta m^2(\bullet, \eta)$ such that $\text{ext}^\varepsilon_* \vec{\nu}_{\beta,N,\varepsilon} \to \nu_{\beta,N}$ weakly as $\varepsilon \to \infty$.*

Proof. The existence of a weak limit of $\text{ext}^\varepsilon_* \tilde{\nu}_{\beta,N,\varepsilon}$ as $\varepsilon \to 0$ was first established in [Par75]. The fact the lattice approximations and the Fourier approximations (i.e. $\nu_{\beta,N,K}$) yield the same limit as the cutoff is removed is not straightforward in 3D because of the mutual singularity of $\nu_{\beta,N}$ and μ_N [BG20]. Previous approaches have relied on Borel summation techniques to show that the correlation functions agree with (resummed) perturbation theory [MS77].

In Section 6.4 we give an alternative proof using stochastic quantisation techniques. The key idea is to view $\nu_{\beta,N}$ as the unique invariant measure for a singular stochastic PDE with a local solution theory that is robust under different approximations. This allows us to show directly that $\text{ext}^\varepsilon_* \tilde{\nu}_{\beta,N,\varepsilon}$ converges weakly to $\nu_{\beta,N}$ and avoids the use of Borel summation and perturbation theory. The strategy is explained in further detail at the beginning of that section. □

Proof of Proposition 6.1 assuming Proposition 6.4. Proposition 6.1 is a direct consequence of Lemma 6.3 and Proposition 6.4 since reflection positivity is preserved under weak limits. □

6.2 Chessboard estimates for $\nu_{\beta,N}$

Let $B \subset \mathbb{B}_N$ be either a unit block or a pair of nearest-neighbour blocks. Recall the natural identification of B with the subset of $\mathbb{T}_N$ given by the union of blocks in B. $\mathbb{T}_N$ can be written as a disjoint union of translates of B. Let $\mathbb{B}_N^B$ be the set of these translates; its elements are also identified with subsets of $\mathbb{T}_N$. Note that if $B = \square \in \mathbb{B}_N$, then $\mathbb{B}_N^B = \mathbb{B}_N$.

We say that $f \in C^\infty(\mathbb{T}_N)$ is B-measurable if supp$f \subset B$ and supp$f \cap \partial B = \varnothing$. We say that $\phi \in S'(\mathbb{T}_N)$ is B-measurable if $\phi(f) = 0$ for every $f \in C^\infty(\mathbb{T}_N)$ unless f is B-measurable. We say that $F \in L^2(\nu_{\beta,N})$ is B-measurable if it is measurable with respect to the σ-algebra generated by $\phi \in S'(\mathbb{T}_N)$ that are B-measurable.

Proposition 6.5. *Let $N \in 4\mathbb{N}$. Let $\{F_{\tilde{B}} : \tilde{B} \in \mathbb{B}_N^B\}$ be a given set of $L^2(\nu_{\beta,N})$-functions such that each $F_{\tilde{B}}$ is $\tilde{B}$-measurable.*

Fix $\tilde{B} \in \mathbb{B}_N^B$ and define an associated set of $L^2(\nu_{\beta,N})$-functions $\{F_{\tilde{B},B'} : B' \in \mathbb{B}_N^B\}$ by the conditions: $F_{\tilde{B},\tilde{B}} = F_{\tilde{B}}$; and, for any $B', B'' \in \mathbb{B}_N^B$ such that B' and B'' share a common face,

$$F_{\tilde{B},B'} = \mathscr{R}_\Pi F_{\tilde{B},B''}$$

where Π is the unique hyperplane of the form (6.1) *containing the shared face between B' and B''.*

Then,

$$\Big| \Big\langle \prod_{\tilde{B} \in \mathbb{B}_N^B} F_{\tilde{B}} \Big\rangle_{\beta,N} \Big| \leqslant \prod_{\tilde{B} \in \mathbb{B}_N^B} \Big| \Big\langle \prod_{B' \in \mathbb{B}_N^B} F_{\tilde{B},B'} \Big\rangle_{\beta,N} \Big|^{\frac{|B|}{N^3}}.$$

Proof. This is a consequence of the reflection positivity of $\nu_{\beta,N}$. The condition $N \in 4\mathbb{N}$ guarantees $F_{\tilde{B},B'}$ is well-defined. See [Shl86, Theorem 2.2]. □

6.3 Proof of Proposition 3.6

In order to be able to apply Proposition 6.5 to the random variables Q_i of Proposition 3.6, we need the following lemma.

Lemma 6.6. *Let $N \in \mathbb{N}$ and $\beta > 0$. Then, for any $\square \in \mathbb{B}_N$, $\exp Q_1(\square), \exp Q_2(\square) \in L^2(\nu_{\beta,N})$ is $\square$-measurable.*

In addition, for any nearest neighbours $\square, \square' \in \mathbb{B}_N$, $\exp Q_3(\square, \square') \in L^2(\nu_{\beta,N})$ is $\square \cup \square'$-measurable.

Proof. The fact that $\exp Q_1(\square), \exp Q_2(\square), \exp Q_3(\square, \square') \in L^2(\nu_{\beta,N})$ follows from estimates obtained in Proposition 5.22. The $\square$ and $\square \cup \square'$ measurability of these observables comes from taking approximations to indicators which are supported on blocks (e.g. using some appropriate regularisation of the distance function) and estimates obtained in Proposition 5.22. □

Proof of Proposition 3.6. Let $B_1, B_2 \subset \mathbb{B}_N$ and B_3 be a set of unordered pairs of nearest neighbour blocks in $\mathbb{B}_N$. Then,

$$\begin{aligned}
&\cosh Q_1(B_1) \cosh Q_2(B_2) \cosh Q_3(B_3) \\
&\quad = 2^{-|B_1|-|B_2|-|B_3|} \prod_{\square_1 \in B_1} \prod_{\square_2 \in B_2} \prod_{\{\square_3, \square_3'\} \in B_3} \left(e^{Q_1(\square_1)} + e^{-Q_1(\square_1)} \right) \\
&\qquad \times \left(e^{Q_2(\square_2)} + e^{-Q_2(\square_2)} \right) \left(e^{Q_3(\square_3, \square_3')} + e^{Q_3(\square_3', \square_3)} \right) \\
&\quad \leqslant 2^{-|B_1|-|B_2|} \sum_{B_1^+, B_1^-, B_2^+, B_2^-} \prod_{i=1}^{2} \left(\prod_{\square_i^+ \in B_i^+} e^{Q_i(\square_i^+)} \prod_{\square_i^- \in B_i^-} e^{-Q_i(\square_i^-)} \right) \\
&\qquad \times \prod_{\{\square_3, \square_3'\} \in B_3} e^{|Q_3(\square_3, \square_3')|}
\end{aligned} \tag{6.3}$$

where $\cosh Q_i(B_i)$ is defined in (3.7) and the sum is over all partitions $B_1^+ \sqcup B_1^- = B_1$ and $B_2^+ \sqcup B_2^- = B_2$.

It suffices to prove that there exists $\tilde{C}_Q > 0$ such that, for any $B_1^\pm$, $B_2^\pm$ and B_3 as above,

$$\begin{aligned}
&\left\langle \prod_{i=1}^{2} \left(\prod_{\square_i^+ \in B_i^+} e^{Q_1(\square_i^+)} \prod_{\square_i^- \in B_i^-} e^{-Q_2(\square_i^-)} \right) \prod_{\{\square_3, \square_3'\} \in B_3} e^{|Q_3(\square_3, \square_3')|} \right\rangle_{\beta,N} \\
&\qquad \leqslant e^{\tilde{C}_Q(|B_1|+|B_2|+|B_3|)}.
\end{aligned} \tag{6.4}$$

Then, taking expectations in (6.3) and using (6.4)

$$\begin{aligned}
&\Big\langle \cosh Q_1(B_1) \cosh Q_2(B_2) \cosh Q_3(B_3) \Big\rangle_{\beta,N} \\
&\quad \leqslant 2^{|B_1|+|B_2|} \sum_{B_1^+, B_1^-} \sum_{B_2^+, B_2^-} \\
&\qquad \left\langle \prod_{i=1}^{2} \left(\prod_{\square_i^+ \in B_i^+} e^{Q_1(\square_i^+)} \prod_{\square_i^- \in B_i^-} e^{-Q_2(\square_i^-)} \right) \prod_{\{\square_3, \square_3'\} \in B_3} e^{|Q_3(\square_3, \square_3')|} \right\rangle_{\beta,N} \\
&\quad \leqslant e^{\tilde{C}_Q(|B_1|+|B_2|+|B_3|)}
\end{aligned}$$

which yields Proposition 3.6 with $C_Q = \tilde{C}_Q$.

To prove (6.4), first fix $B_1^{\pm}$ and $B_2^{\pm}$. Then, by Hölder's inequality,

$$\begin{aligned}\Big\langle \prod_{i=1}^{2}\Big(\prod_{\square_i^+\in B_i^+} e^{Q_1(\square_i^+)} \prod_{\square_i^-\in B_i^-} e^{-Q_2(\square_i^-)}\Big) \prod_{\{\square_3,\square_3'\}\in B_3} e^{|Q_3(\square_3,\square_3')|}\Big\rangle_{\beta,N} \\ \leqslant \prod_{i=1,2}\Bigg(\Big\langle \prod_{\square_i^+\in B_i^+} e^{5Q_i(\square_i^+)}\Big\rangle_{\beta,N}^{\frac{1}{5}} \Big\langle \prod_{\square_i^-\in B_i^-} e^{5Q_i(\square_i^-)}\Big\rangle_{\beta,N}^{\frac{1}{5}}\Bigg) \\ \times \Big\langle \prod_{\{\square_3,\square_3'\}\in B_3} e^{5|Q_3(\square_3,\square_3')|}\Big\rangle_{\beta,N}^{\frac{1}{5}}.\end{aligned} \tag{6.5}$$

Let $i = 1, 2$. Without loss of generality, we use Proposition 6.5 to estimate

$$\Big\langle \prod_{\square\in B_i^+} e^{5Q_i(\square)}\Big\rangle_{\beta,N}.$$

Define $F_\square = e^{5Q_i(\square)}$ if $\square \in B_i^+$ and 1 otherwise. For each $\square \in \mathbb{B}_N$, we generate the family of functions $\{F_{\square,\square'} : \square' \in \mathbb{B}_N\}$ as in Proposition 6.5. Note that for $\square, \square' \in \mathbb{B}_N$ such that $\square$ and $\square'$ are nearest-neighbours,

$$\mathscr{R} e^{5Q_i(\square)} = e^{5Q_i(\square')}.$$

where $\mathscr{R}$ is the reflection across the unique hyperplane containing the shared face of $\square$ and $\square'$. Thus, we have $F_{\square,\square'} = e^{5Q_i(\square')}$ for every $\square \in B_i^+$ and $\square' \in \mathbb{B}_N^B$. If $\square \notin B_i^+$, we have $F_{\square,\square'} = 1$ for every $\square' \in \mathbb{B}_N$.

Lemma 6.6 ensures that $F_\square \in L^2(\nu_{\beta,N})$ is $\square$-measurable for every $\square \in \mathbb{B}_N$. Hence, by Proposition 6.5, we obtain

$$\Big\langle \prod_{\square\in B_i^+} e^{5Q_i(\square)}\Big\rangle_{\beta,N} \leqslant \prod_{\square\in B_i^+}\Big\langle \prod_{\square'\in\mathbb{B}_N} e^{5Q_i(\square')}\Big\rangle_{\beta,N}^{\frac{1}{N^3}}.$$

Therefore, by Proposition 5.1, there exists $C_Q' > 0$ such that, for all β sufficiently large,

$$\Big\langle \prod_{\square\in B_i^+} e^{5Q_i(\square)}\Big\rangle_{\beta,N} \leqslant e^{C_Q'|B_i^+|}. \tag{6.6}$$

For the remaining term involving Q_3, partition $B_3 = \bigcup_{k=1}^{6} B_3^{(k)}$ such that each $B_3^{(k)}$ is a set of disjoint pairs of nearest neighbour blocks, all with same orientation. Then, by Hölder's inequality,

$$\Big\langle \prod_{\{\square,\square'\}\in B_3} e^{5|Q_3(\square,\square')|}\Big\rangle_{\beta,N} \leqslant \prod_{k=1}^{6}\Big\langle \prod_{\{\square,\square'\}\in B_3^{(k)}} e^{30|Q_3(\square,\square')|}\Big\rangle_{\beta,N}^{\frac{1}{6}}. \tag{6.7}$$

Assuming that we have established that there exists $C'_Q > 0$ such that

$$\Big\langle \prod_{\{\square,\square'\}\in B_3^{(k)}} e^{30|Q_3(\square,\square')|} \Big\rangle_{\beta,N} \leqslant e^{C'_Q |B_3^{(k)}|}$$

for every $k \in \{1,\ldots,6\}$, then (6.7) yields

$$\Big\langle \prod_{\{\square,\square'\}\in B_3} e^{5|Q_3(\square,\square')|} \Big\rangle_{\beta,N} \leqslant e^{\frac{C'_Q}{6}|B_3|}.$$

Hence, without loss of generality, we may assume B_3 is a set of disjoint pairs of nearest neighbour blocks, all of the same orientation.

Define $F_B = e^{5|Q_3(\square,\square')|}$ for any $B = \{\square,\square'\} \in B_3$ and 1 otherwise. Note that for any two pairs of nearest-neighbour blocks, $\{\square,\square'\}, \{\tilde\square,\tilde\square'\} \subset \mathbb{B}_N$,

$$\mathscr{R} e^{5|Q_3(\square,\square')|} = e^{5|Q_3(\tilde\square,\tilde\square')|}$$

where $\mathscr{R}$ is the reflection across the unique hyperplane containing the shared face of $\square \cup \square'$ and $\tilde\square \cup \tilde\square'$. Thus, for any $B = \{\square,\square'\} \in B_3$ and $B' = \{\tilde\square,\tilde\square'\} \in \mathbb{B}_N^B$, we have $F_{B,B'} = e^{5|Q_3(\tilde\square,\tilde\square')|}$. If $B \notin B_3$, then we have $F_{B,B'} = 1$ for all $B' \in \mathbb{B}_N^B$.

Lemma 6.6 ensures that $\exp(|Q_3(\square,\square')|)$ is $\square \cup \square'$-measurable. Thus, applying Propositions 6.5 and 5.1, there exists $C'_Q > 0$ such that, for all β sufficiently large,

$$\begin{aligned}
\Big\langle \prod_{B=\{\square,\square'\}\in B_3} e^{5|Q_3(\{\square,\square'\})|} \Big\rangle_{\beta,N}
&\leqslant \prod_{B=\{\square,\square'\}\in B_3} \Big\langle \prod_{B'=\{\tilde\square,\tilde\square'\}\in \mathbb{B}_N^B} e^{5|Q_3(\tilde\square,\tilde\square')|} \Big\rangle_{\beta,N}^{\frac{2}{N^3}} \\
&\leqslant e^{2C'_Q |B_3|}.
\end{aligned} \tag{6.8}$$

Inserting (6.6) and (6.8) into (6.5), and taking into account (6.7), yields (6.4) with $\tilde C_Q = \frac{C'_Q}{15}$, thereby finishing the proof. □

6.4 Equivalence of the lattice and Fourier cutoffs

This section is devoted to a proof of Proposition 6.4 using stochastic quantisation techniques. In Section 6.4.1, we give a rigorous interpretation to (1.3) via the change of variables (6.14). Subsequently, in Section 6.4.2, we establish that $\nu_{\beta,N}$ is the unique invariant measure of (1.3), see Proposition 6.18. In Section 6.4.3, we first establish that local solutions of spectral Galerkin and lattice approximations to (1.3) converge to the same limit (see Propositions 6.13 and 6.19); these approximations admit unique invariant measures given by $\nu_{\beta,N,K}$ and $\tilde\nu_{\beta,N,\varepsilon}$, respectively. Then, using the global existence of solutions and uniqueness of the invariant measure of (1.3), we show that both of these measures converge to $\nu_{\beta,N}$ as the cutoffs are removed.

6.4.1 Giving a meaning to (1.3)

Let ξ be space-time white noise on $\mathbb{T}_N$ defined on a probability space $(\Omega, \mathbb{P})$. This means that ξ is a Gaussian random distribution on Ω satisfying

$$\mathbb{E}[\xi(\Phi)\xi(\Psi)] = \int_0^\infty \int_{\mathbb{T}_N} \Phi\Psi dxdt$$

where $\Phi, \Psi \in C^\infty(\mathbb{R}_+ \times \mathbb{T}_N)$ and $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$. We use the colour blue here to distinguish between the space random processes defined in Section 4 and the space-time random processes that we consider here.

We interpret (1.3) as the limit of renormalised approximations. For every $K \in (0, \infty)$, the Glauber dynamics of $\nu_{\beta,N,K}$ is given by the stochastic PDE

$$\begin{aligned}(\partial_t - \Delta + \eta)\Phi_K = &-\frac{4}{\beta}\rho_K(\rho_K\Phi_K)^3 \\ &+ \Big(4 + \eta + \frac{12}{\beta}\tadpole_K + \frac{2\gamma_K}{\beta^2}\Big)\rho_K^2\Phi_K + \sqrt{2}\xi.\end{aligned} \tag{6.9}$$

Above, ρ_K is as in Section 2 and we recall $\rho_K^2 \neq \rho_K$; $\tadpole_K$ is defined in (2.1); and $\gamma_K = -4^2 \cdot 3 \sunset_K$, where $\sunset_K$ is defined in (4.4).

Remark 6.7. *Recall that the Glauber dynamics for the measure ν with formal density $d\nu(\phi) \propto e^{-\mathscr{H}(\phi)} \prod_{x\in\mathbb{T}_N} d\phi(x)$ is given by the (overdamped) Langevin equation*

$$\partial_t\Phi(t) = \partial_\phi\mathscr{H}(\Phi(t)) + \sqrt{2}\xi$$

where $\partial_\phi\mathscr{H}$ denotes the functional derivative of $\mathscr{H}$.

For fixed K, the (almost sure) global existence and uniqueness of mild solutions to (6.9) is standard (see e.g. [DPZ88, Section III]). Moreover, $\nu_{\beta,N,K}$ is its unique invariant measure (see [Zab89, Theorem 2]). The approximations (6.9), which we call spectral Galerkin approximations, are natural in our context since $\nu_{\beta,N}$ is constructed as the weak limit of $\nu_{\beta,N,K}$ as $K \to \infty$.

The difficulty in obtaining a local well-posedness theory that is stable in the limit $K \to \infty$ lies in the roughness of the white noise ξ. The key idea is to exploit that the small-scale behaviour of solutions to (6.9) is governed by the Ornstein-Uhlenbeck process

$$\ptree = (\partial_t - \Delta + \eta)^{-1}\sqrt{2}\xi.$$

This allows us to obtain an expansion of Φ_K in terms of explicit (renormalised) multilinear functions of $\ptree$, which give a more detailed description of the small-scale behaviour of Φ_K, plus a more regular remainder term. Given the regularities of these explicit stochastic terms, the local solution theory then follows from deterministic arguments.

Remark 6.8. *We are only concerned with the limit $K \to \infty$ in* (6.9)*. We do not try to make sense of the joint $K, N \to \infty$ limit.*

We use the paracontrolled distribution approach of [MW17b], which is modification of the framework of [CC18] (both influenced by the seminal work of [GIP15]). In this approach, the expansion of Φ_K is given by an ansatz, see (6.10), that has similarities to the change of variables encountered in Section 5.4.1. See Remark 6.10. There are also related approaches via regularity structures [Hai14, Hai16, MW18] and renormalisation group [Kup16], but we do not discuss them further.

For every $K \in (0, \infty)$, define

$$
\begin{aligned}
\<1>_K &= \rho_K \<1> \\
\<2>_K &= \<1>_K^2 - \tadpole_K \\
\<3>_K &= \<1>_K^3 - 3\tadpole_K \<1>_K \\
\<20>_K &= (\partial_t - \Delta + \eta)^{-1} \rho_K \<2>_K \\
\<30>_K &= (\partial_t - \Delta + \eta)^{-1} \rho_K \<3>_K \\
\<31>_K &= \<1>_K \circleddash \rho_K \<30>_K \\
\<22>_K &= \<2>_K \circleddash \rho_K \<20>_K - \frac{2}{3} \sunset_K \\
\<32>_K &= \<2>_K \circleddash \rho_K \<30>_K - 2 \sunset_K \<1>_K .
\end{aligned}
$$

We recall that the colour blue is used to distinguish between the above space-time diagrams and the space diagrams of Section 4.1.1.

For any $T > 0$, the vector $\Xi_K = \left(\<1>_K, \<2>_K, \<30>_K, \<31>_K, \<22>_K, \<32>_K \right)$ is space-time stationary and almost surely an element of the Banach space

$$
\begin{aligned}
\mathfrak{X}_T = & \, C([0,T]; \mathscr{C}^{-\frac{1}{2}-\kappa}) \times C([0,T]; \mathscr{C}^{-1-\kappa}) \\
& \times \left(C([0,T]; \mathscr{C}^{\frac{1}{2}-\kappa}) \cap C^{\frac{1}{8}}([0,T]; \mathscr{C}^{\frac{1}{4}-\kappa}) \right) \\
& \times C([0,T]; \mathscr{C}^{-\kappa}) \times C([0,T]; \mathscr{C}^{-\kappa}) \times C([0,T]; \mathscr{C}^{-\frac{1}{2}-\kappa})
\end{aligned}
$$

where the norm on $\mathfrak{X}_T$ is given by the maximum of the norms on the components. Above, for any $s \in \mathbb{R}$, $C([0,T]; \mathscr{C}^s)$ consists of continuous functions $\Phi : [0,T] \to \mathscr{C}^s$ and is a Banach space under the norm $\sup_{t \in [0,T]} \| \cdot \|_{\mathscr{C}^s}$. In addition, for any $\alpha \in (0,1)$, $C^\alpha([0,T]; \mathscr{C}^s)$ consists of α-Hölder continuous functions $\Phi : [0,T] \to \mathscr{C}^s$ and is a Banach space under the norm $\| \cdot \|_{C([0,T];\mathscr{C}^s)} + | \cdot |_{\alpha,T}$ where

$$
|\Phi|_{\alpha,T} = \sup_{0<s<t<T} \frac{\|\Phi(t) - \Phi(s)\|_{\mathscr{C}^s}}{|t-s|^\alpha}.
$$

Proposition 6.9. *There exists a stochastic process $\Xi = (\<1>, \<2>, \<30>, \<31>, \<22>, \<32>)$ such that, for every $T > 0$, $\Xi \in \mathfrak{X}_T$ almost surely and*

$$
\lim_{K \to \infty} \mathbb{E} \|\Xi_K - \Xi\|_{\mathfrak{X}_T} = 0.
$$

Proof. The proof follows from [CC18, Section 4] (see also [MWX17] and [Hai14, Section 10]). The only subtlety is to check that the renormalisation constants Q_K and $\ominus_K$, which were determined by the field theory $\nu_{\beta,N}$, are sufficient to renormalise the *space-time* diagrams appearing in the analysis of the SPDE. Precisely, it suffices to show $\mathbb{E}[\mathsf{V}^2_K(t,x)] = \mathsf{Q}_K$ and $\mathbb{E}\Big[\mathsf{V}_K \rho_K \mathsf{Y}_K(t,x)\Big] = \frac{2}{3}\ominus_K$ for every $(t,x) \in \mathbb{R}_+ \times \mathbb{T}_N$.

There exists a set of complex Brownian motions $\{W^n(\bullet)\}_{n\in(N^{-1}\mathbb{Z})^3}$ defined on $(\Omega,\mathbb{P})$, independent modulo the condition $W^n(\bullet) = \overline{W^{-n}(\bullet)}$, such that

$$\xi(\phi) = \frac{1}{N^3}\sum_{n\in(N^{-1}\mathbb{Z})^3}\int_{\mathbb{R}} \mathcal{F}(\phi)(t,n)N^{\frac{3}{2}}dW^n(t)$$

for every $\phi \in L^2(\mathbb{R}\times\mathbb{T}_N)$.

For $t \geqslant 0$ and $n \in (N^{-1}\mathbb{Z})^3$, let $H(t,n) = e^{-t\langle n\rangle^2}$ be the (spatial) Fourier transform of the heat kernel associated to $(\partial_t - \Delta + \eta)$. For any $K > 0$, define $H_K(t,n) = \rho_K(n)H(t,n)$. We extend both kernels to $t \in \mathbb{R}$ by setting $H(t,\cdot) = H_K(t,\cdot) = 0$ for any $t < 0$. Then

$$\mathcal{F}\mathsf{l}_K(t,n) = \sqrt{2}N^{\frac{3}{2}}\int_{\mathbb{R}} H_K(t-s,n)dW^n(s).$$

By Parseval's theorem and Itô's isometry,

$$\begin{aligned}
&\mathbb{E}\mathsf{l}^2_K(t,x)\\
&= \frac{2}{N^3}\sum_{n_1,n_2\in(N^{-1}\mathbb{Z})^3}\mathbb{E}\Bigg[\bigg(\int_{\mathbb{R}} H_K(t-s,n_1)dW^{n_1}(s)\bigg)\bigg(\int_{\mathbb{R}} H_K(t-s,n_2)dW^{n_2}(s)\bigg)\Bigg]\\
&= \frac{2}{N^3}\sum_{n\in(N^{-1}\mathbb{Z})^3}\rho_K^2(n)\int_{-\infty}^t e^{-2(t-s)\langle n\rangle^2}ds = \mathsf{Q}_K
\end{aligned}$$

for all $(t,x) \in \mathbb{R}_+\times\mathbb{T}_N$. With this observation the convergence of $\mathsf{l}_K, \mathsf{V}_K, \mathsf{Y}'_K$ and Y''_K follows from mild adaptations of [CC18, Section 4].

For the remaining two diagrams, one can show from arguments in [CC18, Section 4] that

$$\rho_K\mathsf{Y}_K \circleddash \mathsf{V}_K - \mathbb{E}\Big[\rho_K\mathsf{Y}_K\mathsf{V}_K\Big] \text{ and } \rho_K\mathsf{Y}'_K \circleddash \mathsf{V}_K - 3\mathbb{E}\Big[\rho_K\mathsf{Y}_K\mathsf{V}_K\Big]\mathsf{l}_K$$

converge to well-defined space-time distributions.

Writing

$$\mathsf{V}(t,x) = \frac{1}{N^3}\sum_{n_1,n_2\in(N^{-1}\mathbb{Z})^3}e_{n_1+n_2}(x)\int_{\mathbb{R}^2}H_K(t-s,n_1)H_K(t-r,n_2)dW^{n_1}(s)dW^{n_2}(r)$$

we have, by Parseval's theorem and Itô's isometry,

$$
\begin{aligned}
&\mathbb{E}\Big[\Upsilon_K \rho_K \vee(t,x)\Big] \\
&= \frac{8}{N^6}\mathbb{E}\Bigg[\sum_{\substack{n_1,n_2,n_3,n_4 \in (N^{-1}\mathbb{Z})^3 \\ n_1+n_3=n_2+n_4=0}} e_{n_1+n_2+n_3+n_4}(x)\rho_K(n_3+n_4) \\
&\qquad \times \int_{\mathbb{R}^5} H_K(t-s,n_1+n_2)H_K(s-u_1,n_1)H_K(s-u_2,n_2)H_K(t-u_3,n_3) \\
&\qquad \times H_K(t-u_4,n_4)dW^{n_1}(u_1)dW^{n_2}(u_2)dW^{n_3}(u_3)dW^{n_4}(u_4)ds\Bigg] \\
&= \frac{8}{N^6}\sum_{\substack{n_1,n_2,n_3,n_4 \in (N^{-1}\mathbb{Z})^3 \\ n_1=-n_3,n_2=-n_4}} \rho_K(n_3+n_4)\int_{\mathbb{R}^3} H_K(t-s,n_1+n_2)H_K(s-u_1,n_1) \\
&\qquad \times H_K(s-u_2,n_2)H_K(t-u_1,n_1)H_K(t-u_2,n_2)du_1du_2ds \\
&= \frac{8}{N^6}\sum_{n_1,n_2 \in (N^{-1}\mathbb{Z})^3} \rho_K^2(n_1+n_2)\rho_K^2(n_1)\rho_K^2(n_2)\int_{\mathbb{R}} H(t-s,n_1+n_2)H(t-s,n_1) \\
&\qquad \times H(t-s,n_2)\int_{\mathbb{R}^2} H(2(s-u_1),n_1)H(2(s-u_2),n_2)du_1du_2ds \\
&= \frac{2}{N^6}\sum_{n_1,n_2 \in (N^{-1}\mathbb{Z})^3} \frac{\rho_K^2(n_1)\rho_K^2(n_2)\rho_K^2(n_1+n_2)}{\langle n_1\rangle^2\langle n_2\rangle^2(\langle n_1+n_2\rangle^2+\langle n_1\rangle^2+\langle n_2\rangle^2)}.
\end{aligned}
$$

By symmetry,

$$
\begin{aligned}
&\frac{2}{N^6}\sum_{n_1,n_2 \in (N^{-1}\mathbb{Z})^3} \frac{\rho_K^2(n_1)\rho_K^2(n_2)\rho_K^2(n_1+n_2)}{\langle n_1\rangle^2\langle n_2\rangle^2(\langle n_1+n_2\rangle^2+\langle n_1\rangle^2+\langle n_2\rangle^2)} \\
&\quad = \frac{2}{3N^6}\sum_{n_1+n_2+n_3=0} \frac{\rho_K^2(n_1)\rho_K^2(n_2)\rho_K^2(n_3)}{\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n_3\rangle^2} \\
&\qquad \times \left(\frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2}+\frac{1}{\langle n_2\rangle^2\langle n_1+n_2\rangle^2}+\frac{1}{\langle n_1+n_2\rangle^2\langle n_1\rangle^2}\right) \\
&\quad = \frac{2}{3}\ominus_K
\end{aligned}
$$

thereby completing the proof. □

We return now to the solution theory for (1.3)/(6.9). Fix $K \in (0,\infty)$. Using the change of variables

$$
\Phi_K = \uparrow - \frac{4}{\beta}\Psi_K + \Upsilon_K + \Theta_K \tag{6.10}
$$

we say that Φ_K is a mild solution of (6.9) with initial data $\phi_0 \in \mathscr{C}^{-\frac{1}{2}-\kappa}$ if (Υ_K, Θ_K) is a mild solution to the system of equations

$$\begin{aligned}(\partial_t - \Delta + \eta)\Upsilon_K &= F_K(\Upsilon_K, \Theta_K; \Xi_K)\\ (\partial_t - \Delta + \eta)\Theta_K &= G_K(\Upsilon_K, \Theta_K; \Xi_K)\end{aligned} \tag{6.11}$$

where

$$\begin{aligned}F_K(\Upsilon_K, \Theta_K; \Xi_K) &= -\frac{4\cdot 3}{\beta}\rho_K\Big\{\mathbf{\vee}_K \varolessthan \rho_K(\Phi_K - \mathbf{\uparrow})\Big\}\\ G_K(\Upsilon_K, \Theta_K; \Xi_K) &= -\frac{4\cdot 3}{\beta}\rho_K\Big\{\mathbf{\vee}_K \circledeq \Big(-\frac{4}{\beta}\rho_K\mathbf{\Psi}_K + \rho_K(\Upsilon_K + \Theta_K)\Big)\Big\}\\ &\quad - \frac{4\cdot 3}{\beta}\rho_K\Big\{\mathbf{\vee}_K \varolessthan \rho_K(\Phi_K - \mathbf{\uparrow}) + \mathbf{\uparrow}_K\big(\rho_K(\Phi_K - \mathbf{\uparrow})\big)^2\Big\}\\ &\quad - \frac{4}{\beta}\rho_K\big(\rho_K(\Phi_K - \mathbf{\uparrow})\big)^3 + \Big(4 + \eta + \frac{2\gamma_K}{\beta^2}\Big)\rho_K\Phi_K\end{aligned}$$

with initial data $(\Upsilon_K(0,\cdot), \Theta_K(0,\cdot)) = \Big(0, \phi_0 + \sqrt{2}(0) - \frac{4\cdot(\sqrt{2})^3}{\beta}\mathbf{\Psi}_K(0)\Big)$.

We split $G_K(\Upsilon_K, \Theta_K; \Xi_K) = G_K^1(\Upsilon_K, \Theta_K; \Xi_K) + G_K^2(\Upsilon_K, \Theta_K; \Xi_K)$,

$$\begin{aligned}G_K^1(\Upsilon_K, \Theta_K; \Xi_K) &= \frac{4^2\cdot 3}{\beta^2}\rho_K\Big\{\mathbf{\Psi}^{\vee}_K + 3\mathbf{\vee}^{\vee}_K\rho_K(\Phi_K - \mathbf{\uparrow})\Big\}\\ &\quad + G_K^{1,a}(\Upsilon_K, \Theta_K; \Xi_K) + G_K^{1,b}(\Upsilon_K, \Theta_K; \Xi_K)\\ G_K^2(\Upsilon_K, \Theta_K; \Xi_K) &= -\frac{4\cdot 3}{\beta}\rho_K\Big\{\mathbf{\vee}_K \circledeq \rho_K\Theta_K + \mathbf{\vee}_K \varolessthan \rho_K(\Phi_K - \mathbf{\uparrow})\\ &\quad + \mathbf{\uparrow}_K\big(\rho_K(\Phi_K - \mathbf{\uparrow})\big)^2\Big\} - \frac{4}{\beta}\rho_K\big(\rho_K(\Phi_K - \mathbf{\uparrow})\big)^3 + (4+\eta)\rho_K\Phi_K\end{aligned}$$

where $G_K^{1,a}(\Upsilon_K, \Theta_K; \Xi_K)$ and $G_K^{2,a}(\Upsilon_K, \Theta_K; \Xi_K)$ are commutator terms defined through the manipulations

$$\begin{aligned}&-\frac{4\cdot 3}{\beta}\rho_K\Big\{\mathbf{\vee}_K \circledeq \rho_K\Upsilon_K\Big\}\\ &= \frac{4^2\cdot 3^2}{\beta^2}\rho_K\Big\{\mathbf{\vee}_K \circledeq \rho_K(\partial_t - \Delta + \eta)^{-1}\big(\rho_K\{\mathbf{\vee}_K \varolessthan \rho_K(\Phi_K - \mathbf{\uparrow})\}\big)\Big\}\\ &= \frac{4^2\cdot 3^2}{\beta^2}\rho_K\Big\{\mathbf{\vee}_K \circledeq \rho_K\Big(\mathbf{Y}_K \varolessthan \rho_K(\Phi_K - \mathbf{\uparrow})\Big)\Big\} + G_K^{1,a}\\ &= \frac{4^2\cdot 3^2}{\beta^2}\rho_K\Big\{\Big(\mathbf{\vee}_K \circledeq \rho_K\mathbf{Y}_K\Big)\rho_K(\Phi_K - \mathbf{\uparrow})\Big\} + G_K^{1,a} + G_K^{1,b}.\end{aligned} \tag{6.12}$$

The precise choice of the splitting of $\Phi_K - \mathbf{\uparrow} + \frac{4}{\beta}\mathbf{\Psi}_K$ into Υ_K and Θ_K is explained in detail in [MW17b, Introduction]. For our purposes, it suffices to note that Υ_K

captures the small-scale behaviour of this difference. On the other hand, Θ_K captures the large-scale behaviour: the term G_K^2 contains a cubic damping term in Θ_K (i.e. with a good sign). Finally, we note that there is a redundancy in the specification of initial condition: any choice such that $\Upsilon_K(0,\cdot) + \Theta_K(0,\cdot) = \phi_0 + \Psi(0) - \frac{4}{\beta}\mathcal{I}[\Psi^3](0)$ is sufficient. Our choice is informed by Remark 1.3 in [MW17b].

Remark 6.10. *Rewriting* (6.10) *as*

$$\Phi_K = \Psi - \frac{4}{\beta}\mathcal{I}[\Psi^3]_K - \frac{4\cdot 3}{\beta}(\partial_t - \Delta + \eta)^{-1}\rho_K\Big\{\Psi^2_K \ogreaterthan \rho_K(\Phi_K - \Psi)\Big\} + \Theta_K$$

we note the similarity between the change of variables for the stochastic PDE given above and for the field theory in (5.21).

Formally taking $K \to \infty$ in (6.11) leads us to the following system:

$$\begin{aligned}(\partial_t - \Delta + \eta)\Upsilon &= F(\Upsilon, \Theta; \Xi)\\ (\partial_t - \Delta + \eta)\Theta &= G(\Upsilon, \Theta; \Xi)\end{aligned} \tag{6.13}$$

where

$$\begin{aligned}
F(\Upsilon,\Theta;\Xi) &= -\frac{4\cdot 3}{\beta}\Psi^2 \ogreaterthan \Big(-\frac{4}{\beta}\mathcal{I}[\Psi^3] + \Upsilon + \Theta\Big)\\
G(\Upsilon,\Theta;\Xi) &= G^1(\Upsilon,\Theta;\Xi) + G^2(\Upsilon,\Theta;\Xi)\\
G^1(\Upsilon,\Theta;\Xi) &= \frac{4^2\cdot 3}{\beta^2}\bigg(\Psi^2\odot\mathcal{I}[\Psi^3] + 3\,\Psi^2\odot\mathcal{I}[\Psi^2]\Big(-\frac{4}{\beta}\mathcal{I}[\Psi^3] + \Upsilon + \Theta\Big)\bigg)\\
&\quad + G^{1,a}(\Upsilon,\Theta;\Xi) + G^{2,b}(\Upsilon,\Theta;\Xi)\\
G^2(\Upsilon,\Theta;\Xi) &= -\frac{4\cdot 3}{\beta}\bigg(\Psi^2 \odot \Theta + \Psi^2 \olessthan \Big(-\frac{4}{\beta}\mathcal{I}[\Psi^3] + \Upsilon + \Theta\Big)\bigg)\\
&\quad - \frac{4\cdot 3}{\beta}\Psi\Big(-\frac{4}{\beta}\mathcal{I}[\Psi^3] + \Upsilon + \Theta\Big)^2\\
&\quad - \frac{4}{\beta}\Big(-\frac{4}{\beta}\mathcal{I}[\Psi^3] + \Upsilon + \Theta\Big)^3\\
&\quad + (4+\eta)\Big(1 - \frac{4}{\beta}\mathcal{I}[\Psi^3] + \Upsilon + \Theta\Big)
\end{aligned}$$

and $G^{1,a}$ and $G^{1,b}$ are commutator terms defined analogously as in (6.12).

For every $T > 0$, define the Banach space

$$\begin{aligned}\mathscr{Y}_T = &\Big[C([0,T];\mathscr{C}^{-\frac{3}{5}}) \cap C((0,T];\mathscr{C}^{\frac{1}{2}+2\kappa}) \cap C^{\frac{1}{8}}((0,T];L^\infty)\Big]\\ &\times \Big[C([0,T];\mathscr{C}^{-\frac{3}{5}}) \cap C((0,T];\mathscr{C}^{1+2\kappa}) \cap C^{\frac{1}{8}}((0,T];L^\infty)\Big]\end{aligned}$$

equipped with the norm

$$\|(\Upsilon, \Theta)\|_{\mathcal{Y}_T} = \max\Bigg\{ \sup_{0\leqslant t\leqslant T} \|\Upsilon(t)\|_{\mathscr{C}^{-\frac{3}{5}}}, \sup_{0<t\leqslant T} t^{\frac{3}{5}}\|\Upsilon(t)\|_{\mathscr{C}^{\frac{1}{2}+2\kappa}}, \sup_{0<s<t\leqslant T} s^{\frac{1}{2}} \frac{\|\Upsilon(t)-\Upsilon(s)\|_{L^\infty}}{|t-s|^{\frac{1}{8}}},$$
$$\sup_{0\leqslant t\leqslant T} \|\Theta(t)\|_{\mathscr{C}^{-\frac{3}{5}}}, \sup_{0<t\leqslant T} t^{\frac{17}{20}}\|\Theta(t)\|_{\mathscr{C}^{1+2\kappa}}, \sup_{0<s<t\leqslant T} s^{\frac{1}{2}} \frac{\|\Theta(t)-\Theta(s)\|_{L^\infty}}{|t-s|^{\frac{1}{8}}} \Bigg\}.$$

Remark 6.11. *The choice of exponents in function spaces in $\mathcal{Y}_T$, as well as the choice of exponents in the blow-up at $t = 0$ in $\|\cdot\|_{\mathcal{Y}_T}$, corresponds to the one made in [MW17b]. It is arbitrary to an extent: it depends on the choice of initial condition, which must have Besov-Hölder regularity strictly better than $-\frac{2}{3}$.*

The local well-posedness of (6.13) follows from entirely deterministic arguments, so we state it with Ξ replaced by any deterministic $\tilde{\Xi}$.

Proposition 6.12. *Let $\tilde{\Xi} \in \mathfrak{X}_{T_0}$ for any $T_0 > 0$, and let $(\Upsilon_0, \Theta_0) \in \mathscr{C}^{-\frac{3}{5}} \times \mathscr{C}^{-\frac{3}{5}}$. Then, there exists $T = T(\|\tilde{\Xi}\|_{\mathfrak{X}_{T_0}}, \|\Upsilon_0\|_{\mathscr{C}^{-\frac{3}{5}}}, \|\Theta_0\|_{\mathscr{C}^{-\frac{3}{5}}}) \in (0, T_0]$ such that there is a unique mild solution $(\Upsilon, \Theta) \in \mathcal{Y}_T$ to (6.13) with initial data (Υ_0, Θ_0).*

In addition, let $\tilde{\Xi}, \Xi' \in \mathfrak{X}_{T_0}$ such that $\|\tilde{\Xi}\|_{\mathfrak{X}_{T_0}}, \|\Xi'\|_{\mathfrak{X}_{T_0}} \leqslant R$ for some $R > 0$, and let $(\Upsilon_0^1, \Theta_0^1), (\Upsilon_0^2, \Theta_0^2) \in \mathscr{C}^{-\frac{3}{5}} \times \mathscr{C}^{-\frac{3}{5}}$. Let the respective solutions to (6.13) be $(\Upsilon^1, \Theta^1) \in \mathcal{Y}_{T_1}$ and $(\Upsilon^2, \Theta^2) \in \mathcal{Y}_{T_2}$ and define $T = \min(T_1, T_2)$. Then there exists $C = C(R) > 0$ such that

$$\|(\Upsilon^1, \Theta^1) - (\Upsilon^2, \Theta^2)\|_{\mathcal{Y}_T} \leqslant C\Big(\|\Upsilon_0^1 - \Upsilon_0^2\|_{\mathscr{C}^{-\frac{3}{5}}} + \|\Theta_0^1 - \Theta_0^2\|_{\mathscr{C}^{-\frac{3}{5}}} + \|\tilde{\Xi} - \Xi'\|_{\mathfrak{X}_{T_0}}\Big).$$

Proof. Proposition 6.12 is proven in Theorem 2.1 [MW17b] (see also Theorem 3.1 [CC18]) by showing that the mild solution map

$$(\Upsilon, \Theta) \mapsto \Big((\partial_t - \Delta + \eta)^{-1}\Upsilon_0, (\partial_t - \Delta + \eta)^{-1}\Theta_0\Big)$$
$$+ \Big((\partial_t - \Delta + \eta)^{-1}F(\Upsilon, \Theta; \tilde{\Xi}), (\partial_t - \Delta + \eta)^{-1}G(\Upsilon, \Theta; \tilde{\Xi})\Big)$$

is a contraction in the ball

$$\mathcal{Y}_{T,M} = \Big\{(\tilde{\Upsilon}, \tilde{\Theta}) \in \mathcal{Y}_T : \|(\tilde{\Upsilon}, \tilde{\Theta})\|_{\mathcal{Y}_T} \leqslant M\Big\}$$

provided that T is taken sufficiently small and M is taken sufficiently large (both depending on the norm of the initial data and of $\|\tilde{\Xi}\|_{\mathfrak{X}_{T_0}}$). □

We say that $\Phi \in C([0,T]; \mathscr{C}^{-\frac{1}{2}-\kappa})$ is a mild solution to (1.3) with initial data $\phi_0 \in \mathscr{C}^{-\frac{1}{2}-\kappa}$ if

$$\Phi = \<1> - \frac{4}{\beta}\<30> + \Upsilon + \Theta \tag{6.14}$$

where $(\Upsilon, \Theta) \in \mathscr{Y}_T$ is a solution to (6.13) with Ξ as in Proposition 6.9 and initial data $\Big(0, \phi_0 + \<1>(0) - \frac{4}{\beta}\<30>(0)\Big)$.

Proposition 6.13. *For any $\phi_0 \in \mathscr{C}^{-\frac{1}{2}-\kappa}$, let $\Phi \in C([0,T]; \mathscr{C}^{-\frac{1}{2}-\kappa})$ be the unique solution of* (1.3) *with initial data ϕ_0 up to time $T > 0$. In addition, for any $K \in (0,\infty)$, let $\Phi_K \in C(\mathbb{R}_+; \mathscr{C}^{-\frac{1}{2}-\kappa})$ be the unique global solution of* (6.9) *with initial data $\rho_K \phi_0$.*

Then,

$$\lim_{K\to\infty} \mathbb{E}\|\Phi - \Phi_K\|_{C([0,T];\mathscr{C}^{-\frac{1}{2}-\kappa})} = 0.$$

Proof. It suffices to show convergence of (Υ_K, Θ_K) to (Υ, Θ) as $K \to \infty$. This follows from Proposition 6.9 and mild adaptations of arguments in [MW17b, Section 2]. □

Proposition 6.13 implies that $\Phi_K \to \Phi$ in probability in $C([0,T]; \mathscr{C}^{-\frac{1}{2}-\kappa})$. Local-in-time convergence is not sufficient for our purposes.

The following proposition establishing global well-posedness of (1.3).

Proposition 6.14. *For every $\phi_0 \in \mathscr{C}^{-\frac{1}{2}-\kappa}$ let $\Phi \in C([0,T^*); \mathscr{C}^{-\frac{1}{2}-\kappa})$ be the unique solution to* (1.3) *with initial condition ϕ_0 and where $T^* > 0$ is the maximal time of existence. Then $T^* = \infty$ almost surely.*

Proof. Proposition 6.14 is a consequence of a strong a priori bound on solutions to (6.13) established in [MW17b, Theorem 1.1]. □

An immediate corollary of Proposition 6.14 is a global-in-time convergence result sufficient for our purposes.

Corollary 6.15. *For every $\phi_0 \in \mathscr{C}^{-\frac{1}{2}-\kappa}$, let $\Phi \in C(\mathbb{R}_+; \mathscr{C}^{-\frac{1}{2}-\kappa})$ be the unique global solution to* (1.3) *with initial condition ϕ_0. For every $K \in (0,\infty)$, let $\Phi_K \in C(\mathbb{R}_+; \mathscr{C}^{-\frac{1}{2}-\kappa})$ be the unique global solution to* (6.9) *with initial condition $\rho_K\phi_0$.*

For every $T > 0$,

$$\lim_{K\to\infty} \mathbb{E}\|\Phi_K - \Phi\|_{C([0,T];\mathscr{C}^{-\frac{1}{2}-\kappa})} = 0.$$

Remark 6.16. *The infinite constant in* (1.3) *represents the renormalisation constants of the approximating equation* (6.9) *going to infinity as $K \to \infty$. Note that there is a one-parameter family of distinct nontrivial "solutions" to* (1.3) *corresponding to taking finite shifts of the renormalisation constants. However, the use of Ξ in the change of variables* (6.14) *fixes the precise solution.*

6.4.2 $\nu_{\beta,N}$ is the unique invariant measure of (6.14)

Denote by $B_b(\mathscr{C}^{-\frac{1}{2}-\kappa})$ the set of bounded measurable functions on $\mathscr{C}^{-\frac{1}{2}-\kappa}$ and by $C_b(\mathscr{C}^{-\frac{1}{2}-\kappa}) \subset B_b(\mathscr{C}^{-\frac{1}{2}-\kappa})$ the set of bounded continuous functions on $\mathscr{C}^{-\frac{1}{2}-\kappa}$.

Let $\Phi(\cdot;\cdot)$ be the solution map to (1.3): for $\phi_0 \in \mathscr{C}^{-\frac{1}{2}-\kappa}$ and $t \in \mathbb{R}_+$, $\Phi(t;\phi_0)$ is the solution at time t to (1.3) with initial condition ϕ_0. For every $t > 0$, define $\mathscr{P}_t^{\beta,N} : B_b(\mathscr{C}^{-\frac{1}{2}-\kappa}) \to B_b(\mathscr{C}^{-\frac{1}{2}-\kappa})$ by

$$(\mathscr{P}_t^{\beta,N} F)(\phi_0) = \mathbb{E} F(\Phi(t;\phi_0))$$

for $F \in B_b(\mathscr{C}^{-\frac{1}{2}-\kappa})$, $\phi_0 \in \mathscr{C}^{-\frac{1}{2}-\kappa}$, and $t \in \mathbb{R}_+$.

Proposition 6.17. *The solution Φ to* (1.3) *is a Markov process and its transition semigroup $(\mathscr{P}_t^{\beta,N})_{t\geqslant 0}$ satisfies the strong Feller property, i.e. $\mathscr{P}_t : B_b(\mathscr{C}^{-\frac{1}{2}-\kappa}) \to C_b(\mathscr{C}^{-\frac{1}{2}-\kappa})$.*

Proof. See [HM18b, Theorem 3.2]. □

Proposition 6.18. *The measure $\nu_{\beta,N}$ is the unique invariant measure of* (1.3).

Proof. By Proposition 2.1 the measures $\nu_{\beta,N,K}$ converge weakly to $\nu_{\beta,N}$ as $K \to \infty$. Hence, by Skorokhod's representation theorem [Bil08, Theorem 25.6] we can assume that there exists a sequence of random variables $\{\phi_K\}_{K\in\mathbb{N}} \subset \mathscr{C}^{-\frac{1}{2}-\kappa}$ defined on the probability space $(\Omega, \mathbb{P})$, independent of the white noise ξ, such that $\phi_K \sim \nu_{\beta,N,K}$ and ϕ_K converges almost surely to a random variable $\phi \sim \nu_{\beta,N}$.

For every $K \in (0,\infty)$, recall that the unique invariant measure of (6.9) is $\nu_{\beta,N,K}$. Let Φ_K denote the solution to (6.9) with random initial data ϕ_K. Hence, $\Phi_K(t) \sim \nu_{\beta,N,K}$ for all $t \in \mathbb{R}_+$.

Denote by Φ the solution to (1.3) with initial condition ϕ. By Proposition 6.14, $\Phi_K(t)$ converges in distribution to $\Phi(t)$ for every $t \in \mathbb{R}$, which implies $\Phi(t) \sim \nu_{\beta,N}$. Thus, $\nu_{\beta,N}$ is an invariant measure of (1.3). As a consequence of the strong Feller property in Proposition 6.17, we obtain that $\nu_{\beta,N}$ is the unique invariant measure of (1.3). □

6.4.3 Proof of Proposition 6.4

The Glauber dynamics of $\tilde{\nu}_{\beta,N,\varepsilon}$ is given by the system of SDEs

$$\begin{aligned} \frac{d}{dt}\tilde{\Phi} &= \Delta^\varepsilon \tilde{\Phi} - \frac{4}{\beta}\tilde{\Phi}^3 + (4 + \delta m^2(\varepsilon,\eta))\tilde{\Phi} + \sqrt{2}\xi_\varepsilon \\ \tilde{\Phi}(0,\cdot) &= \varphi(\cdot) \end{aligned} \tag{6.15}$$

where $\tilde{\Phi} : \mathbb{R}_+ \times \mathbb{T}_N^\varepsilon \to \mathbb{R}$, $\varphi \in \mathbb{R}^{\mathbb{T}_N^\varepsilon}$, and ξ_ε is the lattice discretisation of ξ given by

$$\xi_\varepsilon(t,x) = \frac{4^3}{\varepsilon^3}\int_{\mathbb{T}_N} \xi(t,x')\mathbf{1}_{|x-x'|\leqslant\frac{\varepsilon}{4}} dx'.$$

Note that the integral above means duality pairing between $\xi(t,\cdot)$ and $\mathbf{1}_{|x-\cdot|\leqslant\frac{\varepsilon}{4}}$.

For each $\varepsilon > 0$, the global existence and uniqueness of (6.15), as well as the fact that $\tilde{\nu}_{\beta,N,\varepsilon}$ is its unique invariant measure, is well-known.

The following proposition establishes a global-in-time convergence result for solutions of (6.15) to solutions of (1.3).

Proposition 6.19. *For every $\varepsilon > 0$, denote by $\tilde{\Phi}^\varepsilon$ the unique global solution to* (6.15) *with initial data $\varphi_\varepsilon \in \mathbb{R}^{\mathbb{T}^\varepsilon_N}$. In addition, denote by Φ the unique global solution to* (1.3) *with initial data $\phi \in \mathscr{C}^{-\frac{1}{2}-\kappa}$.*

Then, there exists a choice of constants $\delta m^2(\varepsilon,\eta) \to \infty$ as $\varepsilon \to 0$ such that, for every $T > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{E}\|\Phi - \text{ext}^\varepsilon \tilde{\Phi}^\varepsilon\|_{C([0,T];\mathscr{C}^{-\frac{1}{2}-\kappa})} = 0$$

provided that

$$\lim_{\varepsilon\to 0} \|\phi - \text{ext}^\varepsilon \varphi_\varepsilon\|_{\mathscr{C}^{-\frac{1}{2}-\kappa}} = 0 \qquad (6.16)$$

almost surely.

Proof. See [ZZ18b, Theorem 1.1] or [HM18a, Theorem 1.1]. □

The next proposition establishes that the lattice measures are tight.

Proposition 6.20. *Let $\delta m^2(\bullet,\eta)$ be as in Proposition 6.19. Then, $\text{ext}^\varepsilon_* \tilde{\nu}_{\beta,N,\varepsilon}$ converges weakly to a measure ν as $\varepsilon \to 0$.*

Proof. See [Par75, BFS83, GH18]. □

Proof of Proposition 6.4. For every $\varepsilon > 0$, let $\varphi_\varepsilon \sim \tilde{\nu}_{\beta,N,\varepsilon}$ be a random variable on $(\Omega,\mathbb{P})$ and independent of the white noise ξ. By Proposition 6.20 and in light of Skorokhod's representation theorem [Bil08, Theorem 25.6], we may assume that $\text{ext}^\varepsilon \varphi_\varepsilon$ converges almost surely to $\phi \sim \nu$ as $\varepsilon \to 0$. Reflection positivity is preserved by weak limits hence, by Lemma 6.3, ν is reflection positive.

Denote by $\tilde{\Phi}^\varepsilon$ the solution to (6.15) with initial data φ_ε. Since $\tilde{\nu}_{\beta,N,\varepsilon}$ is the invariant measure of (6.15), $\tilde{\Phi}^\varepsilon(t) \sim \tilde{\nu}_{\beta,N,\varepsilon}$ for every $t \in \mathbb{R}_+$.

Denote by Φ the (global-in-time) solution to (1.3) with initial data ϕ. For every $t > 0$, $\text{ext}^\varepsilon \tilde{\Phi}^\varepsilon(t) \to \Phi(t)$ in distribution as $\varepsilon \to 0$ as a consequence of Proposition 6.19. Hence, $\Phi(t) \sim \nu$ for every $t > 0$. Thus, ν is an invariant measure of (1.3). By Proposition 6.17 the invariant measure of (1.3) is unique. Therefore, $\nu = \nu_{\beta,N}$. □

7 Decay of spectral gap

Proof of Corollary 1.3. The Markov semigroup $(\mathscr{P}_t^{\beta,N})_{t\geqslant 0}$ associated to (1.3) is reversible with respect to $\nu_{\beta,N}$ (see [HM18a, Corollary 1.3] or [ZZ18a, Lemma 4.2]). Thus, one can express $\lambda_{\beta,N}$ as the sharpest constant in the Poincaré inequality

$$\lambda_{\beta,N} = \inf_{F\in D(\mathscr{E}_{\beta,N})} \frac{\mathscr{E}_{\beta,N}(F,F)}{\langle F^2\rangle_{\beta,N} - \langle F\rangle_{\beta,N}^2} > 0 \tag{7.1}$$

where $\mathscr{E}_{\beta,N}$ is the associated Dirichlet form with domain $D(\mathscr{E}_{\beta,N}) \subset L^2(\nu_{\beta,N})$. See [ZZ18a, Corollary 1.5].

The proof of Corollary 1.3 amounts to choosing the right test function in (7.1) and then using the explicit expression for $\mathscr{E}_{\beta,N}$ for sufficiently nice functions due to [ZZ18a, Theorem 1.2].

Let Cyl be the set of $F \in L^2(\nu_{\beta,N})$ of the form

$$F(\cdot) = f\Big(l_1(\cdot), \dots, l_m(\cdot)\Big)$$

where $m \in \mathbb{N}$, $f \in C_b^1(\mathbb{R}^m)$, $l_1, \dots, l_m$ are real trigonometric polynomials, and $l_i(\cdot)$ denotes the (L^2) duality pairing between l_i and elements in $\mathscr{C}^{-\frac{1}{2}-\kappa}$. For any $F \in$ Cyl, let $\partial_{l_i}F$ denote the Gâteaux derivative of F in direction l_i. Let $\nabla F : \mathscr{C}^{-\frac{1}{2}-\kappa} \to \mathbb{R}$ be the unique function such that $\partial_{l_i}F(\phi) = \int_{\mathbb{T}_N} \nabla F(\phi) l_i dx$ for every $\phi \in \mathscr{C}^{-\frac{1}{2}-\kappa}$. In other words, ∇F is the representation of the Gâteaux derivative with respect to the L^2 inner product. Then, for any $F, G \in$ Cyl,

$$\mathscr{E}_{\beta,N}(F,G) = \Big\langle \int_{\mathbb{T}_N} \nabla F \nabla G dx \Big\rangle_{\beta,N}.$$

Now we choose a test function in Cyl to insert into (7.1). Take any $\zeta \in (0,1)$ and $m \in [0, (1-\zeta)\sqrt{\beta})$. Let $\chi_m : \mathbb{R} \to \mathbb{R}$ be a smooth, non-decreasing odd function such that $\chi_m(a) = -1$ for $a \leqslant -m$ and $\chi_m(a) = 1$ for $a \geqslant m$. Define

$$F(\phi) = \chi_m(\mathfrak{m}_N(\phi)).$$

Then, $F \in$ Cyl and $\langle F\rangle_{\beta,N} = 0$. Moreover, its Fréchet derivative DF is supported on the set $\{\mathfrak{m}_N \in [-m,m]\}$.

Thus, inserting F into (7.1), we obtain

$$\lambda_{\beta,N} \leqslant \frac{\mathscr{E}_{\beta,N}(F,F)}{\langle F^2\rangle_{\beta,N}} \leqslant \frac{\left\| \int_{\mathbb{T}_N} |\nabla F|^2 dx \right\|_{L^\infty(\nu_{\beta,N})}}{\langle F^2\rangle_{\beta,N}} \nu_{\beta,N}(\mathfrak{m}_N \in [-m,m]). \tag{7.2}$$

For any $g \in L^2(\mathbb{T}_N)$ and $\varepsilon > 0$, by the linearity of $\mathfrak{m}_N$ and the Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{F(\phi+\varepsilon g) - F(\phi)}{\varepsilon} &\leqslant |\chi_m'|_\infty \Big| \frac{\mathfrak{m}_N(\phi+\varepsilon g) - \mathfrak{m}_N(\phi)}{\varepsilon} \Big| \\ &\leqslant |\chi_m'|_\infty \frac{\int_{\mathbb{T}_N} g dx}{N^3} \leqslant |\chi_m'|_\infty \frac{\left(\int_{\mathbb{T}_N} g^2 dx\right)^{\frac{1}{2}}}{N^{\frac{3}{2}}}\end{aligned}$$

where χ_m' is the derivative of χ_m and $|\cdot|_\infty$ denotes the supremum norm. Note that this estimate is uniform over $\phi \in \mathscr{C}^{-\frac{1}{2}-}$. Then, by duality and the definition of ∇F,

$$\begin{aligned}\Big\| \int_{\mathbb{T}_N} |\nabla F|^2 dx \Big\|_{L^\infty(\nu_{\beta,N})} &= \Big\| \Big(\sup_{g \in L^2 : \int_{\mathbb{T}_N} g^2 dx = 1} \int_{\mathbb{T}_N} \nabla F g dx \Big)^2 \Big\|_{L^\infty(\nu_{\beta,N})} \\ &\leqslant \frac{|\chi_m'|_\infty^2}{N^3}.\end{aligned} \tag{7.3}$$

For the other term in (7.2), using that F^2 is identically 1 on $\{|\mathfrak{m}_N| \geqslant m\}$,

$$\begin{aligned}\langle F^2 \rangle_{\beta,N} &= \nu_{\beta,N}(|\mathfrak{m}_N| \geqslant m) + \langle F^2 \mathbf{1}_{\mathfrak{m}_N \in (-m,m)} \rangle_{\beta,N} \\ &\geqslant 1 - \nu_{\beta,N}(\mathfrak{m}_N \in (-m,m)).\end{aligned} \tag{7.4}$$

We insert (7.3) and (7.4) into (7.2) to give

$$\lambda_{\beta,N} \leqslant \frac{|\chi_m|_\infty^2}{N^3} \frac{\nu_{\beta,N}(\mathfrak{m}_N \in [-m,m])}{1 - \nu_{\beta,N}(\mathfrak{m}_N \in (-m,m))}.$$

By Theorem 1.2, there exists $C = C(\zeta,\eta) > 0$ and $\beta_0 = \beta_0(\zeta,\eta) > 0$ such that, for all $\beta > \beta_0$,

$$\lambda_{\beta,N} \leqslant \frac{|\chi_m'|_\infty^2}{N^3} \frac{e^{-C\sqrt{\beta}N^2}}{1 - e^{-C\sqrt{\beta}N^2}}$$

from which (1.4) follows.

□

A Analytic notation and toolbox

A.1 Basic function spaces on the torus

Let $\mathbb{T}_N = (\mathbb{R}/N\mathbb{Z})^3$ be the 3D torus of sidelength $N \in \mathbb{N}$. Denote by $C^\infty(\mathbb{T}_N)$ the space of smooth functions on $\mathbb{T}_N$ and by $S'(\mathbb{T}_N)$ the space of distributions. For $\phi \in S'(\mathbb{T}_N)$ and $f \in C^\infty(\mathbb{T}_N)$, we write $\int_{\mathbb{T}_N} \phi f dx$ to denote their duality pairing.

For any $p \in [1, \infty]$, let $L^p(\mathbb{T}_N) = L^p(\mathbb{T}_N, đx)$ denote the Lebesgue space with respect to the normalised Lebesgue measure $đx = \frac{dx}{N^3}$.

Let $\mathcal{F}$ denote the Fourier transform, i.e. for any $f \in C^\infty(\mathbb{T}_N)$ and $n \in (N^{-1}\mathbb{Z})^3$,

$$\mathcal{F}f(n) = \int_{\mathbb{T}_N} f e_{-n} dx, \quad f = \frac{1}{N^3} \sum_{n \in (N^{-1}\mathbb{Z})^3} \mathcal{F}f(n) e_n$$

where $e_n(x) = e^{2\pi i n \cdot x}$.

For any $\rho : \mathbb{R}^3 \to \mathbb{R}$, let T_ρ be the Fourier multiplier with symbol $\rho(\cdot)$ defined on smooth functions via

$$T_\rho f = \frac{1}{N^3} \sum_{n \in (N^{-1}\mathbb{Z})^3} \rho(n) \mathcal{F}f(n) e_n.$$

When clear from context, we simply write ρf instead of $T_\rho f$.

For $s \in \mathbb{R}$, the inhomogeneous Sobolev space H^s is the completion of $f \in C^\infty$ with respect to the norm

$$\|f\|_{H^s} = \|\langle \cdot \rangle^s f\|_{L^2}$$

where $\langle \cdot \rangle = \sqrt{\eta + 4\pi^2 |\cdot|^2}$ for a fixed $\eta > 0$ (see Section 2). The norms depend on η but they are equivalent for different choices.

A.2 Besov spaces

In this section, we introduce Besov spaces on $\mathbb{T}_N$ and give some useful estimates. All of the results can be found in [BCD11, Section 2.7] stated for Besov spaces on $\mathbb{R}^3$, but can be adapted to $\mathbb{T}_N$.

Let $B(x, r)$ denote the ball centred at $x \in \mathbb{R}^3$ of radius $r > 0$ and let A denote the annulus $B(0, \frac{4}{3}) \backslash B(0, \frac{3}{8})$. Let $\tilde{\Delta}, \Delta \in C_c^\infty(\mathbb{R}^3; [0, 1])$ be radially symmetric and satisfy

- $\text{supp}\tilde{\chi} \subset B(0, \frac{4}{3})$ and $\text{supp}\chi \subset A$;
- $\sum_{k \geqslant -1} \chi_k = 1$, where $\chi_{-1} = \tilde{\chi}$ and $\chi_k(\cdot) = \chi(2^{-k}\cdot)$ for $k \in \mathbb{N} \cup \{0\}$.

Identify Δ_k with its Fourier multiplier.

$\{\Delta_k\}_{k \in \mathbb{N} \cup \{-1\}}$ are called Littlewood-Paley projectors. For $f \in C^\infty(\mathbb{T}_N)$, we have

$$f = \sum_{k \geqslant -1} \Delta_k f.$$

For $k \geqslant 0$, $\Delta_k f$ contains the frequencies of f order 2^k. Δ_{-1} contains all the low frequencies (i.e. of size less than order 1).

For $s \in \mathbb{R}$, $p, q \in [1, \infty]$, we define the Besov spaces $B^s_{p,q}(\mathbb{T}_N)$ to be the completion of $C^\infty(\mathbb{T}_N)$ with respect to the norm

$$\|f\|_{B^s_{p,q}} = \left\| \left(2^{ks} \|\Delta_k f\|_{L^p} \right)_{k \geqslant -1} \right\|_{l^q}$$

where l^q is the usual space of q-summable sequences, interpreted as a supremum when $q = \infty$. Note that these spaces are separable. Besov-Hölder spaces are denoted $B^s_{\infty,\infty}(\mathbb{T}_N) = \mathscr{C}^s(\mathbb{T}_N)$ and are a strict subset of the usual Hölder spaces (which are not separable) for $s \in \mathbb{R}_+ \backslash \mathbb{N}$. Moreover, the $B^s_{2,2}(\mathbb{T}_N) = H^s(\mathbb{T}_N)$ and their norms are equivalent.

Proposition A.1 (Duality). *Let $s \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in [1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1$. Then,*

$$\left| \int_{\mathbb{T}_N} f g \, dx \right| \leqslant \|f\|_{B^{-s}_{p_1,q_1}} \|g\|_{B^s_{p_2,q_2}} \tag{1.1}$$

for $f, g \in C^\infty(\mathbb{T}_N)$.

Proof. See [GOTW18, Lemma 2.1]. □

Proposition A.2 (Fractional Leibniz estimate). *Let $s \in \mathbb{R}$, $p, p_1, p_2, p_3, p_4, q \in [1, \infty]$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Then, there exists $C = C(s, p_1, p_2, p_3, p_4, q, \eta) > 0$ such that*

$$\|fg\|_{B^s_{p,q}} \leqslant C \|f\|_{B^s_{p_1,q}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{B^s_{p_4,q}} \tag{1.2}$$

for $f, g \in C^\infty(\mathbb{T}_N)$.

Proof. See [GOTW18, Lemma 2.1]. □

Proposition A.3 (Interpolation). *Let $s, s_1, s_2 \in \mathbb{R}$ such that $s_1 < s < s_2$, $p, p_1, p_2, q, q_1, q_2 \in [1, \infty]$ and $\theta \in (0, 1)$ satisfy*

$$\begin{aligned} s &= \theta s_1 + (1 - \theta) s_2 \\ \frac{1}{p} &= \frac{\theta}{p_1} + \frac{1 - \theta}{p_2} \\ \frac{1}{q} &= \frac{\theta}{q_1} + \frac{1 - \theta}{q_2}. \end{aligned} \tag{1.3}$$

Then, there exists $C = C(s, s_1, s_2, p, p_1, p_2, q, q_1, q_2, \theta, \eta) > 0$ such that

$$\|f\|_{B^s_{p,q}} \leqslant C \|f\|^\theta_{B^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{B^{s_2}_{p_2,q_2}} \tag{1.4}$$

for $f \in C^\infty(\mathbb{T}_N)$.

Proof. See [BM18, Proposition 5.7]. □

Proposition A.4 (Bernstein's inequality)**.** *For* $R > 0$*, denote* $B_f(R) = \{n \in (N^{-1}\mathbb{Z})^3 : |n| \leqslant R\}$*. Let* $s_1, s_2 \in \mathbb{R}$ *such that* $s_1 < s_2$*,* $p, q \in [1, \infty]$*. Then, there exists* $C = C(s_2, s_2, p, q, \eta) > 0$ *such that*

$$\|f\|_{B^{s_2}_{p,q}} \leqslant CR^{s_2-s_1}\|f\|_{B^{s_1}_{p,q}} \tag{1.5}$$

$$\|g\|_{B^{s_1}_{p,q}} \leqslant CR^{s_1-s_2}\|g\|_{B^{s_2}_{p,q}} \tag{1.6}$$

for $f, g \in C^\infty(\mathbb{T}_N)$ *such that* $\operatorname{supp}(\mathcal{F}f) \subset B_f(R)$ *and* $\operatorname{supp}(\mathcal{F}g) \subset (N^{-1}\mathbb{Z})^3 \backslash B_f(R)$*.*

Proof. See [BCD11, Lemma 2.1] for a proof on $\mathbb{R}^3$. □

A.3 Paracontrolled calculus

Let $f, g \in C^\infty(\mathbb{T}_N)$. Define the paraproduct

$$f \circledcirc g = \sum_{l<k-1} \Delta_k f \Delta_l g$$

and the resonant product

$$f \circleddash g = \sum_{|k-l|\leqslant 1} \Delta_k f \Delta_l g.$$

Then,

$$fg = f \oslash g + f \circleddash g + f \circledcirc g. \tag{1.7}$$

Proposition A.5 (Paraproduct estimates)**.** *Let* $s \in \mathbb{R}$ *and* $p, p_1, p_2, q \in [1, \infty]$ *be such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$*. Then, there exists* $C = C(s, p, p_1, p_2, q, \eta) > 0$ *such that*

$$\|f \circledcirc g\|_{B^s_{p,q}} \leqslant C\|f\|_{B^s_{p_1,q}}\|g\|_{L^{p_2}} \tag{1.8}$$

for $f, g \in C^\infty(\mathbb{T}_N)$*.*

Proof. See [BCD11, Theorem 2.82] for a proof on $\mathbb{R}^3$. □

Proposition A.6 (Resonant product estimate)**.** *Let* $s_1, s_2 \in \mathbb{R}$ *such that* $s = s_1 + s_2 > 0$*. Let* $p, p_1, p_2, q \in [1, \infty]$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$*. Then, there exists* $C = C(s_1, s_2, p, p_1, p_2, q, \eta) > 0$ *such that*

$$\|f \circleddash g\|_{B^s_{p,q}} \leqslant C\|f\|_{B^{s_1}_{p_1,\infty}}\|g\|_{B^{s_2}_{p_2,q}} \tag{1.9}$$

for $f, g \in C^\infty(\mathbb{T}_N)$*.*

Proof. See [BCD11, Theorem 2.85] for a proof on $\mathbb{R}^3$. $\square$

We now state some useful commutator estimates.

Proposition A.7. *Let $s_1, s_3 \in \mathbb{R}$, $s_2 \in (0,1)$ such that $s_1+s_3 < 0$ and $s_1+s_2+s_3 = 0$. Moreover, let $p, p_1, p_2, q_1, q_2 \in [1,\infty]$ satisfy $\frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2} = 1$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Then, there exists $C = C(s_1, s_2, s_3, p, p_1, p_2, q_1, q_2, \eta) > 0$ such that*

$$\left| \int_{\mathbb{T}_N} (f \otimes g)h - (f \ominus h)g đx \right| \leqslant C\|f\|_{B^{s_1}_{p,\infty}} \|g\|_{B^{s_2}_{p_1,q_1}} \|h\|_{B^{s_3}_{p_2,q_2}} \tag{1.10}$$

for $f, g, h \in C^\infty(\mathbb{T}_N)$.

Proof. This is a modification of [GUZ20, Lemma A.6]. See [BG19, Proposition 7]. $\square$

Proposition A.8. *Let $s_1, s_3 \in \mathbb{R}$, $s_2 \in (0,1)$ such that $s_1+s_3 < 0$ but $s_1+s_2+s_3 > 0$. Morover, let $p, p_1, p_2, p_3 \in [1,\infty]$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$. Then, there exists $C = C(s_1, s_2, s_3, p, p_1, p_2, \eta) > 0$ such that*

$$\|(f \otimes g) \ominus h - (f \ominus h)g\|_{B^{s_1+s_2+s_3}_{p,\infty}} \leqslant C\|f\|_{B^{s_1}_{p_1,\infty}} \|g\|_{B^{s_2}_{p_2,\infty}} \|h\|_{B^{s_3}_{p_3,\infty}} \tag{1.11}$$

for $f, g, h \in C^\infty(\mathbb{T}_N)$.

Proof. This is a modification of [GIP15, Lemma 2.4]. See [BG19, Proposition 6]. $\square$

A.4 Analytic properties of $\mathcal{J}_k$

The family of operators $\{\mathcal{J}_k\}_{k\geqslant 0}$ defined in Section 4.1 satisfies the following estimate: for every multi-index $\alpha \in \mathbb{N}^3$, there exists $C = C(\alpha, \eta)$ such that

$$\left|\partial^\alpha \mathcal{J}_k(x)\right| \leqslant \frac{C}{\langle k \rangle^{\frac{1}{2}} (1 + |x|)^{1+|\alpha|}}. \tag{1.12}$$

Proposition A.9. *Let $s \in \mathbb{R}$, $p, q \in [1,\infty]$. Then, there exists $C = C(s,p,q) > 0$ such that*

$$\|\mathcal{J}_k f\|_{B^{s+1}_{p,q}} \leqslant \frac{C}{\langle k \rangle^{\frac{1}{2}}} \|f\|_{B^s_{p,q}} \tag{1.13}$$

for every $f \in C^\infty(\mathbb{T}_N)$

Proof. This follows from (1.12) and [BCD11, Proposition 2.78]. $\square$

We now state another useful commutator estimate.

Proposition A.10. *Let $s_1 \in \mathbb{R}$, $s_2 \in (0,1)$, $p, p_1, p_2, q, q_1, q_2 \in [1,\infty]$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then, for any $\kappa > 0$, there exists $C = C(s_1, s_2, p, p_1, p_2, q, \kappa, \eta) > 0$ such that*

$$\|\mathcal{J}_k(f \otimes g) - \mathcal{J}_k f \otimes g\|_{B^{s_1+s_2-\kappa}_{p,q}} \leqslant C\|f\|_{B^{s_1}_{p_1,\infty}}\|g\|_{B^{s_2}_{p_2,\infty}} \tag{1.14}$$

for $f, g \in C^\infty(\mathbb{T}_N)$.

Proof. This follows from (1.12) and [BCD11, Lemma 2.99]. □

A.5 Poincaré inequality on blocks

Proposition A.11. *There exists $C_P > 0$ such that, for any $N \in \mathbb{N}$ and $\square \subset \mathbb{T}_N$ a unit block, the following estimate holds for all $f \in C^\infty(\mathbb{T}_N)$:*

$$\int_\square \big(f - f(\square)\big)^2 dx \leqslant C_P \int_\square |\nabla f|^2 dx \tag{1.15}$$

where $f(\square) = \int_\square f dx$.

Proof. See [GT15, (7.45)]. □

A.6 Bounds on discrete convolutions

Lemma A.12. *Let $d \geqslant 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy*

$$\alpha + \beta > d \text{ and } \alpha, \beta < d.$$

Then, there exists $C = C(d, \alpha, \beta) > 0$ such that, uniformly over $n \in (N^{-1}\mathbb{Z})^d$,

$$\frac{1}{N^3} \sum_{\substack{n_1, n_2 \in (N^{-1}\mathbb{Z})^d \\ n_1 + n_2 = n}} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \frac{1}{\langle n \rangle^{\alpha+\beta-d}}$$

Proof. Follows from [MWX17, Lemma 4.1] and by keeping track of N dependence. □

Epilogue

We end Part III with a sketch proof of phase transition for ϕ_3^4 using Glimm, Jaffe, and Spencer's modification of Peierls' argument as in [GJS75].

The starting point is contour bounds $\nu_{\beta,N}$. Recall that $\mathbb{B}_N$ is the natural discretisation of $\mathbb{T}_N$ into unit boxes. For $\phi \sim \nu_{\beta,N}$, let $\sigma^\phi \in \{\pm 1\}^{\mathbb{B}_N}$ be defined by

$$\partial\sigma^\phi(\square) = \begin{cases} +1, & \text{if } \phi(\square) > 0 \\ -1, & \text{otherwise.} \end{cases}$$

As in the case of Ising, each configuration σ^ϕ is in bijection with a configuration of connected contours, and the set of contours $\partial\sigma^\phi$ is called the phase boundary.

Proposition. *There exists $\beta_0 > 0$ such that the following holds: let Γ be a fixed contour. Then, there exists $C = C(\beta_0)$ such that, for all $\beta > \beta_0$,*

$$\nu_{\beta,N}(\Gamma \subset \partial\sigma^\phi) \leqslant e^{-C\sqrt{\beta}|\Gamma|}$$

where $|\Gamma|$ is the number of faces in Γ.

Proof. Each face in Γ occurs as the common face between two nearest-neighbour blocks. We therefore identify Γ with the set of all such pairs of nearest-neighbours. Note that any single block may appear in at most 6 different pairs.

Using this identification, we write

$$\mathbf{1}_{\Gamma\subset\partial\sigma^\phi} = \prod_{\{\square,\square'\}\in\Gamma} \Big(\mathbf{1}_{\phi(\square)>0}\mathbf{1}_{\phi(\square')\leqslant 0} + \mathbf{1}_{\phi(\square)\leqslant 0}\mathbf{1}_{\phi(\square')>0}\Big).$$

We split $\mathbf{1}_{\phi(\square)>0}\mathbf{1}_{\phi(\square')\leqslant 0}$ into three events:

1. $\phi(\square)$ is localised near the potential barrier (Figure 4a);
2. $\phi(\square')$ is localised near the potential barrier (Figure 4b);
3. both $\phi(\square)$ and $\phi(\square')$ are localised away from the potential barrier, but are of opposite spin (Figure 4c).

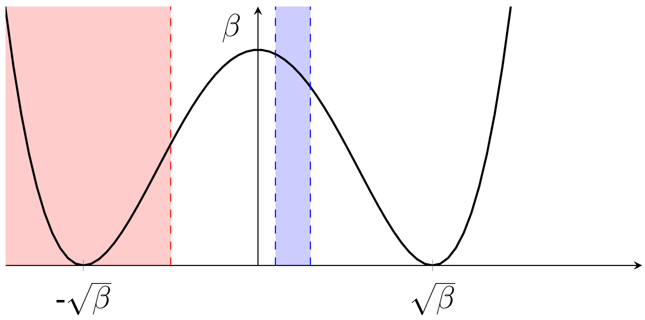
Thus, we can write

$$\mathbf{1}_{\phi(\square)>0}\mathbf{1}_{\phi(\square)} \leqslant \mathbf{1}_{|\phi(\square)|\leqslant\frac{\beta}{2}} + \mathbf{1}_{|\phi(\square')|\leqslant\frac{\beta}{2}} + \mathbf{1}_{\phi(\square)>\frac{\beta}{2}}\mathbf{1}_{\phi(\square)\leqslant\frac{\beta}{2}}.$$

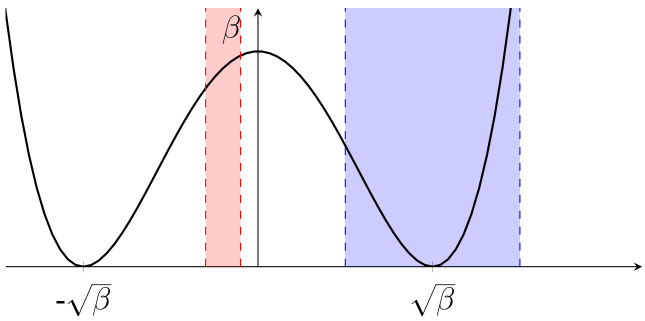
The rest of the proof follows from arguing as in Lemma 3.4 and Proposition 3.2, and then applying the Q-bounds of Proposition 3.6.

$\square$

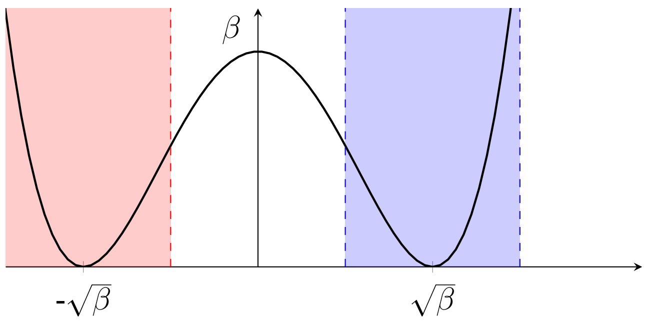
Figure 4



(a) Possibility 1



(b) Possibility 2



(c) Possibility 3

We define infinite volume states by $\langle \cdot \rangle_\beta = \lim_{N\to\infty} \langle \cdot \rangle_{\beta,N}$. Note that the (subsequential) limit can be shown to exist by using reflection positivity. See [Shl86, Theorem 3.1]. With care, one can show contour bounds for $\langle \cdot \rangle_\beta$. Then, by arguing similarly as in the case of Ising, one can show the existence of long range order, i.e. establish the following theorem.

Theorem A.13. *Provided β is sufficiently large,*

$$|\langle \mathbf{1}_{\sigma^\phi(\square)=1}\mathbf{1}_{\sigma^\phi(\square')=-1}\rangle_\beta - \langle \mathbf{1}_{\sigma^\phi(\square)=1}\rangle_\beta \langle \mathbf{1}_{\sigma^\phi(\square')=-1}\rangle_\beta| \geqslant \frac{1}{8}$$

uniformly over unit boxes $\square$ *and* $\square'$.

We now show that the $\phi \mapsto -\phi$ symmetry of $\langle\cdot\rangle_\beta$ is broken for sufficiently large β, which can be upgraded to show spontaneous magnetisation. We do this by introducing an external field $h \in \mathbb{R}$. Define

$$\mathscr{V}_{\beta,h}(\phi(x)) = \mathscr{V}_\beta(\phi(x)) - h\phi(x)$$

and denote by $\nu_{\beta,h,N}$ the corresponding ϕ^4 measure on $\mathbb{T}_N$ associated to this potential. Note that these measures are reflection positive, thus we can define infinite volume states as $\langle\cdot\rangle_{\beta,h} = \lim_{N\to\infty}\langle\cdot\rangle_{\beta,h,N}$ as before.

The following theorem establishes symmetry breaking.

Theorem. *Provided* β *is taken sufficiently large,*

$$\lim_{h\downarrow 0}\langle \mathbf{1}_{\phi(\square)>0}\rangle_{\beta,h} \geqslant 0.8 > \frac{1}{2} = \langle \mathbf{1}_{\phi(\square)>0}\rangle_{\beta,0}$$

for any unit block $\square$.

Proof. For any $h > 0$, the Lee-Yang theorem [SG73] implies

$$|\langle \mathbf{1}_{\phi(\square)>0}\mathbf{1}_{\phi(\square')<0}\rangle_{\beta,h} - \langle \mathbf{1}_{\phi(\square)>0}\rangle_{\beta,h}\langle \mathbf{1}_{\phi(\square')<0}\rangle_{\beta,h}| \to 0$$

as the distance between $\square$ and $\square'$ goes to infinity.

However, one can show that

$$\langle \mathbf{1}_{\phi(\square)>0}\mathbf{1}_{\phi(\square')\leqslant 0}\rangle_{\beta,h} \leqslant \frac{1}{8}$$

provided β is sufficiently large. This follows by developing contour bounds for $\nu_{\beta,h,N}$. One can show that for $|h| < \frac{1}{\beta}$, the external field can be interpreted as an $O(1)$ shift of the minima of the potential $\mathscr{V}_\beta$ provided β is sufficiently large. This is sufficient to extend our analysis, in particular the Q-bounds of Proposition 3.6, to this case.

Thus, by translation invariance,

$$\begin{aligned}\langle \mathbf{1}_{\phi(\square)>0}\rangle_{\beta,h}\langle \mathbf{1}_{\phi(\square')<0}\rangle_{\beta,h} &= \langle \mathbf{1}_{\phi(\square)>0}\rangle_{\beta,h}\langle \mathbf{1}_{\phi(\square)<0}\rangle_{\beta,h} \\ &\leqslant \langle \mathbf{1}_{\phi(\square)>0}\rangle_{\beta,h}(1 - \langle \mathbf{1}_{\phi(\square)>0}\rangle_{\beta,h}) \leqslant \frac{1}{8}.\end{aligned}$$

A (physically possible) solution of this necessarily satisfies $\langle \mathbf{1}_{\phi(\square)>0}\rangle_{\beta,h} > 0.8$. Hence, the limit of this quantity as $h \downarrow 0$, which exists due to correlation inequalities [GRS75, Section V], is strictly greater than $\frac{1}{2}$, thereby establishing symmetry breaking. $\square$

To use this result to show spontaneous magnetisation, it suffices to establish $\langle\phi(\square)\rangle_{\beta,h}$ is of order $\beta\langle \mathbf{1}_{\phi(\square)>0}\rangle_{\beta,h} - \beta\langle \mathbf{1}_{\phi(\square)\leqslant 0}\rangle_{\beta,h}$, i.e. that $\phi(\square)$ localises near the minima of the potential wells. This can be done by using arguments in Proposition 3.7.

IV. Future directions

In summary, in this thesis we have addressed two problems concerning the statistical mechanics of Euclidean field theories in three dimensions. Our first contribution has been to establish quasi-invariance of Gaussian measures under the dynamics of the nonlinear wave equation. Our second contribution has been to establish a surface order large deviation estimate for the average magnetisation of low temperature ϕ^4, and use it to show that the relaxation times of its Glauber dynamics explode in the infinite volume limit. The common theme between these two contributions has been the development of the variational approach to ultraviolet stability of ϕ^4_3 of Barashkov and Gubinelli [BG19] within the context of statistical mechanics arguments.

To conclude, we discuss future directions of our research. There are many interesting open problems, ranging from trivial improvements to science fiction. We restrict our attention to two problems that are fascinating but seem within reach. They both concern the ϕ^4_3 model in the phase coexistence regime and are natural extensions of the work [CGW20].

1 Boundary conditions and Dobrushin states for ϕ^4_3

Due to the presence of phase transition, one expects that ϕ^4_3 is sensitive to boundary conditions at low temperatures. Specifically, one would want to define the analogue of $\nu_{\beta,N}$ on boxes with inhomogeneous Dirichlet boundary conditions and look at the effect of this choice in the infinite volume limit. However, there are already nontrivial technical difficulties in finite volumes. For one, our analysis relies heavily on Fourier analytic techniques which requires working on a torus.

The more serious concern, however, is allowed boundary conditions given the negative regularity of ϕ^4 fields. From the point of view of statistical mechanics, the interesting boundary conditions are functions that are piecewise continuous on blocks. Indeed, these are natural analogs of boundary conditions for lattice spin systems. There are some works in this direction in the case of ϕ^4_2 [Gid79], but none that we know of for ϕ^4_3.

One particularly interesting boundary condition of the above type is $+$ on the top of the box and $-$ on the bottom. These are so-called Dobrushin boundary conditions and are well-studied for the Ising model. Indeed, for Ising, these boundary conditions generate an interface between $+$ and $-$ spins in finite volumes. In $d = 3$, Dobrushin [Dob72] established that this interface exists in the infinite volume limit in some sense: in particular, one obtains a non-translation invariant limit. This is in contrast to $d = 2$, where Aizenman [Aiz80] and Higuchi [Hig79], independently, showed that the interface disappears in the infinite volume limit and translation invariance is recovered. It would be interesting to explore this phenomenon for ϕ^4.

2 A full low temperature expansion for ϕ_3^4

Glimm, Jaffe, and Spencer established a second, more quantitative proof of phase transition for ϕ_2^4 by explicitly constructing two distinct infinite volume measures in [GJS76a, GJS76b]. Their proof combines the Peierls' bounds of [GJS75] with the cluster expansion techniques of [GJS74], resulting in a low temperature expansion for ϕ_3^4.

The two measures that they construct arise as infinite volume limits of measures with a version of $+$ and $-$ boundary conditions, respectively, and satisfy the Osterwalder-Schrader axioms. In order to show that they are distinct, they show that their respective magnetisations do not agree. In fact, they obtain an explicit series (in β) for the spontaneous magnetisation, and obtain corrections due to probabilistic/quantum fluctuations about the classical magnetisation (i.e. $\pm\sqrt{\beta}$, corresponding to the minima of $\mathscr{V}_\beta$). Moreover, they show that these two measures are pure states, in that they exhibit exponential decay of correlations. This implies a mass gap in the corresponding quantum field theories associated to these measures.

Extending the results of [GJS76a, GJS76b] to $d = 3$ would be of great interest. Indeed, results so far have concentrated on high temperatures [FO76]. Thus, gaining a better understanding of the low temperature regime would be a first step towards obtaining a more complete picture of the phase diagram for ϕ_3^4.

References

[ADC20] Michael Aizenman and Hugo Duminil-Copin. Marginal triviality of the scaling limits of critical 4d Ising and φ^4. *arXiv preprint arXiv:1912.07973*, 2020.

[AH87] Michael Aizenman and Richard Holley. Rapid convergence to equilibrium of stochastic Ising models in the Dobrushin-Shlosman regime. In *Percolation theory and ergodic theory of infinite particle systems*, pages 1–11. Springer, 1987.

[Aiz80] Michael Aizenman. Translation invariance and instability of phase coexistence in the two dimensional Ising system. *Communications in Mathematical Physics*, 73(1):83–94, 1980.

[Aiz82] Michael Aizenman. Geometric analysis of φ^4 fields and Ising models. parts i and ii. *Communications in Mathematical Physics*, 86(1):1–48, 1982.

[BCD11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343. Springer Science & Business Media, 2011.

[BCG+80] G Benfatto, M Cassandro, G Gallavotti, F Nicolo, E Olivieri, E Presutti, and E Scacciatelli. Ultraviolet stability in Euclidean scalar field theories. *Communications in Mathematical Physics*, 71(2):95–130, 1980.

[BD98] Michelle Boué and Paul Dupuis. A variational representation for certain functionals of Brownian motion. *The Annals of Probability*, 26(4):1641–1659, 1998.

[BD19] Amarjit Budhiraja and Paul Dupuis. *Analysis and Approximation of Rare Events: Representations and Weak Convergence Methods*, volume 94. Springer, 2019.

[BDH95] David Brydges, Jonathan Dimock, and TR Hurd. The short distance behavior of $(\varphi)^4_3$. *Communications in Mathematical Physics*, 172(1):143–186, 1995.

[BFS83] David C Brydges, Jürg Fröhlich, and Alan D Sokal. A new proof of the existence and nontriviality of the continuum ϕ^4_2 and ϕ^4_3 quantum field theories. *Communications in Mathematical Physics*, 91(2):141–186, 1983.

[BG19] N Barashkov and M Gubinelli. A variational method for ϕ^4_3. *arXiv preprint arXiv:1805.10814*, 2019.

[BG20] Nikolay Barashkov and Massimiliano Gubinelli. The ϕ^4_3 measure via Girsanov's theorem. *arXiv preprint arXiv:2004.01513*, 2020.

[Bil08] Patrick Billingsley. *Probability and measure*. John Wiley & Sons, 2008.

[Bis09] Marek Biskup. Reflection positivity and phase transitions in lattice spin models. In *Methods of contemporary mathematical statistical physics*, pages 1–86. Springer, 2009.

[BIV00] T. Bodineau, D. Ioffe, and Y. Velenik. Rigorous probabilistic analysis of equilibrium crystal shapes. *Journal of Mathematical Physics*, 41(3):1033–1098, 2000.

[BM18] Haïm Brezis and Petru Mironescu. Gagliardo–Nirenberg inequalities and non-inequalities: The full story. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 35, pages 1355–1376. Elsevier, 2018.

[Bod99] T. Bodineau. The Wulff construction in three and more dimensions. *Communications in Mathematical Physics*, 207(1):197–229, 1999.

[Bod02] Thierry Bodineau. Phase coexistence for the Kac-Ising models. In *In and Out of Equilibrium*, pages 75–111. Springer, 2002.

[Bon81] Jean-Michel Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. In *Annales scientifiques de l'École Normale supérieure*, volume 14, pages 209–246, 1981.

[BOP19] Árpád Bényi, Tadahiro Oh, and Oana Pocovnicu. On the probabilistic Cauchy theory for nonlinear dispersive PDEs. In *Landscapes of Time-Frequency Analysis*, pages 1–32. Springer, 2019.

[Bou94] Jean Bourgain. Periodic nonlinear Schrödinger equation and invariant measures. *Communications in Mathematical Physics*, 166(1):1–26, 1994.

[Bou96a] Jean Bourgain. Gibbs measures and quasi-periodic solutions for nonlinear Hamiltonian partial differential equations. In *The Gelfand Mathematical Seminars, 1993–1995*, pages 23–43. Springer, 1996.

[Bou96b] Jean Bourgain. Invariant measures for the 2d-defocusing nonlinear Schrödinger equation. *Communications in Mathematical physics*, 176(2):421–445, 1996.

[BPZ84] Alexander A Belavin, Alexander M Polyakov, and Alexander B Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics B*, 241(2):333–380, 1984.

[BT08] Nicolas Burq and Nikolay Tzvetkov. Random data Cauchy theory for supercritical wave equations i: local theory. *Inventiones Mathematicae*, 173(3):449–475, 2008.

[BT11] Nicolas Burq and Nikolay Tzvetkov. Probabilistic well-posedness for the cubic wave equation. *Journal of the European Mathematical Society (JEMS)*, 16(1):1–30, 2011.

[CC18] Rémi Catellier and Khalil Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. *The Annals of Probability*, 46(5):2621–2679, 2018.

[CCS87] JT Chayes, L Chayes, and Roberto Henrique Schonmann. Exponential decay of connectivities in the two-dimensional Ising model. *Journal of Statistical Physics*, 49(3-4):433–445, 1987.

[CCT03] Michael Christ, James Colliander, and Terence Tao. Ill-posedness for nonlinear Schrodinger and wave equations. *arXiv preprint math/0311048*, 2003.

[CF86] Francis Comets and R Fortet. Grandes déviations pour des champs de Gibbs sur $\mathbb{Z}^d$. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 303(11):511–513, 1986.

[CGMS96] F Cesi, G Guadagni, F Martinelli, and RH Schonmann. On the two-dimensional stochastic Ising model in the phase coexistence region near the critical point. *Journal of Statistical Physics*, 85(1-2):55–102, 1996.

[CGW20] Ajay Chandra, Trishen S. Gunaratnam, and Hendrik Weber. Phase transitions for ϕ^4_3. *arXiv preprint arXiv:2006.15933*, 2020.

[CM49] RH Cameron and WT Martin. The transformation of Wiener integrals by nonlinear transformations. *Transactions of the American Mathematical Society*, 66(2):253–283, 1949.

[CMP95] M Cassandro, R Marra, and E Presutti. Corrections to the critical temperature in 2d Ising systems with kac potentials. *Journal of Statistical Physics*, 78(3-4):1131–1138, 1995.

[CP00] Raphaël Cerf and Ágoston Pisztora. On the Wulff crystal in the Ising model. *The Annals of Probability*, pages 947–1017, 2000.

[Cru83a] Ana Bela Cruzeiro. Équations différentielles ordinaires: non explosion et mesures quasi-invariantes. *Journal of Functional Analysis*, 54(2):193–205, 1983.

[Cru83b] Ana Bela Cruzeiro. Équations différentielles sur l'espace de Wiener et formules de Cameron-Martin non-linéaires. *Journal of Functional Analysis*, 54(2):206–227, 1983.

[DE11] Paul Dupuis and Richard S Ellis. *A weak convergence approach to the theory of large deviations*, volume 902. John Wiley & Sons, 2011.

[DKS89] R. L. Dobrushin, R. Kotecký, and S. B. Shlosman. Equilibrium crystal shapes—a microscopic proof of the Wulff construction. In *Stochastic methods in mathematics and physics (Karpacz, 1988)*, pages 221–229. World Sci. Publ., Teaneck, NJ, 1989.

[DKS92] R. Dobrushin, R. Kotecký, and S. Shlosman. *Wulff construction: A global shape from local interaction*, volume 104 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992.

[DLS78] Freeman J Dyson, Elliott H Lieb, and Barry Simon. Phase transitions in quantum spin systems with isotropic and nonisotropic interactions. In *Statistical Mechanics*, pages 163–211. Springer, 1978.

[Dob65] Roland L Dobrushin. Existence of a phase transition in two-dimensional and three-dimensional Ising models. *Theory of Probability & Its Applications*, 10(2):193–213, 1965.

[Dob72] Roland L'vovich Dobrushin. Gibbs state describing coexistence of phases for a three-dimensional ising model. *Teoriya Veroyatnostei i ee Primeneniya*, 17(4):619–639, 1972.

[DPD03] Giuseppe Da Prato and Arnaud Debussche. Strong solutions to the stochastic quantization equations. *The Annals of Probability*, 31(4):1900–1916, 2003.

[DPZ88] G Da Prato and J Zabczyk. A note on semilinear stochastic equations. *Differential and Integral Equations*, 1(2):143–155, 1988.

[DPZ14] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*. Cambridge university press, 2014.

[Ell85] Richard S Ellis. *Entropy, large deviations, and statistical mechanics*. Springer, 1985.

[Fel74] Joel Feldman. The $\lambda\phi_3^4$ field theory in a finite volume. *Communications in Mathematical Physics*, 37(2):93–120, 1974.

[FILS78] Jürg Fröhlich, Robert Israel, Elliot H Lieb, and Barry Simon. Phase transitions and reflection positivity. i. general theory and long range lattice models. In *Statistical Mechanics*, pages 213–246. Springer, 1978.

[FILS80] Jürg Fröhlich, Robert B Israel, Elliott H Lieb, and Barry Simon. Phase transitions and reflection positivity. ii. lattice systems with short-range and coulomb interactions. In *Statistical Mechanics*, pages 247–297. Springer, 1980.

[FO76] Joel S. Feldman and Konrad Osterwalder. The Wightman axioms and the mass gap for weakly coupled $(\Phi^4)_3$ quantum field theories. *Annals of Physics*, 97(1):80–135, 1976.

[FO88] H Föllmer and M Ort. Large deviations and surface entropy for Markov-fields. *Astérisque*, (157-58):173–190, 1988.

[Föl85] Hans Föllmer. An entropy approach to the time reversal of diffusion processes. In *Stochastic Differential Systems Filtering and Control*, pages 156–163. Springer, 1985.

[FR12] Daniel Faraco and Keith M Rogers. The Sobolev norm of characteristic functions with applications to the Calderón inverse problem. *The Quarterly Journal of Mathematics*, 64(1):133–147, 2012.

[Frö82] Jürg Fröhlich. On the triviality of $\lambda\phi_d^4$ theories and the approach to the critical point in d (-)> 4 dimensions. *Nuclear Physics B*, 200(2):281–296, 1982.

[FS06] Wendell H Fleming and Halil Mete Soner. *Controlled Markov processes and viscosity solutions*, volume 25. Springer Science & Business Media, 2006.

[FSS76] Jürg Fröhlich, Barry Simon, and Thomas Spencer. Infrared bounds, phase transitions and continuous symmetry breaking. *Communications in Mathematical Physics*, 50(1):79–95, 1976.

[FT19] Justin Forlano and William J Trenberth. On the transport of Gaussian measures under the one-dimensional fractional nonlinear Schrödinger equations. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 36, pages 1987–2025. Elsevier, 2019.

[GH18] Massimiliano Gubinelli and Martina Hofmanová. A PDE construction of the Euclidean ϕ_3^4 quantum field theory. *arXiv preprint arXiv:1810.01700*, 2018.

[GH19] Massimiliano Gubinelli and Martina Hofmanová. Global solutions to elliptic and parabolic Φ^4 models in Euclidean space. *Communications in Mathematical Physics*, 368(3):1201–1266, 2019.

[Gid79] Basilis Gidas. The Glimm-Jaffe-Spencer expansion for the classical boundary conditions and coexistence of phases in the φ_2^4 euclidean (quantum) field theory. *Annals of Physics*, 118(1):18–83, 1979.

[GIP15] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. In *Forum of Mathematics, Pi*, volume 3. Cambridge University Press, 2015.

[GJ73] James Glimm and Arthur Jaffe. Positivity of the φ_3^4 Hamiltonian. *Fortschritte der Physik*, 21(7):327–376, 1973.

[GJ85] James Glimm and Arthur Jaffe. Critical problems in quantum fields. In *Quantum Field Theory and Statistical Mechanics*, pages 329–347. Springer, 1985.

[GJ87] James Glimm and Arthur Jaffe. *Quantum physics: a functional integral point of view*. Springer-Verlag, New York, second edition, 1987.

[GJS74] James Glimm, Arthur Jaffe, and Thomas Spencer. The Wightman axioms and particle structure in the $p(\varphi)_2$ quantum field model. *Annals of Mathematics*, pages 585–632, 1974.

[GJS75] James Glimm, Arthur Jaffe, and Thomas Spencer. Phase transitions for ϕ_2^4 quantum fields. *Communication in Mathematical Physics*, 45(3):203–216, 1975.

[GJS76a] James Glimm, Arthur Jaffe, and Thomas Spencer. A convergent expansion about mean field theory. I. The expansion. *Annals of Physics*, 101(2):610–630, 1976.

[GJS76b] James Glimm, Arthur Jaffe, and Thomas Spencer. A convergent expansion about mean field theory. II. Convergence of the expansion. *Annals of Physics*, 101(2):631–669, 1976.

[GJS76c] James Glimm, Arthur Jaffe, and Thomas Spencer. Existence of phase transitions for ϕ_2^4 quantum fields. In *Les méthodes mathématiques de la théorie quantique des champs (Colloq. Internat. CNRS, No. 248, Marseille, 1975)*, pages 175–184. 1976.

[Gli68] James Glimm. Boson fields with the $:\phi^4:$ interaction in three dimensions. *Communications in Mathematical Physics*, 10(1):1–47, 1968.

[GOTW18] Trishen S Gunaratnam, Tadahiro Oh, Nikolay Tzvetkov, and Hendrik Weber. Quasi-invariant Gaussian measures for the nonlinear wave equation in three dimensions. *arXiv preprint arXiv:1808.03158*, 2018.

[Gra09] Loukas Grafakos. *Modern fourier analysis*, volume 250. Springer, 2009.

[Gri64] Robert B Griffiths. Peierls proof of spontaneous magnetization in a two-dimensional Ising ferromagnet. *Physical Review*, 136(2A):A437, 1964.

[GRS75] Francesco Guerra, Lon Rosen, and Barry Simon. The $p(\varphi)_2$ Euclidean quantum field theory as classical statistical mechanics. *Annals of Mathematics*, pages 111–189, 1975.

[GT15] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*. Springer, 2015.

[GUZ20] Massimiliano Gubinelli, B Ugurcan, and Immanuel Zachhuber. Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions. *Stochastics and Partial Differential Equations: Analysis and Computations*, 8(1):82–149, 2020.

[Hai14] Martin Hairer. A theory of regularity structures. *Inventiones Mathematicae*, 198(2):269–504, 2014.

[Hai16] Martin Hairer. Regularity structures and the dynamical Φ^4_3 model, current developments in mathematics 2014. *Int. Press, Somerville, MA*, pages 1–49, 2016.

[HI18] Martin Hairer and Massimo Iberti. Tightness of the Ising–Kac model on the two-dimensional torus. *Journal of Statistical Physics*, 171(4):632–655, 2018.

[Hig79] Y Higuchi. On the absence of non translation invariant Gibbs states for the two dimensional Ising model. In *Colloquia Math. Sociatatis Janos Bolyai*, volume 27, pages 517–534. Random fields, 1979.

[HM18a] Martin Hairer and Konstantin Matetski. Discretisations of rough stochastic PDEs. *The Annals of Probability*, 46(3):1651–1709, 2018.

[HM18b] Martin Hairer and Jonathan Mattingly. The strong Feller property for singular stochastic PDEs. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 54, pages 1314–1340. Institut Henri Poincaré, 2018.

[Hol91] Richard Holley. On the asymptotics of the spin-spin autocorrelation function in stochastic Ising models near the critical temperature. In *Spatial stochastic processes*, pages 89–104. Springer, 1991.

[HS19] Martin Hairer and Philipp Schönbauer. The support of singular stochastic PDEs. *arXiv preprint arXiv:1909.05526*, 2019.

[IS98] Dmitry Ioffe and Roberto H Schonmann. Dobrushin–Koteckỳ–Shlosman theorem up to the critical temperature. *Communications in Mathematical Physics*, 199(1):117–167, 1998.

[Kak48] Shizuo Kakutani. On equivalence of infinite product measures. *Annals of Mathematics*, pages 214–224, 1948.

[Kup16] Antti Kupiainen. Renormalization group and stochastic PDEs. In *Annales Henri Poincaré*, volume 17, pages 497–535. Springer, 2016.

[Leh13] Joseph Lehec. Representation formula for the entropy and functional inequalities. In *Annales de l'IHP Probabilités et statistiques*, volume 49, pages 885–899, 2013.

[Len20] Wilhelm Lenz. Beitrag zum Verständnis der magnetischen Erscheinungen in festen Körpern. *Phys. Zeitschr*, 21:613–615, 1920.

[Lit63] Walter Littman. The wave operator and l_p norms. *Journal of Mathematics and Mechanics*, pages 55–68, 1963.

[LRS88] Joel L Lebowitz, Harvey A Rose, and Eugene R Speer. Statistical mechanics of the nonlinear Schrödinger equation. *Journal of Statistical Physics*, 50(3-4):657–687, 1988.

[LS95] Hans Lindblad and Christopher D Sogge. On existence and scattering with minimal regularity for semilinear wave equations. *Journal of Functional Analysis*, 130(2):357–426, 1995.

[LS12] Eyal Lubetzky and Allan Sly. Critical Ising on the square lattice mixes in polynomial time. *Communications in Mathematical Physics*, 313(3):815–836, 2012.

[MO94] Fabio Martinelli and Enzo Olivieri. Approach to equilibrium of Glauber dynamics in the one phase region. i. the attractive case. *Communications in Mathematical Physics*, 161(3):447–486, 1994.

[MS67] Robert Adol'fovich Minlos and Yakov Grigor'evich Sinai. The phenomenon of "phase separation" at low temperatures in some lattice models of a gas. i. *Mathematics of the USSR-Sbornik*, 2(3):335, 1967.

[MS68] Robert Adol'fovich Minlos and Yakov Grigor'evich Sinai. The phenomenon of "separation of phases" at low temperatures in certain lattice models of a gas. ii. *Trudy Moskovskogo Matematicheskogo Obshchestva*, 19:113–178, 1968.

[MS77] J Magnen and R Sénéor. Phase space cell expansion and Borel summability for the Euclidean φ_3^4 theory. *Communications in Mathematical Physics*, 56(3):237–276, 1977.

[MW17a] Jean-Christophe Mourrat and Hendrik Weber. Convergence of the two-dimensional dynamic ising-kac model to. *Communications on Pure and Applied Mathematics*, 70(4):717–812, 2017.

[MW17b] Jean-Christophe Mourrat and Hendrik Weber. The dynamic Φ_3^4 model comes down from infinity. *Communication in Mathematical Physics*, 356(3):673–753, 2017.

[MW17c] Jean-Christophe Mourrat and Hendrik Weber. Global well-posedness of the dynamic ϕ_2^4 model in the plane. *The Annals of Probability*, 45(4):2398–2476, 2017.

[MW18] Augustin Moinat and Hendrik Weber. Space-time localisation for the dynamic Φ_3^4 model. *arXiv preprint arXiv:1811.05764*, 2018.

[MWX17] Jean-Christophe Mourrat, Hendrik Weber, and Weijun Xu. Construction of Φ_3^4 diagrams for pedestrians. In *From particle systems to partial differential equations*, volume 209 of *Springer Proc. Math. Stat.*, pages 1–46. Springer, Cham, 2017.

[Nel66] Edward Nelson. A quartic interaction in two dimensions. In *Mathematical Theory of Elementary Particles, Proc. Conf., Dedham, Mass., 1965*, pages 69–73. MIT Press, 1966.

[Nel73] Edward Nelson. The free Markoff field. *Journal of Functional Analysis*, 12:211–227, 1973.

[Nua06] David Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[Oll88] Stefano Olla. Large deviations for Gibbs random fields. *Probability Theory and Related Fields*, 77(3):343–357, 1988.

[OOT] Tadahiro Oh, Mamoru Okamoto, and N Tzvetkov. Uniqueness and non-uniqueness of the Gaussian free field evolution under the two-dimensional Wick ordered cubic wave equation. *preprint*.

[OP16] Tadahiro Oh and Oana Pocovnicu. Probabilistic global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on r^3. *Journal de Mathématiques Pures et Appliquées*, 105(3):342–366, 2016.

[OP17] Tadahiro Oh and Oana Pocovnicu. A remark on almost sure global well-posedness of the energy-critical defocusing nonlinear wave equations in the periodic setting. *Tohoku Mathematical Journal*, 69(3):455–481, 2017.

[OS73] Konrad Osterwalder and Robert Schrader. Axioms for Euclidean Green's functions. *Communications in Mathematical Physics*, 31(2):83–112, 1973.

[OS75] Konrad Osterwalder and Robert Schrader. Axioms for Euclidean Green's functions. pt. 2. *Communications in Mathematical Physics*, 42(3):281–305, 1975.

[OST18] Tadahiro Oh, Philippe Sosoe, and Nikolay Tzvetkov. An optimal regularity result on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation. *Journal de l'École polytechnique—Mathématiques*, 5:793–841, 2018.

[OT15] Tadahiro Oh and Nikolay Tzvetkov. On the transport of Gaussian measures under the flow of Hamiltonian pdes. *Séminaire Laurent Schwartz—EDP et applications*, pages 1–9, 2015.

[OT17] Tadahiro Oh and Nikolay Tzvetkov. Quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation. *Probability theory and related fields*, 169(3-4):1121–1168, 2017.

[OT20] Tadahiro Oh and Nikolay Tzvetkov. Quasi-invariant Gaussian measures for the two-dimensional defocusing cubic nonlinear wave equation. *Journal of the European Mathematical Society (JEMS)*, 22(6):1785–1826, 2020.

[OTT19] Tadahiro Oh, Yoshio Tsutsumi, and Nikolay Tzvetkov. Quasi-invariant Gaussian measures for the cubic nonlinear Schrödinger equation with third-order dispersion. *Comptes Rendus Mathematique*, 357(4):366–381, 2019.

[Par75] Yong Moon Park. Lattice approximation of the $(\lambda\varphi^4 - \mu\varphi)_3$ field theory in a finite volume. *Journal of Mathematical Physics*, 16(5):1065–1075, 1975.

[Pei36] Rudolf Peierls. On Ising's model of ferromagnetism. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 32, pages 477–481. Cambridge University Press, 1936.

[Per80] Juan C Peral. Lp estimates for the wave equation. *Journal of functional analysis*, 36(1):114–145, 1980.

[Pfi91] Charles-Edouard Pfister. Large deviations and phase separation in the two-dimensional Ising model. *Helvetica Physica Acta*, 64:953–1054, 1991.

[Pis96] Agoston Pisztora. Surface order large deviations for Ising, Potts and percolation models. *Probability Theory and Related Fields*, 104(4):427–466, 1996.

[Poc] Oana Pocovnicu. Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on $\mathbb{R}^d$.

[PTV19] Fabrice Planchon, Nikolay Tzvetkov, and Nicola Visciglia. Transport of Gaussian measures by the flow of the nonlinear Schrödinger equation. *Mathematische Annalen*, pages 1–35, 2019.

[PW81] Georgio Parisi and Yong Shi Wu. Perturbation theory without gauge fixing. *Sci. Sin*, 24(4):483–496, 1981.

[Ram74] Roald Ramer. On nonlinear transformations of Gaussian measures. *Journal of Functional Analysis*, 15(2):166–187, 1974.

[RY13] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293. Springer Science & Business Media, 2013.

[Sch87] Roberto H Schonmann. Second order large deviation estimates for ferromagnetic systems in the phase coexistence region. *Communications in Mathematical physics*, 112(3):409–422, 1987.

[SG73] Barry Simon and Robert B Griffiths. The $(\varphi^4)_2$ field theory as a classical Ising model. *Communications in Mathematical Physics*, 33(2):145–164, 1973.

[Shl86] Semen Bensionovich Shlosman. The method of reflection positivity in the mathematical theory of first-order phase transitions. *Russian Mathematical Surveys*, 41:83–134, 1986.

[Sog93] Christopher D Sogge. l^p estimates for the wave equation and applications. *Journées équations aux dérivées partielles*, pages 1–12, 1993.

[STX20] Philippe Sosoe, William J Trenberth, and Tianhao Xian. Quasi-invariance of fractional Gaussian fields by the nonlinear wave equation with polynomial nonlinearity. *Differential and Integral Equations*, 33(7/8):393–430, 2020.

[Tho89] Lawrence E Thomas. Bound on the mass gap for finite volume stochastic Ising models at low temperature. *Communications in Mathematical Physics*, 126(1):1–11, 1989.

[TV14] Nikolay Tzvetkov and Nicola Visciglia. Invariant measures and long-time behavior for the benjamin–ono equation. *International Mathematics Research Notices*, 2014(17):4679–4714, 2014.

[TW18] Pavlos Tsatsoulis and Hendrik Weber. Spectral gap for the stochastic quantization equation on the 2-dimensional torus. *Annales de L'Institut Henri Poincaré Probabilité et Statistiques*, 54(3):1204–1249, 2018.

[Tzv08] Nikolay Tzvetkov. Invariant measures for the defocusing nonlinear schrödinger equation. In *Annales de l'Institut Fourier*, volume 58, pages 2543–2604, 2008.

[Tzv15] Nikolay Tzvetkov. Quasi-invariant Gaussian measures for one dimensional Hamiltonian PDE's. *Forum Math. Sigma*, 3, 2015.

[Üst14] Ali Süleyman Üstünel. Variational calculation of Laplace transforms via entropy on Wiener space and applications. *Journal of Functional Analysis*, 267(8):3058–3083, 2014.

[Wig56] Arthur S Wightman. Quantum field theory in terms of vacuum expectation values. *Physical Review*, 101(2):860, 1956.

[Yud63] Victor Iosifovich Yudovich. Non-stationary flows of an ideal incompressible fluid. *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, 3(6):1032–1066, 1963.

[Zab89] Jerzy Zabczyk. Symmetric solutions of semilinear stochastic equations. In *Stochastic Partial Differential Equations and Applications II*, pages 237–256. Springer, 1989.

[ZZ18a] Rongchan Zhu and Xiangchan Zhu. Dirichlet form associated with the ϕ_3^4 model. *Electronic Journal of Probability*, 23, 2018.

[ZZ18b] Rongchan Zhu and Xiangchan Zhu. Lattice approximation to the dynamical ϕ_3^4 model. *The Annals of Probability*, 46(1):397–455, 2018.