On tautological classes of fibre bundles and self-embedding calculus

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution.

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Abstract

In this thesis we study the ring of characteristic classes $H^*(\mathrm{B}\,\mathrm{Diff}^+(M))$ of smooth fibre bundles with a particular focus on tautological classes and the ring that they generate. In the first part we use tools from rational homotopy theory to compute analogues of tautological rings over fibrations, which provides upper bounds on the tautological rings of fibre bundles. In some cases we find an upper bound on the Krull dimension that is sharp.

In the second part, we study the classifying space $\mathrm{B}\,\mathrm{Diff}^+(M)$ using the calculus of embeddings which provides a homotopy theoretic approximation. We construct cohomology classes on the self-embedding tower which extend certain characteristic classes that were introduced by Kontsevich in [Kon94]. This construction is based on introducing configuration space integrals over the tower itself, which also has some consequences for tautological classes that we explore.

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Contents

Chapter 1.

Introduction

1.1. Automorphism spaces of manifolds

The main object of interest in this thesis is the classifying space $\mathrm{B}\,\mathrm{Diff}(M)$ of the topological group of diffeomorphisms of a smooth manifold M, which has deep connections to other fields in mathematics and motivated many important developments in geometric and algebraic topology. Yet despite extensive research, the homotopy type of the classifying space has eluded topologists except for low dimensional manifolds. One basic property of $\mathrm{B}\,\mathrm{Diff}(M)$ is that it classifies smooth fibre bundles with fibre M over a base space B up to bundle isomorphism. More precisely, there exists a universal fibre bundle $\pi : E \to \mathrm{B}\,\mathrm{Diff}(M)$ with fibre M and there is a one-to-one correspondence

$$[B, \mathrm{B}\,\mathrm{Diff}(M)] \xrightarrow{1:1} \{\pi : E \to B \text{ smooth } M\text{-bundle over } B\}/\cong$$

where we associate to a map $f : B \to \mathrm{B}\,\mathrm{Diff}(M)$ the equivalence class of the pullback bundle $\pi : f^*E \to B$ with respect to bundle isomorphism. This justifies the name and is also one of the fundamental tools to studying it. For example, this implies that the cohomology ring $H^*(\mathrm{B}\,\mathrm{Diff}(M))$ is the ring of characteristic classes of smooth fibre bundles with fibre M.

The traditional approach to studying $\mathrm{B}\,\mathrm{Diff}(M)$ is based on successive approximations to the space of diffeomorphisms by different automorphism spaces of M. The first rather coarse approximation to $\mathrm{Diff}(M)$ is the given by the monoid of homotopy self-equivalences $\mathrm{hAut}(M)$, whose homotopy type is accessible via the standard toolbox of homotopy theory. A second finer approximation is given by the semi-simplicial group $\widetilde{\mathrm{Diff}}(M)_\bullet$ with k-simplices given by block diffeomorphisms, i.e. diffeomorphisms $M \times \Delta^k \xrightarrow{\cong} M \times \Delta^k$ that restrict to diffeomorphisms of $\sigma \times M$ for each face $\sigma \in \Delta^k$. To both of these automorphism spaces of M one can associate classifying spaces. It was shown by Stasheff in [Sta63] that $\mathrm{B}\,\mathrm{hAut}(M)$ classifies Hurewicz fibrations, and Rourke and Sanderson showed in [RS71] that $\mathrm{B}\,\widetilde{\mathrm{Diff}}(M)$ classifies block bundles, which over a simplicial complex K is defined as a map $\pi : E \to |K|$ with trivializations $\pi^{-1}(\sigma) \xrightarrow{\cong} M \times \sigma$ for every simplex in $|K|$ that preserve the face structure but not the projection.

One of the great accomplishments of geometric topology from the 70's and 80's is a body of work that provides a strategy to study $\mathrm{B}\,\mathrm{Diff}(M)$ in the so called concordance stable range, following a two step procedure by successively lifting information to $\mathrm{B}\,\mathrm{Diff}(M)$ along the comparison maps

$$\begin{tikzcd}
\frac{\widetilde{\mathrm{Diff}}(M)}{\mathrm{Diff}(M)} \arrow[r, hook] & \mathrm{B}\,\mathrm{Diff}(M) \arrow[d] \\
\frac{\mathrm{hAut}(M)}{\widetilde{\mathrm{Diff}}(M)} \arrow[r, hook] & \mathrm{B}\,\widetilde{\mathrm{Diff}}(M) \arrow[d] \\
& \mathrm{B}\,\mathrm{hAut}(M)
\end{tikzcd}$$

where the left spaces denote the homotopy fibres.

The first step is based on Browder-Novikov-Sullivan-Wall surgery theory to determine the homotopy type of the fibre via a comparison map $\mathrm{hAut}(M)/\widetilde{\mathrm{Diff}}(M) \to \underline{S}^{G/O}(M)$ to Quinn's geometric structure space, which can be shown to be a weak equivalence (with some technicalities involving path components). The structure space sits in a fibre sequence $\underline{S}^{G/O}(M) \to \underline{N}(M) \to \mathbb{L}(M)$ that involves the space of normal invariants $\underline{N}(M) \simeq \mathrm{Map}(M, G/O)$ and the surgery space $\mathbb{L}(M)$, whose homotopy groups are the algebraic L-groups $L_*(\pi_1(M))$; both spaces are computationally accessible (see Quinn's geometric formulation of surgery [Qui70]).

The homotopy fibre $\widetilde{\mathrm{Diff}}(M)/\mathrm{Diff}(M)$ is closely related to the space of concordances $C(M)$, i.e. diffeomorphisms of $M \times I$ that restrict to the identity on $M \times 0 \cup \partial M \times I$. It was shown by Hatcher [Hat78] that there is a spectral sequence with E_1-page given by $E^1_{p,q} = \pi_q(C(M \times I^p))$ converging to $\pi_{p+q+1}(\widetilde{\mathrm{Diff}}(M)/\mathrm{Diff}(M))$. One important feature of concordance spaces is that they behave well with respect to stabilization $C(M) \to C(M \times I)$. For example, the space of stable concordances $\mathscr{C}(M) = \operatorname{colim} C(M \times I^k)$ is an infinite loop space that is closely related to algebraic K-theory, which was observed by Hatcher [Hat75] and made precise in Waldhausen's seminal work on algebraic K-theory of spaces [Wal78, Wal79, WJR13]. Another feature of concordance spaces concerns the connectivity of the stabilization map $C(M) \to C(M \times I)$ that we denote by $\phi(M)$. The concordance stable range $\phi(n) \in \mathbb{N}$ is the maximal integer such that $\phi(M) \geq \phi(n)$ for all compact n-dimensional manifolds M, and a deep result of Igusa [Igu88] gives a lower bound $\phi(n) \geq \min((n-7)/2, (n-4)/3)$. It follows that the homotopy groups in the spectral sequence above agree with those of $\mathscr{C}(M)$ in the concordance stable range which is accessible via algebraic K-theory. This discussion is elegantly encapsulated by a result of Weiss and Williams [WW88] who constructed a map

$$\frac{\widetilde{\mathrm{Diff}}(M)}{\mathrm{Diff}(M)} \longrightarrow \Omega^{\infty}(S^{\infty} \wedge_{\mathbb{Z}/2} \mathscr{C}(M))$$

which is $(\phi(M)+1)$-connected. One prominent example where this strategy has culminated in a computation is the following result of Farrell-Hsiang [FH78]

$$\pi_*(\mathrm{B}\,\mathrm{Diff}_{\partial}(D^k)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } * = 4i \text{ and } k \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

in a range of degrees $* \leq \phi(k)$, which relies on Borel's computation of the algebraic K-theory of the integers.

A quite different approach to $\mathrm{B}\,\mathrm{Diff}(M)$ in the case of surfaces was developed by Madsen and Weiss [MW07] in their proof of the Mumford conjecture and subsequent far reaching generalization to high-dimensional manifolds M^{2n} due to Galatius–Randal-Williams [GRW14, GRW18, GRW17]. For oriented manifolds we denote by $\mathrm{Diff}^+(M)$ the subgroup of orientation preserving diffeomorphisms. The classifying space $\mathrm{B}\,\mathrm{Diff}^+(M)$ can be modelled as a moduli space of manifolds for which one can construct a parametrized version of the Pontrjagin-Thom collapse map

$$\mathrm{B}\,\mathrm{Diff}^+(M^d) \longrightarrow \Omega_0^{\infty}\mathrm{MTSO}(d), \tag{1.2}$$

where $\mathrm{MTSO}(d)$ is the Thom spectrum of the inverse of the universal bundle $\gamma_d \to \mathrm{B}\,\mathrm{SO}(d)$. Madsen and Weiss show that for $d = 2$ this map induces an isomorphism in integral homology in a range of degrees increasing with the genus of the surface M, which reduces the computation to a purely homotopy theoretic problem. Galatius and Randal-Williams have generalized this to simply-connected manifolds of even dimension ($\neq 4$) with a slightly modified Thom spectrum as target. Their seminal result has lead to significant progress in the past years and promises a far reaching understanding of $\mathrm{B}\,\mathrm{Diff}(M)$, especially when combined with embedding calculus.

1.2. Tautological rings

There are characteristic classes of smooth fibre bundles with closed manifold fibres that we can construct without knowing anything specific about the fibre M^d except its dimension as follows. Let $\pi : E \to B$ denote an oriented smooth fibre bundle with fibre M and $P \to B$ its associated principal $\mathrm{Diff}^+(M)$-bundle. We define the *vertical tangent bundle* as

$$T_\pi E := P \times_{\mathrm{Diff}^+(M)} TM \longrightarrow P \times_{\mathrm{Diff}^+(M)} M \cong E.$$

For any characteristic class of oriented vector bundles $c \in H^{|c|}(\mathrm{B}\,\mathrm{SO}(d))$ the fibre integral

$$\kappa_c := \int_\pi c(T_\pi E) \in H^{|c|-d}(B),$$

defines a characteristic class of the fibre bundles by naturality of fibre integration which we call a *generalized Miller-Morita-Mumford class* (MMM-class for short).[1]

Definition 1.2.1. The **tautological ring** $R^*(M)$ is the subring of $H^*(\mathrm{B}\,\mathrm{Diff}^+(M))$ generated by the generalized MMM-classes κ_c of the universal M-bundle $E \to \mathrm{B}\,\mathrm{Diff}^+(M)$ for all $c \in H^*(\mathrm{B}\,\mathrm{SO}(d))$.

Hence, the information in the tautological ring $R^*(M)$ is the same as identifying all relations among MMM-classes. It serves as a suitable object to study the cohomology of $\mathrm{B}\,\mathrm{Diff}^+(M)$ of arbitrary smooth manifolds and as a simple measure to compare their rings of characteristic classes. A related source of universal characteristic classes is given by $H^*(\Omega^\infty \mathrm{MTSO}(d))$ via the parametrized Pontrjagin-Thom map (1.2). With rational coefficients the cohomology of $\Omega^\infty\mathrm{MTSO}(d)$ is a polynomial ring on $H^{*-d}(\mathrm{B}\,\mathrm{SO}(d);\mathbb{Q})$ and the image under (1.2) is precisely the tautological ring.

The tautological ring further plays a distinguished role in connection with the results of Galatius–Randal-Williams because their results imply that a large part of the stable cohomology of $\mathrm{B}\,\mathrm{Diff}^+(M)$ is given by MMM-classes (for example $H^*(\mathrm{B}\,\mathrm{Diff}^+(\#^g S^n \times S^n);\mathbb{Q})$ is described completely by MMM-classes in the stable range). This provides some evidence that the tautological ring captures interesting properties of fibre bundles and is worth studying. In fact, the tautological ring of surfaces Σ_g has been studied extensively because of the close connection of $\mathrm{B}\,\mathrm{Diff}^+(\Sigma_g)$ and the moduli space of Riemann surfaces, which was the original context of the Mumford conjecture. In comparison, tautological rings of high-dimensional manifolds have come into focus much more recently for example in [GGRW17, Gri17, RW18].

We are particularly interested in the ring-theoretic properties of $R^*(M)$ such as the Krull dimension. These large scale ring-theoretic questions cannot be addressed with the results of Galatius–Randal-Williams as some products of tautological classes go beyond the stable range and so necessarily concern the unstable features of $H^*(\mathrm{B}\,\mathrm{Diff}^+(M))$, and this will be the focus of the first part of this thesis.

One important tool to study the tautological ring is a family version of the Hirzebruch signature theorem, which provides an intricate link between the tangent bundle and the

[1] Later on we prefer to denote fibre integration by $\pi_! : H^*(E) \to H^{*-d}(B)$.

topology of a manifold M^{4k}. The signature theorem states that there are polynomials $L_k \in H^{4k}(\mathrm{B}\,\mathrm{SO};\mathbb{Q})$ such that for closed oriented manifold M^{4k} the signature is $\mathrm{sign}(M) = \langle L_k(TM), [M]\rangle$. The first few of these polynomials are given by

$$\begin{aligned} L_1 &= \frac{1}{3}p_1 \\ L_2 &= \frac{1}{45}(7p_2 - p_1^2) \\ L_3 &= \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3). \end{aligned}$$

Representing rational homology classes by rational framed bordism classes, we can define characteristic classes $\sigma_i \in H^i(\mathrm{B}\,\mathrm{Diff}^+(M^d);\mathbb{Q})$ of degrees satisfying $i + d \equiv 0 \bmod 4$ if d is even by evaluating a class $[f : N^i \to \mathrm{B}\,\mathrm{Diff}^+(M), \xi] \in \Omega_i^{\mathrm{fr}}(\mathrm{B}\,\mathrm{Diff}^+(M)) \otimes \mathbb{Q}$ as the signature

$$\langle \sigma_i, [f : N^i \to \mathrm{B}\,\mathrm{Diff}^+(M), \xi]\rangle := \mathrm{sign}(f^*E) \tag{1.3}$$

of the total space of the pullback bundle $f^*E \to N$, which is a closed manifold whose dimension is divisible by 4 by construction. The family signature theorem can then be stated as follows.

Theorem 1.2.2 ([RW19]). *Let $\pi : E \to B$ be a smooth, oriented fibre bundle with fibre M^d a closed, oriented manifold. Then*

$$\kappa_{L_i} = \int_\pi L_i(T_\pi E) = \begin{cases} \sigma_{4i-d} & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd} \end{cases} \tag{1.4}$$

It follows from [Mey72] that the classes $\sigma_i \in H^i(\mathrm{B}\,\mathrm{Diff}^+(M);\mathbb{Q})$ are in the image of the induced map on cohomology of the natural map $\mathrm{B}\,\mathrm{Diff}^+(M) \to \mathrm{B}\,O(H,\lambda)$, where (H,λ) denotes the $(-1)^{d/2}$-symmetric non-degenerate intersection pairing λ on $H = H^{d/2}(M;\mathbb{Z})/\mathrm{tors}$ and $O(H,\lambda)$ denotes the automorphism group of (H,λ). This group is arithmetic and thus its classifying space satisfies strong finiteness conditions, which was first exploited in [GGRW17]. It further implies that for $\mathrm{B}\,\mathrm{Diff}_0(M)$, the classifying space of the connected component over the identity, the signature classes σ_i vanish (this also follows from [CHS57]), which is the version of the family signature theorem that we will use later on.

Theorem 1.2.3. *Let $\pi : E \to B$ be a smooth, oriented fibre bundle with fibre M^d a closed, oriented manifold and trivial fibre transport. Then $\kappa_{L_i} = 0$.*

In recent years, the construction of tautological classes has been extended to other families such as topological fibre bundles and block bundles [ERW14] as well as oriented Hurewicz fibrations with Poincaré fibre [HLLR17], by which we mean a space homotopy equivalent to a (connected) finite CW complex with a choice of fundamental class in $[X] \in H_d(X;\mathcal{D})$ for an orientation system $\mathcal{D}$ such that cap product $-\cap[X]$ induces an isomorphism for all local coefficient systems[2]. The ring of tautological classes of such families is much more computable than $R^*(M)$ because their corresponding classifying spaces are accessible by methods from homotopy theory. Since any relation that holds among tautological classes for block bundles or fibrations holds in particular for fibre bundles, this amounts to obtaining upper bounds to $R^*(M)$ — one of the key ideas of the first part of this thesis that we will elaborate on later as well.

We will focus on the ring of tautological classes of oriented Hurewicz fibrations with Poincaré fibre and *TM-fibrations* introduced in [Ber20a, Ber20b], which is an oriented M-fibration $\pi: E \to B$ with fibre M a smooth closed oriented manifold together with an oriented vector bundle $T_\pi E \to E$ such that the restriction $T_\pi E|_{\pi^{-1}(b)}$ is equivalent to the tangent bundle TM on each fibre $\pi^{-1}(b)$. The tautological classes of TM-fibrations are the fibre integrals of the characteristic classes of this vector bundle.

Although fibrations with Poincaré fibre do not have a vertical tangent vector bundle, one can show that they nonetheless have an Euler class. Given an oriented fibration $\pi: E \to B$ of Poincaré duality spaces with fibre X of formal dimension d, the diagonal $\Delta: E \to E \times_B E$ is a map of Poincaré duality spaces and therefore has an Umkehr map $\Delta_! = D^{-1}_{E\times_B E}\Delta_* D_E : H^*(E) \to H^{*+d}(E \times_B E)$, where D_E and $D_{E\times_B E}$ denote the corresponding Poincaré duality isomorphisms. We define the *fibrewise Euler class* as

$$e^{\mathrm{fw}}(\pi) := \Delta^*\Delta_!(1) \in H^d(E). \tag{1.5}$$

We recognize this as a fibrewise analogue of the description of the Euler class of a Poincaré duality space as the Poincaré dual of the diagonal. This construction has been extended in [HLLR17] to oriented (Hurewicz) fibrations with Poincaré fibre X over general base spaces. In particular, one can associate a fibrewise Euler class to the universal oriented X-fibration

$$X \hookrightarrow E \xrightarrow{\pi} \mathrm{B}\,\mathrm{hAut}^+(X),$$

which classifies oriented fibrations with fibre X.

Definition 1.2.4. The **Euler ring** $E^*(X)$ of an oriented Poincaré duality space X of formal dimension d is defined as the subring of $H^*(\mathrm{B}\,\mathrm{hAut}^+(X))$ generated by the classes

$$\kappa_i := \int_\pi e^{\mathrm{fw}}(\pi)^{i+1} \in H^{i\cdot d}(\mathrm{B}\,\mathrm{hAut}^+(X)),$$

where $e^{\mathrm{fw}}(\pi) \in H^d(E)$ is the fibrewise Euler class of the universal oriented X-fibration.

[2] We give a precise definition in Section 7.1.1

1.3. Embedding calculus and graph complexes

In the past two years a new, promising approach to studying the classifying space $\mathrm{B}\,\mathrm{Diff}(M)$ has emerged based on the calculus of embeddings, which provides homotopy theoretic approximations to embedding spaces $\mathrm{Emb}(M, N)$ of smooth manifolds. More precisely, there are maps

$$\eta_k : \mathrm{Emb}(M, N) \longrightarrow T_k\,\mathrm{Emb}(M, N),$$

for all $k \geq 1$ where the space $T_k\,\mathrm{Emb}(M, N)$ is called the kth Taylor approximation, and these approximations assemble in the so called Taylor tower

$$\begin{tikzcd}
\mathrm{Emb}(M,N) \arrow[d, "\eta_\infty"] \arrow[drr, bend left=10] \arrow[drrr, bend left=10, "\eta_2"] \arrow[drrrr, bend left=10, "\eta_1"] & & & & \\
T_\infty\,\mathrm{Emb}(M,N) := \mathrm{holim}\Big(\dots \arrow[r] & T_3\,\mathrm{Emb}(M,N) \arrow[r, "r_3"] & T_2\,\mathrm{Emb}(M,N) \arrow[r, "r_2"] & T_1\,\mathrm{Emb}(M,N)\Big).
\end{tikzcd} \tag{1.6}$$

By deep theorems of Goodwillie and Goodwillie, Klein and Weiss the connectivity of η_k increases with k if $\dim(N) - \dim(M) \geq 3$, and in particular the limit provides a homotopy theoretic description of embedding spaces

$$\eta_\infty : \mathrm{Emb}(M, N) \xrightarrow{\simeq} T_\infty\,\mathrm{Emb}(M, N).$$

This seminal theorem has been the basis for complete computation of the rational homotopy type of several embedding spaces for example in [ALV07, AT14, FTW17].

If $M = N$ is a closed manifold, then the space of self-embedding agrees with $\mathrm{Diff}(M)$. The Taylor tower is still defined and it induces a map on classifying spaces, but it is not known if $\eta_\infty : \mathrm{Diff}(M) \to T_\infty\,\mathrm{Emb}(M, M)$ is a weak equivalence. It is commonly expected that it is not a weak equivalence but that it is close.[3] This expectation is based on two types of results that have appeared in the past decade:

- In many cases one finds that the rational homotopy type of the limit $T_\infty\,\mathrm{Emb}(M, N)$ can be described in terms of certain graph complexes, chain complexes whose elements and algebraic operations have graphical interpretations. A key input for such results is Kontsevich's proof of formality of the little disks operad, which allows for a computation of a model of the Taylor tower in terms of (derived) mapping spaces of certain right modules over the (framed) little disks operad. Willwacher has recently

[3] This is the impression the author got from eaves-dropping on inspiring conversations by the people who currently advance this exciting research, although this sentiment is not committed to paper yet.

announced such a description of a space that is a mild modification of a delooping of $T_\infty \operatorname{Emb}(M, M)$ and expected to be a close model.

- A second type of result concerns certain characteristic classes of fibre bundles with a (framed) section and whose fibre is an integral homology sphere that were defined by Kontsevich in [Kon94]. These classes arise from a very similar graph complex to the ones that describe the rational homotopy type of embedding spaces, and were shown to be non-trivial by Watanabe in [Wat09a, Wat09b, Wat18]. Watanabe further uses these classes to prove non-triviality of $\pi_*(\mathrm{B}\operatorname{Diff}_\partial(D^{2k+1})) \otimes \mathbb{Q}$ far beyond the concordance stable range.

It is believed that these two topics are closely related, and in combination these results suggest at the very least that the graphical description of the homotopy type of $T_\infty \operatorname{Emb}(M, M)$ and its delooping provides non-trivial information about the classifying space $\mathrm{B}\operatorname{Diff}(M)$ far beyond the ranges involved in the respective approaches to studying it that we mentioned before.

The exact connection between Kontsevich's characteristic classes and (the graphical models from) embedding calculus is however not known. The reason that Kontsevich's name features prominently in both bullet points above is that both rely on the same tool developed by Kontsevich, namely *configuration space integrals*. In the second part of the thesis, we provide a conceptual explanation of the connection of these two themes based on a homotopy theoretic version of configuration space integrals that extends Kontsevich's characteristic classes and can be defined on the self-embedding tower[4].

Finally, there is a fruitful interplay between embedding calculus and the stabilization methods of Galatius–Randal-Williams. This goes back to an idea of Weiss [Wei15], who constructed a fibre sequence

$$\mathrm{B}\operatorname{Diff}_\partial(D^d) \longrightarrow \mathrm{B}\operatorname{Diff}_\partial(M) \longrightarrow \mathrm{B}\operatorname{Emb}^{\cong}_{\frac{1}{2}\partial}(M, M),$$

where M is a smooth manifold with non-empty boundary with a fixed disk $D^{d-1} \hookrightarrow \partial M$ and $\operatorname{Emb}^{\cong}_{\frac{1}{2}\partial}(M, M)$ is the space of embeddings $M \hookrightarrow M$ that are the identity on $\partial M \setminus D^{d-1}$ and isotopic relative to $\partial M \setminus D^{d-1}$ to a diffeomorphism fixing ∂M. If we choose a highly-connected manifold such as $W_{g,1} = \#^g S^n \times S^n \setminus D^{2n-1}$ with large genus then we understand two terms in the above fibre sequence well: the total space $\mathrm{B}\operatorname{Diff}_\partial(W_{g,1})$ by the methods of Galatius–Randal-Williams and the base $\operatorname{Emb}^{\cong}_{\frac{1}{2}\partial}(W_{g,1}, W_{g,1})$ by embedding calculus (the

[4]With the caveat that at present we can only deal with configurations of two particles.

tower converges if $n \geq 3$ because of an improved convergence result based on a notion of codimension using the handle-dimension of the domain and the geometric dimension of the codomain).

This provides a promising tool to obtain information about $\mathrm{B}\,\mathrm{Diff}_\partial(D^d)$ outside the stable range. Weiss has used this to show in [Wei15] that the rational Pontrjagin classes $p_i \in H^{4i}(\mathrm{B}\,\mathrm{Top}(2n);\mathbb{Q})$ can be non-trivial for $i > n$, where $\mathrm{Top}(2n)$ denotes the homeomorphisms of $\mathbb{R}^{2n}$. This is connected to diffeomorphisms of disks by a result from smoothing theory $\mathrm{Diff}_\partial(D^d) \simeq \Omega^{d+1}\mathrm{Top}(d)/O(d))$, and this technique is currently refined by Randal-Williams and Kupers. There are further applications, for example in [Kup19] Kupers has shown that $\pi_*(\mathrm{B}\,\mathrm{Diff}_\partial(D^d))$ is finitely generated in each degree for $d \neq 4, 5, 7$ which was a long standing problem in geometric topology.

Discussion of results

This thesis is divided into two parts with the common theme of tautological classes. In the first part, we use tools from rational homotopy theory to compute the ring of tautological classes for fibrations with Poincaré fibre and TM-fibrations. In the second part we develop a connection between Kontsevich's characteristic classes and the self-embedding tower, and we study some implications for tautological classes.

Part I

The first three chapters are mostly contained in my article [Pri19] with only minor modifications. We start in Chapter 2 with a discussion of the algebraic models for fibrations from rational homotopy theory. Denote by $\mathrm{hAut}_0(X) \subset \mathrm{hAut}(X)$ the connected component of the identity and by

$$X \hookrightarrow E_0 \longrightarrow B\,\mathrm{hAut}_0(X) \tag{1.7}$$

the universal X-fibration with trivial fibre transport. If X is 1-connected then (1.7) is a fibration of 1-connected spaces that we call the universal 1-connected fibration. Its algebraic description from rational homotopy type can be expressed in terms of an algebraic model of X. Our description is based on a Sullivan model $(\Lambda V, d)$ of X, and we show that a model of (1.7) is given by the Chevalley-Eilenberg cochain complex of the differential graded Lie algebra of (positive degree) derivations $(\mathrm{Der}^+(\Lambda V), [d, -])$.

Theorem A (Theorem 2.3.9)**.** *Let X be a 1-connected space of finite type with minimal Sullivan model $(\Lambda V, d)$ and unit $\eta : \mathbb{Q} \to \Lambda V$. Then*

$$C^*_{CE}(\mathrm{Der}^+(\Lambda V); \mathbb{Q}) \xrightarrow{\eta_*} C^*_{CE}(\mathrm{Der}^+(\Lambda V); \Lambda V)$$

is a cdga model of the universal 1-connected fibration (1.7)*.*

Remark. It has been pointed out to the author by his examiners that this result can be extracted from [Laz14]. We will present an elementary proof in this thesis based on a comparison with Tanré's model and explain the relation to [Laz14] in Remark 2.3.6. The theorem is also going to appear as a special case in forthcoming work of Berglund [Ber20b].

In Chapter 3 we discuss models for fibre integration if the fibre X is a rational Poincaré duality space in terms of the algebraic models from rational homotopy theory. By a rational Poincaré duality space X of formal dimension d we mean a space with $\dim H^*(X;\mathbb{Q}) < \infty$ and a linear map $\varepsilon_X : H^d(X;\mathbb{Q}) \to \mathbb{Q}$, called an orientation, that induces a non-degenerate pairing $\varepsilon_X(- \smile -) : H^q(X;\mathbb{Q}) \otimes H^{d-q}(X;\mathbb{Q}) \to \mathbb{Q}$ for all q. Given a cdga model $\pi^* : B \to E$ of a fibration with fibre X, we show that ε_X determines a unique element in $\mathrm{Ext}_B^{-d}(E, B)$ corresponding to a derived B-module homomorphism $\Pi : E \to B[d]$ that induces fibre integration on cohomology (see Proposition 3.1.2). And we use this chain level representative Π to define a representative of the fibrewise Euler class in the algebraic model in Section 4.1.

The above discussion simplifies drastically when one considers rationally positively elliptic spaces, i.e. spaces X with $\dim H^*(X;\mathbb{Q}) < \infty$, $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$ and $\chi(X) > 0$. This is because of strong structural properties which imply that $H^*(X;\mathbb{Q})$ is a complete intersection over $\mathbb{Q}$. Similarly, the algebraic model of the universal 1-connected fibration is a complete intersection over a polynomial ring[5] and we give a simple characterization of the fibrewise Euler class in this case.

Theorem B (Theorem 4.1.10)**.** *If a fibration has an algebraic model given by a complete intersection $E = B[x_1, \dots, x_n]/(\bar{f}_1, \dots, \bar{f}_n)$ over a polynomial ring B (both concentrated in even degrees), then the fibre is a rationally positively elliptic space with a choice of orientation such that the fibrewise Euler class is represented by*

$$e^{\mathrm{fw}}(\pi) = \det\left(\frac{\partial \bar{f}_i}{\partial x_j}\right) \in E.$$

For example, we show in Proposition 4.6 that the universal 1-connected fibration with fibre $X = \mathbb{C}P^n$ is equivalent to the complete intersection $B_n[x]/(x^{n+1} - \sum_{i=2}^{n+1} x_i x^{n+1-i})$ over the polynomial ring $B_n = \mathbb{Q}[x_2, \dots, x_{n+1}]$, where $|x_i| = 2i$ and x restricts to the generator of the fibre $H^2(\mathbb{C}P^n;\mathbb{Q})$. Then fibre integration is $\pi_!(x^n) = 1$ and $\pi_!(x^i) = 0$ for $i < n$ (see (4.2)), and the fibrewise Euler class is represented by $e^{fw}(\pi) = (n+1)x^n - \sum_{i=2}^{n}(n+1-i)x_i x^{n-i}$ by Theorem B. This reduces the computation of the Euler ring to a purely algebraic problem.

Theorem C (Theorem 4.2.7)**.** $E^*(\mathbb{C}P^n) \cong \mathbb{Q}[\kappa_1, \dots, \kappa_{n-1}, \kappa_{n+1}]$.

In this case, we can infer some information about $R^*(\mathbb{C}P^n)$. Using the smooth action of $\mathrm{SU}(n+1)$ on $\mathbb{C}P^n$ we show in Corollary 4.2.10 that the map $\mathrm{B}\,\mathrm{Diff}^+(\mathbb{C}P^n) \to \mathrm{B}\,\mathrm{hAut}^+(\mathbb{C}P^n)$ induces an injection on rational cohomology. In particular $E^*(\mathbb{C}P^n) \hookrightarrow R^*(\mathbb{C}P^n)$.

In Chapter 5 we study the ring of tautological classes of TM-fibrations. This is based on forthcoming work of Alexander Berglund [Ber20b], where he discusses (among other things)

[5] Provided that X satisfies the Halperin conjecture.

the rational homotopy theory of TM-fibrations and constructs cocycle representatives of the characteristic classes of the vector bundle $T_\pi E \to E$ over the total space. He has shared a draft of [Ber20b] with me and has explained his results in a visit to Stockholm in February 2019. We have been working in an ongoing collaboration and many ideas, especially concerning elliptic spaces, have been featured in this thesis, for example Proposition 5.1.5 and Example 5.1.10 for $M = S^2 \times S^2$.

It turns out that even for this simple example when $M = S^2 \times S^2$, the computation of the tautological ring of TM-fibrations is algebraically quite complicated. In comparison, the computation simplifies when we further impose the relations from the family signature theorem and doing this we can compute an upper bound on $R^*(S^2 \times S^2)$ using a computer algebra system. However, its presentation as a $\mathbb{Q}$-algebra in terms of generators and relations is too complicated to print in this thesis[6]. Instead, we extract in Proposition 5.2.2 an upper bound on the Hilbert series and the Krull dimension of $R^*(S^2 \times S^2)$. Another way to extract some palpable information out of our computation is to restrict attention to the much smaller Euler ring and detect elements which are in the kernel of the map $E^*(M) \to R^*(M)$.

Theorem D (Proposition 5.2.3 and 5.2.4)**.** *The kernel* $E^*(M) \to R^*(M)$ *is non-trivial for* $M = \mathbb{C}P^2 \# \mathbb{C}P^2$ *and* $M = S^2 \times S^2$.

This is not merely an existence statement but we give concrete elements in the kernel. In particular, these elements provide computable yet highly non-trivial obstructions to smoothing a fibration with fibre $M = S^2 \times S^2$ or $M = \mathbb{C}P^2 \# \mathbb{C}P^2$.

Another interesting application is the comparison of tautological rings for manifolds that have the same homotopy type but different characteristic classes. We use surgery theory in Appendix C to compute a source of such examples given by the geometric structure set $S(\mathbb{H}P^2)$, whose elements are equivalence classes of homotopy equivalences $f : M \xrightarrow{\simeq} \mathbb{H}P^2$, where M is a smooth closed manifold, up to diffeomorphisms of the domain. We find in Theorem C.1 that the structure set is countably infinite and that the elements can be parametrized (up to finite ambiguity) by the value of $p_1(TM)$.

Theorem E (Theorem 5.2.5)**.** $\operatorname{Kdim} R^*(M) = 0$ *for all but finitely many* $[f, M] \in S(\mathbb{H}P^2)$.

One notable exception that we know is $\mathbb{H}P^2$ itself for which $2 \leq \operatorname{Kdim} R^*(\mathbb{H}P^2) \leq 3$, and we expect that this is the only exception together with $\mathbb{H}P^2 \# \Sigma$, where Σ denotes the exotic 8-sphere. It is quite surprising that the tautological ring of homotopy equivalent

[6] and ultimately not so interesting

manifolds have a common feature at all, and it raises the question how much it depends on the manifold structure of M.

Part II

We begin with a brief review of embedding calculus and configuration space integrals in Chapter 6, which contains no original work. We discuss a description of the Taylor tower which allows for simple description of the delooping of the bottom stages of the self-embedding tower

$$\begin{tikzcd}
\mathrm{B}\,\mathrm{Diff}^+(M) \arrow[d, "\mathrm{B}\,\eta_\infty"] \arrow[rrr, bend left, "\mathrm{B}\,\eta_2"] \arrow[rrrr, bend left, "\mathrm{B}\,\eta_1"] & & & & \\
\mathrm{B}\,T_\infty^\times \operatorname{Emb}(M,M) \arrow[r] & \dots \arrow[rr] & & \mathrm{B}\,T_2^\times \operatorname{Emb}(M,M) \arrow[r, "\mathrm{B}\,r_2"] & \mathrm{B}\,T_1^\times \operatorname{Emb}(M,M),
\end{tikzcd}$$

where the superscript $\times$ indicates that we consider the homotopy-invertible path components of the Taylor approximations (we explain this point in more detail later).

It turns out that $\mathrm{B}\,T_1^\times \operatorname{Emb}(M,M)$ is the classifying space of TM-fibrations, which means that we can pull back the universal TM-fibration over $\mathrm{B}\,T_1^\times \operatorname{Emb}(M,M)$ to the higher stages of the tower, i.e. there are oriented fibrations

$$\pi_k : E_k \to \mathrm{B}\,T_k^\times \operatorname{Emb}(M,M) \tag{1.8}$$

with fibre M with an oriented vector bundle $T_\pi E_k \to E_k$ defined as pull back along $\mathrm{B}\,r_k$.

Remark. From this perspective, we can define tautological rings for each stage of the Taylor tower and the computation of the tautological ring of TM-fibrations is only the first step in a hierarchy of approximations. We expect that one can find many more relations in tautological ring coming from the higher stages and this has been one of the motivations of the author for studying embedding calculus in the first place.

In Section 6.2 we discuss Kontsevich's construction of certain characteristic classes via configuration space integrals. The simplest such class can be constructed as follows.

Let $\pi : E \to B$ be fibre bundle with fibre M and a section $s : B \to E$ so that the bundle is trivialised on a neighbourhood of s and consider the *fibrewise configuration space*

$$EC_2(\pi) := \left\{(x_1, x_2) \in E^2 \mid \pi(x_1) = \pi(x_2) \text{ and } x_1 \neq x_2 \neq s(\pi(x_1))\right\} \longrightarrow B,$$

which is a fibre bundle with fibre the ordered configuration space $C_2(M \setminus *)$. One can construct a fibrewise compactification $E\overline{C}_2(\pi) \to B$ whose fibre is a compact manifold with corners and interior $C_2(M \setminus *)$. If M^d is an odd dimensional homology sphere, one can further construct a so-called propagator $\omega \in H^{d-1}(E\overline{C}_2;\mathbb{Z})$ (this depends on some framing data) whose powers ω^k vanish when restricted to the fibrewise boundary and which have a canonical lift $\tilde{\omega}_k \in H^{k(d-1)}(E\overline{C}_2(\pi), \partial^{\mathrm{fib}} E\overline{C}_2(\pi);\mathbb{Z})$. Then the fibre integral

$$\zeta_2(\pi) := \int_\pi \tilde{\omega}_3 \in H^{d-3}(B;\mathbb{Z}) \tag{1.9}$$

defines a non-trivial characteristic class, originally defined by Kontsevich [Kon94] and shown to be non-trivial by Watanabe in [Wat09a]. The general construction of the characteristic classes is based on similar fibre integrals over fibrewise configuration spaces with more particles.

The goal of Chapter 7 is to extend configuration space integral techniques to study the fibration $E_k \to \mathrm{B}\, T_k^\times \operatorname{Emb}(M, M)$ over the self-embedding tower that we defined in (1.8) above. The fundamental issue we are facing is that the set-theoretic configuration space is not invariant under homotopy equivalences and so there is no meaningful fibrewise configuration space of fibrations that is invariant under fibre homotopy equivalences.

We address this issue in Section 7.1.1 and propose a homotopy theoretic notion of configuration spaces that is robust under homotopy equivalences. This is based on the concept of Poincaré embeddings [Kle99], which is a homotopy theoretically meaningful analogue of an embedding. The technical part in Section 7.1.2 is the construction of fibrewise homotopy configuration spaces for $\pi_2 : E_2 \to \mathrm{B}\, T_2^\times \operatorname{Emb}(M, M)$. An important tool is May's two-sided bar construction [May75], which we review in Appendix A.

Theorem F (Theorem 7.1.8)**.** *The fibration $\pi_2 : E_2 \to \mathrm{B}\, T_2^\times \operatorname{Emb}(M, M)$ admits the structure of a fibrewise homotopy configuration space of two particles.*

Using the construction of the fibrewise homotopy configuration space, we only need to extend the construction of the propagator class to define the simplest characteristic class of Kontsevich over the self-embedding tower.

Theorem G (Theorem 7.3.2)**.** *The construction of Kontsevich's characteristic class ζ_2 from* (1.9) *can be extended to the second stage of the self-embedding tower.*

The statement of this Theorem is quite vague because a precise version involves technicalities related to choices of framings which we discuss in detail in Section 7.3, and we refer to Theorem 7.3.2 for a precise statement. But there is also another issue because ζ_2 is a characteristic class of bundles with structure group $\operatorname{Diff}(M, U)$, i.e. diffeomorphisms that restrict to the identity on a fixed open disk $U \subset M$. This group coincides with $\operatorname{Diff}_\partial(M \setminus U)$

and one should use a version of embedding calculus relative the boundary, which has not yet been described in terms of the Haefliger model which we use. The statement in Theorem G above refers to an ad hoc definition of the Haefliger model in this case. We indicate in Remark 7.3.3 why we expect this to be a valid model for the Taylor approximations for embeddings relative to the boundary.

We can make two observations concerning tautological classes over the self-embedding tower. The first concerns the Euler class. A fact we have not mentioned so far is that for TM-fibrations $\pi : E \to B$ there are a priori two ways to obtain an Euler class: the fibrewise Euler class of the fibration $\pi : E \to B$ or the Euler class of the vector bundle $T_\pi E \to E$. In general, these two Euler classes are not the same for TM-fibrations (Proposition 5.1.5).

We can use the construction of homotopy configuration spaces to show that these classes become identified over $\mathrm{B}\, T_2^\times \operatorname{Emb}(M, M)$. This is because Theorem F in its full strength implies that the vector bundle $T_\pi E_2 \to E_2$ provides a regular neighbourhood of the diagonal map

$$\Delta : E_2 \longrightarrow E_2 \times_{\mathrm{B}\, T_2^\times \operatorname{Emb}(M,M)} E_2,$$

a property which we recognize for the vertical tangent bundle of fibre bundles, which links the global topology of the fibration $\pi_2 : E_2 \to \mathrm{B}\, T_2^\times \operatorname{Emb}(M, M)$ with the vector bundle $T_\pi E_2 \to E_2$.

Theorem H (Theorem 7.2.1). *The fibrewise Euler class of $\pi_2 : E_2 \to \mathrm{B}\, T_2^\times \operatorname{Emb}(M, M)$ agrees with the Euler class of the vector bundle $T_\pi E_2 \to E_2$.*

A second observation involves the family signature theorem. Recall the definition of the signature classes $\sigma_i \in H^i(\mathrm{B}\operatorname{Diff}^+(M); \mathbb{Q})$ in (1.3). As we have pointed out then, these classes can be defined in the cohomology of the classifying space $\mathrm{B}\, O(H, \lambda)$ of the group of automorphisms of the signature pairing, which implies that they are also defined in $H^*(\mathrm{B}\, T_k^\times \operatorname{Emb}(M, M); \mathbb{Q})$ for all $k \geq 1$. For $k = 1$ one can easily check using the rational homotopy theory of TM-fibrations that the family signature theorem does not hold. This can be strengthened to $k = 2$.

Theorem I (Theorem 8.0.1). *The family signature theorem does not hold on $\mathrm{B}\, T_2^\times \operatorname{Emb}(M, M)$. More precisely, for all smooth, closed, oriented manifolds M^{2d} and all $i \in \mathbb{N}$ satisfying $d < 2i \leq 2d-2$ the class $\sigma_{4i-2d} - \kappa_{L_i} \in H^{4i-2d}(\mathrm{B}\, T_2^\times \operatorname{Emb}(M, M); \mathbb{Q})$ does not vanish.*

Part I.

Tautological classes and rational homotopy theory

Chapter 2.

Rational homotopy theory of fibrations

Algebraic models of fibrations $F \to E \to B$ of 1-connected spaces from rational homotopy are well studied and often given in terms of relative Sullivan algebras [Sul77, Hal83, Tho81, FHT01] or semidirect products of dg Lie algebras [Tan83]. The main goal of this chapter is to find an algebraic model of the universal 1-connected fibration

$$X \hookrightarrow E_0 \longrightarrow \mathrm{B}\,\mathrm{hAut}_0(X) \tag{2.1}$$

for a 1-connected space X in terms of its Sullivan model.

There are essentially two different algebraic approaches to rational homotopy theory, using either the Quillen equivalence of the category of 1-connected rational spaces of finite type with the (opposite) category of 1-connected differential graded commutative algebras of finite type (cdga in short), or with the category of connected differential graded Lie algebras of finite type. These two algebraic categories are Quillen equivalent with right adjoint given by the the cdga of Chevalley-Eilenberg cochains $C^*_{CE}(-;\mathbb{Q})$ of a dg Lie algebra.

Tanré gave an algebraic description of (2.1) in [Tan83] in the category of dg Lie algebras whose input is a dg Lie algebra model of the space X. The main result of this chapter, Theorem 2.3.9, gives a similar description of (1.7) in the category of cdga's in terms of a Sullivan model of X. This is based on Sullivan's result from [Sul77] that for a 1-connected space of finite type with minimal Sullivan model $(\Lambda V, d)$, the dg Lie algebra $\mathrm{Der}^+(\Lambda V)$ of (positive degree) derivations is a dg Lie model of $\mathrm{B}\,\mathrm{hAut}_0(X)$. Based on this observation, we will construct an algebraic model of (1.7) as a relative Sullivan algebra with fibre ΛV.

The derivation Lie algebra $\mathrm{Der}^+(\Lambda V)$ is not of finite type in general, which makes it quite technical to discuss its corresponding cdga model. One does not have these difficulties when considering the differential graded coalgebra $C^{CE}_*(\mathrm{Der}^+(\Lambda V);\mathbb{Q})$, and consequently, we start by constructing a coalgebra model of the universal fibration. In the process, we will give an algebraic classification of coalgebra bundles in Section 2.2 that is similar in spirit to Tanré's classification of semidirect products of dg Lie algebras. This classification is not needed in the proof of the main theorem but conceptually quite clarifying.

2.1. Preliminaries on differential graded coalgebras

We recall very briefly some basic terminology and properties of cocommutative coalgebras from [Qui69, App. B] and also [Tan83].

Definition 2.1.1. A **differential graded coalgebra** $C = (C, d_C, \Delta_C, \epsilon_C)$ is a chain complex (C, d_C) over $\mathbb{Q}$ with comultiplication $\Delta_C : C \to C \otimes C$ and augmentation $\epsilon_C : C \to \mathbb{Q}$ that are chain maps satisfying coassociativity $(\Delta_C \otimes \mathrm{Id}_C) \circ \Delta_C = (\mathrm{Id}_C \otimes \Delta_C) \circ \Delta_C$ and counitality $(\mathrm{Id}_C \otimes \epsilon_C) \circ \Delta_C = \mathrm{Id}_C = (\epsilon_C \circ \mathrm{Id}_C) \circ \Delta_C$. A differential graded coalgebra is **cocommutative** if $T \circ \Delta_C = \Delta_C$, where T denotes the isomorphism of $C \otimes C' \to C' \otimes C$ that sends $c \otimes c'$ to $(-1)^{|c| \cdot |c'|} c' \otimes c$.

We denote by dgcc the category of differential graded cocommutative coalgebras over $\mathbb{Q}$ with morphisms given by counital coalgebra maps of degree zero that are chain maps. Often we will omit subscripts of the differential and comultiplication when it is clear from context which coalgebra we are referring to, and sometimes we will use *Sweedler's notation* to express the value of the comultiplication as $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$.

One important example of a coalgebra is the *cofree (conilpotent) cocommutative coalgebra* $\Lambda^c V$ on a graded vector space V. It is defined on the free graded commutative algebra ΛV with comultiplication

$$\Delta(v_1 \wedge \ldots v_n) := \sum_{0 \le k \le n,\, \sigma \in Sh(k, n-k)} \varepsilon(\sigma) \left(v_{\sigma(1)} \wedge \ldots v_{\sigma(k)}\right) \otimes \left(v_{\sigma(k+1)} \wedge \ldots \wedge v_{\sigma(n)}\right)$$

where $Sh(k, n-k)$ denotes k-shuffles and $\varepsilon(\sigma) = \pm 1$ is the sign determined by the Koszul rule.

Definition 2.1.2. A **coderivation** of a coalgebra C is a linear map $\theta : C \to C$ such that $\Delta_C \circ \theta = (\theta \otimes \mathrm{Id} + T \circ (\mathrm{Id} \otimes \theta) \circ T) \circ \Delta_C$. In particular, the differential d_C is a coderivation of degree -1 and the graded vector space of coderivations $\mathrm{CoDer}(C)$ is a dg Lie algebra with bracket given by the commutator and differential $[d_C, -]$.

Completely dual to derivations of free algebras, coalgebra endomorphisms and coderivations $\theta \in \mathrm{CoDer}(\Lambda^c V)$ of a cofree coalgebra $\Lambda^c V$ are determined by their *corestrictions*

$$\theta_n : \Lambda^n V \xrightarrow{\theta} \Lambda V \overset{\pi_V}{\twoheadrightarrow} V.$$

A coalgebra is *coaugmented* if there is a section $\eta : \mathbb{Q} \to C$ of the counit in dgcc and we denote $\eta(1)$ by $1 \in C$. A coaugmented coalgebra C is *conilpotent* if $\overline{C} := \ker(\epsilon)$ is a coalgebra with comultiplication $\overline{\Delta}(c) := \Delta(c) - c \otimes 1 - 1 \otimes c$ such that for all $c \in \overline{C}$ there exists n such that $\overline{\Delta}^n(c) = 0$. We will denote by $\mathsf{dgcc}^{\mathrm{conil}}$ the full subcategory of coaugmented conilpotent coalgebras. Then there is an adjunction [Qui69]

$$\mathsf{dgLie} \underset{\mathcal{L}}{\overset{C}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathsf{dgcc}^{\mathrm{conil}} \tag{2.2}$$

where dgLie is the category of differential graded Lie algebras. We briefly recall the definition of $C : \mathsf{dgLie} \to \mathsf{dgcc}^{\mathrm{conil}}$ from [Qui69, App. B]. For $L \in \mathsf{dgLie}$ there is an acyclic dg Lie algebra $sL \oplus L$ (where sL denotes the suspension $(sL)_* = L_{*-1}$) that contains L as a dg Lie subalgebra with brackets and differential D determined by

$$[sl, sl'] = 0 \qquad [sl, l'] = s[l, l'] \qquad D(sl) = l - sd_L l. \tag{2.3}$$

The enveloping algebra $U(sL \oplus L)$ is an acyclic dg Hopf algebra that is a free $U(L)$-module via the inclusion $L \hookrightarrow sL \oplus L$. Quillen's defintion of the functor C is

$$C(L) := U(sL \oplus L) \otimes_{U(L)} \mathbb{Q}. \tag{2.4}$$

Note that $U(sL \oplus L)$ is a free $U(L)$-module on the enveloping algebra of the abelian Lie algebra sL so that ignoring differentials $C(L) \cong U(sL)$ is isomorphic to the cofree coalgebra $\Lambda^c sL$ as a graded coalgebra. We choose $C(L)$ as our definition of the Chevalley-Eilenberg complex $C^{\mathrm{CE}}_*(L; \mathbb{Q})$. More generally, if M is a left $U(L)$-module then the Chevalley-Eilenberg complex of L with coefficients in M is $C^{\mathrm{CE}}_*(L; M) := U(sL \oplus L) \otimes_{U(L)} M$.

Remark 2.1.3. This definition of C is different than the one used for example in [FHT01] where the differential differs by a sign on the quadratic corestriction. The different choices are isomorphic as dgc coalgebras via a coalgebra isomorphism which only has a non-trivial linear corestriction given by $-\mathrm{Id}_{sL} : sL \to sL$.

Denote by dgLie_c the full subcategory of connected dg Lie algebras, i.e. Lie algebras concentrated in positive degrees; and by dgcc_1 the full subcategory of simply connected coalgebras, i.e. coalgebras whose counit is an isomorphism in degree 0 and $\overline{C}$ is concentrated in degrees >1. Then the adjunction (2.2) induces a Quillen equivalence $\mathcal{L} \dashv C$ between dgLie_c and dgcc_1 according to [Qui69].

Before we end this section, observe that the category dgcc has a symmetric monoidal structure given by the tensor product of the underlying chain complexes: For $C, C' \in \mathsf{dgcc}$ the chain complex $C \otimes C'$ has a comultiplication $\Delta_{C \otimes C'} := (\mathrm{Id}_C \otimes T \otimes \mathrm{Id}_{C'})(\Delta_C \otimes \Delta_{C'})$ and counit $\epsilon_C \otimes \epsilon_{C'}$. The monoidal unit $\mathbf{1}$ is the coalgebra on the 1-dimensional chain complex concentrated in degree zero $(\mathbb{Q}, d = 0)$ with comultiplication determined by $\Delta(1) = 1 \otimes 1$ and counit $\epsilon_{\mathbf{1}} = \mathrm{Id}_{\mathbb{Q}}$. Note that the monoidal unit is also the terminal object in dgcc.

Lemma 2.1.4. *Let $C \in \mathsf{dgcc}$, then the comultiplication $\Delta : C \to C \otimes C$ and counit $\epsilon : C \to \mathbb{Q}$ are maps in dgcc. Hence dgcc is cartesian monoidal, i.e. the monoidal structure is the product in dgcc.*

Remark 2.1.5. The category $\mathsf{dgcc}^{\mathrm{conil}}$ has a model category structure [Qui69] so in particular it has all small limits and colimits. The analogous statement of Lemma 2.1.4 holds in $\mathsf{dgcc}^{\mathrm{conil}}$.

2.2. Classification of coalgebra bundles

We prove a classification of coalgebra bundles which are the algebraic models for fibrations. This classification is inspired by the geometric classification of fibre bundles via associated principal bundles. The counter part of principal bundles in rational homotopy theory are principal L-bundles for a dg Lie algebra L, which have been introduced by Quillen in [Qui69, App. B] where he also describes the classification of such principal L-bundles.

Definition 2.2.1. A **coalgebra bundle** with fibre $C \in \mathsf{dgcc}^{\mathrm{conil}}$ is a map $\pi : E \to B$ in $\mathsf{dgcc}^{\mathrm{conil}}$ such that there exists an isomorphism $\phi : B \otimes C \to E$ of graded coaugmented coalgebras satisfying $\pi\phi|_{B\otimes 1} = \mathrm{Id}_B$ and $\phi|_{1\otimes C} \in \mathrm{Hom}_{\mathsf{dgcc}^{\mathrm{conil}}}(C, E)$. We call such an isomorphism a local trivialization, and a pair $(\pi : E \to B, \phi)$ a trivialized coalgebra bundle with fibre C. Note that a local trivialization does not need to be a chain map. If it is then we say the bundle is trivial.

We want to formulate the classification of such bundles via the pullback of a universal bundle, which is also how the classification of semidirect products of dg Lie algebras in [Tan83] can be phrased. The universal coalgebra bundle with fibre $C \in \mathsf{dgcc}^{\mathrm{conil}}$ is obtained from the universal principal bundles that Quillen described in [Qui69]. In the following, we set

$$L = \mathrm{CoDer}(C) \in \mathsf{dgLie}.$$

Then C and $\mathbb{Q}$ are $U(L)$-modules and the counit $\epsilon : C \to \mathbb{Q}$ is a $U(L)$-module map. We will show that the induced map

$$\mathrm{Id} \otimes \epsilon : U(sL \oplus L) \otimes_{U(L)} C \longrightarrow U(sL \oplus L) \otimes_{U(L)} \mathbb{Q} = \mathcal{C}(L) \tag{2.5}$$

is the universal coalgebra bundle with fibre C and local trivialization

$$\phi_L : \mathcal{C}(L) \otimes C \xrightarrow{\cong} U(sL) \otimes C \xrightarrow{j \otimes \mathrm{Id}_C} U(sL \oplus L) \otimes_{U(L)} C,$$

where j is the map on enveloping algebras induced by inclusion of the abelian Lie algebra $j : sL \to sL \oplus L$. Note that ϕ_L is only a map of graded coalgebras and does not commute with the differential in general because neither the isomorphism $\mathcal{C}(L) \cong U(sL)$ nor the inclusion of Hopf algebras $U(sL) \to U(sL \oplus L)$ are chain maps.

Given a coalgebra bundle $\pi : E \to B$ with fibre C and a map $f : X \to B$ the pullback $f^*E := \lim(X \to B \leftarrow E) \in \mathsf{dgcc}^{\mathrm{conil}}$ exists. We will show that the canonical map $\pi : f^*E \to X$ is a coalgebra bundle with the same fibre. The limit of $X \to B \leftarrow E$ in the category of graded cocommutative coalgebras is the underlying coalgebra of f^*E. Thus for a local trivialization $\phi : B \otimes C \to E$ we get $f\pi_X = \pi \circ \phi \circ (f \otimes \mathrm{Id}_C) : X \otimes C \to B$ and hence there is an induced map of graded coalgebras $f^*\phi : X \otimes C \to f^*E$ which defines a local trivialization of the pullback.

Theorem 2.2.2. *Let $C \in \mathsf{dgcc}^{conil}$ and $L = \mathrm{CoDer}(C) \in \mathsf{dgLie}$. For $B \in \mathsf{dgcc}^{conil}$ denote the set of isomorphism classes of trivialized coalgebra bundles with fibre C over B by $\mathrm{Bun}_B(C)$, then*

$$\begin{aligned} \mathrm{Hom}_{\mathsf{dgcc}}(B, \mathcal{C}(L)) &\longrightarrow \mathrm{Bun}_B(C) \\ f &\longmapsto \left[\pi : f^*(U(sL \oplus L) \otimes_{U(L)} C) \to B, f^*\phi_L\right] \end{aligned}$$

is a one-to-one correspondence, where $f^*\phi_L$ is the induced local trivialization as described above. If $B \in \mathsf{dgcc}_1$ then the above statement is also true if one replaces L by $L^+ = \mathrm{CoDer}^+(C)$.

The transition from principal L-bundles for $L = \mathrm{CoDer}(C)$ to coalgebra bundles with fibre C as in (2.5) corresponds to the construction of associated bundles in geometric topology. Following this analogy, the proof of Theorem 2.2.2 is based on the algebraic analogue of the reverse process, that associates to a smooth fibre bundle $M \to E \to B$ the principal $\mathrm{Diff}(M)$-bundle $\coprod_{b \in B} \mathrm{Diff}(M, E_b) \to B$. We implement this correspondence and construct a principal L-bundle asscociated to a coalgebra bundle with fibre C by using the classification of principal L-bundles due to Quillen.

Let us briefly recall some definitions and important properties from Appendix B in [Qui69].

Definition 2.2.3 ([Qui69])**.** Given $L \in \mathsf{dgLie}$ and $B \in \mathsf{dgcc}^{\mathrm{conil}}$ then a **principal L-bundle** with base B is a triple (E, m, π), where $E \in \mathsf{dgcc}^{\mathrm{conil}}$ is a right $U(L)$-module with module map $m : E \otimes U(L) \longrightarrow E$ that is a map of dgc coalgebras, $\pi : E \to B$ is a map in $\mathsf{dgcc}^{\mathrm{conil}}$ satisfying $\pi(e \cdot u) = \pi(e) \cdot \epsilon(u)$ and there exists a map of graded coalgebras $\rho : B \to E$ so that $\varphi(b \otimes u) = \rho(b) \cdot u$ defines an isomorphism $\varphi : B \otimes U(L) \to E$ of graded cocommutative coalgebras and right $U(L)$-modules. Note however that φ is not a chain map in general. Such a map ρ is called a local trivialization.

Quillen proved that a principal L-bundle over B with a choice of local trivialization is determined by a *twisting function*, a linear map $\tau : B \to L$ of degree -1 such that

$$d_L\tau + \tau d_B + \frac{1}{2}[\,,\,] \circ (\tau \otimes \tau) \circ \Delta = 0 \qquad \text{and} \qquad \tau(1) = 0. \tag{2.6}$$

In particular, the principal L-bundle over B corresponding to a twisting function $\tau : B \to L$ has as total space $E = B \otimes U(L)$ with the obvious module structure and projection and (co)differential on E defined in terms of τ as

$$\begin{aligned} d_E(b \otimes u) &= d_E(b \otimes 1) \cdot u + (-1)^{|b|} b \otimes d_{U(L)}(u) \\ d_E(b \otimes 1) &= d_B(b) + \sum_i (-1)^{|b_{(1)}|} b_{(1)} \otimes \tau(b_{(2)}). \end{aligned} \tag{2.7}$$

Denote by $\mathcal{T}(B, L)$ the set of twisting functions. The functor

$$\mathcal{T}(-,-) : (\mathsf{dgcc}^{\mathrm{conil}})^{\mathrm{op}} \times \mathsf{dgLie} \to \mathsf{Set}$$

is representable for a fixed B or L via $\mathrm{Hom}_{\mathsf{dgLie}}(\mathcal{L}(B), -)$ respectively $\mathrm{Hom}_{\mathsf{dgcc}}(-, \mathcal{C}(L))$, i.e. there is a commutative diagram

$$\begin{tikzcd} & \mathcal{C}(L) \arrow[dr, "\tau_L"] & \\ B \arrow[ur, dashed, "\exists!\,\varphi_B"] \arrow[rr, "\tau"] \arrow[dr, "\tau_B"'] & & L \\ & \mathcal{L}(B) \arrow[ur, dashed, "\exists!\,\varphi_L"'] & \end{tikzcd} \tag{2.8}$$

where φ_B and φ_L are maps in the respective categories. If we consider the principal bundles $E(B, \mathcal{L}(B), \tau_B)$ and $E(\mathcal{C}(L), L, \tau_L) := U(sL \oplus L)$ corresponding to the universal twisting functions (see [Qui69, App. B] for definitions), this can be rephrased as a classification of principal bundles similar to geometric topology: For fixed L, every principal L-bundle over B is obtained as a pullback from the universal bundle $E(\mathcal{C}(L), L, \tau_L)$ along a map $B \to \mathcal{C}(L)$ in $\mathsf{dgcc}^{\mathrm{conil}}$.

We can now implement the construction of the principal $L = \mathrm{CoDer}(C)$-bundle associated to a coalgebra bundle with fibre C.

Proposition 2.2.4. *Let $\pi : E \to B$ be a coalgebra bundle with fibre $C \in \mathsf{dgcc}^{conil}$ and let ϕ be a local trivialization. Then*

$$\begin{aligned} \tau_E|_{\overline{B}} : \overline{B} &\longrightarrow \mathrm{CoDer}(C), \\ b &\longmapsto \left(c \mapsto \pi_C \phi^{-1} d_E \phi(b \otimes c)\right) \end{aligned} \tag{2.9}$$

extends to a twisting function τ_E by setting $\tau_E(1) = 0$

Proof. Since we fix a local trivialization, we can assume that $E = B \otimes C$ as graded coalgebra with differential D such that the projection $\pi = \pi_B : (B \otimes C, D) \to (B, d_B)$ and inclusion

$i : (C, d_C) \to (B \otimes C, D)$ are maps in dgcc. Then $\tau(b)(c) = \pi_C \circ D(b \otimes c)$ and we need to check that this defines in fact a coderivation and that τ satisfies the twisting condition.

Once can check that $\tau(b) := \pi_C \circ D(b \otimes -)$ defines a coderivation of C by using that π_C is a map of graded coalgebras. Moreover, it is clear from the definition that τ has degree -1 so it remains to check $d_L\tau + \tau d_B + \frac{1}{2}[\,,\,] \circ (\tau \otimes \tau) \circ \Delta = 0$. Let us first compute $D(b \otimes c)$. Note that $(\pi_B \otimes \pi_C)\Delta_{B\otimes C} = \mathrm{Id}_{B\otimes C}$, which, combined with D being a coderivation, gives

$$\begin{aligned} D(b \otimes c) &= \sum \pi_B D(b \otimes c_{(1)}) \otimes c_{(2)} + \sum (-1)^{|b_{(1)}|} b_{(1)} \otimes \pi_C D(b_{(2)} \otimes c) \\ &= d_B(b) \otimes c + \sum (-1)^{|b_{(1)}|} b_{(1)} \otimes \pi_C D(b_{(2)} \otimes c) \end{aligned}$$

where the second equality uses that π_B is a chain map. When we apply D again, the first summand gives

$$D(d_B b \otimes c) = \sum d_B(b_{(1)}) \otimes \pi_C D(b_{(2)} \otimes c) + (-1)^{|b_{(1)}|} b_{(1)} \otimes \pi_C D(d_B b_{(2)} \otimes c)$$

as $\Delta_B d_B b = d_B b_{(1)} \otimes b_{(2)} + (-1)^{|b_{(1)}|} b_{(1)} \otimes d_B b_{(2)}$, and the second summand becomes

$$\sum (-1)^{|b_{(1)}|} d_B(b_{(1)}) \otimes \pi_C D(b_{(2)} \otimes c) + \sum (-1)^{|b_{(1,2)}|} b_{(1,1)} \otimes \pi_C D(b_{(1,2)} \otimes \pi_C D(b_{(2)} \otimes c))$$

where we have set $\Delta b_{(1)} = \sum b_{(1,1)} \otimes b_{(1,2)}$. Combining these contributions and applying π_C gives

$$0 = \pi_C D^2(b \otimes c) = \pi_C D(d_B b \otimes c) + \sum (-1)^{|b_{(1)}|} \pi_C D(b_{(1)} \otimes \pi_C D(b_{(2)} \otimes c)). \tag{2.10}$$

We compare with the twisting condition. Its first two contributions are

$$\begin{aligned} [d_C, \tau(b)](c) &= d_C \pi_C D(b \otimes c) - (-1)^{|b|-1} \pi_C D(b \otimes d_C c) \\ &= \pi_C D(1 \otimes \pi_C D(b \otimes c)) + (-1)^{|b|} \pi_C D(b \otimes \pi_C D(1 \otimes c)), \\ \tau(d_B b)(c) &= \pi_C D(d_B b \otimes c), \end{aligned}$$

where we have rewritten d_C in terms of D using that the inclusion $C \to C \otimes B$ is a chain map by assumption; the third and last contribution of the twisting condition is

$$\begin{aligned} &\left(\frac{1}{2} \sum (-1)^{|b_{(1)}|} [\tau(b_{(1)}), \tau(b_{(2)})]\right)(c) = \\ &\qquad \frac{1}{2} \sum_{|b_{(i)}|<|b|} \pi_C D(b_{(1)} \otimes \pi_C D(b_{(2)} \otimes c)) - (-1)^{(|b_{(1)}|-1)(|b_{(2)}|-1)} \pi_C D(b_{(2)} \otimes \pi_C D(b_{(1)} \otimes c)). \end{aligned}$$

Note that $\tau(1) = 0$ so the sum does not include the contribution from $b \otimes 1 + 1 \otimes b$. One can check using the cocommutativity of B that the sum of these three terms gives the right side of (2.10) and thus vanishes. Hence, τ is a twisting function. $\square$

Corollary 2.2.5. *Let C and L be as above and denote by L-Bun_B the set of isomorphism classes of trivialized principal L-bundles over B, then*

$$\begin{aligned} L\text{-}Bun_B &\longrightarrow Bun_B(C) \\ [(E,m,\pi),\rho] &\longmapsto [\mathrm{Id}\otimes\epsilon : E\otimes_{U(L)} C \to E\otimes_{U(L)}\mathbb{Q}, \rho\otimes \mathrm{Id}_C] \end{aligned}$$

is a one-to-one correspondence. We call the coalgebra bundle $\mathrm{Id}\otimes\epsilon : E\otimes_{U(L)} C\to E\otimes_{U(L)}\mathbb{Q}$ *the* ***associated coalgebra bundle****. Hence, we can think of both principal L-bundles and coalgebra bundles just as a choice of local trivialization and a twisting function $\mathcal{T}(B,L)$.*

Proof. We first show that the map $E\otimes_{U(L)} C\to E\otimes_{U(L)}\mathbb{Q}$ defines a coalgebra bundle. Note that the forgetful functor $\mathcal{U} : \mathsf{dgcc}^{\mathrm{conil}}\to\mathsf{dgVect}$ has as right adjoint $\Lambda^c : \mathsf{dgVect}\to\mathsf{dgcc}^{\mathrm{conil}}$ given by the cofree conilpotent cocommutative coalgebra on V with differential determined by its linear part. Hence, $\mathcal{U}$ preserves colimits. If H is a cocommutative dg Hopf algebra and M and N are right respectively left H-modules for which the module maps $M\otimes H\to M$ and $H\otimes N\to N$ are maps in $\mathsf{dgcc}^{\mathrm{conil}}$, then $M\otimes H\otimes N \rightrightarrows M\otimes N$ is a diagram in $\mathsf{dgcc}^{\mathrm{conil}}$, and $\mathcal{U}(\mathrm{coeq}(M\otimes H\otimes N \rightrightarrows M\otimes N)) = M\otimes_H N\in\mathsf{dgVect}$. This should be interpreted as defining a coalgebra structure on the tensor product as a coequalizer in $\mathsf{dgcc}^{\mathrm{conil}}$. If $N_1\to N_2$ is a map in $\mathsf{dgcc}^{\mathrm{conil}}$ and a H-module map, the induced map $M\otimes_H N_1\to M\otimes_H N_2$ is a map in $\mathsf{dgcc}^{\mathrm{conil}}$.

The module map $E\otimes U(L)\to E$ is a map in $\mathsf{dgcc}^{\mathrm{conil}}$ by definition, and the same is true for $\mathbb{Q}$ as a (left) $U(L)$-module for every $L\in\mathsf{dgLie}$. One can check that the $U(L)$-module structure on C given by

$$\begin{aligned} m : U(L)\otimes C &\longrightarrow C \\ (\theta_1\cdot\ldots\cdot\theta_n)\otimes c &\longmapsto \theta_1\circ\ldots\circ\theta_n(c) \end{aligned} \tag{2.11}$$

is a map in $\mathsf{dgcc}^{\mathrm{conil}}$. The counit ϵ is a $U(L)$-module map so that $\mathrm{Id}\otimes\epsilon : E\otimes_{U(L)} C\to E\otimes_{U(L)}\mathbb{Q}$ is a map in $\mathsf{dgcc}^{\mathrm{conil}}$. If $\rho : B\to E$ is a local trivialization, then $\phi : B\otimes C\to E\otimes_{U(L)} C$ given by $\phi(b\otimes c) := \rho(b)\otimes c$ defines a local trivialization of the coalgebra bundle. Hence, the above map is well-defined.

By [Qui69, App.B Prop.5.3], there is a one-to-one correspondence L-$\mathrm{Bun}_B\to\mathcal{T}(B,L)$ that associates to a trivialized principal L-bundle its twisting function. This twisting function is the same as the one obtained from applying Proposition 2.2.4 to the associated bundle. Moreover, given $[\pi : E\to B,\phi]\in\mathrm{Bun}_B(C)$ then there is a trivialized principal L-bundle associated to τ_E and its associated coalgebra bundle is isomorphic to $(\pi : E\to B,\phi)$. □

Corollary 2.2.6. *The map defined in* (2.5) *is a coalgebra bundle with fibre C.*

There is one last technical statement we need for the proof of Theorem 2.2.2 in order to show that the pullback of a coalgebra bundle is a coalgebra bundle.

Lemma 2.2.7. *Let* $(\mathsf{C}, \otimes, \mathbf{1})$ *be a cartesian monoidal category and suppose* $f : X \to B$ *and* $\pi : E \to B$ *are two maps in* C*. Then* $\lim(X \to B \leftarrow E)$ *if it exists is isomorphic to the equalizer of*

$$X \otimes E \underset{X\otimes\pi}{\overset{f\otimes E}{\rightrightarrows}} X \otimes B \otimes E$$

Proof. In a cartesian monoidal category every object is a coalgebra object. So if there are $h : Z \to X$ and $h' : Z \to E$ satisfying $fh = \pi h'$ then $(h \otimes h')\Delta_Z : Z \longrightarrow X \otimes E$ equalizes the above diagram. Hence, there is a unique map $H : Z \longrightarrow \mathrm{eq}(X \otimes E \rightrightarrows X \otimes B \otimes E)$ so that the composition of H with $\mathrm{eq}(X \otimes E \rightrightarrows X \otimes B \otimes E) \to X \otimes E$ and the respective projections to X respectively E commute with h respectively h'. Hence, the equalizer has the universal property of the limit. □

Proof of Theorem 2.2.2. By Corollary 2.2.5, the set $\mathrm{Bun}_B(C)$ is in one-to-one correspondence with the set of twisting functions which is in one-to-one correspondence with $\mathrm{Hom}_{\mathsf{dgcc}}(B, \mathcal{C}(L))$ by [Qui69]. Thus is remains to show that the pullback of coalgebra bundles in Theorem 2.2.2 induces this correspondence. Let $f \in \mathrm{Hom}_{\mathsf{dgcc}}(B, \mathcal{C}(L))$. The pullback in the Theorem is the limit

$$\lim(B \xrightarrow{f} \mathcal{C}(L) \longleftarrow U(sL \oplus L) \otimes_{U(L)} C) \in \mathsf{dgcc}^{\mathrm{conil}}.$$

Since $\mathsf{dgcc}^{\mathrm{conil}}$ is a cartesian monoidal category by Lemma 2.1.4 with all small limits, the above Lemma implies that the pullback in $\mathsf{dgcc}^{\mathrm{conil}}$ of a diagram $X \to B \leftarrow E$ is the cotensor product $X \square_B E = \mathrm{eq}(X \otimes E \rightrightarrows X \otimes B \otimes E)$. If $E \to B$ is a coalgebra bundle with fibre C, then $E \cong B \otimes C$ as graded coalgebras. The B-comodule structure induced by π_B makes E into an extended B-comodule so that $X \square_B E \cong X \otimes C$ as graded coalgebras [EM66, Prop2.1]. This isomorphism is a local trivialization and it coincides with the local trivialization $f^*\phi$ obtained using the universal property of the pullback. In conclusion, $f^*E \to X$ is a coalgebra bundle with fibre C and local trivialization $f^*\phi$. We can apply this to the pullback of the universal coalgebra bundle along $f : B \to \mathcal{C}(L)$. One can check that the twisting function from Proposition 2.2.4 is $\tau_L \circ f$. Hence, the pullback induces a one-to-one correspondence as claimed.

The second part of the statement for simply connected base follows from the following observation about twisting functions $\tau : B \to L = \mathrm{CoDer}(C)$. If $b \in B$ is primitive and $d_B(b) = 0$, then it follows from the twisting equation that $[d_C, \tau(b)] = 0$. Consequently, if $B \in \mathsf{dgcc}_1$, then $\tau : \overline{B} \to \mathrm{CoDer}^+(C)$ as every element in B_2 is primitive and a cycle. In particular, $\mathcal{T}(B, \mathrm{CoDer}(C)) = \mathcal{T}(B, \mathrm{CoDer}^+(C))$ and so we can consider the 1-truncation if the base is simply connected. $\square$

2.3. A coalgebra model of the universal 1-connected fibration

We have set up the classification of coalgebra bundles in complete analogy to the classification of fibrations: A coalgebra C corresponds to a space X, the dg Hopf algebra $U(L)$ for $L = \mathrm{CoDer}^+(C)$ corresponds to $\mathrm{hAut}_0(X)$, and the classification of X-fibrations as pullbacks of the universal X-fibration $E_0 \to \mathrm{B}\,\mathrm{hAut}_0(X)$ corresponds to the classification of coalgebra bundles by Theorem 2.2.2 where $\mathcal{C}(\mathrm{CoDer}^+(\mathrm{C}))$ plays the role of $\mathrm{B}\,\mathrm{hAut}_0(X)$. In the light of this correspondence, it is natural to expect the following theorem for (good) coalgebra models of a space X.

Theorem 2.3.1. *Let X be a 1-connected space of finite type with minimal Sullivan model* $(\Lambda V, d)$ *and dual coalgebra* $(\Lambda^c V^\vee, d^\vee) \in \mathsf{dgcc}_1^{\mathrm{conil}}$. *Then the associated coalgebra bundle*

$$\pi : C_*^{CE}(\mathrm{CoDer}^+(\Lambda^c V^\vee); \Lambda^c V^\vee) \xrightarrow{\epsilon_*} C_*^{CE}(\mathrm{CoDer}^+(\Lambda^c V^\vee); \mathbb{Q}) \tag{2.12}$$

is a coalgebra model for the universal 1-connected fibration (1.7).

A possible strategy of proof is probably to do a classification of coalgebra bundles up to fibre homotopy equivalence similar as in [Tan83], where we say two coalgebra bundles $\pi_i : E_i \to B_i$ for $= 1, 2$ are *fibre homotopy equivalent* if there are quasi-isomorphisms $F : E_1 \to E_2$ and $f : B_1 \to B_2$ in dgcc so that $\pi_2 F = f \pi_1$. But it is much simpler to compare (2.12) with Tanré's model of the universal 1-connected fibration in [Tan83], which we briefly recall.

Let L be a dg Lie algebra and $\mathrm{Der}(L)$ the graded Lie algebra of derivations with differential $[d, -]$. Denote by $\mathrm{Der}^+(L)$ the truncation Lie algebra of positive degree (see (5.6)), then the adjoint $\mathrm{ad} : L \to \mathrm{Der}^+(L)$ is a dg Lie algebra homomorphism and the mapping cone, denoted by $\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\, L$, can be given the structure of a dg Lie algebra with bracket determined by

$$[\theta, sl] = (-1)^{|\theta|} s\theta(l) \qquad\qquad [sl, sl'] = 0.$$

Tanré showed that isomorphism classes of semidirect products of dg Lie algebras $L \rtimes L'$ are in a 1-1 correspondence with maps $\varphi : L' \to \mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\, L$. This is conceptually analogous to our classification of coalgebra bundles in Theorem 2.2.2. Moreover, semidirect products are the models for fibrations in dgLie, and Tanré showed that if $L = (\mathbb{L}V, d)$ is a minimal dg Lie algebra model of X, the semidirect product corresponding to the identity

$$L \longrightarrow L \rtimes_{\mathrm{id}} (\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\, L) \longrightarrow \mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\, L \tag{2.13}$$

is a dg Lie algebra model of (1.7). In particular, applying $\mathcal{C}$ to the above map gives a coalgebra model for the universal 1-connected fibration, which is by inspection also a coalgebra bundle over $\mathcal{C}(\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\ L)$ with fibre $\mathcal{C}(L)$. By Theorem 2.2.2 there is a classifying map

$$\Phi : \mathcal{C}(\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\ L) \to \mathcal{C}(\mathrm{CoDer}^+(\mathcal{C}(L))), \tag{2.14}$$

and we show that if L is a free dg Lie algebra $(\mathbb{L}V, d)$ then Φ is a quasi-isomorphism and induces a fibre homotopy equivalence of the corresponding coalgebra bundles with fibre $\mathcal{C}(\mathbb{L}V, d)$. This almost gives a full proof of Theorem 2.3.1. The last step will be to compare coalgebra bundles with fibre $\mathcal{C}(L)$ for a dg Lie model L of X with coalgebra bundles with fibre $(\Lambda^c V^\vee, d^\vee)$ for the dual of a Sullivan model of X.

Proposition 2.3.2. *Let Φ be the map in* (2.14). *Then the only non-trivial corestriction is linear, i.e. it is induced by a map*

$$\phi : \mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\ L \longrightarrow \mathrm{CoDer}^+(\mathcal{C}(L))$$

of dg Lie algebras. Moreover, ϕ is a quasi-isomorphism if L is a free dg Lie algebra.

Remark 2.3.3. This proposition has been proved in [SS12, Thm 3.17]. We have included it in this discussion because it is derived from the classification of coalgebra bundles, which has the (small) advantage that we do not need to check that it induces a map of dg Lie algebras. Also note that there is a dual statement involving the left adjoint $\mathcal{L}$ in [Gat97].

Proof. The sequence

$$\mathcal{C}(L) \to \mathcal{C}\Big(L \rtimes_{\mathrm{id}} (\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\, L)\Big) \to \mathcal{C}(\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\ L)$$

is a coalgebra bundle with fibre $\mathcal{C}(L)$ and Proposition 2.2.4 describes the corresponding twisting function $\tau : \mathcal{C}(\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\ L) \to \mathrm{CoDer}^+(\mathcal{C}(L))$. Note that the total space can be identified with $C_*^{CE}(\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\ L; \mathcal{C}(L))$. Since the differential on the total space is the differential of the Chevalley-Eilenberg complex of a dg Lie algebra, it only has nontrivial linear and quadratic corestriction. Hence, one can check that $\tau(\chi) = \pi_{\mathcal{C}(L)} D(\chi \otimes -)$ is only non-trivial on elements $\chi \in \Lambda^1 s(\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\ L)$. Denote by $\overline{D}$ the differential on the Chevalley-Eilenberg complex of the semidirect product from [Tan83, VII.2.(11)], then the corestrictions of the coderivation $\tau(\chi)$ of $\mathcal{C}(L)$ for $\chi \in \Lambda^1 s(\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\ L)$ are

$$\begin{aligned} \tau(s^2 l)_n(sl_1 \wedge \ldots \wedge sl_n) &= \pi_{\Lambda^1 sL} \overline{D}(s^2 l \wedge sl_1 \wedge \ldots \wedge sl_n) \\ &= \begin{cases} \pi_{\Lambda^1 sL} \overline{D}(s^2 l) = sl & n = 0 \\ 0 & n > 0 \end{cases} \end{aligned}$$

and

$$\tau(s\theta)_n(sl_1 \wedge \ldots \wedge sl_n) = \pi_{\Lambda^1 sL}\overline{D}(s\theta \wedge sl_1 \wedge \ldots \wedge sl_n)$$
$$= \begin{cases} \pi_{\Lambda^1 sL}\overline{D}(s\theta \wedge sl_1) = (-1)^{|\theta|}s[\theta, l_1] = (-1)^{|\theta|}s\theta(l_1) & n = 1 \\ 0 & n \neq 1 \end{cases}$$

By (2.8) τ factors through a dgc coalgebra map $\Phi : \mathcal{C}(\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\, L) \longrightarrow \mathcal{C}(\mathrm{CoDer}^+(\mathcal{C}(L))$, which in turn has also only a linear part $\Phi_1 : s(\mathrm{Der}^+(L) /\!\!/ \mathrm{ad}\, L) \to s(\mathrm{CoDer}^+(\mathcal{C}(L))$. The desuspension of Φ_1 is described by

$$(s^{-1}\Phi_1)(sl) \in \mathrm{CoDer}^+(\mathcal{C}(L)) \text{ has corestriction :} \qquad \Lambda^0 sL = \mathbb{Q} \xrightarrow{sl} sL$$
$$(s^{-1}\Phi_1)(\theta) \in \mathrm{CoDer}^+(\mathcal{C}(L)) \text{ has corestriction :} \qquad \Lambda^1 sL = sL \xrightarrow{(-1)^{|\theta|}s\theta} sL$$

Since we already know that Φ is a dgc coalgebra map on the Chevalley-Eilenberg complexes of the two dg Lie algebras with only linear part, the desuspension $s^{-1}\Phi_1 =: \phi$ has to be a map of dg Lie algebras, which is also readily checked.

Before we prove the second part of the statement, observe that there are isomorphisms of graded vector spaces

$$\mathrm{CoDer}(\mathcal{C}(L)) \cong \mathrm{Hom}_{\mathsf{grVect}}(\Lambda sL, sL) \cong \mathrm{Hom}_{U(L)}(U(sL \oplus L), sL)$$

where the right side is the graded vector space of $U(L)$-module homomorphisms and sL is the suspension of L considered as a module over itself. We will discuss differential homological algebra and semifree resolutions in more detail in section 3.2 and we refer the reader to that section for the definitions. To finish this proof, we note that $\mathrm{Hom}_{U(L)}(U(sL \oplus L), sL)$ is a differential $U(L)$-module and one can check that the above isomorphism of graded vector spaces is in fact an isomorphism of chain complexes. As $U(sL \oplus L) \to \mathbb{Q}$ is a semifree resolution of $\mathbb{Q}$ [FHT01, Prop. 22.4], the homology of $\mathrm{CoDer}(\mathcal{C}(L))$ can be identified with (a shift of) $H^*_{CE}(L; L) = \mathrm{Ext}_{U(L)}(\mathbb{Q}, L)$.

If $L = (\mathbb{L}V, d)$ is a free dg Lie algebra, there is a small free resolution of $\mathbb{Q}$ as described in [Wei94, Prop. 7.2.4] given by $\mathcal{I} \to U(L) \xrightarrow{\epsilon} \mathbb{Q}$, where $\mathcal{I}$ is the kernel of the augmentation. Then $\mathcal{I}$ is a free $U(L)$-module on a basis of V because L is free. We can build a semifree resolution of $\mathbb{Q}$ from it by setting $P = s\mathcal{I} \oplus U(L)$ with standard differential on $U(L)$ and $d(sl) = l - sd_{U(L)}l$ for $l \in \mathcal{I}$. Hence, $\mathrm{Hom}_{U(L)}(U(sL \oplus L), sL)$ is quasi-isomorphic to $\mathrm{Hom}_{U(L)}(P, sL)$ which is given as a bicomplex by

$$\mathrm{Hom}_{U(\mathbb{L}V)}(s\mathcal{I}, s\mathbb{L}V) \cong \mathrm{Hom}_{\mathsf{grVect}}(V, \mathbb{L}V) \longleftarrow \mathrm{Hom}_{U(\mathbb{L}V)}(U(\mathbb{L}V), s\mathbb{L}V) \cong s\mathbb{L}V,$$

which can be identified with $\mathrm{Der}(\mathbb{L}V) /\!\!/ \mathrm{ad}\,\mathbb{L}V$. There is an inclusion $i : P \to U(s\mathbb{L}V \oplus \mathbb{L}V)$ which is a map between semifree resolutions and therefore induces a quasi-isomorphism $i^* : \mathrm{Hom}_{U(\mathbb{L}V)}(P, s\mathbb{L}V) \to \mathrm{Hom}_{U(\mathbb{L}V)}(U(s\mathbb{L}V \oplus \mathbb{L}V), s\mathbb{L}V)$ [FHT01, Prop.6.7]. This agrees with ϕ under the identification of $\mathrm{Der}(\mathbb{L}V) /\!\!/ \mathrm{ad}\,\mathbb{L}V \cong \mathrm{Hom}_{U(\mathbb{L}V)}(P, s\mathbb{L}V)$ as well as $\mathrm{CoDer}(\mathcal{C}(\mathbb{L}V, d)) \cong \mathrm{Hom}_{U(\mathbb{L}V)}(U(s\mathbb{L}V \oplus \mathbb{L}V), s\mathbb{L}V)$. Hence, the map of 1-truncations in the statement is a quasi-isomorphism. □

Corollary 2.3.4. *The map $\phi_* : C_*^{CE}(\mathrm{Der}^+(\mathbb{L}V) /\!\!/ \mathrm{ad}\,\mathbb{L}V; \mathcal{C}(\mathbb{L}V)) \to C_*^{CE}(\mathrm{CoDer}^+(\mathcal{C}(\mathbb{L}V)); \mathcal{C}(\mathbb{L}V))$ is a quasi-isomorphism in dgcc_1 and is compatible with the map induced by the change of coefficients $\epsilon : \mathcal{C}(\mathbb{L}V) \to \mathbb{Q}$. In particular, ϕ_* is a fibre homotopy equivalence of coalgebra bundles with fibre $\mathcal{C}(\mathbb{L}V)$.*

We need the following lemma to finish the proof of Theorem 2.3.1. It is proved similarly as [BM20, Lem.3.5].

Lemma 2.3.5. *Let $\phi : C \to C'$ be a map in dgcc_1 and define $\mathrm{CoDer}_\phi(C, C')$ as the linear maps $\eta : C \to C'$ satisfying $\Delta_{C'}\eta = (\eta \otimes \phi + T(\eta \otimes \phi)T)\Delta_C$ and differential $D(\eta) = d_{C'}\eta - (-1)^{|\eta|}\eta d_C$. Then*

$$\begin{aligned} \phi_* : \mathrm{CoDer}(C) &\longrightarrow \mathrm{CoDer}_\phi(C, C') \qquad\qquad & \phi^* : \mathrm{CoDer}(C') &\longrightarrow \mathrm{CoDer}_\phi(C, C') \\ \eta &\longmapsto \phi \circ \eta & \eta' &\longmapsto \eta' \circ \phi \end{aligned}$$

are chain maps, and quasi-isomorphisms if ϕ is a quasi-isomorphism and C and C' are cofree.

Proof of 2.3.1. Let $(\mathbb{L}X, d)$ be a minimal Lie model of X and $(\Lambda^c V^\vee, d^\vee) \in \mathsf{dgcc}_1^{\mathrm{conil}}$ the dual of a minimal Sullivan model of X. Fix an injective quasi-isomorphism $\psi : \Lambda^c V^\vee \hookrightarrow \mathcal{C}(\mathbb{L}X)$. Then by the above lemma there is a zig-zag of quasi-isomorphisms

$$\mathrm{CoDer}(\Lambda^c V^\vee) \xrightarrow[\psi_*]{\simeq} \mathrm{CoDer}_\psi(\Lambda^c V^\vee, \mathcal{C}(\mathbb{L}X)) \xleftarrow[\psi^*]{\simeq} \mathrm{CoDer}(\mathcal{C}(\mathbb{L}X)).$$

Note that ψ^* is surjective, and hence by the same argument as in the proof of Theorem 3.12 in [BM20], the pullback $P = \{(\eta, \eta') \in \mathrm{CoDer}^+(\Lambda^c V^\vee) \times \mathrm{CoDer}^+(\mathcal{C}(\mathbb{L}X)) \mid \eta\psi = \psi\eta'\}$ in chain complexes of the above diagram is a differential graded Lie algebra so that the projections of 1-truncations $\pi_1 : P \to \mathrm{CoDer}^+(\Lambda^c V^\vee)$ and $\pi_2 : P^+ \to \mathrm{CoDer}^+(\mathcal{C}(\mathbb{L}X))$ are quasi-isomorphisms in dgLie. Note that both $\Lambda^c V^\vee$ and $\mathcal{C}(\mathbb{L}X)$ are $U(P)$-modules via these projections and importantly, ψ is a $U(P)$-module homomorphism. Hence, we have a sequence of induced maps

$$\begin{tikzcd} C_*^{CE}(\mathrm{CoDer}^+(\mathcal{C}(\mathbb{L}X)); \mathcal{C}(\mathbb{L}X)) & C_*^{CE}(P; \mathcal{C}(\mathbb{L}X)) \arrow[l, "(\pi_2)_*"', "\simeq"] & \\ & C_*^{CE}(P; \Lambda^c V^\vee) \arrow[u, "\psi_*", "\simeq"'] \arrow[r, "(\pi_1)_*", "\simeq"'] & C_*^{CE}(\mathrm{CoDer}^+(\Lambda^c V^\vee); \Lambda^c V^\vee), \end{tikzcd}$$

which are induced by quasi-isomorphisms of dg Lie algebras, and which are all compatible with the map to $C^{CE}(-;\mathbb{Q})$ induced by the counits. Hence, the above describes a sequence of fibre homotopy equivalences of coalgebra bundles. Combined with Corollary 2.3.4, this gives the desired comparison with Tanré's model of the universal fibration. $\square$

Remark 2.3.6. As we remarked in the introduction, Theorem 2.3.1 can also be extracted from [Laz14]. The strategy in this paper is actually quite similar; namely it is based on a classification of extensions of L_∞-algebras up to equivalence. A Sullivan model of a space X can also be interpreted as an L_∞-model and its universal extension from [Laz14] corresponds to the model in Theorem 2.3.1.

2.3.1. A cdga model of the universal 1-connected fibration

The dual of the universal coalgebra fibration in (2.12) determines a cdga model of the universal 1-connected fibration. When we start with the dual coalgebra of a Sullivan model, we can can give simpler description of the dual of (2.12) where we do not need consider coalgebras at all. This also recovers Sullivans dg Lie algebra model $\mathrm{B}\,\mathrm{hAut}_0(X)$ of Sullivan in terms of derivations of a minimal Sullivan model of X.

Let $L \in \mathsf{dgLie}$ and $C \in \mathsf{dgcc}$ be a $U(L)$-module. Recall that if the module map $U(L)\otimes C \to C$ is a map in dgcc or equivalently that C acts by coderivations, then $C_*^{CE}(L;C) := U(sL\oplus L)\otimes_{U(L)} C$ is in dgcc. Thus, the dual $C_*^{CE}(L;C)^\vee$ is a cdga and isomorphic to

$$\begin{aligned} \mathrm{Hom}_{\mathsf{grVect}}\Big(U(sL\oplus L)\otimes_{U(L)} C, \mathbb{Q}\Big) &\cong \mathrm{Hom}_{R-U(L)}\Big(U(sL\oplus L), \mathrm{Hom}_{\mathsf{grVect}}(C,\mathbb{Q})\Big) \\ &\cong \mathrm{Hom}_{L-U(L^{\mathrm{op}})}\Big(U(sL\oplus L), \mathrm{Hom}_{\mathsf{grVect}}(C,\mathbb{Q})\Big), \end{aligned}$$

where the first isomorphism follows by adjunction ($C^\vee := \mathrm{Hom}_{\mathsf{grVect}}(C,\mathbb{Q})$ is a right $U(L)$-module so that we have to consider right $U(L)$-morphisms) and the second isomorphism follows from the fact that every right $U(L)$-module determines a left $U(L^{\mathrm{op}})$-module. Here, L^{op} denotes the opposite dg Lie algebra $(L^{\mathrm{op}}, [\,,]_{L^{\mathrm{op}}}, d_{L^{\mathrm{op}}})$ on the same vector space as L with $[x,y]_{L^{\mathrm{op}}} = [y,x]_L$ and $d_{L^{\mathrm{op}}}(x) = -(-1)^{|x|}d_L(x)$. It is a basic fact that L is isomorphic as dg Lie algebra to its opposite.

Lemma 2.3.7. *Let $(L,[\,,],d) \in \mathsf{dgLie}$, then $(L,[\,,]^{op}, d^{\mathrm{op}})$ is a differential graded Lie algebra and*

$$\begin{aligned} \phi : (L,[\,,],d) &\longrightarrow (L,[\,,]^{op}, d_{L^{\mathrm{op}}}) \\ l &\longmapsto -(-1)^{\lfloor |l|/2 \rfloor} l \end{aligned}$$

is an isomorphism of differential graded Lie algebras.

Consequently, $U(sL \oplus L)$ is isomorphic to $U(sL^{\mathrm{op}} \oplus L^{\mathrm{op}})$ as a left $U(L^{\mathrm{op}})$-module and we get an isomorphism of cdga's

$$\mathrm{Hom}_{\mathsf{grVect}}\left(U(sL \oplus L) \otimes_{U(L)} C, \mathbb{Q}\right) \cong \mathrm{Hom}_{L-U(L^{\mathrm{op}})}\left(U(sL^{\mathrm{op}} \oplus L^{\mathrm{op}}), C^{\vee}\right).$$

This is the definition of $C^*_{CE}(L^{\mathrm{op}}, C^{\vee})$ using left-modules, which is isomorphic to Lie algebra cohomology via right modules by the above Lemma. But it is not necessary to explicitly work out the isomorphism.

Lemma 2.3.8. *Let $C \in \mathsf{dgcc}$ and $L = \mathrm{CoDer}(C) \in \mathsf{dgLie}$. Then $L^{op} \cong \mathrm{Der}(C^{\vee})$ and under this isomorphism the left $U(L^{\mathrm{op}})$-module structure is given by evaluating the derivation.*

Thus, we can avoid dualizing the cdga models of spaces and their derivations completely and we arrive at the following reformulation of Theorem 2.3.1.

Theorem 2.3.9. *Let X be a 1-connected space of finite type with minimal Sullivan model $(\Lambda V, d)$ and unit $\eta : \mathbb{Q} \to \Lambda V$. Then*

$$C^*_{CE}(\mathrm{Der}^+(\Lambda V); \mathbb{Q}) \xrightarrow{\eta_*} C^*_{CE}(\mathrm{Der}^+(\Lambda V); \Lambda V) \tag{2.15}$$

is a cdga model of the universal 1-connected fibration (1.7)*.*

We can give a more explicit description of (2.15). Let L be a connected dg Lie algebra that acts on a connected cgda A of finite type through positive degree derivations. Note that we have to define positive degree derivations of a cdga as derivations that *lower* the degree if we don't fix our convention for chain complexes to be homologically or cohomologically graded. In order to be consistent (for the adjunction as well) we consider a cdga A in this section as negatively graded, so that both grading issues are resolved. In particular, the cgda $\mathrm{Hom}_{U(L)}(U(sL \oplus L), A)$ as the dual of a connected dgc coalgebra is concentrated in negative degrees. If A is of finite type, then

$$\mathrm{Hom}_{L-U(L)}(U(sL \oplus L), A) \cong \bigoplus_{n \le 0} \prod_{k \ge 0} \mathrm{Hom}_{\mathsf{Vect}}((\Lambda sL)^k, A^{k+n}) \cong \mathrm{Hom}_{L-U(L)}(U(sL \oplus L), \mathbb{Q}) \otimes A$$

is an isomorphism of commutative algebras concentrated in negative degrees. We can work out the differential on the right side $C^*_{CE}(L; \mathbb{Q}) \otimes A$ under this isomorphism. Denote by $\{l_i\}$ a basis of L and by $\{a_j\}$ a basis of A, and denote by $\{a_j^*\}$ and $\{(sl_i)^*\}$the corresponding dual bases of $A^{\vee}$ and $\Lambda^1 sL \subset C^*_{CE}(L; \mathbb{Q})$. Furthermore, we denote the differential on $C^*_{CE}(L; \mathbb{Q}) \otimes A$ by D and by d on $C^*_{CE}(L; \mathbb{Q})$. Then under the above isomorphism D is given by

$$\begin{aligned} D(x \otimes 1) &= d(x) \otimes 1 \\ D(1 \otimes a) &= 1 \otimes d_A(a) - \sum_j \prod_i \left(a_j^*(l_i \cdot a) \cdot (sl_i)^*\right) \otimes a_j. \end{aligned} \tag{2.16}$$

for $x \in C^*_{CE}(L;\mathbb{Q})$ and $a \in A$. Finally, note that if A is a simply connected Sullivan algebra, then $\eta_* : C^*_{CE}(L;\mathbb{Q}) \to C^*_{CE}(L;A)$ is a relative Sullivan algebra. In the following, we will always interpret the above cdga's as positively graded.

Example 2.3.10. Consider an even dimensional sphere S^{2n} with Sullivan model

$$A_n = (\Lambda(x,y), |x| = 2n, |y| = 4n-1, d = x^2 \cdot \partial/\partial y).$$

Then $\mathrm{Der}^+(A_n)$ is 3-dimensional with basis $\eta_{2n-1} := x \cdot \partial/\partial y$, $\eta_{2n} := \partial/\partial x$ and $\eta_{4n-1} := \partial/\partial y$ and differential $[d, \eta_{2n}] = -2\eta_{2n-1}$. Hence, the inclusion of the abelian Lie algebra with trivial differential $\mathfrak{a} := \mathbb{Q}\{\eta_{4n-1}\} \hookrightarrow \mathrm{Der}^+(A_n)$ is a quasi-isomorphism of dg Lie algebras and we get a cdga quasi-isomorphism $C^*_{CE}(\mathrm{Der}^+(A_n);\mathbb{Q}) \to C^*_{CE}(\mathfrak{a};\mathbb{Q}) = (\Lambda z_{4n}, d = 0)$ and similarly with coefficients in A_n. Thus, the cgda model of the universal 1-connected S^{2n}-fibration in Theorem 2.3.9 is equivalent to

$$(\Lambda(z_{4n}), d = 0) \longrightarrow (\Lambda(z_{4n}) \otimes \Lambda(x,y), D(x) = 0, D(y) = x^2 - z_{4n}),$$

where we use (2.16) with basis $\{x^k, yx^l\}_{k,l\geq 0}$ of A_n and $\{\eta_{4n-1}\}$ of $\mathfrak{a}$.

Chapter 3.

Fibre integration in rational homotopy theory

We want to introduce fibre integration as a special case of the following more general construction. Let $\pi : E \to B$ be a fibration with fibre F and suppose $H^*(F;k)$ vanishes for $* > d$ and is non-trivial in degree d, where k is some commutative ring. Note that $H^*(F;k)$ is a $k[\pi_1(B)]$-module and let $\phi : H^d(F;k) \to k$ be a $k[\pi_1(B)]$-module homomorphism, where k is the trivial $k[\pi_1(B)]$-module.

Since $H^*(F;k)$ vanishes in degrees higher than d one can project $H^*(E;k)$ onto the d-th row of the E_∞-page of the Serre spectral sequence, and as there are no differentials into this row we have $E_\infty^{*-d,d} \subset E_2^{*-d,d}$. We define **$\phi$-integration** as the composition

$$\Phi_* : H^*(E;k) \twoheadrightarrow E_\infty^{*-d,d} \subset E_2^{*-d,d} = H^{*-d}(B; H^d(F;k)) \xrightarrow{H(\phi)} H^{*-d}(B;k). \tag{3.1}$$

Since the cohomological Serre spectral sequence is compatible with cup product, there is a push-pull identity

$$\phi_*(\pi^*(x) \smile y) = x \smile \phi_*(y) \tag{3.2}$$

for $x \in H^*(B;k)$ and $y \in H^*(E;k)$, and thus ϕ-integration is a $H^*(B;k)$-module map.

Definition 3.0.1. Let $\pi : E \to B$ be a fibration with oriented Poincaré fibre X of formal dimension d and suppose π is oriented, i.e. the orientation $\varepsilon_X : H^d(X;k) \to k$ is an isomorphism of $k[\pi_1(B)]$-modules. Then

$$\pi_! := (\mathcal{E}_X)_* : H^*(E;k) \to H^{*-d}(B;k)$$

is called **fibre integration** of $\pi : E \to B$.

The goal of this chapter is to find formulas for fibre integration and more generally ϕ-integration in terms of the algebraic models that we have introduced in the previous chapter.

3.1. Chain level fibre integration integration

In this section, we will study ϕ-integration using the algebraic models of fibrations $\pi : E \to B$ discussed in the the previous chapter. We will work with the dual concept of coalgebra bundles given by relative (Sullivan) algebras that we denote by $\pi^* : \mathrm{B} \to \mathrm{E}$. The main result is that a homomorphism $\phi : H^d(F;\mathbb{Q}) \to \mathbb{Q}$ as above defines a unique derived chain homotopy class $\Phi : \mathrm{E} \to \mathrm{B}$ of B-modules that induces ϕ-integration. We start by recalling some preliminaries about differential modules and relative Sullivan algebras from [FHT01].

Let $\pi^* : \mathrm{B} \to \mathrm{E}$ be a map of cgda's which gives E the structure of a differential B-module. For differential (left) B-modules M and N we denote by $\mathrm{Hom}_{\mathrm{B}}(M,N)$ the subcomplex of $\mathrm{Hom}_{\mathsf{grVect}}(M,N) = \oplus_i \prod_n \mathrm{Hom}_{\mathsf{Vect}}(M^n, N^{n+i})$ that are B-module homomorphisms, i.e. linear maps that satisfy $f(b \cdot m) = (-1)^{|f|\cdot|b|} b \cdot f(m)$. Then $\mathrm{Hom}_{\mathrm{B}}(M,N)$ is a differential B-module with differential $D(f) := d_N f - (-1)^{|f|} f d_M$ and module structure $(b \cdot f)(m) := b \cdot f(m)$.

We cannot expect to find a representative of ϕ-integration in $\mathrm{Hom}_{\mathrm{B}}(\mathrm{E}, \mathrm{B})$ for every cgda model of π. Instead one should consider suitably derived mapping spaces, which, for our purpose, can be defined ad hoc without reference to model structures via resolutions of dg modules and differential graded Ext groups, as they have sufficient invariance properties under quasi-isomorphisms.[1]

Definition 3.1.1. [FHT01, Ch. 6] A B-module P is called **semifree** if it is the union of an increasing sequence of differential B-submodules $P(0) \subset P(1) \subset \ldots$ such that $P(0)$ and each $P(k)/P(k-1)$ are free B-modules on a basis of cycles. A **semifree resolution** of $M \in \mathsf{Mod}_{\mathrm{B}}$ is a quasi-isomorphism $\varphi : P \to M$ of differential B-modules where P is semifree. The differential Ext groups are defined as $\mathrm{Ext}^*_B(M,N) := H^*(\mathrm{Hom}_{\mathrm{B}}(P,N))$ for a semifree resolution of M.

The most important example of semifree modules in this context are relative Sullivan models of fibrations. A **relative Sullivan algebra** is a cdga of the form $(\mathrm{B} \otimes \Lambda V, D)$ so that $\mathrm{id}_{\mathrm{B}} \otimes 1 : \mathrm{B} \to \mathrm{B} \otimes \Lambda V$ is a map of cdga's, and $V = \oplus_{p \geq 1} V^p$ is a graded vector space with an exhaustive filtration $V(0) \subset V(1) \subset \ldots$ of graded subspaces so that $D|_{1 \otimes V(0)} : V(0) \to \mathrm{B}$ and $D|_{1 \otimes V(k)} : V(k) \to \mathrm{B} \otimes \Lambda V(k-1)$. A relative Sullivan model of $\pi^* : \mathrm{B} \to \mathrm{E}$ is a quasi-isomorphism $\mathrm{E}' \to \mathrm{E}$ of B-algebras where E' is a relative Sullivan algebra.

The following properties of relative Sullivan models can be found in [FHT01] (with some additional assumptions about finiteness): A relative Sullivan algebra is a semifree B-module, so a Sullivan model $\mathrm{E}' \to \mathrm{E}$ is a semifree resolution of E. If B is a simply connected cdga and either the fibre or the base are of finite type then $\mathrm{E}' \otimes_{\mathrm{B}} \mathbb{Q} \cong (\Lambda V, d)$ is a Sullivan model of the fibre. Moreover, every cdga map $\pi^* : \mathrm{B} \to \mathrm{E}$ admits a relative Sullivan model $\varphi : \mathrm{E}' \xrightarrow{\simeq} \mathrm{E}$, which is a cofibrant replacement in the standard model structure on cdga. We denote the relative Sullivan model by $\mathrm{E}' \to \mathrm{E}$ to indicate this fact.

[1] We are following [FMT10] in this point of view.

Proposition 3.1.2. *Let $\pi : E \to B$ be a fibration with fibre F of 1-connected spaces and assume that $H^*(F;\mathbb{Q})$ is of finite type and nontrivial in degree d and vanishes above. Let $\pi^* : \mathrm{B} \to \mathrm{E}$ be a cdga model of π where B is simply connected, and denote by $\mathrm{E}' \xrightarrow{\simeq} \mathrm{E}$ a relative Sullivan model. Then the augmentation of B induces an isomorphism*

$$\mathrm{Ext}_{\mathrm{B}}^{-d}(\mathrm{E},\mathrm{B}) = H^{-d}(\mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B})) \xrightarrow{\cong} H^{-d}(\mathrm{Hom}_{\mathbb{Q}}(\mathrm{E}' \otimes_{\mathrm{B}} \mathbb{Q},\mathbb{Q})) \cong \mathrm{Hom}(H^d(F;\mathbb{Q}),\mathbb{Q})$$
$$[\Phi : \mathrm{E}' \to B] \longmapsto [\Phi \otimes_{\mathrm{B}} \mathbb{Q} : \mathrm{E}' \otimes_{\mathrm{B}} \mathbb{Q} \to \mathrm{B} \otimes_{\mathrm{B}} \mathbb{Q} \cong \mathbb{Q}]$$

In particular, for every $\phi \in \mathrm{Hom}(H^d(F),\mathbb{Q})$ there is a unique element $[\Phi] \in H^{-d}(\mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B}))$ and the composition $H^*(E) \cong H^*(\mathrm{E}') \xrightarrow{H(\Phi)} H^{*-d}(\mathrm{B}) \cong H^{*-d}(B)$ is the same as ϕ-integration.

Proof. We assume that B is simply connected so that $\mathrm{B}^{\geq p} \subset \mathrm{B}$ is a B-submodule. Consider the exhaustive filtration of $\mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B})$ given by $F^p = \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B}^{\geq p})$. According to [Boa99, Thm 9.3] the corresponding spectral sequence is conditionally convergent to the completion

$$\begin{aligned}\varprojlim_p \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B})/\mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B}^{\geq p}) &\cong \varprojlim_p \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B}\,/\,\mathrm{B}^{\geq p}) \\ &\cong \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\varprojlim_p \mathrm{B}\,/\,\mathrm{B}^{\geq p}) \\ &\cong \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B})\end{aligned}$$

where we have used that E' is a projective B-module for the first isomorphism. The E_1-page is $H(\mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathbb{Q})) \otimes B^p$. By assumption, $\mathrm{E}' = (\mathrm{B} \otimes \Lambda V, D)$ is a relative Sullivan algebra so that $\mathrm{E}' \otimes_{\mathrm{B}} \mathbb{Q} = (\Lambda V, d)$ is a cdga model of the fibre. Hence, the E_1-page can be simplified as $E_1^{p,q} = H^q((\Lambda V)^\vee) \otimes \mathrm{B}^p$. Since B is simply connected, the differential on the E_1-page is given by $\mathrm{id} \otimes d_{\mathrm{B}} : E_1^{p,q} \to E_1^{p+1,q}$ and thus the spectral sequence has E_2-page $E_2^{p,q} = H^p(\mathrm{B}) \otimes H^q((\Lambda V)^\vee)$. In particular, the spectral sequence vanishes for $q < -d$. Since the gradings are such that the differentials are $d_r : E_r^{p,q} \to E^{p+r,q-r+1}$, there are only finitely many nontrivial differentials. This implies that the derived E_∞-page is zero and so by [Boa99, Thm 7.1] the spectral sequence converges strongly to

$$E_2^{p,q} = H^p(\mathrm{B}) \otimes H^q((\Lambda V)^\vee) \cong H^p(B) \otimes H_{-q}(F) \Rightarrow H(\mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B})),$$

and we can recover $H(\mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B}))$ from the entries of the E_∞-page. The only contribution with total degree $-d$ comes from $E_\infty^{0,-d} \cong E_2^{0,-d} \cong \mathrm{Hom}(H^d(F;\mathbb{Q}),\mathbb{Q})$ which proves the first part of the statement.

It remains to show that for $\phi = H^d(\Phi \otimes_{\mathrm{B}} \mathbb{Q}) : H^d(\Lambda V, d) \to \mathbb{Q}$ the induced ϕ-integration map coincides with $H(\Phi) : H^*(\mathrm{E}') \to H^*(\mathrm{B})$. First, we note that $\mathrm{E}' = (\mathrm{B} \otimes \Lambda V, D)$ has a filtration $F^p = \mathrm{B}^{\geq p} \otimes \Lambda V$ and that the corresponding spectral sequence converges as $E_2^{p,q} = H^p(\mathrm{B}) \otimes$

$H^q(\Lambda V) \Rightarrow H^{p+q}(\mathrm{E}')$. In fact, B also has an analogous filtration $G^p = \mathrm{B}^{\geq p}$ with only nontrivial differential on the E_1-page. Then Φ induces a map between these two filtrations and the map on E_2-pages is precisely ϕ-integration defined using this spectral sequence. As we have defined ϕ-integration using the Serre spectral sequence, it remains to show that this spectral sequence is isomorphic to the Serre spectral sequence. Grivel has shown in [Gri79] that $A_{PL}(B) \to A_{PL}(E)$ has a filtration which gives rise to the Serre spectral squence, and the construction is based on the construction of the Serre spectral sequence by Dress [Dre67]. Moreover, in the proof of [Gri79, Thm.6.4] he shows that the comparison map of $\mathrm{B} \to \mathrm{E}'$ with $A_{PL}(B) \to A_{PL}(E)$ is compatible with the above filtration on E' and the filtration on $A_{PL}(E)$ and induces an isomorphism on the E_2-pages. This concludes the proof. □

We can use Proposition 3.1.2 to build a representative of fibre integration for an oriented fibration $\pi : E \to B$ and oriented Poincaré fibre (X, ε_X) as follows: Consider a relative Sullivan model $\pi^* : \mathrm{B} \to (\mathrm{B} \otimes \Lambda V, D)$ and pick a chain level representative $\varepsilon_X \in \mathrm{Hom}^{-d}(\Lambda V, \mathbb{Q})$. The above proposition implies that there is a cycle $\Pi \in \mathrm{Hom}^{-d}(\mathrm{E}', \mathrm{B})$ unique up to chain homotopy that satisfies

$$\Pi(1 \otimes \chi) = \varepsilon_X(\chi) \in \mathbb{Q} = \mathrm{B}^0 \tag{3.3}$$

for all $\chi \in (\Lambda V)^d$, and that this chain map induces fibre integration on cohomology

$$\pi_! : H^*(E; \mathbb{Q}) \cong H^*(\mathrm{B} \otimes \Lambda V, D) \xrightarrow{H(\Pi)} H^{*-d}(\mathrm{B}).$$

We demonstrate this technique in the following example.

Example 3.1.3. Recall the relative Sullivan model of the universal 1-connected fibration for an even dimensional sphere $X = S^{2n}$ as discussed in Example 2.3.10. We choose as orientation $\varepsilon_X : H^d(X; \mathbb{Q}) \to \mathbb{Q}$ the homomorphism determined by $\varepsilon_X(x) = 1$. For degree reasons $\Pi(yx^k) = 0$ and since Π has to be a chain map we have $0 = \Pi(D(yx^k)) = \Pi(x^{k+2} - z_{4n}x^k)$. This determines a $\Lambda(z_{4n})$-module map $\Pi : (\Lambda(z_{4n}, x, y), D) \to (\Lambda(z_{4n}), d = 0)$ by

$$\Pi(yx^k) = 0 \qquad \text{and} \qquad \Pi(x^n) = \begin{cases} 0 & n = 2k \\ z_{4n}^k & n = 2k+1 \end{cases}$$

which is a chain map by construction and induces fibre integration on cohomology as it satisfies (3.3).

3.2. Relation to parametrized stable homotopy theory

Fibre integration can also be viewed as a construction in fibrewise stable homotopy theory. More precisely, if $\pi : E \to B$ is a fibration then $\pi_+ := \pi \amalg \mathrm{Id}_B : E_+ = E \coprod B \to B_+ = B \coprod B$

is a map of fibrewise pointed spaces over B. The fibrewise suspension spectra $\Sigma^\infty_B E_+$ is strongly dualizable in the category $\mathsf{Sp}_{/B}$ of parametrized spectra over B if the fibre of π is equivalent to a finite CW complex [MS06]. The fibrewise suspension spectra $\Sigma^\infty_B B_+ = \mathbf{S}_B$ is self dual so that, if the fibre of π is equivalent to a finite CW complex, the dual of $\Sigma^\infty_B \pi_+$ is $D(\Sigma^\infty_B \pi_+) : \Sigma^\infty_B B_+ \to D(\Sigma^\infty_B E_+) = F_B(\Sigma^\infty_B E_+, \mathbf{S}_B)$. If the fibre of $\pi : E \to B$ is Poincaré, we can combine this with

$$F_B(\Sigma^\infty_B E_+, \mathbf{S}_B) \to F_B(\Sigma^\infty_B E_+, H_B\mathbb{Z}) \to \Sigma^{\infty-d}_B E_+ \wedge_B H_B\mathbb{Z},$$

where the second map the fibrewise Poincaré duality equivalence constructed in [HLLR17, Section 3.1], to obtain the version of fibre integration in parametrized stable homotopy theory

$$\pi_! : \Sigma^\infty_B B_+ \to \Sigma^{\infty-d}_B E_+ \wedge_B H_B\mathbb{Z}.$$

Rationally, $H_B\mathbb{Z}$ is equivalent to $\mathbf{S}_B$ so that $\pi_!$ is rationally equivalent to a map of parametrized suspension spectra. The set of homotopy classes of such maps has been computed in [FMT10, Thm 1.1] as a differential Ext group. This is consistent with our result. However, we would like to note that in order to apply Theorem 1.1 from [FMT10] to describe $\pi_!$ as an element in an differential Ext group, it is essential to use the equivalence $F_B(\Sigma^\infty_B E_+, H_B\mathbb{Z}) \to \Sigma^{\infty-d}_B E_+ \wedge_B H_B\mathbb{Z}$ so as to identify $\pi_!$ as a map of fibrewise suspension spectra. The equivalence of parametrized spectra needs Poincaré duality of the fibre. In contrast, we don't need to assume this in order to find a unique representative of $\pi_!$ in the differential Ext groups because the construction works just as well for finding representatives of ϕ-integration in general.

Finally, we want to remark that fibre integration has also been described as elements in differential Ext groups in [FT09, Thm A] for fibrations over Poincaré duality spaces and for pullbacks of such fibrations.

Chapter 4.

The Euler ring of Poincaré duality spaces

In this chapter, we will compute the Euler ring for several examples of Poincaré spaces. The computation is split into two steps. First, we compute the Euler ring of the universal 1-connected fibration over $\mathrm{B}\,\mathrm{hAut}_0(X)$ that we denote by $E_0^*(X)$, and for this computation we can use the tools from rational homotopy theory developed in the previous chapters. In a second step, we will analyse the fibration sequence $\mathrm{B}\,\mathrm{hAut}_0(X) \to \mathrm{B}\,\mathrm{hAut}^+(X) \to \mathrm{B}\,\pi_0(\mathrm{hAut}^+(X))$ to compute the Euler ring itself.

4.1. Algebraic definition of the fibrewise Euler class

We have defined the Euler class of fibrations with Poincaré fibre over Poincaré spaces in the introduction in (1.5). In this section, we will give a definition for arbitrary base spaces using rational homotopy theory and show that it agrees with the definition of the fibrewise Euler class in [HLLR17]. In the following, we denote by $\mathbb{Q}[n]$ the graded vector space with $\mathbb{Q}$ in degree n and for a B-module M define $M[n] := M \otimes \mathbb{Q}[n]$.

Proposition 4.1.1. *Let* $\mathrm{B} \to \mathrm{E}'$ *be a relative Sullivan model of an oriented fibration of 1-connected spaces* $\pi : E \to B$ *with Poincaré fibre* X *of formal dimension* d. *Let* ε_X *be the orientation and* $\Pi \in \mathrm{Hom}_{\mathrm{B}}^{-d}(\mathrm{E}', \mathrm{B})$ *be a corresponding cycle representing fibre integration. Then the map*

$$\begin{aligned} \bar{\Pi} : \mathrm{E}'[-d] &\longrightarrow \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}', \mathrm{B}) \\ e &\longmapsto (e' \mapsto (-1)^{d + d\cdot|e|}\Pi(e \cdot e')) \end{aligned}$$

is a quasi-isomorphism of B*-modules.*

Proof. It is a simple check that $\bar{\Pi}$ defines a B-module homomorphism. By assumption, $\mathrm{E}' = (\mathrm{B} \otimes \Lambda, D)$ is a relative Sullivan algebra and thus has a filtration which induces the Serre spectral sequence as discussed in the proof of Proposition 3.1.2. In the same proof we have described a filtration of $\mathrm{Hom}_{\mathrm{B}}(\mathrm{E}', \mathrm{B})$ which strongly converges because there is a horizontal vanishing line. The map $\bar{\Pi}$ is compatible with the two filtrations and induces a map of the associated spectral sequences. The induced map on the E_2-page is given by

$$E_2^{p,q} = H^p(\mathrm{B}) \otimes H^q(\Lambda V, d) \xrightarrow{\mathrm{Id}\otimes\bar{\varepsilon}_X} H^p(\mathrm{B}) \otimes \mathrm{Hom}(H^{q-d}(\Lambda V), \mathbb{Q})$$

where $\bar{\varepsilon}_X : H^q(\Lambda V) \to \mathrm{Hom}(H^{d-q}(\Lambda V), \mathbb{Q})$ is the adjoint of $H^q(\Lambda V) \otimes H^{d-q}(\Lambda V) \xrightarrow{\cup} H^d(\Lambda V) \xrightarrow{\varepsilon_X} \mathbb{Q}$. Since $(H^*(X;\mathbb{Q}), \varepsilon_X)$ is an oriented Poincaré duality algebra, $\bar{\Pi}$ induces an isomorphism of E_2-pages. □

With this we get an algebraic model of the Umkehr map of $\Delta : E \to E \times_B E$. Let $\mathrm{B} \to \mathrm{E}'$ be a relative Sullivan model of $\pi : E \to B$ as above. Then $\mathrm{E}' \otimes_{\mathrm{B}} \mathrm{E}'$ is a Sullivan model of $E \times_B E$ and $\Pi \otimes \Pi : \mathrm{E}' \otimes_{\mathrm{B}} \mathrm{E}' \to \mathrm{B} \otimes_{\mathrm{B}} \mathrm{B} = \mathrm{B}$ is a chain level representative of fibre integration for the parametrized product as it is a cycle and restricts to the induced orientation of the fibre $X \times X$. If we denote by $\Delta : \mathrm{E}' \otimes \mathrm{E}' \to \mathrm{E}'$ the multiplication, we can provide the Umkehr map as the (dashed) lift in

$$\begin{tikzcd} \mathrm{E}'[-d] \arrow[r, dashed, "\Delta_!"] \arrow[d, "\bar{\Pi}"', "\simeq"] & \mathrm{E}' \otimes_{\mathrm{B}} \mathrm{E}'[-2d] \arrow[d, "\overline{\Pi\otimes\Pi}"', "\simeq"] \\ \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}', \mathrm{B}) \arrow[r, "\Delta^*"] & \mathrm{Hom}_{\mathrm{B}'}(\mathrm{E}' \otimes_{\mathrm{B}} \mathrm{E}', \mathrm{B}) \end{tikzcd} \tag{4.1}$$

uniquely up to homotopy, since E' is semifree. This determines the Umkehr map in the homotopy category of B-modules for any cdga model of the fibration, and enables us to identify a representative of the fibrewise Euler class in terms of the algebraic models.

Definition 4.1.2. Let $\pi : E \to B$ be a 1-connected oriented fibration with Poincaré fibre (X, ε_X) of formal dimension d. Let $\mathrm{B} \to \mathrm{E}'$ be a relative Sullivan model, then $e^{\mathrm{fw}}(\pi) := \Delta^*\Delta_!(1) \in H^d(\mathrm{E}')$.

It remains to show that this coincides with the naive definition from the introduction which was only given when the base is a Poincaré complex. This has been carried out in greater generality in [RW17] and [HLLR17], but in our case there is a simple algebraic proof.

Lemma 4.1.3. *Let $\pi : E \to B$ be an oriented fibration with fibre and base simply connected Poincaré complexes of formal dimension d respectively b. Let $\mathrm{B} \to \mathrm{E}'$ be a relative Sullivan model. Then the Umkehr map $\Delta_! : \mathrm{E}' \to \mathrm{E}' \otimes_{\mathrm{B}} \mathrm{E}'$ agrees on cohomology with $D^{-1}_{E \times_B E} \Delta_* D_E : H^*(E) \to H^{*+d}(E \times_B E)$ where $\Delta : E \to E \times_B E$ and $D_E = [E] \cap$ - and $D_{E\times_B E} = [E \times_B E] \cap$ - denote the Poincaré duality isomorphisms.*

Proof. Since B is a Poincaré space, $B \to *$ is a fibration with Poincaré fibre and we can apply the characterisation of the previous section to get a chain level representative of fibre integration map $\Pi_B \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{B},\mathbb{Q})$ corresponding to evaluating a fundamental class. Suppose $\mathrm{B} \to \mathrm{E}'$ is a Sullivan model of the fibration and let $\Pi \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B})$ be a model of fibre integration of π. It is well known that $([B]\cap -)\circ \pi_! : H^{b+d}(E) \to H^b(B) \to \mathbb{Q}$ is an orientation of the Poincaré algebra $H^*(E;\mathbb{Q})$ itself. In particular, $\Pi_B \circ \Pi \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{E}',\mathbb{Q})$ is a cycle that induces this orientation thus has to be fibre integration of $E \to *$. Denote $\Pi_E := \Pi_B \circ \Pi$ so that

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B}) & \mathrm{E}'[-d] \arrow[l, "\simeq"', "\bar{\Pi}"] \arrow[r, "\simeq", "\bar{\Pi}_E"'] & \mathrm{Hom}_{\mathbb{Q}}(\mathrm{E}',\mathbb{Q}) \arrow[from=1-1, to=1-3, bend right, "(\Pi_B)_*"'] \end{tikzcd}$$

is a commuting diagram, where $(\Pi_B)_*\varphi = \Pi_B \circ \varphi$ for $\varphi \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B})$). We can choose chain level representatives of fibre integration of $E \times_B E \to B$ and $E \times_B E \to *$ as $\Pi \otimes \Pi : \mathrm{E}' \otimes_{\mathrm{B}} \mathrm{E}' \to \mathrm{B} \otimes_{\mathrm{B}} \mathrm{B} \cong \mathrm{B}$ and $\Pi_{E\times_B E} := \Pi_B \circ (\Pi \otimes \Pi)$ by the same arguments as above. We therefore have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Q}}(\mathrm{E}\ \mathbb{Q}) \arrow[r, "\Delta^*"] & \mathrm{Hom}_{\mathbb{Q}}(\mathrm{E}' \otimes_{\mathrm{B}} \mathrm{E}',\mathbb{Q}) \\ \mathrm{E}'[-d] \arrow[u, "\bar{\Pi}_E", "\simeq"'] \arrow[d, "\bar{\Pi}"', "\simeq"] \arrow[r, dashed, shift left] \arrow[r, dashed, shift right] & \mathrm{E}' \otimes_{\mathrm{B}} \mathrm{E}'[-2d] \arrow[u, "\simeq", "\bar{\Pi}_{E\times_B E}"'] \arrow[d, "\simeq"', "\overline{\Pi\otimes\Pi}"] \\ \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}',\mathrm{B}) \arrow[r, "\Delta^*"] \arrow[uu, bend left, "(\Pi_B)_*"] & \mathrm{Hom}_{\mathrm{B}}(\mathrm{E}' \otimes_{\mathrm{B}} \mathrm{E}',\mathrm{B}') \arrow[uu, bend right, "(\Pi_B)_*"'] \end{tikzcd}$$

where the dashed maps denote the maps in the Umkehr maps in the homotopy category from (4.1) and using the Poincaré duality of E. The diagram shows that they induce the same map up to chain homotopy and therefore the same on cohomology. □

This lemma implies that the fibrewise Euler class in Definition 4.1.2 agrees with the construction from the introduction in (1.5) for Poincaré base spaces. This fact also holds for the fibrewise Euler class constructed in [HLLR17], which is sufficient for proving that the two definitions agree.

Corollary 4.1.4. *Let $\pi : E \to B$ be an oriented fibration of 1-connected spaces with Poincaré fibre X and relative Sullivan model* $\mathrm{B} \to \mathrm{E}'$. *Then* $e^{fw}(\pi) = \Delta^*\Delta_![1] \in H^d(\mathrm{E}')$ *agrees with the fibrewise Euler class constructed in [HLLR17].*

Proof. Since rational homology of a space B agrees with rational stably framed bordism, it suffices to evaluate the Euler class on bordism classes $[f : M \to B, \xi] \in \Omega^{fr}(B) \otimes \mathbb{Q}$, where we can assume by performing framed surgery that M is simply connected. The construction in [HLLR17] is natural under pullback $f^* : \mathsf{Sp}_{/B} \to \mathsf{Sp}_{/M}$, and they show in section 3.2 that the Euler class they define coincides for base spaces which are Poincaré with the class defined using the cohomological Umkehr maps from Poincaré duality (see (1.5))).

The class defined Definition 4.1.2 is natural with respect to pullbacks as well: Let $f : B' \to B$ be a map of simply connected spaces and denote the pullback fibration by $\pi' : f^*E \to B'$. Let $f^* : \mathrm{B} \to \mathrm{B}'$ be a cdga model of f. Then $f^*\,\mathrm{E}' := \mathrm{B}' \otimes_{\mathrm{B}} \mathrm{E}'$ is a relative Sullivan model of π'. If Π denotes a chain level representative of fibre integration then $f^*\Pi := \mathrm{B}' \otimes \Pi : \mathrm{B}' \otimes_{\mathrm{B}} \mathrm{E}' \to \mathrm{B}' \otimes_{\mathrm{B}} \mathrm{B} \cong \mathrm{B}'$ is a representative of fibre integration of the pullback and if $\Delta_!$ denotes a choice of lift for the Umkehr map then

$$f^*\Delta_! := \mathrm{B}' \otimes \Delta_! : f^*\,\mathrm{E}' = \mathrm{B}' \otimes_{\mathrm{B}} \mathrm{E}' \longrightarrow \mathrm{B}' \otimes_{\mathrm{B}} \mathrm{E} \otimes_{\mathrm{B}} \mathrm{E}[-d] \cong (f^*\,\mathrm{E}' \otimes_{\mathrm{B}'} f^*\,\mathrm{E}')[-d]$$

is a lift for the pullback which is natural with respect to f^*. If $\mathrm{B}' = M$ is a simply connected manifold, the Euler class agrees with class defined using the cohomological Umkehr maps from Poincaré duality by Lemma 4.1.3. Hence, the definitions of the fibrewise Euler class in Definition 4.1.2 and in [HLLR17] are both natural and agree on rational framed bordism classes. Therefore, our definition of the fibrewise Euler class agrees with that in [HLLR17]. □

It is quite difficult in general to compute a representative of the fibrewise Euler class using Definition 4.1.2 as one has to find a chain level fibre integration map and a lift for $\Delta_!$ in (4.1). In the next sections, we will discuss examples for which there are simpler descriptions of the fibrewise Euler class $e^{\mathrm{fw}}(\pi)$.

4.1.1. The fibrewise Euler class for Leray–Hirsch fibrations

In the case of fibrations $\pi : E \to B$ with oriented Poincaré fibre (X, ε_X) which are Leray–Hirsch, i.e. where the restriction map $H^*(E) \to H^*(X)$ is surjective, the definition of fibre integration and the fibrewise Euler class can be simplified significantly. Surjectivity of the restriction map implies that $H^*(E)$ is a free $H^*(B)$-module, and we may denote by $1, e_1, \ldots, e_k \in H^*(E)$ a $H^*(B)$-basis of the cohomology of E that restricts to a basis $1, x_1 = i^*(e_1), \ldots, x_k = i^*(e_k) \in H^*(X)$ of the fibre. If X is a Poincaré complex of formal dimension d, we can order the basis such that $|e_k| = d$ and all other $|e_i|$ have lower degree. Since fibre integration is a $H^*(B)$-module map. It suffices to determine $\pi_!$ on a basis and for degree reasons $\pi_!(e_i) = 0$ for $i < k$. If we set $\pi_!(e_k) = \varepsilon_X(x_k)$ this restricts to the orientation on the fibre hence determines fibre integration as

$$\pi_!\left(\sum_{i=0}^{k} b_i \cdot e_i\right) = \varepsilon_X(x_k) \cdot b_k \tag{4.2}$$

for $b_i \in H^*(B)$. Since the fibre is Poincaré, the (fibrewise) intersection pairing

$$\langle \text{-}, \text{-} \rangle : H^*(E) \otimes_{H^*(B)} H^*(E) \xrightarrow{\pi_!(-\cup-)} H^*(B) \tag{4.3}$$

is non-degenerate. This enables us to mimic the construction of the Euler class as the dual of the fibrewise diagonal.

Proposition 4.1.5 ([RW18]). *Let $\pi : E \to B$ be an oriented fibration with Poincaré fibre which is Leray–Hirsch. Let $e_0, \ldots, e_k \in H^*(E)$ be an $H^*(B)$-module basis and denote by $e_0^\#, \ldots, e_k^\# \in H^*(E)$ the dual basis under the non-degenerate pairing* (4.3)*. Then the fibrewise Euler class is*

$$e^{\mathrm{fw}}(\pi) = \sum_{i=0}^{k} (-1)^{|e_i|} e_i e_i^\# \in H^d(E;\mathbb{Q}). \tag{4.4}$$

Example 4.1.6. Let $X = S^{2n}$ be an even dimensional sphere and recall the cdga model and fibre integration from the Examples 2.3.10 and 3.1.3. Then 1 and x are a $H^*(\mathrm{B\,hAut}_0(S^{2n});\mathbb{Q})$-basis of the cohomology of the total space that restricts to a basis of $H^*(S^{2n})$ on the fibre, i.e. the fibration is Leray-Hirsch. Note that the formula for fibre integration in (4.2) gives the same result as our construction of Π in Example 3.1.3. We can apply the above Proposition to find a representative of the fibrewise Euler class. The dual basis with respect to the pairing induced by $\pi_! = H(\Pi)$ is $x^\# = 1$ and $1^\# = x$ since $\pi_!(x \cdot 1^\#) = \pi_!(x^2) = 0$, and we find that the fibrewise Euler class is represented by $e^{\mathrm{fw}}(\pi) = 2x$.

4.1.2. The fibrewise Euler class of positively elliptic spaces

A simply-connected space X is called *rationally elliptic space* if the collection of rational homotopy groups is finite dimensional $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$, as well as the cohomology algebra $\dim H^*(X;\mathbb{Q}) < \infty$. If in addition the Euler characteristic is positive, it is called *positively rationally elliptic*. Rationally elliptic spaces are very rigid owing to strong structure theorems about their Sullivan models $(\Lambda V, d)$, which are called elliptic if $\dim V < \infty$ and $\dim H(\Lambda V, d) < \infty$.

Proposition 4.1.7 ([FHT01, Prop. 32.10]). *Let $(\Lambda V, d)$ be an elliptic Sullivan algebra and denote by χ the Euler characteristic of $H(\Lambda V, d)$. Then $\chi \geq 0$ and* $\dim V^{\mathrm{odd}} \geq \dim V^{\mathrm{even}}$. *Moreover, the following are equivalent:*

(i) $\chi(X) > 0$;

(ii) $H(\Lambda V, d)$ *is concentrated in even degrees;*

(iii) $H(\Lambda V, d)$ *is the quotient $\Lambda(x_1, \ldots, x_k)/(f_1, \ldots, f_k)$ of a polynomial algebra on generators x_i of even degree by an ideal generated by a regular sequence;*

(iv) $(\Lambda V, d)$ *is isomorphic to a* pure Sullivan algebra $(\Lambda Q \otimes \Lambda P, d)$, *i.e. $Q = Q^{\mathrm{even}}$ and $V = V^{\mathrm{odd}}$ and $d|_Q = 0$ and $d|_V$ sends a basis of V to a regular sequence in ΛQ;*

(v) $\dim V^{\mathrm{odd}} = \dim V^{\mathrm{even}}$.

A regular sequence in a commutative ring R is a sequence $r_1, \ldots, r_k \in R$ such that r_i is not a zero-divisor in $R/(r_1, \ldots, r_{i-1})$ for all $1 \leq i \leq k$. The third characterisation implies that the rational homotopy theory of positively elliptic spaces is essentially equivalent to the theory of complete intersections over $\mathbb{Q}$.

Definition 4.1.8. Let B be a commutative ring. A B-algebra E is called finite if it is finitely generated as a B-module. A finite B-algebra E is a **complete intersection** if there exists an $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in B[x_1, \ldots, x_n]$ such that E is isomorphic to $B[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$.

From the point of view of fibrations of positively elliptic spaces, we observe that the derivation Lie algebra of an elliptic Sullivan algebra is particularly simple because $\dim V < \infty$. In fact, $\mathrm{Der}^+(\Lambda V)$ is finite dimensional as vector space, which is in sharp contrast to the general case when it is not even finite dimensional degree wise. This is some evidence that the theory of fibrations with positively elliptic fibre is considerably more simple than the general case, which is made precise in the following a conjecture by Halperin.

Halperin Conjecture. Let $E \to B$ be a fibration of simply connected spaces and positively rationally elliptic fibre. Then the fibration is Leray–Hirsch over the $\mathbb{Q}$.

This conjecture is known to be true for a large number of examples [Mei82, ST87, Tho81]. And since positively elliptic spaces always satisfy (rational) Poincaré duality, one can ask about Euler rings for these spaces. Assuming that the conjecture is true (which is a simple check for a given positively elliptic space X), then fibre integration and the fibrewise Euler class of the universal 1-connected fibration $E_0 \to \mathrm{B}\,\mathrm{hAut}_0(X)$ are determined by (4.2) and (4.4). But the conjecture implies even more, namely that $\mathrm{B}\,\mathrm{hAut}_0(X)$ is concentrated in even degrees and that the cohomology of the total space is a complete intersection over $H^*(\mathrm{B}\,\mathrm{hAut}_0(X); \mathbb{Q})$. This has been explained to the author by Alexander Berglund and is explained in more detail in Remark 4.2.5.

This translates the computation of the Euler ring of positively elliptic spaces into one of commutative algebra and complete intersections, and we will study it from this point of view in this section. In the following, we denote by B a commutative ring concentrated in even degrees (corresponding to the cohomology of the base space) and by E a complete intersection over B (corresponding to the cohomology of the total space of a fibration with positively elliptic fibre). For any E-module M there is an E-module structure on the B-linear dual $\mathrm{Hom}_B(M, B)$ given by $(e \cdot \varphi)(m) := \varphi(e \cdot m)$ for $e \in E$, $m \in M$ and $\varphi \in \mathrm{Hom}_B(M, B)$. For a finite B-algebra E there is an isomorphism $\mathrm{Hom}_B(E, E) \cong E \otimes_B \mathrm{Hom}_B(E, B)$ so that we can define the trace of any B-endomorphism of E via the evaluation map $E \otimes_B \mathrm{Hom}_B(E, B) \to B$. In particular, any element in E defines an endomorphism by multiplication of which one can take the trace, giving an element $\mathrm{Tr}_{E/B} \in \mathrm{Hom}_B(E, B)$.

Proposition 4.1.9 ([dSL97]). *Let B be a commutative ring and $f_1, \ldots, f_n \in B[x_1, \ldots, x_n]$ for a non-negative integer n. Assume that $E = B[x_1 \ldots, x_n]/(f_1, \ldots, f_n)$ is a finite B-algebra. Then*

(i) E is a projective B-module;

(ii) $\mathrm{Hom}_B(E, B)$ is a free of rank 1 as E-module;

(iii) there is a generator λ of $\mathrm{Hom}_B(E, B)$ as an E-module such that $\mathrm{Tr}_{E/B} = \det(\partial f_i/\partial x_j) \cdot \lambda$.

The equation in (iii) is the analogue for a complete intersection of the relation $\mathrm{trf}^*_\pi(-) = \pi_!(e^{\mathrm{fw}}(\pi) \cdot -)$ between the (Becker-Gottlieb) transfer $\mathrm{trf}^*_\pi : H^*(E) \to H^*(B)$, fibre integration and the fibrewise Euler class for a fibration $\pi : E \to B$ with Poincaré fibre X.

We make this precise as follows: let B be a polynomial ring over $\mathbb{Q}$ on finitely many generators in positive even degrees and augmentation ε. Consider a complete intersection $E = B[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ where the x_i are all in positive even degree. Then E is a projective B-module by (i) and since B is a connected graded algebra, E is a free B-module. Denote by $\pi : \langle E \rangle \to \langle B \rangle$ the geometric realization [BG76, Ch.5] of a relative Sullivan model of $B \to E$.

Theorem 4.1.10. *Let B and E be as above. Then the homotopy fibre of $\pi : \langle E \rangle \to \langle B \rangle$ has the rational homotopy type of the complete intersection $X = \mathbb{Q}[x_1, \ldots, x_n]/(\bar{f_1}, \ldots, \bar{f_n})$ over $\mathbb{Q}$ with $\bar{f_i} = \epsilon \circ f_i$. In particular, X is a rational Poincaré space with an orientation defined by $\varepsilon_X(\det(\partial \bar{f_i}/\partial x_j)) := \chi(\langle X \rangle)$ and the fibrewise Euler class is represented by*

$$e^{\mathrm{fw}}(\pi) = \det\left(\frac{\partial f_i}{\partial x_j}\right) \in E. \tag{4.5}$$

Proof. Consider the differential B-algebra $E' := (B \otimes \Lambda(x_i, y_i), D)$ with $|y_i| = |f_i| - 1$ and $D(y_i) := f_i$. This is relative version of a pure Sullivan model and the B-algebra map $C : E' \to E$ determined by $C(y_i) = 0$ and $C(x_i) = x_i$ is a quasi-isomorphism of differential B-algebras by the same argument as in the proof of Proposition 4.2.4. Moreover, E' and B are 1-connected Sullivan algebras so that $B \to A_{PL}(\langle B \rangle)$ and $E' \to A_{PL}(\langle E' \rangle)$ are quasi-isomorphisms and hence $B \to E'$ is a relative Sullivan model of the fibration $\pi : \langle E' \rangle \to \langle B \rangle$. Because B is of finite type, a cdga model of $\langle E' \rangle \to \langle B \rangle$ is given by $\mathbb{Q} \otimes_B E'$ [Tho81] which is a pure Sullivan model. Hence, the homotopy fibre has the rational homotopy type of a positively elliptic space [FHT01] with top degree generator given by $\det(\partial \bar{f_i}/\partial x_j) \in X$ [Mur93, ST87], so that ε_X as defined above is a valid choice of orientation.

Note that $\langle E'\rangle \to \langle B\rangle$ is formal so that fibre integration $\pi_! : H(\langle E'\rangle;\mathbb{Q}) \cong E \to H(\langle B\rangle;\mathbb{Q}) \cong B$ determines a map of B-modules with respect to our choice of orientation. According to Proposition 4.1.9, there is a B-module map $\lambda : \mathrm{E} \to \mathrm{B}$ satisfying (iii). It follows for degree reasons that λ and $\pi_!$ agree up to a scalar in $\mathbb{Q}^\times$. One can show that the trace and the transfer agree using the general theory in [DP80], or by directly checking that $\mathrm{Tr}_{E/B}(x) = \pi_!(e^{\mathrm{fw}}(\pi) \cdot x)$, as has been done in [RW18, Lemma 2.3] using that E is a finitely generated free B-algebra. If we evaluate the identity in Proposition 4.1.9(iii) for $x = 1$, we find that $\lambda(\det(\partial f_i/\partial x_j)) = \mathrm{Tr}_{E/B}(1) = \pi_!(e^{\mathrm{fw}}(\pi)) = \chi(\langle X\rangle)$, which agrees with $\pi_!(\det(\partial f_i/\partial x_j))$ by our choice of orientation ε_X. Hence, it follows that $\pi_! = \lambda$ and consequently

$$\det(\partial f_i/\partial x_j) \cdot \pi_! = \det(\partial f_i/\partial x_j) \cdot \lambda = \mathrm{Tr}_{E/B} = \mathrm{trf}^*_\pi = e^{\mathrm{fw}}(\pi) \cdot \pi_!$$

under the identification of $H^*(\langle E'\rangle;\mathbb{Q})$ with E and $H^*(\langle B\rangle;\mathbb{Q})$ with B. The claim follows because B-linear dual of E is a free E-module by (ii). □

4.2. Computations

We will begin by computing the Euler ring of even dimensional spheres, which exhibits some of the algebraic problems we will encounter in this section and combines the examples from the previous sections.

Proposition 4.2.1. *The Euler ring of an even dimensional sphere is $E^*(S^{2n}) \cong \mathbb{Q}[\kappa_2]$ with relations $\kappa_2^k = 2^{k-1}\kappa_{2k}$ and $\kappa_{2i+1} = 0$.*

Proof. We have seen in Example 4.1.6 that $e^{\mathrm{fw}}(\pi) = 2x \in \Lambda(x, y, z_{4n})$ and in Example 3.1.3 that fibre integration is given by $\Pi(x^{2k}) = 0$ and $\Pi(x^{2k+1}) = z_{4n}^k$. Thus $\kappa_{2k} = 2^{1-k}(2^3 z_{4n})^k = 2^{1-k}\kappa_2^k$. Since $\pi_0(\mathrm{hAut}^+(S^{2n}))$ is trivial, the 1-connected universal fibration is the universal fibration and the result follows. □

Remark 4.2.2. This result is of course well known and there are easier ways to prove this. For example, it follows from Example 2.3.10 that $\mathrm{B}\,\mathrm{hAut}^+(S^{2n})$ is rationally equivalent to $K(\mathbb{Q}, 4n)$, which reduces to proving this statement on linear bundles.

Observe that $E^*(S^{2n})$ is finitely generated despite it being defined as a ring with infinitely many generators. This is in fact true in much greater generality, and follows from a direct adaptation of the results in [RW18].

Proposition 4.2.3. *Let X be a Poincaré duality space with $H^*(X;\mathbb{Q})$ concentrated in even degrees and let $N = \dim H^*(X;\mathbb{Q})$. Then the Euler ring $E^*(X)$ is generated by $\kappa_1, \dots, \kappa_{N-2}, \kappa_N$.*

Proof. Let us start by assuming that the universal fibration is Leray-Hirsch and denote by $B{\to}E$ the map on (even degree) cohomology. Consider the B-endomorphism given by $-\cdot e^{\mathrm{fw}}(\pi) : E{\to}E$. Since E is a free B-module, we can apply the Cayley–Hamilton theorem to get that $e^{\mathrm{fw}}(\pi)$ is a root of its characteristic polynomial $p(e)$, which is monic and of degree N. The coefficients are certain polynomials in $\mathrm{Tr}_{E/B}(e^{\mathrm{fw}}(\pi)^j)$. Since $Tr_{E/B}(c) = \pi_!((e^{\mathrm{fw}}(\pi)\cdot c)$ for all $c \in E$ as discussed in the the proof of Theorem 4.1.10, it follows by fibre integrating the characteristic polynomial $p(e^{\mathrm{fw}}(\pi))$ that κ_{N-1} lies in the ring generated by $\kappa_1, \dots, \kappa_{N-2}$. The same does not hold for κ_N because the constant term of $p(e)$ contains a κ_N and fibre integrating $p(e^{\mathrm{fw}}(\pi))\cdot e^{\mathrm{fw}}(\pi)$ will not give a new relation. Fibre integrating $p(e^{\mathrm{fw}}(\pi))\cdot e^{\mathrm{fw}}(\pi)^k$ for $k \geq 2$ expresses κ_{N+k-1} in terms of lower degree κ_i and thus $\kappa_1, \dots, \kappa_{N-2}, \kappa_N$ generate the Euler ring.

In the general case, Randal-Williams shows that under the given assumption the same trace relation from the Cayley-Hamilton theorem still holds for fibre bundles but the argument applies more generally for Hurewicz fibration with finite CW-fibres [RW18, pg. 3843]. □

4.2.1. The Euler ring of complex projective space

We start by applying the results from Chapter 2 to get a model of the universal 1-connected fibration $\pi : E_0{\to} \mathrm{B\,hAut}_0(\mathbb{C}P^n)$ with fibre $\mathbb{C}P^n$. In the following, we denote the minimal model of $\mathbb{C}P^n$ by $P_n := (\Lambda(x,y), |x| = 2, |y| = 2n+1, d = x^{n+1}\partial/\partial y)$.

Proposition 4.2.4. *A cdga model of the universal 1-connected $\mathbb{C}P^n$-fibration is given by*

$$B_n := \Big(\mathbb{Q}[x_2, \dots, x_{n+1}], |x_i| = 2i, d = 0\Big) \xrightarrow{\pi} E_n := \Big(B_n[x]/(x^{n+1} - \textstyle\sum_{i=2}^{n+1} x_i \cdot x^{n+1-i}), |x| = 2, d = 0\Big). \tag{4.6}$$

In particular, the universal fibration is formal.

Proof. Note that $\mathrm{Der}^+(P_n)$ has a (vector space) basis given by $\theta_i := x^{n+1-i}\partial/\partial y$ for $i = 1, \dots, n+1$ of degree $2i-1$ and $\eta := \partial/\partial x$ of degree 2. The only non-trivial differential on the derivation Lie algebra is given by $[d, \eta] = -(n+1)\theta_1$. Denote by $\mathfrak{a}_n \subset \mathrm{Der}^+(P_n)$ the vector space spanned by the cycles $\theta_2, \dots, \theta_{n+1}$ which generate the homology. This is an abelian dg Lie subalgebra with trivial differential and the inclusion $\mathfrak{a}_n \hookrightarrow \mathrm{Der}^+(P_n)$ induces a quasi-isomorphism of dg Lie algebras. Hence, the inclusion induces a quasi-isomorphism of cdga's

$$C^*_{CE}(\mathrm{Der}^+(P_n);\mathbb{Q}) \xrightarrow{\simeq} C^*_{CE}(\mathfrak{a}_n;\mathbb{Q}) \cong (\mathbb{Q}[x_2, \dots, x_{n+1}], d = 0),$$

where $x_i = (s\theta_i)^*$ and has degree $2i$. Combining Theorem 2.3.9 with the above quasi-isomorphism shows that

$$C^*_{CE}(\mathfrak{a}_n; \mathbb{Q}) \xrightarrow{\eta_*} C^*_{CE}(\mathfrak{a}_n; P_n) \cong (C^*_{CE}(\mathfrak{a}_n; \mathbb{Q}) \otimes P_n, D)$$

is a relative Sullivan model of the universal 1-conneted fibration, where $\eta : \mathbb{Q} \to P_n$ is the unit. Here P_n is a left $\mathfrak{a}_n$-module by restricting the $\mathrm{Der}^+(P_n)$-action. With respect to the chosen basis of $\mathfrak{a}_n$ and the basis $\{x^k, yx^l\}_{k,l \geq 0}$, the formulas in (2.16) are

$$\begin{aligned}
D(x) &= d(x) - \sum_{i,k} (x^k)^*(\theta_i(x)) \cdot x_i x^k + (yx^k)^*(\theta_i(x))) \cdot x_i y x^k = 0, \\
D(y) &= d(y) - \sum_{i,k} (x^k)^*((\theta_i(y))) \cdot x_i x^k + (yx^k)^*(\theta_i(y)) \cdot x_i y x^k \\
&= x^{n+1} - \sum_{i=2}^{n+1} x_i \cdot x^{n+1-i}.
\end{aligned}$$

Consider the map of B_n-algebras $C : C^*_{CE}(\mathfrak{a}_n; P_n) \to E_n$ determined by $C(y) = 0$ and $C(x) = x$. The cohomology of $C^*_{CE}(\mathfrak{a}_n; P_n)$ can be computed via the spectral sequence of a relative Sullivan algebra corresponding to the Serre spectral sequence. The spectral sequence degenerates on the E_2-page and thus the cohomology of $C^*_{CE}(\mathfrak{a}_n; P_n)$ is a free module over $H^*_{CE}(\mathfrak{a}_n; \mathbb{Q}) \cong B_n$. Moreover, C sends the B_n-basis $\{1, [x], \dots, [x^n]\}$ of $H^*_{CE}(\mathfrak{a}_n; P_n)$ to a B_n-basis of E_n. Hence, C is a quasi-isomorphism of B_n-algebras and in particular of cgda's. □

Remark 4.2.5. The Halperin conjecture implies that one can always find an abelian Lie subalgebra $\mathfrak{a}$ of the derivations of the minimal model of a positively elliptic space as above that is quasi-isomorphic to it. This has been explained to the author by Alexander Berglund and it gives a shorter proof of the above statement than our original one. In particular, it implies that the spaces in the universal 1-connected fibration are rationally equivalent to $\prod_{i=1}^N K(\mathbb{Q}, 2n_i)$ so that the universal fibration is determined by a map of polynomial rings. This was already observed in [Kur10, Thm 1.1] with different methods. The paper [Kur10, Prop. 1.3] also states the algebraic model for complex projective spaces, which we learned after the completion of this work.

We can apply Theorem 4.1.10 to the complete intersection in Proposition 4.2.4, where we use integral Poincaré duality to fix an orientation on $\mathbb{C}P^n$ by $\varepsilon_{\mathbb{C}P^n}(x^n) = 1$.

Corollary 4.2.6. *The fibrewise Euler class in E_n is represented by*

$$e^{\mathrm{fw}}(\pi) = (n+1) \cdot x^n - \sum_{i=2}^{n} (n+1-i) \cdot x_i \cdot x^{n-i} \in E_n. \tag{4.7}$$

Finally, we are able to determine the Euler ring of complex projective spaces.

Theorem 4.2.7. *The Euler ring of complex projective space is* $E^*(\mathbb{C}P^n) \cong \mathbb{Q}[\kappa_1, \ldots, \kappa_{n-1}, \kappa_{n+1}]$.

Proof. We can compute the Euler ring of the universal 1-connected fibration $E_0^*(\mathbb{C}P^n)$ using the algebraic models and the representative of the fibrewise Euler class and fibre integration. In order to extend this computation to the full Euler ring in $\mathrm{B\,hAut}^+(\mathbb{C}P^n)$, we need to study the spectral sequence of the fibre sequence $\mathrm{B\,hAut}_0(X) \to \mathrm{B\,hAut}^+(X) \to \mathrm{B}\,\pi_0(\mathrm{hAut}^+(X))$ that describes the universal covering of $\mathrm{B\,hAut}^+(X)$. The inclusion $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$ induces a bijection $[\mathbb{C}P^n, \mathbb{C}P^n] \to [\mathbb{C}P^n, \mathbb{C}P^\infty] \cong H^2(\mathbb{C}P^n; \mathbb{Z})$ and the homotopy equivalences correspond to generators of $H^2(\mathbb{C}P^n; \mathbb{Z})$. Thus $\pi_0(\mathrm{hAut}^+(\mathbb{C}P^n))$ is finite and it follows by transfer that $H^*(\mathrm{B\,hAut}^+(\mathbb{C}P^n); \mathbb{Q})$ injects into $H^*(\mathrm{B\,hAut}_0(X); \mathbb{Q})$. In particular, the pullback induces an isomorphism $E^*(\mathbb{C}P^n) \xrightarrow{\cong} E_0^*(\mathbb{C}P^n)$.

The Euler ring is generated by $\kappa_1, \ldots, \kappa_{n-1}, \kappa_{n+1}$ by Proposition 4.2.3, and it remains to identify relations between the polynomials $\kappa_i = \kappa_i(x_2, \ldots, x_{n+1}) \in B_n$ which we get by fibre integrating the representative of the fibrewise Euler class in (4.7). We will show that they are in fact algebraically independent by showing that $\det(\partial\kappa_i/\partial x_j)$ is non-zero. It turns out that the polynomials representing the κ_i are quite complicated so that it is difficult to give a closed formula for the determinant of the Jacobian. We will resolve this issue by focussing on the terms containing x_{n+1} because it is the variable of the highest degree and it is not contained in $e^{\mathrm{fw}}(\pi)$ so that it only arises through fibre integrating x^k for $k > n$. It will be sufficient to consider elements modulo decomposables, i.e. for $x, y \in B_n$ then $x \sim y$ if $x - y \in (B_n^+)^2$. We will start with the following observation about fibre integration.

1. If $k = 2, \ldots, n+1$ then $\pi_!(x^{n+k}) \sim x_k \in B_n$.

Proof. Rewriting x^{n+2} in terms of the module basis $\{1, x, \ldots, x^n\}$, one can see that $\pi_!(x^{n+2}) = x_2$. Then $\pi_!(x^{n+k}) = \pi_!(x^{n+1} \cdot x^{k-1}) = \sum_{i=2}^{n+1} x_i \cdot \pi_!(x^{n+k-i})$ and by induction over k, the only indecomposable contribution is for $i = k$. □

2. For $i = 1, \ldots, n-1$ the highest power of x_{n+1} in $\kappa_i(x_2, \ldots, x_{n+1})$ is $i-1$ and the coefficient $c_i \in \mathbb{Q}[x_2, \ldots, x_n]$ of x_{n+1}^{i-1} satisfies $c_i \sim i(n+1)^i(n-i) \cdot x_{n+1-i}$.

Proof. It follows from degree considerations that the highest power of x_{n+1} is $i-1$ and $c_i = A \cdot x_{n+1-i}$ + decomposables. It remains to determine the coefficient A. When expanding $e^{\mathrm{fw}}(\pi)^{i+1}$ using (4.7), the only relevant contributions are

$$\begin{aligned}
&(n+1)^{i+1} x^{n(i+1)} - (i+1)(n+1-(n+1-i)) x_{n+1-i} x^{n-(n+1-i)} \cdot (n+1)^i x^{ni} \\
=&(n+1)^{i+1} (x^{n+1})^{i-1} \cdot x^{2n-i+1} - i(i+1)(n+1)^i x_{n+1-i} \cdot (x^{n+1})^{i-1} \cdot x^n
\end{aligned}$$

Now rewrite $x^{n+1} = \sum_{i=2}^{n+1} x_i x^{n+1-i}$ and collect all terms containing x_{n+1}^{i-1} and $x_{n+1}^{i-2} x_{n+1-i}$ (we can ignore the rest because it cannot contribute to A) to get

$$(n+1)^{i+1}(x_{n+1}^{i-1} + (i-1)x_{n+1}^{i-2}x_{n+1-i} \cdot x^i) \cdot x^{2n-i+1} - i(i+1)(n+1)^i x_{n+1-i} \cdot x_{n+1}^{i-1} \cdot x^n$$

The statement follows by fibre integrating and discarding decomposables as in **1** above. □

3. The highest contribution of x_{n+1} in κ_{n+1} is $(n+1)^{n+2} \cdot x_{n+1}^n$.

Proof. The expression for $e^{\mathrm{fw}}(\pi)^{n+2}$ contains the summand $(n+1)^{n+2} \cdot x^{n(n+2)} = (n+1)^{n+2} \cdot (x^{n+1})^n \cdot x^n$. This is the only summand that fibre integrates to a multiple of x_{n+1}^n, i.e. $\kappa_{n+1} = (n+1)^{n+2} \cdot x_{n+1}^n + \ldots$ where we can ignore all other terms. □

We can now analyze $\det(\partial\kappa_i/\partial x_j)$ which contains the summand

$$\frac{\partial\kappa_1}{\partial x_n} \cdot \frac{\partial\kappa_2}{\partial x_{n-1}} \cdot \ldots \cdot \frac{\partial\kappa_{n-1}}{\partial x_2} \cdot \frac{\partial\kappa_{n+1}}{\partial x_{n+1}}.$$

It follows from **2** and **3** that the above expression contains $C \cdot x_{n+1}^N$, where C is a non-zero constant and $N = \frac{1}{2}n(n-1)$. This is the only possible way to get a monomial in $\det\left(\frac{\partial\kappa_i}{\partial x_j}\right)$ that contains only x_{n+1}. Hence, the determinant does not vanish and the generating set $\kappa_1, \ldots, \kappa_{n-1}, \kappa_{n+1}$ is algebraically independent. □

Remark 4.2.8. Theorem 4.2.7 has been studied in the smooth case for $n = 2$ in [RW18] by studying the natural smooth 2-torus action on $\mathbb{C}P^2$ and that implies our result in this case as well. This has been extended by Dexter Chua to $n \leq 4$, but for large n the algebra becomes intractable.

Remark 4.2.9. Consider $H^*(\mathrm{B\,hAut}_0(\mathbb{C}P^n);\mathbb{Q})$ as a module over $E^*(\mathbb{C}\mathrm{P}^n)$. Then by [Smi95, Cor 6.7.11] it is finitely generated as a $E^*(\mathbb{C}\mathrm{P}^n)$-module if and only if $\kappa_1, \ldots, \kappa_{n-1}, \kappa_{n+1} \in \mathbb{Q}[x_2, \ldots, x_{n+1}] \cong H^*(\mathrm{B\,hAut}_0(\mathbb{C}P^n);\mathbb{Q})$ is a regular sequence. This can be checked by computing the radical of $(\kappa_1, \ldots, \kappa_{n-1}, \kappa_{n+1})$, as for a parameter ideal the radical is the unique graded maximal ideal $H^+(\mathrm{B\,hAut}_0(\mathbb{C}P^n);\mathbb{Q})$. With this criterion one can check that $H^*(\mathrm{B\,hAut}_0(\mathbb{C}P^n);\mathbb{Q})$ is finite over the Euler ring for $n = 1, 2$ (for $n = 1$ this is obvious) but not for $n = 3, 4$. We don't know how to check this condition for $n \geq 5$ in general but we expect that the cohomology of $\mathrm{B\,hAut}_0(\mathbb{C}P^n)$ is not finite over the Euler ring in this case. This shows that one cannot hope to find a version of Theorem 3.1 in [RW18] with Euler rings instead of tautological rings in order to find a lower bound on the Krull dimensions via Torus actions.

We end this section with an observation about the natural map induced by the action of the projective unitary group $\mathrm{PU}(n+1) \to \mathrm{hAut}_0(\mathbb{C}P^n)$ and the corresponding map between classifying spaces. It is a classical result that the cohomology ring of $\mathrm{B\,PU}(n+1)$ is $\mathbb{Q}[q_2, \ldots, q_{n+1}]$ with generators of degree $|q_i| = 2i$ which are related to Chern classes. Thus $\mathrm{B\,PU}(n+1)$ has the same rational cohomology ring as $\mathrm{B\,hAut}_0(\mathbb{C}P^n)$, and in particular both are rationally equivalent to $\prod_{i=2}^{n+1} K(\mathbb{Q}, 2i)$ by its intrinsic rational formality. Hence, $\mathrm{B\,PU}(n+1) \simeq_{\mathbb{Q}} \mathrm{B\,hAut}_0(\mathbb{C}P^n)$ and it can further be shown that the natural map is a rational equivalence.

Theorem ([Sas74, Kur11]). *The natural map* $\mathrm{B\,PU}(n+1) \longrightarrow \mathrm{B\,hAut}_0(\mathbb{C}P^n)$ *is a rational homotopy equivalence.*

Sasao proves this statement using classical homotopy theory to study the fibration sequence $\mathrm{Map}_{\iota_m}(\mathbb{C}P^m, \mathbb{C}P^n) \to \mathrm{Map}_{\iota_{m-1}}(\mathbb{C}P^{m-1}, \mathbb{C}P^n)$ for $m < n$, where the subscript denote the connected component of the inclusion $\iota_m : \mathbb{C}P^m \hookrightarrow \mathbb{C}P^n$. In particular, he doesn't just obtain results about rational homotopy groups but also statements about the connectivity of the map. For example, he proves that the inclusion $\mathrm{PU}(n+1) \to \mathrm{Map}_{\mathrm{id}}(\mathbb{C}P^n, \mathbb{C}P^n)$ induces an isomorphism on fundamental groups and $\pi_1(\mathrm{PU}(n+1)) \cong \mathbb{Z}/(n+1)\mathbb{Z}$. In contrast, Kuribayashi uses tools from rational homotopy theory as well. His result is based on a rational model of the evaluation map $\mathrm{hAut}_0(X) \times X \to X$.

We want sketch a simple proof of this result that uses the algebraic models of the $\mathbb{C}\mathrm{P}^n$-fibrations over $\mathrm{B\,hAut}_0(\mathbb{C}P^n)$ and $\mathrm{B\,PU}(n+1)$. It will be easier to prove that the natural map $\mathrm{B\,SU}(n+1) \to \mathrm{B\,hAut}_0(\mathbb{C}P^n)$ is a rational equivalence. This implies the theorem as the natural map $\mathrm{B\,SU}(n+1) \to \mathrm{B\,PU}(n+1)$ is a rational equivalence because $\mathrm{SU}(n+1) \to \mathrm{PU}(n+1)$ is an $(n+1)$-fold normal covering.

Proof. The statement is a direct consequence of the projective bundle formula which describes the algebraic structure of the projectivization of a complex vector bundle in terms of the Chern classes. Let $i : \mathrm{B\,SU}(n+1) \to \mathrm{B\,U}(n+1)$ be the natural map induced by inclusion and $\gamma_{n+1} \to \mathrm{B\,U}(n+1)$ the tautological vector bundle. Then there is an algebra isomorphism

$$H^*(\mathbb{P}(i^*\gamma_{n+1}); \mathbb{Q}) \cong H^*(\mathrm{B\,SU}(n+1); \mathbb{Q})[t] \Big/ \Big(t^{n+1} + \sum_{k=2}^{n+1} (-1)^k i^* c_k \cdot t^{n+1-k} \Big)$$

where t is the Euler class of the canonical line bundle $L \to \mathbb{P}(i^*\gamma_{n+1})$ and i^*c_k is the restriction of the k-th Chern class. The projectivization $\mathbb{P}(i^*\gamma_{n+1})$ is the pullback of the universal $\mathbb{C}P^n$-fibration along the natural map $c : \mathrm{B\,SU}(n+1) \to \mathrm{B\,hAut}_0(\mathbb{C}P^n)$. As both E_n and $H^2(\mathbb{P}(i^*\gamma_{n+1}); \mathbb{Q})$ are 1-dimensional in degree 2, the pullback of $x \in E_n^2$ agrees with t up to a scalar, which is 1 for our choice of orientation. It follows that $c^*(x_k) = -(-1)^k i^* c_k$ and hence c^* induces an isomorphism of complete intersections. □

The above theorem has the following non-obvious consequence.

Corollary 4.2.10. *The natural map* $\mathrm{B}\,\mathrm{Diff}^+(\mathbb{C}P^n) \longrightarrow \mathrm{B}\,\mathrm{hAut}^+(\mathbb{C}P^n)$ *induces an injection on rational cohomology.*

Proof. Consider the composition $\mathrm{B}\,\mathrm{PU}(n+1) \to B\,\mathrm{Diff}^+(\mathbb{C}P^n) \to \mathrm{B}\,\mathrm{hAut}^+(\mathbb{C}P^n)$ which factors through the respective connected components of the identity. As $\pi_0(\mathrm{hAut}^+(\mathbb{C}P^n))$ is finite, the cohomology of $\mathrm{B}\,\mathrm{hAut}^+(\mathbb{C}P^n)$ injects into the cohomology of $\mathrm{B}\,\mathrm{hAut}_0(\mathbb{C}P^n)$ as the ring of $\pi_0(\mathrm{hAut}^+(\mathbb{C}P^n))$-invariants. If follows from the previous theorem that $H^*(\mathrm{B}\,\mathrm{hAut}^+(\mathbb{C}P^n);\mathbb{Q})$ injects into $H^*(\mathrm{B}\,\mathrm{PU}(n+1);\mathbb{Q})$ and thus has to inject into $H^*(\mathrm{B}\,\mathrm{Diff}^+(\mathbb{C}P^n);\mathbb{Q})$. □

4.2.2. The Euler ring of products of odd spheres

In the following, let X be simply connected and rationally equivalent to a product of (simply connected) odd dimensional spheres $\prod_{i=1}^{2N} S^{2n_i+1}$. Denote by F the set of all factors, then X has a minimal Sullivan model $A_F = (\Lambda(x_f)_{f\in F}, |x_f| = 2n_f+1, d=0)$. We choose a total ordering $<$ of F so that $f < f'$ implies $|f| \leq |f'|$ where $|f| = 2n_f + 1$ is the dimension of the corresponding sphere. This determines a basis $\{x_S := \prod_{f\in S} x_f\}_{S\subset F}$ where the product $\prod_{f\in S} x_f$ is with respect to the induced total order on S. We define $x_\emptyset := 1$. The dg Lie algebra of derivations of A_F is graded and has the following basis

$$\mathrm{Der}^+(A_F) = \bigoplus_{n\geq -1} \mathbb{Q}\Big\{\eta_S^f \,\Big|\, S \subset (F,<),\ \#S = n+1 \text{ and } |S| < |f|\Big\}$$

where $|S| = \sum_{f\in S} |f|$ and $\eta_S^f = x_S \cdot \frac{\partial}{\partial x_f}$. The bracket is given by

$$[\eta_S^f, \eta_{S'}^{f'}] = \begin{cases} \mathrm{sgn}(S,f,S',f')\eta^{f'}_{S\sqcup(S'\setminus f)} & \text{if } f \in S' \\ -(-1)^{|\eta_S^f|\cdot|\eta_{S'}^{f'}|}\cdot \mathrm{sgn}(S',f',S,f)\eta^{f}_{S'\sqcup(S\setminus f')} & \text{if } f' \in S \\ 0 & \text{otherwise} \end{cases}$$

where the exact form of the signs can easily be worked out but will not matter for the Euler ring, and hence we omit a discussion here. We denote the corresponding dual basis of $(s\,\mathrm{Der}^+(A_F))^\vee$ by y_S^f. In the following, denote by $B_F := C^*_{CE}(\mathrm{Der}^+(A_F);\mathbb{Q})$ the cdga model of the base with differential d and by $E_F := (B_F \otimes A_F, D)$ the cdga model of the total space. Using (2.16), the we find that the relative Sullivan model of the universal fibration is determined by

$$d(y_S^f) = (-1)^{|f|-|S|} \sum_{S_1 \sqcup S_2 = S,\, g \in S_{<f} \setminus S} \operatorname{sgn}(S_1, g, S_2 \cup g, f) \cdot y^f_{S_2 \cup g} y^g_{S_1}$$
$$D(x_f) = - \sum_{S \subset S_{<f} \text{ with } |S|<|f|} y_S^f \wedge x_S.$$

Recall that a chain level representative of fibre integration $\Pi \in \mathrm{Hom}_{B_F}(E_F, B_F)$ is uniquely determined up to homotopy by (3.3). We choose as orientation of X the homomorphism determined by $\varepsilon_X(x_F) = 1$. Since Π lowers degree by $|F|$, it can only be nontrivial on x_F so that setting

$$\Pi(1 \otimes x_S) := \begin{cases} 1 & \text{if } S = F \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

determines the unique element in the 1-dimensional vector space $\mathrm{Hom}^{-d}_{B_F}(E_F, B_F)$ that satisfies (3.3). Hence, by Proposition 3.1.2 is has to be a representative of fibre integration and in particular is a cycle, which can also be easily checked. We construct a representative of $\Delta_! : E_F[-|F|] \to E_F \otimes_{B_F} E_F[-2|F|]$ from section 4 to compute the fibrewise Euler class.

But first, note that the integral fibrewise Euler class of a Poincaré complex of odd formal dimension is a 2-torsion element. From the point of view of rational homotopy theory, there is no structural difference in the algebraic model of the universal fibration of X if $\#F$ is odd or even, so that one can expect the following.

Proposition 4.2.11. *Let X be rationally equivalent to a product of odd dimensional spheres. Then the fibrewise Euler class $e^{fw}(\pi) \in H^{|X|}(E_0; \mathbb{Q})$ vanishes. In particular, $E_0^*(X) = \mathbb{Q}$.*

Proof. Recall that the description of $\Delta_! : E_F[-|F|] \to E_F \otimes_{B_F} E_F[-2|F|]$ uses the quasi-isomorphism $\bar{\Pi} : E_F \to \mathrm{Hom}_{B_F}(E_F, B_F)$. Because A_F is finite dimensional, $\mathrm{Hom}_{B_F}(E_F, B_F)$ is a semifree module itself and it is minimal because A_F is a minimal Sullivan model. Hence, $\bar{\Pi}$ and $\overline{\Pi \otimes \Pi}$ are isomorphisms of B_F-modules by uniqueness of minimal free resolutions [FHT01, Ex.8,Ch.6] so that

$$\Delta_! : E_F[-|F|] \xrightarrow[\cong]{\bar{\Pi}} \mathrm{Hom}_{B_F}(E_F, B_F) \xrightarrow{\Delta^*} \mathrm{Hom}_{B_F}(E_F \otimes_{B_F} E_F, B_F) \xrightarrow[\cong]{(\overline{\Pi\otimes\Pi})^{-1}} E_F \otimes_{B_F} E_F[-2|F|].$$

The composition of $\bar{\Pi}$ with the vector space isomorphism $\mathrm{Hom}_{B_F}(E_F, B_F) \cong (A_F)^\vee \otimes B_F$ is given by $\bar{\varepsilon}_X \otimes \mathrm{Id}_{B_F}$ where $\bar{\varepsilon}_X : A_F \to (A_F)^\vee$ is the adjoint of $\varepsilon_X : A_F \otimes A_F \to \mathbb{Q}$. The same statement holds for $\overline{\Pi \otimes \Pi}$ with the appropriate choice of orientation on $X \times X$ given by $\varepsilon_{X \times X} := \varepsilon_X \otimes \varepsilon_X : (A_F \otimes A_F)^{\otimes 2} \to \mathbb{Q}$. This is very special to the products of odd dimensional spheres and holds because the Sullivan model is finite dimensional, which is only ever true for this situation.

Note that $\Delta^*\bar{\Pi}(1)$ is contained in $(A_F \otimes A_F)^\vee \otimes 1$ so that $(\overline{\Pi \otimes \Pi})^{-1}\Delta^*\bar{\Pi}(1)$ is in $A_F \otimes A_F \otimes 1 \subset E_F \otimes_{B_F} E_F$. A direct computation shows that

$$\Delta_!(1) = \sum_{S_1 \sqcup S_2 = F} \pm x_{S_1} \otimes x_{S_2} \in E_F \otimes_{B_F} E_F$$

for some signs that can be worked out. Hence, the fibrewise Euler class is $e^{\mathrm{fw}}(\pi) = \Delta^* \circ \Delta_!(1) = \sum_{S_1 \sqcup S_2 = F} \pm x_F$ and since $\Pi(e^{fw}(\pi)) = \chi(X) = 0$, the summands must cancel. $\square$

A fact that we have not mentioned so far is that the total space of the universal X-fibration is equivalent to the classifying space of pointed homotopy equivalences $\mathrm{B}\,\mathrm{hAut}_*^+(X)$. In order to understand the fibrewise Euler class in $H^d(\mathrm{B}\,\mathrm{hAut}_*^+(X)); \mathbb{Q})$, we need to study the universal covering spectral sequence of $\mathrm{B}\,\mathrm{hAut}_{0,*}(X) \to \mathrm{B}\,\mathrm{hAut}_*^+(X) \to \mathrm{B}\,\pi_0(\mathrm{hAut}_*^+(X))$. We do not yet know how to do this in general, but in some cases we can deduce that $E^*(X) = \mathbb{Q}$.

Theorem 4.2.12. *Let X be either rationally equivalent to $(S^{2k+1})^{\times n}$ or a finite CW complex rationally equivalent to $S^{2k+1} \times S^{2l+1}$ for $1 < k < l$ and n even. Then $E^*(X) = \mathbb{Q}$.*

Proof. In the second case, we will show that $\pi_0(\mathrm{hAut}^+(X))$ is finite so that the universal covering spectral sequence collapses and the cohomology of $\mathrm{B}\,\mathrm{hAut}_*^+(X)$ injects as the invariants of $H^*(\mathrm{B}\,\mathrm{hAut}_{*,0}(X); \mathbb{Q})$ with respect to the action of $\pi_0(\mathrm{hAut}_0^+(X))$. In particular, the fibrewise Euler class is trivial in $H^{2(k+l+1)}(\mathrm{B}\,\mathrm{hAut}_*^+(X); \mathbb{Q})$ and therefore the Euler ring is $\mathbb{Q}$.

By [Sul77, Thm.10.3] the group $\pi_0(\mathrm{hAut}^+(X))$ is commensurable with an arithmetic subgroup of the homotopy classes automorphisms of A_F. If $X \simeq_{\mathbb{Q}} S^{2k+1} \times S^{2l+1}$ then the group of automorphisms of A_F modulo homotopy is $\mathbb{Q}^\times \times \mathbb{Q}^\times$ and the arithmetic subgroups of this linear algebraic group are finite and hence by commensurability so is $\pi_0(\mathrm{hAut}^+(X))$.

In the first case, the cdga model E_F of $\mathrm{B}\,\mathrm{hAut}_{0,*}(X)$ is a free algebra on $x_1, \ldots, x_n, y^1, \ldots, y^n$ with differential $D(x_i) = -y^i$. Hence $H(E_F) = \mathbb{Q}$ and thus $\mathrm{B}\,\mathrm{hAut}_{0,*}(X)$ is rationally contractible. Therefore, $\mathrm{B}\,\mathrm{hAut}^+(X) \to \mathrm{B}\,\pi_0(\mathrm{hAut}^+(X))$ induces an isomorphism on rational cohomology and thus every class in $H^*(\mathrm{B}\,\mathrm{hAut}_*^+(X); \mathbb{Q})$ is pulled back from $\mathrm{B}\,\pi_0(\mathrm{hAut}_*^+(X))$. In particular, $e^{\mathrm{fw}}(\pi) = H^*e$ for some $e \in H^{n\cdot(2k+1)}(\mathrm{B}\,\pi_0(\mathrm{hAut}^+(X)); \mathbb{Q})$. By commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{B}\,\mathrm{hAut}_*^+(X) \arrow[r, "H", "\simeq_{\mathbb{Q}}"'] \arrow[d, "\pi"] & \mathrm{B}\,\pi_0(\mathrm{hAut}_*^+(X)) \arrow[d, equal] \\
\mathrm{B}\,\mathrm{hAut}^+(X) \arrow[r, "h"] & \mathrm{B}\,\pi_0(\mathrm{hAut}^+(X)),
\end{tikzcd}$$

the fibrewise Euler class $e^{\mathrm{fw}}(\pi) = H^*e = \pi^*h^*e$ is pulled back from the base. Hence, fibre integrating powers of the fibrewise Euler class gives $\pi_!(\pi^*(h^*e)^k) = (h^*e)^k \cdot \pi_!(1) = 0$. $\square$

We expect that one can extend Proposition 4.2.11 to show that $e^{\mathrm{fw}}(\pi) \in H^d(\mathrm{B}\,\mathrm{hAut}_*^+(X); \mathbb{Q})$ always vanishes by studying the universal fibration of $X_{\mathbb{Q}}$ instead. As rationalization is functorial there is a (continuous) map $\mathrm{B}\,\mathrm{hAut}_*^+(X) \to \mathrm{B}\,\mathrm{hAut}_*^+(X_{\mathbb{Q}})$ and the Euler class in the cohomology of $\mathrm{B}\,\mathrm{hAut}_*^+(X_{\mathbb{Q}})$ pulls back to the Euler class in $\mathrm{B}\,\mathrm{hAut}_*^+(X)$.

4.2.3. The Euler ring of some low dimensional positively elliptic spaces

We have already alluded to the rigidity of positively elliptic spaces in Proposition 4.1.7. Another manifestation of this behaviour has been established by Friedlander and Halperin in [FH79], where they prove that for an elliptic pure minimal Sullivan algebra $(\Lambda V, d)$ with homogeneous basis $\{x_i, y_j\}$ of V and degrees $|x_i| = 2a_i$ and $|y_j| = 2b_j - 1$, the sequences of degrees (a_i) and (b_i) – called the exponents – satisfy strong arithmetic conditions.

Theorem 4.2.13 ([FHT01, Thm 32.6]). *Let $(\Lambda V, d)$ be a pure minimal Sullivan model with exponents defined as above and denote by d formal dimension of the Poincaré duality algebra $H(\Lambda V, d)$. Then*

- $\sum_j (2b_j - 1) - \sum_i (2a_i - 1) = d$;
- $\sum_i 2a_i \leq d$;
- $\sum_j (2b_j - 1) \leq 2d - 1$;
- $\dim V \leq d$;
- *V is concentrated in degrees $\leq 2d - 1$ and at most one (odd) basis element can have degree $> d$.*

This can be used to classify rationally elliptic spaces in low dimensions. For example, Herrmann has classified the real homotopy type of elliptic spaces of dimension ≤ 6.

Theorem 4.2.14 ([Her18]). *A closed, simply connected, and rationally elliptic manifold of dimension six or less is*

- *diffeomorphic to S^2 or S^3,*
- *homeomorphic to S^4, $S^2 \times S^2$, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$,*
- *rationally homotopy equivalent to S^5 or $S^2 \times S^3$,*
- *is 6-dimensional with $b_2(M) \leq 2$ and has the real homotopy type of one of the following manifolds*
$$S^6,\ S^3 \times S^3,\ \mathbb{C}P^3, S^2 \times S^4,\ \mathbb{C}P^2 \times S^2,\ \mathrm{SU}(3)/T^2,\ \text{or}\ \mathbb{C}\mathrm{P}^3 \# \mathbb{C}P^3$$

- *is 6-dimensional with $b_2(M) = 3$ and cohomology ring $H^*(M;\mathbb{R})$ isomorphic to a quotient of $\mathbb{R}[x_1, x_2, x_3]$ by one of the following regular sequences*

$$\{x_1^2,\ x_2^2,\ x_3^2\}, \{x_1x_2,\ x_1^2 - x_2^2,\ x_3^2\},\ \{x_1x_2,\ x_1^2 + x_3^2,\ x_2^2 + x_3^2\},\ \{x_2x_3,\ x_1^2 - x_2^2,\ x_1^2 + x_3^2\}$$
$$\{x_2x_3,\ x_1^2 - x_2^2,\ x_1^2 - x_3^2\},\ \{x_1x_2,\ x_3^2,\ x_1^2 - x_1x_3 + x_2^2\},\ \{x_1x_2,\ x_3^2,\ x_1^2 - x_1x_3 - x_2^2\}$$
$$\text{and } \{\sigma x_1^2 - x_2x_3,\ \sigma x_2^2 - x_1x_3,\ \sigma x_3^2 - x_1x_2\} \text{ for } \sigma \neq 0, 1, \frac{1}{2}$$

Remark 4.2.15.

(i) A real homotopy equivalence of a simply connected space induces an equivalence on real homotopy groups $\pi_*(-) \otimes \mathbb{R}$. The reason that the classification theorem above is stated in terms of the real homotopy type is because it relies on the classification of cubic forms over $\mathbb{R}$ which is not available over $\mathbb{Q}$. Of course any regular sequence above with rational coefficients determines a rational homotopy type.

(ii) It is a theorem by Sullivan [Sul77, Thm 13.2] that for any simply connected minimal model $(\Lambda V, d)$ that satisfies Poincaré duality on cohomology there is a simply-connected compact manifold realizing this rational homotopy type if the dimension $4 \nmid d$ and otherwise there are constraints on the intersection form and Pontrjagin classes.

For some of the above examples there are evident manifolds realizing the rational homotopy type. For example the first two regular sequences correspond to $(S^2)^3$ and $S^2 \times (\mathbb{C}P^2 \# \mathbb{C}P^2)$. In general, a construction of a manifold associated to one of the (rational) regular sequence above has been given in [Wal66] via handel decompositions.

The Euler ring $E_0^*(X)$ only depends on the rational homotopy type and we can compute it for all elliptic spaces of dimension ≤ 6 using this classification (although there are some computational limitations).

Theorem 4.2.16. *Let X be a simply connected positively elliptic space with Euler characteristic $\chi(X) = n$. Then*

I *if* $\dim X \leq 4$ *then* $E_0^*(X) \cong \mathbb{Q}[\kappa_1, \ldots, \kappa_{n-2}, \kappa_n]$;

II *if* $\dim X = 6$ *and* $b_2(X) \leq 2$ *then*

a.) if $X \simeq_{\mathbb{Q}} S^2 \times S^4$ *then* $E_0^*(X) \cong \mathbb{Q}[\kappa_2, \kappa_4]$;

b.) otherwise $E_0^*(X) \cong \mathbb{Q}[\kappa_1, \ldots, \kappa_{n-2}, \kappa_n]$.

Proof. The computations will proceed by the same strategy as in Proposition 4.2.4, i.e. by finding a quasi-isomorphic abelian Lie subalgebra with trivial differential $\mathfrak{a} \subset \mathrm{Der}^+(\Lambda)$ of a minimal Sullivan model of X (see Remark 4.2.5). We denote derivations of a free graded commutative algebra ΛV for a graded vector space V with homogeneous with basis $\{x_i\}$ by $p\partial x_i$ for some $p \in \Lambda V$. Also recall that the Euler ring is generated by $\kappa_1, \dots, \kappa_{n-2}, \kappa_n$ by Proposition 4.2.3. Hence, it will suffice to check for relations among these classes.

If $\dim X = 4$ it only remains to check the cases $X = S^2 \times S^2$ and $X = \mathbb{C}P^2 \# \mathbb{C}P^2$ as the computation for spheres and complex projective cases has been settled before and since $S^2 \times S^2 \simeq_{\mathbb{Q}} \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$.

- A minimal Sullivan model for $S^2 \times S^2$ is given by $\Lambda := (\Lambda(x_1, x_2, y_1, y_2), d = \sum x_i^2 \partial y_i)$ with $|x_i| = 2$ and $|y_i| = 3$. Then $\mathrm{Der}^+(\Lambda)$ is the 8-dimensional vector space $\mathbb{Q}\{x_i \partial y_j, \partial x_i, \partial y_j\}$ with the only non-trivial differential given by $[d, \partial x_i] = -2x_i \partial y_i$. Hence, the subspace $\mathfrak{a} := \mathbb{Q}\{x_1 \partial y_2, x_2 \partial y_1, \partial y_1, \partial y_2\}$ is a quasi-isomorphic abelian Lie subalgebra with trivial differential. Then the universal 1-connected fibration is equivalent to $C^*_{CE}(\mathfrak{a}; \mathbb{Q}) \to C^*_{CE}(\mathfrak{a}; \Lambda)$ and using (2.16) we find that this is equivalent to the complete intersection

$$\pi^* : B = \mathbb{Q}[a_1, a_2, b_1, b_2] \longrightarrow E = B[x_1, x_2]/(x_1^2 - a_2 x_2 - b_1, x_2^2 - a_1 x_1 - b_2), \tag{4.9}$$

  where $\{a_1, a_2, b_1, b_2\}$ corresponds to the dual basis of $\mathfrak{a}$ in the same order. The fibrewise Euler class is determined by Theorem 4.1.10 and one can compute that

$$\begin{aligned} \kappa_1 &= 8a_1 a_2 \qquad\qquad \kappa_2 = 28a_1^2 a_2^2 + 64 b_1 b_2 \\ \kappa_4 &= 244 a_1^4 a_2^4 + 1024 a_1^2 b_1^3 + 4736 a_1^2 a_2^2 b_1 b_2 + 1024 b_1^2 b_2^2 + 1024 a_2^2 b_2^3 \end{aligned}$$

  using Macaulay2 (see Example B.1). These elements are algebraically independent and thus the Euler ring is isomorphic to $\mathbb{Q}[\kappa_1, \kappa_2, \kappa_4]$.

- A minimal Sullivan model for $\mathbb{C}P^2 \# \mathbb{C}P^2$ is given by

$$\Lambda := (\Lambda(x_1, x_2, y_1, y_2), d = x_1 x_2 \partial y_1 + (x_1^2 - x_2^2) \partial y_2)$$

  with $|x_i| = 2$ and $|y_i| = 3$. Then $\mathrm{Der}^+(\Lambda)$ is isomorphic as graded vector space to the one above and the only non-trivial differentials are given by

$$[d, \partial x_1] = -x_2 \partial y_1 - 2x_1 \partial y_2 \qquad [d, \partial x_2] = -x_1 \partial y_1 + 2x_2 \partial y_2.$$

  Then $\mathfrak{a} := \{x_1 \partial y_1, x_2 \partial y_1, \partial y_1, \partial y_2\}$ is a quasi-isomorphic abelian Lie subalgebra with trivial differentia and the universal 1-connected fibration is equivalent to the following complete intersection

$$\pi^* : B = \mathbb{Q}[a_1, a_2, b_1, b_2] \longrightarrow E = B[x_1, x_2]/(x_1^2 - x_2^2 - b_2, x_1x_2 - x_1a_1 - x_2a_2 - b_1) \quad (4.10)$$

where $\{a_1, a_2, b_1, b_2\}$ corresponds to a dual basis of $\mathfrak{a}$. Then the fibrewise Euler class is $e^{\mathrm{fw}} = 2(x_1^2 + x_2^2) - 2(x_1a_2 + x_2a_1)$ and we find that

$$\begin{aligned}
\kappa_1 &= 4a_1^2 + 4a_2^2 \\
\kappa_2 &= 8a_1^4 + 48a_1^2a_2^2 + 8a_2^4 + 96a_1a_2b_1 + 64b_1^2 + 8a_1^2b_2 - 8a_2^2b_2 + 16b_2^2
\end{aligned}$$

(see Example B.1) where we omitted κ_4 because it is quite long. But a simple check with Macaulay2 reveals that the polynomials are algebraically independent and thus the Euler ring is isomorphic to $\mathbb{Q}[\kappa_1, \kappa_2, \kappa_4]$.

This covers the first case for $\dim X \le 4$. The strategy remains unchanged in the other cases but the computations get more lengthy so that we have outsourced them to the Appendix in Example B.1. □

It is notable that in all cases except $X \simeq_{\mathbb{Q}} S^2 \times S^4$ there are no other relations in the Euler ring $E_0^*(X)$ other than the trace relations in Proposition 4.2.3. It would be interesting if one could find an algebraic criterion that distinguishes positively elliptic spaces with $\operatorname{Kdim} E_0^*(X) < \chi(X) - 1$.

Remark 4.2.17. In many of the cases above, the computations can be promoted to determine $E^*(X)$ as the group of (homotopy classes) of homotopy self-equivalences $\mathcal{E}(X)$ is known to be finite (see for example [Bau96, Pav99]). The author believes this to be true for all finite CW complexes that are positively elliptic.

Sullivan proved that $\mathcal{E}(X)$ is commensurable to an arithmetic subgroup of $\mathcal{E}(X_{\mathbb{Q}})$ [Sul77, Thm 10.3] and in particular that the map $\mathcal{E}(X) \to \mathcal{E}(X_{\mathbb{Q}})$ has finite kernel. Denote by $\mathcal{E}_0(X)$ the subgroup of those homotopy equivalences that induce an isomorphism on integral homology groups. If X is positively elliptic we can use the structure theory in Proposition 4.1.7 to compute $\mathcal{E}_0(X_{\mathbb{Q}})$ and we claim that it is trivial (but we won't include a proof here). It follows that $\mathcal{E}_0(X)$ is finite. If the automorphism group of the cohomology ring is finite, which is a simple check, the group of homotopy self-equivalences has to be finite as well.

Chapter 5.

Tautological rings of manifolds via rational homotopy theory

The goal of this chapter is provide a systematic way to study relations in the tautological ring using rational homotopy theory. This means that we have to relax the manifold structure of the fibre and instead study fibrations that have an oriented vector bundle on the total space which plays the role of the vertical tangent bundle. In the second section, we refine these techniques by imposing the relations from the family signature theorem which does not hold for such fibrations.

5.1. TM-fibrations and tangential homotopy equivalences

We first introduce the concept of M-fibrations with a vector bundle on the total space whose restriction to a fibre is equivalent to the tangent bundle of M. This vector bundle plays the role of the vertical tangent bundle and we can define MMM-classes as the fibre integrals of its characteristic classes.

Definition 5.1.1. Let M be a smooth oriented manifold. We define a *TM-fibration* as an oriented M-fibration $E \to B$ with fibre M and an oriented vector bundle $T_\pi E \to E$ over the total space whose restriction to a fibre $T_\pi E|_{\pi^{-1}(b)}$ is equivalent to TM for all $b \in B$, i.e. there is a homotopy equivalence $\pi^{-1}(b) \to M$ covered by a vector bundle isomorphism $T_\pi E|_{\pi^{-1}(b)} \to TM$.

Remark 5.1.2. The above definition is a special case of ξ-fibrations due to Berglund [Ber20b, Ber20a]. Given any fibre bundle $\xi \to X$ (in the sense of [Dol63]), one can define a ξ-fibration as an X-fibration $\pi : E \to B$ together with a bundle on the total space (with the same fibre and structure group as ξ) such that the restriction of each fibre of π is weakly equivalent to ξ as in the definition above.

We discuss in Appendix A a general classification theory of fibrations developed by May in [May75], possibly with additional structure, that applies to TM-fibrations. The appropriate category of fibres is introduced in Example A.2(iii), and the automorphisms of the distinguished fibre is the following topological monoid.

Definition 5.1.3. The monoid $\mathrm{hAut}(TM)$ of **tangential homotopy equivalences** is

$$\mathrm{hAut}(TM) := \left\{ \begin{array}{ccc} TM & \xrightarrow[\cong]{\bar{f}} & TM \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array} \;\middle|\; f \in \mathrm{hAut}(M),\ \bar{f} \text{ is linear on each fibre} \right\}$$

topologized as a subspace of $\mathrm{Map}(TM, TM)$. We denote by $\mathrm{hAut}_0(TM)$ the connected components where the underlying map f is homotopic to the identity. If M is oriented, we denote by $\mathrm{hAut}^+(TM)$ the orientation preserving tangential homotopy equivalences.

The classifying space $\mathrm{B}\,\mathrm{hAut}^+(TM)$, constructed by May as the two-sided bar construction $\mathrm{B}(*, \mathrm{hAut}^+(TM), *)$ (see Definition A.3), classifies TM-fibrations up to a suitable equivalence relation by Theorem A.10 (see also [Ber20a, Thm 2.3]). The universal TM-fibration can be described as

$$\mathrm{B}(*, \mathrm{hAut}^+(TM), M) \longrightarrow \mathrm{B}(*, \mathrm{hAut}^+(TM), *) = \mathrm{B}\,\mathrm{hAut}^+(TM), \tag{5.1}$$

where $\mathrm{hAut}^+(TM)$ acts on M by evaluating the underlying map f. There is a good description of the oriented vector bundle over the total space of the universal TM-fibration in (5.1) based on the following observation.

Proposition 5.1.4 ([Ber20a]). *Let $\mathrm{hAut}(M)_{TM}$ denote the connected components of $\mathrm{hAut}^+(M)$ that preserve the tangent bundle under pull back. Then $\mathrm{B}(\mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}, \mathrm{hAut}(M)_{TM}, *)$ and $\mathrm{B}\,\mathrm{hAut}^+(TM)$ are homotopy equivalent. The same is true for $\mathrm{B}\,\mathrm{hAut}_0(TM)$ and the two-sided bar construction with $\mathrm{hAut}_0(M)$ instead.*

The analogous statement also holds for the total space of the universal TM-fibration in (5.1), i.e. it is equivalent to $\mathrm{B}(\mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}, \mathrm{hAut}(M)_{TM}, M)$. The oriented vector bundle on the total space can be described via the map $\mathrm{ev} : \mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM} \times M \to \mathrm{B}\,\mathrm{SO}(d)$, which is equivariant with respect to the action of $\mathrm{hAut}(M)_{TM}$ given by precomposition on the mapping space and evaluation on M, and therefore induces a map

$$E := \mathrm{B}(\mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}, \mathrm{hAut}(M)_{TM}, M) \xrightarrow{\epsilon(\mathrm{ev})} \mathrm{B}\,\mathrm{SO}(d) \tag{5.2}$$

by (A.2). We denote the pull back of the universal vector bundle over $\mathrm{B}\,\mathrm{SO}(d)$ along $\epsilon(\mathrm{ev})$ by $T_\pi E \to E$, and we define κ_c for $c \in H^{|c|}(\mathrm{B}\,\mathrm{SO}(d))$ for the universal TM-fibration $\pi : E \to \mathrm{B}\,\mathrm{hAut}^+(TM)$ by

$$\kappa_c := \int_M c(T_\pi E) \in H^{|c|-d}(\mathrm{B}\,\mathrm{hAut}^+(TM)).$$

There is one important observation to make for even dimensional manifolds. Namely, there are two Euler classes to consider: the fibrewise Euler class of the oriented M-fibration

$E \to \mathrm{B\,hAut}^+(TM)$ and the Euler class of the oriented vector bundle $T_\pi E \to E$. Importantly, these two classes are not the same.

Proposition 5.1.5. *Let M be a smooth, closed, oriented manifold and let $\pi : E \to \mathrm{B\,hAut}^+(TM)$ denote the universal TM-fibration with $T_\pi E \to E$ as above. Then $e^{\mathrm{fw}}(\pi) \neq e(T_\pi E) \in H^d(E;\mathbb{Z})$.*

We will give a proof in the next section. But this statement is hardly surprising because the vector bundle is completely detached from the global topology of the underlying fibration. For example, there are many vector bundles over the trivial fibration $\pi_B : B \times M \to M$ whose restriction to $b \times M$ is isomorphic to TM and whose Euler class is not $e^{\mathrm{fw}}(\pi_B) = 1 \times e(M)$. This is in stark contrast to a fibre bundle $E \to B$, where the vertical tangent bundle provides a regular neighbourhood of the diagonal $\Delta : E \to E \times_B E$ which links the vector bundle and the topology of the bundle.

Basically for this reason, that the fibrewise Euler class is more intricately linked to the global topology of the underlying fibration, we prefer to use $e^{\mathrm{fw}}(\pi)$ instead of $e(T_\pi E)$ in defining a homotopical version of the tautological ring. The advantages of this choice will become more clear later on in computations. With this in mind, we make the following definition.

Definition 5.1.6. The **homotopical tautological ring** $R^*_h(M)$ is defined as the subring of $H^*(\mathrm{B\,hAut}^+(TM);\mathbb{Q})$ generated by $\pi_!((e^{\mathrm{fw}})^i c(T_\pi E)) =: \kappa_{e^i c}$ for all $c \in H^*(\mathrm{B\,SO};\mathbb{Q})$ and all $i \geq 0$. Denote by $R^*_{h,0}(M)$ the image in $H^*(\mathrm{B\,hAut}_0(TM);\mathbb{Q})$.

Observe that a smooth fibre bundle $E \to B$ with fibre M is a TM-fibration and so by the classification theory there is a map on classifying spaces

$$\mathrm{B\,Diff}^+(M) \longrightarrow \mathrm{B\,hAut}^+(TM). \tag{5.3}$$

It can be shown that this map is induced by the map $d : \mathrm{Diff}^+(M) \to \mathrm{hAut}^+(TM)$ that sends a diffeomorphism to its differential and moreover that it preserves κ_c by naturality of fibre integration (see [Ber20b]). Hence, it induces a surjection

$$c : R^*_h(M) \longrightarrow R^*(M) \tag{5.4}$$

of tautological rings. In this sense, the homotopical tautological ring provides an upper bound.

The homotopical tautological ring is more computable, and as in the case for the Euler ring we approach the computation in a two steps. First, we study the analogue of (5.1)

$$\mathrm{B}(*\,\mathrm{hAut}_0(TM), M) \longrightarrow \mathrm{B}(*, \mathrm{hAut}_0(TM), *) = \mathrm{B\,hAut}_0(TM), \tag{5.5}$$

which is an M-fibration with nilpotent base space, using rational homotopy theory. In a second step, we infer $R^*_h(M)$ from the (homotopy) covering $\mathrm{B\,hAut}_0(TM) \to \mathrm{B\,hAut}^+(TM)$. We will discuss the first step in the next section and discuss the second step only in examples where the (homotopy) covering is finite-to-one.

5.1.1. Rational homotopy theory of TM-fibrations

The results of this section are due to Berglund [Ber20a, Ber20b] and are a key ingredient for the applications in Section 5.2. The rational models are based on Proposition 5.1.4 which expresses $\mathrm{B\,hAut}_0(TM)$ in terms of spaces whose rational homotopy type is well known.

Let us introduce some notation first. Let L be a dg Lie algebra, then its n-th Whitehead cover for $n \in \mathbb{Z}$ is defined as

$$L\langle n\rangle_i = \begin{cases} L_i & i > n \\ \ker(d : L_n \to L_{n-1}) & i = n \\ 0 & i < n \end{cases} \tag{5.6}$$

An element $\tau \in L_{-1}$ is called a *Maurer-Cartan element* if $d(\tau) + \frac{1}{2}[\tau, \tau] = 0$. A Maurer-Cartan element determines a new dg Lie algebra L^τ with the same underlying graded Lie algebra and twisted differential $d^\tau = d + [\tau, -]$.

Let A be a cgda and let both L and A of finite type. Then the completed tensor product $A\hat{\otimes}L$, which is the graded vector space in degree n given by $\prod_i A^i \otimes L_{i+n}$, is a dg Lie algebra with bracket $[a \otimes v, a' \otimes v'] = (-1)^{|a'|\cdot|v|} a \cdot a' \otimes [v, v']$. This grading convention is due to the fact that by our convention, Λ is a cochain complex whereas Π is a chain complex, and this choice makes the tensor product into a chain complex.

Let Λ be a Sullivan model for M^{2d} and Π a dg Lie model for $\mathrm{B\,SO}(2d)$, which is given by the dg Lie algebra on $\mathbb{Q}\{q_1, \dots, q_{d-1}, \varepsilon\}$ where $|q_i| = 4i - 1$ and $|\varepsilon| = 2d - 1$ with trivial differential and bracket. The main theorem of [Ber15, Thm 1.5] shows that (rational) homotopy classes of maps $M \to \mathrm{B\,SO}(2d)$ are in one-to-one correspondence with gauge classes of Maurer-Cartan elements in $\Lambda\hat{\otimes}\Pi$. Then the Maurer-Cartan element $\tau(M)$ corresponding to the classifying map of the tangent bundle $TM : M \to \mathrm{B\,SO}(d)$ is given by

$$\tau(M) = e(\Lambda) \otimes \varepsilon + \sum_{i=1}^{d-1} p_i(\Lambda) \otimes q_i \in (\Lambda\hat{\otimes}\Pi)_{-1}, \tag{5.7}$$

where $e(\Lambda), p_i(\Lambda) \in \Lambda$ are cocycle representatives of the Euler and Pontrjagin classes of M, and further that $\mathrm{Map}(M, \mathrm{B\,SO}(2d))_{TM}$ has a dg Lie model given by $(\Lambda\hat{\otimes}\Pi)^{\tau(M)}\langle 0\rangle$.

Remark 5.1.7. Since Π has trivial bracket, the twisting has actually no effect. This is expected since $\mathrm{B}\,\mathrm{SO}(2d)$ is rationally a product of Eilenberg-MacLane spaces and the rational homotopy type of such mapping spaces is independent of the path component.

The derivation Lie algebra $\mathrm{Der}^+(\Lambda)$ acts on $\Lambda\hat{\otimes}\Pi$ by $\theta\cdot(x\otimes q) := \theta(x)\otimes q$, which determines the semi-direct product $\mathrm{Der}^+(\Lambda)\ltimes(\Lambda\hat{\otimes}\Pi)$ see [Tan83, Ch. VII.2]. In this case, $\mathrm{Der}^+(\Lambda)\ltimes(\Lambda\hat{\otimes}\Pi)$ is the dg Lie algebra on the direct sum of chain complexes and bracket determined by $[\theta, x\otimes q] = \theta(x)\otimes q$. Then $\tau(M)$ is a Maurer-Cartan element in this semi-direct product, and we define

$$\mathfrak{g}^{TM} := \left(\mathrm{Der}^+(\Lambda)\ltimes(\Lambda\hat{\otimes}\Pi)\right)^{\tau(M)}\langle 0\rangle, \tag{5.8}$$

which acts on Λ through the projection to the $\mathrm{Der}^+(\Lambda)$.

Theorem 5.1.8 ([Ber20b])**.** *The dg Lie algebra $\mathfrak{g}^{TM}$ is a Lie model for $\mathrm{B}\,\mathrm{hAut}_0(TM)$ and the universal TM-fibration in* (5.5) *is modelled by*

$$C^*_{CE}(\mathfrak{g}^{TM};\mathbb{Q}) \longrightarrow C^*_{CE}(\mathfrak{g}^{TM};\Lambda).$$

Berglund also determines cocycle representatives of the characteristic classes of the oriented vector bundle over the total space of the universal TM-fibration in (5.5) which we again denote by $T_\pi E{\to}E$.

Denote by $P_i \in C^{4i}_{CE}(\mathfrak{g}^{TM};\Lambda)$ the 1-cochain that is determined by

$$P_i(s(q_j\otimes x)) = (-1)^{|x|}\delta_{i,j}\cdot x, \qquad P_i(s(\varepsilon\otimes x)) = 0, \qquad \text{and} \qquad P_i(s\theta) = 0$$

for $x\in\Lambda$ and $\theta\in\mathrm{Der}^+(\Lambda)$. Similarly, we define a 1-cochain $E\in C^{2d}_{CE}(\mathfrak{g}^{TM};\Lambda)$ determined by

$$E(s(q_j\otimes x)) = 0, \qquad E(s(\varepsilon\otimes x)) = (-1)^{|x|}x, \qquad \text{and} \qquad E(s\theta) = 0.$$

Theorem 5.1.9 ([Ber20b])**.** *Let $T_\pi E{\to}E$ denote the vector bundle over the total space of the universal TM-fibration* (5.5)*. Then the characteristic classes of $T_\pi E$ have cocycle representatives in $C^*_{CE}(\mathfrak{g}^{TM};\Lambda)$ given by*

$$e(T_\pi E) = e(\Lambda) + E \qquad \textit{and} \qquad p_i(T_\pi E) = p_i(\Lambda) + P_i, \tag{5.9}$$

where $e(\Lambda)$ and $p_i(\Lambda)$ are considered as 0-cochains in $C^*_{CE}(\mathfrak{g}^{TM};\Lambda)$.

Berglund has used this to compute the homotopical tautological ring in [Ber20b] for spheres and low dimensional $\mathbb{C}P^n$. Another simple example that we study next is for $M = S^2\times S^2$, which is positively elliptic and therefore we can adapt the strategy that we used for the proof of Theorem 4.2.16 to this situation.

Computational strategy for positively elliptic spaces.[1] Assume that M^{2d} is a positively rationally elliptic manifold that satisfies the Halperin conjecture. Let Λ be a pure Sullivan model, then we can find an abelian Lie subalgebra $\mathfrak{a} \hookrightarrow \mathrm{Der}^+(\Lambda)$ with trivial differential such that the inclusion is a quasi-isomorphism of dg Lie algebras. It follows that the induced map of semi-direct products

$$\mathfrak{a} \ltimes (\Lambda \hat{\otimes} \Pi) \hookrightarrow \mathrm{Der}^+(\Lambda) \ltimes (\Lambda \hat{\otimes} \Pi)$$

is a quasi-isomorphism. As Λ is formal with a quasi-isomorphism $\Lambda \to H(\Lambda)$, there is a weak equivalence $\mathfrak{a} \ltimes (\Lambda \hat{\otimes} \Pi) \to \mathfrak{a} \ltimes (H(\Lambda) \hat{\otimes} \Pi)$. Since $\tau(M)$ determines a Maurer-Cartan element in $H(\Lambda) \hat{\otimes} \Pi$, there is a zig-zag of weak equivalences twisted by $\tau(M)$

$$(\mathfrak{a} \ltimes (H(\Lambda) \hat{\otimes} \Pi))^{\tau(M)}\langle 0 \rangle \xleftarrow{\simeq} (\mathfrak{a} \ltimes (\Lambda \hat{\otimes} \Pi))^{\tau(M)}\langle 0 \rangle \xhookrightarrow{\simeq} (\mathrm{Der}^+(\Lambda) \ltimes (\Lambda \hat{\otimes} \Pi))^{\tau(M)}\langle 0 \rangle = \mathfrak{g}^{TM} \tag{5.10}$$

Observe that these weak-equivalences are compatible with the obvious module structure on Λ so that these induce quasi-isomorphisms of the rational model from Theorem 5.1.8 to

$$C^*_{CE}\left((\mathfrak{a} \ltimes (H(\Lambda) \hat{\otimes} \Pi))^{\tau(M)}\langle 0 \rangle; \mathbb{Q}\right) \longrightarrow C^*_{CE}\left((\mathfrak{a} \ltimes (H(\Lambda) \hat{\otimes} \Pi))^{\tau(M)}\langle 0 \rangle; \Lambda\right). \tag{5.11}$$

The key point is that the dg Lie algebra $\mathfrak{a} \ltimes (H(\Lambda) \hat{\otimes} \Pi))^{\tau(M)}$ has trivial differential and bracket so that its Lie algebra cohomology is a polynomial ring.

The triviality of the differential is obvious, and the triviality of the bracket follows from another reformulation of the Halperin conjecture: Let $\Lambda = (\Lambda Q \otimes \Lambda P, d)$ be the pure Sullivan model for M. Then $\mathfrak{a}$ consists of derivations with non-trivial restriction $P \to \Lambda Q$. Since $\tau(M)$ can be represented by a Maurer-Cartan element in $\Lambda Q \hat{\otimes} \Pi$, the action of $\mathfrak{a}$ on $\tau(M)$ is trivial and so is the bracket.

Example 5.1.10. $M = S^2 \times S^2$ with Sullivan model $\Lambda = \left(\Lambda(x_1, x_2, y_1, y_2), d = x_1^2 \partial y_1 + x_2^2 \partial y_2\right)$. We have seen in Chapter 4.2.3 that $\mathfrak{a} := \mathbb{Q}\{x_1 \partial y_2, x_2 \partial y_1, \partial y_1, \partial y_2\}$ is a quasi-isomorphic abelian Lie subalgebra of $\mathrm{Der}^+(\Lambda)$ with trivial differential. Next, we choose a basis of $H(\Lambda) \otimes \Pi\langle 0 \rangle$ say $1 \otimes q_1, x_1 \otimes q_1, x_2 \otimes q_1, 1 \otimes \varepsilon, x_1 \otimes \varepsilon, x_2 \otimes \varepsilon$. Hence, the base of (5.11) is equivalent to a polynomial ring

$$B := C^*_{CE}\left((\mathfrak{a} \ltimes (H(\Lambda) \hat{\otimes} \Pi))^{\tau(M)}\langle 0 \rangle; \mathbb{Q}\right) \cong \mathbb{Q}[a_1, a_2, b_1, b_2, p_{1,0}, p_{1,1}, p_{1,2}, e_0, e_1, e_2]$$

that correspond to a dual basis of $\mathfrak{a} \oplus (H(\Lambda) \otimes \Pi)\langle 0 \rangle$ in the same order as the basis is listed above. Hence, the degrees of the polynomial generators are $\{2, 2, 4, 4, 4, 2, 2, 4, 2, 2\}$. Since

[1] This the starting point of the author's collaboration with Alexander Berglund concerning elliptic spaces.

only $\mathrm{Der}^+(\Lambda)$ acts non-trivially on Λ, the map (5.11) is equivalent to the following complete intersection

$$\pi^* : B \longrightarrow E := B[x_1, x_2]/(x_1^2 - b_1 - a_2x_2, x_2^2 - b_2 - a_1x_1).$$

In this example, we have that $\Lambda \otimes \Pi\langle 0\rangle$ coincides with $H(\Lambda) \otimes \Pi\langle 0\rangle$ and the restriction of P_1 and E with respect to the basis we have chosen is $P_1 = p_{1,0} + p_{1,1}x_1 + p_{1,2}x_2$ and $E = e_0 + e_1x_1 + e_2x_2$. Hence, the cocycle representatives are

$$\begin{aligned} e(T_\pi E) &= 4x_1x_2 + e_0 + e_1x_1 + e_2x_2 \\ p_1(T_\pi E) &= p_{1,0} + p_{1,1}x_1 + p_{1,2}x_2 \\ e^{\mathrm{fw}}(\pi) &= 4x_1x_2 - a_1a_2 \end{aligned}$$

by Theorem 5.1.9 and Theorem 4.1.10 since $e(\Lambda) = 4x_1x_2$ and $p_1(\Lambda) = 0$ as the first Pontrjagin class of $S^2 \times S^2$ vanishes.

Remark 5.1.11.

(i) Observe that cocyle representatives of the two Euler classes $e(T_\pi E)$ and $e^{\mathrm{fw}}(\pi)$ are different. This is in fact true for arbitrary M and gives a proof of Proposition 5.1.5. But there is also a simpler and more direct proof that we give after this remark.

(ii) When computing $R_h^*(M)$ we are using the fibrewise Euler class by our convention, and the reason becomes more apparent in this example. Since fibre integrating polynomials in x_1 and x_2 has image in the subring of B generated by a_1, a_2, b_1, b_2, the homotopical tautological ring $R_h^*(M)$ is a subring of $\mathbb{Q}[a_1, a_2, b_1, b_2, p_{1,0}, p_{1,1}, p_{1,2}]$ by our choice of Euler class. As this subring is smaller, one finds more relations among MMM-classes when compared to the tautological ring defined via the tangential Euler class $e(T_\pi E)$.

It is not hard to see that this is equivalent to computing the tautological ring of the M-fibration over $\mathrm{B}(\prod_{i=1}^{d-1} \mathrm{Map}(M, K(\mathbb{Q}, 4i))_{p_i(M;\mathbb{Q})}, \mathrm{hAut}_0(M), *)$, which is the classifying space of M-fibrations with cohomology classes on the total space of degree $4i$ for $1 \leq i \leq d-1$.

(iii) Some of the relations among MMM-classes that follow from this method could have been deduced using simpler tools, such as a closer analysis of the Serre spectral sequence. The key advantage we see in this approach is that it is systematic and reduces the problem to a straightforward computation.

We will now give a proof of Proposition 5.1.5. It is a direct consequence of Theorem 5.1.9 due to Berglund, but there is also the following more direct argument.

Proof of 5.1.5. Consider a TM-fibration with trivial underlying fibration $\pi_{S^k} : S^k \times M \to S^k$, i.e. a vector bundle over $S^k \times M$ whose restriction to $p \times M$ is equivalent to TM for all $p \in S^k$. A source of such TM-fibrations are elements in $\pi_k(\mathrm{Map}(M, \mathrm{B\,SO}(d))_{TM})$ via the pullback bundle of the adjoint. As we have mentioned before, the rational homotopy groups of this mapping space are $H(\Lambda)\hat{\otimes}\Pi\langle 0\rangle \cong H(M;\mathbb{Q})\hat{\otimes}\Pi\langle 0\rangle$ by [Ber15, Thm 1.5]. There is also a simple description of the algebraic model of the adjoint map $a : S^k \times M \to \mathrm{B\,SO}(d)$ corresponding to $\alpha \otimes \varepsilon \in (H(M;\mathbb{Q})\hat{\otimes}\Pi)_{k-1}$ for any $\alpha \in H^{2d-k}(M;\mathbb{Q})$, and the model of the adjoint sends $e \in H^{2d}(\mathrm{B\,SO}(2d);\mathbb{Q})$ to $\varepsilon_k \times \alpha + 1 \times e(M) \in H^{2d}(S^k \times M;\mathbb{Q})$. For any M we can choose $k = 2d$ and $\alpha = 1 \in H^0(M)$ and we see that for the corresponding TM-fibration the tangential Euler class is different than $e^{\mathrm{fw}}(\pi_{S^k}) = 1 \times e(M)$.

The algebraic model of adjoints is discussed for example in [LS07], but we will revisit this topic in more detail in Chapter 8 which is sufficient to prove this fact about the adjoint. □

Finally, we are at the point when we can make explicit calculations of $R^*_h(M)$. We begin with identifying finite generating sets following [RW18]. The following lemma is a reformulation of [RW18, Thm A] that applies to TM-fibrations as well.

Lemma 5.1.12. *Let M be a closed oriented manifold with cohomology concentrated in even degrees, then $R^*_h(M)$ is a finitely generated $\mathbb{Q}$-algebra.*

The proof is based on an analogue of the Cayley-Hamilton theorem trace relations for automorphisms of finite free algebras (see also Proposition 4.2.3) that holds for the cohomology of the total space of a fibration with fibre M satisfying the assumptions above considered as an algebra over the cohomology of the base. This trace relation [RW18, Cor. 2.7] depends only on $\dim H^*(M;\mathbb{Q})$, and for a TM-fibration $E \to B$ with $M = S^2 \times S^2$ is given by the following identity in $H^*(E)$

$$c^4 = \kappa_{ec}c^3 - \frac{\kappa_{ec}^2 - \kappa_{ec^2}}{2}c^2 + \frac{\kappa_{ec}^3 - 3\kappa_{ec}\kappa_{ec^2} + 2\kappa_{ec^3}}{6}c - \frac{\kappa_{ec}^4 - 6\kappa_{ec}^2\kappa_{ec^2} + 3\kappa_{ec^2}^2 + 8\kappa_{ec}\kappa_{ec^3} - 6\kappa_{ec^4}}{24},$$

where $c = c(T_\pi E)$ for $c \in H^*(\mathrm{B\,SO}(d);\mathbb{Q})$. Randal-Williams identifies a generating set of $R^*_h(S^2 \times S^2)$ from the trace relation above given by $\{\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^2}, \kappa_{e^2p_1}, \kappa_{e^3}, \kappa_{e^3p_1}, \kappa_{e^5}\}$.

Remark 5.1.13. It is also possible to find a generating set of $R^*_h(S^2 \times S^2)$ by studying the reprentatives in the algebraic model $\mathbb{Q}[a_1, a_2, b_1, b_2, p_{1,0}, p_{1,1}, p_{1,2}]$. For example, $\kappa_{ep_1} = 4p_{1,0}$ and $\kappa_{p_1^2} = 2p_{1,1}p_{1,2}$. However, this is not very efficient in most cases as the polynomials become very large very quickly.

One can compute the representatives of the MMM-classes and their relations using a computer algebra system. We have used Macaulay2 for these computations and attached the code in Appendix B. However, it turns out that the computation is too complicated for the author's computer to produce a final answer. There are, however, partial results that one can extract, for example the following identity in $R^*_{h,0}(S^2 \times S^2)$ (see Example B.2):

$$\begin{aligned} 0 =&\kappa_{p_1^2}\kappa_{ep_1}^4\kappa_{e^2} - 2\kappa_{p_1^2}^2\kappa_{ep_1}^2\kappa_{e^2}^2 + \kappa_{p_1^2}^3\kappa_{e^2}^3 - \kappa_{p_1^3}\kappa_{ep_1}^3\kappa_{e^2} - 5\kappa_{p_1^2}\kappa_{ep_1}^2\kappa_{ep_1^2}\kappa_{e^2} + 9\kappa_{p_1^2}\kappa_{p_1^3}\kappa_{ep_1}\kappa_{e^2}^2 - 3\kappa_{p_1^2}^2\kappa_{ep_1^2}\kappa_{e^2}^2 \\ &- 3\kappa_{p_1^2}\kappa_{ep_1}^3\kappa_{e^2p_1} - 5\kappa_{p_1^2}^2\kappa_{ep_1}\kappa_{e^2}\kappa_{e^2p_1} + 9\kappa_{p_1^2}^2\kappa_{ep_1}^2\kappa_{e^3} - \kappa_{p_1^2}^3\kappa_{e^2}\kappa_{e^3} + 4\kappa_{p_1^3}\kappa_{ep_1}\kappa_{ep_1^2}\kappa_{e^2} + 4\kappa_{p_1^2}\kappa_{ep_1^2}^2\kappa_{e^2} \\ &- 7\kappa_{p_1^3}^2\kappa_{e^2}^2 + 4\kappa_{p_1^3}\kappa_{ep_1}^2\kappa_{e^2p_1} + 12\kappa_{p_1^2}\kappa_{ep_1}\kappa_{ep_1^2}\kappa_{e^2p_1} + 6\kappa_{p_1^2}\kappa_{p_1^3}\kappa_{e^2}\kappa_{e^2p_1} + \kappa_{p_1^2}^2\kappa_{e^2p_1}^2 - 24\kappa_{p_1^2}\kappa_{p_1^3}\kappa_{ep_1}\kappa_{e^3} \\ &- 16\kappa_{p_1^3}\kappa_{ep_1^2}\kappa_{e^2p_1} + 16\kappa_{p_1^3}^2\kappa_{e^3} \end{aligned} \tag{5.12}$$

Of course, we don't necessarily learn much having a concrete presentation anyway. Instead, we should focus on properties of $R^*_h(M)$. For example, one can easily extract upper bounds on the Krull dimension from the the rational model.

Lemma 5.1.14. $\operatorname{Kdim} R^*_h(S^2 \times S^2) \leq 7$.

Proof. Since the group of homotopy self-equivalences of $S^2 \times S^2$ is finite [Bau96, Thm 6.3] the homotopical tautological ring is isomorphic to $R^*_{h,0}(S^2 \times S^2)$. It is finitely generated by the trace relation and therefore integral over a polynomial subring on $\operatorname{Kdim} R^*_h(S^2 \times S^2)$ generators by Noether normalization. As it is contained in $\mathbb{Q}[a_1, a_2, b_1, b_2, p_{1,0}, p_{1,1}, p_{1,2}]$, the Krull dimension is ≤ 7 by [Eis95, Cor. 13.5]. $\square$

This is a better upper bound than the one from Randal-Williams using the trace relation (and easier to compute). And of course we can generalize this to produce upper bounds for all positively elliptic manifolds that satisfy the Halperin conjecture, i.e. with the notation introduced above $\operatorname{Kdim} R^*_{h,0}(M) \leq \dim \mathfrak{a} \ltimes H^*(M;\mathbb{Q}) \otimes \Pi'\langle 0\rangle$, where $\Pi' \subset \Pi$ is the subspace spanned by the q_i's.

5.2. Relations from the family signature theorem

We will now change focus back to the smooth tautological ring, for which we know one other source of relations coming from the familiy signature theorem. Denote by $R^*_0(M)$ the image of the tautological ring in $H^*(\mathrm{B}\,\mathrm{Diff}_0(M);\mathbb{Q})$. Then for all $i > \dim M/4$ the MMM-class κ_{L_i} associated to the Hirzebruch L-polynomial $L_i = L_i(p_1, \ldots, p_{d-1}, e) \in H^*(\mathrm{B}\,\mathrm{SO}(2d);\mathbb{Q})$ vanishes by Theorem 1.2.3. It is worth pointing out, that this is the *only source of relations we know that uses the manifold structure of the fibre bundle*.

The goal of this section is to impose these relations in the homotopical tautological ring to improve the upper bound. This turns out to simplify the computational complexity as well.

Definition 5.2.1. We define the **Hirzebruch ideal** as the ideal $I_H \subset H^*(\mathrm{B}\,\mathrm{hAut}_0(TM);\mathbb{Q})$ generated by all κ_{L_i} for $4i > \dim M$ and by $I_H^{\leq n} \subset I_H$ the ideal generated by κ_{L_i} for $\dim M/4 < i \leq n$. Then the surjection (5.4) factors through

$$R^*_{h,0}(M) \longrightarrow R^*_{h,0}(M)/(I_H \cap R^*_{h,0}(M)) \longrightarrow R^*_0(M). \tag{5.13}$$

We will first revisit out computation for $M = S^2 \times S^2$. The quotient $R^*_{h,0}(M)/(I_H \cap R^*_{h,0}(M))$ is generated by fewer elements $\{\kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^2}, \kappa_{e^2p_1}, \kappa_{e^3}, \kappa_{e^3p_1}, \kappa_{e^5}\}$ (see [RW19]) which reduces the computational complexity and we are able to compute it for $I_H^{\leq 12}$ using Macaulay2. However, displaying it would take several pages so instead we extract its Krull dimension and Hilbert series (see the code in Example B.4).

Proposition 5.2.2. *For $M = S^2 \times S^2$ the Hilbert series $P\left(R^*_{h,0}(M)/(I_H^{\leq 12} \cap R^*_{h,0}(M)), T\right)$ is given by*

$$\frac{1 + T^8\left(2 + T^8\left(T^4+1\right)\left(6T^{44} - 8T^{40} - T^{36} - 2T^{32} - T^{28} + 7T^{24} + 4T^{20} + T^{16} - 5T^{12} + T^8 - 4T^4 + 1\right)\right)}{(1-T^4)^2\,(1-T^8)\,(1-T^{16})}$$

and we can read of that $\mathrm{Kdim}\, R^*_{h,0}(M)/(I_H^{\leq 12} \cap R^*_{h,0}(M)) = 4$.

Another way to extract some more palpable information from these computations is to restrict attention to the map $E^*(M) \to R^*(M)$, since the Euler ring is simpler and smaller than $R^*_h(M)$. But also fibrations occur more naturally in nature than TM-fibrations. We are particularly interested in elements of the kernel as these provide obstructions to finding fibre homotopy equivalent fibre bundles.

Proposition 5.2.3. *The following elements are in the kernel of $E^*(M) \to R^*(M)$ for $M = S^2 \times S^2$:*

$$\begin{gathered}
36130625\kappa_1^5\kappa_4 - 1257728\kappa_1\kappa_2^4 - 16765250\kappa_1^3\kappa_2\kappa_4 + 3640975\kappa_1\kappa_2^2\kappa_4 + 171250\kappa_1\kappa_4^2,\\
2854453\kappa_1\kappa_2^5 + 26160025\kappa_1^3\kappa_2^2\kappa_4 - 12274935\kappa_1\kappa_2^3\kappa_4 - 36130625\kappa_1^3\kappa_4^2 + 11410250\kappa_1\kappa_2\kappa_4^2,\\
4913765\kappa_1^3\kappa_2^3 + 883609\kappa_1\kappa_2^4 - 6902125\kappa_1^3\kappa_2\kappa_4 - 3833475\kappa_1\kappa_2^2\kappa_4 + 3631250\kappa_1\kappa_4^2,\\
24568825\kappa_1^5\kappa_2^2 - 561391\kappa_1\kappa_2^4 - 15829250\kappa_1^3\kappa_2\kappa_4 + 756100\kappa_1\kappa_2^2\kappa_4 + 2632500\kappa_1\kappa_4^2,\\
24568825\kappa_1^7\kappa_2 - 242097\kappa_1\kappa_2^4 - 5471475\kappa_1^3\kappa_2\kappa_4 + 262825\kappa_1\kappa_2^2\kappa_4 + 1321250\kappa_1\kappa_4^2,\\
614220625\kappa_1^9 - 1063673\kappa_1\kappa_2^4 - 41486500\kappa_1^3\kappa_2\kappa_4 - 2159400\kappa_1\kappa_2^2\kappa_4 + 15552500\kappa_1\kappa_4^2
\end{gathered}$$

Proof. The code for the computation of the kernel of $E^*_0(M) \to R^*_0(M)/(I_H^{\leq 12} \cap R^*_{h,0}(M))$ is presented in Example B.4. We can promote the computation to the full Euler rings by the standard argument since $\mathcal{E}(S^2 \times S^2)$ is finite, and further since the automorphism group $O(H,\lambda)$ of the intersection form of $S^2 \times S^2$ is finite, its classifying space has trivial rational cohomology so that $\kappa_{L_i} \in R^*(M)$ vanishes by the family signature theorem. □

Finally, we want to discuss one further example to illustrate some disadvantages of the approach presented above. Importantly, we will not rely on the rational homotopy theory of TM-fibrations developed by Berglund and instead use the techniques developed by Randal-Williams in [RW19].

Proposition 5.2.4. *The kernel of* $E^*(\mathbb{C}P^2\#\mathbb{C}P^2)\to R^*(\mathbb{C}P^2\#\mathbb{C}P^2)$ *is non-trivial, and we give some elements in the Appendix in Proposition B.5.*

Proof. The details of the computation can be found in Proposition B.5. Unlike in our previous strategy, we do not try to identify a small generating set but simply choose $\kappa_{e^i p_1^j}$ for $1 \leq i + j \leq 9$. There are many relations among these MMM-classes by fibre integrating the (multiples of the) trace relation (and its polarization)

$$c^4 = \kappa_{ec}c^3 - \frac{\kappa_{ec}^2 - \kappa_{ec^2}}{2}c^2 + \frac{\kappa_{ec}^3 - 3\kappa_{ec}\kappa_{ec^2} + 2\kappa_{ec^3}}{6}c - \frac{\kappa_{ec}^4 - 6\kappa_{ec}^2\kappa_{ec^2} + 3\kappa_{ec^2}^2 + 8\kappa_{ec}\kappa_{ec^3} - 6\kappa_{ec^4}}{24}$$

for various c, and we collect all these identities in an ideal. We also add to this ideal the expressions for κ_{L_i} for $1 \leq i \leq 9$ in terms of the generating set because the automorphism group of the intersection form of $\mathbb{C}P^2\#\mathbb{C}P^2$ is finite so that $\kappa_i = 0$ for all $i > 1$. Finally, one can compute the intersection of this ideal with the subring generated by κ_{e^i} for $i = 2, 3, 5$. This intersection is non-empty and since the Euler ring $E^*(\mathbb{C}P^2\#\mathbb{C}P^2)$ is a polynomial on these generators by Theorem 4.2.16, the kernel is non-trivial. □

We will collect some comments about these two methods and their comparison:

(1) We have also attempted to compute $R_0^*(\mathbb{C}P^2\#\mathbb{C}P^2)/(I_H \cap R_0^*(\mathbb{C}P^2\#\mathbb{C}P^2))$ using the rational homotopy theory of TM-fibrations. On the face of it, this case is quite similar to $M = S^2 \times S^2$ since both have the same Euler characteristic and even $\mathrm{B\,hAut}_0(TM)$ is rationally equivalent. The rational model is the following complete intersection:

$$\begin{aligned} B &= \mathbb{Q}[a_1, a_2, b_1, b_2, p_{1,0}, p_{1,1}, p_{1,2}, e_0, e_1, e_2] \\ E &= B[x_1, x_2]/(x_1^2 - x_2^2 - b_2, x_1x_2 - b_1 - a_2x_1 - a_1x_2) \\ e^{\mathrm{fw}}(\pi) &= 2(x_1^2 + x_2^2) + 2(x_1a_1 + x_2a_2) \\ p_1(T_\pi(E)) &= 3(x_1^2 + x_2^2) + p_{1,0} + p_{1,1}x_1 + p_{1,2}x_2 \end{aligned} \tag{5.14}$$

However, the Macaulay2 computation turned out to be too complicated and was aborted. For the sake of comparison: the computation for $S^2 \times S^2$ was a matter of minutes whereas the computation for $\mathbb{C}P^2\#\mathbb{C}P^2$ was aborted after several days of runtime.

The key difference is that the first Pontrjagin class vanishes for $S^2 \times S^2$ but $p_1(\mathbb{C}P^2 \# \mathbb{C}P^2) = 6$ by the signature theorem. This has the effect that the cocycle representatives of MMM-classes are considerably more complicated polynomials in B for $\mathbb{C}P^2 \# \mathbb{C}P^2$ because one has to fibre integrate polynomials in x_1, x_2 of higher degree. This affects the Gröbner basis computation.

(2) We did check the results we have obtained in Proposition B.5 using the rational model (5.14). More precisely, we have computed with Macaulay2 the image of the last element in Proposition B.5 in the quotient $B/I_H^{\leq 9}$ (corresponding to the number of Hirzebruch relations used in the computation in Proposition B.5). It took over ten days of runtime before it was confirmed that the image was indeed zero.

(3) On the other hand, we also attempted to compute the kernel $E^*(S^2 \times S^2) \to R^*(S^2 \times S^2)$ using Randal-Williams' method. Now in this case the computation did not stop. This begs the question why these two different methods work well in these different yet still similar cases.

The advantage of the approach presented in this thesis is that we can work in small polynomial rings, but on the other hand the MMM-classes are represented by very complicated polynomials which slows down the computation. The disadvantage of Randal-Williams' approach is that we have to use many generators dictated roughly by the dimension of M and $H^*(M; \mathbb{Q})$, and the advantage is that the relations are not very complicated polynomials in the generators.

(4) In all examples we have studied so far the upper bounds on the Krull dimension on the smooth tautological ring obtained by either method agree, even though we can better detect the finer algebra structure through the rational homotopy theory of TM-fibrations.

(5) The computational results that we presented here should be considered only as a starting point, and we expect that they can be improved using more refined algebraic tools.

5.2.1. Tautological rings of fake $\mathbb{H}P^2$

In this section, we study how much the tautological ring changes for smooth closed manifolds that are homotopy equivalent, specifically manifolds homotopy equivalent to $\mathbb{H}P^2$ as a simple first example. The surprising answer turns out to be that the Krull dimension of the tautological ring is almost an invariant of the homotopy type in this case.

We can study the set of manifolds homotopy equivalent to $\mathbb{H}P^2$ via surgery theory. The set of equivalence classes of $f : M \xrightarrow{\simeq} \mathbb{H}P^2$, where M is a smooth closed manifold and equivalence relation $(M_1, f_1) \sim (M_2, f_2)$ if there exists a diffeomorphism $g : M_1 \to M_2$ such that $f_1 \simeq f_2 g$, is called the *geometric structure set* $S(\mathbb{H}P^2)$. It can be determined using Browder-Novikov-Sullivan-Wall surgery exact sequence, which we have analysed in Appendix C. The main statement can be phrased as follows.

Theorem (See Thm C.1)**.** *There are infinitely many smooth closed manifolds M that are homotopy equivalent to $\mathbb{H}P^2$ and they are (up to finite ambiguity) parametrized by the value of the first Pontrjagin class $p_1(TM) \in H^4(M;\mathbb{Z})$.*

The main theorem of this section is the following.

Theorem 5.2.5. $\operatorname{Kdim} R^*(M) = 0$ *for all but finitely many* $[f, M] \in S(\mathbb{H}P^2)$.

Proof. The basic idea of the proof remains the same, i.e. we want to compute the Krull dimension of $R_0^*(M)/(I_H \cap R_0^*(M))$. But in fact it will suffice to compute the dimension of I_H. The algebraic model of the universal TM-fibration is given by

$$\begin{aligned} B &= \mathbb{Q}[x_8, x_{12}, p_{1,0}, p_{2,0}, p_{2,1}, p_{3,0}, p_{3,1}, p_{3,2}, e_0, e_1] \\ E &= B[z]/(z^3 - zx_8 - x_{12}) \end{aligned} \tag{5.15}$$

with coycle representatives of the characteristic classes

$$\begin{aligned} e^{\mathrm{fw}}(\pi) &= 3z^2 - x_8 & p_1(T_\pi E) &= p_1(M) + p_{1,0} \\ p_2(T_\pi E) &= p_2(M) + p_{2,0} + p_{2,1}z & p_3(T_\pi E) &= p_{3,0} + p_{3,1}z + p_{3,2}z^2 \end{aligned}$$

so that we can compute the Hirzebruch ideal $I_H \subset B$ as before. Now the key idea is to treat all manifolds M at the same time by formally adding the parameter p_1 of degree 0 to B that represents the first Pontrjagin class. Denote this polynomial ring by $\overline{B}$ (and omit e_0 and e_1). Then we can define $\overline{E} := E \otimes_B \overline{B}$ as the induced complete intersection over $\overline{B}$ and replace the cocycle representatives for $p_1(T_\pi E)$ and $p_2(T_\pi E)$ by

$$p_1(T_\pi E) = p_1 z + p_{1,0} \in \overline{E} \qquad \text{and} \qquad p_2(T_\pi E) = \frac{45 + p_1^2}{7} z^2 + p_{2,0} + p_{2,1}z \in \overline{E} \tag{5.16}$$

as $p_2(M) = (45 + p_1(M)^2)/7z^2$ by the signature theorem. We further define $\bar{I}_H$ as the ideal generated by the $\kappa_{L_i} \in \overline{B}$ and consider the inclusion $\varphi : \mathbb{Q}[p_1] \to \overline{B}/\bar{I}_H$.

For a homomorphism $\varphi : R \hookrightarrow S$ of commutative rings as above and a prime ideal P of R we call $K(R/P) \otimes_R S$ the fibre over P, where $K(R/P)$ denotes the field of fractions of R/P. It is a classic question in commutative algebra how the fibres vary with P. There are strong structural theorems in case R is Noetherian and S is a finitely generated R-algebra that show that most fibres have common properties. Our statement follows by applying such a structural result about the dimension of the fibres of $\varphi : \mathbb{Q}[p_1] \to \overline{B}/\overline{I}_H$; we use the following statement from [Eis95, Thm 14.8b]:

Suppose S is a positively graded algebra $S = \bigoplus_{i \geq 0} S_i$ which is finitely generated over the Noetherian ring $R = S_0$. Then for every integer n there is an ideal J_n of R such that for any prime P of R

$$\operatorname{Kdim} K(R/P) \otimes_R S \geq n \quad \text{iff} \quad P \supset J_n.$$

This has a geometric interpretation: let $Y \to X$ be the corresponding map of affine schemes where Y is projective over X. Then $K(R/P) \otimes_R S$ is the coordinate ring of the scheme-theoretic fibre and the theorem implies that the dimension of the fibre is upper semi-continuous in the target, i.e. it can only jump upwards when varying in X.

For the morphism $\varphi : \mathbb{Q}[p_1] \to \overline{B}/\overline{I}_H^{\leq 10}$, we have computed the fibre for certain values of p_1 and found that the dimension of the fibre vanishes (see Example B.7). Since the dimension of the fibre can only jump upwards, this implies that the generic fibre has dimension zero. Formulated in terms of the above statement, it implies that for $n = 1$ the ideal J_1 is not zero and thus only finitely many maximal ideals can contain J_1. For $[f : M \xrightarrow{\simeq} \mathbb{H}P^2] \in S(\mathbb{H}P^2)$ the fibre over the maximal ideal $(p_1 - p_1(M)) \subset \mathbb{Q}[p_1]$ is isomorphic to the quotient $B/I_H \cong H^*(\mathrm{B\,hAut}_0(TM); \mathbb{Q})/I_H$ of the corresponding Hirzebruch ideal and thus contains $R^*_{h,0}(M)/(I_H \cap R^*_{h,0}(M))$. Hence, for almost all M we have $\operatorname{Kdim} B/I_H = 0$ and it follows from Noether normalization that the quotient is a finite dimensional vector space. Hence, $R^*_{h,0}(M)/(I_H \cap R^*_{h,0}(M))$ is finite dimensional as well and therefore has vanishing Krull dimension. It follows that $\operatorname{Kdim} R^*_0(M) = 0$, and the general statement follows as both the automorphism of the intersection pairing and $\mathcal{E}(\mathbb{H}P^2)$ are finite (see [Bau96]). □

It is a natural question for which values of p_1 the dimension of the fibre of $\varphi : \mathbb{Q}[p_1] \to \overline{B}/\overline{I}_H$ does vary from the generic fibre. Or to phrase it in a more geometric way, are there $[M, f] \in S(\mathbb{H}P^2)$ for which the upper bound on the Krull dimension of $R^*_{h,0}(M)/(I_H \cap R^*_{h,0}(M))$ is positive. At present, we only know one such exception (and we expect it to be the only one).

Proposition 5.2.6. $\operatorname{Kdim} R^*(\mathbb{H}P^2) \leq 3$.

This upper bound is obtained by computing the dimension of the fibre of $\varphi : \mathbb{Q}[p_1] \to \overline{B}/\overline{I}_H$ over $p_1 = 2$ (see Example B.7), which is the value of the first Pontrjagin class of $\mathbb{H}P^2$ (see Appendix C). The fact that for $\mathbb{H}P^2$ the upper bound is positive is not a coincident but forced by a result of Randal-Williams [RW19] which implies that $\operatorname{Kdim} R^*(\mathbb{H}P^2) > 0$ as $\mathbb{H}P^2$

admits a good circle action. We will briefly discuss Randal-Williams result in the following addendum and discuss its interaction with the commutative algebra perspective.

Addendum – Lower bounds on the Krull dimension and a connection to the $\hat{A}$-genus

Consider a manifold M with a smooth torus action, i.e. a continuous group homomorphism $T^k \to \mathrm{Diff}_0(M)$ which induces a map on classifying spaces. The cohomology ring of $\mathrm{B}\,T^k$ is a polynomial ring $H^*_T := \mathbb{Q}[x_1, \dots, x_k]$ where each generator has degree 2. We denote the image of the tautological ring as $R^*_T(M) \subset H^*_T$. One of the main results of Randal-Williams [RW19, Thm 3.1] provides a criterion on the action that guarantees that H^*_T is integral over $R^*_T(M)$ – a special case of this theorem is given below. If follows from the going-up theorem that $\mathrm{Kdim}\, H^*_T = \mathrm{Kdim}\, R^*_T(M)$ and since $R^*(M)$ surjects onto $R^*_T(M)$, we obtain a lower bound $\mathrm{Kdim}\, R^*(M) \geq \mathrm{Kdim}\, R^*_T(M) = k$.

Theorem 5.2.7 ([RW19, Cor. B]). *Let T^k act effectively on a smooth manifold M such that the fixed set W^T is discrete and non-empty. Then H^*_T is integral over $R^*_T(M)$ and therefore* $\mathrm{Kdim}\, R^*(M) \geq k$.

$\mathbb{H}P^2$ admits a standard $T^2 \subset (\mathbb{C}^\times)^2$ action given by $(\lambda_1, \lambda_2) \cdot [h_0{:}h_1{:}h_2] = [h_0{:}\lambda_1 h_1{:}\lambda_2 h_2]$ with isolated fixed points [1:0:0], [0:1:0] and [0:0:1]. It follows that $\mathrm{Kdim}\, R^*(\mathbb{H}P^2) \geq 2$. This is in contrast to almost all manifolds $[M, f] \in S(\mathbb{H}P^2)$ which cannot admit an effective circle action (that satisfies the condition on the fixed set) by Theorem 5.2.5. This statement can be improved by a result of Atiyah and Hirzebruch using index theory.

Theorem 5.2.8 ([AH70]). *Let M^{4k} be a compact oriented smooth manifold with $w_2(M) = 0$. If a connected compact Lie group G acts non-trivially on M then $\hat{A}(M) = 0$.*

The $\hat{A}$-genus is the ring homomorphism $\hat{A} : \Omega^{\mathrm{SO}}_* \otimes \mathbb{Q} \to \mathbb{Q}$ associated to the multiplicative sequence induced by the power series of the function $\frac{x/2}{\sinh(x/2)}$. Its first three elements as polynomials in the Pontrjagin classes are given by

$$\begin{aligned}
\hat{A}_1(p_1) &= -\frac{1}{24} p_1 \\
\hat{A}_2(p_1, p_2) &= \frac{1}{5760}(-4p_2 + 7p_1^2) \\
\hat{A}_3(p_1, p_2, p_3) &= \frac{1}{967680}(-16p_3 + 44p_2 p_1 - 31p_1^3)
\end{aligned}$$

and we denote by $\hat{A}(M^{4k})$ the evaluation of $\hat{A}_k$ on the fundamental class.

A simple computation shows that a manifold M^8 with the cohomology ring of $\mathbb{H}P^2$ satisfies both $\hat{A}(M) = 0$ and $L(M) = 1$ (where L denotes the signature genus) if and only if $p_1(M) = \pm 2z$. We have computed the Pontrjagin classes of fake quaternionic spaces in (C.3) for $M \simeq \mathbb{H}P^2$, and it follows from (C.2) that there are only two elements in the structure set with vanishing $\hat{A}$-genus: $[\mathbb{H}P^2, \mathrm{Id}]$, which admits a circle action, and $[\mathbb{H}P^2\#\Sigma^8, f]$ where Σ^8 is the unique exotic 8-sphere and f is the homeomorphism with $\mathbb{H}P^2\#S^8 \cong \mathbb{H}P^2$. We do not know if $\mathbb{H}P^2\#\Sigma^8$ admits a good circle action (although it is likely known), but we will see later on that it only really matters if there is a good circle action for some smooth structure.

Now we cannot hope to study a geometric concept like circle actions with the homotopical tautological ring. But it seems interesting that the vanishing of the $\hat{A}$-genus provides an algebraic way to distinguish the fibres over $p_1 = \pm 2$ for $R^*_{h,0}(M)$ for $\varphi : \mathbb{Q}[p_1] \to \overline{B}$, which to the author is completely unobvious from the commutative algebra perspective. In some sense, the $\hat{A}$-genus seems like a good intermediate invariant between the more geometric concept of circle actions and the algebraic theory presented here. In the remainder of this section we want to present some evidence that this is not merely a coincidence.

Consider the complete intersection in (5.15) and instead of formally adding a parameter p_1 as in the proof of Theorem 5.2.5 we now add p_1 and p_2. By abuse of notation, denote by $\overline{B}$ the polynomial ring $H^*(\mathrm{B\,hAut}_0(T\mathbb{H}P^2);\mathbb{Q})[p_1,p_2] = B[p_1,p_2]$ and by $\overline{E}$ the analogous complete intersection over $\overline{B}$ with cocycle representatives

$$p_1(T_\pi E) = p_1 z + p_{1,0} \in \overline{E} \qquad\qquad p_2(T_\pi E) = p_2 z^2 + p_{2,0} + p_{2,1} z \in \overline{E}$$

which allows us to define the Hirzebruch ideal $\bar{I}_H \subset \overline{B}$. Then the homomorphism $\varphi : \mathbb{Q}[p_1,p_2] \to \overline{B}/\bar{I}_H$ now describes a projective scheme over the affine plane whose fibre over a point $(x,y) \in \mathbb{Q}^2$ corresponds to the Hirzebruch ideal with $(p_1,p_2) = (x,y)$. The same result about the dimension of the fibre still applies, i.e. generically the dimension of the fibre vanishes. However, the exceptional fibres are now not isolated points but varieties in the affine plane.

Question. For which points in $\mathbb{A}^2$ is the dimension of the fibre of $\varphi : \mathbb{Q}[p_1,p_2] \to \overline{B}/\bar{I}_H$ positive?

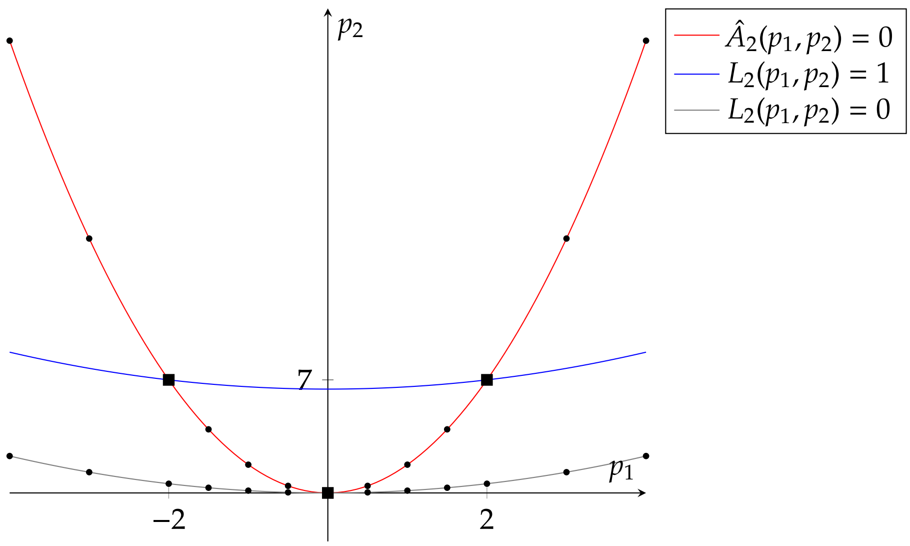
This problem is completely algebraic and detached from the topology of manifolds as we do not require the signature theorem to hold. We have computed the dimension of the fibre of $\varphi : \mathbb{Q}[p_1,p_2] \to \overline{B}/\bar{I}_H^{\leq 10}$ for several points in $\mathbb{A}^2$ using Macaulay2, and as expected "generically" the dimension of the fibre is zero.

The interesting observation is that for all points on $V(-4p_2 + 7p_1^2) \subset \mathbb{A}^2$ for which we have computed the dimension of the fibre of φ, we found that it is at least 1. We have tested this for $p_1 = i$ and for $-4 \leq i \leq 4$ and $p_1 = \pm 1/2, \pm 3/2$. For $p_1 = \pm 2, 0$ the dimension of the fibre is 3. The variety $V(-4p_2 + 7p_1^2)$ describes the values of Pontrjagin classes (p_1,p_2) for which the $\hat{A}$-genus vanishes. So even though this question is detached from the existence of circle actions on manifolds, the vanishing of the $\hat{A}$-genus of this algebraic family still seems to provide a positive lower bound on the Krull dimension of B/I_H for the corresponding value of the Pontrjagin classes. Conceptually, this is similar to the lower bounds obtained from circle actions. However, there are some cautioning remarks we should make:

(i) The computation of the dimension of the fibre of $\varphi : \mathbb{Q}[p_1,p_2] \to \overline{B}/\overline{I}_H$ does not determine the dimension of $R^*_{h,0}(M)/(I_H \cap R^*_{h,0}(M))$ – which is what we are really interested in – but merely provides an upper bound. The computation of $R^*_{h,0}(M)/(I_H \cap R^*_{h,0}(M))$ with Macaulay2 is more complicated yet in principle the same techniques are applicable.

(ii) When writing up this thesis we found by accident another family of examples for which the fibre of $\mathbb{Q}[p_1,p_2] \to \overline{B}/\overline{I}_H^{\leq 10}$ is positive, given by $L_2(p_1,p_2) = 0$.

We have summarized our computations in the following picture of $\mathbb{A}^2$ where every circle represents a point in the affine plane for which the Krull dimension of the fibre of φ : $\mathbb{Q}[p_1,p_2] \to \overline{B}/\overline{I}_H^{\leq 10}$ is 1, and every square corresponds to a point where the dimension of the fibre is 3. We have also indicated the varieties $V(L_2)$ and $V(\hat{A}_2)$, where we expect the dimension of the fibre to be positive, and for completeness also $V(L_2 - 1)$ which contains the values of p_1 and p_2 of fake quaternionic projective spaces by the signature theorem.



Based on these very limited computations and the connection to lower bounds on the Krull dimension of tautological rings, we could make the following extremely optimistic conjecture.

Conjecture 5.2.9. *Let M be a closed, oriented, smooth manifold that is rationally positively elliptic. If $\hat{A}(M) = 0$ then* $\operatorname{Kdim} R^*_{h,0}(M)/(I_H \cap R^*_{h,0}(M)) > 0$.

Clearly, there are many aspects that require further study and in particular we should test this conjecture for more examples. This point can be made for all of Chapter 3 and 4, and therefore this thesis should be considered only as a starting point in the study of tautological rings of rationally elliptic manifolds. We will end Part I with discussing some open questions and further problems in the next section.

5.3. Outlook

In this outlook we will discuss some questions that we hope to address in future work with the techniques developed in the first part of this thesis, as well as questions that naturally arise from the results that we have presented here.

5.3.1. Computational methods

One of the immediate improvements that we intend to work on is a more refined approach to Theorem 5.2.5: Instead of finding an upper bound on the Krull dimension by studying the dimension of the Hirzebruch ideal $I_H \subset H^*(\mathrm{B\,hAut}_0(TM);\mathbb{Q})$ for $M \simeq \mathbb{H}P^2$, we should study the quotient $R^*_{h,0}(M)/(I_H \cap R^*_{h,0}(M))$ directly.

The strategy in the proof of Theorem 5.2.5 can be modified to this situation. Recall the complete intersection $\overline{B} \to \overline{E}$ from the proof which is obtained by formally adding the parameter p_1. Then we can define a tautological ring $\overline{R}^* \subset \overline{B}$ by fibre integrating polynomials of the fibrewise Euler class and $p_i(T_\pi E)$ in (5.16). A better upper bound to the tautological ring of a manifold $M \simeq \mathbb{H}P^2$ is the fibre of $\phi : \mathbb{Q}[p_1] \to \overline{R}^*/(\bar{I}_H \cap \overline{R}^*)$ over the maximal ideal $(p_1 - p_1(M))$. And we can apply the same theorem about common properties of the fibres of ϕ as in the proof of Theorem 5.2.5.

Question 5.3.1. Most fibres of $\phi : \mathbb{Q}[p_1] \to \overline{R}^*/(\bar{I}_H \cap \overline{R}^*)$ share common properties – are there interesting properties beyond the dimension that they share? Are the fibres ever isomorphic?

For both families, $\varphi : \mathbb{Q}[p_1] \to \overline{B}/\bar{I}_H$ and $\phi : \mathbb{Q}[p_1] \to \overline{R}^*/(\bar{I}_H \cap \overline{R}^*)$, there are algorithms to find the exceptional fibres. Roughly, one computes the fibre over the generic point of $\mathrm{Spec}(\mathbb{Q}[p_1])$, which corresponds to computing a Groebner basis of I_H over the field of fractions $\mathbb{Q}(p_1)$ and keeping track of every time one has to divide by a polynomial in $\mathbb{Q}[p_1]$. Such algorithms are described for example [MW10] and they are also implemented in different computer algebra systems. This is particularly interesting as a first test for Conjecture 5.2.9, which we should also test for many more examples.

Goal. Improve the computational methods in this thesis.

5.3.2. Complete intersections

Clearly, the computational results presented in the first part of this thesis are essentially concerned with complete intersections over $\mathbb{Q}$ and many of the statements are of independent interest. We can formulate all the constructions in purely algebraic terms as follows.

Let Λ denote a pure minimal Sullivan model of a connected complete intersection over $\mathbb{Q}$ of formal dimension $2n$. Consider the dg Lie algebra $(\mathrm{Der}^+(\Lambda) \ltimes \Lambda \otimes \Pi'\langle 0\rangle)^\tau$, where $\Pi' = \mathbb{Q}\{q_1, \dots, q_{n-1}\}$ is a trivial dg Lie algebra with $|q_i| = 4i - 1$ and τ is a Maurer-Cartan element in $\Lambda\hat{\otimes}\Pi'$ determined by cohomology classes $p_i(\Lambda) \in H^{4i}(\Lambda)$ for $i = 1, \dots, n-1$. This dg Lie algebra acts on Λ and we define $\mathrm{B}(\Lambda)$ and $\mathrm{E}(\Lambda)$ as the source and target of

$$H^*_{CE}((\mathrm{Der}^+(\Lambda) \ltimes \Lambda \otimes \Pi'\langle 0\rangle)^\tau; \mathbb{Q}) \longrightarrow H^*_{CE}((\mathrm{Der}^+(\Lambda) \ltimes \Lambda \otimes \Pi'\langle 0\rangle)^\tau; \Lambda). \tag{5.17}$$

The reason that we don't record τ in this notation is that if Λ satisfies the Halperin conjecture, the cohomology groups above don't depend on τ. However, there are classes $p_1, \dots, p_{n-1} \in \mathrm{E}(\Lambda)$ defined as in Theorem 5.1.9 and these do depend on τ. We define a tautological ring $R^*(\Lambda, \tau) \subset B^*(\Lambda)$ by fibre integrating polynomials in the fibrewise Euler class and these p_i. Similarly, we define the Hirzebruch ideal $I_H(\Lambda, \tau) \subset \mathrm{B}(\Lambda)$ as the ideal generated by the fibre integrals of the L-polynomials (where we replace p_n by e^2).

Remark 5.3.2. Let X be a simply connected space with Sullivan model Λ. Then the dg Lie algebra $(\mathrm{Der}^+(\Lambda) \ltimes \Lambda \otimes \Pi'\langle 0\rangle)^\tau$ is model of $\mathrm{B}(\prod_{i=1}^{n-1} \mathrm{Map}(X, K(\mathbb{Q}, 4i))_{p_i}, \mathrm{hAut}_0(X), *)$, which is the classifying space of X-fibrations with a choice cohomology classes of degree $4i$ for $i = 1, \dots, n-1$ on the total space whose restriction to the fibre are given by the cohomology classes corresponding to the Maurer-Cartan element τ.

A key part of the strategy to study fake quaternionic projective spaces was to treat all values of the Pontrjagin classes at the same time. This can be done in this algebraic setting analogously: choose a homogeneous basis $\{x_s\}_{s\in S}$ of $H^{4*}(\Lambda)$ where $S = \bigcup S_{4i}$ denotes the corresponding grading. Then we define $\overline{B}(\Lambda, S) := B(\Lambda)[S]$ by formally adding variables of degree zero and $\overline{E}(\Lambda, S) := E(\Lambda) \otimes_{B(\Lambda)} \overline{B}(\Lambda, S)$ the induced complete intersection over it. We define cocycles in $E(\Lambda, S)$ as $p_i = \sum_{s\in S_{4i}} s x_s + P_i$ (compare with the formulas in Theorem 5.1.9 or in the proof of Theorem 5.2.5), which enables us to define the Hirzebruch ideal $\overline{I}_H(\Lambda, S) \subset \overline{B}(\Lambda, S)$.

Question 5.3.3. The fibre of $\varphi : \mathbb{Q}[S] \to \overline{B}(\Lambda, S)/\overline{I}_H(\Lambda, S)$ over a maximal ideal corresponding to a Maurer-Cartan element τ is $B(\Lambda)/I_H(\Lambda, \tau)$. What is the dimension of the generic fibre $\varphi : \mathbb{Q}[S] \to \overline{B}(\Lambda, S)/\overline{I}_H(\Lambda, S)$? Are there examples of complete intersections where it is positive?

Clearly, this construction is quite a bit more complicated for complete intersections Λ whose cohomology ring is not as simple as that of $\mathbb{H}P^2$. It would be interesting to see whether we find similar phenomena in the study of the Hirzebruch ideal in this generality.

Finally, Conjecture 5.2.9 is based on the analogous question for complete intersections.

Question 5.3.4. Suppose Λ is of formal dimension $4n$ and $\tau \in \Lambda \hat{\otimes} \Pi'$ is a Maurer-Cartan element determined by classes $p_i(\Lambda) \in H^{4i}(\Lambda)$. If $0 = \hat{A}_n(p_1(\Lambda), \dots, p_n(\Lambda)) \in H^{4n}(\Lambda)$, then is $\operatorname{Kdim} B(\Lambda)/I_H(\Lambda, \tau)$ positive? Does this hold for $\operatorname{Kdim} R^*(\Lambda, \tau)/(I_H(\Lambda, \tau) \cap R^*(\Lambda, \tau))$ as well?

Phrasing Conjecture 5.2.9 in the required generality for arbitrary complete intersection, makes it obvious how optimistic the conjecture really is. It should rather be motivation for testing it for other complete intersections with more complicated cohomology rings.

5.3.3. Dependence of the smooth structure

It is quite surprising that the tautological rings of generic manifolds $M \simeq \mathbb{H}P^2$ have common properties at all because on the face of it, we don't expect the space of diffeomorphisms of homotopy equivalent manifolds to be similar. This begs the following underlying question.

Question 5.3.5. How does the tautological ring $R^*(M)$ depend on the smooth structure of M?

For example can we distinguish the tautological rings of homeomorphic smooth manifolds $M_1 \approx M_2$ that are not diffeomorphic? The technique presented in this chapter only depends on the rational homotopy type and the rational Pontrjagin classes (which agree rationally for homeomorphic manifolds by results of Novikov). Hence, it is obvious that we cannot hope to distinguish them using the tools developed here. And in fact, we can make a stronger statement based on the following result of Dwyer and Szczarba.

Theorem 5.3.6 ([DS83])**.** *Let $M_1 \approx M_2$ be homeomorphic smooth closed manifolds of dimension $n \neq 4$. Then the classifying spaces* $\mathrm{B}\,\mathrm{Diff}_0(M_1)$ *and* $\mathrm{B}\,\mathrm{Diff}_0(M_2)$ *have the same rational homotopy type.*

By inspection of their proof (see below), we see that the zig-zag of rational homotopy equivalences $\mathrm{B}\,\mathrm{Diff}_0(M_1) \to Z \leftarrow \mathrm{B}\,\mathrm{Diff}_0(M_2)$ constructed in [DS83] is over the classifying space of homeomorphisms $\mathrm{B}\,\mathrm{Homeo}_0(M_i)$. Since tautological classes are defined for topological bundles (see for example [ERW14]), the above zig-zag induces isomorphism between tautological rings.

Corollary 5.3.7. *Let $M_1 \approx M_2$ be homeomorphic smooth closed manifolds of dimension $n \neq 4$. Then $R_0^*(M_1) \cong R_0^*(M_2)$.*

Proof. The proof is basically an inspection of the proof of the above Theorem in [DS83]. For $\dim M_i < 4$ the statement is vacuous because low dimensional manifolds have a unique smooth structure and for $\dim M_i > 4$ the proof relies on smoothing theory. In the following, denote by M the underlying topological manifold and $\mathrm{Homeo}_0(M)$ the connected component of the identity of the group of homeormorphisms of M and by $\mathrm{Diff}'(M_i)$ the collection of components of diffeomorphisms that are topologically isotopic to the identity. By a result from smoothing theory the $\mathrm{Homeo}_0(M)$-equivariant homotopy type of the quotient $\mathrm{Homeo}_0(M)/\mathrm{Diff}'(M_i)$ is equivalent to that of the connected component Γ_i of the space of sections of the topological tangent bundle τ_M

$$\begin{tikzcd} & \mathrm{B\,Gl}(n) \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "\tau_M"'] & \mathrm{B\,Top}(n) \end{tikzcd}$$

that contains the lift corresponding to the tangent bundle of M_i (to be precise one should consider simplicial mapping spaces). Denote by $\Lambda_{\mathbb{Q}} \mathrm{B\,Gl}(n) \to \mathrm{B\,Top}(n)$ the fibrewise rationalization (see [BK72]) and by $\Gamma^{\mathbb{Q}}$ the space of sections of the topological tangent bundle to $\Lambda_{\mathbb{Q}} \mathrm{B\,Gl}(n)$. Then Dwyer and Szczarba show that Γ_1 and Γ_2 map to the same path component $\Gamma_c^{\mathbb{Q}} \subset \Gamma^{\mathbb{Q}}$ and that the inclusions are rational equivalences (spaces of sections such as Γ_i and $\Gamma_c^{\mathbb{Q}}$ are nilpotent so this statement is meaningful) which are equivariant with respect to the action of $\mathrm{Homeo}_0(M)$. Hence, there is a zig-zag

$$\mathrm{B\,Diff}'(M_1) \simeq \Gamma_1 /\!\!/ \mathrm{Homeo}_0(M) \xrightarrow{\simeq_{\mathbb{Q}}} \Gamma_c^{\mathbb{Q}} /\!\!/ \mathrm{Homeo}_0(M) \xleftarrow{\simeq_{\mathbb{Q}}} \Gamma_2 /\!\!/ \mathrm{Homeo}_0(M) \simeq \mathrm{B\,Diff}'(M_2)$$

of spaces over $\mathrm{B\,Homeo}_0(M)$ via the obvious map. Since MMM-classes are defined in $\mathrm{B\,Homeo}_0(M)$, the above zig-zag induces an isomorphism of tautological rings defined over $\mathrm{B\,Diff}'(M_i)$. The claim follows by considering the corresponding zig-zag of universal covers which are given by $\mathrm{B\,Diff}_0(M_i)$. □

Remark 5.3.8.

(i) One interesting consequence is that there are lower bounds on the Krull dimension on the tautological ring obtained by continuous torus actions that satisfy the conditions of Theorem 5.2.7 with respect to some smooth structure. For example, it follows that $2 \leq \mathrm{Kdim}\, R^*(\mathbb{H}P^2 \# \Sigma^8) \leq 3$.

(ii) A weaker result holds for the tautological rings $R^*(M)$ and $R^*(M\#\Sigma)$ for simply-connected manifolds M^{2n} and some exotic sphere Σ by work of [Kra18] which implies that $R^*(M) \cong R^*(M\#\Sigma)$ as vector spaces in a range of degrees that depends on the manifold M.

Question 5.3.9. Is the tautological ring a homeomorphism invariant? Are there relations that hold in $R^*(M_1)$ but not in $R^*(M_2)$ for homeomorphic manifolds $M_1 \approx M_2$?

Coming back to our result about the tautological ring of $M \simeq \mathbb{H}P^2$, we note that we don't actually infer much information about the cohomology of $\mathrm{B}\,\mathrm{Diff}^+(M)$ itself beyond the fact the most MMM-classes are zero or more precisely that the image of $H^*(\mathrm{B}\,\mathrm{hAut}^+(TM);\mathbb{Q})$ is a finite dimensional vector space. From this point of view, our Theorem 5.2.5 is similar in spirit to results in [HLLR17] and [BFJ16] where it is shown that for bundles with fibres certain aspherical manifolds the MMM-classes are trivial.

These results are in a certain tension with the description of the stable cohomology of $\mathrm{B}\,\mathrm{Diff}^+(M^{2n}\#^g S^n \times S^n)$ by Galatius–Randal-Williams which is inherently given by tautological classes. Of course, the homological stability results have no consequences for the large scale ring theoretic structure of the the tautological ring. But maybe this discrepancy can also be attributed to the fact that the manifolds we happen to study (and create under stabilization with $\#^g S^n \times S^n$) usually have many symmetries: We naturally consider $\mathbb{H}P^2$ and not the element in the structure set $S(\mathbb{H}P^2)$ with $p_1(M) = -6718z \in H^4(M;\mathbb{Z})$. This leads to the following question.

Question 5.3.10. Are there manifolds M that do not admit smooth effective actions of compact connected Lie groups (with respect to any smooth structure) but for which the Krull dimension of $R^*(M)$ is positive?

Finally, it would be very interesting to find a source of relations in the tautological ring that requires the manifold structure in the fibre beyond the family signature theorem – either for a specific manifold M or for arbitrary manifolds. For example, it would be interesting if we could distinguish the smooth tautological rings of manifolds in $S(\mathbb{H}P^2)$ beyond $\mathbb{H}P^2$.

5.3.4. Block bundles and geometric conditions on the fibre

There are many more situations where one can define versions of the tautological ring. For example, we can study fibre bundles with fibrewise curvature conditions or where the structure group is a subgroup of $\mathrm{Diff}(M)$ such as the group of symplectomorphisms $\mathrm{Symp}(M,\omega)$ for a symplectic manifold (M,ω). The symplectic case is particularly interesting because it is essentially a cohomological condition that seems well suited to study with tautological classes.

Question 5.3.11. Denote by $R^*_{\mathrm{Symp}}(M, \omega)$ the ring of tautological classes of the universal bundle over $\mathrm{B}\,\mathrm{Symp}(M, \omega)$. Is there a universal source of relations among MMM-classes for the symplectic tautological ring $R^*_{\mathrm{symp}}(M, \omega)$ that uses the symplectic structure?

There is some evidence that this is the case. This relies on deep theorems of Gromov that describe the connected component of the identity $\mathrm{Symp}_0(M, \omega)$ for some 4-dimensional symplectic manifolds such as $(S^2 \times S^2, \omega)$, where ω is the symplectic form so that both $p \times S^2$ and $S^2 \times p$ have area 1, and $(\mathbb{C}P^2, \omega)$ with symplectic form induced by the Fubini-Study Kähler metric. According to [KM05], there are deformation retractions

$$\mathrm{SO}(3) \times \mathrm{SO}(3) \xhookrightarrow{\simeq} \mathrm{Symp}_0(S^2 \times S^2, \omega)$$
$$\mathrm{PSU}(3) \xhookrightarrow{\simeq} \mathrm{Symp}_0(\mathbb{C}P^2, \omega)$$

and so we can compare the tautological rings $R^*_0(M)$ and $R^*_{\mathrm{symp},0}(M)$ in these cases.

Example 5.3.12.

(i) For $S^2 \times S^2$ we can show that $R^*_0(S^2 \times S^2) \to R^*_{\mathrm{symp},0}(S^2 \times S^2, \omega)$ has a non-trivial kernel using a family of torus actions $\phi_k : T^2 \to \mathrm{Diff}_0(S^2 \times S^2)$ for $k \in \mathbb{N}$. The image of the MMM-classes in $H^*(\mathrm{B}\,T^2; \mathbb{Q})$ under ϕ_k^* is computed in [RW19] and for $k = 0$ this action is symplectic with respect to the standard form. It follows from the formulas of the MMM-classes in $H^*(\mathrm{B}\,\mathrm{SO}(3) \times \mathrm{B}\,\mathrm{SO}(3); \mathbb{Q}) \hookrightarrow H^*(\mathrm{B}\,T^2; \mathbb{Q})$ discussed in [RW19, pg. 3871] that $0 = \kappa^2_{e^3 p_1} - \kappa_{ep_1^2}\kappa_{e^5} \in R^*_{\mathrm{symp},0}(S^2 \times S^2, \omega)$. On the other hand, the image of $\kappa^2_{e^3 p_1} - \kappa_{ep_1^2}\kappa_{e^5}$ in $H^*(\mathrm{B}\,T^2; \mathbb{Q})$ with respect to the action ϕ_1 is non-trivial. Hence, it is an element in the kernel.

In fact, Randal-Williams uses these torus actions to show that $\mathrm{Kdim}\, R^*(S^2 \times S^2) \geq 3$. Since the symplectic tautological ring is contained in $H^*(\mathrm{B}\,\mathrm{SO}(3) \times \mathrm{B}\,\mathrm{SO}(3); \mathbb{Q})$ which has Krull dimension 2, it follows that there has to be a kernel. So the above is simply an explicit description of one element in the kernel.

(ii) The tautological ring of the $\mathbb{C}\mathrm{P}^2$-bundle over $\mathrm{B}\,\mathrm{SU}(3)$ is isomorphic to that over $\mathrm{B}\,\mathrm{PSU}(3)$ as $\mathrm{SU}(3) \to \mathrm{PSU}(3)$ is a finite covering. Let $\gamma_3 \to \mathrm{B}\,\mathrm{SU}(3)$ be the universal bundle, then the cohomology of the $\mathbb{C}P^2$-bundle $\pi : \mathbb{P}(\gamma_3) \to \mathrm{B}\,\mathrm{SU}(3)$ is described by the projective bundle formula $H^*(\mathbb{P}(\gamma_3); \mathbb{Q}) = H^*(\mathrm{B}\,\mathrm{SU}(3); \mathbb{Q})[x]/(x^3 + c_2 x - c_3)$ where $x = c_1(L)$ of the canonical line bundle $L \to \mathbb{P}(\gamma_3)$ and c_2, c_3 are the Chern classes in $H^*(\mathrm{B\,SU}(3);\mathbb{Q}) = \mathbb{Q}[c_2, c_3]$ of γ_3. The vertical tangent bundle is determined by $T_\pi E \oplus \mathbb{C} \cong \overline{L} \otimes \pi^*\gamma_3$ and by using the splitting principle we can compute that $c_1(T_\pi E) = -3x$ and $c_2(T_\pi E) = 3x^2 + c_2$. Observe that $c_2(T_\pi E) = e(T_\pi E)$ and so this also follows from Theorem 4.1.10. Since $p_1(T_\pi E) = c_1^2(T_\pi E) - 2c_2(T_\pi E)$, one can compute with Macaulay2 that

$$R^*_{\mathrm{symp},0}(\mathbb{C}P^2, \omega) \cong \frac{\mathbb{Q}[\kappa_{e^2}, \kappa_{ep_1}, \kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{p_1^4}]}{(7\kappa_{ep_1} - 4\kappa_{p_1^2}, 7\kappa_{e^2} - \kappa_{p_1^2}, 13\kappa^2_{p_1^2} - 49\kappa_{p_1^3})}.$$

This agrees with the lower bound computed in [RW19] as the maximal torus $T^2 \to \mathrm{PSU}(3)$ corresponds to the action studied [RW19]. Finally, we compute

$$\ker\left(R^*_{h,0}(\mathbb{C}P^2)/(I_H \cap R^*_{h,0}(\mathbb{C}P^2)) \to R^*_{\mathrm{symp},0}(\mathbb{C}P^2, \omega)\right) = (7\kappa_{ep_1} - 4\kappa_{p_1^2}, 13\kappa^2_{p_1^2} - 49\kappa_{p_1^3}),$$

but we don't know if these elements are non-trivial in $R^*_0(\mathbb{C}P^2)$.

Obviously, this relies on knowing the homotopy type of $\mathrm{Symp}_0(M, \omega)$. But it would be interesting if there is there is a source of relations that uses the symplectic structure which accounts for the difference.

Another situation in which tautological rings have been defined are (smooth) block bundles that we have introduced in the introduction as the analogues of fibre bundles over simplicial complexes $E \to |K|$, i.e. over each simplex $\sigma \subset K$ there is a trivialization $E|_\sigma \to \sigma \times M$ which preserves the face structure $E|_\tau \to \tau \times M$ for each face $\tau \subset \sigma$ but which does not commutes with the projection π. Recently, Ebert and Randal-Williams have defined generalization of MMM-classes for block bundles [ERW14] by constructing a stable vertical tangent bundle, and defined a tautological ring $\widetilde{R}^*(M) \subset H^*(\mathrm{B}\,\widetilde{\mathrm{Diff}}^+(M); \mathbb{Q})$. Since the space of block diffeomorphisms sits between diffeomorphisms and homotopy automorphisms, there is a factorization of tautological rings $E^*(M) \to \widetilde{R}^*(M) \to R^*(M)$ and we can ask whether the elements in the kernel described in Proposition 5.2.3 and Proposition 5.2.4 are in the kernel of $E^*(M) \to \widetilde{R}^*(M)$.

Question 5.3.13. What is the kernel $E^*(M) \to \widetilde{R}^*(M)$? How does it compare to the kernel $E^*(M) \to R^*(M)$?

We suspect that the Euler ring $E^*(M)$ injects into $\widetilde{R}^*(M)$ for $M = S^2 \times S^2$ or $M = \mathbb{C}P^2 \# \mathbb{C}P^2$, even though the total space of a block bundle over a smooth manifold is a smooth manifold and so the family signature theorem holds. However, the vertical tangent bundle for block bundle is stable and therefore the square of the fibrewise Euler class is not identified with the corresponding Pontrjagin class (see [RW17, Prop. 3.1]). Furthermore, by analysing the space level surgery exact sequence

$$\underline{S}^{G/O}(M) \longrightarrow \mathrm{Map}(M, G/O) \longrightarrow \mathbb{L}(M)$$

for $M = S^2 \times S^2$ and $M = \mathbb{C}P^2 \# \mathbb{C}P^2$ we see that the structure space has rational homotopy groups concentrated in even degrees. Since the connected of the base point of $\underline{S}^{G/O}(M)$ is equivalent to $\mathrm{hAut}_0(M)/\widetilde{\mathrm{Diff}}_0(M)$, it follows that the Serre spectral sequence of $\mathrm{hAut}_0(M)/\widetilde{\mathrm{Diff}}_0(M) \to \mathrm{B}\,\widetilde{\mathrm{Diff}}_0(M) \to \mathrm{B}\,\mathrm{hAut}_0(M)$ collapses at the E_2-page. Hence, the cohomology of the base injects and there cannot be a kernel for $E_0^*(M) \to \widetilde{R}_0^*(M)$. However, there is an issue as the surgery exact sequence is only a fibration for high-dimensional manifolds. In the above case, the argument might still work because one can do surgery for simply-connected 4-manifolds but a more careful analysis is required (or for some fake quaternionic projective space instead).

Another approach is to study the rational homotopy type of $\mathrm{B}\,\widetilde{\mathrm{Diff}}_0(M)$ so that can simply compute $\widetilde{R}^*(M)$. This has been initiated by Berglund and Madsen who have described the rational homotopy type of $\mathrm{B}\,\widetilde{\mathrm{Diff}}_{\partial,0}(M)$ of highly-connected $2n$-manifolds whose boundary is a sphere [BM13, BM20]

Question 5.3.14. What is the adaptation of the rational model $\mathrm{B}\,\widetilde{\mathrm{Diff}}_0(M)$ in [BM13, BM20] to closed manifolds M and specifically rationally elliptic manifolds.

Part II.

Tautological classes and self-embedding calculus

Chapter 6.

Embedding calculus and configuration space integrals

In the second part of this thesis we will study the space of diffeomorphisms of a smooth closed manifold M considered as the space of self-embeddings using the calculus of embeddings, which provides a homotopy theoretic approximation to $\mathrm{Diff}(M)$. There has been a lot of progress recently in the study of the rational homotopy type of these homotopy theoretic models based on the connection with the homotopy theory of modules over operads. At the heart of this progress lies the geometric notion of configuration space integrals, which originated as invariants of links and knots and has been essential in Kontsevich's proof of the formality of the little disk operad.

In this chapter, we briefly review some of the basic ideas and constructions of embedding calculus and an example of how configuration space integrals can be used to study diffeomorphism groups. These are two separate topics yet it is commonly believed that they are closely related and we will comment on the conjectured connection at the end of the chapter.

6.1. Embedding calculus

The material discussed in this section is based on [Wei96, Wei99, GW99]. We will not attempt to describe the historical development of embedding calculus, but the following reformulation of Smale-Hirsch immersion theory serves as a good motivation. Let M^m and N^n and be smooth manifolds and $\mathrm{Imm}(M,N)$ the space of smooth immersions and

$$\mathrm{Mono}(TM,TN) = \left\{ \begin{array}{ccc} TM & \xhookrightarrow{\bar{f}} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array} \middle| \, \bar{f} \text{ fibrewise linear injection} \right\}$$

the space of vector bundle monomorphisms. The main result of immersion theory is that the natural map $\mathrm{Imm}(M,N) \to \mathrm{Mono}(TM,TN)$ that sends an immersion to its differential is a homotopy equivalence if the codimension $\dim(N) - \dim(M)$ is positive [Sma59, Hir59, HP64].

We can reformulate this theorem by considering $\mathrm{Imm}(-,N)$ as a contravariant functor from the poset $\mathcal{O}(M)$ of open subsets of M ordered by inclusion to Top, the category of compactly generated weak Hausdorff spaces. In fact, this defines a sheaf $\mathrm{Imm}(-,N) : \mathcal{O}(M)^{\mathrm{op}} \to \mathsf{Top}$, and the main result of immersion theory is equivalent to the following pullback square

$$\begin{tikzcd} \mathrm{Imm}(V_1 \cup V_2, N) \arrow[r, "|_{V_1}"] \arrow[d, "|_{V_2}"] & \mathrm{Imm}(V_1, N) \arrow[d] \\ \mathrm{Imm}(V_2, N) \arrow[r] & \mathrm{Imm}(V_1 \cap V_2, N) \end{tikzcd}$$

being a homotopy pullback square, i.e. the map

$$\mathrm{Imm}(V_1 \cup V_2, N) \longrightarrow \mathrm{holim}(\mathrm{Imm}(V_2, N) \to \mathrm{Imm}(V_1 \cap V_2, N)) \leftarrow \mathrm{Imm}(V_1, N))$$

is a weak homotopy equivalence. A functor with this property is called *1-excisive* and this property implies that the value of this functor is determined locally, i.e. we could compute the homotopy groups of $\mathrm{Imm}(M, N)$ by picking an open cover of M and using squares as above which have a Mayer-Vietoris sequence for the homotopy groups.

We can recover the main result of immersion theory by observing that $\mathrm{Mono}(-, TN) : \mathcal{O}(M)^{\mathrm{op}} \to \mathsf{Top}$ is a 1-excisive sheaf as well (this is a lot easier to check) so that it suffices to prove that the map of sheaves $\mathrm{Imm}(-,N) \to \mathrm{Mono}(-, TN)$ is a homotopy equivalence for open subsets $V \subset M$ that are either empty or diffeomorphic to $\mathbb{R}^m$, which again is a lot easier to check.

This can be generalized to the notion of k-excisive functors $F : \mathcal{O}^{\mathrm{op}}(M) \to \mathsf{Top}$ which in this context corresponds to functors whose value is determined multi-locally, i.e. by the value of F on tubular neighbourhoods of not more than k points in M. One can construct for any functor $F : \mathcal{O}(M)^{\mathrm{op}} \to \mathsf{Top}$ that satisfies the following conditions:

1.) for open sets $U \subset V \in \mathcal{O}(M)$ with embedding $j : V \to U$ such that both $j : V \to U \subset V$ and $j|_U$ are isotopic to the identity, the map $F(V) \xrightarrow{\simeq} F(U)$ is a homotopy equivalence;

2.) for every sequence of open sets $V_0 \subset V_1 \subset \ldots$ of open sets in $\mathcal{O}(M)$ the canonical map $F(\cup_i V_i) \longrightarrow \mathrm{holim}_i\, F(V_i)$ is a weak homotopy equivalence;

the *best approximation from the right*

$$T_k F : \mathcal{O}(M)^{\mathrm{op}} \to \mathsf{Top}$$

which is k-excisive, i.e. there is a natural transformation $\eta_k : F \to T_k F$ such that $\eta_k(V) : F(V) \to T_k F(V)$ is a homotopy equivalence for all open subsets $V \subset M$ diffeomorphic to $\coprod_l \mathbb{R}^m$ for $l \leq k$. The functors $T_k F$ are called the k-th Taylor approximation of F and assemble into a the so-called *Taylor tower* via a sequence of natural transformations $r_k : T_kF \to T_{k-1}F$

$$\begin{tikzcd} F \arrow[d, "\eta_1"] \arrow[dr, "\eta_2"] \arrow[drr, bend left, "\eta_3"] & & & \\ T_1F & T_2F \arrow[l, "r_2"] & T_3F \arrow[l, "r_3"] & \dots \arrow[l] \end{tikzcd} \tag{6.1}$$

If we now come back to the embedding functor $\mathrm{Emb}(-,N) : \mathcal{O}(M)^{\mathrm{op}} \to \mathsf{Top}$, it is in general not k-excisive for any $k \in \mathbb{N}$. However, if $\dim(N) - \dim(M) \geq 3$ then the natural transformations $\eta_k(M) : \mathrm{Emb}(M,N) \to T_k \,\mathrm{Emb}(M,N)$ are $(k(n-m-2)+1-m)$-connected [Wei96, Thm 4.4], which implies that the limit

$$\eta_\infty : \mathrm{Emb}(M,N) \xrightarrow{\simeq} \mathrm{holim}_k\, T_k \,\mathrm{Emb}(M,N)$$

is a weak homotopy equivalence.

This is a groundbreaking result in geometric topology and has been essential in studying the homotopy type of embedding spaces (see for example [ALV07, AT14, FTW17]) as the construction of the Taylor tower is homotopy theoretic and thus more approachable. The proof depends on deep theorems of Goodwillie and Klein and we will not discuss them here as they are not applicable for self-embeddings.

There are many different, more modern constructions of the Taylor approximations, for example in [BdBW13] where they omit reference to the fixed manifold M, or in terms of operads and modules for example in [Tur13, BdBW13]. These have some technical advantages. We will discuss a model for the Taylor approximations $T_k \,\mathrm{Emb}(M,N)$ in the next section in terms of equivariant function spaces, which we will use in the next chapter.

6.1.1. The Haefliger model

This description of the embedding calculus tower in terms of equivariant function spaces is due to Goodwillie, Klein and Weiss [GKW03]. For M and N smooth closed manifolds, the model for $T_2 \,\mathrm{Emb}(M,N)$ recovers an approximation to the space of embeddings that was originally due to Haefliger [Hae63] which motivates the name. We will begin by discussing the first two stages of the tower before giving the description for $k \geq 3$.

The value of an embedding determined at a point is given by its differential and thus the first Taylor approximation is equivalent to $\mathrm{Mono}(TM, TN)$ which we essentially take as definition

$$T_1 \,\mathrm{Emb}(M,N) := \mathrm{Mono}(TM, TN). \tag{6.2}$$

For the second stage, consider the space $\mathrm{Map}^{S_2}(M^2, N^2)$ of smooth[1] S_2-equivariant maps where the symmetric group acts by permuting the factors. Denote by $\mathrm{IvMap}(M^2, N^2)$ the subspace of *strongly isovariant* maps, i.e. S_2-equivariant maps that satisfy the tangential isovariance condition $(DF)^{-1}(T\Delta_N) = T\Delta_M$. Observe in particular, that this condition for the zero vectors implies that $F^{-1}(\Delta_N) = \Delta_M$, i.e. F preserves all isotropy groups and hence is isovariant. Then the second Taylor approximation $T_2 \operatorname{Emb}(M, N)$ is defined as the homotopy limit of the following diagram

$$\begin{tikzcd} & \mathrm{IvMap}(M^2, N^2) \arrow[d] \\ \mathrm{Map}(M, N) \arrow[r, "f\mapsto f\times f"] & \mathrm{Map}^{S_2}(M^2, N^2). \end{tikzcd} \tag{6.3}$$

An explicit model for $\mathrm{holim}(A \to B \leftarrow C)$ is given by paths $\beta \in B^I$ such that $\beta(0) \in A$ and $\beta(1) \in C$, which is the literal pullback if we turn any of the two maps into a fibration. Consequently, we can choose the following specific model

$$T_2 \operatorname{Emb}(M, N) := \left\{ (f, H, G) \in \mathrm{Map}(M, N) \times \mathrm{Map}^{S_2}(M^2, N^2)^I \times \mathrm{IvMap}(M^2, N^2) \;\middle|\; \begin{matrix} H(0) = f \times f \\ H(1) = G \end{matrix} \right\}. \tag{6.4}$$

Remark 6.1.1. Haefliger first studied embeddings using a version of the diagram (6.3) that does not require smooth maps and strongly isovariant maps. He proved that for $\dim(M) + 1 < 2\dim(N)/3$ the map $\eta_2 : \operatorname{Emb}(M, N) \to T_2 \operatorname{Emb}(M, N)$ (see (6.6)) is 1-connected [Hae63]. Haefliger's statement was later improved to show that η_2 is $(2\dim(N) - 3 - 3\dim(M))$-connected. See [GKW01, Sect. 1.2] for details and references.

Next we give explicit models from [GKW03] for the maps of the bottom part of the embedding tower (1.6)

$$\begin{tikzcd} \operatorname{Emb}(M, N) \arrow[r, "\eta_1"] \arrow[dr, "\eta_2"'] & T_1 \operatorname{Emb}(M, N) \\ & T_2 \operatorname{Emb}(M, N). \arrow[u, "r_2"'] \end{tikzcd} \tag{6.5}$$

The map η_1 sends an embedding to its differential. To define η_2, we observe that for a map $f : M \to N$ the induced map $f \times f : M^2 \to N^2$ is isovariant if and only if f is injective and strongly isovariant if f is an embedding. Hence, we can define

$$\begin{aligned} \eta_2 : \operatorname{Emb}(M, N) &\longrightarrow T_2 \operatorname{Emb}(M, N) \\ i &\longmapsto (i, \mathrm{const}_{i\times i}, i \times i). \end{aligned} \tag{6.6}$$

[1]In this section all mapping spaces we consider are spaces of smooth maps.

The restriction $r_2(f, H, G) \in T_1 \operatorname{Emb}(M, N)$ should be thought of as the (infinitesimal) restriction of the isovariant map $G : M^2 \to N^2$ to the diagonal, which induces a map between the tubular neighbourhoods. In order to make this precise, we identify the space of S_2-equivariant maps $\operatorname{Map}^{S_2}(M^2, N^2)$ with $\operatorname{Map}(M^2, N)$ as every S_2-equivariant map $G : M^2 \to N^2$ is of the form $G(m_1, m_2) = (g(m_1, m_2), g(m_2, m_1))$ for $g := \pi_1 \circ G \in \operatorname{Map}(M^2, N)$. Using this identification and notation, we can define r_2 as

$$\begin{aligned} r_2 : T_2 \operatorname{Emb}(M, N) &\longrightarrow T_1 \operatorname{Emb}(M, N) \\ (f, H, G) &\longmapsto (v_p \in T_pM \mapsto D_{(p,p)}g(v_p, -v_p) \in T_{g(p,p)}N). \end{aligned} \tag{6.7}$$

This defines a bundle monomorphism because $D_{(p,p)}g(-v_p, v_p) = 0$ implies that

$$0 = DG_{(p,p)}(v_p, -v_p) = (D_{(p,p)}g(v_p, -v_p), D_{(p,p)}g(-v_p, v_p)) \in T_{(p,p)}\Delta$$

which in turn implies that $(v_p, -v_p) \in T\Delta_M$ which can only happen if $v_p = 0$.

Lemma 6.1.2. *The diagram* (6.5) *commutes. Moreover, for self-embeddings of a closed manifold M it is a diagram of topological monoids.*

Proof. Let $i : M \to N$ be an embedding, then $\eta_2(i) = (i, \mathrm{const}_{i\times i}, i \times i)$ and thus $r_2(\eta_2(i))(v_p) = D_p i(v_p) = r_1(i)(v_p)$ showing that (6.5) commutes.

For $M = N$ the space of embeddings coincides with $\operatorname{Diff}(M)$ and $T_1 \operatorname{Emb}(M, M)$ coincides with the space of bundle maps of the tangent bundle TM, both of which are monoids under composition. The second stage $T_2 \operatorname{Emb}(M, M)$ is a monoid under pointwise composition of the S_2-equivariant homotopy, i.e. for $(f, H, G), (f', H', G') \in T_2 \operatorname{Emb}(M, M)$ there is a composition

$$(f, H, G) \cdot (f', H', G') := (f \circ f', H \circ H', G \circ G') \in T_2 \operatorname{Emb}(M, M)$$

where $(H \circ H')_t := H_t \circ H'_t \in \operatorname{Map}^{S_2}(M^2, M^2)$ is the usual composition. The maps in the diagram are compatible with composition by the chain rule. $\square$

The Taylor approximation for $k \geq 3$ can be expressed similarly as a homotopy limit of certain mapping spaces. Let $R \subset S$ be finite sets, define the space of *admissible maps* $\operatorname{aMap}(M^S, N^R)$ to be the subspace of smooth maps $f : M^S \to N^R$ satisfying $(Df)^{-1}(TN^{R/\rho}) = TM^{S/\rho}$ for every equivalence relations ρ on R. Here, R/ρ is the quotient set and we consider the corresponding diagonal $N^{R/\rho} \subset N^R$ as a subspace, and similarly for the induced equivalence relation on S.

There are maps between the spaces of admissible spaces induced by pre- and post-composition with projections of the products encoded by inclusions of the finite sets, i.e. for $R_1 \supset R_2$ and $S_1 \subset S_2$ there are maps

$$\begin{aligned} \mathrm{aMap}(M^{S_1}, N^{R_1}) &\longrightarrow \mathrm{aMap}(M^{S_2}, N^{R_2}). \\ f &\longmapsto \pi_{R_2} \circ f \circ \pi_{S_1} \end{aligned}$$

If we fix an ambient set $[k] = \{1, \dots, k\}$, we can can encode the spaces of admissible maps and maps between them as a functor from a poset P_k whose elements are pairs (R, S) with objects pairs of non-empty sets $R \subset S \subset [k]$ and partial order $(R_1, S_1) \leq (R_2, S_2)$ if $R_2 \subset R_1$ and $S_1 \subset S_2$.

Proposition 6.1.3 ([GKW03])**.** *The Taylor approximation $T_k \operatorname{Emb}(M, N)$ for $k \geq 2$ can be described as*

$$T_k \operatorname{Emb}(M, N) := \left(\operatorname*{holim}_{(R,S) \in P_k} \mathrm{aMap}(M^S, N^R) \right)^{S_k}. \tag{6.8}$$

Here, the symmetric group S_k acts as on the category P_k and with natural isomorphisms $\eta_g : \mathrm{aMap} \Rightarrow \mathrm{aMap} \circ g$ for every $g \in S_k$ such that $\eta_{g \cdot g'} = \eta_{g'} \circ \eta_g$. Hence, it determines an action on the homotopy limit. Goodwillie, Klein and Weiss also give a simplified description $T_k \operatorname{Emb}(M, N)$ by choosing a specific model of homotopy limits and computing the invariants.

Proposition 6.1.4 ([GKW03, Prop. 2.7])**.** *The definition of $T_k \operatorname{Emb}(M, N)$ in* (6.8) *is homeomorphic to the totatlization of the following incomplete cosimiplicial space*

$$[i] \longmapsto \prod_{1 \leq k_0 < k_1 < \ldots k_i \leq k} \mathrm{aMap}(M^{k_i}, N^{k_0})^{st(k_0, \ldots, k_i)}, \tag{6.9}$$

where $st(k_0, \dots, k_i) := S_{k_0} \times S_{k_1 - k_0} \times \ldots \times S_{k_i - k_{i-1}}$ acts on M^{k_i} via the inclusion $st(k_0, \dots, k_i) \subset S_{k_i}$ and on N^{k_0} through the projection onto the first factor.

The face maps d_j on the space of i-cosimplices for $0 \leq j \leq i+1$ of the incomplete cosimplicial space above are given by

$$\begin{aligned} d_j : \prod_{1 \leq k_0 < \ldots k_i \leq k} \mathrm{aMap}(M^{k_i}, N^{k_0})^{\mathrm{st}(k_0, \ldots, k_i)} &\longrightarrow \prod_{1 \leq k_0 < \ldots k_{i+1} \leq k} \mathrm{aMap}(M^{k_{i+1}}, N^{k_0})^{\mathrm{st}(k_0, \ldots, k_{i+1})} \\ (x)_{k_0 < \ldots < k_i} &\longmapsto (d_i x)_{k_0 < \ldots < k_{i+1}} := x_{k_0 < \ldots < \hat{k}_j < \ldots < k_{i+1}}. \end{aligned}$$

Remark 6.1.5. The topological group $\operatorname{Diff}(M)$ acts continuously on the incomplete cosimplicial space in (6.9) by precomposition and hence acts on the totalization $T_k \operatorname{Emb}(M, N)$. This was one of the motivations for the construction of the Haefliger style model [BdBW13, pg. 1]. Another practical advantage of the Haefliger model is that it is quite small in comparison with other models of $T_k \operatorname{Emb}(M, N)$.

6.1.2. On delooping the self-embedding tower

Finally, the focus of this work is not on the group $\mathrm{Diff}(M)$ but its classifying space. Therefore, we are interested in a delooping of the Taylor tower. It follows from the description in [BdBW13] of $T_k \operatorname{Emb}(M,M)$ as a derived endomorphism space that it is a monoid under composition. However, a delooping of the tower with its restriction maps is more complicated in this model although possible in general and we refer to [KR19, Sect. 3.2] for the constructions.[2]

Fixing a model of the self-embedding tower that we can deloop, a final precursory remark involves the set of path components $\pi_0(T_k \operatorname{Emb}(M,M))$ of the self-embedding Taylor, which forms a monoid under composition but is in general not a group. Denote by $T_k^\times \operatorname{Emb}(M,M)$ the union of the *homotopy invertible path components*, i.e. path components that have an inverse up to homotopy under composition. Then η_k has image in $T_k^\times \operatorname{Emb}(M,M)$ because $\operatorname{Emb}(M,M)$ is a group for a closed manifold M, and it suffices to focus on $T_k^\times \operatorname{Emb}(M,M)$ instead. For example, the homotopy invertible elements $T_1^\times \operatorname{Emb}(M,M)$ in the Haefliger model are tangential homotopy equivalences $\mathrm{hAut}(TM)$. We want to further restrict the connected components when M is an oriented manifold.

Definition 6.1.6. Let M be a closed oriented manifold. Denote by $T_k^\times(M) \subset T_k \operatorname{Emb}(M,M)$ the union of path components that are homotopy invertible under composition and whose image in $T_1 \operatorname{Emb}(M,M)$ under $r_2 \circ \ldots \circ r_k$ is in $\mathrm{hAut}^+(TM)$.

It will be convenient to add a 0th stage to the tower by defining $T_0 \operatorname{Emb}(M,M) := \operatorname{Map}(M,M)$ with $r_1 : T_1 \operatorname{Emb}(M,M) \to T_0 \operatorname{Emb}(M,M)$ given in the Haefliger model by forgetting the vector bundle map. If M is oriented then $T_0^\times(M) := \mathrm{hAut}^+(M)$.

With these conventions, the delooping of the self-embedding tower is given by

$$\begin{tikzcd}
& \mathrm{B}\,\mathrm{Diff}^+(M) \arrow[d, "\mathrm{B}\,\eta_\infty"] \arrow[rr, "\mathrm{B}\,\eta_2"] \arrow[rrr, bend left, "\mathrm{B}\,\eta_1"] \arrow[rrrr, bend left, "\mathrm{B}\,\eta_0"] & & & & \\
\mathrm{B}\,T_\infty^\times(M) := \operatorname{holim}\Big(\ldots \arrow[rr] & & \mathrm{B}\,T_2^\times(M) \arrow[r, "\mathrm{B}\,r_2"] & \mathrm{B}\,T_1^\times(M) \arrow[r, "\mathrm{B}\,r_1"] & \mathrm{B}\,T_0^\times(M)\Big)
\end{tikzcd} \tag{6.10}$$

which gives a sequence of approximations of the classifying space $\mathrm{B}\,\mathrm{Diff}^+(M)$. The first two stages of this approximation are spaces that we have encountered in the first part of this thesis: the classifying space of oriented M-fibrations $\mathrm{B}\,T_0^\times(M) = \mathrm{B}\,\mathrm{hAut}^+(M)$ and the classifying space of oriented TM-fibrations $\mathrm{B}\,\mathrm{hAut}^+(TM)$.

[2] In the next chapter, we will use the monoid structure for the bottom stages of the self-embedding tower discussed in Lemma 6.1.2.

Remark 6.1.7. Observe that tautological classes are already defined in $\mathrm{B}\,T_1^\times(M)$. In the first part of this thesis, we have studied relations in $R^*(M)$ enforced by relations of the homotopical tautological ring $R_h^*(M)$ in $H^*(\mathrm{B}\,T_1^\times(M))$. From this perspective, we see that this is only the first step of a hierarchy of relations in the tautological ring enforced by the higher stages of the Taylor tower, and this provides the conceptual connection between the two parts.

So far, we do not know any relations in the tautological ring $R^*(M)$ that are enforced from the higher stages of the tower. But we expect that there is a source of such relations coming from some graphical calculus and this has been one of the main motivations for the author to study the self-embedding calculus tower.

We can pull back the universal oriented M-fibration $E_0 \to \mathrm{B}\,\mathrm{hAut}^+(M) = \mathrm{B}\,T_0^\times(M)$ along compositions of the restriction maps $\mathrm{B}\,r_k$ in the tower to obtain oriented M-fibrations

$$\pi_k : E_k \longrightarrow \mathrm{B}\,T_k^\times(M) \tag{6.11}$$

that form a sequence of fibrations

$$\begin{tikzcd}
E_\infty \arrow[r] \arrow[d] & \dots \arrow[r] & E_k \arrow[r, "\bar{r}_k"] \arrow[d] & E_{k-1} \arrow[r, "\bar{r}_{k-1}"] \arrow[d] & \dots \arrow[r] & E_1 \arrow[r, "\bar{r}_1"] \arrow[d] & E_0 \arrow[d] \\
\mathrm{B}\,T_\infty^\times(M) \arrow[r] & \dots \arrow[r] & \mathrm{B}\,T_k^\times(M) \arrow[r, "\mathrm{B}\,r_k"] & \mathrm{B}\,T_{k-1}^\times(M) \arrow[r, "\mathrm{B}\,r_{k-1}"] & \dots \arrow[r] & \mathrm{B}\,T_1^\times(M) \arrow[r, "\mathrm{B}(r_1)"] & \mathrm{B}\,T_0^\times(M)
\end{tikzcd} \tag{6.12}$$

Denote by $T_\pi E_1 \to E_1$ the vector bundle over the universal TM-fibration $E_1 \to \mathrm{B}\,T_1^\times(M) = \mathrm{B}\,\mathrm{hAut}^+(TM)$ (see (5.2)), then we define $T_\pi E_k \to E_k$ as the pullback $(\bar{r}_k \circ \dots \circ \bar{r}_2)^* T_\pi E_1$.

The vertical tangent bundle of a fibre bundle is intricately linked to the global topology of the bundle. In subsequent chapters we will study if similar statements hold for the TM-fibration $T_\pi E_k \to E_k$ over $\mathrm{B}\,T_k^\times(M)$.

6.2. Configuration space integrals

We will give an idea of configuration space integrals by examining their role in Kontsevich's construction of characteristic classes of certain fibre bundles following the exposition in [Wat09a] and [Wat09b].

Let M^d be an odd dimensional integral homology sphere with basepoint $\infty \in M$ and neighbourhood $\infty \in U_\infty \cong \mathbb{R}^d$, and let $\mathrm{Diff}(M, U_\infty)$ be the group of diffeomorphisms that are the identity on U_∞. The complement $M \setminus \infty$ is parallelizable and we consider the space of orientation preserving framings

$$\mathrm{Fr}^+(M;\tau) := \left\{ \bar{f} : T(M \setminus \infty) \xrightarrow{\cong} M \times \mathbb{R}^d \,\middle|\, f|_{T(U_\infty \setminus \infty)} = \tau \right\}$$

that agree with a fixed framing $\tau : T(U_\infty \setminus \infty) \xrightarrow{\cong} (U_\infty \setminus \infty) \times \mathbb{R}^d$. Let $g \in \mathrm{Diff}(M, U_\infty)$ and $\tau_M \in \mathrm{Fr}^+(TM; \tau)$, then $\tau_M \cdot g := (g^{-1} \times \mathrm{Id}_{\mathbb{R}^d}) \circ \tau_M \circ Dg$ determines a right action, and

$$\mathrm{B}\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty; \tau) := \mathrm{Fr}^+(M; \tau) \times_{\mathrm{Diff}^+(M,U_\infty)} E\,\mathrm{Diff}^+(M, U_\infty) \tag{6.13}$$

is the classifying space of smooth M-bundles $\pi : E \to B$ with a trivial subbundle $B \times U_\infty \subset E$ determining a section $s_\infty(b) := (b, \infty)$, and a framing of the vertical tangent bundle of $E \setminus s_\infty(B) \to B$ that agrees with the fixed framing τ on $B \times (U_\infty \setminus \infty)$.

We want to construct characteristic classes of such fibre bundles $\pi : E \to B$. Consider the *fibrewise configuration space* of n particles

$$EC_n(\pi) := \left\{(e_1, \ldots, e_n) \in E^n \,\middle|\, \forall i \neq j\ \pi(e_i) = \pi(e_j),\ e_i \neq e_j,\ \text{and}\ e_i \neq s_\infty(\pi(e_i))\right\} \xrightarrow{\pi_n} B \tag{6.14}$$

which is a smooth fibre bundle over B with fibre $C_n(M \setminus \infty)$. The construction of the characteristic classes relies on fibre integrating certain differential forms over the fibrewise configuration spaces introduced above. In order to guarantee convergence of such integrals, one seeks a compactify $EC_n(\pi)$ fibrewise to obtain a smooth bundle whose fibres are manifolds with corners. If we can extend the differential forms to the compactification such that they vanish on the boundary, fibre integration is well-defined.

On a single fibre, this amounts to compactifying the configuration spaces $C_n(M \setminus \infty)$ to a manifold with corners. Such a compactification was first described by Fulton-MacPherson [FM94] for non-singular algebraic varieties using real blow-ups of the diagonals and adapted to differential geometry in [AS94]. We will only discuss the compactification for $n = 2$ and refer to [Wat09b] and [Les20, Ch. 8] for the general case.

For a smooth manifold N and submanifold M which is transverse to ∂N and satisfies $\partial M = \partial N \cap M$, there is a smooth manifold $Bl(N, M)$ with corners that is called the *blow-up* which is unique up to diffeomorphism and has smooth blowdown map $p : Bl(N, M) \to N$ satisfying

- the restriction $p|_{p^{-1}(N \setminus M)} : p^{-1}(N \setminus M) \to N \setminus M$ is a diffeomorphism,
- there is a canonical identification $p^{-1}(M) \cong S(\nu_{M \subset N}) = (TN|_M/TM)^{\#} / \mathbb{R}_{>0}$ such that p restricts to the projection $S(\nu_{M \subset N}) \to M$ under the identification (the subscript # above denotes the complement of the zero section);
- any chart adapted to $M \hookrightarrow N$ provides an embedding $S(\nu_{M \subset N}) \times [0, \infty) \to Bl(N, M)$.

See [Les20, Def. 3.1] for more details (in particular the last point). The compactification of $C_2(M \setminus \infty)$ is defined as a sequence of blowups. Consider the subspace

$$\Sigma_M := (\Delta \cup \infty \times M \cup M \times \infty) \setminus \infty \times \infty \subset M^2.$$

Then Σ_M is diffeomorphic to a submanifold of $Bl(M^2, \infty \times \infty)$ which is transverse to the boundary of $Bl(M^2, \infty \times \infty)$, so that we can define

$$\overline{C}_2(M, \infty) := Bl(Bl(M^2, \infty \times \infty), \Sigma_M) \tag{6.15}$$

which is a smooth manifold with corners. A similar procedure is applied to configuration of more particles. It follows from the properties of the blow-up above that any diffeomorphism of M fixing a neighbourhood of ∞ determines a diffeomorphism of the compactification $\overline{C}_2(M, \infty)$ and thus this can can be carried out fibrewise for the fibrewise configuration space in (6.14) to obtain a smooth fibre bundle $E\overline{C}_n(\pi) \to B$ with fibre $\overline{C}_n(M \setminus \infty)$ and interior $EC_2(\pi)$.

From now on we assume that the base is a smooth manifold so that we can introduce the differential forms on $E\overline{C}_n(\pi)$ that we fibre integrate. The basic case is for $n = 2$.

Lemma 6.2.1 ([Wat09a, Sect. 2.2]). *A framing $\tau_M \in \mathrm{Fr}^+(M; \tau)$ determines a map on the boundary $p(\tau_M) : \partial\overline{C}_2(M \setminus \infty) \to S^{d-1}$.*

The idea of the construction is that a point in the boundary corresponds to two particles colliding, say along two smooth paths $\gamma_i : I \to M$ with $\gamma_1(0) = \gamma_2(0)$, and the image $p(\tau_M)$ is the normalized difference $\tau_M(\dot\gamma_1(0) - \dot\gamma_2(0))$. Since $C_2(M \setminus \infty)$ has the same homology as S^{d-1}, it follows from Poincaré-Lefschetz duality that the cohomology class $p(\tau_M)^*(u) \in H^{d-1}(\partial\overline{C}_2(M, \infty); \mathbb{Z})$, where $u \in H^{d-1}(S^{d-1}; \mathbb{Z})$ denotes a generator, extends uniquely to a class in $H^{d-1}(\overline{C}_2(M, \infty); \mathbb{Z})$. For a bundle $E \to B$ classified by a map to $B\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty; \tau)$, this construction generalizes to give a map $p(\tau_E) : \partial E\overline{C}_2(\pi) \to S^{d-1}$ for some choice of vertical framing τ_E.

Lemma 6.2.2 ([Wat09b, Lem. 2.2]). *Let $E \to B$ be as above with vertical framing τ_E, and assume that B is a smooth manifold. Then there exists a closed form $\omega \in \Omega^{d-1}_{dR}(E\overline{C}_2(\pi))$ such that $\omega|_{\partial E\overline{C}_2(\pi)} = p(\tau_E)^* Vol_{S^{d-1}}$.*

The class ω is called the *propagator* and depends on the choice of vertical framing τ_E. Let Γ be a directed graph with vertices $\{1, \dots, n\}$, then for every edge $e \in E(\Gamma)$ there are maps $\phi_e : E\overline{C}_n(\pi) \to E\overline{C}_2(\pi)$ by forgetting all particles but the end points of e. We define

$$\omega(\Gamma) := \bigwedge_{e \in E(\Gamma)} \phi_e^* \omega \in \Omega^{|E(\Gamma)| \cdot (d-1)}_{dR}(E\overline{C}_n(\pi)) \tag{6.16}$$

and the *configuration space integral*

$$I(\Gamma) := \int_{\overline{C}_n(M, \infty)} \omega(\Gamma) \in \Omega^{|E(\Gamma)| \cdot (d-1) - n \cdot d}_{dR}(B). \tag{6.17}$$

This by itself does not determine a characteristic class of $E \to B$ because $I(\Gamma)$ need not be closed. The exterior derivative of $I(\Gamma)$ can be computed using a generalized Stokes' theorem for a fibre bundle $F \hookrightarrow E \to B$ for a manifold F with non-empty boundary or corners

$$d \int_F \alpha = \int_F d\alpha \pm \int_{\partial F} \alpha \tag{6.18}$$

for $\alpha \in \Omega^*_{dR}(E)$ (see [Wat09b, App. A.1]). Hence, $dI(\Gamma) = \int_{\partial \overline{C}_n(M,\infty)} \omega(\Gamma)|_{\partial E\overline{C}_n(\pi)}$ because ω is closed.

The key insight is that the boundary of $\overline{C}_n(M \setminus \infty)$ can be described by pieces corresponding to configuration spaces of $n-1$ particles corresponding to two of the n-particles colliding (encoded by $A \subset \{1, \dots, n\}$ of size 2). It turns out that the fibre integrals over the piece of the boundary corresponding to A agrees with $I(\Gamma')$, where Γ' is a graph with vertices $\{1, \dots, n\}/A$ induced by Γ. This means that we can encode the differential directly on the vector space spanned by all suitable graphs. This naturally leads to the concept of graph complex.

The graph complex

Let $\overline{\mathcal{G}}_{n,m}$ be the set of isomorphism classes of graphs Γ of valence at least 3, with bijections $V(\Gamma) \cong \{1, \dots, n\}$ and $E(\Gamma) \cong \{1, \dots, m\}$, and with an orientation o considered as an orientation of the vector space of $\mathbb{R}^V \oplus \mathbb{R}^{H(E)}$ where $H(E)$ is the set of half-edges $H(E) = \{e^+, e^-\}_{e \in E}$. Then $\overline{\mathcal{G}}_{n,m}$ generates a vector space over $\mathbb{Q}$ and we denote by $\mathcal{G}_{n,m}$ the quotient induced from the relation $(\Gamma, -o) \sim -(\Gamma, o)$. There is a differential $d : \mathcal{G}_{n,m} \to \mathcal{G}_{n-1,m-1}$ defined as

$$d(\Gamma, o) = \sum_{e \in E(\Gamma)} (\Gamma/e, o_e)$$

where Γ/e is the graph obtained by contracting the edge e, i.e. the graph on the quotient of V where we identified the endpoints of e and edges $E(\Gamma) \setminus e$, and o_e is the induced orientation (see [Wat09b, pg. 627] for the definition). Denote by $\mathcal{G}$ the chain complex $(\bigoplus_{n,m} \mathcal{G}_{n,m}, d)$.

Theorem 6.2.3 ([Kon94, Wat09b]). *Let M be an odd dimensional homology sphere and $\pi : E \to B$ a smooth M^d-bundle over a smooth manifold B that is classified by a map to* $\mathrm{B\,Diff}^{\mathrm{fr}}(M, U_\infty; \tau)$. *Let $\omega \in \Omega^{d-1}_{dR}(E\overline{C}_2(\pi))$ be the propagator form, then*

$$I : \mathcal{G} \longrightarrow \Omega^*_{dR}(B), \qquad I(\Gamma) := \int_{\overline{C}_{|V(\Gamma)|}(M,\infty)} \omega(\Gamma) \tag{6.19}$$

is a chain map and every class in $H(\mathcal{G})$ defines characteristic class of $\pi : E \to B$.

The main contribution of [Wat09b] is to construct examples of framed M^{2k+1}-bundles with $B = S^{2n(k-1)}$ for some $n > 1$ which have non-trivial characteristic classes in the image of I.

This has consequences for unframed $\mathrm{Diff}(M, U_\infty)$ bundles as well since the homotopy fibre of $\mathrm{B}\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty) \to \mathrm{B}\,\mathrm{Diff}(M, U_\infty)$ is equivalent to $\Omega^d\,\mathrm{SO}(d)$ so that

$$\pi_i(\mathrm{B}\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty)) \cong \pi_i(\mathrm{B}\,\mathrm{Diff}(M, U_\infty))$$

for even i and $> d-3$.

Theorem 6.2.4 ([Wat09b, Cor. 3.2])**.** *For $n \geq 2$ and $k \geq 3$ odd there is a lower bound on the rational homotopy groups* $\dim \pi_{2n(k-1)}(\mathrm{B}\,\mathrm{Diff}_\partial(D^{2k+1}) \otimes \mathbb{Q} \geq \dim \mathcal{A}_{2n,3n}$*, where $\mathcal{A}_{2n,3n}$ is a certain quotient of $\mathcal{G}_{2n,3m}$.*

The way we have presented it may obscure that $\mathcal{A}_{2n,3n}$ is a combinatorial object, and for small n the dimensions are known and positive. Hence, this proves the non-triviality of $\pi_{2n(k-1)}(\mathrm{B}\,\mathrm{Diff}_\partial(D^{2k+1}))$ which far exceeds the known results that we presented in the introduction beyond the concordance stable range.

The simplest graph homology class

The simplest graph characteristic class is associated to the θ-graph, i.e. the graph on 2 vertices and 3 edges having distinct end points that is a cycle in the graph complex. The construction of the corresponding characteristic class can be generalized to smooth M^d-bundles $\pi : E \to B$ classified by a map $B \to \mathrm{B}\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty; \tau)$ over arbitrary base spaces B. However, it does not arise from a graph complex in this case.

Let τ_E be a vertical framing and $p(\tau_E) : \partial E\overline{C}_2(\pi) \to S^{d-1}$ the induced map on the boundary, then there exists a unique class $\omega \in H^{d-1}(E\overline{C}_2(\pi); \mathbb{Z})$ such that $\omega_{\partial E\overline{C}_2(\pi)} = p(\tau_E)^* u$ for a generator $u \in H^{d-1}(S^{d-1}; \mathbb{Z})$ by the same cohomological argument as before. Then we want to set $I(\theta) = \fint_{\overline{C}_2(M,\infty)} \omega^3$. However, unlike for de-Rham forms, we can only fibre integrate relative classes $H^*(E\overline{C}_2(\pi), \partial E\overline{C}_2(\pi); \mathbb{Z})$. Since $\omega^2|_{\partial E\overline{C}_2(\pi)} = p(\tau_E)^*(u^2) = 0$, there exists a choice of relative class $\tilde{\omega}_2 \in H^{d-1}(E\overline{C}_2(\pi), \partial E\overline{C}_2(\pi); \mathbb{Z})$. This choice can be made universally as the choice of null-homotopy of

$$S^{d-1} \xrightarrow{\Delta} S^{d-1} \times S^{d-1} \xrightarrow{u \times u} K(\mathbb{Z}, d-1) \times K(\mathbb{Z}, d-1) \xrightarrow{\mu} K(\mathbb{Z}, 2d-2),$$

where $u : S^{d-1} \to K(\mathbb{Z}, d-1)$ satisfies $u^*(\iota_{d-1}) = u$ and μ is the map such that $\mu^*(\iota_{2d-2}) = \iota_{d-1} \times \iota_{d-1}$. Define $\tilde{\omega}_n := \tilde{\omega}_2 \cdot \omega^{n-2} \in H^{n\cdot(d-1)}(E\overline{C}_2(\pi), \partial E\overline{C}_2(\pi); \mathbb{Z})$.

Theorem 6.2.5 ([Kon94, Wat09a])**.** *The class $\zeta_2(\pi, \tau_E) := \fint_{\overline{C}_2(M,\infty)} \tilde{\omega}_3 \in H^{d-3}(B; \mathbb{Z})$ is a characteristic class of bundles classified by* $\mathrm{B}\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty; \tau)$*.*

The main contribution of [Wat09a] is to refine this construction to give an invariant of unframed bundles over $B = S^{d-3}$ when $M = S^d$ and $d = 2k+1$ is odd. In this case, any bundle classified by a map to $\mathrm{B}\,\mathrm{Diff}(D^d, \partial)$ gives rise to a framed bundle by pulling back along a map $q_k : S^{2k-2} \to S^{2k-2}$ of large degree. Watanabe constructs a correction term to $\zeta_2(q_k^* \pi, \tau_E)$ that correct for the choice of q_k and vertical framing to obtain a rational invariant $\hat{Z}_2 : \pi_{2k-2}(\mathrm{B}\,\mathrm{Diff}(D^d, \partial)) \otimes \mathbb{Q} \to \mathbb{Q}$ (see [Wat09a, Thm 2]). He then shows that $\hat{Z}_2$ evaluates non-trivially on certain elements constructed in [ABK72].

6.3. Possible connections

There are two overarching questions of current research interest. The first is whether there is a generalization of Kontsevich's characteristic classes from a graph complex to more general manifolds. The second is whether Kontsevich's classes are related to embedding calculus, or more precisely whether they are defined over the delooping of the self-embedding tower. The recent results that express the Taylor tower in terms of automorphisms of framed configuration spaces considered as modules over the framed little discs operad [BdBW13, Tur13], combined with results of Willwacher et. al. that give graphical models for such automorphism spaces in terms of graph complexes that are similar to those for Kontsevich's characteristic classes suggest that the answer to both questions is yes.

In the following chapter, we will work on the second question and attempt to define fibrewise configuration spaces for the fibrations $E_k \to \mathrm{B}\,T_k^{\times}(M)$ as a natural precursor to defining characteristic classes via configuration space integrals.

Chapter 7.

A geometric approach to self-embedding calculus

In this chapter, we aim to introduce configuration spaces over the self-embeddings tower $E_k \to \mathrm{B}\, T_k^\times(M)$. The key difficulty is that the set-theoretic fibrewise configuration spaces of an M-fibration is not invariant under fibre-homotopy equivalences. This is because configuration spaces are not a homotopy invariant of a space, and in particular homotopy equivalences of a space do not determine homotopy equivalences of corresponding the configuration spaces. Instead, we consider a homotopy theoretic definition of configuration spaces as the complement of a Poincaré embedding structure of the fat diagonal. This homotopy theoretic notion of embeddings was developed by Klein [Kle99, Kle02] and we review his definition in Section 7.1.1. We show that such a homotopy theoretic fibrewise configuration spaces exists for the fibration $E_2 \to \mathrm{B}\, T_2^\times(M)$ in Section 7.1.2 and this enables us to define the simplest characteristic class associated to the θ-graph from Theorem 6.2.5 on the second stage of the tower in Section 7.3. Moreover, it has several interesting consequences related to tautological classes that we discuss in Section 7.2 and Chapter 8.

7.1. Fibrewise configuration space over $\mathrm{B}\, T_2^\times(M)$

This section is conceptually the first step in constructing graph cohomology classes on the self-embedding tower directly and at the heart of our geometric approach to embedding calculus. The basic idea is to find a homotopy theoretic replacement of configuration spaces that is robust under homotopy equivalences, and show that $E_2 \to \mathrm{B}\, T_2^\times(M)$ has this structure for each fibre.

7.1.1. Poincaré embeddings and configuration spaces

We will briefly review some concepts and results from [Kle99]. A space K is *homotopy finite* if it is homotopy equivalent to a finite complex, and of *homotopy dimension* $\leq k$ if is homotopy equivalent to a k-dimensional CW complex. Klein defines a Poincaré duality space X (or PD space in short) of formal dimension d as a homotopy finite space equipped with a local coefficient system (considered as a functor from the fundamental groupoid

$\mathcal{D} : \Pi_1(X) \to \mathsf{AbGrp}$ which is pointwise free abelian of rank 1), and a fundamental class $[X] \in H_d(X; \mathcal{D})$ such that the cap product

$$- \cap [X] : H^*(X; \mathcal{M}) \xrightarrow{\cong} H_{d-*}(X; \mathcal{M} \otimes \mathcal{D})$$

induces an isomorphism for every local system $\mathcal{M}$. Similarly, a Poincaré duality pair $(X, \partial X)$ satisfies a Poincaré-Lefschetz isomorphism.

Definition 7.1.1. Let $f : K \to X$ be a map from a connected homotopy finite space K to a PD space X of formal dimension d. Then a **Poincaré embedding** structure for f (PD embedding in short) is a diagram of spaces

$$\begin{tikzcd} G \arrow[r] \arrow[d] & C \arrow[d, "j"] \\ K \arrow[r, "f"'] \arrow[ur, Rightarrow, "H"] & X \end{tikzcd}$$

commuting up to a homotopy H such that

(i) The square is homotopy cocartesian, i.e. the map $\mathrm{hocolim}(K \leftarrow G \to C) \to X$ is a homotopy equivalence[1];

(ii) The spaces G and C are homotopy finite;

(iii) The image of the fundamental class under the composite

$$H_d(X; \mathcal{D}) \longrightarrow H_d(X, C; \mathcal{D}) \xleftarrow{\cong} H_d(K, G; \mathcal{D})$$

equips (the mapping cylinder of) (K, G) with the structure of a Poincaré duality pair and similarly for (C, G);

(iv) If $\mathrm{hodim}\, K \leq k$, then $G \to K$ is $(d - k - 1)$-connected.

The space C is called the **complement** and G is called the **gluing space**. If such a structure exists we say that f **Poincaré embeds**.

The prototypical example of a PD embedding is an embedding of smooth closed manifolds $M \hookrightarrow N$, where we can choose the literal complement $C = N \setminus M$ and $G = \partial \nu_M$ for a tubular neighbourhood $\nu_M \subset N$.

Remark 7.1.2. We have slightly modified Klein's definition here: The diagram in Definition 7.1.1 commutes up to a *fixed homotopy*, i.e. the homotopy is part of the data, whereas for

[1] We can use the double mapping cylinder as a model of the homotopy colimit so that the given homotopy determines a map $\mathrm{hocolim}(K \leftarrow G \to C) \to X$.

Klein the square needs to strictly commute. We prefer our definition since the (Poincaré) complement of an embedding of smooth closed manifolds $M \hookrightarrow N$ is the literal complement $N \setminus M$ as we have seen above. These two definitions are equivalent: A strict PD embedding in the sense of Klein is a PD embedding in our definition as well, and a Poincaré embedding in our sense determines a strict Poincaré embedding in the sense of Klein for $\mathrm{cyl}(G \to K) \to X$, where the map to X is determined by the fixed homotopy.

Ordinary configuration spaces $C_k(X)$ are defined as the complement of the inclusion of the fat diagonal $\Delta_k X := \{(x_1, \ldots, x_k) \in X^k \mid \exists\, i \neq j \text{ with } x_i = x_j\} \hookrightarrow X^k$. Now suppose that the inclusion $\Delta_k X \hookrightarrow X$ is a PD embedding with

$$\begin{tikzcd} G \arrow[r] \arrow[d] & C \arrow[d] \\ \Delta_k X \arrow[r] \arrow[ur, Rightarrow, "H"] & X^k, \end{tikzcd} \tag{7.1}$$

then C can be interpreted as a homotopy theoretic version of the ordinary configuration space of k particles. We will refer to the complement C of a Poincaré embedding structure of the fat diagonal as a *homotopy configuration space*. For closed, smooth or PL manifolds the inclusion of the fat diagonal is a Poincaré embedding with complement $C = C_k(M)$ as the fat diagonal admits a regular neighbourhood.

The key advantage of this homotopy theoretic notion of configuration spaces is that it is better behaved with respect to homotopy equivalences. More precisely, a homotopy equivalence $f : X \to X'$ induces homotopy equivalences

$$\begin{tikzcd} \Delta_k X \arrow[r, hook] \arrow[d, "\simeq"] & X^k \arrow[d, "\simeq"', "f^k"] \\ \Delta_k X' \arrow[r, hook] & X'^k \end{tikzcd}$$

and therefore any Poincaré embedding structure of $\Delta_k X \hookrightarrow X^k$ will determine a PD embedding for $\Delta_k X' \hookrightarrow X'^k$ and vice versa. In particular, a homotopy configuration space for X will also be one for X'.

However, the Poincaré embedding structure might not exist and even if it does, the homotopy type of the complement is not necessarily unique. This follows for example from the results of Longoni and Salvatore [LS05] who show that there are homotopy equivalent smooth closed manifolds whose configuration spaces are not homotopy equivalent. But either configuration spaces defines a homotopy configuration space (for both manifolds). But it can be shown that the stable homotopy type of complements is unique (assuming that the codimension is ≥ 3).

Proposition 7.1.3 ([AK04, Cor. 6.4]). *Assume that $f : K \to X$ Poincaré embeds and that the homotopy dimension of K is $\leq \dim X - 3$. Then the stable homotopy type of any complement C is unique.*

It follows that the cohomology groups of the complement of a PD embedding are unique. This can also be seen more directly from an application of excision.

Lemma 7.1.4. *Let $f : K \to X$ be a Poincaré duality embedding with $d = \dim X$ and complement C. Let $\mathcal{D}$ and $\mathcal{M}$ be local coefficient systems where $\mathcal{D}$ is the PD duality coefficient system of X. Then there is a long exact sequence*

$$\ldots \longrightarrow H_{d-*}(K; \mathcal{M} \otimes \mathcal{D}) \xrightarrow{f_*} H_{d-*}(X; \mathcal{M} \otimes \mathcal{D}) \longrightarrow H^*(C; \mathcal{M}) \longrightarrow H_{d-*-1}(K; \mathcal{M} \otimes \mathcal{D}) \to \ldots .$$

If K is a Poincaré space itself of dimension k then we can replace the corresponding terms in the above long exact sequence with the Umkehr map $f_! : H^{*-d+k}(K; f^*(\mathcal{M} \otimes \mathcal{D}_X) \otimes \mathcal{D}_K^{-1}) \to H^*(X; \mathcal{M})$.

Proof. Let G be the gluing space of the Poincaré embedding structure of f. Then we can replace X by $\mathrm{hocolim}(K \leftarrow G \to C)$ and K and C by the mapping cylinders of $G \to K$ and $G \to C$, which can be considered as subspaces of the homotopy colimit. Consider the long exact sequence of the pair (X, C) and note that we can now use excision for the homotopy colimit and Poincaré duality for the pair (K, G) to obtain the following diagram

$$\begin{tikzcd} \ldots \arrow[r] & H^*(X, C; \mathcal{M}) \arrow[r] \arrow[d, "\mathrm{exc.}"', "\simeq"] & H^*(X; \mathcal{M}) \arrow[r] \arrow[dd, "\mathrm{PD.}"', "\simeq"] & H^*(C; \mathcal{M}) \arrow[r] & \ldots \\ & H^*(K, G; \mathcal{M}) \arrow[d, "\mathrm{PD.}"', "\simeq"] & & & \\ & H_{d-*}(K; \mathcal{M} \otimes \mathcal{D}) \arrow[r, dashed, "i_*"] & H_{d-*}(X; \mathcal{M} \otimes \mathcal{D}) & & \end{tikzcd} \tag{7.2}$$

Here, the bottom map is induced by the inclusion of (the mapping cylinder) $i : K \to X$. The fundamental class of X can be decomposed $[X] = i_*[K, G] + j_*[C, G]$ for the fundamental classes of the pairs (K, G) and (C, G) (the above equation is true for chain representatives of the corresponding fundamental classes). Then for $[x] \in H^*(X, C; \mathcal{M})$ we find

$$i_*([K, G] \cap i^* x) = i_*[K, G] \cap x = (i_*[K, G] + j_*[C, G]) \cap x,$$

where the last equality follows since the cochain representative x evaluates trivially on chains in C. Therefore the square commutes, and the claim follows as the inclusion of the mapping cylinder is equivalent to f. ∎

A simple application is the computation of the cohomology ring of the complement C of a PD embedding structure of the diagonal $\Delta : X \to X \times X$ for an orientable Poincaré duality space (i.e. with trivial PD duality system $\mathcal{D}$). This has been determined for ordinary configuration spaces in [CT78] and will be relevant later on.

Corollary 7.1.5. *Let X be an orientable Poincaré duality space and suppose $\Delta : X \to X \times X$ Poincaré embeds with complement C. Then*

$$H^*(C;\mathbb{Z}) \cong H^*(X \times X;\mathbb{Z})/(\Delta_!(1)).$$

Proof. Note that $\Delta_* : H_*(X;\mathbb{Z}) \to H_*(X \times X;\mathbb{Z})$ is injective as it as a left inverse induced by projection to either factor. Hence, $\Delta_!$ is injective and thus $H^*(C) = \mathrm{Coker}(\Delta_! : H^*(X) \to H^{*+d}(X^2))$. Since $\Delta_!$ is a Δ^*-module map, meaning that $\Delta_!(\Delta^*(a) \cdot b) = a \cdot \Delta_!(b)$ for any $a \in H^*(X \times X)$ and $b \in H^*(X)$, the claim follows as $\Delta_!(x) = \Delta_!((\Delta^* \pi_i^* x) \cdot 1) = \pi_i^* x \cdot \Delta_!(1)$. □

Remark 7.1.6. The condition in Corollary 7.1.5 on the diagonal being a Poincaré embedding is rather weak. For example, it is true for topological manifolds or for 2-connected Poincaré spaces [Kle99, Cor.B]. A complete answer for when the diagonal is a Poincaré embedding is given in [Kle08].

With this robust homotopy theoretic notion of configuration spaces, we can now state the goal of this section. Namely, we want to show that the diagonal $\Delta : E_2 \to E_2 \times_{\mathrm{B}\, T_2^\times(M)} E_2$ admits in a natural way the structure of a *fibrewise Poincaré embedding* over $\mathrm{B}\, T_2^\times(M)$, by which we mean a diagram of fibrations over $\mathrm{B}\, T_2^\times(M)$

$$\begin{tikzcd} G \arrow[r] \arrow[d] & C_2 \arrow[d] \\ E_2 \arrow[r, "\Delta"'] \arrow[ur, Rightarrow, "H"] & E_2 \times_{\mathrm{B}\, T_2^\times(M)} E_2 \end{tikzcd} \tag{7.3}$$

together with a fibrewise homotopy H such that the restriction to each fibre is a Poincaré embedding structure of the diagonal. Then the space $C_2 \to \mathrm{B}\, T_2^\times(M)$ is a homotopy theoretic model for the fibrewise configuration space of two particles of $E_2 \to \mathrm{B}\, T_2^\times(M)$.

7.1.2. A fibrewise Poincaré embedding structure via the bar construction

It will be important to have good models of the oriented M-fibration $E_2 \to \mathrm{B}\, T_2^\times(M)$ at hand in order to find an explicit Poincaré embedding structure of the diagonal. We will make extensive use of May's work on the classification of fibrations as explained in [May75]. We have discussed the results we need in Appendix A and have outsourced all technical details there as well with the exception of the definition of the two-sided bar construction, which will appear throughout this section.

Definition 7.1.7. Let G be a topological monoid and let X and Y be left and right G-spaces. Then the two-sided bar construction $\mathrm{B}_\bullet(X,G,Y)$ is the simplicial space with n-simplices given by $X\times G^n\times Y$ and face and degeneracy operators given by

$$d_i(y[g_1,\dots,g_n]x) = \begin{cases} yg_1[g_2,\dots,g_n]x & i=0 \\ y[g_1,\dots,g_{i-1},g_ig_{i+1},\dots,g_n]x & 1\le i<n \\ y[g_1,\dots,g_{n-1}]g_nx & i=n \end{cases}$$

$$\text{and}\qquad s_i(y[g_1,\dots,g_n]x) = y[g_1,\dots,g_i,e,g_{i+1},\dots,g_n]x \qquad 0\le i\le n.$$

Denote by $\mathrm{B}(Y,G,X)$ the geometric realization of this simplicial space. The classifying space of G is defined as $\mathrm{B}\,G := \mathrm{B}(*,G,*)$.

The bar construction is functorial with respect to maps $(k,f,j):(Y,G,X)\to(Y',G',X')$ of triples as above, where $f:G\to G'$ is a map of topological monoids and $k:X\to X'$ and $j:Y\to Y'$ are f-equivariant. Then (k,f,j) induces a map of the simplicial spaces above and thus a map of geometric realizations

$$\mathrm{B}(k,f,j):\mathrm{B}(Y,G,X)\longrightarrow \mathrm{B}(Y',G',X').$$

The monoid that is relevant for the classification of oriented Hurewicz fibration and fibre M is the monoid of oriented homotopy self-equivalences $\mathrm{hAut}^+(M)$ which acts on M through evaluation. The main theorem of [May75] implies that

$$\Gamma\,\mathrm{B}(*,\mathrm{hAut}^+(M),M)\longrightarrow \mathrm{B}(*,\mathrm{hAut}^+(M),*) \tag{7.4}$$

is the universal oriented M-fibration, where Γ is the functor that replaces a map by a fibration (see Definition A.3). More precisely, for a space $B\in \mathsf{Top}^{CW}$, the full subcategory of Top of spaces that are equivalent to CW complexes, there is a bijection between $[B,\mathrm{B}\,\mathrm{hAut}^+(M)]$ and the set of oriented M-fibrations over B up to fibre-homotopy equivalence given by pulling back the universal fibration (7.4) (see Theorem A.10).

Recall that we have defined $E_k\to \mathrm{B}\,T_k^\times(M)$ in (6.11) as the pullback of the universal fibration over $\mathrm{B}\,T_0^\times(M)=\mathrm{B}\,\mathrm{hAut}^+(M)$ along the maps $\mathrm{B}(r_1)\circ\dots\circ\mathrm{B}(r_k):\mathrm{B}\,T_k^\times(M)\to\mathrm{B}\,T_0^\times(M)$, i.e.

$$E_k := (\mathrm{B}(r_1)\circ\dots\circ\mathrm{B}(r_k))^*\Gamma\,\mathrm{B}(*,\mathrm{hAut}^+(M),M)\xrightarrow{\pi_k}\mathrm{B}\,T_k^\times(M).$$

as well as vector bundles $T_\pi E_k\to E_k$ as pullbacks $T_\pi E_k := (\bar r_2\circ\dots\circ\bar r_k)^*T_\pi E_1$ of the universal vector bundle $T_\pi E_1\to E_1$ whose restriction to each fibre is equivalent to $TM\to M$.

We will use the Haefliger model for the Taylor tower, in part because the bottom part of the tower

$$T_2^\times(M) \xrightarrow{r_2} T_1^\times(M) \xrightarrow{r_1} T_0^\times(M) = \mathrm{hAut}^+(M),$$

which we have discussed in Section 6.1.1, is a diagram of topological monoids and monoid maps by Lemma 6.1.2. But more importantly, the Haefliger models $T_2^\times(M)$ and $T_1^\times(M)$ act on M via $(f,H,G)\cdot p := \pi_1 \circ G(p,p)$ for $(f,H,G) \in T_2^\times(M)$ and $p \in M$ and $(\bar{f}, f)\cdot p := f(p)$ for $(\bar{f}, f) \in T_1^\times(M)$. Since the identity map of M is equivariant with respect to r_2 and r_1, there are maps of triples

$$(*, T_2^\times(M), M) \xrightarrow{(\mathrm{Id}_*, r_2, \mathrm{Id}_M)} (*, T_1^\times(M), M) \xrightarrow{(\mathrm{Id}_*, r_1, \mathrm{Id}_M)} (*, \mathrm{hAut}^+(M), M)$$

that give rise to the following commutative diagram

$$\begin{tikzcd}
\mathrm{B}(*, T_2^\times(M), M) \arrow[r] \arrow[d] & \mathrm{B}(*, T_1^\times(M), M) \arrow[r] \arrow[d] & \mathrm{B}(*, \mathrm{hAut}^+(M), M) \arrow[d] \\
\mathrm{B}(*, T_2^\times(M), *) \arrow[r] & \mathrm{B}(*, T_1^\times(M), *) \arrow[r] & \mathrm{B}(*, \mathrm{hAut}^+(M), *).
\end{tikzcd} \tag{7.5}$$

All vertical maps are quasifibrations by Theorem A.8, i.e. the inclusion of the fibre of the projection into the homotopy fibre is a weak equivalence for all points in the base (see Definition A.6), and every square in (7.5) is a pullback square by Proposition A.11. These two statements imply that every square in (7.5) is a homotopy pullback square, and in particular the natural comparison map from (7.5) to the first three stages of (6.12) is a fibre-homotopy equivalence:

$$\begin{aligned} &\mathrm{B}(*, T_2^\times(M), M) \xrightarrow{\simeq_{\mathrm{fw}}} E_2 \text{ over } \mathrm{B}\,T_2^\times(M) \\ \text{and} \quad &\mathrm{B}(*, T_1^\times(M), M) \xrightarrow{\simeq_{\mathrm{fw}}} E_1 \text{ over } \mathrm{B}\,T_1^\times(M). \end{aligned} \tag{7.6}$$

Therefore, it suffices to construct the data for a fibrewise Poincaré embedding of the fibrewise diagonal of $\mathrm{B}(*, T_2^\times(M), M) \to \mathrm{B}\,T_2^\times(M)$.

The key property of the Haefliger model that we use is that $T_2^\times(M)$ acts naturally through the projection to $\mathrm{IvMap}(M^2, M^2)$ not only on M^2 and the diagonal $\Delta \subset M^2$ but also on the complement $M^2 \setminus \Delta$. Hence, we can for example consider the following diagram of quasi-fibrations over $\mathrm{B}\,T_2^\times(M)$

$$\mathrm{B}(*, T_2^\times(M), M) \xrightarrow{\mathrm{B}(\Delta)} \mathrm{B}(*, T_2^\times(M), M^2) \xleftarrow{\mathrm{B}(\mathrm{incl})} \mathrm{B}(*, T_2^\times(M), M^2 \setminus \Delta)$$

induced by the $T_2^\times(M)$-equivariant diagram $M \xrightarrow{\Delta} M^2 \hookleftarrow M^2 \setminus \Delta$, where $T_2^\times(M)$ acts on M as above and on M^2 and $M^2 \setminus \Delta$ through the projection to $\mathrm{IvMap}(M^2, M^2)$ and evaluation.

This coincides with the map on fibres over the 0-skeleton of $\mathrm{B}\,T_2^\times(M)$ and therefore up to homotopy on all fibres of the projection to $\mathrm{B}\,T_2^\times(M)$. Hence, $\mathrm{B}(*, T_2^\times(M), M^2 \setminus \Delta)$ is the natural candidate for the fibrewise complement of a fibrewise Poincaré embedding structure of the diagonal.

We can now state our main theorem of this section.

Theorem 7.1.8. *The diagonal* $\Delta : E_2 \to E_2 \times_{\mathrm{B}\,T_2^\times(M)} E_2$ *has the structure of a fibrewise Poincaré embedding with complement* $\mathrm{B}(*, T_2^\times(M), M^2 \setminus \Delta)$ *and gluing space* $S(T_\pi E_2)$.

There are three problems we need to address. First, we need to compare $\mathrm{B}(\Delta)$ with the fibrewise diagonal $\Delta : \mathrm{B}(*, T_2^\times(M), M) \to \mathrm{B}(*, T_2^\times(M), M) \times_{\mathrm{B}\,T_2^\times(M)} \mathrm{B}(*, T_2^\times(M), M)$, and secondly we need to construct a candidate for the fibrewise gluing space. In the last step, we will show that the candidate for the gluing space has a vector bundle reduction given by $S(T_\pi E_2)$.

Let us start with the second point, and observe that we would like to use the gluing space of the standard Poincaré embedding structure of $\Delta : M \to M^2$ with complement $M^2 \setminus \Delta$ given by the sphere bundle of a tubular neighbourhood $S(\nu_\Delta)$. However, $\mathrm{IvMap}(M^2, M^2)$ does not act naturally on a tubular neighbourhood of the diagonal. Instead we consider a homotopy theoretic version of the tubular neighbourhood (following the terminology in [Qui88]).

Definition 7.1.9. Let (X, Y) be a pair of spaces. The **homotopy link** is the space of paths starting in Y and leaving immediately for positive times

$$\mathrm{holink}(X, Y) := \{\gamma \in \mathrm{Map}(I, X) \,|\, \gamma(0) \in Y,\ \gamma(t) \in X \setminus Y \text{ for } t > 0\} \subset \mathrm{Map}(I, X).$$

A version of this construction was first considered by Nash [Nas55] and later used by Hu to construct analogues of normal bundles (see [HR96, App. B] for some historical references). Fadell has shown in [Fad65] that $\mathrm{ev}_0 : \mathrm{holink}(N, M) \to M$ for a smooth manifold N and submanifold $M \subset N$ is a spherical fibration equivalent to the normal sphere bundle.

Hence, we can consider $\mathrm{holink}(M^2, \Delta)$ as natural candidate for the gluing space of the Poincaré embedding structure of $\Delta : M \to M^2$. This space has an action of $\mathrm{T}_2^\times(M)$ through the projection to $\mathrm{IvMap}(M^2, M^2)$ by post-composition.

Proposition 7.1.10. *Let M be a smooth closed manifold, then the following diagram of quasifibrations over* $\mathrm{B}\,T_2^\times(M)$

$$\begin{tikzcd}
\mathrm{B}(*, T_2^\times(M), \mathrm{holink}(M^2, \Delta)) \arrow[r, "\mathrm{B}(\mathrm{ev}_1)"] \arrow[d, "\mathrm{B}(\mathrm{ev}_0)"] & \mathrm{B}(*, T_2^\times(M), M^2 \setminus \Delta) \arrow[d] \\
\mathrm{B}(*, T_2^\times(M), M) \arrow[r, "\mathrm{B}(\Delta)"] & \mathrm{B}(*, T_2^\times(M), M^2),
\end{tikzcd} \tag{7.7}$$

which commutes up to homotopy $\mathrm{B}(\mathrm{ev}_t) : \mathrm{B}(*, T_2^\times(M), \mathrm{holink}(M^2, \Delta)) \times I \to \mathrm{B}(*, T_2^\times(M), M^2)$*, determines a fibrewise Poincaré embedding structure of* $\mathrm{B}(\Delta)$*.*

Proof. The induced projection $\pi : \mathrm{hocolim}(E_1 \leftarrow E_0 \to E_2) \to B$ from the homotopy colimit of quasifibrations $\pi_i : E_i \to B$ is a again a quasifibration over B since

$$\begin{aligned} \mathrm{hofib}_b(\pi) &\xleftarrow{\simeq} \mathrm{hocolim}\Big(\mathrm{hofib}_b(\pi_1) \leftarrow \mathrm{hofib}_b(\pi_0) \to \mathrm{hofib}_b(\pi_2)\Big) \\ &\xleftarrow{\simeq} \mathrm{hocolim}\Big(\pi_1^{-1}(b) \leftarrow \pi_0^{-1}(b) \to \pi_2^{-1}(b)\Big) = \pi^{-1}(p). \end{aligned} \tag{7.8}$$

Hence, it suffices to check that the above diagram is a Poincaré embedding for the fibre over the 0-skeleton $F_0\,\mathrm{B}\,T_2^\times(M)$, where (7.7) restricts to

$$\begin{array}{ccc} \mathrm{holink}(M^2, \Delta) & \longrightarrow & M^2 \setminus \Delta \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad\Delta\quad} & M^2 \end{array}$$

But since $\mathrm{holink}(M^2, M)$ is equivalent to the normal sphere bundle [Fad65, Prop. 4.8], this is equivalent to the standard Poincaré embedding structure of the diagonal. □

We will now address the first problem and compare the bottom horizontal map $\mathrm{B}(\Delta)$ in (7.7) with the fibrewise diagonal.

Lemma 7.1.11. *There is a zig-zag of fibre-homotopy equivalences from the map* $\mathrm{B}(\Delta)$ *in* (7.7) *to the fibrewise diagonal*

$$\Delta : \mathrm{B}(*, T_2^\times(M), M) \longrightarrow \mathrm{B}(*, T_2^\times(M), M) \times_{\mathrm{B}\,T_2^\times(M)} \mathrm{B}(*, T_2^\times(M), M). \tag{7.9}$$

Proof. It will be important for the proof to keep track of the (left) action $\varphi : T_2^\times(M) \times X \to X$ of $T_2^\times(M)$ on a space X by denoting the pair (X, φ) for a left G-space X.

The map $\mathrm{B}(\Delta)$ in (7.7) is induced by the diagonal map $\Delta : M \to M^2$ which is equivariant with respect to the following actions of $T_2^\times(M)$

$$\begin{aligned} \phi_1 &: T_2^\times(M) \times M \longrightarrow M & \phi_1((f, H, G), m) &:= g(m, m), \\ \phi_2 &: T_2^\times(M) \times M^2 \longrightarrow M^2 & \phi_2((f, H, G), (m, m')) &:= G(m, m'), \end{aligned}$$

where we use the identification $\mathrm{Map}^{S_2}(M^2, M^2) \approx \mathrm{Map}(M^2, M)$ that sends an S_2-equivariant map G to $g := \pi_1 \circ G$ (since $G(m_1, m_2) = (g(m_1, m_2), g(m_2, m_1))$ by equivariance). Observe that $B(*, T_2^\times(M), M)$ in (7.9) is $\mathrm{B}(*, T_2^\times(M), (M, \phi_1))$ when we record the action. We have defined $\mathrm{B}(\Delta)$ as the map induced by the following map of triples

$$(*, T_2^\times(M), (M, \phi_1)) \xrightarrow{(\mathrm{Id}_*, \mathrm{Id}_{T_2^\times(M)}, \Delta)} (*, T_2^\times(M), (M^2, \phi_2)).$$

Since a point $(f,H,G) \in T_2^\times(M)$ contains a homotopy H of equivariant maps, we can consider families of actions, considered as actions on $M \times I$ and $M^2 \times I$ respectively, in the same way

$$\begin{aligned} \varphi_1 : T_2^\times(M) \times (M \times I) &\longrightarrow M \times I & \varphi_1((f,H,G),(m,t)) &:= (\pi_1 \circ H_t(m,m),t), \\ \varphi_2 : T_2^\times(M) \times (M^2 \times I) &\longrightarrow M^2 \times I & \varphi_2((f,H,G),(m,m',t)) &:= (H_t(m,m'),t), \end{aligned}$$

so that there is a map of triples $(*, T_2^\times(M), (M \times I, \varphi_1)) \to (*, T_2^\times(M), (M^2 \times I, \varphi_2))$ induced by $\Delta \times \mathrm{Id}_I : M \times I \to M^2 \times I$.

Since $H_1 = G$ for a point $(f,H,G) \in T_2^\times(M)$, the inclusions $i_1 : (M, \phi_1) \to (M \times I, \varphi_1)$ and $i_1 : (M^2, \phi_2) \to (M^2 \times I, \varphi_2)$ are equivariant and induce maps of triples. The restriction for $t = 0$ is given by $H_0 = f \times f$, and thus there are equvariant maps $i_0 : (M, \varphi) \to (M \times I, \varphi_1)$ and $i_0 : (M^2, \varphi \times \varphi) \to (M^2 \times I, \varphi_2)$, where the action $\varphi : T_2^\times(M) \times M \to M$ is defined as $\varphi((f,H,G),m) := f(m)$ and $\varphi \times \varphi$ denotes the product action on M^2.

We arrive at the following commutative diagram

$$\begin{tikzcd} \mathrm{B}(*, T_2^\times(M), (M, \phi_1)) \arrow[r, "\mathrm{B}(i_1)", "\simeq"'] \arrow[d, "\mathrm{B}(\Delta)"] & \mathrm{B}(*, T_2^\times(M), (M \times I, \varphi_1)) \arrow[d, "\mathrm{B}(\Delta \times \mathrm{Id})"] & \mathrm{B}(*, T_2^\times(M), (M, \varphi)) \arrow[l, "\mathrm{B}(i_0)"', "\simeq"] \arrow[d, "\mathrm{B}(\Delta)"] \\ \mathrm{B}(*, T_2^\times(M), (M^2, \phi_2)) \arrow[r, "\mathrm{B}(i_1)", "\simeq"'] & \mathrm{B}(*, T_2^\times(M), (M^2 \times I, \varphi_2)) & \mathrm{B}(*, T_2^\times(M), (M^2, \varphi \times \varphi)), \arrow[l, "\mathrm{B}(i_0)"', "\simeq"] \end{tikzcd} \tag{7.10}$$

where the horizontal maps are induced by the inclusions and homotopy equivalences by Proposition A.4. The vertical maps are all induced by diagonal maps, and the left vertical map is $\mathrm{B}(\Delta)$ from (7.7).

By Corollary A.13, the equivariant projections $\pi_i : (M^2, \varphi \times \varphi) \to (M, \varphi)$ induce a homeomorphism of the lower right corner

$$\mathrm{B}(*, T_2^\times(M), (M^2, \varphi \times \varphi)) \xrightarrow{\mathrm{B}(\pi_1) \times \mathrm{B}(\pi_2)} \mathrm{B}(*, T_2^\times(M), (M, \varphi)) \times_{\mathrm{B}\, T_2^\times(M)} \mathrm{B}(*, T_2^\times(M), (M, \varphi)).$$

The (pre-)composition with $\mathrm{B}(\Delta)$ is given by $\mathrm{B}(\pi_1 \circ \Delta) \times \mathrm{B}(\pi_2 \circ \Delta) = \mathrm{B}(\mathrm{Id}) \times \mathrm{B}(\mathrm{Id})$ so that under this identification the right column is the fibrewise diagonal of $\mathrm{B}(*, T_2^\times(M), (M, \varphi)) \to \mathrm{B}\, T_2^\times(M)$. The claim now follows, as the top row shows that the two quasi-fibrations

$$\begin{aligned} \mathrm{B}(*, T_2^\times(M), (M, \phi_1)) &\longrightarrow \mathrm{B}\, T_2^\times(M) \\ \mathrm{B}(*, T_2^\times(M), (M, \varphi)) &\longrightarrow \mathrm{B}\, T_2^\times(M) \end{aligned}$$

are fibre-homotopy equivalent. We can also build an explicit zig-zag of fibre-homotopy equvialences between their respective diagonals by considering the concordance of quasi-fibrations given by the diagonal $M \times I \to M^2 \times I$ with respect to the product action of φ_1 on $M^2 \times I$

$$\mathrm{B}(*, T_2^\times(M), (M \times I, \varphi_1)) \xrightarrow{\mathrm{B}\,\Delta} \mathrm{B}(*, T_2^\times(M), (M^2 \times I, \varphi_1 \times \varphi_1)).$$

□

Before we give the proof of Theorem 7.1.8, we need to introduce a version of the homotopy link which is better suited to constructing a vector bundle reduction of the gluing space.

Definition 7.1.12. Let M^d, N^{n+k} be smooth closed manifolds and $M \subset N$ an embedded submanifold. Then we can define the **transversal homotopy link** as

$$\operatorname{holink}^{\pitchfork}(N, M) := \left\{\gamma \in \operatorname{holink}(N, M) \,\middle|\, \gamma \text{ is smooth and } \dot{\gamma}(0) \notin TM\right\}$$

topologized as a subspace of smooth maps $\operatorname{Map}_{C_\infty}(I, N)$.

Lemma 7.1.13. *Let M, N be smooth closed manifolds and $M \subset N$ an embedded submanifold. Then the inclusion $\operatorname{holink}^{\pitchfork}(N, M) \xrightarrow{\simeq} \operatorname{holink}(N, M)$ is a fibrewise homotopy equivalence over M via ev_0.*

Proof. We will first prove that $\mathrm{ev}_0 : \operatorname{holink}^{\pitchfork}(N, M) \to M$ has the weak covering homotopy property (WCHP) [Dol63]. By a version of Hurewicz's theorem [Dol63, Thm 5.12], this can be checked on a numerable covering and we may choose a covering $\{U_i\}_{i\in I}$ of M by open disks $U_i \cong \mathbb{R}^d$. Let $\{V_i\}_{i\in I}$ be a covering of M of open disks in N such that there are charts such that $(V_i, U_i) \cong (\mathbb{R}^{d+k}, \mathbb{R}^d)$. Then $\operatorname{holink}^{\pitchfork}(N, M)|_{U_i}$ is fibre-homotopy equivalent to $\operatorname{holink}^{\pitchfork}(V_i, U_i)$ by shrinking the paths until the image is contained in V_i. Since the WCHP is preserved under fibre-homotopy equivalences, it suffices to check that $\operatorname{holink}^{\pitchfork}(V_i, U_i) \to U_i$ has the WCHP. Using the charts, this is homeomorphic to the map $\mathrm{ev}_0 : \operatorname{holink}^{\pitchfork}(\mathbb{R}^{d+k}, \mathbb{R}^d) \to \mathbb{R}^d$ for which one can easily find a path lifting function.

Let ν_M be a tubular neighbourhood, then there is a map of the sphere bundle of a tubular neighbourhood $S(\nu_M) \to \operatorname{holink}^{\pitchfork}(N, M)$ over M by sending $v_p \in S(\nu_M)$ to the path $t \mapsto t \cdot v_p$. By the same argument as before, the fibre of $\operatorname{holink}^{\pitchfork}(N, M) \to M$ over a point $p \in U_i$ deformation retracts to the fibre of $\operatorname{holink}^{\pitchfork}(\mathbb{R}^{d+k}, \mathbb{R}^d) \to \mathbb{R}^d$ using the identification $(V_i, U_i) \cong (\mathbb{R}^{d+k}, \mathbb{R}^d)$. This fibre is equivalent to S^{d-1} since we can first homotope a path and replace it by its differential at zero, and then get rid of the possible part in the $\mathbb{R}^d$ direction. Hence, the fibre of $\mathrm{ev}_0 : \operatorname{holink}^{\pitchfork}(\mathbb{R}^{d+k}, \mathbb{R}^d) \to \mathbb{R}^d$ deformation retracts onto the fibre of $S(\nu_{\mathbb{R}^d \subset \mathbb{R}^{d+k}}) \to \mathbb{R}^d$.

Therefore, the map of fibrations $S(\nu_M) \to \operatorname{holink}^{\pitchfork}(N, M)$ over M induces an equivalence of the fibres and by Dold's theorem [Dol63, Thm 6.3] it is a fibre-homotopy equivalence. The claim then follows from [Fad65, Prop.4.8] which shows that $S(\nu_M) \to \operatorname{holink}(N, M)$ is a fibre-homotopy equivalence. This map factors through the transversal homotopy link

$$\begin{tikzcd} S(\nu_M) \arrow[r, "\simeq_{\mathrm{fw}}"] \arrow[dr, "\simeq_{\mathrm{fw}}"'] & \operatorname{holink}^{\pitchfork}(N, M) \arrow[d] \\ & \operatorname{holink}(N, M). \end{tikzcd} \tag{7.11}$$

and therefore the inclusion is a fibre-homotopy equivalence as well. □

Remark 7.1.14. The first part of the proof can be simplified by observing that $\mathrm{ev}_0 : \mathrm{holink}^{(\pitchfork)}(N,M)\to M$ is a locally trivial bundle (for the homotopy link this is [Fad65, Prop. 4.1]). This is proved by picking adapted charts $(V_i,U_i) \cong (\mathbb{R}^{d+k},\mathbb{R}^d)$ and observing that one can construct maps $\gamma : U_i\times U_i\to \mathrm{Diff}_0(D^{d+k},D^d)$, where this denotes diffeomorphisms of the disk isotopic to the identity that restrict to diffeomorphisms of $D^d \subset D^{d+k}$, satisfying $\gamma(x,y)(x) = y$, $\gamma(x,x) = \mathrm{Id}$ and $\gamma(x,y)|_{\partial D^{d+k}} = \mathrm{Id}$ for all $x,y \in U_i$. The maps γ are constructed using flows of vector fields pointing from x to y and cut off by bump functions. It follows that the evaluation is a fibration by [Dol63, Thm 4.8].

Proof of Theorem 7.1.8. It follows from Lemma 7.1.11 that the fibrewise Poincaré embedding structure of $\mathrm{B}(\Delta)$ discussed in Proposition 7.1.10 determines a fibrewise Poincaré embedding structure of the fibrewise diagonal

$$\Delta : \mathrm{B}(*,T_2^\times(M),M) \longrightarrow \mathrm{B}(*,T_2^\times(M),M) \times_{\mathrm{B}\,T_2^\times(M)} \mathrm{B}(*,T_2^\times(M),M)$$

with complement $\mathrm{B}(*,T_2^\times(M),M^2\setminus\Delta)$ and gluing space $\mathrm{B}(*,T_2^\times(M),\mathrm{holink}(M^2,\Delta))$. It remains to prove that $\mathrm{B}(*,T_2^\times(M),\mathrm{holink}(M^2,\Delta))$ is equivalent to $S(T_\pi E_2)$.

We can replace the gluing space by using the transversal homotopy link instead by Lemma 7.1.13. The transversal homotopy link has a map to $\left(\frac{TM^2|_\Delta}{T\Delta}\right)^\#$, the complement of the zero section of the normal bundle, given by

$$\begin{aligned} d/dt : \mathrm{holink}^{\pitchfork}(M^2,\Delta) &\longrightarrow \left(\frac{TM^2|_\Delta}{T\Delta}\right)^\# . \\ \gamma &\longmapsto [\dot\gamma(0)] \end{aligned}$$

We can identify the normal bundle with TM via the isomorphism $TM\to\frac{TM^2|_\Delta}{T\Delta}$ that sends v_p to $[(v_p,-v_p)]$, which equips the normal bundle with an action of $T_1^\times(M)$ by conjugation. Then the map d/dt is equivariant with respect to $r_2 : T_2^\times(M)\to T_1^\times(M)$.

To see this, consider an isovariant map $G \in \mathrm{IvMap}(M^2,M^2)$ and $\gamma \in \mathrm{holink}^{\pitchfork}(M^2,\Delta)$ and write $\dot\gamma(0) = (v_p + w_p, -v_p + w_p) \in T_{(p,p)}M^2$ for unique $v_p,w_p \in T_pM$ and $p = \gamma_i(0)$. Since $[\dot\gamma(0)] = [(v_p,-v_p)]$, we have

$$r_2(G)\cdot[\dot\gamma(0)] = [D_{(p,p)}g(v_p,-v_p), -D_{(p,p)}g(v_p,-v_p)] \in \left(\frac{TM^2|_\Delta}{T\Delta}\right)_{(g(p,p),g(p,p))},$$

which agrees with

$$
\begin{aligned}
\left[\frac{d}{dt}(G\circ\gamma)|_{t=0}\right] &= \left[\frac{d}{dt}(g(\gamma_1,\gamma_2))|_{t=0}, \frac{d}{dt}(g(\gamma_2,\gamma_1))|_{t=0}\right] \\
&= \left[D_{(p,p)}g(v_p+w_p,-v_p+w_p), D_{(p,p)}g(-v_p+w_p,v_p+w_p)\right] \\
&= \left[D_{(p,p)}g(v_p,-v_p)+D_{(p,p)}g(w_p,w_p), D_{(p,p)}g(-v_p,v_p)+D_{(p,p)}g(w_p,w_p)\right] \\
&= \left[D_{(p,p)}g(v_p,-v_p), D_{(p,p)}g(-v_p,v_p)\right] \in \left(\frac{TM^2|_\Delta}{T\Delta}\right)_{(g(p,p),g(p,p))}.
\end{aligned}
$$

Hence, we obtain the following diagram

$$
\begin{tikzcd}
\mathrm{B}(*,T_2^\times,\mathrm{holink}(M^2,\Delta)) \arrow[d, "\mathrm{B}(\mathrm{ev}_0)"] & \mathrm{B}(*,T_2^\times,\mathrm{holink}^{\pitchfork}(M^2,\Delta)) \arrow[l, "\simeq"'] \arrow[r, "*"] \arrow[d, "\mathrm{B}(\mathrm{ev}_0)"] & \mathrm{B}(*,T_1^\times(M),TM^\#) \arrow[d] \\
\mathrm{B}(*,T_2^\times(M),M) \arrow[r, equal] & \mathrm{B}(*,T_2^\times(M),M) \arrow[r, "\mathrm{B}\, r_2=\bar{r}_2"] & \mathrm{B}(*,T_1^\times(M),M)
\end{tikzcd}
\tag{7.12}
$$

where the top map labelled by $*$ is given by $\mathrm{B}(\mathrm{Id}, r_2, d/dt)$. The left horizontal map in the first row is induced by the inclusion of the transversal homotopy link, and it is a homotopy equivalence by Lemma 7.1.13 and Proposition A.4. All vertical maps are quasifibrations with spherical fibres by Theorem A.8, and, when restricted to the 0-skeleton of $\mathrm{B}(*,T_i^\times(M),M)$, the horizontal maps are homotopy equivalences by (7.11). Hence, the right square is a homotopy pullback and it follows that the gluing space $\mathrm{B}(*,T_2^\times(M),\mathrm{holink}(M^2,\Delta))$ in (7.7) is equivalent to $\bar{r}_2^*\,\mathrm{B}(*,T_1^\times(M),TM^\#)$. The statement of the theorem then follows from the next lemma which proves that $\mathrm{B}(*,T_1^\times(M),TM^\#)$ is equivalent to $S(T_\pi E_1)$. □

Lemma 7.1.15. *The restriction of* $T_\pi E_1 \to E_1$ *along* $\mathrm{B}(*,T_1^\times(M),M) \xrightarrow{\simeq_{\mathrm{fw}}} E_1$ *from* (7.6) *is equivalent to the following vector bundle*

$$
\mathrm{B}(\pi_{TM}) : \mathrm{B}(*,\mathrm{hAut}^+(TM),TM) \longrightarrow \mathrm{B}(*,\mathrm{hAut}^+(TM),M) \tag{7.13}
$$

induced by the tangent bundle $\pi_{TM} : TM \to M$ *and the evident action of* $\mathrm{hAut}^+(TM)$*.*

Proof. Recall that $T_1^\times(M) = \mathrm{hAut}^+(TM)$ and that $E_1 \to \mathrm{B}\,T_1^\times(M)$ is the universal TM-fibration defined in (5.1) with a vector bundle $T_\pi E_1 \to E_1$ obtained in (5.2) as the pull back along the evaluation map

$$
\mathrm{B}(\mathrm{Map}(M,\mathrm{B}\,\mathrm{SO}(d))_{TM}, \mathrm{hAut}(M)_{TM}, M) \xrightarrow{\epsilon(\mathrm{ev})} \mathrm{B}\,\mathrm{SO}(d).
$$

Part of this statement is Proposition 5.1.4 that shows that

$$
\mathrm{B}(\mathrm{Map}(M,\mathrm{B}\,\mathrm{SO}(d))_{TM}, \mathrm{hAut}(M)_{TM}, M) \simeq \mathrm{B}(*,\mathrm{hAut}^+(TM),M) \tag{7.14}
$$

which is our model for E_1. Hence, we need to prove two things: First, we need to show (7.13) is a vector bundle, and secondly we need to show that it is equivalent to the definition of the vertical tangent bundle defined via the evaluation map $\epsilon(\mathrm{ev})$ from (5.2) under the equivalence (7.14).

We will start with the first problem and prove that $\mathrm{B}(\pi_{TM})$ is a vector bundle by induction over the skeleta $\mathrm{B}(*, \mathrm{hAut}^+(TM), TM) = \bigcup_{j\geq 0} F_j$ and $\mathrm{B}(*, \mathrm{hAut}^+(TM), M) = \bigcup_{j\geq 0} F'_j$ (see [May72, Def. 11.1]) which is really an adaptation of the proof of Theorem A.8. In the following, we denote $\mathrm{B}(\pi_{TM})$ by p. Then

$$\begin{aligned} F_0 &= TM && \text{and} && F_j \setminus F_{j-1} = (F_j \mathrm{B}\,\mathrm{hAut}^+(TM) \setminus F_{j-1} \mathrm{B}\,\mathrm{hAut}^+(TM)) \times TM, \\ F'_0 &= M && \text{and} && F'_j \setminus F'_{j-1} = (F_j \mathrm{B}\,\mathrm{hAut}^+(TM) \setminus F_{j-1} \mathrm{B}\,\mathrm{hAut}^+(TM)) \times M \end{aligned}$$

and therefore the restrictions

$$p|_{F_0} : F_0 \to F'_0 \qquad \text{and} \qquad p|_{F_j \setminus F_{j-1}} : F_j \setminus F_{j-1} \to F'_j \setminus F'_{j-1}$$

are vector bundles. Assume by induction that $p|_{F_{j-1}} : F_{j-1} \to F'_{j-1}$ is a vector bundle. Following the proof of Theorem A.8, we see that there are neighbourhoods U of $F'_{j-1} \subset F'_j$ and deformation retractions $H : p^{-1}(U) \times I \to p^{-1}(U)$ and $h : U \times I \to U$ compatible with the projection p. Moreover, the pair (H_1, h_1) defines a linear isomorphism on each fibre, proving that $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a vector bundle. Hence, $p|_{F_j} : F_j \to F'_j$ is a vector bundle for each j. The colimit $p : \bigcup_j F_j \to \bigcup_j F'_j$ is a vector bundle by using sequences of deformation retractions as above which deformation retracts open neighbourhoods of points onto the finite skeleta similar to the contraction of a mapping telescope.

It remains to prove in the second part that the two vector bundles over E_1 are equivalent. The equivalence in (7.14) is based on the zig-zag of equivalences constructed in [Ber20a]

$$\mathrm{B}(\mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}, \mathrm{hAut}(M)_{TM}, *) \xleftarrow{\simeq} \mathrm{B}(\mathrm{Map}(TM, \gamma_d), \mathrm{hAut}^+(TM), *) \xrightarrow{\simeq} \mathrm{B}\,\mathrm{hAut}^+(TM),$$

where $\mathrm{Map}(TM, \gamma_d)$ denotes the space of pairs of maps $(\bar{f}, f)$ for a bundle map $\bar{f} : TM \to \gamma_d$ covering a map $f : M \to \mathrm{B}\,\mathrm{SO}(d)$, and $\mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}$ denotes the path-component of the tangent bundle and $\mathrm{hAut}(M)_{TM}$ denotes the collection of path-components of $\mathrm{hAut}^+(M)$ that preserve the tangent bundle, i.e. act on $\mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}$ by pre-composition. Then consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{B}(*, \mathrm{hAut}^+(TM), TM) \arrow[r] & \mathrm{B}(*, \mathrm{hAut}^+(TM), M) \\
\mathrm{B}(\mathrm{Map}(TM, \gamma_d), \mathrm{hAut}^+(TM), TM) \arrow[u] \arrow[r] \arrow[dd, "\epsilon(\mathrm{ev})"] & \mathrm{B}(\mathrm{Map}(TM, \gamma_d), \mathrm{hAut}^+(TM), M) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
 & \mathrm{B}(\mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}, \mathrm{hAut}(M)_{TM}, M) \arrow[d, "\epsilon(\mathrm{ev})"] \\
\gamma_d \arrow[r] & \mathrm{B}\,\mathrm{SO}(d)
\end{tikzcd}$$

where the horizontal maps are vector bundles (the middle one by the same argument as above) and the left vertical maps are linear isomorphisms on the fibres. It follows that the vector bundle obtained as pull back along $\epsilon(\mathrm{ev})$ is equivalent to (7.13) under the zig-zag of equivalences on the right which concludes the proof. □

Addendum – The action of the symmetric group

In this brief addendum, we will discuss a feature of ordered configuration spaces that is seemingly missing in the discussion above, namely the action of the symmetric group that permutes particles. It is straightforward to construct suitable actions of S_2 for the models from the previous actions based on the following observation about the bar construction.

Observation. Let (Y, G, X) be a triple as in Definition 7.1.7 and suppose further that X also has a left action by a group H *such that the action of G and H commute*. Then H acts on the simplicial space $\mathrm{B}_\bullet(Y, G, X)$ and hence on the geometric realization $\mathrm{B}(Y, G, X)$.

The symmetric group S_2 acts naturally on M^2 and $M^2 \setminus \Delta$ and since $T_2^\times(M)$ acts through S_2-equivariant maps, the actions of $T_2^\times(M)$ and S_2 commute. Hence, the associated bar constructions are S_2-spaces over $\mathrm{B}\,T_2^\times(M)$ (i.e. the action preserves the projection to $\mathrm{B}\,T_2^\times(M)$). Moreover, S_2 acts on $S(T_\pi E_2)$ by the (linear) antipodal map on the fibre of $T_\pi E_2 \to E_2$. Since $\mathrm{B}(*, T_2^\times(M), M)$ corresponds to the fibrewise diagonal we equip it with the trivial S_2 action.

Theorem 7.1.16. *The Poincaré embedding structure from Theorem 7.1.8 gives an S_2-equivariant fibre-homotopy equivalence*

$$\mathrm{hocolim}\Big(E_2 \leftarrow S(T_\pi E_2) \to \mathrm{B}(*, T_2^\times(M), M^2 \setminus \Delta)\Big) \longrightarrow E_2 \times_{\mathrm{B}\,T_2^\times(M)} E_2$$

of S_2-equivariant quasifibrations.

We have discussed the theory of equivariant fibrations and quasifibrations in Appendix A.1. The theorem shows that the Poincaré embedding structure we have constructed in Theorem 7.1.8 is in fact compatible with a fibrewise action of S_2 which agrees with standard action on the complement $M^2 \setminus \Delta$ on each fibre.

Proof. We need to check that all of our previous arguments behave well with respect to the natural action of S_2. Let us start with the fibrewise Poincaré embedding structure from Proposition 7.1.10 given by the square below

$$\begin{array}{ccc} \mathrm{B}(*, T_2^{\times}(M), \mathrm{holink}(M^2, \Delta)) & \xrightarrow{\mathrm{B}(ev_1)} & \mathrm{B}(*, T_2^{\times}(M), M^2 \setminus \Delta) \\ \big\downarrow{\scriptstyle \mathrm{B}(\mathrm{ev}_0)} & & \big\downarrow \\ \mathrm{B}(*, T_2^{\times}(M), M) & \xrightarrow{\mathrm{B}(\Delta)} & \mathrm{B}(*, T_2^{\times}(M), M^2). \end{array}$$

As observed above, M^2 and $M^2 \setminus \Delta$ have natural actions of S_2 commuting with the action of $T_2^{\times}(M)$ and the same is true for $\mathrm{holink}(M^2, \Delta)$ and M (where the latter is equipped with the trivial action). It follows that all spaces in the square above are S_2-equivariant quasifibrations over $\mathrm{B}\,T_2^{\times}(M)$ by Theorem A.16. The homotopy colimit of the upper left triangle is a quasi-fibration by the same argument as in the proof of Proposition 7.1.10, and it is an equivariant quasi-fibration because the map in (7.8) is an equivariant homotopy equivalence as the homotopy colimit in Top of diagrams of G-spaces is the homotopy colimit in the category of G-spaces. Therefore, it is sufficient to check that the restriction of the square over the 0-simplex of $\mathrm{B}\,T_2^{\times}(M)$ induces an equivariant homotopy equivalence from the homotopy colimit to M^2.

Over the 0-simplex, we can replace the homotopy link by the sphere bundle of a tubular neighbourhood. If we choose an S_2-equivariant tubular neighbourhood, the inclusion $S(\nu_\Delta) \to \mathrm{holink}(M^2, \Delta)$ is a map of S_2-equivariant fibrations over M. Since the inclusion induces equivalences between the fibres, which have a free S_2-action, it follows from an equivariant Dold Theorem [Wan80b, Thm 1.11] that $S(\nu_\Delta) \to \mathrm{holink}(M^2, \Delta)$ is a fibrewise S_2-homotopy equivalence. Hence, we can replace the homotopy colimit by the standard Poincaré embedding structure of the diagonal, which does induce an equivariant homotopy equivalence.

We can then check that the comparison of $\mathrm{B}(\Delta)$ in Lemma 7.1.11 with the fibrewise diagonal is valid S_2-equivariantly as (7.10) is a diagram of S_2-equivariant quasifibrations and fibrewise S_2-homotopy equivalences. Finally, it remains to show that the gluing space is equivariantly equivalent to $S(T_\pi E_2)$. This follows again from the existence of equivariant tubular neighbourhoods and the fact that the map

$$d/dt : \mathrm{holink}^{\pitchfork}(M^2, \Delta) \to \left(\frac{TM^2|_\Delta}{T\Delta}\right)^{\#}$$

which is used in (7.12) is a fibrewise S_2-homotopy equivalence with respect to the antipodal action on the normal bundle. $\square$

7.2. An application to tautological classes

Theorem 7.1.8 provides the diagonal $\Delta : E_2 \to E_2 \times_{\mathrm{B}\, T_2^\times(M)} E_2$ with a regular neighbourhood which is equivalent to the vector bundle $T_\pi E_2 \to E_2$. This links the vector bundle structure with the global topology of the fibration in an intricate way, and we recognize this property from fibre bundles where the vertical tangent bundle also provides a regular neighbourhood of the diagonal. This is the basic reason why for fibre bundles the Euler class of the vertical tangent bundle and the fibrewise Euler classes coincide (see [HLLR17, Sect. 3.2]), and hence the same is true for $E_2 \to \mathrm{B}\, T_2^\times(M)$.

Theorem 7.2.1. *The fibrewise Euler class of $E_2 \to \mathrm{B}\, T_2^\times(M)$ agrees with the Euler class of the vector bundle $T_\pi E_2 \to E_2$.*

We could follow the strategy in [HLLR17] to show that $e(T_\pi E_2)$ agrees with the fibrewise Euler class. But instead we use a fibrewise version of Corollary 7.1.5, which we expect to be useful as a characterization of the propagator form in the rational homotopy theory of embedding calculus.

Proposition 7.2.2. *Let $E \to B$ be an oriented fibration with Poincaré fibre X of formal dimension d such that the diagonal $\Delta : E \to E \times_B E$ admits the structure of a Poincaré embedding with complement C. Then $H^*(C) \cong H^*(E \times_B E)/(\Delta_!(1))$, where $\Delta_!$ is the fibrewise Umkehr map on cohomology.*

The proof is completely analogous but relies on a version of Poincaré duality for fibrations with Poincaré fibre. This is best expressed using parametrized stable homotopy theory and spectra and we will briefly recall the necessary objects following the discussion in [HLLR17, Sect. 3], which are also the crucial ingredients to define the fibrewise Euler class and Euler ring that we studied in the first part of this thesis.

Let $E \to B$ be a fibration with Poincaré fibre and $\Sigma^\infty_B E_+$ the parametrized suspension spectrum. Denote by $H_B\mathbb{Z}$ and $\mathbb{S}_B$ the trivial parametrized Eilenberg-MacLane and sphere spectrum, and by $F_B(\Sigma^\infty_B E_+, H_B\mathbb{Z})$ the internal parametrized mapping spectrum whose homotopy groups $[\mathbb{S}_B^{-d}, F_B(\Sigma^\infty_B E_+, H_B\mathbb{Z})]_B = H^d(E;\mathbb{Z})$ recover the cohomology of E. Then *fibrewise Poincaré duality* is an equivalence of $H_B\mathbb{Z}$-modules

$$D_E^{\mathrm{fw}} : \Sigma_B^d F_B(\Sigma_B^\infty E_+, H_B\mathbb{Z}) \xrightarrow{\simeq} \Sigma_B^\infty E_+ \wedge_B H_B\mathbb{Z}$$

which is essential in defining the Umkehr map $\Delta_! : H^*(E) \to H^{*+d}(E \times_B E)$ on the level of spectra as $(D^{\mathrm{fw}}_{E\times_B E})^{-1} \circ (\Delta \wedge_B \mathrm{Id}) \circ D_E^{\mathrm{fw}} : F_B(\Sigma_B^\infty E_+, H_B\mathbb{Z}) \to \Sigma_B^d F_B(\Sigma_B^\infty E \times_B E_+, H_B\mathbb{Z})$ and the fibrewise Euler class as $e^{\mathrm{fw}}(\pi) := \Delta^*\Delta_!(1) \in H^d(E)$.

Remark 7.2.3. Observe that we have proved in Proposition 4.1.1 a version of fibrewise Poincaré duality in rational homotopy theory and that Definition 4.1.2 of the fibrewise Euler class is exactly the same.

We will briefly recall the construction of the fibrewise duality map D_E^{fw}. We have to shorten the notation and denote the suspension spectrum $\Sigma_B^\infty E_+$ by E and omit subscripts although all constructions are over B. Then the fundamental classes of the fibres $[E_b] \in H_d(E_b;\mathbb{Z}) \cong H^0(b, (E \wedge H\mathbb{Z})|_b^{-d})$ assemble in the Atiyah-Hirzebruch spectral to give a fundamental class in $(E \wedge H\mathbb{Z})^{-d}(B)$ that we denote by $[E]_B : \mathbb{S}^d \to E \wedge H\mathbb{Z}$. Then D_E^{fw} is defined as the composition

$$\begin{aligned} \mathbb{S}^d \wedge F(E, H\mathbb{Z}) \xrightarrow{[E]_B \wedge \mathrm{Id}} E \wedge H\mathbb{Z} \wedge F(E, H\mathbb{Z}) &\xrightarrow{\Delta\wedge \mathrm{Id}} E \wedge E \wedge H\mathbb{Z} \wedge F(E, H\mathbb{Z}) \\ &\xrightarrow{\mathrm{Id}\wedge \mathrm{ev} \wedge \mathrm{Id}} E \wedge H\mathbb{Z} \wedge H\mathbb{Z} \longrightarrow E \wedge H\mathbb{Z} \end{aligned}$$

and it is an equivalence [HLLR17] which can be checked fibrewise where it reduces to ordinary Poincaré duality. Moreover, there is also a relative version of fibrewise Poincaré duality: Let $E' \subset E$ a pair of fibrations over B such that the fibres $F' \to F$ are Poincaré duality pairs. Then the fundamental classes $[E_b, E'_b] \in H_d(E_b, E'_b; \mathbb{Z}) \cong H^0(b, (E/E' \wedge H\mathbb{Z})|_b^{-d})$ assemble to give a fundamental class $[E, E']_B : \mathbb{S}^d \to E/E' \wedge H\mathbb{Z}$ and we define

$$D^{\mathrm{fw}}_{(E,E')} : \mathbb{S}^d \wedge F_B(E/E', H\mathbb{Z}) \xrightarrow{\simeq} E \wedge_B H\mathbb{Z}.$$

using the diagonal $\Delta_{E/E'} : E/E' \to E \wedge E/E'$. Here, $E/E' = \mathrm{cofib}(E' \to E)$ is the cofibre in the category of parametrized spectra; if the underlying map of fibrations is a cofibration then it is the suspension spectrum of the fibrewise quotient. Then $D^{\mathrm{fw}}_{E,E'}$ is an equivalence by the same fibrewise argument and relative Poincaré duality.

Proof of 7.2.2. We will first prove a fibrewise version of Lemma 7.1.4 to show that for a map of fibrations $f : K \to X$ over a base space B which admits a fibrewise Poincaré embedding structure $X = \mathrm{hocolim}(K \leftarrow G \to C)$, the cohomology of the complement C is determined by the map $f : K \to X$ on the fibrewise analogue of homology $[\mathbb{S}_B, - \wedge H\mathbb{Z}]$.

Denote the induced pushout of parametrized suspension spectra over B the same. We mimic the proof of Lemma 7.1.4 and study the long exact sequence of the pair (X, C) on cohomology, which on the level of spectra this corresponds to studying the long exact sequence of homotopy groups of $F_B(X/C, H\mathbb{Z}) \to F_B(X, H\mathbb{Z}) \to F_B(C, H\mathbb{Z})$. Without loss of generality we may assume that $C \to X$ and $G \to K$ are cofibrations so that the homeomorphism $f : K/G \to X/C$ of spaces over B induces an isomorphism of suspension spectra. Let $[X]_B : S^d_B \to X$ be a representative of the fundamental class of X. By assumption X is a pushout hocolim$(K \leftarrow G \to C)$ via the fibrewise Poincaré embedding structure and the fundamental class $[X]_B$ induces a fundamental class of (K, G) via the collapse

$$\mathbb{S}^d \xrightarrow{[X]_B} X \wedge H\mathbb{Z} \xrightarrow{c \wedge \mathrm{Id}} X/C \wedge H\mathbb{Z} \overset{\approx}{\leftarrow} K/G \wedge H\mathbb{Z}$$

(and similarly for (C, G)). Then the following diagram is analogous to (7.2)

$$\begin{tikzcd}
S^d \wedge F(X/C, H\mathbb{Z}) \arrow[rr, "\mathrm{Id} \wedge c^*"] \arrow[d, "{[X]_B \wedge \mathrm{Id}}"] & & S^d \wedge F(X, H\mathbb{Z}) \arrow[d, "{[X]_B \wedge \mathrm{Id}}"] \\
X \wedge H\mathbb{Z} \wedge F(X/C, H\mathbb{Z}) \arrow[rr, "\mathrm{Id} \wedge c^*"] \arrow[dr, "\Delta_X \wedge \mathrm{Id}"] \arrow[dd, "c \wedge \mathrm{Id}"] & & X \wedge H\mathbb{Z} \wedge F(X, H\mathbb{Z}) \arrow[d, "\Delta_X \wedge \mathrm{Id}"] \\
{} \arrow[r, phantom, "\mathrm{II}"] & X \wedge X \wedge H\mathbb{Z} \wedge F(X/C, H\mathbb{Z}) \arrow[r, "\mathrm{Id} \wedge c^*"] \arrow[d, "\mathrm{Id} \wedge c \wedge \mathrm{Id}"] & X \wedge X \wedge H\mathbb{Z} \wedge F(X, H\mathbb{Z}) \arrow[d, "\mathrm{ev}"] \\
X/C \wedge H\mathbb{Z} \wedge F(X/C, H\mathbb{Z}) \arrow[r, "\Delta_{X/C}"] & X \wedge X/C \wedge H\mathbb{Z} \wedge F(X/C, H\mathbb{Z}) \arrow[r, "\mathrm{ev}"] & X \wedge H\mathbb{Z} \wedge H\mathbb{Z} \arrow[dd, equal] \\
K/G \wedge H\mathbb{Z} \wedge F(K/G, H\mathbb{Z}) \arrow[u, "\approx"] \arrow[d, "\Delta_{K/G}"] \arrow[r, phantom, "\mathrm{I}"] & {} & \\
K \wedge K/G \wedge H\mathbb{Z} \wedge F(K/G, H\mathbb{Z}) \arrow[r, "f \wedge \mathrm{Id}"] \arrow[d, "\mathrm{ev}"] & X \wedge K/G \wedge H\mathbb{Z} \wedge F(K/G, H\mathbb{Z}) \arrow[r, "\mathrm{ev}"] \arrow[uu, "\approx"] \arrow[d, "\mathrm{ev}"] & X \wedge H\mathbb{Z} \wedge H\mathbb{Z} \arrow[dd] \\
K \wedge H\mathbb{Z} \wedge H\mathbb{Z} \arrow[r, "f \wedge \mathrm{Id}"] \arrow[d] & X \wedge H\mathbb{Z} \wedge H\mathbb{Z} \arrow[d] & \\
K \wedge H\mathbb{Z} \arrow[r, "f \wedge \mathrm{Id}"] & X \wedge H\mathbb{Z} \arrow[r, equal] & X \wedge H\mathbb{Z}
\end{tikzcd} \tag{7.15}$$

The commutativity of most squares are obvious and commutativity of squares I and II follow from the the following commutative diagram of diagonals

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "\Delta_X"] & X/C \arrow[d, "\Delta_{X/C}"] & K/G \arrow[l, "\approx", "f"'] \arrow[d, "\Delta_{K/G}"] & K \arrow[l] \arrow[d, "\Delta_K"] \\
X \wedge X \arrow[r, "c \wedge \mathrm{Id}"] & X \wedge (X/C) & K \wedge (K/G) \arrow[l, "f \wedge f"'] & K \wedge K. \arrow[l]
\end{tikzcd}$$

Observe that the composition of vertical maps on the right in (7.15) is the fibrewise Poincaré duality map D^{fw}_X and that under the identification $F(X/C, H\mathbb{Z}) \approx F(K/G, H\mathbb{Z})$ the composition of vertical maps on the left is the relative Poincaré duality map $D^{\mathrm{fw}}_{K,G}$. Hence, the long exact sequence in cohomology of (X, C) is determined by the map f.

We now apply this to a fibrewise Poincaré embedding structure of the diagonal

$$\begin{tikzcd} G \arrow[r] \arrow[d] & C \arrow[d] \\ E \arrow[r, "\Delta"] & E \times_B E \end{tikzcd}$$

and use D^{fw}_E to obtain the following commutative diagram

$$\begin{tikzcd} \left[\mathbb{S}^{-k}, \mathbb{S}^{2d} \wedge F(E \times_B E/C, H\mathbb{Z})\right]_B \arrow[r] \arrow[d, "\cong"] & \left[\mathbb{S}^{-k}, \mathbb{S}^{2d} \wedge F(E \times_B E, H\mathbb{Z})\right]_B \arrow[d, "\cong", "D^{\mathrm{fw}}_{E\times_B E}"'] \\ \left[\mathbb{S}^{-k}, E \wedge H\mathbb{Z}\right]_B \arrow[r, "\Delta"] & \left[\mathbb{S}^{-k}, E \times_B E \wedge H\mathbb{Z}\right]_B \\ \left[\mathbb{S}^{-k}, \mathbb{S}^{d} F(E, H\mathbb{Z})\right]_B \arrow[r, dashed, "\Delta_!"] \arrow[u, "D^{\mathrm{fw}}_E", "\cong"'] & \left[\mathbb{S}^{-k}, \mathbb{S}^{2d} \wedge F(E \times_B E, H\mathbb{Z})\right]_B \arrow[u, "D^{\mathrm{fw}}_{E\times_B E}", "\cong"'] \end{tikzcd}$$

where the bottom map is the definition of the fibrewise Umkehr map $\Delta_!$. The corners of this diagram are identified with cohomology groups and the first row is equivalent to the restriction $H^{k+2d}(E \times_B E, C) \to H^{k+2d}(E \times_B E)$. Hence, we have shown that the long exact sequence in cohomomlogy of the pair $(E \times_B E, C)$ is equivalent to

$$\ldots \longrightarrow H^{*-d}(E) \xrightarrow{\Delta_!} H^*(E \times_B E) \xrightarrow{i^*} H^*(C) \longrightarrow \ldots$$

The rest of the claim follows in exactly the same way as before: the Umkehr map $\Delta_!$ is injective as the diagonal has a left-inverse given by projection, and since every element $x \in H^*(E)$ can be written as $\Delta^*\pi_i^*(x)$ it follows that the image of $\Delta_!$ is the ideal generated by $\Delta_!(1)$ because $\Delta_!$ is a Δ^*-module map. □

Proof of Theorem 7.2.1. Consider the fibrewise Poincaré embedding structure of the diagonal constructed in Theorem 7.1.8

$$\begin{tikzcd} S(T_\pi E_2) \arrow[r, "i"] \arrow[d, "\pi"] & C \arrow[d, "j"] \\ E_2 \arrow[r, "\Delta"] & E_2 \times_{\mathrm{B}\, T_2^\times(M)} E_2, \end{tikzcd}$$

where we denote the complement by C and the maps are the one we have constructed before. Then $i^* j^* \Delta_!(1)$ vanishes by Proposition 7.2.2 and therefore $0 = i^* j^* \Delta_!(1) = \pi^* \Delta^*(\Delta_!(1)) = \pi^*(e^{\mathrm{fw}})$. Then it follows from the Gysin sequence of $S(T_\pi E_2) \to E_2$ that the fibrewise Euler class is a multiple of $e(T_\pi E_2)$, and since they agree when restricted to a fibre of $E_2 \to \mathrm{B}\, T_2^\times(M)$, the two Euler classes must coincide. □

7.3. The first non-trivial Kontsevich class on $\mathrm{B}\,T_2^{\times}(M)$

The goal of this section is to develop a proof of concept that one can define Kontsevich's characteristic classes from Theorem 6.2.3 via configurations space integrals on the embedding calculus tower directly. As we haven't constructed fibrewise homotopy configuration spaces for more than two particles, we have to limit ourselves to graph classes involving not more than two vertices, which effectively leaves only the simplest such class from Theorem 6.2.5 obtained by fibre integrating the class associated to the θ-graph. Observe that this cohomology class can be constructed and studied without reference to a graph complex as has been done in [Wat09a], and we follow this point of view here. It is an interesting open problem whether one can extend the graph complex construction over the self-embedding tower.

There is a further complication because the graph characteristic classes are defined for diffeomorphisms that are the identity on some neighbourhood of a fixed point $\infty \in M$, which is equivalent to $\mathrm{Diff}(M, U_\infty) = \mathrm{Diff}_\partial(M \setminus U_\infty)$ for some fixed neighbourhood $\infty \in U_\infty \cong \mathbb{R}^d$. One can develop embedding calculus for manifolds with boundaries but the Haefliger model has not been generalized to this context so far. In the following, we give an ad hoc modification of the Haefliger model to this situation and construct that graph class for it.

Let M be an odd dimensional integral homology sphere and fix a point $\infty \in M$. Denote by $\mathrm{Map}_\infty(M, M) \subset \mathrm{Map}_{C^\infty}(M, M)$ the space of smooth maps $f : M \to M$ such that $f^{-1}(\infty) = \infty$ and $f|_U = \mathrm{Id}$ for some neighbourhood U of $\infty \in M$. Similarly, define $\mathrm{Map}_\infty^{S_2}(M^2, M^2) \subset \mathrm{Map}_{C^\infty}^{S_2}(M^2, M^2)$ as the subspace of smooth equivariant maps $F : M^2 \to M^2$ satisfying

- $F^{-1}(\infty \times \infty) = \infty \times \infty$ and $F|_{U \times U} = \mathrm{Id}$ for some neighbourhood U of $\infty \in M$,
- $F^{-1}(\infty \times M) = \infty \times M$ and $D(\pi_1 \circ F)|_{T_\infty M} : T_\infty M \subset T_{(\infty,p)} M^2 \to T_\infty M$ is the identity for all $p \in M$.

where the second condition has similar consequences for $M \times \infty$ by equivariance. The product map $f \mapsto f \times f$ determines a map

$$\mathrm{Map}_\infty(M, M) \longrightarrow \mathrm{Map}_\infty^{S_2}(M^2, M^2).$$

Finally, we define $\mathrm{IvMap}_\infty(M^2, M^2) \subset \mathrm{Map}_\infty^{S_2}(M^2, M^2)$ as the subspace of strongly isovariant maps.

Definition 7.3.1. We define a model for the second pointed Taylor approximation as

$$T_2(M, \infty) := \mathrm{holim}\left(\mathrm{Map}_\infty(M, M) \to \mathrm{Map}_\infty^{S_2}(M^2, M^2) \leftarrow \mathrm{IvMap}_\infty(M^2, M^2)\right).$$

Then $T_2(M, \infty)$ is a submonoid of $T_2(M)$ via the inclusion $T_2(M, \infty) \to T_2(M)$. If we consider the first two stages of the Taylor tower in (6.5), observe that $\eta_2|_{\mathrm{Diff}(M, U_\infty)}$ has image in $T_2(M, \infty)$ and that $r_2|_{T_2(M,\infty)}$ has image in tangential homotopy equivalences of TM that are the identity on $TM|_U$ for some neighbourhood of $\infty \in M$ and such that the preimage of ∞ is only ∞. This can be taken as ad hoc definition for $T_1(M, \infty)$.

Let $\tau : T(M \setminus \infty) \to (M \setminus \infty) \times \mathbb{R}^d$ be a framing and consider the space of framings $\mathrm{Fr}^+(M; \tau)$ introduced in Section 6.2. There is a homeomorphism from $\mathrm{Fr}^+(TM; \tau)$ to the space of bundle maps of $T(M \setminus \infty)$ to the trivial bundle over a point $\mathbb{R}^d \to *$ given by $\tau_M \to \pi_{\mathbb{R}^d} \circ \tau_M$ that agrees with $\pi_{\mathbb{R}^d} \circ \tau$ on a neighbourhood of ∞. We denote this space by $\mathrm{Bun}^+_\infty(TM, \mathbb{R}^d; \tau)$ (for $M = S^d$ this can be identified with linear maps $T\mathbb{R}^d \to \mathbb{R}^d$ that are standard outside a compact set). Under this homeomorphism, the action of $\mathrm{Diff}(M, U_\infty)$ on $\mathrm{Bun}^+_\infty(TM, \mathbb{R}^d; \tau)$ is given by precomposition with the derivative and thus extends to an action of $T_1(M, \infty)$ and consequently by $T_2(M, \infty)$ via the restriction r_2. Then the classifying space of pointed framed M-bundles is $\mathrm{B}\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty; \tau) = \mathrm{B}(\mathrm{Bun}^+_\infty(TM, \mathbb{R}^d; \tau), \mathrm{Diff}(M, U_\infty), *)$ and we can make an analogous definition

$$\mathrm{B}\, T_2^{\mathrm{fr}}(M, \infty; \tau) := \mathrm{B}(\mathrm{Bun}^+_\infty(TM, \mathbb{R}^d; \tau), T_2^\times(M, \infty), *) \tag{7.16}$$

so that there is a map

$$\mathrm{B}(*, \eta_2, *) : \mathrm{B}\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty; \tau) \to \mathrm{B}\, T_2^{\mathrm{fr}}(M, \infty; \tau). \tag{7.17}$$

Theorem 7.3.2. *There exists a cohomology class $\bar{\zeta}_2 \in H^{d-3}(\mathrm{B}\, T_2^{\mathrm{fr}}(M, \infty; \tau); \mathbb{Z})$ such that $\mathrm{B}(\eta_2)^*(\bar{\zeta}_2)$ agrees with the first non-trivial Kontsevich class $\zeta_2(\pi_{univ}, \tau) \in H^{d-3}(\mathrm{B}\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty; \tau); \mathbb{Z})$ from Theorem 6.2.5 of the universal vertically framed M-bundle.*

Proof. It suffices to construct a propagator class ω in the cohomology of the fibrewise configuration space $\mathrm{B}(\mathrm{Bun}^+_\infty(TM, \mathbb{R}^d; \tau), T_2^\times(M, \infty), C_2(M \setminus \infty))$, where $T_2(M, \infty)$ acts on the configuration space $C_2(M \setminus \infty)$ through projection to $\mathrm{IvMap}_\infty(M^2, M^2)$. The strategy of the construction of ω is analogous as to the case of fibre bundles, and we could again use (transversal) homotopy links as a homotopy theoretic replacement for the boundary of the respective compactifications. Instead, we will describe an action of $T_2(S^d, \infty)$ on the compactification $\overline{\mathrm{C}}_2(M, \infty)$ directly.

Observe that the action of $T_2(M, \infty)$ on $C_2(M \setminus \infty)$ is compatible with the action of $\mathrm{Diff}(M, U_\infty)$ via η_2, and we will describe how this can be extended to the compactification $\overline{C}_2(M,\infty)$. Recall from (6.15) that the boundary $\partial\overline{C}_2(M,\infty)$ is the union of four pieces

$$\begin{aligned} S_\infty &\cong p^{-1}(\infty\times\infty)\setminus\Sigma_M \subset Bl(M^2,\infty\times\infty)\\ S_{\infty\times M} &= p^{-1}(\infty\times(M\setminus\infty))\subset\overline{C}_2(M,\infty)\\ S_{M\times\infty} &= p^{-1}((M\setminus\infty)\times\infty)\subset\overline{C}_2(M,\infty)\\ S_\Delta &= p^{-1}(\Delta_{M\setminus\infty})\subset\overline{C}_2(M,\infty), \end{aligned}$$

where p here denotes the blowdown maps. By definition, all maps in $\mathrm{Map}^{S_2}_\infty(M^2,M^2)$ are required to fix a neighbourhood of $\infty\times\infty$ so that the actions extends trivially to $S_{\infty\times\infty}$. And similarly, since the equivariant maps act trivially on the normal bundle of $\infty\times M$ and $M\times\infty$ by assumption, the action extends trivially to $S_{\infty\times M}$ and $S_{M\times\infty}$. Lastly, S_Δ is identified with the unit normal bundle of $T(M\setminus\infty)$ and under this identification, $T_2(S^d,\infty)$ acts via $r_2: T_2(S^d,\infty)\to T_1(S^d,\infty)$ which captures the tangential behaviour on the normal bundle of the diagonal. Since $T_1(S^d,\infty)$ acts trivially close to ∞, this describes a compatible action on their union.

Recall from Lemma 6.2.1 that the propagator is constructed via $p(\tau_M):\partial\overline{C}_2(M,\infty)\to S^{d-1}$ which depends on $\tau_M\in\mathrm{Fr}^+(TM;\tau)\approx\mathrm{Bun}^+_\infty(TM,\mathbb{R}^d;\tau)$. From the proof in [Wat09a] we see that $p(\tau_M)$ only depends on τ_M on S_Δ. Hence, the map

$$p(-):\mathrm{Bun}^+_\infty(TM,\mathbb{R}^d;\tau)\times\partial\overline{C}_2(M,\infty)\longrightarrow S^{d-1}$$

factors through the quotient with respect to the action of $T_2(M,\infty)$ and there is a commutative diagram

$$\begin{tikzcd} \mathrm{B}(\mathrm{Bun}^+_\infty(TM,\mathbb{R}^d;\tau),\mathrm{Diff}(M,U_\infty),\partial\overline{C}_2(M,\infty)) \arrow[r, "\epsilon(p(-))"] \arrow[d, "\mathrm{B}(*,\eta_2,*)"'] & S^{d-1} \\ \mathrm{B}(\mathrm{Bun}^+_\infty(TM,\mathbb{R}^d;\tau),T_2^\times(M,\infty),\partial\overline{C}_2(M,\infty)) \arrow[ur, bend right, "\epsilon(p(-))"'] & \end{tikzcd} \tag{7.18}$$

Then by the same cohomological argument as for fibre bundles, there exits a unique propagator class

$$\omega\in H^{d-1}(\mathrm{B}(\mathrm{Bun}^+_\infty(TM,\mathbb{R}^d),T_2^\times(M,\infty),\overline{C}_2(M,\infty));\mathbb{Z})$$

whose square has a canonical choice of relative class $\tilde\omega_2$ in

$$H^{d-1}(\mathrm{B}(\mathrm{Bun}^+_\infty(TM,\mathbb{R}^d;\tau),T_2^\times(M,\infty),\overline{C}_2(M,\infty)),\mathrm{B}(\mathrm{Bun}^+_\infty(TM,\mathbb{R}^d;\tau),T_2^\times(M,\infty),\partial\overline{C}_2(M,\infty));\mathbb{Z}).$$

Finally, for $\tilde\omega_3:=\omega\cdot\tilde\omega_2$ we can define

$$\bar\zeta_2:=\pi_!(\tilde\omega_3)\in H^{d-3}(\mathrm{B}\,T_2^{\mathrm{fr}}(M,\infty;\tau);\mathbb{Z})$$

where we fibre integrate over the pair of quasifibrations over $\mathrm{B}\, T_2^{\mathrm{fr}}(M, \infty)$ given by the two-sided bar construction above and with fibres the Poincaré pair $(\overline{C}_2(M, \infty), \partial\overline{C}_2(M, \infty))$.

Crucially, the propagator ω pulls back to the propagator constructed over $\mathrm{B}\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty; \tau)$ by the diagram (7.18), so that $\bar{\zeta}_2$ pulls back to $\zeta_2(\pi_{\mathrm{univ}}, \tau) \in H^{d-3}(\mathrm{B}\,\mathrm{Diff}^{\mathrm{fr}}(M, U_\infty; \tau); \mathbb{Z})$ by naturality of fibre integration. $\square$

Remark 7.3.3. The author expects that the ad hoc construction $T_2(M, \infty)$ is equivalent to the second Taylor approximation of $\mathrm{Emb}_\partial(-, M \setminus \mathrm{int}\, D^d)$ evaluated on $M \setminus \mathrm{int}\, D^d$. This is because the homotopy fibre of $r_2 : T_2(M, \infty) \to T_1(M, \infty)$ has a similar description of the usual Layers of the tower in terms of a space of sections of a fibrations with base and fibre related to configuration spaces of $M \setminus \infty$. This shows that $T_2(M, \infty)$ considered as a suitable functor is polynomial of degree ≤ 2 and so it only remains to show that it agrees with $\mathrm{Emb}_\partial(-, M \setminus \mathrm{int}\, D^d)$ on a union of at most two discs. But we also expect that one can simply generalize the construction and proofs in [GKW03] to include manifolds with boundaries.

Chapter 8.

The failure of the family signature theorem over $\mathrm{B}\, T_2^{\times}(M)$

It is unknown how good the Taylor approximation

$$\mathrm{Diff}(M) \longrightarrow T_\infty \,\mathrm{Emb}(M, M)$$

is in the codimension 0 case. One possible way to check the connectivity of the above map is via the family signature theorem, which describes an intricate link between the vertical tangent bundle and the global topology of the bundle. It is a natural question whether the same is true for the TM-fibrations $T_\pi E_k \to E_k$ over the tower. The main result of this section shows that this does not hold on $\mathrm{B}\, T_2^{\times}(M)$.

Theorem 8.0.1. *The family signature theorem does not hold universally on $\mathrm{B}\, T_2^{\times}(M)$. More precisely, for all smooth closed oriented manifolds M^{2d} and all $i \in \mathbb{N}$ satisfying $d < 2i \leq 2d-2$ the class $\sigma_{4i-2d} - \kappa_{L_i} \in H^{4i-2d}(\mathrm{B}\, T_2^{\times}(M); \mathbb{Q})$ does not vanish.*

We will prove this theorem by constructing elements in the homotopy groups of $\mathrm{B}\, T_2^{\times}(M)$ for which we can control the signature and MMM-classes. In the following, we denote by $\mathcal{G}(TM)$ the space of orientation preserving orthogonal bundle maps of TM covering the identity for some choice of metric, i.e. $\mathcal{G}(TM)$ is isomorphic to the gauge group of the frame bundle $\mathrm{Fr}^+(TM)$. Similarly, we denote by $h\mathcal{G}^{S_2}(S(TM))$ the space of orientation preserving fibrewise S_2-homotopy equivalences of $S(TM)$ that cover the identity.

Definition 8.0.2. For an oriented smooth manifold M we define

$$F(M, 2) := \mathrm{hofib}_{\mathrm{Id}}\left(\mathcal{G}(TM) \longrightarrow h\mathcal{G}^{S_2}(S(TM))\right). \tag{8.1}$$

A point $(\bar{f}, H) \in F(M, 2)$ consists of a bundle map $\bar{f} : TM \to TM$ covering the identity together with a fibrewise S_2-homotopy $H : S(TM) \times I \to S(TM)$ from the map of the sphere bundle induced by $\bar{f}$ to the identity. Then $F(M, 2)$ is a monoid under point-wise composition of H, and we will show later on that $\mathrm{B}\, F(M, 2)$ is equivalent to the space defined in (8.5). The key advantage in considering $F(M, 2)$ is the following.

Theorem 8.0.3. *There is a continuous map* $\mathrm{B}\,\bar{G} : \mathrm{B}\,F(M,2) \longrightarrow \mathrm{B}\,T_2^{\times}(M)$ *such that the composition* $\mathrm{B}(r_2) \circ \mathrm{B}\,\bar{G} : \mathrm{B}\,F(M,2) \to \mathrm{B}\,T_1^{\times}(M)$ *is homotopic to the delooping of the composition* $F(M,2) \to \mathcal{G}(TM) \hookrightarrow T_1^{\times}(M)$ *that forgets the* S_2*-homotopy. In particular, the composition* $\mathrm{B}(r_1 \circ r_2) \circ \mathrm{B}\,\bar{G} : \mathrm{B}\,F(M,2) \to \mathrm{B}\,T_0^{\times}(M)$ *is null homotopic.*

The idea of the construction of such a map is simple. For a point $(\bar{f}, H) \in F(M,2)$ we can define a map

$$G : F(M,2) \longrightarrow \mathrm{Map}^{S_2}(M^2, M^2)^I$$

by fixing an S_2-equivariant tubular neighbourhood of the diagonal $\varphi : TM \hookrightarrow M^2$ and defining $(G_t)|_{TM}$ as

$$G_t(\bar{f}, H)(v_p) := \begin{cases} 0_p & |v_p| = 0 \\ |v_p| \cdot H_{1-t}(v_p/|v_p|) & 0 < |v_p| \leq 1 \\ |v_p| \cdot H_{1-t+(|v_p|-1)\cdot t}(v_p/|v_p|) & 1 \leq |v_p| \leq 2 \\ v_p & |v_p| > 2 \end{cases} \tag{8.2}$$

and $(G_t)|_{M^2 \setminus TM} = \mathrm{Id}$ outside the tubular neighbourhood. Observe that $G_t : M^2 \to M^2$ is an isovariant map S_2-homotopic to the identity $\mathrm{Id}_{M^2} = G_0$, and that $G_1 : M^2 \to M^2$ restricted to the disk bundle of length ≤ 1 is the linear map $\bar{f}$ on the fibres. Hence, the triple (Id_M, G, G_1) is almost an element in $T_2\,\mathrm{Emb}(M,M)$ were it not for the lack of smoothness of the maps defined. We will therefore find a weakly equivalent model of the Haefliger model where we relax the smoothness conditions to require only smoothness on a tubular neighbourhood of the diagonal.

In the following, we will pick a Riemannian metric on M which will also determine a metric on an S_2-equivariant tubular neighbourhood of the diagonal $\nu(\Delta_M) \subset M^2$. But the choice will not matter for the constructions.

Definition 8.0.4. Let $\mathrm{IvMap}_{C_0}(M^2, N^2)$ be the space of isovariant maps with the subspace topology of the space of continuous equivariant maps. For $\epsilon > 0$ we define $\mathrm{IvMap}^{\epsilon}(M^2, N^2)$ as the subset of $\mathrm{IvMap}^{C_0}(M^2, M^2)$ of maps whose restriction to $\nu_{<\epsilon}(\Delta_M) \subset M^2$ is smooth and satisfies the strong isovariance condition on $\nu_{<\epsilon}(\Delta_M)$, and topologized as the subspace of the inclusion

$$\mathrm{IvMap}^{\epsilon}(M^2, N^2) \xrightarrow{\text{incl.}\times\text{rest.}} \mathrm{IvMap}_{C_0}(M^2, N^2) \times \mathrm{Map}^{S_2}_{C^\infty}(\nu_{<\epsilon}(\Delta_M), N^2).$$

Then we define $\mathrm{IvMap}^{\Delta}(M^2, N^2)$, the space of isovariant maps that are strongly isovariant near the diagonal, as the colimit $\bigcup_{n \geq 0} \mathrm{IvMap}^{1/n}(M^2, N^2)$. Finally, we define

$$\overline{T}_2\,\mathrm{Emb}(M,N) := \mathrm{holim}\left(\mathrm{Map}_{C_0}(M,N) \to \mathrm{Map}^{S_2}_{C_0}(M^2, N^2) \leftarrow \mathrm{IvMap}^{\Delta}(M^2, N^2)\right), \tag{8.3}$$

where $\mathrm{Map}_{C_0}(M,N)$ and $\mathrm{Map}^{S_2}_{C_0}(M^2,N^2)$ are the spaces of continuous maps with the usual compact-open topology.

By inspection, the construction in (8.2) defines a map of topological monoids

$$G : F(M,2) \longrightarrow \overline{T}_2 \,\mathrm{Emb}(M,M). \tag{8.4}$$

Proposition 8.0.5. *The inclusion*

$$T_2 \,\mathrm{Emb}(M,N) \longrightarrow \overline{T}_2 \,\mathrm{Emb}(M,N)$$

is a weak homotopy equivalence.

Proof. Consider the comparison of homotopy pullback squares

$$\begin{tikzcd}
& \overline{T}_2 \,\mathrm{Emb}(M,N) \arrow[rr] \arrow[dd] & & \mathrm{IvMap}^{\Delta}(M^2,N^2) \arrow[dd] \\
T_2 \,\mathrm{Emb}(M,N) \arrow[rr] \arrow[ur] \arrow[dd] & & \mathrm{IvMap}(M^2,N^2) \arrow[ur] \arrow[dd] & \\
& \mathrm{Map}_{C_0}(M,N) \arrow[rr] & & \mathrm{Map}^{S_2}_{C_0}(M^2,N^2) \\
\mathrm{Map}_{C_\infty}(M,N) \arrow[rr] \arrow[ur] & & \mathrm{Map}^{S_2}_{C_\infty}(M^2,N^2) \arrow[ur] &
\end{tikzcd}$$

induced by the natural inclusions. The inclusion of the space of smooth maps into the space of continuous maps is a weak equivalence since one can approximate all continuous maps by smooth maps and continuous homotopies between smooth maps by smooth homotopies (see for example [Hir94, Ch. 2]).

Similarly, the comparison $\mathrm{IvMap}(M^2,N^2) \to \mathrm{IvMap}^{\Delta}(M^2,N^2)$ is a weak equivalence. To see this, observe that any element in $[f] \in \pi_k(\mathrm{IvMap}^{\Delta}(M^2,N^2))$ is represented by a map f whose image is contained in $\mathrm{IvMap}^{1/N}(M^2,N^2)$ for some $N \in \mathbb{N}$. This follows from an adaptation of the proof of [Hat02, Prop. A.1] applied to $\mathrm{IvMap}^{\Delta}(M^2,N^2)$ as the colimit of metrizable spaces $\mathrm{IvMap}^{1/n}(M^2,N^2)$. We can then approximate the adjoint of f by smooth maps arbitrarily closely away from $S^k \times \nu_{<1/N}(\Delta_M)$ where the adjoint is already smooth. This can be done equivariantly, for example by observing that $\mathrm{Map}^{S_2}(M^2,N^2) = \mathrm{Map}(M^2,N)$ and then using ordinary smooth approximations, and preserving the isovariance condition since the image of the complement $S^k \times \nu_{<1/N}(\Delta_M)$ is contained in some complement of a tubular neighbourhood of the diagonal $\Delta_N \subset N^2$ so that this remains true if the smoothing is close enough. This shows that $\mathrm{IvMap}(M^2, N^2) \to \mathrm{IvMap}^{\Delta}(M^2, N^2)$ is surjective on homotopy groups and a relative argument shows injectivity.

It follows that the map on homotopy limits $T_2 \operatorname{Emb}(M,N) \longrightarrow \overline{T}_2 \operatorname{Emb}(M,N)$ is a weak homotopy equivalence as the comparison maps in the diagram above are weak equivalences. $\square$

Remark 8.0.6. An alternative strategy for a proof is to follow the argument in [GKW03] and check that it extends to this construction of $\bar{T}_2 \operatorname{Emb}(M,N)$ to prove it is a valid model for the second Taylor approximation of the embedding functor. In fact, it generalizes for all $k \geq 2$ for a weaker notion of admissable maps given by isovariant maps that are only smooth on a neighbourhood of the fat diagonal and satisfy the strong isovariance condition. Everything in section 3 and 4 of [GKW03] goes through, and the only adaption of the proof is their Proposition 5.1, which holds also for the weakened version of admissable maps as it only requires the existence of jets on the fat diagonal.

In some sense, this already constitutes a proof Theorem 8.0.3 because the Taylor tower is only well-defined up to weak equivalence so we might as well use $\overline{T}_2^{\times}(M)$, where the statement of Theorem 8.0.3 follows by inspection of the definition of $G : F(M,2) \to \overline{T}_2^{\times}(M)$. On the other hand, if $F(M,2)$ is in Top^{CW} we can also find lifts (up to homotopy) along the weak equivalence $T_2(M) \xrightarrow{\simeq} \overline{T}_2^{\times}(M)$ so that the Theorem also holds for the Haefliger model.

Proof of Theorem 8.0.3. By [May75, Thm 6.4] the space $F(M,2)$ is in Top^{CW} if both $\mathcal{G}(TM)$ and $h\mathcal{G}^{S_2}(S(TM))$ are. We can express both spaces as equivariant mapping spaces

$$\begin{aligned} \mathcal{G}(TM) &= \mathrm{Map}^{\mathrm{SO}(d)}(\mathrm{Fr}^+(TM), \mathrm{SO}(d)) \\ h\mathcal{G}^{S_2}(S(TM)) &= \mathrm{Map}^{\mathrm{SO}(d)}(\mathrm{Fr}^+(TM), \mathrm{hAut}^{S_2}(S^{d-1})) \end{aligned}$$

where $\mathrm{SO}(d)$ acts by conjugation on the targets. It follows directly from an equivariant version of Milnor's theorem [Wan80c, Cor. 4.13] that $\mathcal{G}(TM)$ has the homotopy type of a CW complex if M is compact. For $h\mathcal{G}^{S_2}(S(TM))$ note that $\mathrm{hAut}(S^{d-1})$ has the homotopy type of a $\mathrm{O}(d)$-CW complex by [Wan80c, Thm 4.12]. The symmetric group $S_2 < \mathrm{O}(d)$ acts by the antipodal map and the fixed points $\mathrm{hAut}^{S_2}(S^{d-1})$ have the homotopy type of an $\mathrm{SO}(d)$-CW complex. It follows as above from [Wan80c, Cor. 4.13] that $h\mathcal{G}^{S_2}(S(TM)) \in \mathsf{Top}^{CW}$.

Therefore $F(M,2) \in \mathsf{Top}^{CW}$ and there exist lifts (up to homotopy) along weak equivalences

$$\begin{tikzcd} & T_2 \operatorname{Emb}(M,N) \arrow[d, "\simeq_w"] \\ F(M,2) \arrow[ur, dashed, "\bar{G}"] \arrow[r, "G"] & \overline{T}_2 \operatorname{Emb}(M,N) \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{B}\, T_2 \operatorname{Emb}(M,N) \arrow[d, "\simeq_w"] \\ \mathrm{B}\, F(M,2) \arrow[ur, dashed] \arrow[r, "\mathrm{B}\, G"] & \mathrm{B}\, \overline{T}_2 \operatorname{Emb}(M,N). \end{tikzcd}$$

Since $\mathcal{G}(TM)$ is a group, it follows that $F(M,2)$ is a grouplike topological monoids so that the image of G lands in the homotopy invertible components of $\bar{T}_2\operatorname{Emb}(M,N)$.

The second part of the theorem follows since r_2 extends to $\bar{T}_2\operatorname{Emb}(M,N)$ and $r_2G(\bar{f},H) = \bar{f}$ and $r_1r_2G(\bar{f},H) = \mathrm{Id}_M$ as the bundle map $\bar{f}$ covers the identity. □

We can now turn to computing the (rational) homotopy groups of $F(M,2)$ in order to find counter examples with base space $B = S^k$ for which the family signature theorem fails. By definition, this involves the computation of rational homotopy groups of $\mathcal{G}(TM)$ and $h\mathcal{G}^{S_2}(S(TM))$. Equivalently, we can study the classifying space $\mathrm{B}\,F(M,2)$, which reduces to problem to computing the homotopy type of non-equivariant mapping spaces by the following lemma.

Lemma 8.0.7. *The delooping* $\mathrm{B}\,F(M,2)$ *is equivalent to the homotopy fibre*

$$\operatorname{hofib}_{S(TM)}\Big(\operatorname{Map}(M,\mathrm{B}\,\mathrm{SO}(d))_{TM} \longrightarrow \operatorname{Map}(M,\mathrm{B}\,\mathrm{hAut}^{S_2}(S^{d-1}))_{S(TM)}\Big). \tag{8.5}$$

Proof. Observe that $\mathrm{B}\,\mathcal{G}(TM) = \operatorname{Map}(M,\mathrm{B}\,\mathrm{SO}(d))_{TM}$ by [Hus94, Ch.7 Cor. 3.5] and the analogous statement holds for $\mathrm{B}\,h\mathcal{G}^{S_2}(S(TM)) \simeq \operatorname{Map}(M,\mathrm{B}\,\mathrm{hAut}^{S_2}(S^{d-1}))_{S(TM)}$ by [BHMP84, Prop. 4.3]. Furthermore, by [BHMP84, Thm 3.3] there are homotopy equivalences of H-spaces

$$\Omega\operatorname{Map}(M,\mathrm{B}\,\mathrm{SO}(d))_{TM} \longrightarrow \mathcal{G}(TM)$$
$$\Omega\operatorname{Map}(M,\mathrm{B}\,\mathrm{hAut}^{S_2}(S^{d-1})_{S(TM)} \longrightarrow h\mathcal{G}^{S_2}(S(TM))$$

so that the inclusion $\mathcal{G}(TM)\to h\mathcal{G}^{S_2}(S(TM))$ corresponds to the map induced by the inclusion $\mathrm{B}\,\mathrm{SO}(d)\to\mathrm{B}\,\mathrm{hAut}^{S_2}(S^{d-1})$. The claim then follows. □

We can compute the homotopy groups of mapping spaces via the *Federer spectral sequence* [Fed56]. For a map $f : X\to Y$ from a finite dimensional CW complex X to a simple space Y, there is an *extended spectral sequence* (see [BK72, Ch. 9.4]) with E^2-page

$$E^2_{p,q} = \begin{cases} H^{-p}(X;\pi_q(Y)) & p+q\geq 0 \\ 0 & p+q<0 \end{cases}$$

that converges to $\pi_{p+q}(\operatorname{Map}(X,Y),f)$. We will use a more general version available for the space of sections of a fibration $\pi : E\to B$ with path connected and simple fibre $F = \pi^{-1}(b_0)$ for $b_0\in B$.

Theorem 8.0.8 ([Sch73, Thm 1.1])**.** *If B is a finite dimensional CW complex and $s : B\to E$ a section of π. Then there is a functorial spectral sequence with $E^2_{p,q} = H^{-p}(B;\underline{\pi_q}(F,s(b_0)))$ which converges* *to $\pi_{p+q}(\Gamma(\pi), s)$, where $\Gamma(\pi)$ is the space of sections. Here, $\underline{\pi_q}(F, s(b_0))$ is a local coefficient system by lifting loops in B to the identity Id_F, which gives a homomorphism $\pi_1(F) \to \mathrm{Aut}(\pi_q(F, s(b_0)))$.*

Remark 8.0.9. The fact that F is simple implies that one can ignore base points so that $\pi_1(F) \to \mathrm{Aut}(\pi_q(F, s(b_0)))$ is well-defined. Alternatively, one can describe the coefficient system as a functor

$$\underline{\pi_q}(\pi) : \Pi(B) \to \mathsf{Set}_*, \ \mathsf{Gr}, \text{ or } \mathsf{Ab}, \qquad b \longmapsto \pi_q(\pi^{-1}(b), s(b)).$$

Note that the E^2-page of the spectral sequence is independent of the section s, which enters only in computing the differentials $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

This recovers Federers spectral sequence for the trivial fibration $X \times Y \to X$ and the section defined by $f : X \to Y$. One can construct this spectral sequence from an exact couple associated to the long exact sequence of the homotopy groups associated to a Postnikov tower of Y. Alternatively, following [KR19] it arises as a special case of the Bousfield-Kan spectral sequence [BK72, Ch. 10.6] associated to the totalization of the cosimplicial space with k-cosimplices the subspace of $\mathrm{Map}(\Delta^k \times \mathrm{Sing}_k(B), E)$ of pairs $(f : \Delta^k \to E, \tau : \Delta^k \to B)$ which satisfy $\pi \circ f = \tau$. The totalization is the space of sections of $\epsilon^* E \to |\mathrm{Sing}(B)|$ where $\epsilon : |\mathrm{Sing}(B)| \to B$ is the weak equivalence that replaces a space by its singular complex. This section space is weakly equivalent to $\Gamma(\pi)$ by [KR19, Lem. 5.1].

Proposition 8.0.10. *Let d be an even natural number, then $\mathrm{B\,hAut}^{S_2}(S^{d-1})$ is simply connected and rationally equivalent to $K(\mathbb{Q}, d)$.*

Proof. The fundamental group of $\mathrm{B\,hAut}^{C_2}(S^{d-1})$ is trivial by an equivariant version of the Hopf degree theorem [tD79, Thm 8.4.1]. Alternatively, it follows from the fact that the group of orientation preserving homotopy self-equivalences $\mathcal{E}^+(\mathbb{R}P^{d-1})$ is trivial [BG73, Cor. 6]. This implies the claim as $\mathrm{hAut}^{S_2}(S^{d-1}) \to \mathrm{hAut}^+(\mathbb{R}P^{d-1})$ is an S_2-covering with deck transformation given by composing with the antipodal map, which is S_2-homotopic to the identity for d even.

We will compute the rational homotopy groups using yet another version of the Federer spectral sequence for maps of principal bundles [Sch73, Thm 2.1]. This version is based on the observation that maps in $\mathrm{hAut}^{S_2}(S^{d-1})$ are self-maps of the principal bundle $S^{d-1} \to \mathbb{R}P^{d-1}$. For principal G-bundles $\pi : E \to B$ and $\pi' : E' \to B'$ one can identify the space of G-equviariant maps with the space of sections of $\pi : E \times_G E' \to B$ which has fibre E', whose homotopy groups can be computed via the normal Federer spectral sequence.

In this case, the bundle $\pi_1 : (S^{d-1} \times S^{d-1})/S_2 \to \mathbb{R}P^{d-1}$ has fibre S^{d-1} and the action of $\pi_1(\mathbb{R}P^d)$ on $\pi_{d-1}(S^{d-1})$ is given by the antipodal map which is trivial for d even. Hence, the

E^2-page is $E^2_{p,q} = H^{-p}(\mathbb{R}P^{d-1}; \underline{\pi_q}(S^{d-1}))$ whose only infinite contribution for $p+q>0$ is

$$E^2_{0,d-1} = \pi_{d-1}(S^{d-1})^{\pi_1(\mathbb{R}P^{d-1})} = \mathbb{Z}.$$

Therefore, $E^\infty_{0,d-1} \cong \mathbb{Z}$ and all other entries are finite groups. The resulting extensions for $\pi_*(\mathrm{hAut}^{S_2}(S^{d-1}) \otimes \mathbb{Q}$ are trivial for all degrees except $* = d-1$ (note that the fundamental group is abelian so that $- \otimes \mathbb{Q}$ is defined) and the result follows. □

We can compute the rational homotopy groups of $\mathrm{Map}(M^{2d}, \mathrm{B}\,\mathrm{hAut}^{C_2}(S^{2d-1}))_{S(TM)}$ by replacing the target with its rationalization $K(\mathbb{Q}, d)$. In general, for a finite CW complex X and a nilpotent space Y with rationalization $r : Y \to Y_{\mathbb{Q}}$, the induced map on mapping spaces

$$\mathrm{Map}(X, Y)_f \xrightarrow{r_*} \mathrm{Map}(X, Y_{\mathbb{Q}})_{rf}$$

is a rational equivalence for all maps $f : X \to Y$ by [HMR75, Thm 3.11]. We then need to understand the space of maps into an Eilenberg-MacLane space. This problem has been studied originally by Thom and he found that

$$\mathrm{Map}(X, K(G, n))_f \simeq \prod_{i=1}^{n} K(H^{n-i}(X; G), i)$$

for any map $f : X \to K(G, n)$ (see [Smi10] for references). Combining these two statement, we find that for $q \geq 1$

$$\pi_q(\mathrm{Map}(M^{2d}, \mathrm{B}\,\mathrm{hAut}^{C_2}(S^{2d-1}))_{S(TM)}) \otimes \mathbb{Q} \cong H^{2d-q}(M; \mathbb{Q}). \tag{8.6}$$

The same strategy applies to $\mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}$ since the rationalization $\mathrm{B}\,\mathrm{SO}(d)_{\mathbb{Q}}$ is a product of Eilenberg-MacLane space determined by Pontrjagin classes and possibly the Euler class depending on the parity of d, i.e. in even dimension this decomposition is $\mathrm{B}\,\mathrm{SO}(2d)_{\mathbb{Q}} \simeq \prod_{i=1}^{d-1} K(\mathbb{Q}, 4i) \times K(\mathbb{Q}, 2d)$. We can combine these computations in the following statement.

Proposition 8.0.11. $\mathrm{B}\,F(M^{2d}, 2) \simeq_{\mathbb{Q}} \prod_{i=1}^{d-1} \left(\prod_{k=1}^{4i} K(H^{4i-k}(M; \mathbb{Q}), k) \right)$.

Proof. Rationally, the composition $\mathrm{B}\,\mathrm{SO}(2d) \to \mathrm{B}\,\mathrm{hAut}^{S_2}(S^{2d-1}) \to \mathrm{B}\,\mathrm{hAut}^{+}(S^{2d-1})$ corresponds to the projection onto the Euler class. Hence, $\mathrm{hofib}(\mathrm{B}\,\mathrm{SO}(2d) \to \mathrm{B}\,\mathrm{hAut}^{S_2}(S^{2d-1}))$ is rationally given by a product of Eilenberg-MacLane spaces corresponding to the Pontrjagin classes. The claim follows since $\mathrm{B}\,F(M, 2)$ is equivalent to $\mathrm{Map}(M, \mathrm{hofib}(\mathrm{B}\,\mathrm{SO}(2d) \to \mathrm{B}\,\mathrm{hAut}^{S_2}(S^{2d-1}))$ by Lemma 8.0.7. □

In a last step, we need to determine the characteristic classes of the vector bundle associated to the adjoint maps of non-trivial elements in $\pi_*(\mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM})$. We will first discuss the adjoint of homotopy classes in general for $\mathrm{Map}(X, K(G, n))_f$.

The independence of the homotopy type of the path component $f : X \to K(G, n)$ is due to the grouplike monoid structure of $K(G, n) \times K(G, n) \to K(G, n)$, which induces a homotopy equivalence $- \cdot f : \mathrm{Map}(X, K(G, n))_{\mathrm{const}_e} \to \mathrm{Map}(X, K(G, n))_f$. A homotopy class of maps $\bar{a} : S^k \to \mathrm{Map}(X, K(G, n))_{\mathrm{const}}$ is equivalent to a cohomology class $\alpha \in H^{n-k}(X; G) \cong H^n(S^k \wedge X; G)$ by considering the adjoint $a : S^k \times X \to K(G, n)$ via the correspondence $a^* \iota_n = \epsilon_k \times \alpha$ for a generator $\epsilon_k \in H^k(S^k; G)$. If we consider the map $\bar{a} \cdot f : S^k \to \mathrm{Map}(X, K(G, n))_f$, it has as adjoint

$$S^k \times X \xrightarrow{a \times (f \circ \pi_X)} K(G, n) \times K(G, n) \longrightarrow K(G, n)$$

and hence ι_n is pulled back to

$$\epsilon_k \times \alpha + 1 \times f^*(\iota_n) \in H^n(S^k \times X; G). \tag{8.7}$$

We can now come to the proof of the main theorem of this section.

Proof of 8.0.1. For $d/2 < i \leq d - 1$ consider an element $[\alpha] \in \pi_{4i-2d}(\mathrm{B}\,F(M, 2))$ such that for some $\lambda \in \mathbb{Q}^\times$ the element $[\alpha] \otimes \lambda$ corresponds to the generator of the fundamental class $\epsilon_M \in H^{2d}(M; \mathbb{Z}) \to H^{2d}(M; \mathbb{Q}) \cong \pi_{4i-2d}(K(H^{2d}(M; \mathbb{Q}), 4i - 2d))$ in the decomposition from Proposition 8.0.11. Then the adjoint of the composition $S^{4i-2d} \to \mathrm{B}\,F(M, 2) \to \mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}$ gives a vector bundle $T_\pi E \to S^{4i-2d} \times M$ whose rational characteristic classes can be computed by post-composing with the rationalization $\mathrm{B}\,\mathrm{SO}(2d) \to \mathrm{B}\,\mathrm{SO}(2d)_{\mathbb{Q}}$. Hence, the ith Pontrjagin class is $p_i(T_\pi E) = \lambda \cdot \epsilon_{4i-2d} \times \epsilon_M + 1 \times p_i(M)$ by (8.7).

By Theorem 8.0.3, the following square commutes up to homotopy

$$\begin{array}{ccc} \mathrm{B}\,F(M, 2) & \longrightarrow & \mathrm{B}\,\mathcal{G}(TM) \\ \downarrow{\scriptstyle \mathrm{B}\,\bar{G}} & & \downarrow \\ \mathrm{B}\,T_2^\times(M) & \xrightarrow{\mathrm{B}(r_2)} & \mathrm{B}\,T_1^\times(M) \end{array}$$

and since the upper right corner is equivalent to $\mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}$ it follows that the vector bundle $T_\pi E \to S^{4i-2d} \times M$ is equivalent to the pullback of the TM-fibration $T_\pi E_2 \to E_2$ along

$$a : S^{4i-2d} \xrightarrow{\alpha} \mathrm{B}\,F(M, 2) \xrightarrow{\mathrm{B}\,\overline{G}} \mathrm{B}\,T_2^\times(M).$$

Therefore the MMM-class κ_{p_i} pulls back to $\lambda \cdot \epsilon_{4i-2d}$, and is a non-zero multiple of κ_{L_i} since the fibre integrals of products of lower Pontrjagin classes vanish by construction. Since the composition $\mathrm{B}\,F(M, 2) \to \mathrm{B}\,T_2^\times(M) \to \mathrm{B}\,T_0^\times(M)$ is null-homotopic by Theorem 8.0.3, the signature classes vanish and the family signature theorem fails for the map a above. □

Appendix A.

Classifying spaces and fibrations

Let us start by briefly motivating the classification theory of May guided by the classification theory of fibre bundles we have discussed in the introduction. This appendix is based on [May75] and the goal is to give an account of May's unified approach to the classification theory of fibrations, and some of the techniques and concepts that have been used extensively in the second part of this thesis.

Definition A.1 ([May75, Def. 4.1]). A **category of fibres** $(\mathcal{F}, F)$ is a category $\mathcal{F}$ with a faithful underlying space functor $\mathcal{F} \to \mathsf{Top}$ and a distinguished object F, and mapping spaces for $X, X' \in \mathcal{F}$ given as subspace $\mathrm{Map}_{\mathcal{F}}(X, X') \subset \mathrm{Map}(X, X')_{\text{w.eq.}}$ of the collection of path components of weak homotopy equivalences such that for all $X \in \mathcal{F}$ the space $\mathrm{Map}_{\mathcal{F}}(F, X)$ is non-empty and for every $\phi \in \mathrm{Map}_{\mathcal{F}}(F, X)$ post-composition induces a weak equivalence $\phi_* : \mathrm{Map}_{\mathcal{F}}(F, F) \to \mathrm{Map}_{\mathcal{F}}(F, X)$.

The objects should be thought of as the allowed preimages of an *$\mathcal{F}$-fibration* $\pi : E \to B$ [May75, Def. 2.1] for any point b and the morphisms should be thought of as the quality of homotopy lifting functions.

Example A.2. (i) Let $F \in \mathsf{Top}^{CW}$ and $(\mathsf{Top}^{CW}(F), F)$ be the category of fibres with objects all spaces homotopy equivalent to F and $\mathrm{Map}_{\mathsf{Top}^{CW}(F)}(X, X') = \mathrm{Map}(X, X')_{\text{h.eq.}}$ the space of all homotopy equivalences.

(ii) Let $F \in \mathsf{Top}^{CW}$ and $(\mathsf{Top}(F), F)$ be the category of fibres with objects all spaces of the same weak homotopy type as F and $\mathrm{Map}_{\mathsf{Top}(F)}(X, X') = \mathrm{Map}(X, X')_{\text{w.eq.}}$ the space of all weak equivalences.

(iii) Let $E \to B$ be an oriented vector bundle and $(\mathsf{Top}^{CW}(E \to B), E)$ be the category of fibres with objects all orientated vector bundles $E' \to B'$ for which there exists maps

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B' \end{array}$$

where f is a homotopy equivalence and $\bar{f}$ is an fibrewise linear orientation preserving isomorphism. The space of maps is given by tangential homotopy equivalences.

The key step in the classification theory is exactly the same as for fibre bundles, namely the construction of a contractible principal G-bundle $\mathrm{E}\,G \to \mathrm{B}\,G$, where $G = \mathrm{Map}_{\mathcal{F}}(F, F)$ is the topological monoid of automorphisms of the distinguished fibre in $\mathcal{F}$. Then there is a formally equivalent classification theory of G-Prin(B), equvialence classes of principal G-bundles over reasonable base spaces B, as well as equvialence classes of $\mathcal{F}$-fibrations $\mathrm{Bun}^{\mathcal{F}}(B)$

$$[B, \mathrm{B}\,G] \xrightarrow{1:1} G\text{-Prin}(B) \xrightarrow{1:1} \mathrm{Bun}^{\mathcal{F}}(B)$$

where the maps are induced by pull back of the universal principal G-bundle and the associated $\mathcal{F}$-fibration. We will briefly discuss the construction of the contractible principal G-bundle, some properties of the bar construction and the classification theory. The constructions are based on the notion of simplicial sets and spaces (see for example [May67]).

Definition A.3. Let G be a topological monoid and let X and Y be left and right G-spaces. Then the two-sided bar construction $\mathrm{B}_\bullet(X, G, Y)$ is the simplicial space with n-simplices given by $X \times G^n \times Y$ and face and degeneracy operators given by

$$d_i(y[g_1, \dots, g_n]x) = \begin{cases} yg_1[g_2, \dots, g_n]x & i = 0 \\ y[g_1, \dots, g_{i-1}, g_i g_{i+1}, \dots, g_n]x & 1 \leq i < n \\ y[g_1, \dots, g_{n-1}]g_n x & i = n \end{cases}$$

$$\text{and} \qquad s_i(y[g_1, \dots, g_n]x) = y[g_1, \dots, g_i, e, g_{i+1}, \dots, g_n]x \qquad 0 \leq i \leq n.$$

Denote by $\mathrm{B}(Y, G, X)$ the geometric realization of this simplicial space.

The bar construction is functorial with respect to maps $(k, f, j) : (Y, G, X) \to (Y', G', X')$ of triples as above, where $f : G \to G'$ is a map of topological monoids and $k : X \to X'$ and $j : Y \to Y'$ are f-equivariant. Then (k, f, j) induces a map of the simplicial spaces above and thus a map of geometric realizations

$$\mathrm{B}(k, f, j) : \mathrm{B}(Y, G, X) \longrightarrow \mathrm{B}(Y', G', X').$$

With this, we can already define the contractible principal fibration as

$$\mathrm{E}\,G := \mathrm{B}(*, G, G) \xrightarrow{\mathrm{B}(\mathrm{Id}_*, \mathrm{Id}_G, \mathrm{const})} \mathrm{B}(*, G, *) =: \mathrm{B}\,G. \tag{A.1}$$

We will now discuss some properties of the bar construction that we need later-on and that will provide some justification of the classification property of (A.1).

There is an additional technical condition on the pair (G, e) that we have to assume throughout, namely that it is a strong NDR-pair [May72, Def. A.1]. One can easily make any topological monoid well-pointed [May72, App. A.8] by adding an interval to the identity element and giving the whiskered space G' the obvious multiplication such that $G' \to G$ is a map of monoids with new unit the endpoint of the interval. In particular, any G-space will be a G'-space and so we can largely ignore this technical subtlety for our discussion.

Proposition A.4 ([May75, Prop. 7.3]). *If k, f and j are homotopy equivalences, then so is* $\mathrm{B}(k, f, j)$.

For maps $f : Z \to Y \times X$ and $h : Y \times X \to Z$ satisfying $h(yg, x) = h(y, gx)$ for all $g \in G$, there are induced maps

$$\tau(f) : Z \longrightarrow \mathrm{B}(Y, G, X) \quad \text{and} \quad \epsilon(h) : \mathrm{B}(Y, G, X) \longrightarrow Z. \tag{A.2}$$

These maps can be defined as maps of simplicial spaces $\tau(f)_\bullet : Z_\bullet \to \mathrm{B}(Y, G, X)$ respectively $\epsilon(h)_\bullet : \mathrm{B}(Y, G, X) \to Z_\bullet$ where $Z_\bullet$ is the constant simplicial space $Z_q = Z$ with all face and degeneracy maps being identities, and where the simplicial maps are given by $\tau(f)_q := s_0^q \circ f$ and $\epsilon(h)_q := h \circ d_0^q$. Evidently, $\tau(f)$ factors through the inclusion of the space of 0-simplices and $\epsilon(h)$ factors through the quotient map $Y \times_G X \to Z$.

Now consider the triple (G, G, X) for a left G-space X. It has maps as above with $f = \mathrm{const}_e \times \mathrm{Id}_X : X \to G \times X$ and h given by the action, and we denote the induced maps of the geometric realization as $\tau = \tau(f)$ and $\epsilon = \epsilon(h)$.

Proposition A.5 ([May75, Prop. 7.5]). *$\epsilon : \mathrm{B}(G, G, X) \to X$ is a map of left G-spaces and a strong deformation retraction with inverse τ.*

This implies for example that $\mathrm{E}\, G$ is contractible. The next theorem, which is a mild generalization of [May75, Thm 7.6], implies that the inclusion of the fibre of (A.1) into the homotopy fibre is a homotopy equivalence. This property has been formalized by Dold and Thom and studied in [DT58].

Definition A.6. A map $\pi : E \to B$ is a **quasi-fibration** if it is surjective and the inclusion $\pi^{-1}(b) \hookrightarrow \mathrm{hofib}_b(\pi)$ is a weak homotopy equivalence for all $b \in B$. A subspace is called distinguished if the restriction $\pi|_{\pi^{-1}(A)} : \pi^{-1}(A) \to A$ is a quasi-fibration.

There is a standard inductive procedure to check whether a map is a quasifibration due to Dold and Thom.

Proposition A.7 ([May72, Lem. 7.2]). *Let $p : E \to B$ be a map onto a filtered space $B = \cup F_j B$. Then each $F_j B$ is distinguished and p is a quasi-fibration provided that*

(1) *F_0B and every open subset of $F_jB \setminus F_{j-1}B$ for $j > 0$ is distinguished.*

(2) *For each $j > 0$, there is an open subset U of F_jB which contains $F_{j-1}B$ and there are homotopies $h_t : U \to U$ and $H_t : p^{-1}(U) \to p^{-1}(U)$ such that*

   (a) *$h_0 = \mathrm{Id}$, $h_t(F_{j-1}B) \subset F_{j-1}B$ and $h_1(U) \subset F_{j-1}B$;*

   (b) *$H_0 = \mathrm{Id}$ and H covers h, i.e. $pH_r = h_t p$; and*

   (c) *$H_1 : p^{-1}(x) \to p^{-1}(h_1(x))$ is a weak equivalence for all $x \in U$.*

Note that a pullback square of quasi-fibrations is a homotopy pullback square. In particular, it is fibre-homotopy equivalent to the square obtained by applying Γ to the projections.

Theorem A.8. *Let G be a grouplike topological monoid and $p : X \to Y$ be a left G-map that is a fibration and Z a right G-space. Then $P := \mathrm{B}(\mathrm{Id}, \mathrm{Id}, p) : \mathrm{B}(Z, G, X) \to \mathrm{B}(Z, G, Y)$ is a quasifibration.*

Remark A.9. Before we give the proof, we should point out that May proved this statement for $Y = *$ and the argument below is exactly the same but only mildly modified. One can identify the homotopy fibre of P using May's original statement by combining it with [May75, Prop. 7.8]. However, we find it convenient to have this more general statement at our disposal and it seems quite natural to prove the statement in this generality.

Proof. We closely follow May's argument here, which is based on the work of Dold and Thom about gluing conditions for quasifibrations [DT58] (see [May72, Lem. 7.2]).

Consider the filtration $F_i\,\mathrm{B}(Z, G, X)$ and $F_i\,\mathrm{B}(Z, G, Y)$ by skeleta [May72, Def. 11.1] satisfying $P^{-1}F_i\,\mathrm{B}(Z, G, Y) = F_i\,\mathrm{B}(Z, G, X)$. In the following, we occasionally abbreviate the filtration by F_i if it is clear from context if we talk about the filtration of the source or the target of P. Then

$$P|_{F_0} : F_0\,\mathrm{B}(Z, G, X) = Z \times X \xrightarrow{\mathrm{Id} \times p} F_0\,\mathrm{B}(Z, G, , Y) = Z \times Y,$$

and the restriction to $F_i\,\mathrm{B}(Z, G, X) \setminus F_{i-1}\,\mathrm{B}(Z, G, X) = (F_i\,\mathrm{B}(Z, G, *) \setminus F_{i-1}\,\mathrm{B}(Z, G, *)) \times X$ is given by

$$P|_{F_i - F_{i-1}} : (F_i\,\mathrm{B}(Z, G, *) \setminus F_{i-1}\,\mathrm{B}(Z, G, *)) \times X \xrightarrow{\mathrm{Id} \times p} (F_i\,\mathrm{B}(Z, G, *) \setminus F_{i-1}\,\mathrm{B}(Z, G, *)) \times Y.$$

Hence, the first condition of the gluing condition of quasifibrations [May72, Lem. 7.2] is satisfied. It remains to check the second condition, i.e. that for each $i > 0$ there is a neighbourhood U of $F_i\,\mathrm{B}(Z, G, Y) \setminus F_{i-1}\,\mathrm{B}(Z, G, Y)$ and deformation retractions h and H of U and $P^{-1}U$ onto $F_{i-1}\,\mathrm{B}(Z, G, Y)$ respectively $F_{i-1}\,\mathrm{B}(Z, G, X)$ compatible with P so that $H_1 : P^{-1}(y) \to P^{-1}(h_1(y))$ is a weak equvialence for all $y \in U$.

Such deformation retractions exists by precisely the same argument as [May72, Lem. 7.2], and it is exactly at this point that the strong NDR condition is needed. It remains to check that they induce weak equvialences of the fibres. To check this, let $[y] = [z[g_1,\dots,g_i]y,a] \in U\backslash F_{i-1}$ so that $g_i \neq e$ for all i. Then $h_1([y])$ is represented by a non-degenerate representative $[z'[g'_1,\dots,g'_j]y',a']$ with $j < i$. Since we can choose constant homotopies representing the strong NDR pairs $(X,\emptyset)$, $(Y,\emptyset)$ and $(Z,\emptyset)$, the non-degenerate representative of $h_1([y])$ is determined through the pair $(k : G^i{\times}\Delta^i{\times}I{\to}G^i{\times}\Delta^i, v : G^i{\times}\Delta^i{\to}I)$ representing the strong NDR pair $(G,e)^i \times (\Delta^j,\partial\Delta^j)$: If $k_1(g_1,\dots,g_i,a) = (g''_1,\dots,g''_i,a'')$ then $h_1([y]) = (z[g''_1,\dots,g''_i]y,a'')$ and the corresponding non-degenerate is determined in [May72, Lem. 11.3], and it follows that $y' = gy$ for $g \in G$ independent of y and z. Hence, we get a diagram as

$$\begin{tikzcd}
p^{-1}(y) \subset X \arrow[r, dashed, "g"] \arrow[d, "\iota"] & p^{-1}(gy) \subset X \arrow[d, "\iota'"] \\
P^{-1}([y]) \arrow[r, "H_1"] & P^{-1}(h_1[y])
\end{tikzcd}$$

where ι and ι' are homeomorphisms that send $x \in p^{-1}(y)$ to $[z[g_1,\dots,g_i]x,a] \in P^{-1}([y])$ and similarly for ι'. The key observation is that the non-degenerate representative of $H_1(\iota(x))$ agrees with $\iota'(gx)$. This is because it is determined again basically through $k_1(g_1,\dots,g_i,a)$. But then we can complete the square as indicated, and since g restricts to a homotopy equivalence of the fibres, the claim follows. ◻

Combining the previous two statements, we see that $\mathrm{E}\,G{\to}\mathrm{B}\,G$ is a principal quasi-fibration with fibre G and with contractible total space. This is essentially all the necessary properties needed to mimic the classification property of $\mathrm{E}\,G{\to}\mathrm{B}\,G$ in the classification theory for principal bundles for groups, and thus justifies that one should think of (A.1) as the *universal G-quasifibration*.

One last technical point is that quasi-fibrations are not preserved under pullbacks. Hence, in order to state the classification theorem via pullbacks of universal bundles, we need to replace quasi-fibrations by $\mathcal{F}$-fibrations. This is done via the usual construction that replaces a map $\pi : E{\to}B$ by a fibration $\Gamma\pi : \Gamma E{\to}B$ defined as

$$\Gamma E := \Big\{(\gamma,s,e) \in \mathrm{Map}([0,\infty],B) \times [0,\infty] \times E \,\Big|\, \gamma(0) = \pi(e),\ \gamma(t) = \gamma(s)\ \forall t \geq s\Big\} \xrightarrow{\mathrm{ev}_s} B \tag{A.3}$$

(see [May75, Def. 3.2]). In this generality, we need as additional assumption that Γ turns $\mathcal{F}$-quasifibration into $\mathcal{F}$-fibrations. A category of fibres for which this is true is called Γ-complete [May75, Def. 5.1]. In practice, we are mostly interested in the categories of fibre corresponding to Hurewicz fibrations with a given fibre which are Γ-complete [May75, Lem. 6.8]. With this in mind, the universal $\mathcal{F}$-fibration is given by

$$\pi : \Gamma\,\mathrm{B}(*, G, F) \longrightarrow \mathrm{B}(*, G, *) = \mathrm{B}\,G \tag{A.4}$$

where $G = \mathrm{Map}_{\mathcal{F}}(F, F)$ acts on F by evaluation. We can now state May's main theorem.

Theorem A.10 ([May75, Thm 9.2])**.** *Let $(\mathcal{F}, F)$ be a Γ-complete category of fibres with $G = \mathrm{Map}_{\mathcal{F}}(F, F)$. For a space B of the homotopy type of a CW complex, there is a natural bijection*

$$[B, \mathrm{B}\,G] \xrightarrow{1:1} Bun^{\mathcal{F}}(B).$$
$$[f : B \to \mathrm{B}\,G] \longmapsto [f^*\Gamma\,\mathrm{B}(*, G, F) \to B]$$

We will need two more results. The first is useful for comparing quasifibration sequences obtained from bar constructions.

Proposition A.11 ([May75, Prop. 7.5])**.** *Let $(k, f, \mathrm{Id}) : (Z, H, X) \to (Y, G, X)$ be a morphism of triples above. Then the following diagrams are pullbacks*

$$\begin{tikzcd} \mathrm{B}(Z,H,X) \arrow[r, "\mathrm{B}(k,f,\mathrm{Id})"] \arrow[d] & \mathrm{B}(Y,G,X) \arrow[d] \\ \mathrm{B}(Z,H,*) \arrow[r, "\mathrm{B}(k,f,\mathrm{Id})"] & \mathrm{B}(Y,G,*) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathrm{B}(Y,G,X) \arrow[r] \arrow[d] & \mathrm{B}(*,G,X) \arrow[d] \\ \mathrm{B}(Y,G,*) \arrow[r] & \mathrm{B}(*,G,*) \end{tikzcd},$$

where all unlabeled arrows are induced by the (equivariant) maps to a point.

The second statement is useful to study product actions and diagonal maps.

Proposition A.12 ([May75, Prop. 7.4])**.** *For triples (Y, G, X) and (Y', G', X') we can form the triple $(Y \times Y', G \times G', X \times X')$ via the product actions, and the projections of simplicial spaces define a natural homeomorphism*

$$\mathrm{B}(Y \times Y', G \times G', X \times X') \xrightarrow{\approx} \mathrm{B}(Y, G, X) \times \mathrm{B}(Y', G', X')$$

Corollary A.13. *Consider two triples (Y, G, X) and (Y, G, X'). Then $\mathrm{B}(Y, G, X \times X')$ is naturally homeomorphic to $\mathrm{B}(Y, G, X) \times_{\mathrm{B}(Y,G,*)} \mathrm{B}(Y, G, X')$ where we equip $X \times X'$ with the product action. The homeomorphism is induced by the projections $\pi_X : X \times X' \to X$ and $\pi_{X'} : X \times X' \to X'$.*

Proof. Consider the cube

$$\begin{tikzcd} & \mathrm{B}(Y,G,X\times X') \arrow[dl] \arrow[dd] & & & \mathrm{B}(Y,G,X)\times_{\mathrm{B}(Y,G,*)}\mathrm{B}(Y,G,X') \arrow[dl] \arrow[dd] \\ \mathrm{B}(Y^2,G^2,X\times X') \arrow[rr, "\approx"] \arrow[dd] & & \mathrm{B}(Y,G,X)\times\mathrm{B}(Y,G,X') \arrow[dd, "\mathrm{B}(\pi)\times\mathrm{B}(\pi')"] & & \\ & \mathrm{B}(Y,G,*) \arrow[dl, "\mathrm{B}(\Delta)"'] \arrow[rrr, equal] & & & \mathrm{B}(Y,G,*) \arrow[dll, "\Delta"] \\ \mathrm{B}(Y^2,G^2,*) \arrow[rr, "\approx"] & & \mathrm{B}(Y,G,*)\times\mathrm{B}(Y,G,*) & & \end{tikzcd}$$

where

- the left face is induced by maps of triples given by the diagonal map $\Delta : G \to G \times G$ and projections,
- the horziontal homeomorphisms are those of Proposition A.12,
- the right face is the pullback square of the diagonal Δ and the vertical map $\mathrm{B}(\pi_1)\times\mathrm{B}(\pi_2)$ induced by the projections $\pi : X \to *$ and $\pi' : X' \to *$.

The commutativity of all square can be checked on the level of simplicial spaces. Since the right face is a pullback by definition, it follows from commutativity that there is a map

$$\mathrm{B}(Y, G, X \times X') \longrightarrow \mathrm{B}(Y, G, X) \times_{\mathrm{B}(Y,G,*)} \mathrm{B}(Y, G, X')$$

induced by the G-equivariant projectons $\pi_X : X \times X' \to X$ and $\pi_{X'} : X \times X' \to X'$. Moreover, since the left square is a pullback by Proposition A.11 and all horizontal maps are homeomorphisms, this map is a homeomorphism. □

A.1. Equivariant classifying spaces and fibrations

We will discuss here a generalization of [May75] to the equivariant setting due to Waner [Wan80a, Wan80b, Wan80c]. In the following, we fix a compact Lie group H and work in the category $H\mathsf{Top}$ of compactly generated weak Hausdorff spaces with an action of H.

We will give the definition of equivariant fibrations and quasifibrations, and for the latter we will need to introduce equivariant homotopy groups defined for a pair $(X, Y) \in H\mathsf{Top}$ as equivariant homotopy classes of maps

$$\pi_n^{H'}(X, Y; \phi; H) := \pi_n(X^{H'}, Y^{H'}, Q)$$

where $H' \subset H$ is an arbitrary closed subgroup and $\phi : H/H' \to Y$ is an H-map that serves as basepoint and Q denotes the image (this is slightly modified from [Wan80c, Def. 2.1] but equivalent).

Definition A.14 ([Wan80b]). A map $p : E \to B$ in $H\mathsf{Top}$ is an H-**equivariant fibration** if it satisfies the H-covering homotopy property. It has fibre F if for each $b \in B$ there is some action of the isotropy group H_b on F such that it is H_b-homotopy equivalent to $p^{-1}(b)$ which has an H_b-action by restriction. The map p is called an H-**quasifibration** if

$$p_* : \pi_n^{H'}(E, p^{-1}(b); \phi; H_b) \longrightarrow \pi_n^{H'}(B, b; p\phi; H_b)$$

is an isomorphism for each $b \in B$, closed subgroup $H' \subset H_b$ and basepoint $\phi : H_b/H' \to p^{-1}(b)$.

Remark A.15. The homotopy fibre of an equivariant map $p : E \to B$ has a natural action and the definition of an equivariant quasi-fibration is equivalent to the inclusion $p^{-1}(b) \to \mathrm{hofib}_b\, p$ inducing an isomorphism on equivariant homotopy groups for all $p \in B$ and subgroups $H' \subset H_b$ of the corresponding isotropy group.

Since we are only interested in base spaces with a trivial action of H so that the isotropy groups above always coincide with H, this leads to a considerable simplification in that we do not need to introduce equivariant categories of fibres. Instead, we can define a category of fibres for $F \in H\mathsf{Top}^{CW}$ as in [Wan80b, Def. 1.3.2] (for the special case of $\alpha = H$) which is a category of fibres in the sense of Definition A.1:

- Let $(H\mathsf{Top}(F), F)$ be the category of fibres with objects $X \in H\mathsf{Top}(F)$ of the same weak H-equivariant homotopy type and as mapping spaces all equivariant weak equivalences.
- Let $(H\mathsf{Top}(F)^{CW}, F)$ be the category of fibres with objects $X \in H\mathsf{Top}(F)$ of the same H-equivariant homotopy type and as mapping space all equivariant homotopy equivalences.

Proposition A.16. *Let* $\mathrm{hAut}^H(F)$ *denote the space of H-equivariant homotopy equivalences of F. Then*

$$\mathrm{B}(*, \mathrm{hAut}^H(F), F) \longrightarrow \mathrm{B}\,\mathrm{hAut}^H(F)$$

is an H-equivariant quasifibration.

Proof. This is a special case of Proposition 2.2.3 in [Wan80a] for the equivariant category of fibres ([Wan80a, Def. 1.1.1]) with only one distinguished object $F \to H/H$, so that Λ is a point and the Λ-monoid (i.e. ordinary monoid) is $\mathrm{hAut}^H(F)$.

Alternatively, we can yet again check the details in the proof of Theorem A.8 and observe that the in this situation, we can apply an equivariant version of Lemma A.7 (see [Wan80b, Lem. 2.3]) to conclude the statement. □

Waner proves an equivariant Γ-completeness which implies that Γ applied to the map in Proposition A.16 above gives an H-equivariant fibration which is universal.

Theorem A.17 ([Wan80a, Thm 2.4.1])**.** $\mathrm{B}\,\mathrm{hAut}^H(F)$ *is the classifying space of* $(H\mathsf{Top}^{CW}(F), F)$*-fibrations over H-trivial base spaces.*

Appendix B.

Computations

In this appendix we collect all computational results about the Euler ring and homotopical tautological ring involving computations in Macaulay2.

Example B.1. In this example we finish the proof of Theorem 4.2.16. First, we give the Macaulay2 code for the complete intersections for universal 1-connected fibrations with fibre $S^2 \times S^2$ respectively $\mathbb{C}P^2 \# \mathbb{C}P^2$ from (4.9) respectively (4.10).

```
   X=S2xS2
B=QQ[a1,a2,b1,b2,Degrees=>{2,2,4,4}];
E=B[x1,x2]/ideal(x1^2-b1-x2*a2,x2^2-b2-x1*a1);
e=4*x1*x2-a1*a2;
T=QQ[k1,k2,k4]
f=map(B,T,matrix{{coefficient(x1*x2,e^2),coefficient(x1*x2,e^3),coefficient(x1*x2,e^5)}});

   X=CP2#CP2
B=QQ[a1,a2,b1,b2,Degrees=>{2,2,4,4}];
E=B[x1,x2]/ideal(x1*x2-b1-x2*a2-x1*a1,x1^2-x2^2-b2);
e=2*(x1^2+x2^2)-2*(x1*a2+x2*a1);
T=QQ[k1,k2,k4]
f=map(B,T,matrix{{coefficient(x2^2,e^2),coefficient(x2^2,e^3),coefficient(x2^2,e^5)}});
```

In both cases, the kernel of f is trivial so that $\kappa_1, \kappa_2, \kappa_4$ are algebraically independent.

Next we compute the Euler ring for six dimensional positively elliptic spaces. The case $b_2(X) = 1$ corresponds to S^6, $\mathbb{C}P^3$ or $S^2 \times S^4$, where the first two cases are settled already and it remains to compute $E_0^*(S^2 \times S^4)$.

- The minimal model is $\Lambda := (\Lambda(x_2, x_4, y_3, y_7), d = x_2^2 \partial y_3 + x_4^2 \partial y_7)$ with $|x_i| = i$ and $|y_i| = i$.
- The derivation Lie algebra $\mathrm{Der}^+(\Lambda)$ is bigraded by total degree and the restrictions $V \to \Lambda V \twoheadrightarrow \Lambda^i V$ corresponding to product length. We say a derivation θ has bidegree (n, m) if it lowers degree by n and $\theta(V) \subset \Lambda^{m+1} V$. Then $\mathrm{Der}^+(\Lambda)$ has a homogeneous

basis with respect to bidegree give by

$$
\begin{array}{cccccccc}
 & -1 & & 0 & & 1 & & 2 \\
1 & & & x_2\partial y_3 & & x_2x_4\partial y_7 & & x_2^3\partial y_7 \\
 & & \nearrow & & \nearrow & & \nearrow & \\
2 & \partial x_2 & & x_2\partial x_4 & & x_2y_3\partial y_7 & & \\
3 & \partial y_3 & & x_4\partial y_7 & & x_2^2\partial y_7 & & \\
 & & \nearrow & & \nearrow & & & \\
4 & \partial x_4 & & y_3\partial y_7 & & & & \\
5 & & & x_2\partial y_7 & & & & \\
7 & \partial y_7 & & & & & &
\end{array}
$$

where the rows correspond to the total degree and the columns to product length. For $S^2 \times S^4$ the differential d is homogeneous of bidegree $(-1, 1)$ and thus $[d, -]$ is a differential of the same bidegree and we have indicated above the non-trivial differentials for $\mathrm{Der}^+(\Lambda)$.

Hence, $\mathfrak{a} := \mathbb{Q}\{\partial y_3, x_2\partial y_7, \partial y_7\}$ is a quasi-isomorphic abelian Lie subalgebra with trivial differential.

- Hence, the complete intersection describing the universal 1-connected fibration is

$$B = \mathbb{Q}[z_4, z_6, z_8] \longrightarrow E = B[x_2, x_4]/(x_2^2 - z_4, x_4^2 - x_2z_6 - z_8)$$

  with fibrewise Euler class $e^{\mathrm{fw}} = 4x_2x_4$ by Theorem 4.1.10. One can easily compute by hand that $\kappa_1 = 0$ and $\kappa_2 = 64z_4z_8$ and with only slightly more effort that $\kappa_4 = 1024z_4^3z_6^2 + 1024z_4^2z_8^2$. Hence, κ_2 and κ_4 are algebraically independent and $E_0^*(S^2 \times S^2) \cong \mathbb{Q}[\kappa_2, \kappa_4]$.

For $b_2(X) = 2$ there are three cases corresponding to $\mathbb{C}P^2 \times S^2$, $\mathbb{C}P^3\#\mathbb{C}P^3$ and $\mathrm{SU}(3)/T^2$. The underlying graded commutative algebra of a minimal model in all cases is given by $\Lambda := \Lambda(x_1, x_2, y_3, y_5)$ with $|x_i| = 2$ and $|y_i| = i$. The differentials are not homogeneous with respect to the bidegree yet we find it convenient to display a basis of $\mathrm{Der}^+(\Lambda)$ with respect to bidegree:

$$
\begin{array}{cccc}
 & -1 & 0 & 1 \\
1 & & x_i\partial y_3 & x_i^2\partial y_5,\ x_1x_2\partial y_5 \\
2 & \partial x_i & y_3\partial y_5 & \\
3 & \partial y_3 & x_i\partial y_5 & \\
5 & \partial y_5 & &
\end{array}
$$

- The minimal model of $S^2 \times \mathbb{C}P^2$ is $(\Lambda, d = x_1^2 \partial y_3 + x_2^3 \partial y_5)$. The non-trivial differentials in $(\mathrm{Der}^+(\Lambda), [d, -])$ are

$$[d, \partial x_1] = -2x_1 \partial y_3 \qquad [d, \partial x_2] = -3x_2^2 \partial y_5 \qquad [d, y_3 \partial y_5] = x_1^2 \partial y_5$$

and one can see that $\mathfrak{a} := \mathbb{Q}\{x_2 \partial y_3, x_1 x_2 \partial y_5, x_1 \partial y_5, x_2 \partial y_5, \partial y_3, \partial y_5\}$ defines a quasi-isomorphic abelian Lie subalgebra with trivial differential. Hence, the complete intersection describing the universal 1-connected fibration is

$$\pi^* : B = \mathbb{Q}[a_2, b_2, a_4, b_4, z_4, z_6] \longrightarrow E = B[x_1, x_2]/(x_1^2 - x_2 a_2 - z_4, x_2^3 - x_1 x_2 b_2 - x_1 a_4 - x_2 b_4 - z_6)$$

where the generators of B correspond to a dual basis of $\mathfrak{a}$ in the same order. The fibrewise Euler class is $e^{\mathrm{fw}} = 2x_1(3x_2^2 - x_1 b_2 - b_4) - a_2(x_2 b_2 + a_4)$.

- The minimal model of $\mathbb{C}P^3 \# \mathbb{C}P^3$ is $(\Lambda, d = x_1 x_2 \partial y_3 + (x_1^2 - x_2^2) \partial y_5)$. The non-trivial differentials in $(\mathrm{Der}^+(\Lambda), [d, -])$ are

$$\begin{aligned} [d, \partial x_1] &= -x_2 \partial y_3 - 2x_1^2 \partial y_5 \\ [d, \partial x_2] &= -x_1 \partial y_3 + 2x_2^2 \partial y_5 \\ [d, y_3 \partial y_5] &= x_1 x_2 \partial y_5 \end{aligned}$$

and one can see that $\mathfrak{a} := \mathbb{Q}\{x_1 \partial y_3, x_2 \partial y_3, x_1 \partial y_5, x_2 \partial y_5, \partial y_3, \partial y_5\}$ defines a suitable Lie subalgebra. Hence, the complete intersection describing the universal 1-connected fibration is

$$\pi^* : B = \mathbb{Q}[a_2, b_2, a_4, b_4, z_4, z_6] \longrightarrow E = B[x_1, x_2]/(x_1^3 - x_2^3 - x_1 a_4 - x_2 b_4 - z_6, x_1 x_2 - x_1 a_2 - x_2 b_2)$$

and the fibrewise Euler class is $e^{\mathrm{fw}} = (3x_1^2 - a_4)(x_1 - b_2) - (-3x_2^2 - b_4)(x_2 - a_2)$.

- The minimal model of $\mathrm{SU}(3)/T^2$ is $(\Lambda, d = (x_1^2 + x_1 x_2 + x_2^2) \partial y_3 + (x_1^2 x_2 + x_1 x_2^2) \partial y_5)$. The non-trivial differentials in $(\mathrm{Der}^+(\Lambda), [d, -])$ are

$$\begin{aligned} [d, \partial x_1] &= -(2x_1 + x_2) \partial y_3 - (2x_1 x_2 + x_2^2) \partial y_5 \\ [d, \partial x_2] &= -(x_1 + 2x_2) \partial y_3 - (x_1^2 + 2x_1 x_2) \partial y_5 \\ [d, y_3 \partial y_5] &= (x_1^2 + x_1 x_2 + x_2^2) \partial y_5 \end{aligned}$$

and one can see that $\mathfrak{a} := \mathbb{Q}\{x_1^2 \partial y_5, x_2^2 \partial y_5, x_1 \partial y_5, x_2 \partial y_5, \partial y_3, \partial y_5\}$ defines a suitable Lie subalgebra. Hence, the complete intersection over $B = \mathbb{Q}[a_2, b_2, a_4, b_4, z_4, z_6]$ describing the universal 1-connected fibration is

$$E = B[x_1, x_2]/(x_1^2 + x_1 x_2 + x_2^2 - z_4, x_1^2 x_2 + x_1 x_2^2 - x_1^2 a_2 - x_2^2 b_2 - x_1 a_4 - x_2 b_4 - z_6)$$

with $e^{\mathrm{fw}} = (2x_1 + x_2)(x_1^2 + 2x_1 x_2 - 2x_2 b_2 - b_4) - (x_1 + 2x_2)(2x_1 x_2 + x_2^2 - 2x_1 a_2 - a_4)$.

We check algebraic independence using Jacobian criterion (adding a_2 as a generator so that we can compute determinants) via the following Macaulay2 code.

```
   X=CP2xS^2
B=QQ[a2,b2,a4,b4,z4,z6,Degrees=>{2,2,4,4,4,6}];
E=B[x1,x2]/ideal(x1^2-x2*a2-z4,x2^3-x1*x2*b2-x1*a4-x2*b4-z6);
e=2*x1*(3*x2^2-x1*b2-b4)-a2*(x2*b2+a4);
T=matrix{{coefficient(x1*x2^2,e^2),coefficient(x1*x2^2,e^3),
          coefficient(x1*x2^2,e^4),coefficient(x1*x2^2,e^5)
          coefficient(x1*x2^2,e^7),a2}};
det(jacobian T)

   X=CP3#CP3
B=QQ[a2,b2,a4,b4,z4,z6,Degrees=>{2,2,4,4,4,6}];
E=B[x1,x2]/ideal(x1^3-x2^3-x1*a4-x2*b4-z6,x1*x2-x1*a2-x2*b2-z4);
e=(3*x1^2-a4)*(x1-b2)-(-3*x2^2-b4)*(x2-a2);
T=matrix{{coefficient(x2^3,e^2),coefficient(x2^3,e^3),
          coefficient(x2^3,e^4),coefficient(x2^3,e^5),
          coefficient(x2^3,e^7)a2X}};
det(jacobian T)

   X=SU(3)/T2
B=QQ[a2,b2,a4,b4,z4,z6,Degrees=>{2,2,4,4,4,6}];
E=B[x1,x2]/ideal(x1^2+x1*x2+x2^2-z4,x1^2*x2+x1*x2^2-x1^2*a2-x2^2*b2-x1*a4-x2*b4-z6);
e=-(2*x1+x2)*(x1^2+2*x1*x2-2*x2*b2-b4)+(x1+2*x2)*(2*x1*x2+x2^2-2*x1*a2-a4);
T=matrix{{coefficient(x1*x2^2,e^2),coefficient(x1*x2^2,e^3),
          coefficient(x1*x2^2,e^4),coefficient(x1*x2^2,e^5),
          coefficient(x1*x2^2,e^7),a2}};
det(jacobian T)
```

Example B.2. Macaulay2 code for producing partial results about $R^*_{h,0}(S^2 \times S^2)$ based on the algebraic model in Example 5.1.10. The generating set of the tautological ring is given by $\{\kappa_{p_1^2}, \kappa_{p_1^3}, \kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^2}, \kappa_{e^2p_1}, \kappa_{e^3}, \kappa_{e^3p_1}, \kappa_{e^5}\}$ (see [RW19]). We compute the ideal of relations among the first seven generators as the computation of the ideal of relations of all generators does not stop otherwise. The result of the computation is that the ideal of relations – defined as the ideal I below – is generated by one element displayed in (5.12).

```
B=QQ[a1,a2,b1,b2,p10,p11,p12,Degrees=>{2,2,4,4,4,2,2}];
E=B[x1,x2]/ideal(x1^2-b1-x2*a2,x2^2-b2-x1*a1);
e=4*x1*x2-a1*a2;
p=p10+p11*x1+p12*x2;
T=QQ[k02,k03,k11,k12,k20,k21,k30];
f=map(B,T,matrix{{coefficient(x1*x2,p^2),coefficient(x1*x2,p^3),
```

```
                  coefficient(x1*x2,e*p),coefficient(x1*x2,e*p^2),
                  coefficient(x1*x2,e^2),coefficient(x1*x2,e^2*p),
                  coefficient(x1*x2,e^3)}});
gbTrace=2;
I=ker f;
```

Remark B.3. We have used the mathematica notebook in [McT14] to compute the Hirzebruch L-polynomials.

```
K[Q_,n_Integer]:=Module[{z,x},SymmetricReduction[SeriesCoefficient[
                  Product[ComposeSeries[Series[Q[z],{z,0,n}],
                  Series[x[i]z,{z,0,n}]],{i,1,n}],n],Table[x[i],{i,1,n}],
                  Table[Subscript[c,i],{i,1,n}]][[1]]//FactorTerms]
# replace n by natural number to compute n-th Hirzebruch polynomial
K[Sqrt[#]/Tanh[Sqrt[#]]&,n]/.c->p
```

It produces the L-polynomial in the infinite polynomial ring $\mathbb{Q}[p_1, p_2, \ldots]$, and we manually set $p_i = 0$ for $i > n$ and $e^2 = p_n$ in computations for a manifold of dimension $2n$.

Example B.4. We compute $R^*_{h,0}(S^2 \times S^2)/(I_H^{\leq 12} \cap R^*_{h,0}(S^2 \times S^2))$. We use the same model as in the previous example and add the MMM-classes associated to the Hirzebruch L-classes.

```
B=QQ[a1,a2,b1,b2,P10,P11,P12,Degrees=>{2,2,4,4,4,2,2}];
E=B[x1,x2]/ideal(x1^2+b1+x2*a2,x2^2+b2+x1*a1);
e=4*x1*x2-a1*a2;
p1=P10+P11*x1+P12*x2;
p2=e^2;

# the following terms are necessary for the L-classes but vanish for degree reason
p3=0;
# repeat until i=12
p12=0;

L1=p1/3;
L2=(7*p2-p1^2)/45;
L3=(2*p1^3-13*p2*p1+62*p3)/945;
L4=(-3*p1^4+22*p2*p1^2-71*p3*p1-19*p2^2+381*p4)/14175;
# we omit printing the formulas for L5,...,L12

I12=ideal(coefficient(x1*x2,L2),coefficient(x1*x2,L3),
          coefficient(x1*x2,L4),coefficient(x1*x2,L5),
          coefficient(x1*x2,L6),coefficient(x1*x2,L7),
          coefficient(x1*x2,L8),coefficient(x1*x2,L9),
          coefficient(x1*x2,L10),coefficient(x1*x2,L11),
```

```
          coefficient(x1*x2,L12));
H=QQ[a1,a2,b1,b2,P10,P11,P12,k20,k03,k11,k12,k30,k31,k50,MonomialOrder => Eliminate 7];
i=map(H,B,matrix{{a1,a2,b1,b2,P10,P11,P12}});
J1=ideal(k20-i(coefficient(x1*x2,e^2)),k03-i(coefficient(x1*x2,p1^3)),
         k11-i(coefficient(x1*x2,e*p1)),k12-i(coefficient(x1*x2,e*p1^2)),
         k30-i(coefficient(x1*x2,e^3)),k31-i(coefficient(x1*x2,e^3*p1)),
         k50-i(coefficient(x1*x2,e^5)));
J2=i(I12);
J=J1+J2;
I=selectInSubring(1,gens gb J)
```

This computation produces a presentation of $R_0^*(S^2 \times S^2)/(I_H^{\leq 12} \cap R_0^*(S^2 \times S^2))$. We will not display the relations here but instead only compute the Hilbert Series, which is given by

$$((1-T^{16})(1-T^{12})(1-T^8)^3(1-T^4)^2)^{-1}\Big(1 - T^{12} - 2T^{16} - 3T^{20} - 3T^{24} + 4T^{28} + 6T^{32} + 13T^{36} + 14T^{40} - 11T^{44} \\ -39T^{48} - 26T^{52} + 16T^{56} + 39T^{60} + 31T^{64} - 7T^{68} - 31T^{72} - 23T^{76} + 5T^{80} + 21T^{84} + 2T^{88} - 6T^{92}\Big)$$

We have displayed a simplified expression for it in Proposition 5.2.2.

We can then compute the intersection with the Euler ring for Proposition 5.2.3 with the following Macaulay2 code:

```
S=B/I;
H=QQ[k1,k2,k4];
f=map(S,H,matrix{{k20,k30,k50}});
K=ker f;
```

Proposition B.5. *The following elements are in the kernel of* $E^*(\mathbb{C}P^2\#\mathbb{C}P^2)\to R^*(\mathbb{C}P^2\#\mathbb{C}P^2)$*:*

- $380305504\kappa_1^3\kappa_2^4 - 760611008\kappa_1\kappa_2^5 - 850381875\kappa_1^7\kappa_4 + 5015695500\kappa_1^5\kappa_2\kappa_4 - 11175572425\kappa_1^3\kappa_2^2\kappa_4 + 9091417850\kappa_1\kappa_2^3\kappa_4 + 1139406250\kappa_1^3\kappa_4^2 - 2278812500\kappa_1\kappa_2\kappa_4^2,$
- $380305504\kappa_1^5\kappa_2^3 - 1521222016\kappa_1\kappa_2^5 - 3904612875\kappa_1^7\kappa_4 + 22320565900\kappa_1^5\kappa_2\kappa_4 - 47604525745\kappa_1^3\kappa_2^2\kappa_4 + 37163690890\kappa_1\kappa_2^3\kappa_4 + 4392856250\kappa_1^3\kappa_4^2 - 8785712500\kappa_1\kappa_2\kappa_4^2,$
- $380305504\kappa_1^7\kappa_2^2 - 3042444032\kappa_1\kappa_2^5 - 13504851875\kappa_1^7\kappa_4 + 75386090700\kappa_1^5\kappa_2\kappa_4 - 154656805337\kappa_1^3\kappa_2^2\kappa_4 + 115808062874\kappa_1\kappa_2^3\kappa_4 + 12529026250\kappa_1^3\kappa_4^2 - 25058052500\kappa_1\kappa_2\kappa_4^2,$
- $6485\kappa_1^8\kappa_4 - 39440\kappa_1^6\kappa_2\kappa_4 + 75327\kappa_1^4\kappa_2^2\kappa_4 - 34154\kappa_1^2\kappa_2^3\kappa_4 - 21240\kappa_2^4\kappa_4 + 9370\kappa_1^4\kappa_4^2 - 39980\kappa_1^2\kappa_2\kappa_4^2 + 42480\kappa_2^2\kappa_4^2,$
- $2090068808\kappa_1^2\kappa_2^5 - 4180137616\kappa_2^6 - 382708575\kappa_1^6\kappa_2\kappa_4 - 904779340\kappa_1^4\kappa_2^2\kappa_4 + 1162163739\kappa_1^2\kappa_2^3\kappa_4 + 4356458482\kappa_2^4\kappa_4 + 4962029000\kappa_1^4\kappa_4^2 - 13927874750\kappa_1^2\kappa_2\kappa_4^2 + 8007633500\kappa_2^2\kappa_4^2,$
- $204075\kappa_1^7\kappa_2\kappa_4 - 1354060\kappa_1^5\kappa_2^2\kappa_4 + 2976401\kappa_1^3\kappa_2^3\kappa_4 - 2169162\kappa_1\kappa_2^4\kappa_4 + 236000\kappa_1^5\kappa_4^2 - 960250\kappa_1^3\kappa_2\kappa_4^2 + 976500\kappa_1\kappa_2^2\kappa_4^2,$
- $337417972\kappa_1^4\kappa_2^3\kappa_4 - 1559822964\kappa_1^2\kappa_2^4\kappa_4 + 1769974040\kappa_2^5\kappa_4 + 23556375\kappa_1^6\kappa_4^2 - 768426800\kappa_1^4\kappa_2\kappa_4^2 + 3600505265\kappa_1^2\kappa_2^2\kappa_4^2 - 4315754330\kappa_2^3\kappa_4^2 - 775806250\kappa_1^2\kappa_4^3 + 1551612500\kappa_2\kappa_4^3,$
- $84354493\kappa_1^6\kappa_2^2\kappa_4 - 1274923245\kappa_1^2\kappa_2^4\kappa_4 + 1875010546\kappa_2^5\kappa_4 - 144681875\kappa_1^6\kappa_4^2 - 93270600\kappa_1^4\kappa_2\kappa_4^2 + 3160618621\kappa_1^2\kappa_2^2\kappa_4^2 - 4790699842\kappa_2^3\kappa_4^2 - 1040678750\kappa_1^2\kappa_4^3 + 2081357500\kappa_2\kappa_4^3$

Proof. The proof is based on just two sources of relations among MMM-classes in this situation: the trace relation and the family signature theorem.

Recall that the trace relation [RW18, Cor. 2.7] for a TM-fibration $E\to B$ with fibre M whose cohomology is concentrated in even degrees only depends on $\dim H^*(M;\mathbb{Q})$, and for $M = \mathbb{C}P^2\#\mathbb{C}P^2$ is given by

$$c^4 = \kappa_{ec}c^3 - \frac{\kappa_{ec}^2 - \kappa_{ec^2}}{2}c^2 + \frac{\kappa_{ec}^3 - 3\kappa_{ec}\kappa_{ec^2} + 2\kappa_{ec^3}}{6}c - \frac{\kappa_{ec}^4 - 6\kappa_{ec}^2\kappa_{ec^2} + 3\kappa_{ec^2}^2 + 8\kappa_{ec}\kappa_{ec^3} - 6\kappa_{ec^4}}{24} \tag{B.1}$$

where $c \in H^{|c|}(\mathrm{B\,SO}(2d);\mathbb{Q})$ denotes the characteristic class $c(T_\pi E) \in H^{|c|}(E;\mathbb{Q})$. It implies that we can rewrite any polynomial in $H^*(\mathrm{B\,SO}(4);\mathbb{Q})$ as polynomials of degree ≤ 5 and coefficients in the tautological ring. In particular, the set $\{\kappa_{e^a p_1^b}\}_{a+b\leq 5}$ generates $R^*(M)$ but we will consider instead the following larger generating set $T := \{\kappa_{e^a p_1^b}\}_{a+b\leq 9}$ because we can express more easily the κ_{L_i} for $i \leq 9$ in terms of generators in T. We then simply add all relations among them obtained from fibre integrating multiples $- \cdot e^i p_1^j$ of (B.1) for all i, j with $i + j + 4 \leq 9$. In fact, this can be refined by partially polarizing the trace identity for $c = se + tp_1$ and using the coefficients of $s^n t^{4-n}$ individually (see [RW18, Sect. 4.4]). Finally, we add the relations from the family signature theorem, which implies that $\kappa_{L_i} = 0$ for $i > 2$ since the automorphism group of the intersection of $\mathbb{C}P^2\#\mathbb{C}P^2$ is $\mathrm{SO}(2,\mathbb{Z})$ which is finite.

We limit ourselves to adding $\kappa_{L_2}, \ldots, \kappa_{L_9}$ because they are linear relations of generators of T directly.

We have used to following maple code to produce the ideal of relations of generators of S:

```
with(PolynomialIdeals);
#set maximal degree of polynomial in e and p1 and characteristic numbers
N := 9;
K00 := 0;
K10 := 4;
K01 := 6;

# function that converts e^a*p1^b into Kab
pf := proc (f)
q := 0;
   for a from 0 to N do
   for b from 0 to N do
     if a+b <= N then q := q+convert(cat('K', a, b), symbol)*coeff(coeff(f, e, a), p1, b)
     end if
   end do
   end do
ans := q
end proc;

# recursive function that rewrites powers of p1 using the trace relation
rec1 := proc (n::nonnegint)
          if n = 0 then 1
          elif n = 1 then p1
          elif n = 2 then p1^2
          elif n = 3 then p1^3
          elif 3 < n then
          K11*rec1(n-1)-((1/2)*K11^2-(1/2)*K12)*rec1(n-2)
          +(1/6)*(K11^3-3*K11*K12+2*K13)*rec1(n-3)
          +(1/24)*(-K11^4+6*K11^2*K12-8*K11*K13-3*K12^2+6*K14)*rec1(n-4)
          end if
        end proc;

# recursive function that rewrites powers of e using the trace relation
rece := proc (n::nonnegint)
          if n = 0 then 1
          elif n = 1 then e
          elif n = 2 then e^2
```

```
          elif n = 3 then e^3 elif 3 < n then
          K20*rece(n-1)-((1/2)*K20^2-(1/2)*K30)*rece(n-2)
          +(1/6)*(K20^3-3*K20*K30+2*K40)*rece(n-3)
          +(1/24)*(-K20^4+6*K20^2*K30-8*K20*K40-3*K30^2+6*K50)*rece(n-4)
          end if
        end proc;

# function for (s*e+t*p1)^n
pd:=proc (n::nonnegint)
    add(factorial(n)*s^k*rece(k)*t^(n-k)*rec1(n-k)/(factorial(k)*factorial(n-k)), k = 0 .. n)
    end proc;

# Trace identity for s*e+t*p1
Td:=-pd(4)+pf(e*pd(1))*pd(3)-((pf(e*pd(1))^2-pf(e*pd(2)))*(1/2))*pd(2)
    +((2*pf(e*pd(3))-3*pf(e*pd(2))*pf(e*pd(1))+pf(e*pd(1))^3)*(1/6))*pd(1)
    +(6*pf(e*pd(4))-8*pf(e*pd(3))*pf(e*pd(1))-3*pf(e*pd(2))^2+
      6*pf(e*pd(2))*pf(e*pd(1))^2-pf(e*pd(1))^4)*(1/24);

# polarization
S31 := simplify(coeff(coeff(Td, t^3), s));
S22 := simplify(coeff(coeff(Td, t^2), s^2));
S13 := simplify(coeff(coeff(Td, t), s^3));
# Trace identity for e and p1
T1 := -e^4+K20*e^3+((-K20^2+K30)*(1/2))*e^2+((K20^3-3*K20*K30+2*K40)*(1/6))*e
      +(-K20^4+6*K20^2*K30-8*K20*K40-3*K30^2+6*K50)*(1/24);
T2 := -p1^4+K11*p1^3+((-K11^2+K12)*(1/2))*p1^2+((K11^3-3*K11*K12+2*K13)*(1/6))*p1
      +(-K11^4+6*K11^2*K12-8*K11*K13-3*K12^2+6*K14)*(1/24);

# the Hirzebruch ideal
H := PolynomialIdeal(K02-7*K20, K03-13/2*K21, -19*K40+22*K22-3*K04, 127*K41-83*K23+10*K05,
                     8718*K60-27635*K42+12842*K24-1382*K06,
                     -7978*K61+11880*K43-4322*K25+420*K07,
                     -68435*K80+423040*K62-407726*K44+122508*K26-10851*K08,
                     11098737*K81-29509334*K63+20996751*K45-5391213*K27+438670*K09);

# add relations from the polarized trace relation
S := Add(H, PolynomialIdeal(0));

for a from 0 to N do
for b from 0 to N do
  if a+b+4 <= N then
    S := Add(S, PolynomialIdeal(simplify(pf(e^a*p1^b*S31))))
  end if;
```

```
  if a+b+4 <= N then
    S := Add(S, PolynomialIdeal(simplify(pf(e^a*p1^b*S22))))
  end if;
  if a+b+4 <= N then
    S := Add(S, PolynomialIdeal(simplify(pf(e^a*p1^b*S13))))
  end if;
  if a+b+4 <= N then
    S := Add(S, PolynomialIdeal(simplify(pf(e^a*p1^b*T1))))
  end if;
  if a+b+4 <= N then
    S := Add(S, PolynomialIdeal(simplify(pf(e^a*p1^b*T2))))
  end if
end do
end do
```

We then save this ideal S of the polynomial ring $\mathbb{Q}[T]$ (see generating set T above) as a text file and compute the intersection with the subring generated by $\kappa_{e^2}, \kappa_{e^3}, \kappa_{e^5}$ using Macaulay2. We have displayed the elements in the Proposition above; in particular it is non-empty and so gives elements in the kernel $E^*(\mathbb{C}P^2\#\mathbb{C}P^2) \to R^*(\mathbb{C}P^2\#\mathbb{C}P^2)$ because by Theorem 4.2.16 the Euler ring is $E^*(\mathbb{C}P^2\#CP^2) = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_4]$. $\square$

Remark B.6. This list of elements in the kernel in Proposition B.5 above is in fact not extensive. There are two more relations but the coefficients of the polynomials are so large that we can't display them properly.

Example B.7 (Fake quaternionic projective spaces)**.** We present the code used for computing the dimension of the Hirzebruch ideal for some different choices of p_1.

```
P1= *insert rational number*
B=QQ[x8,x12,P10,P20,P21,P30,P31,P32,Degrees=>{8,12,4,8,4,12,8,4}];
E=B[z]/ideal(z^3-x8*z-x12);
e=3*z^2-x8;
p1=P1*z+P10;
p2=(45+P1^2)/7*z^2+P20+P21*z;
p3=P30+P31*z+P32*z^2;
p4=e^2;
p5=0;
# same for p6=...=p10=0

L3=(2*p1^3-13*p2*p1+62*p3)/945;
L4=(-3*p1^4+22*p2*p1^2-71*p3*p1-19*p2^2+381*p4)/14175;
#omit the formulas for L5,...,L10
```

```
I10=ideal(coefficient(z^2,L3),coefficient(z^2,L4),coefficient(z^2,L5),
          coefficient(z^2,L6),coefficient(z^2,L7),coefficient(z^2,L8),
          coefficient(z^2,L9),coefficient(z^2,L10));
dim I10
```

We have collected the results of a few computations in the following table:

p_1	1	2	3	−6718
$\mathrm{Kdim}\, B/I_H^{\leq 10}$	0	3	0	0

but we have computed the dimension of $I_H^{\leq 10}$ for many more values of p_1 but have always found zero. The case $p_1 = -6718$ is included as there is a manifold in the structure set with this Pontrjagin class.

Appendix C.

The geometric structure set of $\mathbb{H}P^2$

In this appendix we analyse the Browder-Novikov-Sullivan-Wall surgery exact sequence of $\mathbb{H}P^2$ and compute the geometric structure set $S(\mathbb{H}P^2)$ as well as the characteristic classes of the elements $[M, f] \in S(\mathbb{H}P^2)$. Our references on surgery theory are [Bro72] and [CLM15].

Theorem C.1. *The structure set $S(\mathbb{H}P^2)$ is countably infinite and is parametrized up to finite ambiguity by the value of the first Pontrjagin class. More precisely, the possible values of Pontrjagin classes of elements $[M, f] \in S(\mathbb{H}P^2)$ are $p_1(M) = (2 - 48l)z$ for $l = 140n$ or $l = 80 + 140n$ and any integer $n \in \mathbb{Z}$ where $z \in H^4(M;\mathbb{Z})$ is the generator corresponding to the second Chern class of the tautological line bundle over $\mathbb{H}P^2$, and for each possible value of $p_1(M)$ there are two elements in the structure set.*

Let us briefly introduce the necessary terminology for the surgery exact sequence for simply connected manifolds M^{4d}. The geometric structure set $S(M)$ is the set of equivalence classes of orientation preserving homotopy equivalences $f : M' \to M$ where M' is a smooth closed oriented manifold, and equivalence relation $(f_1 : M_1' \to M) \sim (f_2 : M_2' \to M)$ if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that $f_2 \circ \phi \simeq f_1$. Denote by G/O be the homotopy fibre of $\mathrm{B}\,\mathrm{O} \to \mathrm{B}\,G$, where $\mathrm{O} = \operatorname{colim} \mathrm{O}(d)$ and $G := \operatorname{colim} \mathrm{hAut}(S^{d-1})$. It is the classifying space of stable vector bundles whose associated stable spherical fibration is trivial.

The surgery exact sequence simplifies (see for example [Bro72, Thm II.3.1]) to

$$\begin{aligned} S(M^{4d}) \longrightarrow [M, G/O] &\xrightarrow{\sigma} L_{4d}(\mathbb{Z}) \cong \mathbb{Z} \\ [\xi \to M] &\longmapsto \frac{1}{8}\left(\operatorname{sign}(M) - \langle L_d(TM - \xi), [M]\rangle\right) \end{aligned}$$

which is *not* a homomorphism of groups but $S(M^{4d})$ is identified with the preimage of $\sigma^{-1}(0)$, i.e. elements in the set of degree one normal invariants via the Pontrjagin-Thom construction whose surgery obstruction vanishes.

We compute $[M, G/O]$ for $M = \mathbb{H}P^2$ via the coexact Puppe sequence $S^7 \to S^4 \to \mathbb{H}P^2 \to S^8 \to S^5$ of the attaching map of the top cell of $\mathbb{H}P^2$. This induces a long exact sequence of groups when applying $[-, G/O]$ (as G/O is an infinite loop space). Since the surgery obstruction factors through $[M, \mathrm{B}O]$, we compare the exact sequences associated to $[-,G/O]{\to}[-,\mathrm{B}O]$ which gives

$$\begin{tikzcd}
\pi_5(G/O) \arrow[r] \arrow[d] & \pi_8(G/O) \arrow[r] \arrow[d] & \left[\mathbb{H}P^2, G/O\right] \arrow[r] \arrow[d] & \pi_4(G/O) \arrow[r] \arrow[d] & \pi_7(G/O) \arrow[d] \\
\pi_5(BO) \arrow[r] & \pi_8(BO) \arrow[r] & \left[\mathbb{H}P^2, BO\right] \arrow[r] & \pi_4(BO) \arrow[r] & \pi_7(BO).
\end{tikzcd} \tag{C.1}$$

The computation of these groups is a classical problem in homotopy theory and well understood. For $k > 0$ the homotopy groups of G are isomorphic to π_k^s the stable homotopy groups of spheres (and $\pi_0(G) = \mathbb{Z}/2$) which can be found in [Rav86, pg. 3]. The homotopy groups of O are known by Bott periodicity [Bot59]. Since $\Omega\mathrm{B}G \simeq G$ and $\Omega\mathrm{B}O \simeq O$ we can use the fibration sequence $O{\to}G{\to}G/O$ to compute $\pi_*(G/O)$, and furthermore the map on homotopy groups $O{\to}G$ can be identified with the J-homomorphism. The image of the J-homomorphism is completely understood, and the following account is from [Rav86] but really due to Adams and Quillen.

Theorem ([Rav86, Thm 1.1.13])**.** *If $n \neq 3 \bmod 4$, then the J-homomorphism $J_n : \pi_n(SO){\to}\pi_n^s$ is injective. The order of the image of the J-homomorphism $J_{4k-1} : \pi_{4k-1}(SO) \longrightarrow \pi_{4k-1}^s$ is a cyclic group of order* denominator$(B_k/4k)$*, where B_k is the k-th Bernoulli number.*

We combine these results in the following table, which allows for a computation of $\pi_*(G/O)$ in the relevant range.

k	0	1	2	3	4	5	6	7	8
$\pi_k(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$\pi_k(G)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$\mathrm{Im}(J_k)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/24$	0	0	0	$\mathbb{Z}/240$	$\mathbb{Z}/2$
$\pi_k(G/O)$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$

We use this result in (C.1) and obtain

$$\begin{tikzcd}
0 \arrow[r] & \pi_8(G/O) = \mathbb{Z}\oplus\mathbb{Z}/2 \arrow[r] \arrow[d, "\cdot 240"] & \left[\mathbb{H}P^2, G/O\right] \arrow[r] \arrow[d] & \pi_4(G/O) = \mathbb{Z} \arrow[r] \arrow[d, "\cdot 24"] & 0 \\
0 \arrow[r] & \pi_8(BO) = \mathbb{Z} \arrow[r] & \left[\mathbb{H}P^2, BO\right] \arrow[r] & \pi_4(BO) = \mathbb{Z} \arrow[r] & 0
\end{tikzcd} \tag{C.2}$$

which determines $[\mathbb{H}P^2, G/O] \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2$ and the image in $[\mathbb{H}P^2, \mathrm{B}O]$. According to [Bae02], the generators of $\pi_4(\mathrm{B}O)$ and $\pi_8(\mathrm{B}O)$ are given by the tautological line bundles $L_H{\to}\mathbb{H}P^1 = S^4$ and $L_O{\to}\mathbb{O}P^1 = S^8$ over the quaternions and octonions. Denote by abuse of notation $L_H \to \mathbb{H}P^2$ the tautological line bundle over $\mathbb{H}P^2$ which restricts to $L_H \to \mathbb{H}P^1$ and thus provides a splitting of the lower short exact sequence in the above diagram, i.e. $[\mathbb{H}P^2, \mathrm{B}\,O] \cong \mathbb{Z}\{L_H, L_O\}$ (where $L_O \to \mathbb{H}P^2$ denotes the pullback of $L_O \to S^8$ along the collapse map).

It remains to determine the characteristic classes of $T\mathbb{H}P^2, L_H$ and L_O in order to compute the surgery obstruction map.

(1) The sphere bundle of $L_H \to \mathbb{H}P^n$ is diffeomorphic to S^{4n+3} and it follows from the Gysin sequence that the Euler class $e(L_H)$ is a generator of $H^4(\mathbb{H}P^n; \mathbb{Z})$ that we denote by z. Since L_H is a complex vector bundle of rank 2, it follows that $c_2(L_H) = e(L_H) = z$ and consequently

$$p(L_H) = (1-z)^2 = 1 - 2z + z^2 \quad \text{and} \quad L(L_H) = 1 - \frac{2}{3}z + \frac{1}{15}z^2.$$

(2) Even though the sphere bundle of the octonionic line bundle over $\mathbb{O}P^1$ is diffeomorphic to S^{15}, we cannot compute the Pontrjagin classes in this way because L_O is not a complex vector bundle. Instead we use the results in [BM58, Ker59] that show that the kth Pontrjagin class of vector bundles $E \to S^{4k}$ is divisible by $(2k-1)!a_k$ where $a_k = 1$ for k even and 2 if k is odd. Moreover, it follows from [Ker59] and the construction of characteristic classes as primary obstructions that these values are obtained by the generators of $\pi_{4k}(\mathrm{B}\,O)$. However, we do not know the sign of $p_2(L_O)$ in terms of $c_2(L_H)$ but we can simply choose a generator $L_O' \to S^8$ whose pullback over $\mathbb{H}P^2$ satisfies

$$p(L_O') = 1 + 6z^2 \quad \text{and} \quad L(L_O') = 1 + \frac{14}{15}z^2.$$

(3) The characteristic classes of $\mathbb{H}P^n$ have been determined in [Szc64] (see also [MS74, Prob. 20.4]). Using the fibre bundle $\pi : \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ with fibre $S^3/S^1 = \mathbb{C}P^1$, the tangent bundle satisfies $T\,\mathbb{C}P^{2n+1} \cong \pi^* T\mathbb{H}P^n \oplus T_\pi E$ and the Chern classes of the vertical tangent bundle are determined over a single fibre as those of $T\mathbb{C}P^1$. From this one can work out that $p(\mathbb{H}P^n) = (1+z)^{2n+2}/(1+4z)$ where $z = c_2(L_H)$. For $n = 2$ this implies

$$p(\mathbb{H}P^2) = 1 + 2z + 7z^2 \quad \text{and} \quad L(\mathbb{H}P^2) = 1 + \frac{2}{3}z + z^2.$$

Proof of Theorem C.1. Consider an element in $\xi \in [\mathbb{H}P^2, G/O]$ and denote its image in $[\mathbb{H}P^2, \mathrm{B}\,O]$ by $-24l \cdot L_H - 240k \cdot L_O'$. The L-class of $T\mathbb{H}P^2 - \xi$ is

$$\begin{aligned} L(T\mathbb{H}P^2 + 24l \cdot L_H + 240k \cdot L_O) &= (1 + \frac{2}{3}z + z^2)(1 - \frac{2}{3}z + \frac{1}{15}z^2)^{24l}(1 + \frac{14}{15}z^2)^{240k} \\ &= 1 + (\frac{2}{3} - 16l)z + (1 + 224k + \frac{8}{5}l(80l - 9))z^2. \end{aligned}$$

The surgery obstruction vanishes if $140k + l(80l - 9) = 0$. Since $20 \mid 140k$ and $20 \nmid 80l - 9$ it follows that $20 \mid l$ and we substitute $l = 20l'$. The equation becomes $7k + l'(1600l' - 9) = 0$. Hence, either $7 \mid l'$ or $7 \mid 1600l' - 9$ which implies $l' = 4\,(7)$. Combining these results, we find solutions for $n \in \mathbb{Z}$ given by

$$\begin{array}{lll} \text{Case I:} & l = 140n & k = -n(11200n - 9) \\ \text{Case II:} & l = 80 + 140n & k = -(7n + 4)(1600n + 913). \end{array}$$

These two families of solutions provide infinitely many elements in $[\mathbb{H}P^2, G/O]$ whose surgery obstruction vanishes. Moreover, the total Pontrjagin class is

$$p(T\mathbb{H}P^2 + 24l \cdot L_H + 240k \cdot L'_O) = 1 + (2 - 48l)z + (1440k + 24l(48l - 5) + 7)z^2 \tag{C.3}$$

which varies with n and we see that the structure set contains infinitely many different diffeomorphism types of manifolds. □

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