

ON EXISTENCE AND UNIQUENESS OF A MODIFIED CARRYING SIMPLEX FOR DISCRETE KOLMOGOROV SYSTEMS

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ABSTRACT. For a C^1 map T from $C = [0, +\infty)^N$ to C of the form $T_i(x) = x_i f_i(x)$, the dynamical system $x(n) = T^n(x)$ as a population model is competitive if $\frac{\partial f_i}{\partial x_j} \leq 0$ ($i \neq j$). A well known theorem for competitive systems, presented by Hirsch (J. Bio. Dyn. 2 (2008) 169–179) and proved by Ruiz-Herrera (J. Differ. Equ. Appl. 19 (2013) 96–113) with various versions by others, states that, under certain conditions, the system has a compact invariant surface $\Sigma \subset C$ that is homeomorphic to $\Delta^{N-1} = \{x \in C : x_1 + \dots + x_N = 1\}$, attracting all the points of $C \setminus \{0\}$, and called carrying simplex. The theorem has been well accepted with a large number of citations. In this paper, we point out that one of its conditions requiring all the N^2 entries of the Jacobian matrix $Df = (\frac{\partial f_i}{\partial x_j})$ to be negative is unnecessarily strong and too restrictive. We prove the existence and uniqueness of a modified carrying simplex by reducing that condition to requiring every entry of Df to be nonpositive and each f_i is strictly decreasing in x_i . As an example of applications of the main result, sufficient conditions are provided for vanishing species and dominance of one species over others.

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1. INTRODUCTION

In this paper, we are concerned with the global asymptotic behaviour of the discrete dynamical system

$$(1) \quad x(n) = T^n(x), \quad x \in C, \quad n \in \mathbb{N},$$

where $C = \mathbb{R}_+^N = [0, +\infty)^N$, $\mathbb{N} = \{0, 1, 2, \dots\}$ and the map $T : C \rightarrow C$ has the form

$$(2) \quad T_i(x) = x_i f_i(x), \quad i \in I_N = \{1, 2, \dots, N\}$$

and $f \in C^1(C, C)$ with $f_i(x) > 0$ for all $x \in C$ and $i \in I_N$. System (1) is a typical mathematical model for the population dynamics of a community of N species, where each $x_i(n)$ represents the population size or density at time n (at the end of n th time period), and the function $f_i(x)$ denotes the per capita growth rate, of the i th species. If $\frac{\partial f_i}{\partial x_j} \leq 0$

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for all $i, j \in I_N$ with $i \neq j$, then increase of the j th population reduces the per capita growth rate of the i th species, so (1) models the population dynamics of a community of competitive species.

System (1) and its various particular instances as models have attracted huge interests from researchers in the last two decades. One of the important and influential developments is the existence of a *carrying simplex* $\Sigma \subset C$: a compact invariant hypersurface homeomorphic to $\Delta^{N-1} = \{x \in C : x_1 + \cdots + x_N = 1\}$ such that every trajectory except the origin is asymptotic with a trajectory in Σ . Since Σ attracts all the points of $C \setminus \{0\}$, the dynamics of (1) on C is essentially described by the dynamics on Σ . The carrying simplex theory was originally established by Hirsch [10] (see [13] for latest update) for competitive Kolmogorov systems of differential equations. Since then the idea of a carrying simplex for discrete systems gradually appeared in literature (see [19], [20], [14] for example). But a more accepted theorem for existence and uniqueness of a carrying simplex for (1) was given by Hirsch [11] without proof. Then Ruiz-Herrera [9] presented a more general theorem covering Hirsch's result with a complete proof.

For any $x, y \in C$, we write $x \leq y$ or $y \geq x$ if $x_i \leq y_i$ for all $i \in I_N$; $x < y$ or $y > x$ if $x \leq y$ but $x \neq y$; $x \ll y$ or $y \gg x$ if $x_i < y_i$ for all $i \in I_N$. The map T given by (2) is said to be *retrotone* in a subset $X \subset C$ if for any $x, y \in X$, $T(x) < T(y)$ implies $x_i < y_i$ for all $i \in I(y) = \{j \in I_N : y_j \neq 0\}$. Let $[0, r] = \{x \in C : 0 \leq x \leq r\}$. The theorem below is Theorem 6.1 in [9].

Theorem 1.1. *Assume that T with $T([0, r]) \subset [0, r]$ for some $r \gg 0$ satisfies the following conditions:*

- (i) *For each $i \in I_N$, the map T restricted to the positive half x_i -axis has a fixed point $q_i e_i$ with $q_i > 0$, e_i the i th standard unit vector and $q \ll r$.*
- (ii) *T is retrotone and locally one to one in $[0, r]$.*
- (iii) *For any $x, y \in [0, r]$, if $T(x) < T(y)$ then, for each $j \in I_N$, either $x_j = 0$ or $f_j(x) > f_j(y)$.*

Then the map admits a carrying simplex Σ .

Note that Theorem 1.1 can be only applied to the system restricted to the space $[0, r] \subset C$ if no condition for T on $C \setminus [0, r]$ is provided. However, if for any compact set $S \subset C$ there is a $k \in \mathbb{N}$ such that $T^k(S) \subset [0, r]$, then Theorem 1.1 can be applied directly to the system on C .

When f on C is a C^1 map, T is also a C^1 map with Jacobian matrix

$$(3) \quad DT(x) = \text{diag}(f_1(x), \dots, f_N(x))(I - M(x)),$$

where I is the identity matrix and

$$(4) \quad M(x) = (M_{ij}(x)) = \left(-\frac{x_i}{f_i(x)} \frac{\partial f_i}{\partial x_j}(x) \right)_{N \times N}.$$

Then, by Lemma 4.1, Corollary 6.1 and Remark 6.4 in [9], Theorem 1.1 has the following version with easily checkable conditions.

Theorem 1.2. *Assume that T satisfies the following conditions:*

- (i) *For each $i \in I_N$, the map T restricted to the positive half x_i -axis has a fixed point $q_i e_i$ with $q_i > 0$, e_i the i th standard unit vector and $q \ll r$ for some $r \in C$.*
- (ii) *All entries of the Jacobian Df are negative.*
- (iii) *The spectral radius of $M(x)$ satisfies $\rho(M(x)) < 1$ for all $x \in [0, q] \setminus \{0\}$.*

Then the map admits a carrying simplex Σ .

A more user-friendly variation of Theorem 1.2 given by Jiang and Niu [16, Theorem 3.1] is the above theorem with simply a replacement of condition (iii) by (iii)' below:

- (iii)' For each $x \in [0, q] \setminus \{0\}$ with $I(x) = \{j \in I_N : x_j > 0\}$, either

$$f_i(x) + \sum_{j \in I(x)} x_j \frac{\partial f_i}{\partial x_j}(x) > 0 \quad \forall i \in I(x)$$

or

$$f_i(x) + \sum_{j \in I(x)} x_i \frac{\partial f_i}{\partial x_j}(x) > 0 \quad \forall i \in I(x).$$

A carrying simplex Σ has the important and interesting features: compact, invariant, unordered ($p \leq q$ implies $p = q$ for $p, q \in \Sigma$), homeomorphic to Δ^{N-1} by radial projection, and attracting all the points of $C \setminus \{0\}$. Therefore, if (1) admits a carrying simplex, the dynamics of the system on the N -dimensional space C is essentially described by the dynamics on this $(N - 1)$ -dimensional hypersurface Σ . Due to these features of Σ , Theorem 1.2 is phenomenal and lays the foundation for further investigations. There are a large number of applications of this theorem, the following are just a few examples.

Ruiz-Herrera [9] investigated exclusion and dominance utilizing the existence of a carrying simplex. Jiang and Niu [15, 16] and Gyllenberg et al. [8, 6] dealt with some well known three-dimensional competitive models. Based on the existence of a carrying simplex, they classified the systems into 33 topologically equivalent classes and gave a phase portrait on Σ for each class. Jiang, Niu and Wang [17] studied heteroclinic cycles via carrying simplex. Balreira et al. [4] and Gyllenberg et al. [7] provided criteria for global stability of an interior fixed point based on the existence of a carrying simplex. Baigent [1, 2] investigated the geometric feature of a carrying simplex and found conditions for Σ to be convex. Baigent and Hou [3] and Hou [12] provided split Lyapunov function method

and geometric method for global stability. Although these methods were not based on the existence of a carrying simplex, comments and comparisons with those using carrying simplex were made there.

We note that condition (ii) in Theorem 1.2 is very restrictive; it excludes the possibility of applying the theorem to systems with some zero entries of Df . But actually, condition (ii) is too strong and unnecessary, a compact invariant set attracting all the points of $C \setminus \{0\}$ with most of the features of a carrying simplex may still exist even if $\frac{\partial f_i}{\partial x_j} = 0$ for some distinct $i, j \in I_N$.

The aim of this paper is to prove the existence and uniqueness of a modified carrying simplex under a much weaker condition than (ii): instead of (ii) requiring all N^2 entries of Df to be negative, we require each entry of Df to be nonpositive, with each f_i strictly decreasing in x_i , on a compact set. We shall present the main results in section 2 and leave the proofs to section 5. In section 3, we present some results on dominant species and vanishing species as an application of the main results. In section 4, we deal with some known models as examples. We finally conclude the paper in section 6.

2. NOTATION AND MAIN RESULTS

For $C = \mathbb{R}_+^N$ we let $\dot{C} = \{x \in C : \forall i \in I_N, x_i > 0\}$ and $\partial C = C \setminus \dot{C}$. Then \dot{C} is the interior of C and ∂C is the boundary of C . The part of ∂C restricted to the i th coordinate plane and the part restricted to the positive half x_i -axis are denoted by π_i and X_i respectively, i.e.

$$\begin{aligned}\pi_i &= \{x \in C : x_i = 0\}, i \in I_N, \\ X_i &= \{x \in C : x_i > 0, \forall j \in I_N \setminus \{i\}, x_j = 0\}, i \in I_N.\end{aligned}$$

Denote the i th standard unit vector by e_i , i.e. the i th component of e_i is 1 and others are 0. For any nonempty subset $I \subset I_N$, define

$$\begin{aligned}C_I &= \{x \in C : \forall j \in I_N \setminus I, x_j = 0\}, \\ \dot{C}_I &= \{x \in C_I : \forall i \in I, x_i > 0\}.\end{aligned}$$

For any $x, y \in C_I$, by writing $x \leq_I y$ we mean $x_i \leq y_i$ for all $i \in I$; we write $x <_I y$ if $x \leq_I y$ but $x \neq y$; and we write $x \ll_I y$ if $y - x \in \dot{C}_I$. We may also use $y \geq_I x$, $y >_I x$ and $y \gg_I x$ for $x \leq_I y$, $x <_I y$ and $x \ll_I y$ respectively. If $I = I_N$, we simply drop the subscript “ I ” from these inequalities. For any $a, b \in C$ with $a \leq b$, we let $[a, b] = \{x \in C : a \leq x \leq b\}$. Then $[a, b]$ is a k -dimensional cell if $b - a$ has exactly k positive components. For each $x \in C$, the positive limit set $\omega(x)$ of $T^n(x)$ is defined by

$$\omega(x) = \bigcap_{n=1}^{\infty} \overline{\{T^k(x) : k \geq n\}},$$

where \bar{A} denotes the closure of any set A . If T is invertible and $T^{-n}(x)$ exist for all $n \in \mathbb{N}$, the negative limit set $\alpha(x)$ is defined by

$$\alpha(x) = \bigcap_{n=1}^{\infty} \overline{\{T^{-k}(x) : k \geq n\}}.$$

Also, the whole trajectory of x is denoted by $\gamma(x) = \{T^n(x) : n \in \mathbb{Z}\}$.

Suppose a simply connected closed set $S \subset C \setminus \{0\}$ is an $(N-1)$ -dimensional hypersurface which divides C into three mutually exclusive subsets S^- , S and S^+ with $0 \in S^-$ and $C = S^- \cup S \cup S^+$. A point $p \in C$ is said to be *below* (*on* or *above*) S if $p \in S^-$ (S or S^+). For any nonempty subset $S_0 \subset C$, S_0 is said to be *below* (*above*) S if $S_0 \subset S^- \cup S$ ($S \cup S^+$); S_0 is said to be *strictly below* (*strictly above*) S if $S_0 \subset S^-$ (S^+).

Let B be either C or a positively invariant $[0, r]$ for some $r \in \dot{C}$. For convenience, we define the concept of a modified carrying simplex as follows.

Definition 2.1. A nonempty set $\Sigma \subset B \setminus \{0\}$ is called a modified carrying simplex of (1) if Σ meets the following requirements.

- (i) Σ is compact, invariant and homeomorphic to Δ^{N-1} by radial projection.
- (ii) Σ attracts all the points of $B \setminus \{0\}$, i.e. $\omega(x) \subset \Sigma$ for each $x \in B \setminus \{0\}$.

Moreover, if x is below Σ with a nonempty support $I(x) \subset I_N$, then there is a $y \in \Sigma$ with $I(y) = I(x)$ such that $\lim_{n \rightarrow +\infty} (T^n(x) - T^n(y)) = 0$.

Note that the “unordered” property of Σ is not mentioned in the above definition. We shall see in Remark 2.1 (f) below that the unordered property of Σ here is slightly different from that for carrying simplex in Hirsch [11], Ruiz-Herrera [9] and the literature. But the main difference between modified carrying simplex and the carrying simplex in literature is that the latter requires every trajectory in $B \setminus \{0\}$ to be asymptotic to one in Σ whereas the former requires every nontrivial trajectory below Σ to be asymptotic to one in Σ and Σ to attract all the points of $B \setminus \{0\}$. Obviously, the concept of a modified carrying simplex is more general and it includes carrying simplex as a particular class.

Definition 2.2. The map $T : C \rightarrow C$ defined by (2) is said to be weakly retrotone in a subset $X \subset C$ if for $x, y \in X$ with $T(x) > T(y)$ and $T(x) - T(y) \in \dot{C}_I$ for some $I \subset I_N$, then $x > y$ and $x_i > y_i$ for all $i \in I$.

Comparing this with the definition of retrotone given in section 1 we see that if T is retrotone then it is weakly retrotone, but not vice versa.

Theorem 2.3. *Assume that T defined by (2) with $T([0, r]) \subset [0, r]$ for some $r \in \dot{C}$ satisfies the following conditions:*

- (i) For each $i \in I_N$, the map T restricted to X_i has a fixed point $q_i e_i$ with $q_i > 0$ and $q \ll r$.
- (ii) T is weakly retrotone and locally one to one in $[0, r]$.
- (iii) For any $x, y \in [0, r]$, if $T(x) < T(y)$ and $T(y) - T(x) \in \dot{C}_I$ for some $I \subset I_N$ then, for each $j \in I$, either $x_j = 0$ or $f_j(x) > f_j(y)$.

Then 0 is a repellor with the basin of repulsion $\mathcal{B}(0) \subset [0, r]$, (1) has a unique modified carrying simplex Σ and $\Sigma = \overline{\mathcal{B}(0)} \setminus (\{0\} \cup \mathcal{B}(0))$. Moreover, for each $p \in \Sigma$ and every $q \in [0, r] \setminus \{0\}$ with $q < p$, we have $\alpha(q) \subset \pi_i$ provided $q_i < p_i$.

Remark 2.1

- (a) Condition (i) of Theorem 2.3 is the same as that of Theorem 1.1 but conditions (ii) and (iii) are weaker than those of Theorem 1.1.
- (b) Condition (ii) and the definition (2) imply that $T : [0, r] \rightarrow T([0, r])$ is a homeomorphism. This follows from the local one to one property of T on $[0, r]$, $T(x) = 0$ if and only if $x = 0$, and Lemma 4.1 in [9].
- (c) Condition (ii) implies that, for each $i \in I_N$, the function $T_i(se_i)$ is strictly increasing for $s \in [0, r_i]$. Indeed, from (b) above we know that T is one to one on $[0, r]$. As $T_j(se_i) = 0$ and $T_i(se_i) > 0$ for all $j \in I_N \setminus \{i\}$ and $s \in (0, r_i]$, the one to one property of T ensures that $T_i(s_1 e_i) \neq T_i(s_2 e_i)$ for $0 < s_1 < s_2 \leq r_i$. By (ii) we must have $T_i(s_1 e_i) < T_i(s_2 e_i)$ for $0 < s_1 < s_2 \leq r_i$. By continuity, $T_i(se_i)$ is strictly increasing for $s \in [0, r_i]$.
- (d) Conditions (ii) and (iii) imply that, for each $i \in I_N$, $f_i(se_i)$ is strictly decreasing for $s \in [0, r_i]$. Indeed, for $0 < s_1 < s_2 \leq r_i$, from (c) above we see that $0 < T_i(s_1 e_i) < T_i(s_2 e_i)$ and $T(s_1 e_i) < T(s_2 e_i)$. From (iii) we have $f_i(s_1 e_i) > f_i(s_2 e_i)$. By continuity of f , $f_i(se_i)$ is strictly decreasing for $s \in [0, r_i]$.
- (e) The conclusion that the origin is a repellor immediately follows from conditions (i)–(iii). In fact, condition (i) implies that $f_i(q_i e_i) = 1$ for all $i \in I_N$. From (d) above we have $f_i(0) > 1$ for all $i \in I_N$. As each $f_i(0)$ is an eigenvalue of $DT(0)$, all eigenvalues of $DT(0)$ are greater than 1 so 0 is a repellor.
- (f) From the conclusion we see that for each $p \in \Sigma$, there is a nonempty $I \subset I_N$ such that $p \in \dot{C}_I$. Then, for each $q \in [0, r]$ with $q \ll_I p$, we have $\alpha(q) \subset \pi_i$ for all $i \in I$. As $q \ll_I p$ and $p \in \dot{C}_I$ imply that $q \in C_I$, we have $\alpha(q) \subset \bigcap_{i=1}^N \pi_i = \{0\}$, so $\alpha(q) = \{0\} \not\subset \Sigma$. Since $\alpha(q) \subset \Sigma$ if $q \in \Sigma$ by the invariance and compactness of Σ , we must have $q \notin \Sigma$. This shows that Σ is unordered in a strict sense: for any nonempty $I \subset I_N$, any $p \in \dot{C}_I$ and any $q \ll_I p$, we cannot have both $p \in \Sigma$ and $q \in \Sigma$. In other words, Σ is unordered in the sense of \ll_I for any nonempty $I \subset I_N$: there are no distinct points $p, q \in \Sigma \cap C_I$ such that $p \ll_I q$. However, due to the possibility of $\frac{\partial f_i(x)}{\partial x_j} = 0$ for some $i \neq j$ and some x , Σ does allow ordered

points on it in the sense of $<$, i.e. $p, q \in \Sigma$ with $p < q$. This is demonstrated by the trivial example below.

Example Consider the system (1) with T given by

$$(5) \quad T_i(x) = x_i g_i(x_i), i \in I_N,$$

where each $g_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ is positive, continuous, $0 < g_i(u) < 1$ for $u \geq r_i > q_i > 0$, $g_i(q_i) = 1$, $g_i \in C^1([0, r_i], \mathbb{R})$, $g_i'(u) < 0$, and $g_i(u) + u g_i'(u) > 0$ for $u \in [0, r_i]$. Then T satisfies all the conditions of Theorem 2.3, so it has a unique modified carrying simplex Σ . Note that system (1) with T defined by (5) is a trivial case of (1) with T defined by (2) when there is no interaction between distinct component equations of the system. Since q_i is the globally attracting equilibrium of the i th component equation on the positive x_i -axis, Σ is the upper boundary surface of the cell $[0, q]$, i.e.

$$\Sigma = \{x \in [0, q] : x_i = q_i \text{ for some } i \in I_N\}.$$

Clearly, $q \in \Sigma$ and for each $p \in \Sigma \setminus \{q\}$, we have $p < q$. Thus, ordered points in the sense of $<$ are permitted on Σ .

Now utilising DT and Df , we obtain conditions which guarantee conditions (ii) and (iii) and the following version of Theorem 2.3 with easily checkable conditions. Consider the matrix $M(x)$ given by (4) and

$$(6) \quad \tilde{M}(x) = (\tilde{M}_{ij}(x)) = \left(-\frac{x_j}{f_i(x)} \frac{\partial f_i}{\partial x_j}(x) \right)_{N \times N}.$$

Theorem 2.4. *Assume that T given by (2) satisfies the following conditions:*

- (i) *For each $i \in I_N$, the map T restricted to X_i has a fixed point $q_i e_i$ with $q_i > 0$ and $q \ll r$ for some $r \in \dot{C}$.*
- (ii) *The entries of the Jacobian Df satisfy*

$$(7) \quad \forall x \in [0, r], \forall i, j \in I_N, \frac{\partial f_i}{\partial x_j}(x) \leq 0,$$

and f_i is strictly decreasing in $x_i \in [0, r_i]$ for $x \in [0, r]$.

- (iii) *For each $x \in [0, q] \setminus \{0\}$, either $\rho(M(x)) < 1$ for $M(x)$ given by (4) or $\rho(\tilde{M}(x)) < 1$ for $\tilde{M}(x)$ given by (6).*

Then 0 is a repeller with the basin of repulsion $\mathcal{B}(0) \subset [0, r]$, (1) has a unique modified carrying simplex Σ and $\Sigma = \overline{\mathcal{B}(0)} \setminus (\{0\} \cup \mathcal{B}(0))$. Moreover, for each $p \in \Sigma$ and every $q \in [0, r] \setminus \{0\}$ with $q < p$, we have $\alpha(q) \subset \pi_i$ provided $q_i < p_i$.

Remark 2.2

- (a) When $\frac{\partial f_i}{\partial x_i} \leq 0$, a sufficient condition for f_i to be strictly decreasing for $x_i \in [0, r_i]$, $x \in [0, r]$ with x_j fixed for all $j \in I_N \setminus \{i\}$, is that the set Z_i of zeros of $\frac{\partial f_i}{\partial x_i}$ in $[0, r_i]$ is either empty or finite or infinite with only a finite number of accumulation points. In particular, when each Z_i is empty, condition (ii) in Theorem 2.4 can be replaced by

(ii)* For all $i, j \in I_N$, the entries of the Jacobian Df satisfy

$$(8) \quad \forall x \in [0, r], \quad \frac{\partial f_i}{\partial x_i}(x) < 0, \quad \frac{\partial f_i}{\partial x_j}(x) \leq 0.$$

- (b) Comparing Theorem 2.4 with Theorem 1.2, we see that condition (i) of Theorem 2.4 is the same as (i) of Theorem 1.2 and (iii) of Theorem 2.4 has one more choice than (iii) of Theorem 1.2, but condition (ii) of Theorem 2.4 only requires each entry of Df to be nonnegative instead of N^2 entries of Df to be strictly negative in Theorem 1.2, plus the strictly decreasing requirement of each f_i in x_i . Even if (ii) is replaced by the stronger condition (ii)* above, it only requires N diagonal entries of Df to be negative. From this point of view, with a trade off of having a modified carrying simplex rather than the well known carrying simplex, we have significantly reduced the cost and generalised the existing results.

- (c) Under condition (ii) of Theorem 2.4, if

$$(9) \quad f_i(x) + \sum_{j=1}^N x_j \frac{\partial f_i}{\partial x_j}(x) > 0 \quad \forall i \in I_N,$$

using one type of matrix norm we have

$$\|M(x)\| = \max_{i \in I_N} \sum_{j=1}^N \left| \frac{x_j}{f_i(x)} \frac{\partial f_i}{\partial x_j}(x) \right| < 1.$$

By Theorem 6.1.3 in [18], $\rho(M(x)) \leq \|M(x)\|$. Thus, (9) is a sufficient condition for $\rho(M(x)) < 1$. By the same reason, if

$$(10) \quad f_i(x) + \sum_{j=1}^N x_j \frac{\partial f_i}{\partial x_j}(x) > 0 \quad \forall i \in I_N,$$

then $\rho(\tilde{M}(x)) \leq \|\tilde{M}(x)\| < 1$. Therefore, condition (iii) of Theorem 2.4 is met if (9) or (10) holds for each $x \in [0, q] \setminus \{0\}$.

Corollary 1. *Under the conditions of Theorem 2.3 or Theorem 2.4, the following conclusions hold.*

- (i) *For any periodic orbit $\gamma \subset \Sigma$, the points on γ are unordered, i.e. if $p, q \in \gamma$ with $p \leq q$ then $p = q$.*

- (ii) For any $x \in \Sigma$, if there are two points $p, q \in \gamma(x)$ satisfying $p < q$ then $\alpha(x)$ consists of either a single fixed point or a periodic orbit.

Proof. (i) Suppose there are two points $p, q \in \gamma$ satisfying $p < q$. Then there is at least one $i \in I_N$ such that $p_i < q_i$. From Theorem 2.3 we have $\alpha(p) \subset \pi_i$, so $q \notin \alpha(p)$, a contradiction to $q \in \gamma = \alpha(p)$ due to the periodicity of γ . Therefore, γ is unordered.

(ii) By $x \in \Sigma$ we have $\gamma(x) \subset \Sigma$ and $\alpha(x) \subset \Sigma$. Since $p, q \in \gamma(x)$ with $p < q$, we have $T(T^{-1}(p)) = p < q = T(T^{-1}(q))$. Then the weakly retrotone property of T implies that $T^{-1}(p) < T^{-1}(q)$ and $T^{-n}(p) < T^{-n}(q)$ for all $n \in \mathbb{N}$. For each $i \in I_N$, if there is an $n \in \mathbb{N}$ such that $(T^{-n}(p))_i < (T^{-n}(q))_i$, by Theorem 2.3 we have $\alpha(x) = \alpha(p) \subset \pi_i$; otherwise, we have $(T^{-n}(p))_i = (T^{-n}(q))_i$ for all $n \in \mathbb{N}$. Thus, there is a proper subset $I \subset I_N$ such that $\alpha(x) \subset \pi_i$ for each $i \in I$ and $(T^{-n}(p))_j = (T^{-n}(q))_j$ for all $n \in \mathbb{N}$ and $j \in I_N \setminus I$. As p and q are two distinct points on $\gamma(x)$, there is an $n_1 > 0$ such that either $T^{n_1}(p) = q$ or $T^{n_1}(q) = p$. Hence, since the component $(T^n(p))_j$ is an n_1 -periodic function for $n \in \mathbb{N}$ for each $j \in I_N \setminus I$, we obtain

$$\begin{aligned} \alpha(x) &= \{T^k(y) : k \in \{0, 1, \dots, n_1 - 1\}, (T^k(y))_i = 0, i \in I; \\ &\quad (T^k(y))_j = (T^{k-n_1}(p))_j, j \in I_N \setminus I\} \end{aligned}$$

Therefore, $\alpha(x)$ consists of either a single fixed point or a periodic orbit. \square

Remark 2.3 Just as we mentioned after Theorem 1.1, Theorems 2.3 and 2.4 can be only applied to systems on the space $[0, r] \subset C$ if no condition for T on $C \setminus [0, r]$ is given. However, a simple additional condition

$$(11) \quad \forall i \in I_N, \forall x \in C \text{ with } x_i \geq r_i, 0 < f_i(x) < 1$$

guarantees that for each compact set $S \subset C$ there is a $k \in \mathbb{N}$ such that $T^k(S) \subset [0, r]$, so that Theorems 2.3 and 2.4 can be applied directly to systems on C .

In general, for any topological space X , a system $x(n) = F^n(x)$ for $x \in X, n \in \mathbb{N}$ with a map $F : X \rightarrow X$, and a compact invariant set $A \subset X$, A is called a global attractor of the system if A attracts the points of any bounded set $B \subset X$ uniformly. For our system (1) with (2) on C under the conditions of Theorem 2.3 or Theorem 2.4, since 0 is a repelling fixed point, by saying that Σ is a global attractor of the system in $[0, r] \setminus \{0\}$ ($C \setminus \{0\}$), we mean Σ uniformly attracts the points of any bounded set $B \subset [0, r] \setminus \{0\}$ ($B \subset C \setminus \{0\}$) that is bounded away from 0, i.e. $\bar{B} \subset [0, r] \setminus \{0\}$ ($\bar{B} \subset C \setminus \{0\}$).

Corollary 2. *Under the conditions of Theorem 2.3 or Theorem 2.4, the modified carrying simplex Σ is a global attractor in $[0, r] \setminus \{0\}$. In addition, if (11) holds, then Σ is a global attractor of the system in $C \setminus \{0\}$.*

Before we prove Theorem 2.3, Theorem 2.4 and Corollary 2 in section 5, we present an application of Theorem 2.4 in next section.

3. CRITERIA FOR DOMINANCE AND VANISHING SPECIES

In this section, we consider (2) and give sufficient conditions for dominance of some species under the assumption that the conditions of Theorem 2.4 are met.

Viewing (2) as a population model for N competitive species, we say that the j th species is *dominated* or *vanishing* if $\lim_{n \rightarrow +\infty} x_j(n) = 0$ for all $x \in \dot{C}$; we say that the j th species is *dominant* if $\liminf_{n \rightarrow \infty} x_j(n) > \delta > 0$ for all $x \in \dot{C}$ and all other species are vanishing. Let

$$(12) \quad \Gamma_i = \{x \in C : f_i(x) = 1\}, i \in I_N.$$

Under the general assumptions for (2), each Γ_i is a closed set and an $(N - 1)$ -dimensional hypersurface. In this section, we assume that each Γ_i is simply connected and divides C into three mutually exclusive subsets Γ_i^+ , Γ_i and Γ_i^- with $0 \in \Gamma_i^-$. Then the closure of Γ_i^- is $\overline{\Gamma_i^-} = \Gamma_i^- \cup \Gamma_i$. But if we consider the restriction of Γ_i to $[0, r]$, this assumption is met if the conditions of Theorem 2.4 hold: each $\Gamma_i \cap [0, r]$ is a simply connected closed set and an $(N - 1)$ -dimensional hypersurface such that $\Gamma_i^- \cap [0, r]$ is strictly below Γ_i and $\Gamma_i^+ \cap [0, r]$ is strictly above Γ_i .

Under the conditions of Theorem 2.4, let $Q_i = q_i e_i$, the fixed point of T on X_i , the positive half x_i -axis.

Theorem 3.1. *Assume that (11) and the conditions of Theorem 2.4 hold.*

- (a) *If for some $i \in I_N$ and all $j \in I_N \setminus \{i\}$, $\frac{\partial f_i}{\partial x_i}(Q_i) < 0$ and $\Gamma_i \cap [0, r]$ is strictly below Γ_j , then $\lim_{n \rightarrow +\infty} x_i(n) = 0$ for all $x \in C \setminus X_i$ so the i th species is vanishing.*
- (b) *If for some $i \in I_N$ and all $j \in I_N \setminus \{i\}$, $\frac{\partial f_i}{\partial x_i}(Q_i) < 0$ and $\Gamma_i \cap [0, r]$ is strictly above Γ_j , then the i th species is dominant and the axial fixed point Q_i is globally asymptotically stable.*

Proof. By Theorem 2.4 the system has a unique modified carrying simplex Σ .

(a) Under the assumption that $\Gamma_i \cap [0, r]$ is strictly below Γ_j for all $j \in I_N \setminus \{i\}$, we first claim that

$$(13) \quad \Gamma_i^- \cap \Sigma = \emptyset$$

so that $\Gamma_i^- \cap [0, r]$ is strictly below Σ and Σ is above Γ_i . Indeed, if (13) were not true then we would have a point $p \in (\Gamma_i^- \cap \Sigma)$. As $0 \notin \Sigma$, we have $p \neq 0$ and a nonempty $J \subset I_N$ such that $p_j > 0$ if and only if $j \in J$. Since p is below Γ_i and $\Gamma_i \cap [0, r]$ is strictly below Γ_j for all $j \in I_N \setminus \{i\}$, p is below Γ_j for all $j \in I_N$. Let $u = T(p)$. Then

$$\forall j \in J, u_j = T_j(p) = p_j f_j(p) > p_j; \forall k \in I_N \setminus J, u_k = p_k = 0,$$

so $p \ll_J u$. As Σ is invariant and $p \in \Sigma$, we have $u \in \Sigma$. Then, by Theorem 2.4, we have $\alpha(p) = \{0\}$ so $p \in \mathcal{B}(0)$, a contradiction to $p \in \Sigma = \overline{\mathcal{B}(0)} \setminus (\{0\} \cup \mathcal{B}(0))$. This shows our claim (13).

Since the axial fixed point Q_i is below Γ_j for all $j \in I_N \setminus \{i\}$, the Jacobian matrix $DT(Q_i)$ has $N - 1$ eigenvalues $f_j(Q_i) > 1$ for $j \in I_N \setminus \{i\}$ and one eigenvalue $1 + q_i \frac{\partial f_i}{\partial x_i}(Q_i)$. By assumption, (4) and (6), the only nonzero eigenvalue of $M(Q_i)$ and $\tilde{M}(Q_i)$ is $-q_i \frac{\partial f_i}{\partial x_i}(Q_i) > 0$, so $\rho(M(Q_i)) = \rho(\tilde{M}(Q_i)) = -q_i \frac{\partial f_i}{\partial x_i}(Q_i)$. By condition (iii) of Theorem 2.4, we have $0 < 1 + q_i \frac{\partial f_i}{\partial x_i}(Q_i) < 1$. So Q_i is a saddle point in C with X_i as its one-dimensional stable manifold and a repeller on Σ . Thus, to show that $\lim_{n \rightarrow +\infty} x_i(n) = 0$ for all $x \in C \setminus X_i$, by the definition of modified carrying simplex, we need only show that $\lim_{n \rightarrow +\infty} x_i(n) = 0$ for all $x \in C \setminus X_i$ on or above Σ , i.e. $x \in (\Sigma \cup \Sigma^+) \setminus X_i$.

Now for any $x \in C$ with $x_i > r_i$, the assumption (11) ensures that $x(n) \in [0, r]$ for large enough $n \in \mathbb{N}$. Without loss of generality, we only consider $x \in (\Sigma \cup (\Sigma^+ \cap [0, r])) \setminus X_i$. We first show that the set $(\Sigma \cup (\Sigma^+ \cap [0, r])) \setminus X_i$ is positively invariant. From the proof of Theorem 2.4 given in section 5 we shall see that the conditions of Theorem 2.4 imply the conditions of Theorem 2.3. Thus, $[0, r]$ is positively invariant and, by Remark 2.3 (b), $T : [0, r] \rightarrow T([0, r])$ is a homeomorphism. As 0 is a repeller with the basin of repulsion $\mathcal{B}(0) \subset [0, r]$, we shall see in section 5 (Lemma 5.4) that $\overline{\mathcal{B}(0)}$ is invariant. Thus, T maps the set

$$[0, r] \setminus \overline{\mathcal{B}(0)} = [0, r] \setminus (\Sigma \cup \mathcal{B}(0) \cup \{0\}) = [0, r] \setminus (\Sigma \cup \Sigma^-) = [0, r] \cap \Sigma^+$$

into itself. As Σ is invariant, $\Sigma \cup (\Sigma^+ \cap [0, r])$ is positively invariant. For each $x \in \Sigma \cup (\Sigma^+ \cap [0, r]) \setminus X_i$, there is a $j \in I_N \setminus \{i\}$ such that $x_j > 0$, so $T_j(x) = x_j f_j(x) > 0$. Thus, $T(x) \in \Sigma \cup (\Sigma^+ \cap [0, r]) \setminus X_i$. This shows the positive invariance of $\Sigma \cup (\Sigma^+ \cap [0, r]) \setminus X_i$.

By (13), Σ is above Γ_i , so $\Sigma \cup (\Sigma^+ \cap [0, r]) \setminus X_i$ is above Γ_i . Thus, for $x \in (\Sigma \cup (\Sigma^+ \cap [0, r])) \setminus X_i$, $x(n) = T^n(x) \in (\Sigma \cup (\Sigma^+ \cap [0, r])) \setminus X_i$, so $x(n)$ is on or above Γ_i for all $n \in \mathbb{N}$. Hence,

$$\forall n \in \mathbb{N}, x_i(n+1) = T_i(x(n)) = x_i(n) f_i(x(n)) \leq x_i(n).$$

This shows that $\{x_i(n)\}$ is a bounded monotone nonincreasing sequence, so there is an $x_0 \geq 0$ such that $\lim_{n \rightarrow +\infty} x_i(n) = x_0$. Suppose $x_0 > 0$. Then, for each $y \in \omega(x) \subset \Sigma$, we have $y_i = x_0$. As $T^n(y) \in \omega(x)$ for all integer n , we have $T_i(y) = y_i f_i(y) = x_0 = y_i$ so $f_i(y) = 1$ and $y \in \Gamma_i$. Therefore, $\omega(x) \subset \Gamma_i \cap \Sigma$. If $\omega(x) = \{Q_i\}$, as Q_i is below Γ_j for all $j \in I_N \setminus \{i\}$, there is a $\delta > 0$ such that the closure $\overline{O(Q_i, \delta)} \cap [0, r]$ of the open ball centred at Q_i with radius δ restricted to $[0, r]$, i.e. $O(Q_i, \delta) \cap [0, r]$, is strictly below Γ_j for all $j \in I_N \setminus \{i\}$. Let

$$m_0 = \min\{f_j(u) : u \in \overline{O(Q_i, \delta)} \cap [0, r], j \in I_N \setminus \{i\}\}.$$

Then $m_0 > 1$. Since $\lim_{n \rightarrow +\infty} x(n) = Q_i$, there is $n_1 \in \mathbb{N}$ such that $x(n) \in O(Q_i, \delta) \cap [0, r]$ for $n \geq n_1$. As $x \notin X_i$, we have $x_j > 0$ for some $j \in I_N \setminus \{i\}$. Then, for this j and all $n \geq 1$,

$$x_j(n+n_1) = T_j(x(n-1+n_1)) = x_j(n-1+n_1) f_j(x(n-1+n_1)) \geq m_0 x_j(n-1+n_1),$$

so $x_j(n+n_1) \geq m_0^n x_j(n_1) \rightarrow +\infty$ as $n \rightarrow +\infty$, a contradiction to the boundedness of $\{x(n)\}$. This contradiction shows the existence of a point $y \in \omega(x) \setminus \{Q_i\}$. Since Q_i is the

unique intersection point of Σ with X_i and $y \in \Sigma \setminus \{Q_i\}$, we have $y \notin X_i$ so $y_j > 0$ for some $j \in I_N \setminus \{i\}$. Since $\Gamma_i \cap [0, r]$ is strictly below Γ_j and $\omega(x) \subset \Gamma_i \cap \Sigma \subset \Gamma_i \cap [0, r]$, $\omega(x)$ is strictly below Γ_j . Let

$$\rho = \min_{u \in \omega(x)} f_j(u).$$

Then, by the continuity of f and the compactness of $\omega(x)$, $\rho > 1$ and

$$y_j(n+1) = T_j(y(n)) = y_j(n)f_j(y(n)) \geq \rho y_j(n).$$

Thus,

$$y_j(n) \geq \rho^n y_j \rightarrow +\infty \quad (n \rightarrow +\infty),$$

a contradiction to the boundedness of $\omega(x)$. This contradiction shows that we must have $x_0 = 0$, i.e. $\lim_{n \rightarrow +\infty} x_i(n) = 0$ for all $x \in C \setminus X_i$.

(b) Under the condition that $\Gamma_i \cap [0, r]$ is strictly above Γ_j for every $j \in I_N \setminus \{i\}$, we first show that Σ is below Γ_i by assuming the opposite: there is a point $p \in \Sigma \cap \Gamma_i^+$. As $p \neq 0$, there is a nonempty $J \subset I_N$ as the support of p . As p is above Γ_i and $\Gamma_i \cap [0, r]$ is strictly above Γ_j for all $j \in I_N \setminus \{i\}$, p is above Γ_j for all $j \in I_N$. Thus,

$$\forall j \in J, T_j(p) = p_j f_j(p) < p_j,$$

so $T(p) \ll_J p$. By Theorem 2.4, $\alpha(T(p)) = \{0\}$ so $T(p) \in \mathcal{B}(0)$, a contradiction to $T(p) \in \Sigma = \overline{\mathcal{B}(0)} \setminus (\{0\} \cup \mathcal{B}(0))$. This shows that Σ must be below Γ_i .

We need only show that Q_i is stable and attracts all the points of $C \setminus \pi_i$ as the dominance of the i th species is implied by the global attraction of Q_i . As Q_i is above Γ_j for all $j \in I_N \setminus \{i\}$, we have $f_j(Q_i) \in (0, 1)$ for all $j \in I_N \setminus \{i\}$. By the assumption $\frac{\partial f_i}{\partial x_i}(Q_i) < 0$ and condition (iii) of Theorem 2.4, $1 + q_i \frac{\partial f_i}{\partial x_i}(Q_i) \in (0, 1)$. Thus, every eigenvalue of $DT(Q_i)$ is in $(0, 1)$ so Q_i is asymptotically stable. To show the global attraction of Q_i in $C \setminus \pi_i$, by the assumption (11) and the definition of a modified carrying simplex, we need only show that $\lim_{n \rightarrow +\infty} x(n) = Q_i$ for all $x \in (\Sigma \cup (\Sigma^+ \cap [0, r])) \setminus \pi_i$.

If $x \in \Sigma \setminus \pi_i$, as $x(n) \in \Sigma$ for all $n \in \mathbb{N}$ and Σ is below Γ_i , the sequence $\{x_i(n)\}$ is bounded and monotone nondecreasing. Thus, there is a $\beta > 0$ such that $\lim_{n \rightarrow +\infty} x_i(n) = \beta$. For any $y \in \omega(x)$, we have $T_i(y) = y_i f_i(y) = \beta = y_i$ so $f_i(y) = 1$. Thus, $y \in \Gamma_i$ and $\omega(x) \subset \Gamma_i$. We claim that $\omega(x) = \{Q_i\}$. To verify this claim, as $\omega(x)$ is compact, $\omega(x) \subset \Gamma_i$ and $\Gamma_i \cap [0, r]$ is strictly above Γ_j for all $j \in I_N \setminus \{i\}$, there is a $\delta > 0$ such that the closure $\overline{O(\omega(x), \delta)} \cap [0, r]$ of the open set $O(\omega(x), \delta) \cap [0, r]$ with

$$O(\omega(x), \delta) = \{z \in C : \|z - u\| < \delta \text{ for some } u \in \omega(x)\}$$

is strictly above Γ_j for all $j \in I_N \setminus \{i\}$. Let

$$\mu = \max\{f_j(u) : u \in \overline{O(\omega(x), \delta)} \cap [0, r], j \in I_N \setminus \{i\}\}.$$

By the continuity of f and the compactness of $\overline{O(\omega(x), \delta)} \cap [0, r]$, we have $0 < \mu < 1$. By the definition of $\omega(x)$, there is an integer $N_1 \geq 0$ such that $x(n) \in O(\omega(x), \delta) \cap [0, r]$ for all $n > N_1$. Let $J \subset I_N$ be the support of x . Then $x_j(n) > 0$ for all $n \in \mathbb{N}$ if and only if $j \in J$.

If $J = \{i\}$ then $x = Q_i$ and the above claim is obviously true. Now suppose $\{i\} \subset J \neq \{i\}$. Then,

$$\forall n > N_1, \forall j \in J \setminus \{i\}, x_j(n+1) = x_j(n)f_j(x(n)) \leq \mu x_j(n),$$

so $x_j(n + N_1) \leq \mu^n x_j(N_1) \rightarrow 0$ as $n \rightarrow +\infty$. This shows that $\omega(x) = \Sigma \cap X_i = \{Q_i\}$.

Now suppose $x \in (\Sigma^+ \cap [0, r]) \setminus \pi_i$. By the asymptotic stability of Q_i there is a $\delta > 0$ such that every point $z \in O(Q_i, \delta) \cap C$ is attracted to Q_i , i.e. $\lim_{n \rightarrow +\infty} z(n) = Q_i$. Thus, as long as $Q_i \in \omega(x)$, there is an $m \in \mathbb{N}$ such that $x(m) \in O(Q_i, \delta) \cap C$ so that $\omega(x) = \{Q_i\}$. We now prove $Q_i \in \omega(x)$ by contradiction. If $Q_i \notin \omega(x)$ and there is a $y \in \omega(x) \subset \Sigma$ with $y_i > 0$, by the previous paragraph we have $\lim_{n \rightarrow +\infty} y(n) = Q_i$. As $\omega(x)$ is compact and $y(n) \in \omega(x)$ for all $n \in \mathbb{N}$, we have $Q_i \in \omega(x)$, which contradicts the condition $Q_i \notin \omega(x)$. Thus, if $Q_i \notin \omega(x)$ then $\omega(x) \subset \Sigma \cap \pi_i$. If $\omega(x)$ is strictly below Γ_i , by definition of $\omega(x)$ there is a $K \in \mathbb{N}$ such that $x(n)$ is below Γ_i for all $n \geq K$. Thus, $\{x_i(n)\}$ is an increasing sequence for $n \geq K$ so that $x_i(n) \geq x_i(K) > 0$ for $n \geq K$ and each $y \in \omega(x)$ satisfies $y_i \geq x_i(K) > 0$, a contradiction to the assumption $\omega(x) \subset \pi_i$. If there is a point $p \in \omega(x) \subset (\Sigma \cap \pi_i)$ on or above Γ_i , there is a nonempty $J \subset I_N \setminus \{i\}$ as the support of p . As $\Gamma_i \cap [0, r]$ is strictly above Γ_j for all $j \in I_N \setminus \{i\}$, p is above Γ_j for all $j \in I_N \setminus \{i\}$ so $T(p) \ll_J p$. This leads us to $\alpha(T(p)) = \{0\}$ by Theorem 2.4, so $T(p) \in \mathcal{B}(0)$, a contradiction to $T(p) \in \Sigma = \overline{\mathcal{B}(0)} \setminus (\{0\} \cup \mathcal{B}(0))$. These contradictions show that we must have $Q_i \in \omega(x)$ so $\omega(x) = \{Q_i\}$. \square

We note that Theorem 2.3 in [9] is consistent with our Theorem 3.1 (b) but under the stronger conditions of Theorem 1.1. While Theorem 3.1 used one surface $\Gamma_i \cap [0, r]$ comparing with the other $N - 1$ surfaces Γ_j to obtain one species vanishing, our next result repeat such a condition several times to get multiple species vanishing.

Theorem 3.2. *Assume that (11) and the conditions of Theorem 2.4 hold. Assume also the existence of an integer $k \in I_N \setminus \{N\}$ such that for all $i \in \{1, \dots, k\}$, $\frac{\partial f_i}{\partial x_i}(Q_i) < 0$ and*

$$(14) \quad \forall j \in \{i+1, \dots, N\}, (\cap_{\ell=1}^{i-1} \pi_\ell) \cap \Gamma_i \cap [0, r] \text{ is strictly below } \Gamma_j.$$

Then the i th species is dominated for all $i \in \{1, \dots, k\}$. In addition, if $k = N - 1$ and $\frac{\partial f_N}{\partial x_N}(Q_N) < 0$, then the N th species is dominant and the N th axial fixed point Q_N is globally asymptotically stable in C .

Remark 3.1 Here the symbol $\cap_{i \in \emptyset} \pi_i$ is deemed as C . So, (14) for $i = 1$ is simplified as

$$(15) \quad \forall j \in \{2, \dots, N\}, \Gamma_1 \cap [0, r] \text{ is strictly below } \Gamma_j.$$

Proof of Theorem 3.2. For $k > 1$ we first prove that

$$(16) \quad \forall i \in \{2, \dots, k\}, (\cap_{\ell=1}^{i-1} \pi_\ell) \cap \Gamma_i^- \cap \Sigma = \emptyset.$$

The proof of (16) is similar to that of (13). Suppose (16) is not true. Then, for some $i \in \{2, \dots, k\}$, there exists a point $u \in (\cap_{\ell=1}^{i-1} \pi_\ell) \cap \Gamma_i^- \cap \Sigma$, so u is below Γ_i . By (14),

$(\cap_{\ell=1}^{i-1} \pi_\ell) \cap [0, r] \cap \Gamma_i$ is strictly below Γ_j for all $j \in \{i+1, \dots, N\}$. Thus, u is below Γ_j for all $j \in \{i, \dots, N\}$. Note that $u \in (\cap_{\ell=1}^{i-1} \pi_\ell) \cap \Sigma$ implies $u > 0$ and $u_j = 0$ for all $j \in \{1, \dots, i-1\}$. Thus, there is a nonempty $J \subset \{i, \dots, N\}$ such that $u_j > 0$ if and only if $j \in J$. Then,

$$\forall j \in J, T_j(u) = u_j f_j(u) > u_j,$$

so $u \ll_J T(u)$. As $T(u) \in \Sigma$, by Theorem 2.4 we obtain $u \in \mathcal{B}(0)$, a contradiction to $u \in \Sigma = \overline{\mathcal{B}(0)} \setminus (\{0\} \cup \mathcal{B}(0))$. This contradiction shows the truth of (16).

For $k \geq 1$ with $i = 1$, from Remark 3.1 and Theorem 3.1 we know that the first species is vanishing, i.e. $\omega(x) \subset \Sigma \cap \pi_1$ for all $x \in C \setminus \overline{X_1}$.

Then for $k \geq 2$ with $i = 2$, from (14) we see that $\Gamma_2 \cap \pi_1 \cap [0, r]$ is strictly below Γ_j for all $j \in \{3, \dots, N\}$. As $\omega(x) \subset \Sigma \cap \pi_1$ for all $x \in C \setminus \overline{X_1}$, we can prove that $\omega(x) \subset \Sigma \cap (\pi_1 \cap \pi_2)$ for all $x \in C \setminus (\overline{X_1} \times \overline{X_2})$ (the proof is included in the general case below).

Then for $k \geq 3$ with $i = 3$, from (14) we see that $\Gamma_3 \cap (\pi_1 \cap \pi_2) \cap [0, r]$ is strictly below Γ_j for all $j \in \{4, \dots, N\}$. As $\omega(x) \subset \Sigma \cap (\pi_1 \cap \pi_2)$ for all $x \in C \setminus (\overline{X_1} \times \overline{X_2})$, we can prove that $\omega(x) \subset \Sigma \cap (\cap_{\ell=1}^3 \pi_\ell)$ for all $x \in C \setminus (\overline{X_1} \times \overline{X_2} \times \overline{X_3})$ (the proof is included in the general case below).

In general, for $k > 1$ with $i = k$, from (14) we see that $\Gamma_k \cap (\cap_{\ell=1}^{k-1} \pi_\ell) \cap [0, r]$ is strictly below Γ_j for all $j \in \{k+1, \dots, N\}$. Suppose we know that $\omega(x) \subset \Sigma \cap (\cap_{\ell=1}^{k-1} \pi_\ell)$ for all $x \in C \setminus (\overline{X_1} \times \dots \times \overline{X_{k-1}})$. We need to prove that

$$(17) \quad \forall x \in C \setminus (\overline{X_1} \times \dots \times \overline{X_k}), \omega(x) \subset \Sigma \cap (\cap_{\ell=1}^k \pi_\ell).$$

From condition (11), Theorem 2.4 and the definition of a modified carrying simplex, instead of (17) we need only prove that

$$(18) \quad \forall x \in ([0, r] \setminus \Sigma^-) \setminus (\overline{X_1} \times \dots \times \overline{X_k}), \omega(x) \subset \Sigma \cap (\cap_{\ell=1}^k \pi_\ell).$$

The proof of (18) is divided into the following two steps.

Step 1. We show that

$$(19) \quad \forall x \in (\cap_{\ell=1}^{k-1} \pi_\ell) \cap ([0, r] \setminus \Sigma^-) \setminus X_k, \omega(x) \subset \Sigma \cap (\cap_{\ell=1}^k \pi_\ell).$$

From (16) we know that $(\cap_{\ell=1}^{k-1} \pi_\ell) \cap \Sigma$ is above Γ_k , so $(\cap_{\ell=1}^{k-1} \pi_\ell) \cap ([0, r] \setminus \Sigma^-) \setminus X_k$ is above Γ_k . Note that π_ℓ and $C \setminus \pi_\ell$ are positively invariant for any $\ell \in I_N$. Thus, for any $x \in (\cap_{\ell=1}^{k-1} \pi_\ell) \cap ([0, r] \setminus \Sigma^-) \setminus X_k$, if $x_k = 0$ then $x \in (\cap_{\ell=1}^k \pi_\ell)$, so $x(n) \in (\cap_{\ell=1}^k \pi_\ell)$ for all $n \in \mathbb{N}$ and $\omega(x) \subset \Sigma \cap (\cap_{\ell=1}^k \pi_\ell)$. If $x_k > 0$, as both $\cap_{\ell=1}^{k-1} \pi_\ell$ and $([0, r] \setminus \Sigma^-) \setminus X_k = \Sigma \cup (\Sigma^+ \cap [0, r]) \setminus X_k$ (from the proof of Theorem 3.1) are positively invariant and $(\cap_{\ell=1}^{k-1} \pi_\ell) \cap ([0, r] \setminus \Sigma^-) \setminus X_k = (\cap_{\ell=1}^{k-1} \pi_\ell) \cap (([0, r] \setminus \Sigma^-) \setminus X_k)$, we have $x(n) \in (\cap_{\ell=1}^k \pi_\ell) \cap ([0, r] \setminus \Sigma^-) \setminus X_k$ for all $n \in \mathbb{N}$, so each $x(n)$ is on or above Γ_k for all $n \in \mathbb{N}$. Hence,

$$\forall n \in \mathbb{N}, x_k(n+1) = T_k(x(n)) = x_k(n) f_k(x(n)) \leq x_k(n).$$

As $\{x_k(n)\}$ is a positive monotone nonincreasing sequence, there is a $\mu \geq 0$ such that $\lim_{n \rightarrow +\infty} x_k(n) = \mu$. Suppose $\mu > 0$. Then, for each $y \in \omega(x) \subset \Sigma$ we have $y_k = \mu$. As

$T(y) \in \omega(x)$, we have $\mu = T_k(y) = y_k f_k(y) = \mu f_k(y)$ so $f_k(y) = 1$ and $y \in \Gamma_k$. Therefore, $\omega(x) \subset \Gamma_k$. By the positive invariance of $(\cap_{\ell=1}^{k-1} \pi_\ell) \cap [0, r]$, $\omega(x) \subset \Gamma_k \cap (\cap_{\ell=1}^{k-1} \pi_\ell) \cap [0, r]$. Since the set $\Gamma_k \cap (\cap_{\ell=1}^{k-1} \pi_\ell) \cap [0, r]$ is strictly below Γ_j for all $j \in \{k+1, \dots, N\}$, $\omega(x)$ is strictly below Γ_j for all $j \in \{k+1, \dots, N\}$. As $\omega(x)$ is compact, there is a $\delta > 0$ such that $\overline{O(\omega(x), \delta)} \cap [0, r]$ is strictly below Γ_j for all $j \in \{k+1, \dots, N\}$. For this δ , there is an $m \in \mathbb{N}$ such that $x(n) \in O(\omega(x), \delta) \cap [0, r]$ for all $n \geq m$. Note that $x \in (\cap_{\ell=1}^{k-1} \pi_\ell) \cap ([0, r] \setminus \Sigma^-) \setminus X_k$ implies $x_j > 0$ for some $j \in \{k+1, \dots, N\}$. For this j , let

$$\eta = \min\{f_j(u) : u \in \overline{O(\omega(x), \delta)} \cap [0, r]\}.$$

Then $\eta > 1$ and for all $n \geq 1$,

$$x_j(n+m) = T_j(x(n+m-1)) = x_j(n+m-1)f_j(x(n+m-1)) \geq \eta x_j(n+m-1).$$

It follows from this that $x_j(n+m) \geq \eta^n x_j(m) \rightarrow +\infty$ as $n \rightarrow +\infty$, a contradiction to the boundedness of $\{x(n)\}$. This contradiction shows that $\mu = 0$ and (19) follows.

Step 2. Now we prove (18). For $x \in ([0, r] \setminus \Sigma^-) \setminus (\overline{X_1} \times \dots \times \overline{X_k})$, we show that $\omega(x) \subset \Sigma \cap (\cap_{\ell=1}^k \pi_\ell)$. From the supposition we know that $\omega(x) \subset \Sigma \cap (\cap_{\ell=1}^k \pi_\ell)$. Suppose $\omega(x) \not\subset \Sigma \cap (\cap_{\ell=1}^k \pi_\ell)$. Then either $\omega(x) = \{Q_k\}$ or there is a $y \in \omega(x) \setminus \{Q_k\}$ with $y_k > 0$.

In the former case, $\lim_{n \rightarrow +\infty} x(n) = Q_k$. Since Q_k is below Γ_j for all $j \in \{k+1, \dots, N\}$, there is an $\varepsilon > 0$ such that $\overline{O(Q_k, \varepsilon)} \cap [0, r]$ is strictly below Γ_j for all $j \in \{k+1, \dots, N\}$. For this $\varepsilon > 0$, there is an $m \in \mathbb{N}$ such that $x(n) \in O(Q_k, \varepsilon) \cap [0, r]$ for all $n \geq m$. That $x \notin \overline{X_1} \times \dots \times \overline{X_k}$ ensures the existence of some $j \in \{k+1, \dots, N\}$ with $x_j > 0$ so that $x_j(n) > 0$ for all $n \in \mathbb{N}$. For this j , let

$$\eta_0 = \min\{f_j(u) : u \in \overline{O(Q_k, \varepsilon)} \cap [0, r]\}.$$

Then $\eta_0 > 1$ and for all $n \geq 1$,

$$x_j(n+m) = T_j(x(n+m-1)) = x_j(n+m-1)f_j(x(n+m-1)) \geq \eta_0 x_j(n+m-1).$$

This leads to $x_j(n+m) \geq \eta_0^n x_j(m) \rightarrow +\infty$ as $n \rightarrow +\infty$, a contradiction to the boundedness of $\{x(n)\}$.

In the latter case, from Step 1 we see that $\lim_{n \rightarrow +\infty} y_k(n) = 0$. Without loss of generality, we may assume that $0 < y_k < q_k$, where $Q_k = q_k e_k$. Since the whole trajectory $\gamma(y)$ is contained in $\omega(x)$ and $\omega(x)$ is compact, from Step 1 we derive that $\omega(y) \subset \omega(x) \cap (\cap_{\ell=1}^{k-1} \pi_\ell) \cap \Sigma$. Let

$$\eta_1 = \max\{f_k(u) : u \in [0, r]\}$$

and take a small $\varepsilon \in (0, \frac{y_k}{3\eta_1})$. Since the set

$$S = \{z \in (\cap_{\ell=1}^{k-1} \pi_\ell) \cap \Sigma : \varepsilon \leq z_k \leq y_k\}$$

is compact and $\omega(z) \subset (\cap_{\ell=1}^k \pi_\ell) \cap \Sigma$ for all $z \in S$ from Step 1, by continuous dependence there is a $\delta \in (0, \min\{\frac{\varepsilon}{2}, \frac{1}{6}y_k\})$ such that

$$(20) \quad \begin{aligned} \forall u \in O(S, \delta) \cap [0, r], \quad \exists z \in S, \quad \exists n_1(z) \in \mathbb{N}, \text{ such that } z_k(n_1) < \frac{\varepsilon}{2}, \\ \forall n \in \{0, \dots, n_1\}, \|u(n) - z(n)\| < \frac{\varepsilon}{2}. \end{aligned}$$

For this δ , there is an $m_0 \in \mathbb{N}$ such that $x(n) \in O((\cap_{\ell=1}^{k-1} \pi_\ell) \cap \Sigma, \delta)$ for all $n \geq m_0$. As $\emptyset \neq \omega(y) \subset \omega(x) \cap (\cap_{\ell=1}^k \pi_\ell) \cap \Sigma$, there is an $m_1 \geq m_0$ such that $0 < x_k(m_1) < \varepsilon$. Then

$$x_k(m_1 + 1) = T_k(x(m_1)) = x_k(m_1) f_k(x(m_1)) \leq \eta_1 \varepsilon < \frac{1}{3} y_k.$$

Thus, either (i) $x_k(m_1 + 1) < \varepsilon$ or (ii) $x(m_1 + 1) \in O(S, \delta)$. In case (i), by taking $m_2 = m_1 + 1$ we have $x_k(m_2 + 1) < \frac{1}{3} y_k$. In case (ii), by (20) there is a $z \in S$ and $n_1 \in \mathbb{N}$ such that $z_k(n_1) < \frac{1}{2} \varepsilon$ and $\|x(m_1 + 1 + n) - z(n)\| < \frac{1}{2} \varepsilon$ for all $0 \leq n \leq n_1$. By the positive invariance of $(\cap_{\ell=1}^{k-1} \pi_\ell) \cap \Sigma$, for any $z \in (\cap_{\ell=1}^{k-1} \pi_\ell) \cap \Sigma$, we have $z(n) \in (\cap_{\ell=1}^{k-1} \pi_\ell) \cap \Sigma$ for all $n \in \mathbb{N}$. As (16) implies that $(\cap_{\ell=1}^{k-1} \pi_\ell) \cap \Sigma$ is above Γ_k and $S \subset (\cap_{\ell=1}^{k-1} \pi_\ell) \cap \Sigma$, $z_k(n)$ is monotone nonincreasing in n for each $z \in S$. Then,

$$x_k(m_1 + 1 + n) < z_k(n) + \frac{1}{2} \varepsilon \leq z_k + \frac{1}{2} \varepsilon < x_k(m_1 + 1) + \varepsilon < \frac{2}{3} y_k, 0 \leq n \leq n_1$$

and $x_k(m_1 + 1 + n_1) < z_k(n_1) + \frac{1}{2} \varepsilon < \varepsilon$. Take $m_2 = m_1 + 1 + n_1$. In either (i) or (ii), we see that $x_k(n) < \frac{2}{3} y_k$ for all $m_1 \leq n \leq m_2$. Repeating the above process we obtain $x_k(n) < \frac{2}{3} y_k$ for all $n \geq m_1$, a contradiction to $y \in \omega(x)$.

The contradictions in both cases above show that $\omega(x) \subset (\cap_{\ell=1}^k \pi_\ell) \cap \Sigma$. Then (18) follows.

Finally, if $k = N - 1$, we have $\lim_{n \rightarrow +\infty} x(n) = Q_N$ for all $x \in C$ with $x_N > 0$. As Q_N is above Γ_j for all $j \in \{1, \dots, N - 1\}$ and $\frac{\partial f_N}{\partial x_N}(Q_N) < 0$, every eigenvalue of the Jacobian matrix $DT(Q_N)$ is in the interval $(0, 1)$. Thus, Q_N is globally asymptotically stable. \square

Note that the statement of Theorem 3.2 used the natural ascending order of numbers for the species. Obviously, the statement is still true after a permutation from ascending order of numbers.

Corollary 3. *Assume that (11) and the conditions of Theorem 2.4 hold. Assume also the existence of a permutation $p : I_N \rightarrow I_N$ and an integer $k \in I_N \setminus \{N\}$ such that $\frac{\partial f_{p(i)}}{\partial x_{p(i)}}(Q_{p(i)}) < 0$ for all $i \in \{1, \dots, k\}$ and*

$$(21) \quad \forall j \in \{i + 1, \dots, N\}, (\cap_{\ell=1}^{i-1} \pi_{p(\ell)}) \cap \Gamma_{p(i)} \cap [0, r] \text{ is strictly below } \Gamma_{p(j)}.$$

Then the $p(i)$ th species is dominated for all $i \in \{1, \dots, k\}$. In addition, if $k = N - 1$ and $\frac{\partial f_{p(N)}}{\partial x_{p(N)}}(Q_{p(N)}) < 0$ then the $p(N)$ th species is dominant and the $p(N)$ th axial fixed point $Q_{p(N)}$ is globally asymptotically stable in C .

4. SOME EXAMPLES

In this section, we apply our results obtained in sections 2 and 3 to some known models as examples. All these models fit well into our system (1) for maps $T : C \rightarrow C$ of the form (2), where the sign of each entry $\frac{\partial f_i(x)}{\partial x_j}$ of the Jacobian Df is completely determined by the corresponding entry a_{ij} of a constant matrix $A = (a_{ij})_{N \times N}$:

$$\forall i, j \in I_N, \forall x \in C, \frac{\partial f_i(x)}{\partial x_j} = -\sigma_{ij}(x)a_{ij}, \sigma_{ij}(x) > 0.$$

All entries of the matrix A are assumed positive in most of the references cited here either due to their particular meaning in the original model or due to convenience of theoretical analysis by using available results such as Theorem 1.1, Theorem 1.2 and their variations mentioned in section 1. Under such an assumption, each system models the population dynamics of a community of N competing species where the population of the j th species directly affects the growth rate of the population of the i th species in a negative way as $a_{ij} > 0$.

Unfortunately, as far as the author knows, not much has been found about such systems modelling competing species where the population of the j th species affects the growth rate of the population of the i th species in a negative way, directly or indirectly, due to $a_{ij} > 0$ or $a_{ij} = 0$. With the help of our Theorem 2.3 and Theorem 2.4, we are now able to deal with these models under the relaxed assumption:

$$\forall i, j \in I_N, a_{ij} \geq 0, a_{ii} > 0.$$

Since our results obtained in section 3 and this section below are all based on the assumptions of Theorem 2.4, if $\frac{\partial f_i(x)}{\partial x_j} = 0$ ($a_{ij} = 0$ for the models below) for at least one pair of indices i, j at some point $x \in [0, r]$, then these results are not achievable by using Theorem 1.2 and its variations as their conditions are not fully met. This demonstrates that the class of systems to which Theorem 2.4 is applicable is broader than that for Theorem 1.2 and its variations. Hence, our main results are a significant improvement of those available in literature.

4.1. The competitive Leslie-Gower models. The competitive Leslie-Gower models are system (1) for maps $T : C \rightarrow C$ of the form (2): $T_i(x) = x_i f_i(x)$, where

$$(22) \quad \forall i, j \in I_N, f_i(x) = \frac{c_i}{1 + \sum_{k=1}^N a_{ik} x_k}, c_i > 1, a_{ij} \geq 0, a_{ii} > 0.$$

Under the condition that $a_{ij} > 0$ for all $i, j \in I_N$, Jiang and Niu [16] have shown that each Leslie-Gower model admits a carrying simplex.

I. Following the same lines as those in [16], we check that each Leslie-Gower model with $a_{ij} \geq 0$ and $a_{ii} > 0$ has a unique modified carrying simplex Σ by Theorem 2.4. Indeed, for $x \in X_i$, $f_i(x) = 1$ if and only if $x_i = \frac{c_i - 1}{a_{ii}} = q_i$, so T restricted to X_i has a unique fixed

point $Q_i = q_i e_i$. For $i, j \in I_N$,

$$\frac{\partial f_i}{\partial x_j} = -\frac{c_i a_{ij}}{[1 + \sum_{k=1}^N a_{ik} x_k]^2},$$

so $\frac{\partial f_i}{\partial x_j} \leq 0$ and $\frac{\partial f_i}{\partial x_i} < 0$. Also, for all $x \in C$,

$$\forall i \in I_N, f_i(x) + \sum_{j=1}^N x_j \frac{\partial f_i}{\partial x_j} = \frac{c_i}{[1 + \sum_{k=1}^N a_{ik} x_k]^2} > 0.$$

By Remark 2.2 (c), conditions (i)–(iii) of Theorem 2.4 and (11) are all met for any $r \gg q$.

The surfaces Γ_i are now $(N - 1)$ -dimensional planes in C :

$$\forall i \in I_N, \Gamma_i = \{x \in C : a_{i1}x_1 + \cdots + a_{iN}x_N = c_i - 1\}.$$

II. If for some $i \in I_N$, the following inequalities hold:

$$(23) \quad \forall j \in I_N \setminus \{i\}, \forall k \in I_N, a_{jk}(c_i - 1) < a_{ik}(c_j - 1),$$

then the intersection point of Γ_i with each positive half axis X_k is below Γ_j for every $j \in I_N \setminus \{i\}$. So Γ_i is strictly below Γ_j in C for all $j \in I_N \setminus \{i\}$. By Theorem 3.1 (a), the i th species is dominated.

III. If for some $i \in I_N$,

$$(24) \quad \forall j \in I_N \setminus \{i\}, \forall k \in I_N, \text{ either } a_{ik} = a_{jk} = 0 \text{ or } a_{ik}(c_j - 1) < a_{jk}(c_i - 1),$$

then either X_k is parallel to both Γ_i and Γ_j or the intersection point of Γ_j with X_k is below Γ_i , so Γ_i is strictly above Γ_j for all $j \in I_N \setminus \{i\}$. By Theorem 3.1 (b), the i th species is dominant and the fixed point Q_i is globally asymptotically stable.

IV. Note that (24) is a sufficient condition for Γ_i to be strictly above Γ_j for all $j \in I_N \setminus \{i\}$ in C . But the condition in Theorem 3.1 (b) only requires the relationship of such planes restricted to $[0, r]$. So (24) is much stronger than the requirement of Theorem 3.1 (b). For example, let us consider the three-dimensional Leslie-Gower model with

$$(25) \quad f_1(x) = \frac{2}{1 + x_1 + 0.25x_3}, f_2(x) = \frac{2}{1 + 2x_1 + x_2 + 0.2x_3}, f_3(x) = \frac{2}{1 + 2x_1 + x_3}.$$

Clearly, e_1 , e_2 and e_3 are the axial fixed points. Take $r = (1.1, 1.1, 1.1) \gg (1, 1, 1) = q$. The intersection points of Γ_3 with the X_1 and X_3 are $(0.5, 0, 0)$ and e_3 respectively. As $f_1(0.5, 0, 0) = \frac{2}{1.5} > 1$ and $f_1(e_3) = \frac{2}{1.25} > 1$, both $(0.5, 0, 0)$ and e_3 are below Γ_1 . Since X_2 is parallel to both Γ_1 and Γ_3 , Γ_1 is strictly above Γ_3 in \mathbb{R}_+^3 . The intersection points of Γ_2 with the axes are $(0.5, 0, 0)$, e_2 and $(0, 0, 5)$. We know that $(0.5, 0, 0)$ is below Γ_1 already. As $f_1(e_2) = 2 > 1$ but $f_1(0, 0, 5) = \frac{2}{2.25} < 1$, e_2 is below Γ_1 but $(0, 0, 5)$ is above Γ_1 . So Γ_1 is not above Γ_2 on \mathbb{R}_+^3 and (24) is not met. However, restricted to $[0, r]$, Γ_2 intersects one of the edges of $[0, r]$ at $(0.39, 0, 1.1)$ and $f_1(0.39, 0, 1.1) = \frac{2}{1.665} > 1$. So $\Gamma_2 \cap \pi_2 \cap [0, r]$ is strictly below Γ_1 . This, together with e_2 below Γ_1 , implies that $\Gamma_1 \cap [0, r]$ is strictly above

Γ_2 . By Theorem 3.1 (b), the first species is dominant and the fixed point e_1 is globally asymptotically stable.

V. Now suppose the following inequalities hold:

$$(26) \quad \forall i \in I_N \setminus \{N\}, \forall j, k \in \{i+1, \dots, N\}, \\ a_{ji}(c_i - 1) < a_{ii}(c_j - 1), a_{(i+1)k}(c_i - 1) < a_{ik}(c_{i+1} - 1).$$

Then, for each $i \in I_N \setminus \{N\}$, the intersection point of Γ_i with X_i is below Γ_j for all $j \in \{i+1, \dots, N\}$ and $\Gamma_i \cap (X_{i+1} \times \dots \times X_N)$ is strictly below Γ_{i+1} . Thus, $\Gamma_{N-1} \cap (X_{N-1} \times X_N)$ is strictly below Γ_N , $\Gamma_{N-2} \cap (X_{N-2} \times X_{N-1} \times X_N)$ is strictly below Γ_{N-1} and Γ_N, \dots , $\Gamma_i \cap (X_i \times \dots \times X_N)$ is strictly below Γ_j for all $j \in \{i+1, \dots, N\}$. By Theorem 3.2, the N th species is dominant and $Q_N = \frac{c_N-1}{a_{NN}}e_N$ is globally asymptotically stable.

4.2. The generalised competitive Atkinson-Allen models. The generalised competitive Atkinson-Allen models are systems (1) for maps $T : C \rightarrow C$ of the form (2): $T_i(x) = x_i f_i(x)$, where

$$(27) \quad \forall i, j \in I_N, f_i(x) = c_i + \frac{(1+u_i)(1-c_i)}{1 + \sum_{k=1}^N a_{ik}x_k}, 0 < c_i < 1, u_i > 0, a_{ij} \geq 0, a_{ii} > 0.$$

Under the condition that $a_{ij} > 0$ for all $i, j \in I_N$, Gyllenberg et al [6] have shown that each such model admits a carrying simplex.

I. Following the same lines as those in [6], we check that each generalised Atkinson-Allen model with $a_{ij} \geq 0$ and $a_{ii} > 0$ has a unique modified carrying simplex Σ by Theorem 2.4. Indeed, for $x \in X_i$, $f_i(x) = 1$ if and only if $x_i = \frac{u_i}{a_{ii}} = q_i$, so T restricted to X_i has a unique fixed point $Q_i = q_i e_i$. For $i, j \in I_N$,

$$\frac{\partial f_i}{\partial x_j} = -\frac{(1+u_i)(1-c_i)a_{ij}}{[1 + \sum_{k=1}^N a_{ik}x_k]^2},$$

so $\frac{\partial f_i}{\partial x_j} \leq 0$ and $\frac{\partial f_i}{\partial x_i} < 0$. Also, for all $x \in C$,

$$\forall i \in I_N, f_i(x) + \sum_{j=1}^N x_j \frac{\partial f_i}{\partial x_j} = c_i + \frac{(1+u_i)(1-c_i)}{[1 + \sum_{k=1}^N a_{ik}x_k]^2} > 0.$$

By Remark 2.2 (c), conditions (i)–(iii) of Theorem 2.4 and (11) are all met for any $r \gg q$.

The surfaces Γ_i are now $(N-1)$ -dimensional planes in C :

$$(28) \quad \forall i \in I_N, \Gamma_i = \{x \in C : a_{i1}x_1 + \dots + a_{iN}x_N = u_i\}.$$

II. If for some $i \in I_N$, the following inequalities hold:

$$(29) \quad \forall j \in I_N \setminus \{i\}, \forall k \in I_N, a_{jk}u_i < a_{ik}u_j,$$

then the intersection point of Γ_i with each positive half axis X_k is below Γ_j for every $j \in I_N \setminus \{i\}$. So Γ_i is strictly below Γ_j in C for all $j \in I_N \setminus \{i\}$. By Theorem 3.1 (a), the i th species is dominated.

III. If for some $i \in I_N$,

$$(30) \quad \forall j \in I_N \setminus \{i\}, \forall k \in I_N, \text{ either } a_{ik} = a_{jk} = 0 \text{ or } a_{ik}u_j < a_{jk}u_i,$$

then either X_k is parallel to both Γ_i and Γ_j or the intersection point of Γ_j with X_k is below Γ_i , so Γ_i is strictly above Γ_j for all $j \in I_N \setminus \{i\}$. By Theorem 3.1 (b), the i th species is dominant and the fixed point Q_i is globally asymptotically stable.

IV. Note that (30) is much stronger than the requirement of Theorem 3.1 (b). Similar to (25), we can easily construct a three-dimensional generalised Atkinson-Allen model as an example which fails (30) but satisfies the condition of Theorem 3.1 (b).

V. Now suppose the following inequalities hold:

$$(31) \quad \forall i \in I_N \setminus \{N\}, \forall j, k \in \{i+1, \dots, N\}, a_{ji}u_i < a_{ii}u_j, a_{(i+1)k}u_i < a_{ik}u_{i+1}.$$

Then, for each $i \in I_N \setminus \{N\}$, the intersection point of Γ_i with X_i is below Γ_j for all $j \in \{i+1, \dots, N\}$ and $\Gamma_i \cap (X_{i+1} \times \dots \times X_N)$ is strictly below Γ_{i+1} . Thus, $\Gamma_{N-1} \cap (X_{N-1} \times X_N)$ is strictly below Γ_N , $\Gamma_{N-2} \cap (X_{N-2} \times X_{N-1} \times X_N)$ is strictly below Γ_{N-1} and $\Gamma_N, \dots, \Gamma_i \cap (X_i \times \dots \times X_N)$ is strictly below Γ_j for all $j \in \{i+1, \dots, N\}$. By Theorem 3.2, the N th species is dominant and $Q_N = \frac{u_N}{a_{NN}}e_N$ is globally asymptotically stable.

VI. The standard Atkinson-Allen models are systems (1) for maps $T : C \rightarrow C$ of the form (2): $T_i(x) = x_i f_i(x)$, where

$$(32) \quad \forall i, j \in I_N, f_i(x) = c + \frac{2(1-c)}{1 + \sum_{k=1}^N a_{ik}x_k}, 0 < c < 1, a_{ij} \geq 0, a_{ii} > 0.$$

Note that f defined by (32) is a special case of (25) with $c_i = c$ and $u_i = 1$ for all $i \in I_N$. Thus, the results obtained above for generalised Atkinson-Allen models can be applied to the standard Atkinson-Allen models with simplified conditions (u_i, u_j replaced by 1 in (29)–(31)). For these models with $a_{ij} > 0$ for all $i, j \in I_N$, see [5], [15] and the references therein for further results.

4.3. The competitive Ricker models. The competitive Ricker models are systems (1) for maps $T : C \rightarrow C$ of the form (2): $T_i(x) = x_i f_i(x)$, where

$$(33) \quad \forall i, j \in I_N, f_i(x) = \exp \left[u_i \left(1 - \sum_{k=1}^N a_{ik}x_k \right) \right], u_i > 0, a_{ij} \geq 0, a_{ii} > 0.$$

Under the conditions that $a_{ij} > 0$ for all $i, j \in I_N$ and

$$(34) \quad \forall i \in I_N, u_i < a_{ii} / \sum_{j=1}^N a_{ij}, \text{ or } \forall i \in I_N, u_i < 1 / \sum_{j=1}^N \frac{a_{ij}}{a_{jj}},$$

Gyllenberg et al [8] have shown that each such model admits a carrying simplex.

I. We check that, under (34), each Ricker model with $a_{ij} \geq 0$ and $a_{ii} > 0$ has a unique modified carrying simplex Σ by Theorem 2.4. (i) For $x \in X_i$, $f_i(x) = 1$ if and only if $x_i = \frac{1}{a_{ii}} = q_i$, so T restricted to X_i has a unique fixed point $Q_i = q_i e_i$. (ii) For $i, j \in I_N$,

$$\frac{\partial f_i}{\partial x_j} = -u_i a_{ij} f_i(x),$$

so $\frac{\partial f_i}{\partial x_j} \leq 0$ and $\frac{\partial f_i}{\partial x_i} < 0$. (iii) For all $x \in [0, q]$, we have

$$\forall i \in I_N, f_i(x) + \sum_{j=1}^N x_j \frac{\partial f_i}{\partial x_j} = f_i(x) \left[1 - u_i \sum_{j=1}^N a_{ij} x_j \right] \geq f_i(x) \left[1 - u_i \sum_{j=1}^N \frac{a_{ij}}{a_{jj}} \right] > 0,$$

or

$$\forall i \in I_N, f_i(x) + x_i \sum_{j=1}^N \frac{\partial f_i}{\partial x_j} = f_i(x) \left[1 - u_i x_i \sum_{j=1}^N a_{ij} \right] \geq f_i(x) \left[1 - \frac{u_i}{a_{ii}} \sum_{j=1}^N a_{ij} \right] > 0.$$

By Remark 2.2 (c), conditions (i)–(iii) of Theorem 2.4 and (11) are all met for any $r \gg q$. Then, by Theorem 2.4, each Ricker model with (34) has a modified carrying simplex Σ .

The surfaces Γ_i are now $(N - 1)$ -dimensional planes in C :

$$\forall i \in I_N, \Gamma_i = \{x \in C : a_{i1}x_1 + \cdots + a_{iN}x_N = 1\}.$$

II. If for some $i \in I_N$, the following inequalities hold:

$$(35) \quad \forall j \in I_N \setminus \{i\}, \forall k \in I_N, a_{jk} < a_{ik},$$

then the intersection point of Γ_i with each positive half axis X_k is below Γ_j for every $j \in I_N \setminus \{i\}$. So Γ_i is strictly below Γ_j in C for all $j \in I_N \setminus \{i\}$. By Theorem 3.1 (a), the i th species is dominated.

III. If for some $i \in I_N$,

$$(36) \quad \forall j \in I_N \setminus \{i\}, \forall k \in I_N, \text{ either } a_{ik} = a_{jk} = 0 \text{ or } a_{ik} < a_{jk},$$

then either X_k is parallel to both Γ_i and Γ_j or the intersection point of Γ_j with X_k is below Γ_i , so Γ_i is strictly above Γ_j for all $j \in I_N \setminus \{i\}$. By Theorem 3.1 (b), the i th species is dominant and the fixed point Q_i is globally asymptotically stable.

IV. Note that (36) is much stronger than the requirement of Theorem 3.1 (b). Similar to (25), we can easily construct a three-dimensional Ricker model as an example which fails (36) but satisfies the condition of Theorem 3.1 (b).

V. Now suppose the following inequalities hold:

$$(37) \quad \forall i \in I_N \setminus \{N\}, \forall j, k \in \{i + 1, \dots, N\}, a_{ji} < a_{ii}, a_{(i+1)k} < a_{ik}.$$

Then, for each $i \in I_N \setminus \{N\}$, the intersection point of Γ_i with X_i is below Γ_j for all $j \in \{i + 1, \dots, N\}$ and $\Gamma_i \cap (X_{i+1} \times \cdots \times X_N)$ is strictly below Γ_{i+1} . Thus, $\Gamma_{N-1} \cap (X_{N-1} \times X_N)$

is strictly below Γ_N , $\Gamma_{N-2} \cap (X_{N-2} \times X_{N-1} \times X_N)$ is strictly below Γ_{N-1} and Γ_N, \dots , $\Gamma_i \cap (X_i \times \dots \times X_N)$ is strictly below Γ_j for all $j \in \{i+1, \dots, N\}$. By Theorem 3.2, the N th species is dominant and $Q_N = \frac{1}{a_{NN}}e_N$ is globally asymptotically stable.

4.4. General competitive models with plane nullclines. In [12], the competitive models given by system (1) for maps $T : C \rightarrow C$ of the form (2) $T_i(x) = x_i f_i(x)$ are considered, where

$$(38) \quad \forall i \in I_N, f_i(x) = G_i((Ax)_i)$$

with

$$(39) \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}$$

satisfying $a_{ii} > 0$ and $a_{ij} \geq 0$, $(Ax)_i$ denoting the i th component of Ax . Assume that the functions $G_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfy the following conditions:

(a1) Each G_i is positive and strictly decreasing with $G_i(u_i) = 1$ and $G'_i(u_i) < 0$ for some $u_i > 0$.

(a2) For $x \in C$ and each $i \in I_N$, $\frac{\partial T_i}{\partial x_i} > 0$ for $0 \leq x_i \leq \frac{u_i}{a_{ii}} = q_i$.

Then each nullcline surface Γ_i is a hyperplane given by (28). Under (a1), (a2) and another condition, criteria are established in [12] for global stability of a fixed point by geometric method of using the relative positions of the nullcline planes in $[0, q]$. The matrix $M(x)$ defined by (4) is $M(x) = -\text{diag}(\frac{x_i G'_i((Ax)_i)}{G_i((Ax)_i)})A$.

I. Assume that

$$(40) \quad \forall x \in [0, q], \rho(M(x)) < 1.$$

We check that each such model has a unique modified carrying simplex Σ by Theorem 2.4.

(i) For $x \in X_i$, $G_i((Ax)_i) = 1$ if and only if $(Ax)_i = u_i$, i.e. $x_i = \frac{u_i}{a_{ii}} = q_i$, so T restricted to X_i has a unique fixed point $Q_i = q_i e_i$. (ii) For $i, j \in I_N$,

$$\frac{\partial G_i((Ax)_i)}{\partial x_j} = a_{ij} G'_i((Ax)_i).$$

By $a_{ij} \geq 0$, $a_{ii} > 0$ and (a1), $\frac{\partial G_i((Ax)_i)}{\partial x_j} \leq 0$ and $G_i((Ax)_i)$ is strictly decreasing in x_i . Condition (iii) of Theorem 2.4 follows from (40). Then, by Theorem 2.4, each model with (38) has a modified carrying simplex Σ .

By the same reasoning as that given for generalised Atkinson-Allen models, we obtain the following conclusions.

II. If (29) holds, then the i th species is dominated.

III. If (30) holds, then the i th species is dominant and Q_i is globally asymptotically stable.

IV. If (31) holds, then the N th species is dominant and Q_N is asymptotically stable.

5. PROOF OF THE MAIN THEOREMS

In this section, we aim at providing complete proofs for Theorem 2.3, Theorem 2.4 and Corollary 2. Although some of the ideas used here are credited to [9] and [19, 20], for readers' convenience we present an independent proof rather than citing some lemmas and theorems and modifying their proofs bit by bit. However, this does not mean that the proofs are trivial modifications from those in the references. Actually, the author's main contribution in this paper is the sharp observation that the system permits a modifies carrying simplex if the retrotone property for T is relaxed to weakly retrotone, which leads to the dramatic relaxation of the conditions of Theorem 1.2 to those of Theorem 2.4. To prove these results, in addition to inheriting some techniques shown in Lemma 5.2, Lemma 5.3, the main part of the proof of Theorem 2.3 and a small part of the proof of Theorem 2.4, the author's own methods and techniques are reflected in Lemma 5.1, Lemma 5.4, Lemma 5.5, the main part of the proof of Theorem 2.4 and Corollary 2.

Lemma 5.1. *Assume that T satisfies the conditions of Theorem 2.3. Then, for any $x \in [0, r]$, $[0, T(x)] \subset T([0, r])$.*

Proof. By Remark 2.1 (b) we know that T is a homeomorphism from $[0, r]$ to $T([0, r])$. Thus, T maps an open set of $[0, r]$ to an open set of $T([0, r])$. Clearly, the set

$$[0, r) = \{x \in [0, r] : 0 \leq x \ll r\}$$

is open in $[0, r]$, so $T([0, r))$ is also open in $T([0, r])$. We first show that

$$(41) \quad \forall x \in [0, r), [0, T(x)] \subset T([0, r)).$$

Suppose (41) is not true. Then, for some $x \in [0, r) \setminus \{0\}$, there is a y satisfying $0 < y < T(x)$ but $y \notin T([0, r))$. Since $T([0, r))$ is open and $T(x) \in T([0, r))$, there is an $s_0 \in [0, 1)$ such that $y(s) = y + s(T(x) - y) \in T([0, r))$ for $s \in (s_0, 1]$ but $y(s_0) = y + s_0(T(x) - y) \notin T([0, r))$. As T is weakly retrotone, we have $0 < z(s) = T^{-1}(y(s)) < x$ for $s \in (s_0, 1)$ and $z(s_1) < z(s_2)$ for any $s_0 < s_1 < s_2 \leq 1$. Thus, $\lim_{s \rightarrow s_0+} z(s)$ exists. Define $z(s_0) = \lim_{s \rightarrow s_0+} z(s)$. Then $z(s_0) \in [0, x] \subset [0, r)$ so $T(z(s_0)) \in T([0, r))$. By continuity of T ,

$$T(z(s_0)) = \lim_{s \rightarrow s_0+} T(z(s)) = \lim_{s \rightarrow s_0+} y(s) = y(s_0),$$

a contradiction to $y(s_0) \notin T([0, r))$. This contradiction shows the truth of (41).

Now we show that $[0, T(x)] \subset T([0, r])$ for all $x \in [0, r]$. This is true by (41) if $x \in [0, r)$, so we suppose $x \in [0, r] \setminus [0, r)$. Then $T(x) \in T([0, r] \setminus [0, r))$ and $x(s) = sx \in [0, r)$ for all $s \in [0, 1)$ with $\lim_{s \rightarrow 1-} x(s) = x$. Moreover, by (41), $[0, T(x(s))] \subset T([0, r)) \subset T([0, r])$ for all $s \in [0, 1)$. For each $y \in [0, T(x)]$, if $y \in [0, T(x(s))]$ for some $s \in [0, 1)$ then $y \in T([0, r])$; if $y \notin [0, T(x(s))]$ for any $s \in [0, 1)$ then there is an increasing sequence $\{s_n\} \subset [0, 1)$ with

$s_n \uparrow 1$ and a sequence $\{y_n\}$ with $y_n \in [0, T(x(s_n))] \subset T([0, r])$ such that $\lim_{n \rightarrow \infty} y_n = y$. So $y \in \overline{T([0, r])}$. But since $T([0, r]) \subset T([0, r])$, $T([0, r])$ is open and $T([0, r])$ is closed, we have $\overline{T([0, r])} \subset T([0, r])$ so $y \in T([0, r])$. This shows that $[0, T(x)] \subset T([0, r])$ for all $x \in [0, r]$. \square

For any $x \in [0, r]$, we denote the image of x under $(T^{-1})^k$ by $x(-k)$ if $(T^{-1})^k(x) = T^{-k}(x)$ exists.

Lemma 5.2. *Assume that the conditions of Theorem 2.3 hold. Suppose $x \in [0, r] \setminus \{0\}$ such that $(T^{-1})^k(x)$ exists and $x(-k) \in [0, r]$ for all $k \in \mathbb{N}$. Then, for any $y \in [0, r]$ with $y < x$ and $x - y \in \dot{C}_I$ for some nonempty $I \subset I_N$, $y(-k)$ exists in $[0, r]$ for all $k \in \mathbb{N}$ and*

$$(42) \quad \forall i \in I, \lim_{k \rightarrow \infty} y_i(-k) = 0.$$

Proof. By the existence of $x(-1) \in [0, r]$ we have $x = T(T^{-1}(x)) = T(x(-1)) \in T([0, r])$. Thus, by Lemma 5.1, $[0, x] \subset T([0, r])$. As $y \in [0, r]$ and $y < x$, we have $y \in T([0, r])$ so $y(-1)$ exists and $y(-1) \in [0, r]$. It then follows from the weak retrone property of T that $y(-1) < x(-1)$ and $y_i(-1) < x_i(-1)$ for all $i \in I$. If $y(-k) = (T^{-1})^k(y)$ exists, $y(-k) < x(-k)$ and $y_i(-k) < x_i(-k)$ for all $i \in I$ and some $k \in \mathbb{N}$, by the same reasoning as above we obtain the existence of $y(-k-1) = (T^{-1})^{k+1}(y)$, $y(-k-1) < x(-k-1)$ and $y_i(-k-1) < x_i(-k-1)$ for all $i \in I$. By induction, we see the existence of $y(-k) \in [0, r]$ with $y(-k) < x(-k)$ and $y_i(-k) < x_i(-k)$ for all $i \in I$ and all $k \in \mathbb{N}$.

To prove (42) by contradiction, we suppose the existence of $i \in I$ such that $0 < y_i < x_i$ and $y_i(-k) \not\rightarrow 0$ as $k \rightarrow \infty$. As $x(-k), y(-k) \in [0, r]$ for all $k \in \mathbb{N}$ and $[0, r]$ is compact, we can select a subsequence $\{\sigma(k)\} \subset \{k\}$ such that

$$\lim_{k \rightarrow \infty} x(-\sigma(k)) = \bar{x}, \lim_{k \rightarrow \infty} y(-\sigma(k)) = \bar{y}, \bar{y}_i > 0.$$

By $0 < y_i < x_i$ we have $0 < y_i(-k) < x_i(-k)$ for all $k \in \mathbb{N}$. Now define

$$\Delta(k) = \frac{y_i(-k)}{x_i(-k)}, k \in \mathbb{N}.$$

Then

$$1 > \Delta(k) = \frac{T_i(y(-k-1))}{T_i(x(-k-1))} = \frac{y_i(-k-1)f_i(y(-k-1))}{x_i(-k-1)f_i(x(-k-1))} = \Delta(k+1) \frac{f_i(y(-k-1))}{f_i(x(-k-1))}.$$

By condition (iii) of Theorem 2.3, $\frac{f_i(y(-k-1))}{f_i(x(-k-1))} > 1$. So

$$\forall k \in \mathbb{N}, 0 < \Delta(k+1) < \Delta(k) < 1.$$

This shows the existence of a $\beta \in [0, 1)$ such that $\lim_{k \rightarrow \infty} \Delta(k) = \beta$. In particular,

$$1 > \beta = \lim_{k \rightarrow \infty} \Delta(\sigma(k)) = \lim_{k \rightarrow \infty} \frac{y_i(-\sigma(k))}{x_i(-\sigma(k))} = \frac{\bar{y}_i}{\bar{x}_i} > 0,$$

so $0 < \bar{y}_i = \beta \bar{x}_i < \bar{x}_i$. By continuity of T , $T(x(-\sigma(k))) \rightarrow T(\bar{x})$ and $T(y(-\sigma(k))) \rightarrow T(\bar{y})$ as $k \rightarrow \infty$. Thus,

$$\beta = \lim_{k \rightarrow \infty} \Delta(\sigma(k) - 1) = \lim_{k \rightarrow \infty} \frac{y_i(-\sigma(k) + 1)}{x_i(-\sigma(k) + 1)} = \lim_{k \rightarrow \infty} \frac{T_i(y(-\sigma(k)))}{T_i(x(-\sigma(k)))} = \frac{T_i(\bar{y})}{T_i(\bar{x})}.$$

From this we obtain $T_i(\bar{y}) = \beta T_i(\bar{x}) < T_i(\bar{x})$. As $y(-k) < x(-k)$ for all $k \in \mathbb{N}$, we have $T(y(-\sigma(k))) = y(-\sigma(k) + 1) < x(-\sigma(k) + 1) = T(x(-\sigma(k)))$ and, as $k \rightarrow \infty$, $T(\bar{y}) < T(\bar{x})$. By condition (iii) of Theorem 2.3, we have $\frac{f_i(\bar{y})}{f_i(\bar{x})} > 1$. As $\lim_{k \rightarrow \infty} \frac{f_i(y(-\sigma(k)))}{f_i(x(-\sigma(k)))} = \frac{f_i(\bar{y})}{f_i(\bar{x})}$ and $\frac{f_i(y(-k))}{f_i(x(-k))} > 1$ for all $k \in \mathbb{N}$, there is an $\eta > 1$ such that $\frac{f_i(y(-\sigma(k)))}{f_i(x(-\sigma(k)))} \geq \eta$ for all $k \in \mathbb{N}$. Thus,

$$\Delta(\sigma(k) - 1) = \frac{T_i(y(-\sigma(k)))}{T_i(x(-\sigma(k)))} = \Delta(\sigma(k)) \frac{f_i(y(-\sigma(k)))}{f_i(x(-\sigma(k)))} \geq \eta \Delta(\sigma(k)).$$

From this and $\sigma(k) - 1 \geq \sigma(k - 1)$ for $k > 1$ we obtain

$$\forall k > 1, \Delta(\sigma(k)) \leq \frac{1}{\eta} \Delta(\sigma(k) - 1) \leq \frac{1}{\eta} \Delta(\sigma(k - 1)).$$

This implies that

$$\forall k \in \mathbb{N}, \beta < \Delta(\sigma(k + 1)) \leq \frac{1}{\eta^k} \Delta(\sigma(1)).$$

Letting $k \rightarrow \infty$, we obtain $\beta = 0$, a contradiction to $\beta = \frac{\bar{y}_i}{\bar{x}_i} > 0$. This contradiction shows the truth of (42). \square

Lemma 5.3. *Suppose the existence of $x, y \in [0, r] \setminus \{0\}$ with support $I(x) = I(y) \subset I_N$ satisfying $x(k) \leq y(k)$ for all $k \in \mathbb{N}$. Then, under the conditions of Theorem 2.3,*

$$(43) \quad \lim_{k \rightarrow \infty} (y(k) - x(k)) = 0.$$

Proof. If $x(k_0) = y(k_0)$ for some $k_0 \in \mathbb{N}$ then $y(k) = x(k)$ for all $k \geq k_0$ so (43) holds. Now assume that $x(k) < y(k)$ for all $k \in \mathbb{N}$. For each $i \in I(x)$, if there is a $k_1 \in \mathbb{N}$ such that $x_i(k_1) = y_i(k_1)$, then we must have $x_i(k_1 + 1) = y_i(k_1 + 1)$, for the inequality $T_i(x(k_1)) = x_i(k_1 + 1) < y_i(k_1 + 1) = T_i(y(k_1))$ and condition (ii) of Theorem 2.3 would imply $x_i(k_1) < y_i(k_1)$. Thus, $x_i(k) = y_i(k)$ for all $k \geq k_1$ so $\lim_{k \rightarrow \infty} (y_i(k) - x_i(k)) = 0$. Now suppose for a fixed $i \in I(x)$,

$$\forall k \in \mathbb{N}, 0 < x_i(k) < y_i(k).$$

Define $\delta(k) = \frac{x_i(k)}{y_i(k)}$ for all $k \in \mathbb{N}$. Then

$$1 > \delta(k + 1) = \frac{T_i(x(k))}{T_i(y(k))} = \delta(k) \frac{f_i(x(k))}{f_i(y(k))}.$$

As $\frac{f_i(x(k))}{f_i(y(k))} > 1$ by condition (iii) of Theorem 2.3, $\{\delta(k)\}$ is a positive increasing sequence bounded above by 1. If $\lim_{k \rightarrow \infty} \delta(k) = 1$ then

$$y_i(k) - x_i(k) = y_i(k)[1 - \delta(k)] \rightarrow 0 \quad (k \rightarrow \infty).$$

Suppose $\lim_{k \rightarrow \infty} \delta(k) = \beta$ for some $\beta \in (0, 1)$. If $\lim_{k \rightarrow \infty} (y_i(k) - x_i(k)) \neq 0$, there must be a subsequence $\{\sigma(k)\} \subset \{k\}$ such that

$$\lim_{k \rightarrow \infty} x(\sigma(k)) = \bar{x}, \lim_{k \rightarrow \infty} y(\sigma(k)) = \bar{y}, \bar{x} < \bar{y}, \bar{x}_i < \bar{y}_i.$$

Then,

$$(44) \quad 1 > \delta(\sigma(k+1)) \geq \delta(\sigma(k) + 1) = \frac{T_i(x(\sigma(k)))}{T_i(y(\sigma(k)))} = \delta(\sigma(k)) \frac{f_i(x(\sigma(k)))}{f_i(y(\sigma(k)))}.$$

By condition (iii) of Theorem 2.3 again, $\frac{f_i(x(\sigma(k)))}{f_i(y(\sigma(k)))} > 1$. As

$$T(x(\sigma(k))) = x(\sigma(k) + 1) < y(\sigma(k) + 1) = T(y(\sigma(k))),$$

$\lim_{k \rightarrow \infty} T(x(\sigma(k))) = T(\bar{x})$ and $\lim_{k \rightarrow \infty} T(y(\sigma(k))) = T(\bar{y})$, we have $T(\bar{x}) \leq T(\bar{y})$. If $T_i(\bar{x}) < T_i(\bar{y})$ then $T(\bar{x}) < T(\bar{y})$. By condition (iii) of Theorem 2.3, we obtain $\frac{f_i(\bar{x})}{f_i(\bar{y})} > 1$. If $T_i(\bar{x}) = T_i(\bar{y})$, then

$$1 = \frac{T_i(\bar{x})}{T_i(\bar{y})} = \frac{\bar{x}_i f_i(\bar{x})}{\bar{y}_i f_i(\bar{y})}, \quad \frac{f_i(\bar{x})}{f_i(\bar{y})} = \frac{\bar{y}_i}{\bar{x}_i} > 1.$$

Therefore, there is an $\eta > 1$ such that $\frac{f_i(x(\sigma(k)))}{f_i(y(\sigma(k)))} \geq \eta$ for all $k \in \mathbb{N}$. Then, from (44) we obtain

$$\delta(\sigma(k+1)) \geq \eta \delta(\sigma(k)) \geq \eta^k \delta(\sigma(1)) \rightarrow +\infty \quad (k \rightarrow \infty),$$

a contradiction to $\delta(k) < 1$. This contradiction shows the conclusion (43). \square

Under the assumptions of Theorem 2.3, $T([0, r]) \subset [0, r]$. By Remark 2.1 (b), $T : [0, r] \rightarrow T([0, r])$ is a homeomorphism, so T maps open sets to open sets and closed sets to closed sets. As

$$\forall n \in \mathbb{N}, T^{n+1}([0, r]) \subset T^n([0, r])$$

and $[0, r]$ is compact, $T^n([0, r])$ is compact for all $n \in \mathbb{N}$. From Remark 2.1 (e) we know that 0 is a repeller with basin of repulsion $\mathcal{B}(0) \subset [0, r]$.

Lemma 5.4. *Assume that the conditions of Theorem 2.3 hold. Let*

$$(45) \quad A_0 = \bigcap_{n=0}^{\infty} T^n([0, r]).$$

Then A_0 is nonempty, compact, invariant and $A_0 = \overline{\mathcal{B}(0)}$.

Proof. That $A_0 \neq \emptyset$ is obvious as $0, q_i e_i$ and all fixed points of T are in A_0 . As each $T^n([0, r])$ is compact and any nonempty intersection of compact sets is compact, by (45) A_0 is compact. The invariance of A_0 follows from (45) and $T([0, r]) \subset [0, r]$. Clearly, by (45) we see that A_0 is the largest invariant set of T in $[0, r]$. As $\mathcal{B}(0)$ is an open subset of $[0, r]$ and invariant, we have $\mathcal{B}(0) \subset A_0$. To show that $A_0 = \overline{\mathcal{B}(0)}$, we take an arbitrary point $x \in A_0 \setminus \mathcal{B}(0)$ and show that $x \in \overline{\mathcal{B}(0)}$. This is trivial if $x = 0$ as $0 \in \overline{\mathcal{B}(0)}$. If $x \neq 0$ then there is a nonempty $I \subset I_N$ such that $x \in \dot{C}_I$. Moreover, $u_s = sx \ll_I x$ for all $s \in (0, 1)$. By the invariance of A_0 , $x(-k)$ exists in A_0 for all $k \in \mathbb{N}$. Then, by Lemma

5.2, $u_s(-k)$ exists in $[0, r]$ for all $s \in (0, 1)$ and all $k \in \mathbb{N}$ and $\lim_{k \rightarrow \infty} u_s(-k) = 0$. Thus, $u_s \in \mathcal{B}(0)$ for all $s \in (0, 1)$. Since $x = \lim_{s \rightarrow 1^-} u_s$, we have $x \in \overline{\mathcal{B}(0)}$. \square

With the help of Lemmas 5.1–5.4 we are now in a position to prove Theorem 2.3.

Proof of Theorem 2.3. Let $\Sigma = \overline{\mathcal{B}(0)} \setminus (\{0\} \cup \mathcal{B}(0))$. We verify that Σ is a modified carrying simplex. Clearly, $\Sigma \subset [0, r] \setminus \{0\}$ and $\Sigma \neq \emptyset$ as all the nontrivial fixed points are in Σ . From Lemma 5.4 we see that $\overline{\mathcal{B}(0)}$ is compact and invariant. As $\{0\} \cup \mathcal{B}(0)$ is open and invariant, Σ is compact and invariant. That $T : \Sigma \rightarrow \Sigma$ is a homeomorphism follows from T being a homeomorphism from $[0, r]$ to $T([0, r])$.

To show that Σ is homeomorphic to Δ^{N-1} by radial projection, we define a map $m : \Delta^{N-1} \rightarrow \Sigma$ as follows. For each $x \in \Delta^{N-1}$, as $\lambda x \in \mathcal{B}(0)$ for sufficiently small $\lambda > 0$ and $\mathcal{B}(0)$ is open, there is a unique $\lambda_0 = \lambda_0(x) > 0$ such that $\lambda x \in \mathcal{B}(0)$ for all $0 < \lambda < \lambda_0$ but $\lambda_0 x \notin \mathcal{B}(0)$. Since $\lim_{\lambda \rightarrow \lambda_0} \lambda x = \lambda_0 x \neq 0$, we have $\lambda_0 x \in \overline{\mathcal{B}(0)}$ so $\lambda_0 x \in \Sigma$. We claim that $\lambda x \notin A_0$ for $\lambda > \lambda_0$, where A_0 is given by (45). Indeed, if there is a $\lambda_1 > \lambda_0$ such that $u = \lambda_1 x \in A_0$, then $u(-k) \in A_0$ for all $k \in \mathbb{N}$. By Lemma 5.2, we would have $\lambda_0 x \in \mathcal{B}(0)$, a contradiction to $\lambda_0 x \notin \mathcal{B}(0)$. Thus,

$$\forall x \in \Delta^{N-1}, \Sigma \cap \{\lambda x : \lambda > 0\} = \{\lambda_0(x)x\}.$$

Then the map $m : \Delta^{N-1} \rightarrow \Sigma$ defined by $m(x) = \lambda_0(x)x$ is a bijection. The map m is a homeomorphism if m and m^{-1} are continuous.

To show that m is continuous, we need only show that $\lambda_0 : \Delta^{N-1} \rightarrow \mathbb{R}_+$ is continuous. Suppose λ_0 is not continuous at a point $x_0 \in \Delta^{N-1}$, i.e. $\lim_{x \rightarrow x_0} \lambda_0(x) \neq \lambda_0(x_0)$. Since λ_0 is obviously bounded, there is a sequence $\{x_k\} \subset \Delta^{N-1}$ such that

$$x_k \rightarrow x_0 \text{ and } \lambda_0(x_k) \rightarrow \mu \neq \lambda_0(x_0) \text{ as } k \rightarrow \infty.$$

Then $\{m(x_k)\} \subset \Sigma$ and $m(x_k) = \lambda_0(x_k)x_k \rightarrow \mu x_0$ as $k \rightarrow \infty$. Since Σ is compact, we have $\mu x_0 \in \Sigma$. This contradicts $\Sigma \cap \{\lambda x : \lambda > 0\} = \{\lambda_0(x_0)x_0\} \neq \{\mu x_0\}$. This contradiction shows the continuity of m on Δ^{N-1} .

To show that $m^{-1} : \Sigma \rightarrow \Delta^{N-1}$ is continuous, since the continuity of m implies that Σ is a continuous surface, for each $y \in \Sigma$, there is a unique $\mu = \mu(y) > 0$ such that $\mu(y)y \in \Delta^{N-1}$ so that $m^{-1}(y) = \mu(y)y$. Then the continuity of m^{-1} follows from showing the continuity of $\mu : \Sigma \rightarrow \mathbb{R}_+$ by the same technique as above. Therefore, Σ is homeomorphic to Δ^{N-1} by radial projection.

Next, we show that for each $x \in [0, r] \setminus \{0\}$, if x is above Σ then $\omega(x) \subset \Sigma$; if x is below Σ then there is a $y \in \Sigma$ with support $I(y) = I(x)$ such that

$$(46) \quad \lim_{k \rightarrow +\infty} [x(k) - y(k)] = 0.$$

Now suppose x is above Σ . By Lemma 5.4 we have $\omega(x) \subset A_0$. As $\mathcal{B}(0)$ does not contain any positive limit point and $0 \notin \omega(x)$, we must have $\omega(x) \subset \Sigma$.

Next, suppose x is below Σ with support $I(x) \subset I_N$. As $\overline{\mathcal{B}(0)} = \{0\} \cup \mathcal{B}(0) \cup \Sigma$, we have $\Sigma^- = \{0\} \cup \mathcal{B}(0)$. By $x > 0$, we must have $x \in \mathcal{B}(0)$. Define sets

$$\forall k \in \mathbb{N}, U(k, x) = \{y \in \Sigma : I(y) = I(x), x(k) < y(k)\}.$$

Note that $T(\mathcal{B}(0)) = \mathcal{B}(0)$ and $T(\Sigma) = \Sigma$. For each fixed $k \in \mathbb{N}$, $x(k)$ is below Σ so there is $\mu_0 > 1$ such that $\mu_0 x(k) \in \Sigma$ but $\mu x(k) \in \mathcal{B}(0)$ for $1 \leq \mu < \mu_0$. Taking $y = T^{-k}(\mu_0 x(k))$ we have $y \in \Sigma$, $I(y) = I(x)$ and $y(k) = \mu_0 x(k) > x(k)$. Thus, $y \in U(k, x)$ so $U(k, x) \neq \emptyset$. For each $z \in U(k+1, x)$ we have

$$T(x(k)) = x(k+1) < z(k+1) = T(z(k)).$$

As T is weakly retrotone, we must have $x(k) < z(k)$ so $z \in U(k, x)$. This shows that

$$\forall k \in \mathbb{N}, U(k+1, x) \subset U(k, x).$$

From the definition we see that each $U(k, x)$ is compact. Then $\emptyset \neq \bigcap_{k=0}^{\infty} U(k, x) \subset \Sigma$. Taking any $y \in \bigcap_{k=0}^{\infty} U(k, x)$ we obtain $x(k) < y(k)$ for all $k \in \mathbb{N}$ so (46) follows from Lemma 5.3.

So far we have proved that $\Sigma = \overline{\mathcal{B}(0)} \setminus (\{0\} \cup \mathcal{B}(0))$ is a modified carrying simplex. Now for each $p \in \Sigma$ and every $q \in [0, r] \setminus \{0\}$ with $q < p$, by Lemma 5.2 we know that $q(-k) \in [0, r]$ exists for all $k \in \mathbb{N}$ and $\lim_{k \rightarrow +\infty} q_i(-k) = 0$, so $\alpha(q) \subset \pi_i$, for any $i \in I_N$ with $q_i < p_i$.

Finally, we show the uniqueness of the modified carrying simplex Σ . Suppose we have another modified carrying simplex $\Sigma_1 \neq \Sigma$. Then, on a half line starting from the origin we have two distinct points $p \in \Sigma$ and $q \in \Sigma_1$ so there is a positive number $\lambda \neq 1$ such that $p = \lambda q$. Clearly p and q have the same support $I \subset I_N$ so we have either $p \ll_I q$ or $q \ll_I p$. In the first case, by Lemma 5.2 we would have $\alpha(p) = \{0\}$, a contradiction to $\alpha(p) \subset \Sigma$ as $0 \notin \Sigma$. In the second case, by Lemma 5.2 again we would have $\alpha(q) = \{0\}$, a contradiction to $\alpha(q) \subset \Sigma_1$ as $0 \notin \Sigma_1$. This shows that Σ is the unique modified carrying simplex. \square

To prove Theorem 2.4, we need the following lemma.

Lemma 5.5. *Let U be a small neighbourhood of $[0, r]$ and $T \in C^1(U, U)$. Assume that the Jacobian matrix $DT(x)$ is invertible on $[0, r]$ with $(DT(x))^{-1} = (t_{ij})$. If*

$$\forall x \in U, \forall i, j \in I_N, t_{ii}(x) > 0 \text{ and } t_{ij}(x) \geq 0,$$

then T from U to $T(U)$ is one-to-one and is weakly retrotone on $[0, r]$.

Proof. Since $DT(x)$ is continuous on U and invertible on $[0, r]$, there is a small neighbourhood $U_1 \subset U$ of $[0, r]$ such that $DT(x)$ is invertible on U_1 . Without loss of generality, we assume that $U_1 = U$. By the inverse function theorem, T from U to $T(U)$ is one-to-one and invertible. Moreover, T^{-1} on $T(U)$ is differentiable. As $g(u) = u = T^{-1}(T(u))$ for $u \in U$, by the chain rule of differentiation we have

$$I = Dg(u) = [D(T^{-1})(T(u))][DT(u)],$$

so

$$(47) \quad D(T^{-1})(T(u)) = (DT(u))^{-1} = (t_{ij}(u)).$$

Now for any $x, y \in [0, r]$ with $T(x) < T(y)$ and $T(y) - T(x) \in \dot{C}_I$ for some nonempty $I \subset I_N$, we have

$$\begin{aligned} y - x &= T^{-1}(T(y)) - T^{-1}(T(x)) \\ &= T^{-1}(T(x) + s(T(y) - T(x)))|_0^1 \\ &= \int_0^1 \frac{d}{ds} T^{-1}(T(x) + s(T(y) - T(x))) ds \\ &= \int_0^1 D(T^{-1})(T(x) + s(T(y) - T(x))) ds (T(y) - T(x)). \end{aligned}$$

By the assumption on the entries of $(DT(u))^{-1}$ and (47), the diagonal entries of the matrix $D(T^{-1})(T(x) + s(T(y) - T(x)))$ are positive and other entries are nonnegative. As the matrix

$$\int_0^1 D(T^{-1})(T(x) + s(T(y) - T(x))) ds$$

maintains the same feature as $D(T^{-1})(T(x) + s(T(y) - T(x)))$, for each $i \in I$, $T(y) - T(x) > 0$ and $T_i(y) - T_i(x) > 0$ imply $y - x > 0$ and $y_i - x_i > 0$. Thus, T on $[0, r]$ is weakly retrotone. \square

Proof of Theorem 2.4. We need only show that conditions (ii) and (iii) of Theorem 2.4 imply conditions of (ii) and (iii) of Theorem 2.3. Since

$$DT(x) = \text{diag}(f_1(x), \dots, f_N(x))(I - M(x)),$$

where I is the identity matrix and $M(x)$ is given by (4), if $\rho(M(x)) < 1$ by (iii) of Theorem 2.4, then $DT(x)$ is invertible with

$$\begin{aligned} (DT(x))^{-1} &= (I - M(x))^{-1} \text{diag}\left(\frac{1}{f_1(x)}, \dots, \frac{1}{f_N(x)}\right) \\ &= \left(I + \sum_{k=1}^{\infty} M^k(x)\right) \text{diag}\left(\frac{1}{f_1(x)}, \dots, \frac{1}{f_N(x)}\right). \end{aligned}$$

From this it is clear that each diagonal entry of $(DT(x))^{-1}$ is positive and other entries are nonnegative. Then condition (ii) of Theorem 2.3 follows from Lemma 5.5.

Now suppose $\rho(\tilde{M}(x)) < 1$ holds instead of $\rho(M(x)) < 1$, where $\tilde{M}(x)$ is given by (6). If $x \gg 0$, then

$$\text{diag}\left(\frac{1}{x_1}, \dots, \frac{1}{x_N}\right) DT(x) \text{diag}(x_1, \dots, x_N) = \text{diag}(f_1(x), \dots, f_N(x))(I - \tilde{M}(x)),$$

so $DT(x)$ is invertible with

$$\begin{aligned} (DT(x))^{-1} &= \text{diag}(x_1, \dots, x_N)(I - \tilde{M}(x))^{-1} \text{diag}\left(\frac{1}{x_1 f_1(x)}, \dots, \frac{1}{x_N f_N(x)}\right) \\ &= \text{diag}(x_1, \dots, x_N)\left(I + \sum_{k=1}^{\infty} \tilde{M}^k(x)\right) \text{diag}\left(\frac{1}{x_1 f_1(x)}, \dots, \frac{1}{x_N f_N(x)}\right). \end{aligned}$$

From this we see that each diagonal entry of $(DT(x))^{-1}$ is positive and other entries are nonnegative. Then condition (ii) of Theorem 2.3 follows from Lemma 5.5.

If $\rho(\tilde{M}(x)) < 1$ holds with $x \gg 0$, then there is a proper subset $J_1 \subset I_N$ as the support of x . Without loss of generality, we assume that $J_1 = \{1, \dots, k\}$ for some positive integer $k < N$ (as we can always rearrange the order of the components). Let $J_2 = \{k+1, \dots, N\}$ and $U = \text{diag}(x_1, \dots, x_k, 1, \dots, 1)$. Then

$$U^{-1}DT(x)U = \text{diag}(f_1(x), \dots, f_N(x)) \begin{pmatrix} I_1 - M_1(x) & -M_2(x) \\ 0 & I_2 \end{pmatrix},$$

where I_1 and I_2 are $k \times k$ and $(N-k) \times (N-k)$ identity matrices respectively, and

$$\begin{aligned} M_1(x) &= \left(-\frac{x_j}{f_i(x)} \frac{\partial f_i}{\partial x_j}(x) \right)_{k \times k}, \text{ for } i, j \in J_1, \\ M_2(x) &= \left(-\frac{1}{f_i(x)} \frac{\partial f_i}{\partial x_j}(x) \right)_{k \times (N-k)}, \text{ for } i \in J_1, j \in J_2. \end{aligned}$$

Note that

$$\tilde{M}(x) = \begin{pmatrix} M_1(x) & 0 \\ M_3(x) & 0 \end{pmatrix},$$

where $M_3(x)$ is an $(N-k) \times k$ matrix. Then $\rho(M_1(x)) = \rho(\tilde{M}(x)) < 1$, so $DT(x)$ is invertible with

$$\begin{aligned} (DT(x))^{-1} &= U \begin{pmatrix} (I - M_1(x))^{-1} & (I - M_1(x))^{-1} M_2(x) \\ 0 & I_2 \end{pmatrix} \\ &\quad \times \text{diag}\left(\frac{1}{f_1(x)}, \dots, \frac{1}{f_N(x)}\right) U^{-1}. \end{aligned}$$

As $(I - M_1(x))^{-1} = I + \sum_{n=1}^{\infty} M_1^n(x)$ with positive diagonal entries and nonnegative other entries, each diagonal entry of $(DT(x))^{-1}$ is positive and other entries are nonnegative. Then condition (ii) of Theorem 2.3 follows from Lemma 5.5.

For any $x, y \in [0, r]$, if $T(x) < T(y)$ and $T(y) - T(x) \in \dot{C}_J$ for some $J \subset I_N$, by the weakly retrotone property of T we have $x < y$ and $x_j < y_j$ for all $j \in J$. By condition (ii) of Theorem 2.4, each f_i is nonincreasing in every x_j but strictly decreasing in x_i for $x \in [0, r]$. Then we have $f(x) > f(y)$ and $f_j(x) > f_j(y)$ for all $j \in J$, so condition (iii) of Theorem 2.3 holds.

Finally, we check that $[0, r]$ is positively invariant. Note that Remark 2.1 (b) and (c) do not reply on the positive invariance of $[0, r]$. Then, for each $x \in [0, r]$ and every $i \in I_N$, by (ii) of Theorem 2.4 we have $T_i(x) \leq T_i(x_i e_i)$. By Remark 2.1 (c), $T_i(x_i e_i)$ is increasing for $x_i \in [0, r_i]$, so

$$T_i(x) \leq T_i(x_i e_i) \leq T_i(r_i e_i) = r_i f_i(r_i e_i) < r_i.$$

This shows that $T(x) \ll r$ and $T([0, r]) \subset [0, r]$. \square

Proof of Corollary 2. For any bounded set $B \subset [0, r] \setminus \{0\}$ with $\overline{B} \subset [0, r] \setminus \{0\}$, there is a small $\delta > 0$ such that $O(0, \delta) \cap \overline{B} = \emptyset$ and $O(0, \delta) \cap [0, r]$ is strictly below Σ . Since 0 is a repeller, $\overline{\mathcal{B}(0)}$ is invariant by Lemma 5.4, $[0, r]$ is positively invariant, and T from $[0, r]$ to $T([0, r])$ is a homeomorphism by Remark 2.1 (b), for $\delta > 0$ small enough the set $[0, r] \setminus O(0, \delta)$ is positively invariant with $\overline{B} \subset [0, r] \setminus O(0, \delta)$ and $O(0, \delta) \cap [0, r]$ is strictly below Σ . Then, for each $n \in \mathbb{N}$, $T^n([0, r] \setminus O(0, \delta))$ is compact and

$$\Sigma \subset T^{n+1}([0, r] \setminus O(0, \delta)) \subset T^n([0, r] \setminus O(0, \delta)).$$

From this follows

$$\Sigma \subset \bigcap_{n=0}^{\infty} T^n([0, r] \setminus O(0, \delta)).$$

We claim that

$$(48) \quad \Sigma = \bigcap_{n=0}^{\infty} T^n([0, r] \setminus O(0, \delta)).$$

Indeed, from Lemma 5.4 we know that

$$\bigcap_{n=0}^{\infty} T^n([0, r] \setminus O(0, \delta)) \subset A_0 = \overline{\mathcal{B}(0)} = \{0\} \cup \mathcal{B}(0) \cup \Sigma.$$

If (48) is not true, then there is a point $p \in (\bigcap_{n=0}^{\infty} T^n([0, r] \setminus O(0, \delta))) \setminus \Sigma$, so $T^{-n}(p) \in [0, r] \setminus O(0, \delta)$ for all $n \in \mathbb{N}$. This shows that $\lim_{n \rightarrow \infty} T^{-n}(p) \neq 0$. On the other hand, however, as $p \notin \{0\} \cup \Sigma$, we must have $p \in \mathcal{B}(0)$ so $\lim_{n \rightarrow \infty} T^{-n}(p) = 0$. This contradiction shows the truth of (48).

Now from (48) we see that Σ attracts the points of $[0, r] \setminus O(0, \delta)$ uniformly. As $B \subset [0, r] \setminus O(0, \delta)$, Σ attracts the points of B uniformly. Therefore, Σ is a global attractor in $[0, r] \setminus \{0\}$ under the conditions of Theorem 2.3 or Theorem 2.4.

Under the additional condition (11), for any bounded set $B \subset C \setminus \{0\}$ with $\overline{B} \subset C \setminus \{0\}$, from Remark 2.3 we know the existence of an integer $k > 0$ such that $T^k(\overline{B}) \subset [0, r]$. By the definition of T , $T^k(x) = 0$ if and only if $x = 0$ on C . As $0 \notin \overline{B}$, $0 \notin T^k(\overline{B})$ so $T^k(\overline{B}) \subset [0, r] \setminus \{0\}$. From the previous paragraph we know that Σ attracts the points of $T^k(\overline{B})$ uniformly. Thus, Σ attracts the points of B uniformly. Hence, Σ is a global attractor in $C \setminus \{0\}$. \square

6. CONCLUSION

We have so far considered the discrete dynamical system (1) with the maps T defined by (2). Recall that the current available carrying simplex theory is about the existence of an $(N - 1)$ -dimensional surface that is a compact invariant set attracting all the points in $C \setminus \{0\}$. With the existing concept of a carrying simplex for the system and the available criteria on existence of carrying simplex as the main concern of this paper, we have successfully achieved our goal of extending this theory to a broader class of systems: We have first defined the concept of a modified carrying simplex, which is a slight relaxation from the concept of a carrying simplex and is still an $(N - 1)$ -dimensional surface that is a compact invariant global attractor of the system in $C \setminus \{0\}$. We then have established our criteria for existence and uniqueness of a modified carrying simplex.

In comparison with the existing criteria for existence of a carrying simplex, our criteria for existence and uniqueness of a modified carrying simplex have the following main virtue: Instead of requiring all the entries of the Jacobian $Df(x)$ to be negative for all $x \in [0, q]$, we only require each entry of $Df(x)$ to be nonpositive and each $f_i(x)$ to be strictly decreasing in x_i . Thus, we have significantly reduced the cost of having an $(N - 1)$ -dimensional surface as a compact invariant global attractor of the system in $C \setminus \{0\}$. In other words, our criteria can be applied to a broader class of systems as competitive models.

The significance of the carrying simplex theory lies in that the global dynamics of the system in C can be described by the dynamics on the modified carrying simplex Σ . As one application of this theory, we have investigated vanishing species and dominance of one species over others. Assuming the existence of a modified carrying simplex, we have obtained sufficient geometric conditions for one or more species to die out. We have also obtained conditions for one species to dominate all others and one axial fixed point to be globally asymptotically stable.

Above all, with our theorems for modified carrying simplex Σ , we have laid the foundation for exploring the global dynamics of the system. We expect future research work will be flourishing based on modified carrying simplex.

Open Problem Suppose system (1) with T defined by (2) satisfies the conditions of Theorem 2.4, so the system permits a modified carrying simplex Σ . Is it possible to construct a sequence $\{T^{[k]}\}$ satisfying the following conditions?

- (i) For each integer $k > 0$, the map $T^{[k]}$ from $[0, r]$ to C has the form (2).
- (ii) Each $T^{[k]}$ on $[0, r]$ meets the requirements of Theorem 1.2, so system (1) with T replaced by $T^{[k]}$ permits a carrying simplex $\Sigma^{[k]}$.
- (iii) As $k \rightarrow \infty$, $T^{[k]}(x) \rightarrow T(x)$ uniformly for $x \in [0, r]$
- (iv) As $k \rightarrow \infty$, $\Sigma^{[k]} \rightarrow \Sigma$ in the following sense:

$$\forall \varepsilon > 0, \exists K > 0, \forall k \geq K, \Sigma^{[k]} \subset O(\Sigma, \varepsilon).$$

If the answer is YES, then our Theorem 2.4 can be viewed as the result of a limit process from Theorem 1.2, i.e. system (1) with (2) satisfying the conditions of Theorem 2.4 can be approximated by systems satisfying the conditions of Theorem 1.2.

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