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A CLASSIFICATION OF SOME ALMOST α-PARA-KENMOTSU MANIFOLDS *

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Abstract. In this paper, we mainly study local structures and curvatures of the almost α-para-Kenmotsu manifolds. In particular, locally symmetric almost α-para-Kenmotsu manifolds satisfying certain nullity conditions are classified.
Key words: curvatures; α-para-Kenmotsu manifolds; nullity conditions.

1. Introduction

One of the recent topics in the theory of almost contact metric manifolds is the study of the so-called nullity distributions. In [5], E. Boeckx studied the full classification of contact (κ, μ)-spaces, later in [11] and [12], P. Dacko and Z. Olszak gave a systematic study of almost cosymplectic (κ, μ, ν)-spaces and almost cosymplectic $(-1, \mu, 0)$-spaces. G. Dileo and A. M. Pastore in [8] studied nullity distributions on almost Kenmotsu manifolds. In recent years, many authors have turned to the study of almost paracontact geometry due to an unexpected relationship between contact (κ, μ)-spaces and paracontact geometry that was found in [3].

The study of almost paracontact geometry was introduced by Kaneyuki and Williams in [14] and then it was continued by many other authors. A systematic study of almost paracontact metric manifolds was carried out in [16] by Zamkovoy. In fact, such manifolds were studied earlier in [17],[18],[6],[15] and in these papers the authors called such structures almost para-cohermitian. The curvature identities for different classes of almost paracontact metric manifolds were obtained in [13],[10],[16].

In [2], a complete study of paracontact metric manifolds satisfying a certain nullity condition has been carried out, later, in [9], the authors gave a complete study of almost α-cosymplectic manifolds, where α is a function, basic properties of such manifolds are obtained and general curvature identities are proved. It is

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also showed that almost α-para-Kenmotsu (κ, μ, ν)-spaces have para-Kähler leaves. Motivated by [7], [8] and [9], the aim of this paper is devoted to investigate local symmetry and nullity distributions on almost α-para-Kenmotsu manifolds.

This paper is organized in the following way. In section 2, some preliminaries and properties about almost α-para-Kenmotsu manifolds are given. In section 3, we characterize almost paracontact metric manifolds which are $\mathcal{CR}$-integrable almost α-para-Kenmotsu through the existence of a suitable linear connection, and in section 4, we investigate almost α-para-Kenmotsu manifolds which are locally symmetric and give some properties. In section 5, we study almost α-para-Kenmotsu manifolds satisfying some nullity distributions and give some properties and classification theorems of them.

2. Almost α-para-Kenmotsu manifolds

Now, we recall some basic notions of almost paracontact manifold (see [9]). A $2n+1$-dimensional smooth manifold M is said to have an almost paracontact structure if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying the following conditions:

(*i*) $\varphi^2 = \mathrm{Id} - \eta \otimes \xi, \quad \eta(\xi) = 1$,

(*ii*) the tensor field φ induces to an almost paracomplex structure on each fibre of $\mathcal{D}$ =Ker(η), i.e. the $\pm 1-$eigendistributions $\mathcal{D}^{\pm} := \mathcal{D}_{\varphi}(\pm 1)$ of φ have equal dimension n.

From the definition, it follows that $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$ and $\mathrm{rank}(\varphi) = 2n$. When the tensor field $\mathcal{N}_{\varphi} := [\varphi, \varphi] - 2d\eta \otimes \xi$ vanishes identically the almost paracontact manifold is said to be normal. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.1}$$

for any vector fields $X, Y \in \Gamma(TM)$, then we say that $(M^{2n+1}, \varphi, \xi, \eta, g)$ is an almost paracontact metric manifold. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n, n+1)$. For an almost paracontact metric manifold, there always exists an orthogonal basis$\{\xi, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ such that $g(X_i, X_j) = \delta_{ij}, g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \varphi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a $\varphi-$basis. Moreover,we can define a skew-symmetric tensor field 2-form Φ by $\Phi(X, Y) := g(X, \varphi Y)$,which is usually called the fundamental form.

Lemma 2.1. ([*16*]) *For an almost paracontact structure* (φ, ξ, η, g)*, the covariant derivative* $\nabla\varphi$ *of* φ *with respect to the Levi-Civita connection* ∇ *is given by*

$$\begin{aligned} 2g((\nabla_X \varphi)Y, Z) = &- 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) - g(\mathcal{N}^{(1)}(Y, Z), \varphi X) \\ &+ \mathcal{N}^{(2)}(Y, Z)\eta(X) + 2d\eta(\varphi Y, X)\eta(Z) - 2d\eta(\varphi Z, X)\eta(Y). \end{aligned}$$

Definition 2.1. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold, if it satisfies

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi,$$

where α=const.$\neq 0$, then M^{2n+1} is called an almost α-para-Kenmotsu manifold.

Let M be an almost α-para-Kenmotsu manifold with structure (φ, ξ, η, g). Since the 1-form η is closed, then the distribution $\mathcal{D} = \ker(\eta)$ is integrable, we have $L_\xi \eta = 0$, and $[X, \xi] \in \mathcal{D}$ for any $X \in \mathcal{D}$. Then, using Lemma 2.1, the Levi-Civita connection is given by

$$2g((\nabla_X \varphi)Y, Z) = -2\alpha g(\eta(Y)\varphi X + g(X, \varphi Y)\xi, Z) - g(\mathcal{N}(Y, Z), \varphi X) \tag{2.2}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$. If we replace X by ξ, it follows $\nabla_\xi \varphi = 0$, which implies that $\nabla_\xi \xi = 0$ and $\nabla_\xi X \in \mathcal{D}$ for any $X \in \mathcal{D}$.

The tensor fields $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ and $h^{'} = h \cdot \varphi$ are symmetric operators anticommuting with φ and $h\xi = 0 = h^{'}\xi$, and we note that $\nabla_\xi h^{'} = 0$ if and only if $\nabla_\xi h = 0$. Let $Y = \xi$ in (2.2) we obtain

$$\nabla_X \xi = \alpha \varphi^2 X + \varphi h X \tag{2.3}$$

Proposition 2.1. *An almost α-para-Kenmotsu manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ has para-Kähler leaves if and only if*

$$(\nabla_X \varphi)Y = g(\alpha\varphi X + hX, Y) - \eta(Y)(\alpha\varphi X + hX).$$

Theorem 2.1. *([9]) Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold with para-Kähler leaves. Then M^{2n+1} is para-Kenmotsu $(\alpha = 1)$ if and only if $\nabla_X \xi = \varphi^2 X$.*

Proposition 2.2. *([9]) Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold. Then, for any $X, Y, Z \in \Gamma(TM^{2n+1})$,*

$$R(\xi, X)\xi = \alpha^2 \varphi^2 X + 2\alpha\varphi h X - h^2 X + \varphi(\nabla_\xi h)X, \tag{2.4}$$

$$\frac{1}{2}(R(\xi, X)\xi + \varphi R(\xi, \varphi X)\xi) = \alpha^2 \varphi^2 X - h^2 X, \tag{2.5}$$

$$\begin{aligned} R(X, Y)\xi =& \alpha\eta(X)(\alpha Y + \varphi h Y) \\ & - \alpha\eta(Y)(\alpha X + \varphi h X) + (\nabla_X \varphi h)Y - (\nabla_Y \varphi h)X, \end{aligned} \tag{2.6}$$

$$\begin{aligned} & g(R(\xi, X)Y, Z) + g(R(\xi, X)\varphi Y, \varphi Z) - g(R(\xi, \varphi X)\varphi Y, Z) \\ & - g(R(\xi, \varphi X)Y, \varphi Z) = 2(\nabla_{hX}\Phi)(Y, Z) + 2\alpha^2\eta(Y)g(X, Z) \\ & - 2\alpha^2\eta(Z)g(X, Y) - 2\alpha\eta(Z)g(\varphi h X, Y) + 2\alpha\eta(Y)g(\varphi h X, Z). \end{aligned} \tag{2.7}$$

Proposition 2.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold. Then, for any $X, Y \in \Gamma(TM^{2n+1})$,*

$$g(\mathcal{N}(\varphi X, Y), \xi) = 0. \tag{2.8}$$

Proof. By direct computations one has

$$g(\mathcal{N}(\varphi X,Y),\xi)=g([\varphi X,\varphi Y],\xi)=g((\nabla_X\varphi)Y-(\nabla_{\varphi Y}\varphi)\varphi X,\xi),$$

which implies (2.8) by using (2.2) and $[\xi,X]=-2\varphi hX$.

Theorem 2.2. *([9]) Let M^{2n+1} be an almost α-para-Kenmotsu manifold with $h=0$. Then, M^{2n+1} is locally a warped product $M_1\times_{f^2}M_2$, where M_2 is an almost para-Kähler manifold, M_1 ia an open interval with coordinate t and $f^2=we^{2\alpha t}$ for some positive constant.*

3. $\mathcal{CR}$-integrability

For an almost α-para-Kenmotsu manifold we have $[X,Y]-[\varphi X,\varphi Y]\in\mathcal{D}$ for any $X,Y\in\mathcal{D}$, since $d\eta=0$ and thus $\mathcal{D}$ is integrable. Hence, the structure (φ,ξ,η,g) is $\mathcal{CR}$-integrable if and only if $\mathcal{N}(X,Y)=[X,Y]+[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]=0$ on $\mathcal{D}$, that is to the request that the integral manifolds of $\mathcal{D}$ are para-Kähler.

Theorem 3.1. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost paracontact metric manifold. Then, M^{2n+1} is a $\mathcal{CR}$-integrable almost α-para-Kenmotsu manifold if and only if there exists a linear connection $\tilde\nabla$ such that*

1) $\tilde\nabla\varphi=0$, $\tilde\nabla g=0$, $\tilde\nabla\eta=0$.

2) the torsion $\tilde T$ satisfies

a) $\tilde T(X,Y)=0$ for any $X,Y\in D$,

b) $\tilde T(\xi,X)=X+h^{'}X$ for any $X\in D$,

c) $\tilde T_\xi$ is selfadjoint.

Moreover, such a connection is uniquely determined by

$$\tilde\nabla_XY=\nabla_XY+g(\alpha X-h^{'}X,Y)\xi-\eta(Y)(\alpha X-h^{'}X), \tag{3.1}$$

∇ being the Levi-Civita connection.

Proof. Let M^{2n+1} is a $\mathcal{CR}$-integrable almost α-para-Kenmotsu manifold. We put $\tilde\nabla=\nabla+H$, where the tensor field H of type (1,2) is defined by

$$H(X,Y)=g(\alpha X-h^{'}X,Y)\xi-\eta(Y)(\alpha X-h^{'}X).$$

Since $H(X,\varphi Y)-\varphi(H(X,Y))=-(g(\alpha\varphi X+hX,Y)-\eta(Y)(\alpha\varphi X+hX))=-(\nabla_X\varphi)Y$, owing to Proposition 2.1. By direct calculations, we get $g(H(X,Y),Z)+g(H(X,Z),Y)=0$ and $(\nabla_X\eta)Y-\eta(H(X,Y))=0$, moreover, we get $\tilde\nabla\varphi=0$, $\tilde\nabla g=0$, $\tilde\nabla\eta=0$. Since $\tilde T(X,Y)=\eta(X)(\alpha Y-h^{'}Y)-\eta(Y)(\alpha X-h^{'}X)=0$ for any $X,Y\in\mathcal{D}$, and $\tilde T(\xi,X)=\alpha X-h^{'}X$ for any $X\in\mathcal{D}$, hence $\tilde T_\xi$ is selfadjoint. As for the uniqueness and the vice versa part, the proof is similar with Theorem 3.1 in [8]. □

Corollary 3.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a $\mathcal{CR}$-integrable almost α-para-Kenmotsu manifold. Then M^{2n+1} is a α-para-Kenmotsu manifold if and only if the linear connection $\tilde{\nabla}$ verifies $\tilde{T}_\xi \circ \varphi = \varphi \circ \tilde{T}_\xi$.*

Proof. Since $\tilde{T}_\xi \varphi X - \varphi \tilde{T}_\xi X = \tilde{T}(\xi, \varphi X) - \varphi \tilde{T}(\xi, X) = -2hX$ for any $X \in \mathcal{D}$, hence, Corollary 3.1 is easily followed by Theorem 3.1. □

4. Almost α-para-Kenmotsu manifolds and local symmetrys

In this section, we investigate almost α-para-Kenmotsu manifolds which are locally symmetric, that is, almost α-para-Kenmotsu manifolds satisfying the condition $\nabla R = 0$, which is a natural generalization of almost α-para-Kenmotsu manifold of constant curvature.

By similar proof as that of proposition 6 in [7], we get the following lemma

Lemma 4.1. *Let M^{2n+1} be a locally symmetric almost α-para-Kenmotsu manifold. Then, $\nabla_\xi h = 0$.*

Theorem 4.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost α-para-Kenmotsu manifold. Then, M^{2n+1} is an α-para-Kenmotsu manifold if and only if $h = 0$. Moreover, if any of the above equivalent conditions holds, M^{2n+1} has constant sectional curvature $c = -\alpha^2$.*

Proof. First, assuming that M^{2n+1} is an α-para-Kenmotsu manifold, by Theorem 2.1. we have $\nabla_X \xi = \alpha \varphi^2 X$, comparing with (2.3) it follows that $h = 0$ and by (2.6), we easily obtain $R(X,Y)\xi = -\alpha^2(\eta(Y)X - \eta(X)Y)$, let ∇_Z acting on the above equation and by the local symmetry, we have $R(X,Y)Z = -\alpha^2(g(Y,Z)X - g(X,Z)Y)$, it follows then M is of constant sectional curvature $c = -\alpha^2$. Now, supposing M' is the integral manifold of $\mathcal{D}$ and ∇' is the corresponding connection on M'. Then $\nabla_X Y = \nabla'_X Y + h(X,Y)$, then $h(X,Y) = g(\nabla_X Y, \xi)\xi = -\alpha g(X,Y)\xi$, this implies $H = -\alpha\xi$ thus $h(X,Y) = g(X,Y)H$, and M' is a totally umbilical submanifold of M^{2n+1}. What is more, it is not difficult to see that $R'(X,Y) = R(X,Y) + \alpha^2(g(Y,Z)X - g(X,Z)Y) = 0$, we know that M' is flat and the sectional curvature of M' vanishes. This means that M' is a flat para-Kähler manifold. For another part of the proof, noticing that $\nabla_Z \xi = \alpha \varphi^2 Z = \alpha Z$ if and only if $h = 0$, by Theorem 2.1 we prove that M^{2n+1} is an α-para-Kenmotsu manifold. At last, it is obvious from the proof of the equivalence that if any of the above equivalent conditions holds, M^{2n+1} has constant sectional curvature $c = -\alpha^2$. Thus, we complete the proof. □

Theorem 4.2. *An almost α-para-Kenmotsu manifold of constant curvature c is an α-para-Kenmotsu manifold and $c = -\alpha^2$.*

Proof. Supposing M^{2n+1} is an almost α-para-Kenmotsu manifold of constant sectional curvature c, it is obvious that

$$R(X,Y)Z = c(\eta(Y)X - \eta(X)Y). \tag{4.1}$$

∇_W acting on (4.1) we get $\nabla_W R = 0$, thus, M^{2n+1} is locally symmetric, by Lemma 4.1, we get $\nabla_\xi h = 0$. Comparing (2.6) with (4.1), we obtain

$$(c+\alpha^2)(\eta(Y)X-\eta(X)Y)+\alpha(\eta(Y)\varphi hX-\eta(X)\varphi hY)-(\nabla_X\varphi h)Y+(\nabla_Y\varphi h)X=0.$$

Choosing $X=\xi$ and $Y\in\mathcal{D}$ and by Lemma 4.1, we get

$$-(c+\alpha^2)Y-2\alpha\varphi hY+h^2X=0. \tag{4.2}$$

Now, if Y is an eigenvector of h with eigenvalue λ, then (4.2) becomes $-(c+\alpha^2)Y-2\alpha\lambda\varphi Y+\lambda^2X=0$. We get $\lambda=0$ and $c=-\alpha^2$ since Y and φY are linearly independent. Hence $h=0$ and $c=-\alpha^2$, by Theorem 4.1, we know M^{2n+1} is an α-para-Kenmotsu manifold of constant curvature $c=-\alpha^2$. Thus, we complete the proof. □

5. Almost α-para-Kenmotsu manifolds and nullity distributions

In this section, we study almost α-para-Kenmotsu manifolds under the assumption that ξ belongs to the (κ,μ)-nullity distribution and $(\kappa,\mu)'$-nullity distribution.

First, we consider the (κ,μ)-nullity distribution. if ξ belongs to the (κ,μ)-nullity distribution, $(\kappa,\mu)\in R^2$, denoted by $\mathcal{N}(\kappa,\mu)$,which is given by putting for each $p\in M^{2n+1}$,

$$\begin{aligned}\mathcal{N}_p(\kappa,\mu)=\{&Z\in\Gamma(T_pM^{2n+1})|R(X,Y)Z\\&=\kappa(g(Y,Z)X-g(X,Z)Y)+\mu(g(Y,Z)hX-g(X,Z)hY)\}.\end{aligned}$$

So, if $\xi\in\mathcal{N}(\kappa,\mu)$, that is, for any X,Y$\in\Gamma(TM^{2n+1})$

$$R(X,Y)\xi=\kappa(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY).$$

Proposition 5.1. *([9]) Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost α-para-Kenmotsu (κ,μ)-space. Then the following identities hold:*

$$h^2X=(\kappa+\alpha^2)\varphi^2X, \tag{5.1}$$

$$R(\xi,X)Y=\kappa(g(X,Y)\xi-\eta(Y)X)+\mu(g(X,hY)\xi-\eta(Y)hX), \tag{5.2}$$

$$Q\xi=-2nk\xi, \tag{5.3}$$

$$(\nabla_X\varphi)Y=g(\alpha\varphi X+hX,Y)-\eta(Y)(\alpha\varphi X+hX). \tag{5.4}$$

Theorem 5.1. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost α-para-Kenmotsu manifold. Let us suppose that $\xi\in\mathcal{N}(\kappa,\mu)$. Then, $\kappa=-1,h=0$ and M^{2n+1} is locally a warped product of an almost paraKähler manifold and an open interval. Moreover, assuming the local symmetry, M^{2n+1} is locally isometric to the hyperbolic space $H^{2n+1}(-\alpha^2)$ of constant curvature $-\alpha^2$.*

Proof. $\xi \in \mathcal{N}(\kappa, \mu)$ means that $R(X,\xi)\xi = \kappa X + \mu hX$, for any unit vector field X orthogonal to ξ. Combining with (2.5), it follows that $h^2 X = (\alpha^2 + \kappa)X$. Now, if X is a unit eigenvector of h with eigenvalue λ, we get $\lambda^2 = \alpha^2 + \kappa \geq 0$. It follows that $\kappa \geq -\alpha^2$ and $Spec(h) = \{0, \lambda, -\lambda\}$. Computing $R(X,\xi)\xi$ by means of (2.6), we easily obtain

$$R(X,\xi)\xi = -\alpha^2 X - 2\alpha\lambda\varphi X + \lambda^2 X - \lambda\varphi\nabla_\xi X + \varphi h \nabla_\xi X,$$

thus we have

$$(\kappa + \lambda\mu + \alpha^2 - \lambda^2)X + 2\alpha\lambda\varphi X + \lambda\varphi\nabla_\xi X - \varphi h\nabla_\xi X = 0,$$

and taking the scalar product with φX, we obtain $\alpha\lambda = 0$. Since $\alpha = const. \neq 0$, it follows that $\lambda = 0, h = 0, \kappa = -\alpha^2$ and thus $K(X,\xi) = -\alpha^2$.

Being $h = 0$, Theorem 2.2 ensures that M^{2n+1} is locally a warped product of an almost para-Kähler manifold and an open interval. Furthermore, if M^{2n+1} is locally symmetric, by Theorem 4.1, it is an α-para-Kenmotsu manifold locally isometric to $H^{2n+1}(-\alpha^2)$. Thus, we complete the proof. □

From Theorem 5.1 we know for almost α-para-Kenmotsu manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, if $\xi \in \mathcal{N}(\kappa, \mu)$, then $\kappa = -1, h = 0$ and M^{2n+1} is locally a warped product of an almost para-Kähler manifold and an open interval. Therefore, we consider the $(\kappa,\mu)'$-nullity distribution, $(\kappa,\mu)' \in R^2$, as the distribution $\mathcal{N}(\kappa,\mu)'$ is given by putting for each $p \in M^{2n+1}$,

$$\begin{aligned} \mathcal{N}_p(\kappa,\mu)' &= \{Z \in \Gamma(T_pM^{2n+1}) | R(X,Y)Z \\ &= \kappa(g(Y,Z)X - g(X,Z)Y) + \mu(g(Y,Z)h'X - g(X,Z)h'Y)\}. \end{aligned} \tag{5.5}$$

So, if $\xi \in \mathcal{N}(\kappa,\mu)'$, that is, for any X,Y$\in \Gamma(TM^{2n+1})$

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y). \tag{5.6}$$

Theorem 5.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa,\mu)'$ and $h' \neq 0$. Then, $\kappa < -\alpha^2, \mu = 2\alpha$ and $Spec(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigenvalue and $\lambda = \sqrt{-(\alpha^2+\kappa)}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves.*

Proof. Let X be a unit vector field orthogonal to ξ, we have $R(X,\xi)\xi = kX + \mu h'X$ and if we suppose $X \in [\lambda]'$, since $h'^2 = -h^2$, combing with (2.5), we get $\lambda^2 = -(\kappa + \alpha^2) \geq 0$, then $\kappa \leq -\alpha^2$. $Spec(h') = \{0, \lambda, -\lambda\}$. Using (2.6) to compute $R(X,\xi)\xi$, we have

$$(\kappa + \lambda\mu + \alpha^2 - 2\alpha\lambda + \lambda^2)X - \lambda\nabla_\xi X + h'\nabla_\xi X = 0. \tag{5.7}$$

let (5.7) take the scalar product with X and φX respectively, we get $\lambda(\mu - 2\alpha) = 0$ and $\lambda g(\nabla_\xi X, \varphi X) = 0$. If $\lambda = 0$, then $h' = 0$ or equivalently $h = 0$, $N(\kappa,\mu) =$

$N(\kappa,\mu)'$ and Theorem 5.1 applies. Therefore, assuming $\lambda \neq 0$, it follows that $\kappa < -\alpha^2$ and $\mu = 2\alpha$, $g(\nabla_\xi X, \varphi X) = 0$ for any unit $X \in [\lambda]'$. Let (5.7) take the scalar product with any $Y \in [-\lambda]'$, we get $g(\nabla_\xi X, Y) = 0$ and thus $\nabla_\xi X \in [\lambda]'$. Analogously $\nabla_\xi Y \in [-\lambda]'$ and we obtain $\nabla_\xi h' = 0$. Comparing (5.6) with (2.6) for any $X, Y \in \mathcal{D}$, we have

$$(\nabla_X h')Y - (\nabla_Y h')X = 0. \tag{5.8}$$

If $X \in [\lambda]'$, by (2.3) we have $\nabla_X \xi = \alpha X - h'X = (\alpha - \lambda)X \in [\lambda]'$, and since $\nabla_\xi h' = 0$, we easily get $\nabla_\xi X \in [\lambda]'$. By (5.8) we have

$$0 = (\nabla_X h')Z - (\nabla_Z h')X = -\lambda\nabla_X Z - h'\nabla_X Z - \lambda\nabla_Z X + h'\nabla_Z X. \tag{5.9}$$

let (5.9) take the scalar product with $Y \in [-\lambda]'$, we get $g(\nabla_Z X, Y) = 0$, therefore $\nabla_Z X \in [\lambda]'$ since $g(\nabla_X Z, \xi) = 0$. For any $X, W \in [\lambda]'$, $Y, Z \in [-\lambda]'$ it follows that $\nabla_X W \in [\xi] \oplus [\lambda]'$ since $g(\nabla_X W, \xi) = (\lambda - \alpha)g(X, W)$. Hence, we get $g([X, W], \xi) = g(\nabla_X W - \nabla_W X, \xi) = 0$ and $g([X, W], Y) = g(\nabla_X W - \nabla_W X, Y) = 0$, thus $[X, W] \in [\lambda]'$. Similarly, it holds $[Y, Z] \in [-\lambda]'$. Therefore, the distributions $[\xi] \oplus [\lambda]'$,$[\xi] \oplus [-\lambda]'$, $[\lambda]'$ and $[-\lambda]'$ are integrable. It is easy to see that the distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are totally geodesic leaves. Now we prove the distribution $[\lambda]'$ is totally umbilical, we choose a local orthonormal frame $\{\xi, e_i, \varphi e_i\}$, with $e_i \in [\lambda]'$. The second fundamental form $h(e_i, e_j) = g(\nabla_{e_i} e_j, \xi)\xi = (\lambda - \alpha)\delta_{ij}\xi$, so the mean curvature vector field is $H = (\lambda - \alpha)\xi$, hence $h(X, W) = g(X, W)H$ and thus $[\lambda]'$ is totally umbilical. Similarly, we can get $[-\lambda]'$ is also totally umbilical with the mean curvature vector field is $H' = (\lambda + \alpha)\xi$ and $h'(Y, Z) = g(Y, Z)H'$. Thus, we complete the proof. □

Theorem 5.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa,\mu)'$ and $h' \neq 0$. Then, the integral manifolds of $\mathcal{D}$ are para-Kähler manifolds.*

Proof. For any $X, Y, Z \in \mathcal{D}$, if $\xi \in \mathcal{N}(\kappa,\mu)'$, then $R(X, Y)\xi = 0$, (2.7) in Proposition 2.2 gives that $(\nabla_{hX}\Phi)(Y, Z) = 0$. Replacing X by hX, we get $(\nabla_{h^2X}\Phi)(Y, Z) = 0$ or equivalently, $-\lambda^2(\nabla_X\Phi)(Y, Z) = 0$ since $h^2X = -h'^2X = -\lambda^2X$ if X is a unit eigenvector of h' with eigenvalue λ. Being $\lambda \neq 0$, we get $(\nabla_X\Phi)(Y, Z) = 0$. Using (2.2) we obtain $g(N(Y, Z), \varphi X) = 0$, which together with (2.8) in Proposition 2.3 gives $\mathcal{N}(Y, Z) = 0$ for any $Y, Z \in \mathcal{D}$, therefore the integral manifolds of $\mathcal{D}$ are para-Kähler. Thus, we complete the proof. □

Corollary 5.1. *Any almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa,\mu)'$, $\kappa < -\alpha^2$, is a $\mathcal{CR}$-manifold.*

Theorem 5.4. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa,\mu)'$ and $h' \neq 0$. Then, M^{2n+1} is locally isometric to $H^{n+1}(-(\lambda - \alpha)^2) \times R^n$.*

Proof. As proved in Theorem 5.2, the distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves and the distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. It follows that M^{2n+1} is locally isometric to the product of an integral manifold M_1^{n+1} of $[\xi] \oplus [\lambda]'$ and an integral manifold M_2^n of $[-\lambda]'$. Therefore, we can choose coordinates $(u^0, \ldots, u^{2n})$ such that $\frac{\partial}{\partial u^0} \in [\xi]$, $\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n} \in [\lambda]'$ and $\frac{\partial}{\partial u^{n+1}}, \ldots, \frac{\partial}{\partial u^{2n}} \in [-\lambda]'$. Now, we set $X_i = \frac{\partial}{\partial u^i}$ for any $i \in \{1, \ldots, n\}$, so that the distribution $[-\lambda]'$ is spanned by the vector fields $\varphi X_1, \ldots, \varphi X_n$. it is easy to see that $X_i \in [\lambda]'$ is projectable and $\varphi X_i \in [-\lambda]'$ is vertical, then $[X_i, \varphi X_j]$ is vertical by [1], hence $[X_i, \varphi X_j] \in [-\lambda]'$. Taking the scalar product with any $Z \in [\lambda]'$, since $\nabla_{X_i}\varphi X_j \in [-\lambda]'$, we get $g(\nabla_{\varphi X_j} X_i, Z) = 0$ and then $\nabla_{\varphi X_j} X_i = 0$. Applying $(\nabla_{\varphi X}\varphi)\varphi Y - (\nabla_X \varphi)Y = \alpha(\eta(Y)\varphi X + 2g(X, \varphi Y)\xi) + \eta(Y)hX$ (appeared in [9]), we have $(\nabla_{X_i}\varphi)X_j + \varphi(\nabla_{\varphi X_i}\varphi X_j) = 0$, which implies $(\nabla_{X_i}\varphi)X_j = 0$, $\nabla_{\varphi X_i}\varphi X_j = 0$, since the two part belong to $[-\lambda]'$ and $[\lambda]'$ respectively. $\nabla_{\varphi X_i}\varphi X_j = 0$ means that M_2^n of $[-\lambda]'$ is flat. Now we compute the curvature of M_1^{n+1}. Applying φ to $(\nabla_{X_i}\varphi)X_j = 0$ gives

$$\nabla_{X_i} X_j - \varphi \nabla_{X_i} \varphi X_j = (\lambda - \alpha) g(X_i, X_j)\xi.$$

Derivating with respect to X_k yields:

$$\begin{aligned} & \nabla_{X_k}\nabla_{X_i} X_j - (\nabla_{X_k}\varphi)(\nabla_{X_i}\varphi X_j) - \varphi \nabla_{X_k}\nabla_{X_i}\varphi X_j \\ = \; & (\lambda - \alpha) X_k(g(X_i, X_j))\xi - (\lambda - \alpha)^2 g(X_i, X_j) X_k. \end{aligned}$$

taking the scalar product with X_l on both sides of the above equality and taking into account $g((\nabla_{X_k}\varphi)(\nabla_{X_i}\varphi X_j), X_l) = -g(\nabla_{X_i}\varphi X_j, (\nabla_{X_k}\varphi)X_l) = 0$, we obtain

$$g(\nabla_{X_k}\nabla_{X_i} X_j, X_l) + g(\nabla_{X_k}\nabla_{X_i}\varphi X_j, \varphi X_l) = -(\lambda - \alpha)^2 g(X_i, X_j) g(X_k, X_l).$$

Interchanging i and k, subtracting and being $[X_i, X_k] = 0$ we have

$$\begin{aligned} & g(R(X_k, X_i)X_j, X_l) + g(R(X_k, X_i)\varphi X_j, \varphi X_l) \\ = \; & -(\lambda - \alpha)^2 g(X_i, X_j) g(X_k, X_l) + (\lambda - \alpha)^2 g(X_k, X_j) g(X_i, X_l). \end{aligned}$$

Since $\nabla_{\varphi X_i}\varphi X_j = 0$ and $[\varphi X_i, \varphi X_j] = 0$, by a straightforward calculation we obtain

$$g(R(X_k, X_i)\varphi X_j, \varphi X_l) = g(R(\varphi X_j, \varphi X_l)X_k, X_i) = 0,$$

and thus

$$g(R(X_k, X_i)X_j, X_l) = -(\lambda - \alpha)^2 [g(X_i, X_j) g(X_k, X_l) - g(X_k, X_j) g(X_i, X_l)].$$

Moreover, since $R(X, Y)\xi = 0$ for any $X, Y \in \mathcal{D}$, we get $g(R(X_i, X_j)\xi, X_k) = 0$. By (2.4) in Proposition 2.2, and $\nabla_\xi h = 0$ because of the symmetry, we get $g(R(X_i, \xi)\xi, X_j) = -(\lambda - \alpha)^2 g(X_i, X_j)$. Therefore, we conclude that the integral manifold M_1^{n+1} of $[\xi] \oplus [\lambda]'$ is a space of constant curvature $-(\lambda - \alpha)^2$. Thus, we complete the proof. □

Lemma 5.1. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost α-para-Kenmotsu manifold such that $\xi\in N(\kappa,2\alpha)'$ and $h'\neq 0$. Then, for any $X,Y\in\Gamma(TM^{2n+1})$,*

$$(\nabla_X h')Y = g(h'^2X-\alpha h'X,Y)\xi+\eta(Y)(h'^2X-\alpha h'X). \tag{5.10}$$

Proof. We choose a local orthonormal frame $\{\xi,e_i,\varphi e_i\}$ with $e_i\in[\lambda]'$.

1) If $X,Y\in[\lambda]'$, we know that $\nabla_XY\in[\xi]\oplus[\lambda]'$ from Theorem 5.2. It is easy to get

$$\nabla_XY=g(\nabla_XY,e_i)e_i+g(\nabla_XY,\xi)\xi=(\lambda-\alpha)g(X,Y)\xi+g(\nabla_XY,e_i)e_i,$$

and thus

$$(\nabla_Xh')Y=\nabla_Xh'Y-h'\nabla_XY=\lambda\nabla_XY-\lambda g(\nabla_XY,e_i)e_i=\lambda(\lambda-\alpha)g(X,Y)\xi.$$

2) If $X,Y\in[-\lambda]'$, we know that $\nabla_XY\in[\xi]\oplus[-\lambda]'$ from Theorem 5.2. Similarly we have

$$\nabla_XY=g(\nabla_XY,\varphi e_i)\varphi e_i+g(\nabla_XY,\xi)\xi=-(\lambda+\alpha)g(X,Y)\xi+g(\nabla_XY,\varphi e_i)\varphi e_i,$$

and

$$(\nabla_Xh')Y=\lambda(\lambda+\alpha)g(X,Y)\xi.$$

3) If $X\in[\lambda]'$, $Y\in[-\lambda]'$, since $g(\nabla_XY,\xi)=(\lambda-\alpha)g(X,Y)=0$, and for any $Z\in[\lambda]'$, $g(\nabla_XY,Z)=Xg(Y,Z)-g(Y,\nabla_XZ)=0$, thus we get $\nabla_XY\in[-\lambda]'$ and $(\nabla_Xh')Y=\nabla_Xh'Y-h'\nabla_XY=0$, therefore we have $(\nabla_Yh')X=0$ since $(\nabla_Xh')Y-(\nabla_Yh')X=0$.

Therefore, for any $X\in\Gamma(TM^{2n+1})$, we write $X=X_\lambda+X_{-\lambda}+\eta(X)\xi$, with $X_\lambda\in[\lambda]'$ and $X_{-\lambda}\in[-\lambda]'$, since $\nabla_\xi h'=0$, we get

$$\begin{aligned}(\nabla_Xh')Y &= (\nabla_{X_\lambda}h')Y_\lambda+\eta(Y)(\nabla_{X_\lambda}h')\xi+(\nabla_{X_{-\lambda}}h')Y_{-\lambda}+\eta(Y)(\nabla_{X_{-\lambda}}h')\xi\\ &= \lambda(\lambda-\alpha)g(X_\lambda,Y_\lambda)\xi+\lambda(\lambda-\alpha)\eta(Y)X_\lambda+\lambda(\lambda+\alpha)g(X_{-\lambda},Y_{-\lambda})\xi\\ &\quad+\lambda(\lambda+\alpha)\eta(Y)X_{-\lambda}\\ &= -\alpha\lambda\{g(X_\lambda,Y_\lambda)-g(X_{-\lambda},Y_{-\lambda})\}\xi+\lambda^2\{g(X_\lambda,Y_\lambda)+g(X_{-\lambda},Y_{-\lambda})\}\xi\\ &\quad+\eta(Y)(-\alpha\lambda X_\lambda+\alpha\lambda X_{-\lambda}+\lambda^2X_\lambda-\lambda^2X_{-\lambda})\\ &= g(h'^2X-\alpha h'X,Y)\xi+\eta(Y)(h'^2X-\alpha h'X).\end{aligned}$$

Lemma 5.2. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost α-para-Kenmotsu manifold such that $\xi\in\mathcal{N}(\kappa,2\alpha)'$ and $h'\neq 0$. Then, for any $X,Y\in\mathcal{D}$,*

$$R(X,Y)h'Z-h'R(X,Y)Z=(\kappa+2\alpha^2)[g(Y,Z)h'X-g(X,Z)h'Y-g(h'Y,Z)X+g(h'X,Z)Y].$$

Proof. We know from Lemma 5.1 that $(\nabla_X h')Y = g(h'^2X - \alpha h'X, Y)\xi$ for any $X, Y \in \mathcal{D}$, by direct calculation we obtain

$$\begin{aligned}
&R(X,Y)h'Z - h'R(X,Y)Z \\
&= \nabla_X\nabla_Y h'Z - \nabla_Y\nabla_X h'Z - \nabla_{[X,Y]}h'Z - h'R(X,Y)Z \\
&= g((\nabla_X h'^2)Y - (\nabla_Y h'^2)X - \alpha((\nabla_X h')Y - (\nabla_Y h')X), Z) + g(h'^2Y - \alpha h'Y, Z)\nabla_X\xi \\
&- g(h'^2X - \alpha h'X, Z)\nabla_Y\xi - g(\nabla_Y\xi, Z)(h'^2X - \alpha h'X) + g(\nabla_X\xi, Z)(h'^2Y - \alpha h'Y).
\end{aligned}$$

It follows that for any $X, Y \in \mathcal{D}$, we know from $h'^2X = -h^2X = -(\kappa + \alpha^2)X$, and $(\nabla_X h'^2)Y = -(\kappa + \alpha^2)\eta(\nabla_X Y)\xi$, hence, $(\nabla_X h'^2)Y - (\nabla_Y h'^2)X = 0$ since $\mathcal{D}$ is integrable, and from Lemma 5.1, we get $(\nabla_X h')Y - (\nabla_Y h')X = 0$. Lemma 5.2 is followed by direct computation. Thus, we complete the proof. □

Lemma 5.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa, 2\alpha)'$ and $h' \neq 0$. Then, for any $X, Y, Z \in \mathcal{D}$, we have*

$$\begin{aligned}
& R(X,Y)\varphi Z - \varphi R(X,Y)Z \\
= \; & g(\alpha X - h'X, \varphi Z)(\alpha Y - h'Y) - g(\alpha X - h'X, Z)(\alpha\varphi Y - \varphi h'Y) \\
& + g(\alpha Y - h'Y, Z)(\alpha\varphi X - \varphi h'X) - g(\alpha Y - h'Y, \varphi Z)(\alpha X - h'X).
\end{aligned}$$

Proof. Since the Weingarten operator for an integral manifold M' of $\mathcal{D}$ is given by

$$AX = -\nabla_X\xi = -(\alpha X - h'X),$$

by Theorem 2.3 in [4] we get the Guass equation

$$R(X,Y)Z = R'(X,Y)Z + g(\alpha X - h'X, Z)(\alpha Y - h'Y) - g(\alpha Y - h'Y, Z)(\alpha X - h'X).$$

By Theorem 5.3, the integral manifolds of $\mathcal{D}$ are para-Kähler manifolds, and from Lemma 10.1 of [4], we know $R'(X,Y)\varphi Z - \varphi R'(X,Y)Z = 0$. Combining with the above two equations, we get the required formula for R and φ. Thus, we complete the proof. □

Proposition 5.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa, 2\alpha)'$ and $h' \neq 0$. Then, for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the curvature tensor R satisfies:*

$$\begin{aligned}
R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0, \\
R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0, \\
R(X_\lambda, Y_{-\lambda})Z_\lambda &= (\kappa + 2\alpha^2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \\
R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(\kappa + 2\alpha^2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\
R(X_\lambda, Y_\lambda)Z_\lambda &= (\kappa + 2\alpha\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (\kappa - 2\alpha\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{aligned}$$

Proof. For any $X \in [\lambda]'$, and $Y, Z \in [-\lambda]'$, by Lemma 5.2 we have

$$-\lambda R(X,Y)Z - h'R(X,Y)Z = 2\lambda(\kappa + 2\alpha^2)g(Y,Z)X.$$

Taking the scalar product with $W \in [\lambda]'$, we obtain

$$g(R(X,Y)Z,W) = -(\kappa + 2\alpha^2)g(Y,Z)g(X,W). \tag{5.11}$$

Lemma 5.2 implies that $R(X,Y)Z \in [\lambda]'$ for any $X, Y, Z \in [\lambda]'$ and $R(X,Y)Z \in [-\lambda]'$ for any $X, Y, Z \in [-\lambda]'$. Now, in order to compute $R(X_\lambda, Y_\lambda)Z_{-\lambda}$, we consider a local orthonormal frame $\{\xi, e_i, \varphi e_i\}$, with $e_i \in [\lambda]'$. Condition $\xi \in N(\kappa, 2\alpha)'$ means that $g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, \xi) = g(R(X_\lambda, Y_\lambda)\xi, Z_{-\lambda}) = 0$, and since $R(X_\lambda, Y_\lambda)e_i \in [\lambda]'$, thus $g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, e_i) = 0$. Using the first Bianchi identity and (5.11), we have

$$\begin{aligned} g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, \varphi e_i) &= g(R(Y_\lambda, Z_{-\lambda})\varphi e_i, X_\lambda) - g(R(X_\lambda, Z{-}\lambda)\varphi e_i, Y_\lambda) \\ &= -(\kappa + 2\alpha^2)[g(Z_{-\lambda}, \varphi e_i)g(X_\lambda, Y_\lambda) - g(Z_{-\lambda}, \varphi e_i)g(X_\lambda, Y_\lambda)] \\ &= 0, \end{aligned}$$

so that $R(X_\lambda, Y_\lambda)Z_{-\lambda}{=}0$. The terms $R(X_{-\lambda}, Y_{-\lambda})Z_\lambda$, $R(X_\lambda, Y_{-\lambda})Z_\lambda$ and $R(X_\lambda, Y_{-\lambda})Z_{-\lambda}$ are computed in a similar manner. By Lemma 5.3, using $R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0$, we get

$$R(X_\lambda, Y_\lambda)\varphi Z_\lambda = -(\alpha - \lambda)^2[g(Y_\lambda, \varphi Z_{-\lambda})X_\lambda - g(X_\lambda, \varphi Z_{-\lambda})Y_\lambda]$$

Replacing $Z_{-\lambda}$ by $\varphi Z_\lambda \in [\lambda]'$, and since $-(\alpha - \lambda)^2 = \kappa + 2\alpha\lambda$, we have

$$R(X_\lambda, Y_\lambda)Z_\lambda = R(X_\lambda, Y_\lambda)\varphi(\varphi Z_\lambda) = (\kappa + 2\alpha\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda].$$

In the same manner, we obtain $R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (\kappa - 2\alpha\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}]$. Thus, we complete the proof. □

Proposition 5.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa, 2\alpha)'$ and $h' \neq 0$. Then,we have*

1) $K(X, \xi) = \kappa + 2\alpha\lambda$, if $X \in [\lambda]'$;
$K(X, \xi) = \kappa - 2\alpha\lambda$, if $X \in [-\lambda]'$;
2) $K(X, Y) = \kappa + 2\alpha\lambda$, if $X, Y \in [\lambda]'$;
$K(X, Y) = \kappa - 2\alpha\lambda$, if $X, Y \in [-\lambda]'$;
$K(X, Y) = -(\kappa + 2\alpha^2)$, if$X \in [\lambda]', Y \in [-\lambda]'$.
3) $r = 8\alpha\lambda n - 4\alpha^2 n^2 - 2kn$.

Proof. The proof for the sectional curvature is easily followed by Proposition 5.2. In order to compute the scalar curvature, we choose a orthonormal frame $\{\xi, e_i, \varphi e_i\}$ with $e_i \in [\lambda]'$, by direct calculations we have

$$Ric(\xi, \xi) = \sum_{i=1}^{n} R(\xi, e_i, e_i, \xi) - \sum_{i=1}^{n} R(\xi, \varphi e_i, \varphi e_i, \xi) = 4\alpha\lambda n,$$

$$\begin{aligned} Ric(e_i,e_i) &= \sum_{i=1}^{n} R(e_i,\xi,\xi,e_i) + \sum_{j\neq i=1}^{n} R(e_i,e_j,e_j,e_i) - \sum_{j=1}^{n} R(e_i,\varphi e_j,\varphi e_j,e_i) \\ &= n(\kappa+2\alpha\lambda) + n(\kappa+2\alpha^2), \end{aligned}$$

$$Ric(\varphi e_i,\varphi e_i) = (\kappa-2\alpha\lambda)(2-n) + n(\kappa+2\alpha^2),$$

and it is easy to get the scalar curvature $r = 8\alpha\lambda n - 4\alpha^2 n^2 - 2kn$. $\square$

Proposition 5.4. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa,2\alpha)'$ and $h' \neq 0$. Then, M^{2n+1} is locally isometric to the warped products*

$$S^{n+1}(\kappa+2\alpha\lambda) \times_f R^n, \quad \textit{or} \quad B^{n+1}(\kappa-2\alpha\lambda) \times_{f'} R^n,$$

where $S^{n+1}(\kappa+2\alpha\lambda)$ is a space of constant positive curvature $\kappa+2\alpha\lambda$, $B^{n+1}(\kappa-2\alpha\lambda)$ is a space of constant negative curvature $\kappa-2\alpha\lambda$, $f = ce^{-(\lambda+\alpha)t}$, $f' = c'e^{(\alpha-\lambda)t}$, with c, c' positive constants.

Proof. By Theorem 5.2, we get that the distributions $[\xi]\oplus[\lambda]'$ and $[\xi]\oplus[-\lambda]'$ are integrable with totally geodesic leaves, the distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. First, we consider that M^{2n+1} is locally a warped product $S\times_f F$ such that $TS = [\xi]\oplus[\lambda]'$ and $TF = [-\lambda]'$. Now, we compute the function f. We have denoted by $\check{g}$ and $\hat{g}$ the pseudo-Riemannian metrics on S and F, respectively, such that the warped metric is given by $\check{g} + f^2\hat{g}$. Then, the projection $\pi : S\times_f F \to S$ is a submersion with horizontal distribution $[\xi]\oplus[\lambda]'$ and vertical distribution $[-\lambda]'$. From Theorem 5.2 we know that the mean curvature vector field for the immersed submanifold $(F,\hat{g})$ is $H' = (\lambda+\alpha)\xi$. By Proposition 4.1 in [4], we get for any $Y,Z \in [-\lambda]'$, $nor(\nabla_Y Z) = h(Y,Z) = -\frac{g(Y,Z)}{f}\, grad_{\check{g}} f$. And since $h(Y,Z) = g(Y,Z)H'$, we get $-(\lambda+\alpha)f\xi = grad_{\check{g}} f$. We choose local coordinates $\{t,x^1,\ldots,x^n\}$ on B such that $\xi = \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x^i} \in [\lambda]'$ for any $i = 1,\ldots,n$. After direct computation we get $f = ce^{-(\lambda+\alpha)t}$, $c>0$. Since $\xi \in \mathcal{N}(\kappa,2\alpha)'$, we have $R(X,Y)\xi = 0$, and $R(X,\xi)\xi = (\kappa+2\alpha\lambda)X$, also by $\xi \in \mathcal{N}(\kappa,2\alpha)'$, we get $R(\xi,X)Y = \kappa(g(X,Y)\xi - \eta(Y)X) + 2\alpha(g(h'X,Y)\xi - \eta(Y)'hX)$, thus, we get $R(\xi,X)Y = (\kappa+2\alpha\lambda)g(X,Y)\xi$. Applying Proposition 5.2, we get $R(X,Y)Z = (\kappa+2\alpha\lambda)[g(Y,Z)X - g(X,Z)Y]$, hence, we conclude that S is a space of constant curvature $\kappa+2\alpha\lambda > 0$. Next, we compute the curvature R^F of $(F,\hat{g})$, by Proposition 4.2 in [4], for any $U,V,W \in [-\lambda]'$, it holds

$$R^F(V,W)U = R(V,W)U - \frac{g(\mathrm{grad} f, \mathrm{grad} f)}{f^2}\{g(V,U)W - g(W,U)V\}.$$

Since $\mathrm{grad} f = -(\lambda+\alpha)f\xi$, we get that $g(\mathrm{grad} f,\mathrm{grad} f) = (\lambda+\alpha)^2 f^2 = (2\alpha\lambda-\kappa)f^2$, and by Proposition 5.2, we get $R(V,W)U = (2\alpha\lambda-\kappa)\{g(V,U)W - g(W,U)V\}$. Then, $R^F(V,W)U = 0$, and thus the fibers of the warped product are flat spaces.

Similar discussions for horizontal distribution $[\xi]\oplus[-\lambda]'$ and vertical distribution $[\lambda]'$. In this case, the mean curvature vector field for the immersed submanifold

$(F, \hat{g})$ is $H' = (\lambda - \alpha)\xi$ and computing the warping function, we obtain $f' = c'e^{(\alpha-\lambda)t}, c' > 0$. Moreover, we can also prove that the base manifold of the warped product is a space of constant curvature $\kappa - 2\alpha\lambda < 0$ and the fibers are flat spaces. Thus, we complete the proof. □

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