FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 4 (2020), 1231–1237
https://doi.org/10.22190/FUMI2004231M

GRUNDY DOMINATION SEQUENCES IN GENERALIZED CORONA PRODUCTS OF GRAPHS

Seyedeh Maryam Moosavi Majd and Hamid Reza Maimani* *

Abstract. For a graph $G = (V, E)$, a sequence $S = (v_1, \dots, v_k)$ of distinct vertices of G it is called a *dominating sequence* if $N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \varnothing$ and is called a *total dominating sequence* if $N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$ for each $2 \leq i \leq k$. The maximum length of (total) dominating sequence is denoted by (γ_{gr}^t) $\gamma_{gr}(G)$. In this paper we compute (total) dominating sequence numbers for generalized corona products of graphs.
Keywords: dominating sequence; total dominating sequence; generalized corona products.

1. Introduction

In this paper, G is a simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. For notation and graph theoretical terminology, we generally follow [8]. The order $|V|$ and the size $|E|$ of G is denoted by $n = n(G)$ and $m = m(G)$, respectively. For every vertex $v \in V$, the *open neighborhood* $N_G(v)$ of v is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = d_G(v) = |N_G(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write P_n for the path of order n, C_n for the cycle of order n, and K_n for the complete graph of order n. A subset D of $V(G)$ is called a *dominating set* of G if every vertex of G is either in D or adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the number of vertices in a smallest dominating set of G. A *total dominating set* of G is a set D of vertices of G such that every vertex is adjacent to a vertex in D. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. A dominating set of cardinality $\gamma(G)$ ($\gamma_t(G)$) is called a γ-*set*

Received July 12, 2020; accepted August 21, 2020
2020 *Mathematics Subject Classification.* Primary 05C69; Secondary 05C76
*Corresponding Author

(γ_t-*set*). For further information about various domination sets in graphs, we refer reader to [9, 10].

Let G be a graph of order n and let $H_1, H_2, \ldots, H_n$, be n graphs. The *generalized corona product*, is the graph obtained by taking one copy of graphs $G, H_1, H_2, \cdots, H_n$ and joining the ith vertex of G to every vertex of H_i. This product is denoted by $G \circ \wedge_{i=1}^{n} H_i$. If each H_i is isomorphic to a graph H, then generalized corona product is called the *corona product* of G and H and is denoted by $G \circ H$.

Let G be a graph of size m and H be a graph. The *edge corona product*, denoted by $G \diamond H$, is the graph obtained by taking one copy of G and m copies of H, and then joining two end-vertices of the ith edge e_i of G to every vertex of ith copy of H. The *neighborhood corona*, denoted by $G \star H$, is the graph obtained by taking n copies of H and for each $i, 1 \leq i \leq n$, the ith copy of H being adjacent to vertices of $N_G[v_i]$. It is not difficult to see that $G \diamond H$ is the same as $G \circ \wedge_{i=1}^{n} H_i$, where each H_i is a disjoint union of $\deg(v_i)$ copies of H and $G \star H$ is the same as $G \circ \wedge_{i=1}^{n} H_i$, where each H_i is a disjoint union of $\deg(v_i) + 1$ copies of H.

Based on the domination number and the total domination number, various Grundy domination invariants have been introduced in recent years by some authors [1, 5, 6] and then they continued the study of these concepts in [3, 2, 4, 7].

In [5] the first type of Grundy dominating sequence was introduced. Let $S = (v_1, \ldots, v_k)$ be a sequence of distinct vertices of a graph G. The corresponding set $\{v_1, \ldots, v_k\}$ of vertices from the sequence S will be denoted by $\widehat{S}$. A sequence $S = (v_1, \ldots, v_k)$ is called a *closed neighborhood sequence* if, for each i,

$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$

If for a closed neighborhood sequence S, the set $\widehat{S}$ is a dominating set of G, then S is called a *dominating sequence* of G. Clearly, if $S = (v_1, v_2, \ldots, v_k)$ is a dominating sequence for G, then $k \geq \gamma(G)$. We call the maximum length of a dominating sequence in G the *Grundy domination number* of G and denote it by $\gamma_{gr}(G)$. The corresponding sequence is called a Grundy dominating sequence of G or γ_{gr}-sequence of G.

Total dominating sequences were introduced in [6], when G is a graph without isolated vertices. Using the same notation as in the previous paragraph, we say that a sequence $S = (v_1, \ldots, v_k)$ is called an *open neighborhood sequence* if, for each $2 \leq i \leq k$,

$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \emptyset.$$

Any open neighborhood sequence S, where $\widehat{S}$ is a total dominating set is called a *total dominating sequence*. The maximum length of a total dominating sequence in G is called the *Grundy total domination number* of G and denoted by $\gamma_{gr}^{t}(G)$.

The corresponding sequence is called a *Grundy total dominating sequence* of G or a γ_{gr}^{t}-total sequence.

An additional variant of the Grundy (total) domination number was introduced in [1]. Let G be a graph without isolated vertices. A sequence $S = (v_1, \dots, v_k)$, where $v_i \in V(G)$, is called a $Z-$ *sequence* if for each i,

$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$

Then the Z-Grundy domination number $\gamma_{gr}^{Z}(G)$ of the graph G is the length of a longest Z-sequence.

Let $S_1 = (v_1, \dots, v_n)$ and $S_2 = (u_1, \dots, u_m), n, m \geq 1$, be two sequences in G, with $\widehat{S_1} \cap \widehat{S_2} = \emptyset$. The *concatenation* of S_1 and S_2 is defined as the sequence $S_1 \oplus S_2 = (v_1, \dots, v_n, u_1, \dots, u_m)$. Clearly $\oplus$ is an associative operation on the set of all sequences, but is not commutative. If $S_2 = \{v\}$, then $S_1 \oplus S_2$ is denoted by $S_1 \oplus v$.

In the next section, we compute Grundy domination numbers for generalized corona products of graphs and based on, we find Grundy domination numbers of edge and neighborhood corona products of graphs.

2. Main Results

In this section we give the exact value of (total) Grundy domination numbers for generalized corona products, and compute them for corona product of some special graphs. First we state two necessary known propositions.

Proposition 2.1. *[6] For $n \geq 4$ even, $\gamma_{gr}^{t}(P_n) = n$ and $\gamma_{gr}^{t}(C_n) = n - 2$, while for $n \geq 3$ odd, $\gamma_{gr}^{t}(P_n) = \gamma_{gr}^{t}(C_n) = n - 1$.*

Proposition 2.2. *[5, 1] For $n \geq 3$, $\gamma_{gr}(C_n) = \gamma_{gr}^{Z}(C_n) = n - 2$, while for $n \geq 2$, $\gamma_{gr}(P_n) = \gamma_{gr}^{Z}(P_n) = n - 1$.*

we are now state and proof the our first main result.

Theorem 2.1. *Let G and $H_1, H_2, \dots, H_n$ be $n+1$ graphs without isolated vertices. Then*

$$\gamma_{gr}(G \circ \wedge_{i=1}^{n} H_i) = \sum_{i=1}^{n} \gamma_{gr}(H_i) + \gamma_{gr}^{Z}(G).$$

Proof. Set $K = G \circ \wedge_{i=1}^{n} H_i$. Let $S = (v_1, \dots, v_k)$ be a Z-Grundy sequence of G and S_i be a γ_{gr}-sequence of H_i for $1 \leq i \leq n$. It is not difficult to see that

$$S_1 \oplus v_1 \oplus S_2 \oplus v_2 \oplus \dots \oplus S_k \oplus v_k \oplus S_{k+1} \oplus S_{k+2} \oplus \dots \oplus S_n$$

is a dominating sequence for K. This implies that $\gamma_{gr}(K) \geq \sum_{i=1}^{n} \gamma_{gr}(H_i) + \gamma_{gr}^{Z}(G)$.

Let T be a γ_{gr}-sequence of K such that $|\widehat{T} \bigcap V(G)|$ is minimum among all γ_{gr}-sequences. Suppose that $\widehat{T} \bigcap V(G) = \{v_1, \ldots, v_t\}$, where $(v_1, \ldots, v_t)$ is a subsequence of T. If $t > \gamma_{gr}^{Z}(G)$, then $(v_1, \ldots, v_t)$ is not a Z-sequence for G and thus, there exists $1 \leq l \leq t$ such that $N_G(v_l) \backslash \bigcup_{i=1}^{l-1} N_G[v_i] = \varnothing$. But $N_K[v_l] \backslash \bigcup_{i=1}^{l-1} N_K[v_i] \neq \varnothing$, since $(v_1, \ldots, v_t)$ is a sub-sequence of T. If $\widehat{T} \bigcap V(H_l) \neq \varnothing$, then there exists an element $z \in V(H_l)$ such that one of the $(v_1, \ldots, v_l, z)$ or $(v_1, \ldots, v_{i-1}, z, v_i, \ldots, v_l)$ is a subsequence of T. If $(v_1, \ldots, v_l, z)$ is a subsequence of T, then $N_K[z] \backslash \bigcup_{i=1}^{l} N_K[v_i] = \emptyset$, which is a contradiction. Hence $(v_1, \ldots, v_{i-1}, z, v_i, \ldots, v_l)$ is a subsequence of T. Therefore there exists $x \in N_K[v_l] \setminus \bigcup_{i=1}^{l-1} N_K[v_i] \bigcup N_K[z]$. Since $v_l \in N_K[z]$ and $N_G(v_l) \setminus \bigcup_{i=1}^{l-1} N_G[v_i] = \emptyset$, we conclude that $x \neq v_l$. In addition, $x \in V(H_l)$ and x, z are not adjacent vertices, and $x \notin \widehat{T}$. Now, by replacing v_l by x in T, we obtain a γ_{gr}-sequence T', such that $|\widehat{T'} \bigcap V(G)| < |\widehat{T} \bigcap V(G)|$, which is a contradiction. Hence $\widehat{T} \bigcap V(H_l) = \emptyset$. Again consider a vertex $x \in V(H_l)$ and put x instead of v_l in T. Then we obtain a γ_{gr}-sequence T' such that the size of intersection of $\widehat{T'}$ and $V(G)$ is less than the size of intersection of $\widehat{T}$ and $V(G)$. This is a contradiction and so we conclude that $|\hat{T} \bigcap V(G)| \leq \gamma_{gr}^{Z}(G)$. It is not difficult to see $|\widehat{T} \bigcap V(H_i)| \leq \gamma_{gr}(H_i)$ for $1 \leq i \leq n$ and thus $\gamma_{gr}(K) \leq \sum_{i=1}^{n} \gamma_{gr}(H_i) + \gamma_{gr}^{Z}(G)$. □

The following corollary is an easy consequence of Theorem 2.1 and Proposition 2.2.

Corollary 2.1. *For $n, m \geq 3$*

$$\gamma_{gr}(C_n \circ C_m) = n(m-1) - 2, \gamma_{gr}(P_n \circ P_m) = mn - 1,$$

$$\gamma_{gr}(C_n \circ P_m) = nm - 2, \gamma_{gr}(P_n \circ C_m) = n(m-1) - 1.$$

we are now stat and proof our second main result.

Theorem 2.2. *Let G and $H_1, H_2, \ldots, H_n$ be graphs without isolated vertices. Then*

$$\gamma_{gr}^{t}(G \circ \wedge_{i=1}^{n} H_i) = \sum_{i=1}^{n} \gamma_{gr}^{t}(H_i) + \gamma_{gr}^{Z}(G).$$

Proof. Consider the sequence

$$T = S_1 \oplus v_1 \oplus S_2 \oplus v_2 \oplus \ldots \oplus S_k \oplus v_k \oplus S_{k+1} \oplus S_{k+2} \oplus \ldots \oplus S_n,$$

where $S = (v_1, \ldots, v_k)$ is a Z-Grundy sequence of G and S_i's are γ_{gr}^{t}-sequences of H_i's for $1 \leq i \leq n$. We show that T is a γ_{gr}^{t}-sequence for $K = G \circ \wedge_{i=1}^{n} H_i$. Let $x \in \widehat{T}$. Hence there exists either $1 \leq i \leq n$ such that $x \in \widehat{S_i}$ or $1 \leq j \leq k$ for which $x = v_j$. If $x = v_j$, then there exists $y \in N_G(v_j) \setminus \bigcup_{t=1}^{j-1} N_G[v_t]$. Hence $y \neq v_t$ for

$1 \le t \le j-1$ and therefore $y \in N_K(v_j) \setminus \bigcup_{t=1}^{j-1} N_K[v_t] \bigcup (\bigcup_{t=1}^{j} N_k[S_t])$. This implies that

$$N_K(v_j) \setminus \bigcup_{t=1}^{j-1} N_K[v_t] \bigcup (\bigcup_{t=1}^{j} N_K[S_t]) \neq \emptyset.$$

The same argument can be apply when $x \in \widehat{S}_i$. Since clearly $\widehat{T}$ is a total dominating set, we conclude that T is a total dominating sequence of G. Hence

$$\gamma_{gr}^t(K) \ge \sum_{i=1}^{n} \gamma_{gr}^t(H_i) + \gamma_{gr}^Z(G).$$

Now suppose that T is a γ_{gr}^t-sequence of K such that $|\widehat{T} \bigcap V(G)|$ is minimum among all γ_{gr}^t-sequences of G. Suppose that $\widehat{T} \bigcap V(G) = \{v_1, \ldots, v_t\}$ and $t > \gamma_{gr}^Z(G)$. Hence $(v_1, \ldots, v_t)$ is not a Z-sequence for G. Therefore, there exists $1 \le l \le t$ such that $N_G(v_l) \setminus \bigcup_{i=1}^{l-1} N_G[v_i] = \emptyset$. If $\widehat{T} \bigcap V(H_l) = \emptyset$, then by replacing v_l by $x \in V(H_l)$, we can construct a γ_{gr}^t-sequence T' such that $|\widehat{T'} \bigcap V(G)| < |\widehat{T} \bigcap V(G)|$, which is a contradiction. Hence $\widehat{T} \bigcap V(H_l) \neq \emptyset$. If there exists $x \in \widehat{T} \bigcap V(H_l)$ such that x appears after v_l in the sequence T, then (v_l, x) is a subsequence of T and $N_K(x) \setminus N_K(v_l) \neq \emptyset$. Since $N_G(v_l) \setminus \bigcup_{i=1}^{l-1} N_G[v_i] = \emptyset$, we conclude that $N_K(x) \setminus N_K(v_l) = \{v_l\}$ and hence $\widehat{T} \bigcap V(H_l) = \{x\}$. Now choose $y \in N(x)$ and replace v_l by y in T. Again we obtain a γ_{gr}^t-sequence T' such that $|\widehat{T'} \bigcap V(G)| < |\widehat{T} \bigcap V(G)|$, which is a contradiction. Hence all elements of $\widehat{T} \bigcap V(H_l)$ appear before v_l in the sequence T. Hence there exists $y \in V(H_l)$ such that $y \in N_K(v_l) \setminus \bigcup_{x \in \widehat{T} \bigcap V(H_l)} N_K(x)$. Since $\deg_{H_l}(y) \ge 1$, there exists $z \in V(H_l)$ which is adjacent to y. Clearly $z \notin \widehat{T}$ and by changing v_l with z, we get a γ_{gr}^t-sequence T' such that $|\widehat{T'} \bigcap V(G)| < |\widehat{T} \bigcap V(G)|$, which is a contradiction. This argument implies that $|\widehat{T} \bigcap V(G)| \le \gamma_{gr}^Z(G)$. One can easily check that $|\widehat{T} \bigcap V(H_i)| \le \gamma_{gr}^t(H_i)$ for $1 \le i \le n$ and so we conclude that $\gamma_{gr}^t(K) \le \sum_{i=1}^{n} \gamma_{gr}^t(H_i) + \gamma_{gr}^Z(G)$.
□

Corollary 2.2. *Let G be a graph of order n and size m and H be a graph without isolated vertices. Then $\gamma_{gr}(G \diamond H) = 2m\gamma_{gr}(H) + \gamma_{gr}^Z(G)$ and $\gamma_{gr}^t(G \star H) = (2m + n)\gamma_{gr}^t(H) + \gamma_{gr}^Z(G)$.*

Proof. Note that $G \diamond H$ is the same as $G \circ \wedge_{i=1}^{n} H_i$, where H_i is the disjoint union of $\deg(v_i)$ copies of H. Hence by Theorem 2.1,

$$\gamma_{gr}(G \diamond H) = \gamma_{gr}(G \circ \wedge_{i=1}^{n} H_i) = \sum_{i=1}^{n} \gamma_{gr}(H_i) + \gamma_{gr}^Z(G) = 2m\gamma_{gr}(H) + \gamma_{gr}^Z(G).$$

The proof of the second part of the corollary is similar. □

Corollary 2.3. *Let G be a connected graph of order n. Then $\gamma_{gr}^t(G \circ K_1) = 2n$.*

Proof. Suppose that $V(G) = \{v_1, \ldots, v_n\}$ is the vertex set of G. It is not difficult to see that sequence $(u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n)$, where u_i is the vertex of K_1, which is adjacent to v_i, is a Grundy total domination sequence of $G \circ K_1$. □

Corollary 2.4. *Let G be a nontrivial connected graph of order n. Then $\gamma_{gr}^t(G \circ H) = n\gamma_{gr}^t(H) + \gamma_{gr}^Z(G)$, for any nontrivial connected graph H.*

As a similar argument to proof of Theorem 2.1, we can find the Z-Grundy domination number of corona product of graphs.

Theorem 2.3. *Let G and $H_1, H_2, \ldots, H_n$ be $n+1$ graphs without isolated vertices. Then*

$$\gamma_{gr}^Z(G \circ \wedge_{i=1}^n H_i) = \sum_{i=1}^n \gamma_{gr}^Z(H_i) + \gamma_{gr}^Z(G).$$

REFERENCES

1. B. Brešar, Cs. Bujtas, T. Gologranc, S. Klavzar, G. Kosmrlj, B. Patkos, Z. Tuza and M. Vizer, Grundy dominating sequences and zero forcing sets, *Discrete Optim.* 26 (2017), 66–77.
2. B. Brešar, C. Bujtas, T. Gologranc, S. Klavzar, G. Kosmrlj, B. Patkos, Z. Tuza, M. Vizer, Dominating sequences in grid-like and toroidal graphs *Electron. J. Combin.*, 23 (2016), P4.34 (19 pages).
3. B. Brešar, T. Gologranc and T. Kos, Dominating sequences under atomic changes with applications in Sierpinski and interval graphs, *Appl. Anal. Discrete Math.* 10 (2016), 518–531.
4. B. Brešar, Kos and Terros, Grundy domination and zero forcing in Kneser graphs, *Ars Math. Contemp.*, 17(2019), 419-430.
5. B. Brešar, T. Gologranc, M. Milanič, D. F. Rall, R. Rizzi, Dominating sequences in graphs. *Discrete Math.* 336 (2014), 22-36.
6. B. Brešar, M. A. Henning, D. F. Rall, Total dominating sequences in graphs. *Discrete Math.* 339 (2016) 1165-1676.
7. B. Brešar, T. Kos, G. Nasini, P. Torres, Total dominating sequences in trees, split graphs, and under modular decomposition, *Discrete Optim.*, 28(2018), 16-30.
8. G. Chartrand, L. Lesniak, Graphs and digraphs, Third Edition, CRC Press,(1996).
9. T. W. Haynes, S. Hedetniemi, P. Slater, Fundamentals of Domination in Graphs, CRC Press, (1998).
10. M. A. Henning and A. Yeo, Total domination in graphs, *(Springer Monographs in Mathematics.)* ISBN-13: 987-1461465249 (2013).

Seyedeh Maryam Moosavi Majd
Department of Mathematics
Science and Research Branch, Islamic Azad University
Tehran, Iran
moosavi.majd@gmail.com

Hamid Reza Maimani
Mathematics Section, Department of Basic Sciences
Shahid Rajaee Teacher Training University
P.O. Box 16785-163
Tehran, Iran
maimani@ipm.ir