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ON (p, q)-STANCU-SZÁSZ-BETA OPERATORS AND THEIR APPROXIMATION PROPERTIES

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Abstract. In the present paper, we have introduced the generalized form of (p, q)-analogue of the Szász-Beta operators with Stancu type parameters. We have studied the local approximation properties of these operators and obtained the convergence rate and weighted approximation.
Keywords: Szász-Beta operators; Stancu type parameters; weighted approximation.

1. Introduction and preliminaries

In the last two decades, the applications of q-calculus emerged as a new area in the field of approximation theory. The development of q-calculus has led to the discovery of various modifications of Bernstein polynomials involving q-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

In 1987, Lupaş [11] introduced the first q-analogue of the classical Bernstein operators and investigated its approximating and shape preserving properties. Another q-generalization of the classical Bernstein polynomial is due to Phillips [20]. Several generalization of well known positive linear operators based on q-integers were introduced and their approximation properties have been studied by several researchers.

Recently, Mursaleen *et al* introduced the use of (p, q)-calculus in approximation theory and constructed the (p, q)-analogue of Bernstein operators [13] and (p, q)-analogue of Bernstein-Stancu operators [15]. Most recently, the (p, q)-analogue of

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some more operators have been studied in [1]- [3], [5], [12], [14], [16], [17], [18] and [19].

The (p,q)-integer was introduced to generalize or unify several forms of q-oscillator algebras well known in the Physics literature related to the representation theory of single parameter quantum algebras. The (p,q)-integer is defined by

$$(1.1)\quad [n]_{p,q} = p^{n-1} + qp^{n-2} + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n-q^n}{p-q} & (p \neq q \neq 1) \\ \frac{1-q^n}{1-q} & (p=1) \\ n & (p=q=1) \end{cases}$$

The (p,q)-binomial expansion is

$$(ax+by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(x+y)^n_{p,q} := (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y),$$

$$(1-x)^n_{p,q} := (1-x)(p-qx)(p^2-q^2x)\cdots(p^{n-1}-q^{n-1}x).$$

The (p,q)-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The definite integral of a function f is defined by

$$\int_0^a f(t) d_{p,q}t = (q-p)a \sum_{k=0}^{\infty} f\left(\frac{p^k}{q^{k+1}}a\right) \frac{p^k}{q^{k+1}}, \qquad if \mid \frac{p}{q} \mid < 1,$$

$$\int_0^a f(t) d_{p,q}t = (p-q)a \sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}a\right) \frac{q^k}{p^{k+1}}, \qquad if \mid \frac{q}{p} \mid < 1.$$

There are two (p,q)-analogues of the classical exponential function defined as follows

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

which satisfy the equality $e_{p,q}(x)E_{p,q}(-x) = 1$. For $p = 1$, $e_{p,q}(x)$ and $E_{p,q}(x)$ reduce to q-exponential functions.

For $m, n \in \mathbb{N}$, the (p,q)-Beta and the (p,q)-Gamma functions are defined by

$$B_{p,q}(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} d_{p,q}x,$$

and

$$\Gamma_{p,q}(n) = \int_0^{\infty} p^{\frac{n(n-1)}{2}} E_{p,q}(-qx) d_{p,q}x, \qquad \Gamma_{p,q}(n+1) = [n]_{p,q}!$$

respectively. The two functions are connected through

$$B_{p,q}(m,n) = q^{\frac{2-m(m-1)}{2}} p^{\frac{-m(m+1)}{2}} \frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.$$

For $p = 1$, all the notions of the (p,q)-calculus reduce to those of q-calculus.

Based on (p,q)-calculus, very recently Acar [1] defined the (p,q) analogue of Szász operators as

$$S_{n,p,q}(f;x) = \sum_{k=0}^{n} s_{n,k}^{p,q}(x) f\left(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\right) \tag{1.2}$$

for $x \in [0,\infty), 0 < q < p \leq 1$, where

$$s_{n,k}^{p,q}(x) = \frac{q^{\frac{k(k-1)}{2}}}{E_{p,q}([n]_{p,q}x)} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!}.$$

Gupta and Noor [9] proposed Szász-Beta operators and obtained some direct results in simultaneous approximation. Gupta and Aral [8] extended the studies and they proposed the q-analogue of Szász-Beta operators. Later on Aral and Gupta [4] introduced the (p,q)-analogue of the Szász-Beta operators as follows

$$D_n^{(p,q)}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} f(p^{k+1}qt) d_{p,q}t \tag{1.3}$$

where $s_{n,k}^{p,q}(x)$ is defined in (1.2). In this paper, we have generalized this operator (1.3) with Stancu type parameters. Assuming that $0 \leq \alpha \leq \beta$, for $x \in [0,\infty), 0 < q < p \leq 1$, we define

$$D_{n,p,q}^{\alpha,\beta}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} f\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right) d_{p,q}t. \tag{1.4}$$

2. Auxiliary results

Lemma 2.1. *For $x \in [0,\infty), 0 < q < p \leq 1$, we have*

$$\begin{aligned}
(i) D_n^{p,q}(1;x) &= 1, \\
(ii) D_n^{p,q}(t;x) &= x, \\
(iii) D_n^{p,q}(t^2;x) &= \frac{[2]_{p,q}qx}{p[n-1]_{p,q}} + \frac{p[n]_{p,q}x^2}{[n-1]_{p,q}}, \\
(iv) D_n^{p,q}(t^3;x) &= \frac{p^3[n]_{p,q}^2}{q^6[n-1]_{p,q}[n-2]_{p,q}}x^3 \\
&+ \left(\frac{(p[2]_{p,q}+p^2)[n]_{p,q}}{p^2q^6([n-1]_{p,q}[n-2]_{p,q}} + \frac{(p^2q+2pq^2)[n]_{p,q}}{q^6([n-1]_{p,q}[n-2]_{p,q}}\right)x^2 \\
&+ \left(\frac{[2]_{p,q}}{p^3q^5([n-1]_{p,q}[n-2]_{p,q}} + \frac{(p[2]_{p,q}+p^2)}{p^3q^5([n-1]_{p,q}[n-2]_{p,q}}\right)x, \\
(v) D_n^{p,q}(t^4;x) &= \frac{p^6[n]_{p,q}^3}{q^{12}[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^4 \\
&+ \frac{[n]_{p,q}^2(p^5+3p^3q^2+2p^3q+2p^2q^3+pq^4+q^3)}{q^{11}[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^3 \\
&+ \frac{[n]_{p,q}}{p^5q^9[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\Big(p^8+3p^7q+5p^6q^2 \\
&+5p^5q^3+2p^4q^4+p^4q^2+p^3q^4+2p^3q^3+2p^2q^4+pq^5\Big)x^2 \\
&+ \frac{(p^6+2p^5q+p^4q^2+p^3q^3+p^3q^2+p^3q+2p^2q^4+2p^2q^2+2pq^5+pq^3+q^6)}{p^6q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x.
\end{aligned}$$

Lemma 2.2. *Let $e_r(t) = t^r$, $r \in \mathbb{N} \cup \{0\}$. For $x \in [0,\infty)$, $0 < q < p \leq 1$, $0 \leq \alpha \leq \beta$, we have*

$$\begin{aligned}
(i) D_{n,p,q}^{(\alpha,\beta)}(e_0;x) &= 1, \\
(ii) D_{n,p,q}^{(\alpha,\beta)}(e_1;x) &= \frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta},
\end{aligned}$$

$$(iii) D_{n,p,q}^{(\alpha,\beta)}(e_2;x) = \frac{p[n]_{p,q}^3}{[n-1]_{p,q}([n]_{p,q}+\beta)^2}x^2 + \frac{[n]_{p,q}(q(p+q)[n]_{p,q}+2\alpha p[n-1]_{p,q})}{p([n]_{p,q}+\beta)^2[n-1]_{p,q}}x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2},$$

$$(iv) D_{n,p,q}^{(\alpha,\beta)}(e_3;x) = \frac{p^3[n]_{p,q}^5}{q^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}x^3 + \frac{[n]_{p,q}^3([n]_{p,q}(p^3q+2p^2q^2+2p+q)+3p^2q^6\alpha[n-2]_{p,q})}{pq^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}x^2 + \frac{[n]_{p,q}}{([n]_{p,q}+\beta)^3}\left(\frac{[n]_{p,q}^2([2]_{p,q}+p[2]_{p,q}+p^2)}{q^5p^3[n-1]_{p,q}[n-2]_{p,q}} + \frac{3q\alpha[2]_{p,q}[n]_{p,q}}{p[n-1]_{p,q}} + 3\alpha^3\right)x + \frac{\alpha^3}{([n]_{p,q}+\beta)^3},$$

$$(v) D_{n,p,q}^{(\alpha,\beta)}(e_4;x) = \frac{p^6[n]_{p,q}^7}{q^{12}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^4 + \frac{[n]_{p,q}^5([n]_{p,q}(p^5+3p^3q^2+2p^3q+2p^2q^3+pq^4+q^3)+4\alpha p^3q^5[n-3]_{p,q})}{q^{11}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^3 + \frac{[n]_{p,q}^3}{p^5q^9([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\Big([n]_{p,q}^2(p^8+3p^7q+5p^6q^2$$

$$+ 5p^5q^3+2p^4q^4+p^4q^2+p^3q^4+2p^3q^3+2p^2q^4+pq^5) + (4\alpha[n]_{p,q}[n-3]_{p,q}(p^7q^4+2p^6q^5+2p^5q^3+p^4q^4)) + (6\alpha^2p^6q^9[n-2]_{p,q}[n-3]_{p,q})\Big)x^2$$

$$+ \frac{[n]_{p,q}}{p^6q^6([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\Big([n]_{p,q}^3(p^6+2p^5q+p^4q^2 + p^3q^3+p^3q^2+p^3q+2p^2q^4+2p^2q^2+2pq^5+pq^3+q^6) + 4\alpha[n]_{p,q}^2[n-3]_{p,q} (2p^5q+p^4q^2+p^4q+p^3q^2) + 6\alpha^2[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}(p^6q^7+p^5q^8) + \alpha^3[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}p^6q^6\Big)x + \frac{\alpha^4}{([n]_{p,q}+\beta)^4}.$$

Proof. Using Lemma 2.1, we can easily say, $(i) D_{n,p,q}^{(\alpha,\beta)}(e_0;x) = 1$. Moreover

$$(ii) D_{n,p,q}^{(\alpha,\beta)}(e_1;x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right)d_{p,q}t$$

$$= \frac{[n]_{p,q}}{[n]_{p,q}+\beta}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\frac{p^{k+1}q}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^k}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t + \frac{\alpha}{[n]_{p,q}+\beta}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t$$

$$
\begin{aligned}
&= \frac{[n]_{p,q}}{[n]_{p,q}+\beta} D_n^{p,q}(e_1;x) + \frac{\alpha}{[n]_{p,q}+\beta} D_n^{p,q}(e_0;x) \\
&= \frac{[n]_{p,q}}{[n]_{p,q}+\beta} x + \frac{\alpha}{[n]_{p,q}+\beta}.
\end{aligned}
$$

$$
\begin{aligned}
(iii) D_{n,p,q}^{(\alpha,\beta)}(e_2;x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \left(\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q}+\beta} \right)^2 d_{p,q}t \\
&= \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{2k+2}q^2}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k+1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&\quad + \frac{2\alpha [n]_{p,q}}{([n]_{p,q}+\beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1}q}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&\quad + \frac{\alpha^2}{([n]_{p,q}+\beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&+ \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2} D_n^{p,q}(e_2;x) + \frac{2\alpha [n]_{p,q}}{([n]_{p,q}+\beta)^2} D_n^{p,q}(e_1;x) \\
&\quad + \frac{\alpha^2}{([n]_{p,q}+\beta)^2} D_n^{p,q}(e_0;x)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{p[n]_{p,q}^3}{[n-1]_{p,q}([n]_{p,q}+\beta)^2} x^2 + \frac{[n]_{p,q}(q(p+q)[n]_{p,q} + 2\alpha p[n-1]_{p,q})}{p([n]_{p,q}+\beta)^2 [n-1]_{p,q}} x \\
&\quad + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}.
\end{aligned}
$$

$$
\begin{aligned}
(iv) D_{n,p,q}^{(\alpha,\beta)}(e_3;x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \left(\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q}+\beta} \right)^3 d_{p,q}t \\
&= \frac{[n]_{p,q}^3}{([n]_{p,q}+\beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{3k+3}q^3}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k+2}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&\quad + \frac{3\alpha [n]_{p,q}^2}{([n]_{p,q}+\beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{2k+2}q^2}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k+1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&\quad + \frac{3\alpha^2 [n]_{p,q}}{([n]_{p,q}+\beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1}q}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&\quad + \frac{\alpha^3}{([n]_{p,q}+\beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&+ \frac{[n]_{p,q}^3}{([n]_{p,q}+\beta)^3} D_n^{p,q}(e_3;x) + \frac{3\alpha [n]_{p,q}^2}{([n]_{p,q}+\beta)^3} D_n^{p,q}(e_2;x)
\end{aligned}
$$

$$+\frac{3\alpha^2[n]_{p,q}}{([n]_{p,q}+\beta)^3}D_n^{p,q}(e_1;x)+\frac{\alpha^3}{([n]_{p,q}+\beta)^3}D_n^{p,q}(e_0;x)$$

$$= \frac{p^3[n]_{p,q}^5}{q^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}x^3$$

$$+\frac{[n]_{p,q}^3([n]_{p,q}(p^3q+2p^2q^2+2p+q)+3p^2q^6\alpha[n-2]_{p,q})}{pq^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}x^2$$

$$+\frac{[n]_{p,q}}{([n]_{p,q}+\beta)^3}\left(\frac{[n]_{p,q}^2([2]_{p,q}+p[2]_{p,q}+p^2)}{q^5p^3[n-1]_{p,q}[n-2]_{p,q}}+\frac{3q\alpha[2]_{p,q}[n]_{p,q}}{p[n-1]_{p,q}}+3\alpha^3\right)x$$

$$+\frac{\alpha^3}{([n]_{p,q}+\beta)^3}.$$

$$(v)D_{n,p,q}^{(\alpha,\beta)}(e_4;x) = \sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right)^4 d_{p,q}t$$

$$= \frac{[n]_{p,q}^4}{([n]_{p,q}+\beta)^4}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{p^{4k+4}q^4}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k+3}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t$$

$$= \frac{4\alpha[n]_{p,q}^3}{([n]_{p,q}+\beta)^4}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{p^{3k+3}q^3}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k+2}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t$$

$$+\frac{6\alpha^2[n]_{p,q}^2}{([n]_{p,q}+\beta)^4}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{p^{2k+2}q^2}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k+1}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t$$

$$+ \frac{4\alpha^3[n]_{p,q}}{([n]_{p,q}+\beta)^4}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{p^{k+1}q}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t$$

$$+\frac{\alpha^4}{([n]_{p,q}+\beta)^4}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t$$

$$+ \frac{[n]_{p,q}^4}{([n]_{p,q}+\beta)^4}D_n^{p,q}(e_4;x)+\frac{4\alpha[n]_{p,q}^3}{([n]_{p,q}+\beta)^4}D_n^{p,q}(e_3;x)+\frac{6\alpha^2[n]_{p,q}^2}{([n]_{p,q}+\beta)^4}D_n^{p,q}(e_2;x)$$

$$+\frac{4\alpha^3[n]_{p,q}}{([n]_{p,q}+\beta)^4}D_n^{p,q}(e_1;x)+\frac{\alpha^4}{([n]_{p,q}+\beta)^4}D_n^{p,q}(e_0;x)$$

$$= \frac{p^6[n]_{p,q}^7}{q^{12}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^4$$

$$+\frac{[n]_{p,q}^5([n]_{p,q}(p^5+3p^3q^2+2p^3q+2p^2q^3+pq^4+q^3)+4\alpha p^3q^5[n-3]_{p,q})}{q^{11}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^3$$

$$+\frac{[n]_{p,q}^3}{p^5q^9([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\Big([n]_{p,q}^2(p^8+3p^7q+5p^6q^2$$

$$+ 5p^5q^3+2p^4q^4+p^4q^2+p^3q^4+2p^3q^3+2p^2q^4+pq^5)$$

$$
\begin{aligned}
&+(4\alpha[n]_{p,q}[n-3]_{p,q}(p^7q^4+2p^6q^5+2p^5q^3+p^4q^4))\\
&+(6\alpha^2p^6q^9[n-2]_{p,q}[n-3]_{p,q})\Big)x^2\\
&+\frac{[n]_{p,q}}{p^6q^6([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\Big([n]_{p,q}^3(p^6+2p^5q+p^4q^2\\
&+p^3q^3+p^3q^2+p^3q+2p^2q^4+2p^2q^2+2pq^5+pq^3+q^6)+4\alpha[n]_{p,q}^2[n-3]_{p,q}\\
&(2p^5q+p^4q^2+p^4q+p^3q^2)+6\alpha^2[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}(p^6q^7+p^5q^8)\\
&+\alpha^3[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}p^6q^6\Big)x+\frac{\alpha^4}{([n]_{p,q}+\beta)^4}.
\end{aligned}
$$

□

We readily obtain the following lemma.

Lemma 2.3. *For $x\in[0,\infty)$, $0<q<p\leq 1$, $0\leq\alpha\leq\beta$, we have*

$$
\begin{aligned}
(i)D_{n,p,q}^{\alpha,\beta}((t-x);x) &= \left(\frac{[n]_{p,q}}{([n]_{p,q}+\beta)}-1\right)x+\frac{\alpha}{([n]_{p,q}+\beta)},\\
(ii)D_{n,p,q}^{\alpha,\beta}((t-x)^2;x) &= \left(\frac{p[n]_{p,q}^3}{[n-1]_{p,q}([n]_{p,q}+\beta)^2}-\frac{2[n]_{p,q}}{([n]_{p,q}+\beta)}+1\right)x^2\\
&+\left(\frac{[n]_{p,q}}{([n]_{p,q}+\beta)^2}\left(\frac{2[2]_{p,q}[n]_{p,q}}{p[n-1]_{p,q}}+2\alpha\right)-\frac{2\alpha}{([n]_{p,q}+\beta)}\right)x\\
&+\frac{\alpha^2}{([n]_{p,q}+\beta)^2},\\
(iii)D_{n,p,q}^{\alpha,\beta}((t-x)^4;x) &= \left(\frac{p^6[n]_{p,q}^7}{q^{12}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\right.
\end{aligned}
$$

$$
\begin{aligned}
-\;&\frac{4p^3[n]_{p,q}^5}{q^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}\\
&+\frac{6p[n]_{p,q}^3}{([n]_{p,q}+\beta)^2[n-1]_{p,q}}-\frac{4[n]_{p,q}}{([n]_{p,q}+\beta)}+1\Big)x^4\\
&+\Big(\frac{[n]_{p,q}^5}{q^{11}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\\
&\Big[[n]_{p,q}(p^5+3p^3q^2+2p^3q+2p^2q^3+pq^4+q^3)+4\alpha p^3q^5[n-3]_{p,q}\Big]\\
&-\frac{4[n]_{p,q}^3([n]_{p,q}(p^3q+2p^2q^2+2p+q)+3p^2q^6\alpha[n-2]_{p,q})}{pq^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}\\
&+\frac{6[n]_{p,q}([n]_{p,q}(pq+q^2)+2\alpha p[n-1]_{p,q})}{p([n]_{p,q}+\beta)^2[n-1]_{p,q}}-\frac{4\alpha}{([n]_{p,q}+\beta)}\Big)x^3
\end{aligned}
$$

$$+\Bigg(\frac{[n]^3_{p,q}}{p^5q^9([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}$$

$$\Bigg[[n]^2_{p,q}(p^8+3p^7q+5p^6q^2+5p^5q^3+2p^4q^4+p^4q^2+p^3q^4+2p^3q^3$$

$$+2p^2q^4+pq^5)+4\alpha[n]_{p,q}[n-3]_{p,q}(p^7q^4+2p^6q^5+2p^5q^3+p^4q^4)$$

$$+\quad 6\alpha^2p^6q^9[n-2]_{p,q}[n-3]_{p,q}\Bigg]-\frac{4[n]_{p,q}}{p^3q^5([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}$$

$$\Bigg[[n]^2_{p,q}(2p^2+pq+p+q)+3\alpha[n-2]_{p,q}(p^3q^6+p^2q^7)$$

$$+3\alpha^2p^3q^5[n-1]_{p,q}[n-2]_{p,q}\Bigg]+\frac{6\alpha^2}{([n]_{p,q}+\beta)^2}\Bigg)x^2$$

$$+\Bigg(\frac{[n]_{p,q}}{p^6q^6([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}$$

$$\Bigg[[n]^3_{p,q}(p^6+2p^5q+p^4q^2+p^3q^3+p^3q^2+p^3q+2p^2q^4+2p^2q^2+2pq^5$$

$$+pq^3+q^6)+4\alpha[n]^2_{p,q}[n-3]_{p,q}(2p^5q+p^4q^2+p^4q+p^3q^2)$$

$$+6\alpha^2[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}(p^6q^7+p^5q^8)$$

$$+\alpha^3p^6q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}\Bigg]-\frac{4\alpha^3}{([n]_{p,q}+\beta)^3}\Bigg)x$$

$$+\frac{\alpha^4}{([n]_{p,q}+\beta)^4}.$$

3. Local approximation

In this section, we present local approximation theorem for operators $D^{\alpha,\beta}_{n,p,q}$. By $C_B[0,\infty)$, we denote the space of all real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \| = \sup_{0\le x<\infty} | f(x) | .$$

Further, let us consider the following K-functional:

$$K_2(f,\delta)=\inf_{g\in W^2}\{\| f-g \| +\delta \| g^{''} \|\}$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty): g^{'},g^{''}\in C_B[0,\infty)\}$. By Theorem 2.4 of [6], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta}) \tag{3.1}$$

where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)} | f(x+2h)-2f(x+h)+f(x) |$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\le\delta} \sup_{x\in[0,\infty)} \mid f(x+h) - f(x) \mid .$$

Theorem 3.1. *Let $f \in C_B[0,\infty)$ and $0 < q < p \le 1$, $0 \le \alpha \le \beta$. Then for all $n \in \mathbb{N}$, there exists an absolute constant $C > 0$ such that*

$$\mid D^{\alpha,\beta}_{n,p,q}(f;x) - f(x) \mid \le C\omega_2(f,\delta_n(x)) + \omega(f,\alpha_n(x)),$$

where

$$\delta_n(x) = \sqrt{D^{\alpha,\beta}_{n,p,q}((t-x)^2;x) + (\alpha_n(x))^2}, \quad \alpha_n(x) = \frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta} - x.$$

Proof. For $x \in [0,\infty)$, we consider the auxiliary operators $\bar{D}^*_n$ defined by

$$\bar{D}^*_n(f;x) = D^{\alpha,\beta}_{n,p,q}(f;x) + f(x) - f\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta}\right).$$

From Lemma 2.2 (i), (ii) and Lemma 2.3 (i), we observe that the operators $\bar{D}^*_n(f;x)$ are linear and reproduce the linear functions. Hence

$$\begin{aligned}
\bar{D}^*_n(1;x) &= D^{\alpha,\beta}_{n,p,q}(1;x) + 1 - 1 = 1,\\
\bar{D}^*_n(t;x) &= D^{\alpha,\beta}_{n,p,q}(t;x) + x - \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta}\right) = x,\\
\bar{D}^*_n((t-x);x) &= \bar{D}^*_n(t;x) - x\bar{D}^*_n(1;x) = 0.
\end{aligned}$$

Let $x \in [0,\infty)$ and $g \in W^2$. Using the Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u.$$

Applying $\bar{D}^*_n$ to both sides of the above equation, we have

$$\begin{aligned}
\bar{D}^*_n(g;x) - g(x) &= g'(x)\bar{D}^*_n((t-x);x) + \bar{D}^*_n\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\\
&= D^{\alpha,\beta}_{n,p,q}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\\
&\quad - \int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}} \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta} - u\right) g''(u)\mathrm{d}u.
\end{aligned}$$

On the other hand, since

$$\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right| \le \int_x^t \mid t-u \mid\mid g''(u) \mid \mathrm{d}u \le \parallel g'' \parallel \int_x^t \mid t-u \mid \mathrm{d}u \le (t-x)^2 \parallel g'' \parallel$$

and

$$\left|\int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}}\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}-u\right)g''(u)\mathrm{d}u\right|$$

$$\leq \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}-x\right)^2 \parallel g'' \parallel .$$

We conclude that

$$\begin{aligned}\left|\bar{D}_n^*(g;x)-g(x)\right| &\leq \left|D_{n,p,q}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\right.\\ &\quad \left.-\int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}}\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}-u\right)g''(u)\mathrm{d}u\right|\\ &\leq \parallel g'' \parallel D_{n,p,q}^{\alpha,\beta}((t-x)^2;x)+\parallel g'' \parallel \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}-x\right)^2\\ &= \parallel g'' \parallel \delta_n^2(x).\end{aligned}$$

Now, taking into account boundedness of $\bar{D}_n^*$, we have

$$\mid \bar{D}_n^*(f;x) \mid \;\leq\; \mid D_{n,p,q}^{\alpha,\beta}(f;x) \mid +2 \parallel f \parallel \leq 3 \parallel f \parallel .$$

Therefore

$$\begin{aligned}\mid D_{n,p,q}^{\alpha,\beta}(f;x)-f(x) \mid &\leq \mid \bar{D}_n^*(f-g;x)-(f-g)(x) \mid +\left|f\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right|\\ &\quad +\mid \bar{D}_n^*(g;x)-g(x) \mid\\ &\leq \mid \bar{D}_n^*(f-g;x) \mid + \mid (f-g)(x) \mid +\left|f\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right|\\ &\quad +\mid \bar{D}_n^*(g;x)-g(x) \mid\\ &\leq 4 \parallel f-g \parallel +\omega(f,\alpha_n(x))+\delta_n^2(x) \parallel g'' \parallel .\end{aligned}$$

Hence, taking the infimum on the right-hand side over all $g \in W^2$, we have the following result

$$\mid D_{n,p,q}^{\alpha,\beta}(f;x)-f(x) \mid \;\leq\; 4K_2(f,\delta_n^2(x))+\omega(f,\alpha_n(x)).$$

In view of the property of K-functional, we get

$$\mid D_{n,p,q}^{\alpha,\beta}(f;x)-f(x) \mid \;\leq\; C\omega_2(f,\delta_n(x))+\omega(f,\alpha_n(x)).$$

This completes the proof of the theorem. □

4. Approximation properties in weighted spaces

Let $B_\rho[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $\mid f(x) \mid\leq M_f\rho(x)$, where M_f is a constant depending only on f and $\rho(x)$ is a weight function.

Let $C_\rho[0,\infty)$ be the space of all continuous functions in $B_\rho[0,\infty)$ with the norm

$\|f\|_\rho = \sup\limits_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$ and

$$C_\rho^0 = \left\{ f \in C_\rho[0,\infty) : \lim_{x\to\infty} \frac{\mid f(x) \mid}{\rho(x)} < \infty \right\}.$$

In what follows, we assume the weight function as $\rho(x) = 1 + x^2$.

Theorem 4.1. *Let $0 < q = q_n < p = p_n \leq 1$ such that $q_n \to 1$, $p_n \to 1$, as $n \to \infty$. For each $f \in C_\rho^0$, we have*

$$\lim_{n\to\infty} \|D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x)\|_\rho = 0.$$

Proof. With elementary calculations, it can be easily followed that $\lim\limits_{n\to\infty} \|D_{n,p_n,q_n}^{\alpha,\beta}(e_i;\cdot) - e_i\|_\rho = 0$, where $e_i(x) = x^i$, $i = 0,1,2$. By weighted Korovkin theorem given in [7], we get the required result. □

Next we give the following theorem to approximate all functions in C_ρ^0. This type of result is discussed in [10] for locally integrable functions.

Theorem 4.2. *Let $0 < q = q_n < p = p_n \leq 1$ such that $q_n \to 1$, $p_n \to 1$, $q_n^n \to 1$, $p_n^n \to 1$ as $n \to \infty$. For each $f \in C_\rho^0$ and $a > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\mid D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}
\sup_{x\in[0,\infty)} \frac{\mid D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} &\leq \sup_{x\leq x_0} \frac{\mid D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} + \sup_{x\geq x_0} \frac{\mid D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} \\
&\leq \|D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,x_0]} \\
&\quad + \|f\|_\rho \sup_{x\geq x_0} \frac{\mid D_{n,p_n,q_n}^{\alpha,\beta}(1+t^2;x) \mid}{(1+x^2)^{1+a}} \\
&\quad + \sup_{x\geq x_0} \frac{\mid f(x) \mid}{(1+x_0^2)^{1+a}}
\end{aligned}$$

$$= I_1 + I_2 + I_3. \tag{4.1}$$

Since $| f(x) | \leq \|f\|_\rho (1+x^2)$, we have

$$I_3 = \sup_{x \geq x_0} \frac{| f(x) |}{(1+x^2)^{1+a}} \leq \sup_{x \geq x_0} \frac{\|f\|_\rho}{(1+x^2)^a} \leq \frac{\|f\|_\rho}{(1+x_0^2)^a}$$

Let $\epsilon > 0$ be arbitrary. There exists $n_1 \in \mathbb{N}$ such that

$$\|f\|_\rho \frac{| D^{\alpha,\beta}_{n,p_n,q_n}(1+t^2;x) |}{(1+x^2)^{1+a}} < \frac{1}{(1+x^2)^{1+a}} \|f\|_\rho \left((1+x^2) + \frac{\epsilon}{3\|f\|_\rho} \right), \qquad \forall n \geq n_1$$

$$< \frac{\|f\|_\rho}{(1+x^2)^a} + \frac{\epsilon}{3} \qquad \forall n \geq n_1. \tag{4.2}$$

Hence

$$\|f\|_\rho \sup_{x \geq x_0} \frac{| D^{\alpha,\beta}_{n,p_n,q_n}(1+t^2;x) |}{(1+x^2)^{1+a}} < \frac{\|f\|_\rho}{(1+x_0^2)^a} + \frac{\epsilon}{3}, \qquad \forall n \geq n_1.$$

Thus

$$I_2 + I_3 < \frac{2\|f\|_\rho}{(1+x_0^2)^a} + \frac{\epsilon}{3}, \qquad \forall n \geq n_1.$$

Now, let us choose x_0 to be so large that $\frac{\|f\|_\rho}{(1+x^2)^a} < \frac{\epsilon}{6}$.

Then,

$$I_2 + I_3 < \frac{2\epsilon}{3}, \qquad \forall n \geq n_1. \tag{4.3}$$

$$I_1 = \|D^{\alpha,\beta}_{n,p_n,q_n}(f) - f\|_{C[0,x_0]} < \frac{\epsilon}{3}, \qquad \forall n \geq n_2. \tag{4.4}$$

Let $n_0 = \max(n_1, n_2)$. Then, combining (4.1)-(4.4), we get

$$\sup_{x \in [0,\infty)} \frac{| D^{\alpha,\beta}_{n,p_n,q_n}(f;x) - f(x) |}{(1+x^2)^{1+a}} < \epsilon, \qquad \forall n \geq n_0.$$

This completes the proof. □

Now we present ordinary approximation in terms of Lipschitz constant defined by

$$lip_M(\gamma) = \left\{ f \in C_B[0,\infty) : | f(t) - f(x) | \leq M \frac{| t-x |^\gamma}{(t+x)^{\frac{\gamma}{2}}} \right\}, \tag{4.5}$$

where M is a positive constant and $0 < \gamma \leq 1$.

Theorem 4.3. *Let be $f \in C_B[0,\infty)$, $0 < q < p \leq 1$, $0 \leq \alpha \leq \beta$, then for any $x \in (0,\infty)$, the following inequality holds:*

$$|D_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| \quad \leq \quad M\left(\frac{\varphi_{n,p,q}^{(\alpha,\beta)}(x)}{x}\right)^{\frac{\gamma}{2}},$$

where $\varphi_{n,p,q}^{(\alpha,\beta)}(x) = D_{n,p,q}^{\alpha,\beta}((e_1 - x)^2; x)$.

Proof. First, we prove that the result is true for $\gamma = 1$. Then, for $f \in lip_M(\gamma)$, we obtain

$$\begin{aligned}
|D_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| &\leq \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^\infty \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \\
&\quad \times \left| f\left(\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta}\right) - f(x) \right| d_{p,q}t \\
&\leq M \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^\infty \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \\
&\quad \times \frac{\left| \frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} - x \right|}{\sqrt{\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} + x}} d_{p,q}t.
\end{aligned}$$

Using $\sqrt{x} < \sqrt{\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} + x}$ and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|D_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| &\leq \frac{M}{\sqrt{x}} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^\infty \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \\
&\quad \times \left| \frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} - x \right| d_{p,q}t \\
&= \frac{M}{\sqrt{x}} D_{n,p,q}^{\alpha,\beta}((e_1 - x)^2; x) \leq M \sqrt{\frac{\varphi_{n,p,q}^{(\alpha,\beta)}(x)}{x}}.
\end{aligned}$$

Therefore, the result is true for $\gamma = 1$. We prove that the result is true for $0 < \gamma \leq 1$, applying Hölder's inequality with $p = \frac{2}{\gamma}, q = \frac{1}{2-\gamma}$,

$$\begin{aligned}
|D_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| &\leq \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^\infty \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \\
&\quad \times \left| f\left(\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta}\right) - f(x) \right| d_{p,q}t \\
&\leq \sum_{k=0}^{\infty} \Bigg\{ s_{n,k}^{p,q}(x) \Bigg(\frac{1}{B_{p,q}(k,n+1)} \int_0^\infty \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}
\end{aligned}$$

$$\times\left|f\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right|d_{p,q}t\Bigg)^{\frac{2}{\gamma}}\Bigg\}^{\frac{\gamma}{2}}$$

$$\times\left\{\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t\right\}^{\frac{2-\gamma}{2}}$$

$$\leq \left\{\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}\right.$$

$$\left.\times\left|f\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right|^{\frac{2}{\gamma}}d_{p,q}t\right\}^{\frac{\gamma}{2}}.$$

Since $f\in lip_M(\gamma)$, we have

$$\begin{aligned}|D_{n,p,q}^{\alpha,\beta}(f;x)-f(x) &\leq \frac{M}{x^{\frac{\gamma}{2}}}\left\{\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}\right.\\ &\left.\times\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}-x\right)^2 d_{p,q}t\right\}^{\frac{\gamma}{2}}\\ &= \frac{M}{x^{\frac{\gamma}{2}}}\left(D_{n,p,q}^{\alpha,\beta}((e_1-x))^2;x)\right)^{\frac{\gamma}{2}}\leq M\left(\sqrt{\frac{\varphi_{n,p,q}^{(\alpha,\beta)}(x)}{x}}\right)^{\gamma}.\end{aligned}$$

Therefore, the proof is completed. □

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