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Calculating Drag and Moments in an Unsteady Free Stream Using Unified Airloads Theory

William Hurley
Washington University in St. Louis

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by William Hurley
whurley@wustl.edu
Advisor: Dr. Dave Peters

Abstract

The Unified Finite State Airloads theory (Unified Theory) developed by Dr. Dave Peters and his former students was used to produce curves for moment and drag in an unsteady free stream. Previous work has been done to show that the Unified theory exactly matches Issacs for lift and C_L in an unsteady free stream with a time varying angle of attack (α). This report will demonstrate in detail how to calculate lift, drag, moment about mid chord, and moment about quarter cord and their coefficients ($C_L, C_d, C_m, C_m(3/4)$) using the Unified theory. A useful approximation to the Unified Theory, the Greenberg Approximation, neglects the effect of the time varying free stream velocity on the wake. The accuracy of the Greenberg approximation is quantified over the 3 cases of angle of attack for each of the calculated loads.

Solution Procedure

The Unified theory obtains its equations for loads from satisfaction of the non-penetration condition at the surface of the airfoil. Equations for circulatory lift, L_c, and total bound vorticity Γ are necessary for loads calculations. For rigid body airfoil motions and a free stream velocity that is not dependant on spacial position, the non-circulatory lift L_{nc} is identical to both Issacs and Greenberg. Therefore the issue becomes calculating circulatory lift. All of the following equations come from the work of Peters and Johnson [1].

$$L_c = 2\pi\rho b u_0(w_0 + \frac{1}{2}w_1 - w_0) \tag{1}$$

$$\Gamma = 2\pi b(w_0 + \frac{1}{2}w_1 - \lambda_0 - \frac{1}{2}\lambda_1) \tag{2}$$

Where b is the semi chord (m), ρ is the air density ($\frac{kg}{m^3}$), u_0 is the unsteady velocity parallel to airfoil chord ($\frac{m}{s}$), w_0 and w_1 are the first two terms in the Glauert series for flow due to airfoil motion, and λ_0 and λ_1 are the first two terms in the Glauert series for induced flow due to shed vorticies and gusts. Note that both L_n and Γ are both per unit length from the assumptions of thin airfoil theory.

The Gluert induced flow coefficients λ_n must satisfy the differential equations given in eq. 3 and eq. 4. For $n = 1$, eq. 3 is used, and for $n \geq 2$, eq. 4 is used.

$$b(\dot{\lambda_0} - \frac{1}{2}\dot{\lambda_2}) + u_0\lambda_1 = \dot{\Gamma}/\pi \tag{3}$$

$$\frac{b}{2n}(\dot{\lambda_{n-1}} - \frac{1}{2}\dot{\lambda_{n+1}}) + u_0\lambda_n = \dot{\Gamma}/(n\pi) \tag{4}$$

These differential equations only generate N equations for $N+1$ unknowns. Eq. 5 defines the final equation required to eliminate the last unknown, λ_0, from the above equations.

$$\lambda_0 = \sum b_n\lambda_n \tag{5}$$

Eq. 6 defines the values for b_n when $n \neq N$.

$$b_n = (-1)^{n+1}\frac{(N+n-1)!}{(N-n-1)!}\frac{1}{(n!)^2} \tag{6}$$

Eq. 7 defines the value for b_n for when $N = n$.

$$b_N = (-1)^{N+1} \tag{7}$$

The previous equations provide a complete set of dimensional differential equations. Following the analysis of ref [1], $N = 8$ was chosen for the number of states. Increasing the number of states N increases the fidelity of the calculation. It is most useful to non-dimensionalize the differential equations above and put them into matrix form as shown in eq. 8.

$$[A]\bar{\lambda}_n^* = -\bar{u}_0(t)\bar{\lambda}_n + [C](\bar{w}_0 + \bar{w}_1/2)^* \tag{8}$$

Where $\bar{u}_0, \bar{\lambda}_n$ and $\bar{w}_n$ have been normalized by the average free stream velocity (v_0) and $()^*$ is a derivative with respect to a non-dimensional time $\tau = v_0t/b$. For reference, the $[A]$ and $[C]$ matrices are provided in the appendix for $N = 8$. An unsteady free stream u_0 and a time varying angle of attack α are inputted into eq. 8. The equation for the free stream is shown in eq. 9.

$$\bar{u}_0 = 1 + \mu sin(\omega t) = 1 + \mu sin(k\tau) \tag{9}$$

$$k = \omega b/v_0 \tag{10}$$

μ is the percent of the free stream velocity of the oscillations, k is the reduced frequency, and ω is the angular velocity of the blade. 3 cases of angle of attack (α) were used. Steady α, in phase α, and out of phase α.

$$\alpha = 1 \tag{11}$$

$$\alpha = sin(k\tau) \tag{12}$$

$$\alpha = cos(k\tau) \tag{13}$$

For small motions in α and plunge h, where h is measured at a point "a" near the airfoil center we generate the following equations.

$$\bar{\omega}_0 = \bar{u}_0\alpha + h^* - a\alpha^* \tag{14}$$

$$\bar{\omega}_0^* = \bar{u}_0{}^*\alpha + h^{**} - a\alpha^{**} \tag{15}$$

$$\bar{\omega}_1 = \alpha^* \tag{16}$$

$$\bar{\omega}_1^* = \alpha^{**} \tag{17}$$

Since oscillations are about the center, both h and a are 0. With the induced flow coefficients solved for, one has everything needed to solve for lift, drag, moments and their coefficients. Note that it could take multiple periods for the solution to come to steady state. When defining the tspan for ODE45 in MATLAB, be sure that it is large enough so that the solution is able to reach steady state. The final period should always be taken to ensure accuracy. The following equations define the normalized versions of circulatory lift L_c and circulatory lift coefficient C_{Lc}.

$$\bar{L}_c = \bar{u}_0(\bar{\omega}_0 + \frac{1}{2}\bar{\omega}_1 - \bar{\lambda}_0) \tag{18}$$

$$\bar{C}_{Lc} = (\bar{\omega}_0 + \frac{1}{2}\bar{\omega}_1 - \bar{\lambda}_0)/\bar{u}_0 \tag{19}$$

The following two equations define the normalized versions of drag D and drag coefficient C_d.

$$\bar{D} = \bar{\lambda}_0(\alpha\bar{u}_0 - \bar{\lambda}_0) \tag{20}$$

$$\bar{C}_D = \frac{D}{\bar{u}_0^2} = \frac{\bar{\lambda}_0}{\bar{u}_0}(\alpha - \frac{\bar{\lambda}_0}{\bar{u}_0}) \tag{21}$$

Unlike moments and lift, Drag is non-linear so α was multiplied by 0.1 so that α stays in the linear regime. The next two equations define the normalized versions moment about the mid cord, $M_{1/2}$, and its coefficient $C_{m_{1/2}}$.

$$\bar{M}_{1/2} = -\frac{1}{8}\omega_1 \bar{u}_0 \tag{22}$$

$$\bar{C}_{M_{1/2}} = -\frac{1}{8}\omega_1 / \bar{u}_0 \tag{23}$$

The final two equations define the normalized versions moment about the quarter cord, $M_{3/4}$, and its coefficient $C_{m_{3/4}}$.

$$\bar{M}_{3/4} = -\frac{1}{8}\alpha^* \bar{u}_0 \tag{24}$$

$$\bar{C}_{M_{3/4}} = -\frac{1}{8}\alpha^* / \bar{u}_0 \tag{25}$$

Quantifying the Greenberg Approximation

If $\bar{u}_0$ is set to 1 in on the left hand side of eqs. (3 and 4) so that the affect of the unsteady free stream on the wake is neglected, one recovers the results of the Greenberg Approximation. The Greenberg Approximation is computationally more efficient however it has varying levels of accuracy depending on the percentage of the free stream velocity of the oscillations (μ), and the reduced frequency (k). The Greenberg approximation was compared to the exact calculation of the Unified theory using a two norm. The formula for a two norm is given in eq. 26.

$$error = \sqrt{\frac{\int (Greenberg - Unified)^2 dt}{\int (Unified)^2 dt}} \tag{26}$$

The Greenberg approximation was quantified over a trade space of values of μ and k for each individual force (Lift, Drag, Moment), and each case of α. Moment about the quarter cord is always the same for Greenberg and the Unified theory so that data was left out. Only data for normalized forces is included since they provide the same information as the data for the coefficients.

Table 1: Lift Tables

$\alpha = 1$	$\mu = 0.2$	0.4	0.6	0.8
k = 0.2	0.005	0.018	0.036	0.057
0.4	0.007	0.024	0.048	0.076
0.6	0.007	0.026	0.052	0.082
0.8	0.007	0.027	0.053	0.084

$\alpha = \sin(k\tau)$	$\mu = 0.2$	0.4	0.6	0.8
k = 0.2	0.043	0.070	0.087	0.099
0.4	0.062	0.097	0.116	0.129
0.6	0.068	0.106	0.125	0.138
0.8	0.069	0.107	0.127	0.140

$\alpha = \cos(k\tau)$	$\mu = 0.2$	0.4	0.6	0.8
k = 0.2	0.046	0.079	0.107	0.131
0.4	0.066	0.102	0.134	0.163
0.6	0.082	0.117	0.152	0.183
0.8	0.097	0.133	0.169	0.203

Table 2: Drag Tables

$\alpha = 1$	$\mu = 0.2$	0.4	0.6	0.8
k = 0.2	0.121	0.248	0.388	0.551
0.4	0.130	0.266	0.412	0.575
0.6	0.131	0.267	0.413	0.572
0.8	0.130	0.265	0.409	0.564

$\alpha = \sin(k\tau)$	$\mu = 0.2$	0.4	0.6	0.8
k = 0.2	0.111	0.219	0.324	0.425
0.4	0.105	0.206	0.303	0.396
0.6	0.106	0.205	0.296	0.381
0.8	0.111	0.209	0.294	0.370

$\alpha = \cos(k\tau)$	$\mu = 0.2$	0.4	0.6	0.8
k = 0.2	0.070	0.139	0.207	0.282
0.4	0.061	0.121	0.182	0.247
0.6	0.067	0.129	0.189	0.250
0.8	0.079	0.152	0.217	0.279

Table 3: Moment(mid) Tables

$\alpha = 1$	$\mu = 0.2$	0.4	0.6	0.8
k = 0.2	0.005	0.018	0.036	0.057
0.4	0.007	0.024	0.048	0.076
0.6	0.007	0.026	0.052	0.082
0.8	0.007	0.027	0.053	0.084

$\alpha = \sin(\mathrm{k}\tau)$	$\mu = 0.2$	0.4	0.6	0.8
k = 0.2	0.042	0.069	0.086	0.098
0.4	0.059	0.094	0.114	0.128
0.6	0.064	0.102	0.122	0.136
0.8	0.065	0.103	0.124	0.138

$\alpha = \cos(\mathrm{k}\tau)$	$\mu = 0.2$	0.4	0.6	0.8
k = 0.2	0.045	0.076	0.104	0.129
0.4	0.061	0.095	0.127	0.156
0.6	0.073	0.106	0.139	0.171
0.8	0.084	0.116	0.151	0.187

References

[1]Peters, David A. and Johnson, Mark J., “Finite State Airloads for Deformable Airfoils on Fixed and Rotating Wings,” ASME Symposium on Aeroelasticity and Fluid/Structure Interaction, Chicago, Illinois, November 6-11, 1994.

Appendix

Matrix [A] and [C] for eq. 8. The number of states was chosen to be N = 8.

$$[A] = \begin{bmatrix} \frac{3}{2}b_1 + 1 & \frac{3}{2}b_2 - 1/2 & \frac{3}{2}b_3 & \frac{3}{2}b_4 & \frac{3}{2}b_5 & \frac{3}{2}b_6 & \frac{3}{2}b_7 \\ \frac{1}{2}b_1 + \frac{3}{4} & \frac{1}{2}b_2 & \frac{1}{2}b_3 - \frac{1}{4} & \frac{1}{2}b_4 & \frac{1}{2}b_5 & \frac{1}{2}b_6 & \frac{1}{2}b_7 \\ \frac{1}{3}b_1 + \frac{1}{3} & \frac{1}{3}b_2 + \frac{1}{6} & \frac{1}{3}b_3 & \frac{1}{3}b_4 - \frac{1}{6} & \frac{1}{3}b_5 & \frac{1}{3}b_6 & \frac{1}{3}b_7 \\ \frac{1}{4}b_1 + \frac{1}{4} & \frac{1}{4}b_2 & \frac{1}{4}b_3 + \frac{1}{8} & \frac{1}{4}b_4 & \frac{1}{4}b_5 - \frac{1}{8} & \frac{1}{4}b_6 & \frac{1}{4}b_7 \\ \frac{1}{5}b_1 + \frac{1}{5} & \frac{1}{5}b_2 & \frac{1}{5}b_3 & \frac{1}{5}b_4 + \frac{1}{10} & \frac{1}{5}b_5 & \frac{1}{5}b_6 - \frac{1}{10} & \frac{1}{5}b_7 \\ \frac{1}{6}b_1 + \frac{1}{6} & \frac{1}{6}b_2 & \frac{1}{6}b_3 & \frac{1}{6}b_4 & \frac{1}{6}b_5 + \frac{1}{12} & \frac{1}{6}b_6 & \frac{1}{6}b_7 - \frac{1}{12} \\ \frac{1}{7}b_1 + \frac{1}{7} & \frac{1}{7}b_2 & \frac{1}{7}b_3 & \frac{1}{7}b_4 & \frac{1}{7}b_5 & \frac{1}{7}b_6 + \frac{1}{14} & \frac{1}{7}b_7 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 2 & 1 & \frac{2}{3} & \frac{1}{2} & \frac{2}{5} & \frac{1}{3} & \frac{2}{7} \end{bmatrix}^T$$