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# Induced Flow in Coaxial Rotor Systems

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# Summary

In coaxial rotor helicopter systems, both rotors of the helicopter spin in opposite directions on the same vertical axis, requiring the bottom rotor to spin faster with a higher blade pitch angle and the top rotor to spin slower with a lower blade pitch angle. This was proven by Peters and Seidel [1]. As both blades are spinning, there is an induced flow of air between the rotors. The solution of the potential flow equations shows that the flow induced by the upper rotor on the lower is related to the adjoint velocity above the rotors and the time delays in the system. Therefore, the model used to determine the effects of the inflow will have both velocity states and co-states with included time delays. The time delays result from the effects of the vortex shed by the upper rotor, causing the induced flow on the upper rotor to decay with time as the flow on the lower rotor increases with time while the vortex moves downward. After the vortex has moved past the lower rotor, it decays again with an added time delay that matches that of the time it took to move between the two rotors. The main objective of this study is to determine how the velocity co-states affect the time delays on the Bode Plots for the transfer functions between rotor inputs and induced velocity on the two rotors. The computer language MATLAB and complex arithmetic are utilized in conjunction with Fourier Transforms to predict how the air will flow between the rotors when they are coupled through blade-element theory.

# Variables and Definitions

| | |
|---|---|
| $A(s)$ | Laplace Transform of velocity state for both rotors |
| $A_U(s)$, $A_L(s)$ | Laplace Transform of velocity states for upper rotor and lower rotor |
| $d$ | Distance between upper rotor and lower rotor |
| $E(s)$ | Transfer Function, $E(s) = \frac{W(s)}{F(s)} = \frac{e^{-sd}}{\lambda+s} + \frac{e^{-sd}-e^{-\lambda d}}{\lambda-s}$ |
| $F(s)$ | Laplace Transform of forces on both rotors |
| $F_U(s)$, $F_L(s)$ | Laplace Transform of forces on upper rotor and lower rotor |
| $F(t)$ | Force on both rotors as a function of time |
| $F_U(t)$, $F_L(t)$ | Force on upper rotor and lower rotor as a function of time |
| $G(s)$ | Transfer Function, $G(s) = \frac{\Delta(s)}{F(s)} = \frac{1}{\lambda-s}$ |
| $H(s)$ | Transfer Function, $H(s) = \frac{V(s)}{F(s)} = \frac{e^{-sd}-e^{-\lambda d}}{\lambda-s}$ |
| $\mathbf{i}$ | $\sqrt{-1}$ |
| $p$ | Rotating flap frequency divided by $\Omega$ |
| $s$ | Laplace Transform variable |
| $u$ | Frequency State, $\frac{\text{rad.}}{\text{s}}$ |
| $V(s)$ | Laplace Transform of the adjoint velocity contribution |
| $V(t)$ | Velocity contribution to the lower rotor velocity related to time |
| $W(s)$ | Laplace Transform of the total velocity due to both $\alpha$ and $\delta$ |
| $w(t)$ | Total velocity due to both $\alpha$ and $\delta$ as a function of time |
| $\alpha(t)$ | Velocity state for both rotors |
| $\alpha_U(t)$, $\alpha_L(t)$ | Velocity states for upper rotor and lower rotor |
| $\beta_U$, $\beta_L$ | Flap angles on upper rotor and lower rotor |
| $\gamma$ | Lock number |
| $\Delta(s)$ | Laplace Transform of $\delta$ |
| $\delta(t)$ | Adjoint velocity co-state for upper rotor |
| $\theta_U$, $\theta_L$ | Step input from upper rotor and lower rotor |
| $\lambda$ | Induced velocity, $\lambda = 2.09$ |

# Transfer Function with Zero Initial Conditions

The equation for the velocity co-state experienced by the upper rotor, δ, is written[1]:

$$-\dot{\delta}(t) + \lambda\delta(t) = F(t)$$

Taking the Laplace Transform of the co-state equation with zero initial conditions gets $F_U(s)$ in terms of $\Delta_U(s)$:

$$(-s + \lambda)\Delta(s) = F(s)$$

Dividing both sides by $F(s) \cdot (-s + \lambda)$ yields the Transfer Function with initial conditions set to zero. In this scenario, the Laplace Transform variable, s, is equivalent to **i**u.

$$G(s) = \frac{\Delta(s)}{F(s)} = \frac{1}{\lambda - s}$$

$$G(\mathbf{i}u) = \frac{1}{\lambda - \mathbf{i}u}$$

The following five plots show the real portion, imaginary portion, magnitude, and phase angle of G(s) plotted against the frequency variable, u. Note that the value of λ used was 2.09.

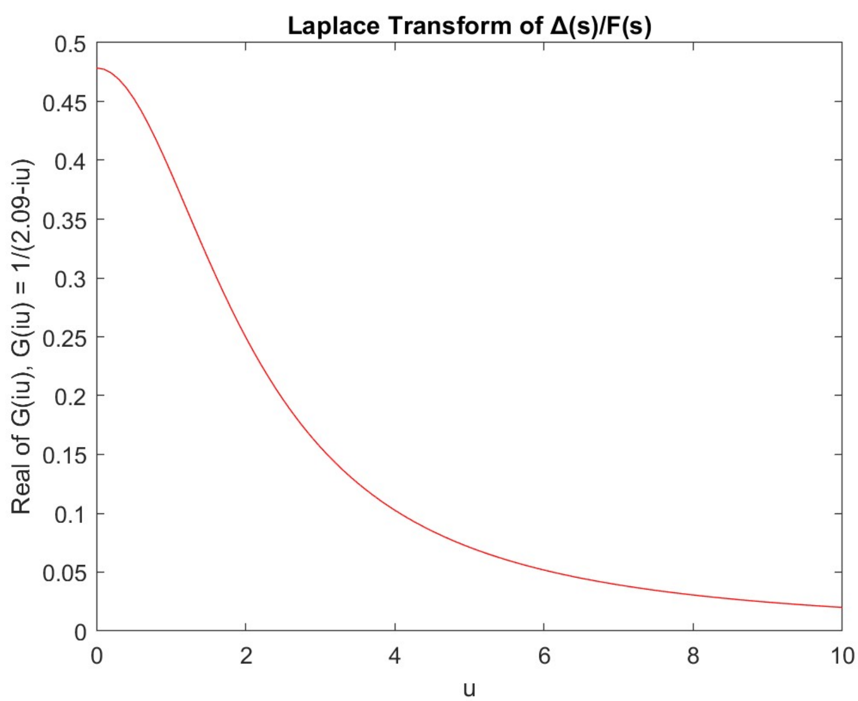


**Figure 1: Real Component of the Laplace Transform of Δ(s)/F(s)**

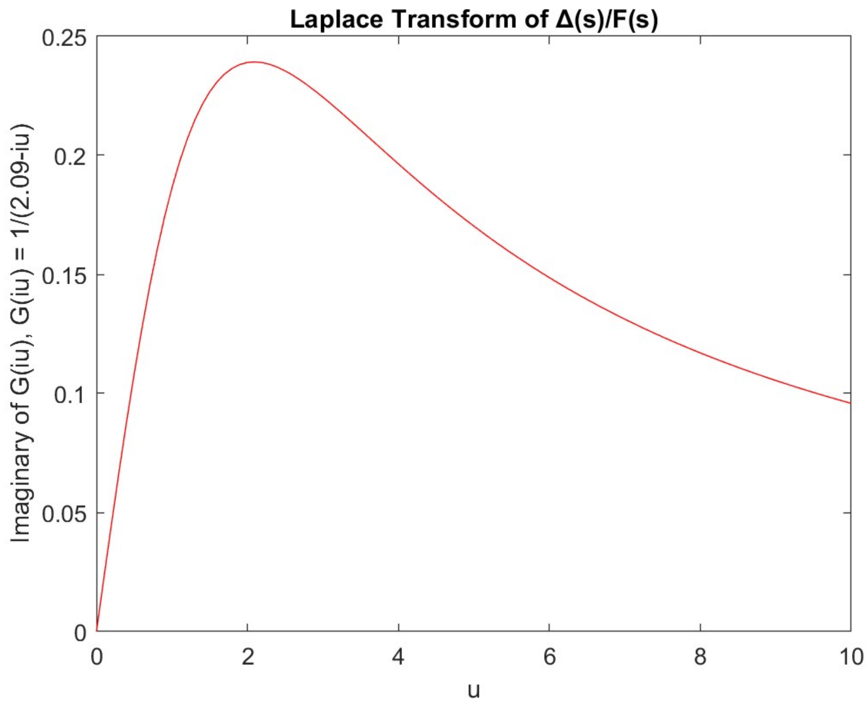


**Figure 2: Imaginary Component of the Laplace Transform of $\Delta(s)/F(s)$**

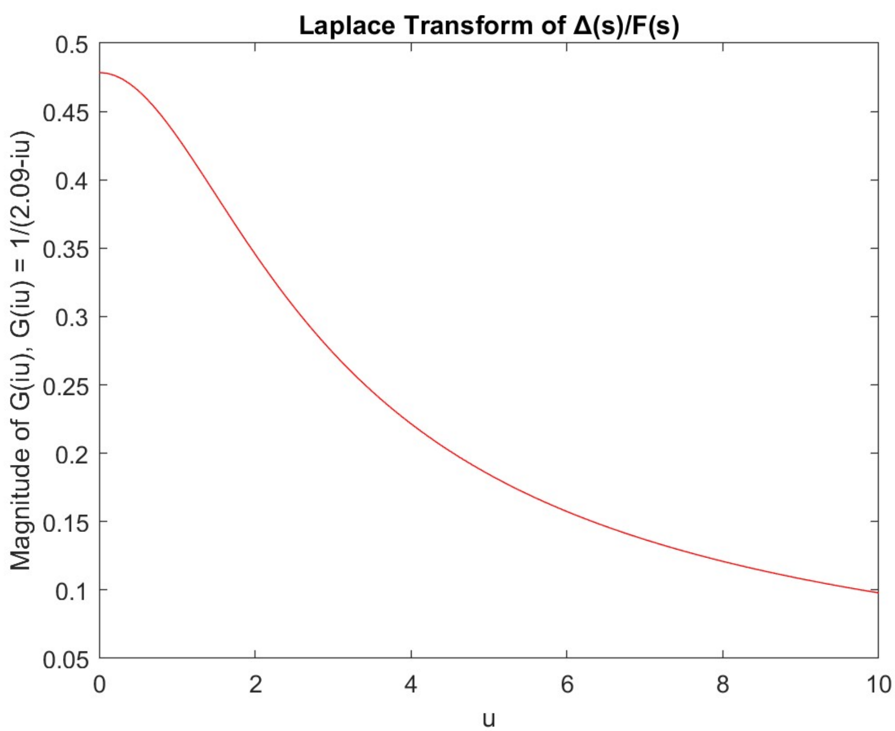


**Figure 3: Magnitude of the Laplace Transform of $\Delta(s)/F(s)$**

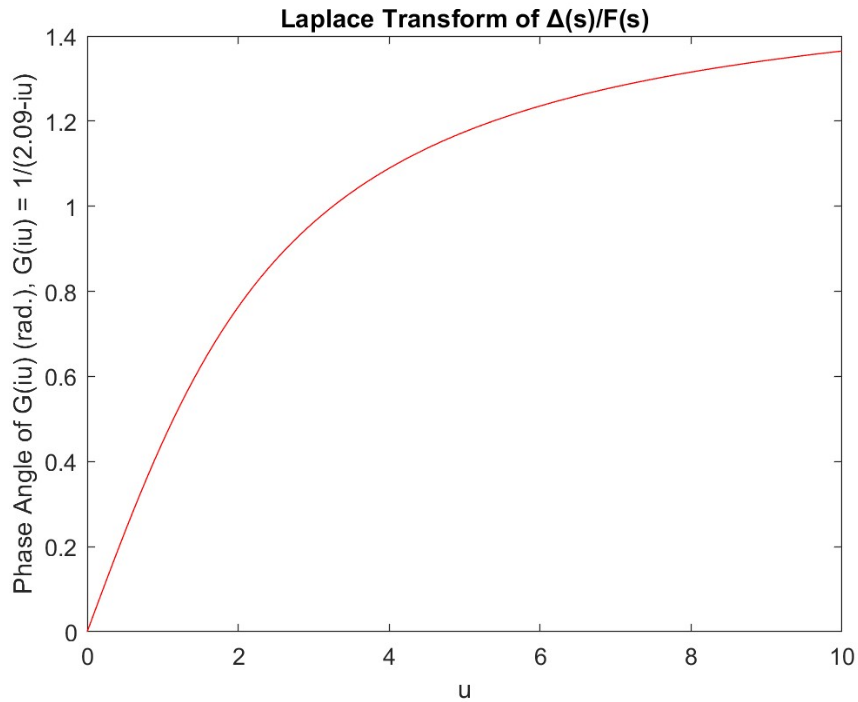


**Figure 4: Phase Angle (rad.) of the Laplace Transform of Δ(s)/F(s)**

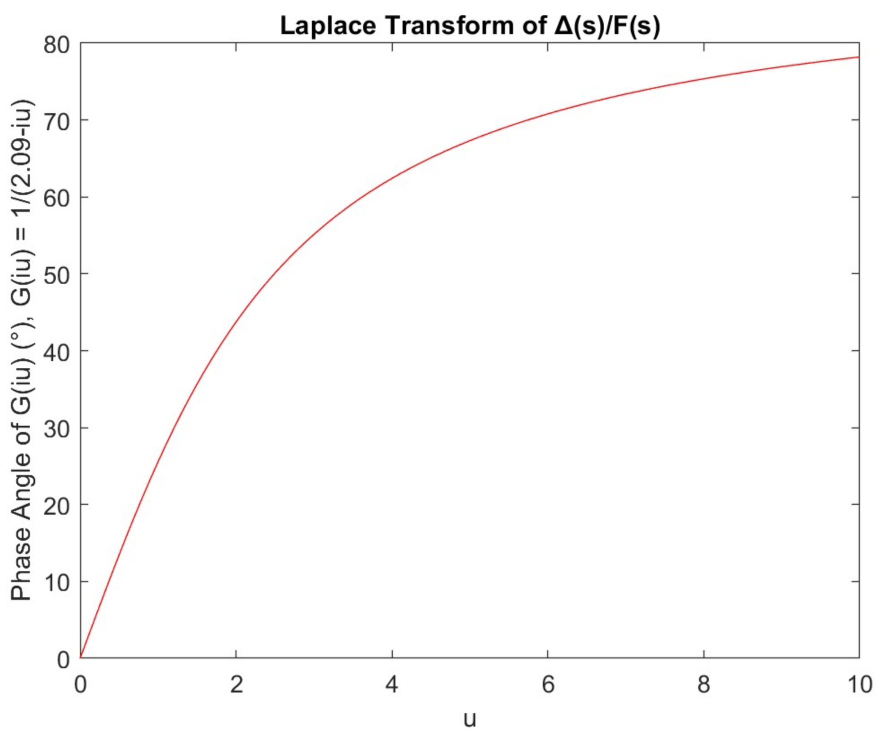


**Figure 5: Phase Angle (°) of the Laplace Transform of Δ(s)/F(s)**

# Transfer Function with Nonzero Initial Conditions

Instead of equating F(t) with Δ(t), for nonzero conditions, the equation for V(t) was used:

$$\delta(\mathrm{t}-\mathrm{d})-\delta(\mathrm{t})e^{-\lambda\mathrm{d}}=\mathrm{V}(\mathrm{t})$$

Using the same methods as before, V(t) became V(s) after taking the Laplace Transform:

$$e^{-\mathrm{s}}\ \Delta(\mathrm{s})-e^{-\lambda\mathrm{d}}\Delta(\mathrm{s})=\mathrm{V}(\mathrm{s})$$

Dividing both sides by $\mathrm{F(s)}=\Delta(\mathrm{s})(\lambda-\mathrm{s})$ yielded the Transfer Function with nonzero initial conditions. In this case, the Laplace Transform variable, s, is equivalent to **i**u.

$$\mathrm{H(s)}=\frac{\mathrm{V(s)}}{\mathrm{F(s)}}=\frac{e^{-\mathrm{sd}}-e^{-\lambda\mathrm{d}}}{\lambda-\mathrm{s}}$$

$$\mathrm{H}(\mathbf{i}\mathrm{u})=\frac{e^{-\mathbf{i}\mathrm{ud}}-e^{-\lambda\mathrm{d}}}{\lambda-\mathbf{i}\mathrm{u}}$$

The following five plots show the real portion, imaginary portion, magnitude, and phase angle of H(s) plotted against the frequency variable, u. Note that the value of λ used was 2.09 and the value of d used varied for the following five plots.

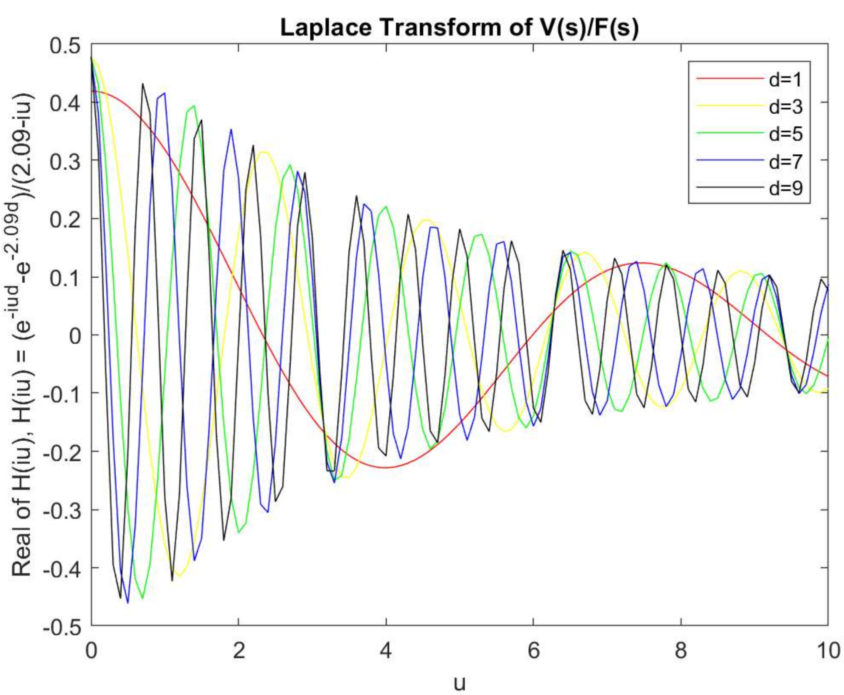


**Figure 6: Real Component of the Laplace Transform of V(s)/F(s)**

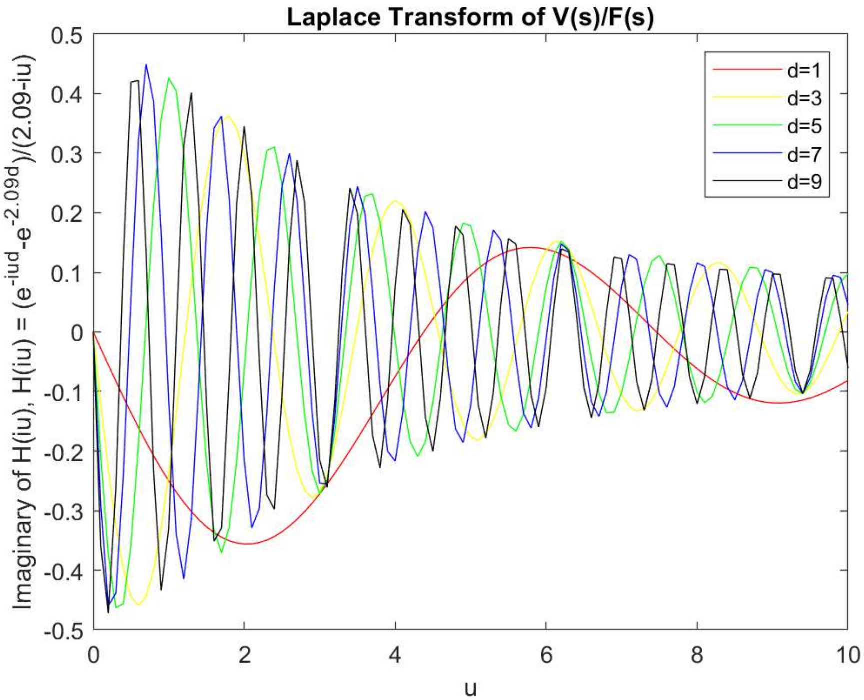


**Figure 7: Imaginary Component of the Laplace Transform of V(s)/F(s)**

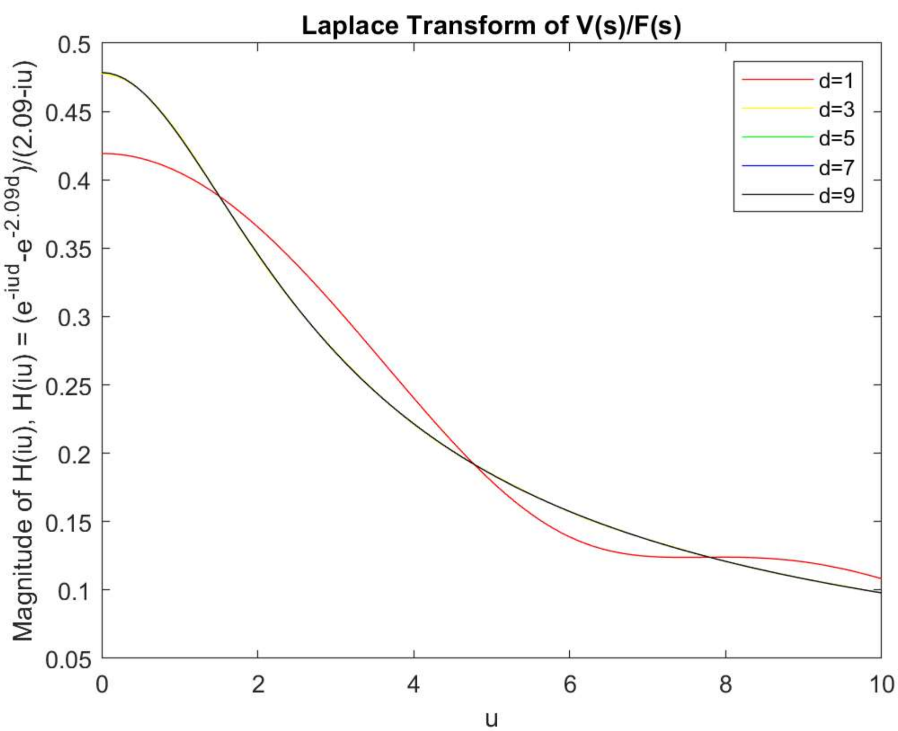


**Figure 8: Magnitude of the Laplace Transform of V(s)/F(s)**

**(black curve actually has yellow, green, and blue right on top of it)**

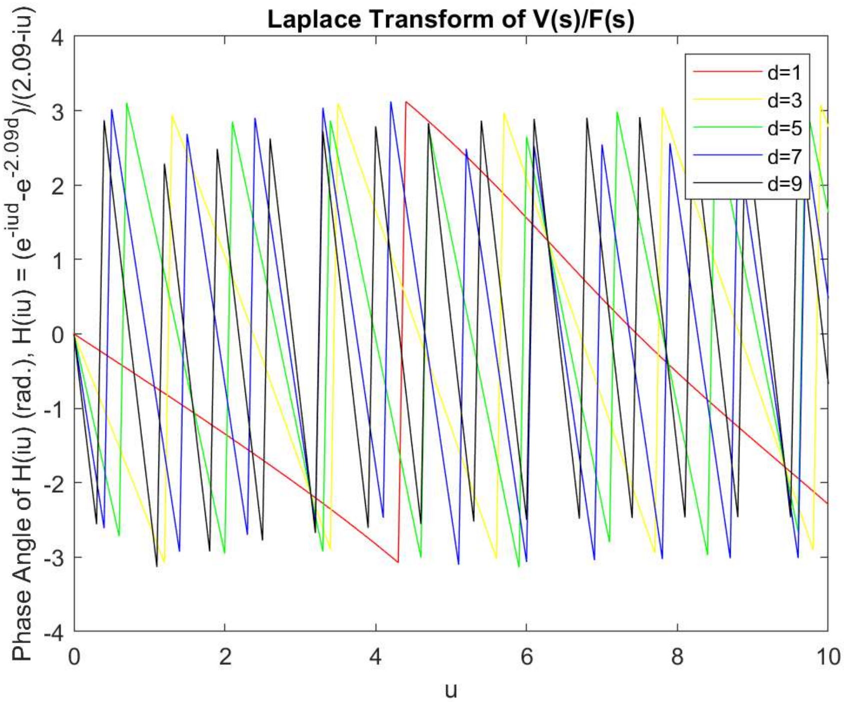


**Figure 9: Phase Angle (rad.) of the Laplace Transform of V(s)/F(s)**

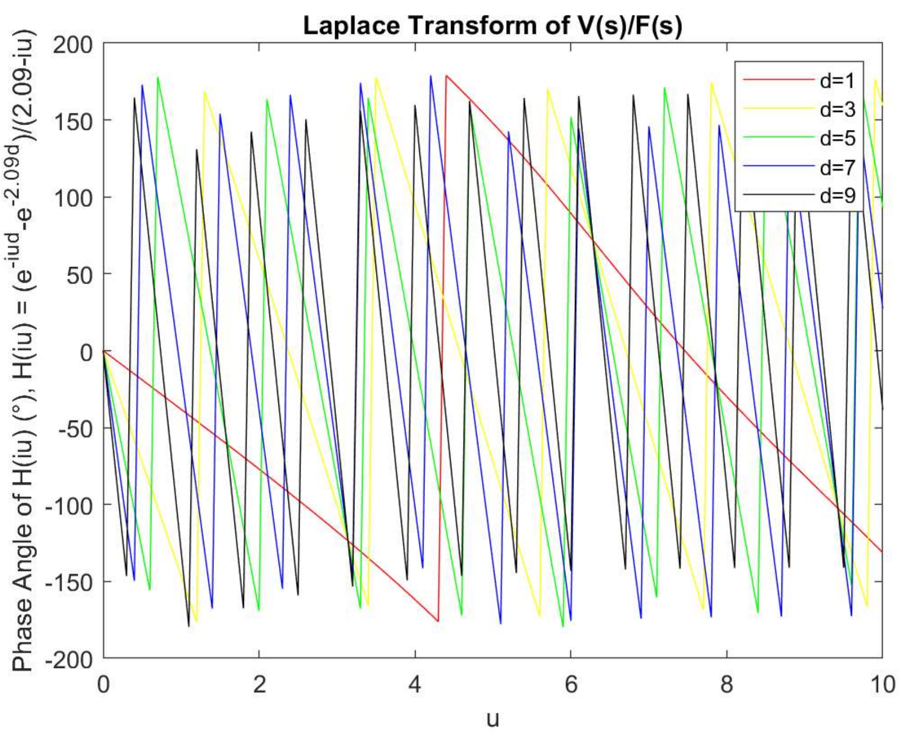


**Figure 10: Phase Angle (°) of the Laplace Transform of V(s)/F(s)**

# Transfer Function with Velocity States and Co-States

For this section, the velocity states, represented by $\alpha$, were considered for both rotors in addition to the velocity co-state for the upper rotor[1]:

$$\dot{\alpha}(t) + \lambda\alpha(t) = F(t)$$

$$-\dot{\delta}(t) + \lambda\delta(t) = F(t)$$

Note that the Laplace transform of the first equation with $\alpha$ is:

$$A(s)(s + \lambda) = F(s)$$

The pair of equations above were combined together to obtain the total velocity caused by the states of both rotors and the co-state of the upper rotor, represented by w(t):

$$\alpha(t - d) + \delta(t - d) - \delta(t)e^{-\lambda d} = w(t)$$

Taking the Laplace Transform of this equation gives the following equation for W(s):

$$A(s)e^{-sd} + \Delta(s)e^{-sd} - \Delta(s)e^{-\lambda d} = W(s)$$

Dividing both sides by $F(s) = A(s)(\lambda + s) = \Delta(s)(\lambda - s)$ yielded the following Transfer Function:

$$E(s) = \frac{W(s)}{F(s)} = \frac{e^{-sd}}{\lambda + s} + \frac{e^{-sd} - e^{-\lambda d}}{\lambda - s}$$

$$E(\mathbf{i}u) = \frac{e^{-\mathbf{i}ud}}{\lambda + \mathbf{i}u} + \frac{e^{-\mathbf{i}ud} - e^{-\lambda d}}{\lambda - \mathbf{i}u}$$

The following five plots show the real portion, imaginary portion, magnitude, and phase angle of E(s) plotted against the frequency variable, u. Note that the value of $\lambda$ used was 2.09 and the value of d used varied for the following five plots.

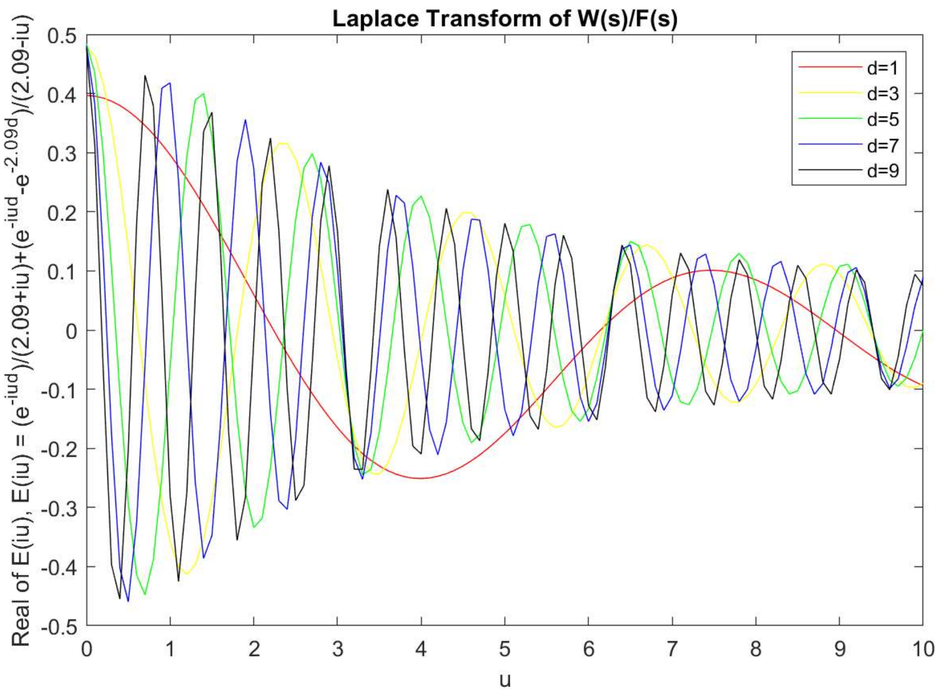


**Figure 11: Real Component of the Laplace Transform of E(s)/F(s)**

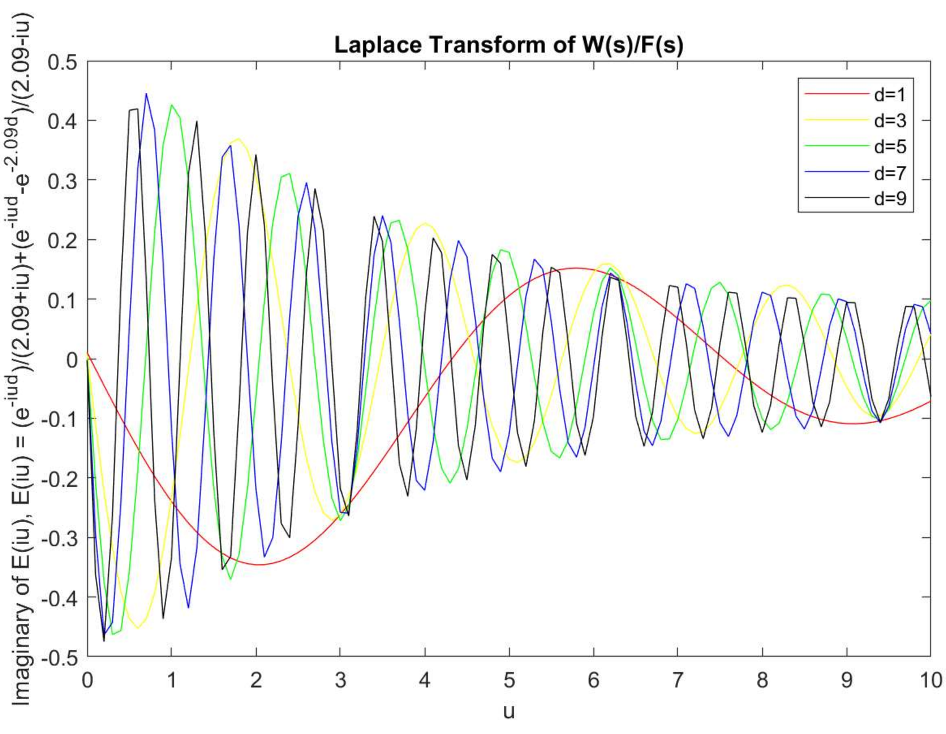


**Figure 12: Imaginary Component of the Laplace Transform of E(s)/F(s)**

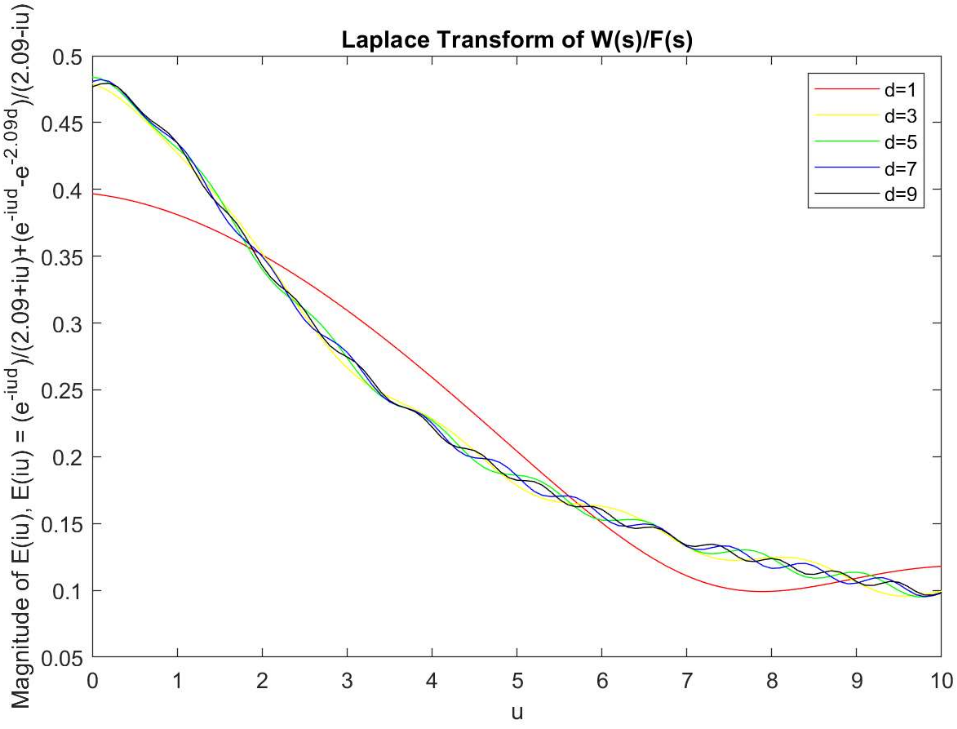


**Figure 13: Magnitude of the Laplace Transform of E(s)/F(s)**

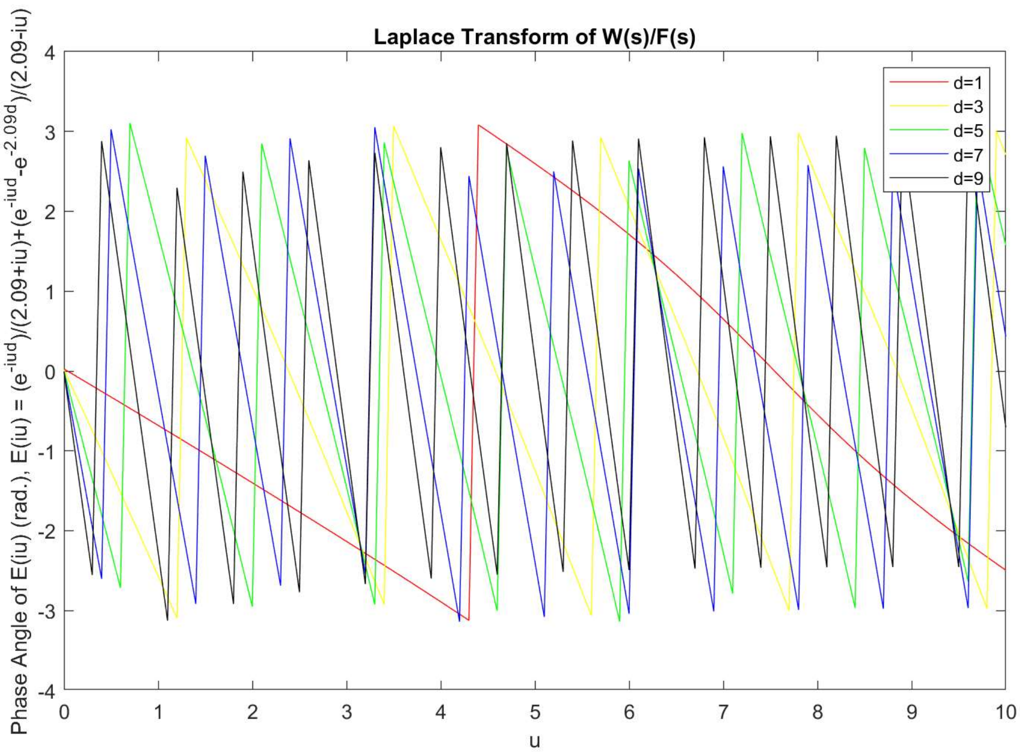


**Figure 14: Phase Angle (rad.) of the Laplace Transform of E(s)/F(s)**

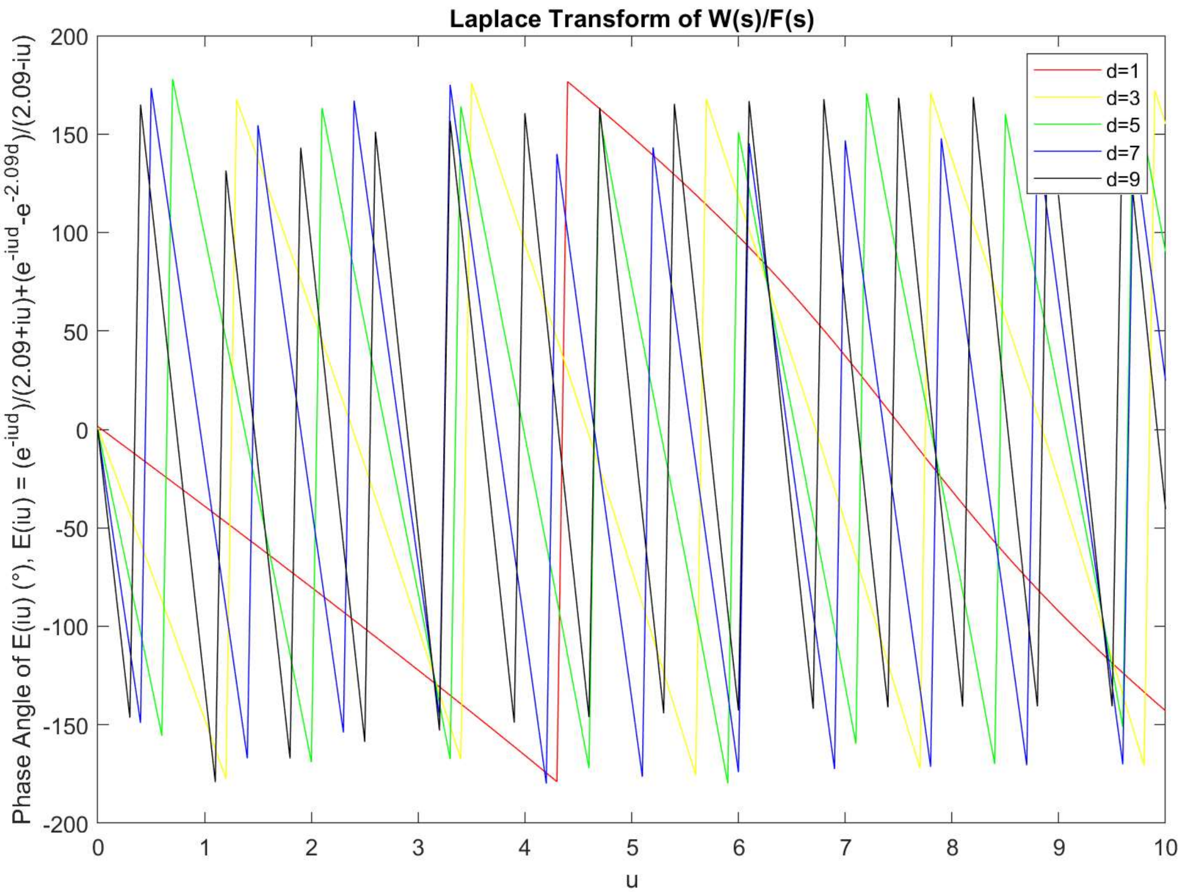


**Figure 15: Phase Angle (°) of the Laplace Transform of E(s)/F(s)**

# References

[1] Seidel, Cory and Peters, David A., "Coupled Inflow and Structural Dynamics of a Coaxial Rotor with Time Delays and Adjoint Variables," Proceedings of the VFS Conference on Transformative Vertical Flight, San Jose, CA, January 21-23, 2020.