Highly Nonlinear Ising Model and Social Segregation

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Abstract: The usual interaction energy of the random field Ising model in statistical physics is modified by complementing the random field by adding to the energy of the usual Ising model a nonlinear term S^n , where S is the sum of the neighbor spins, and n=0,1,3,5,7,9,11.

Within the Schelling model of urban segregation, this modification corresponds to housing prices depending on the immediate neighborhood. Simulations at different temperatures (T), lattice size (L), magnetic field (h), number of neighbors (m) and different time intervals (number of iterations) showed that results for all n are similar, expect for n=3 in violation of the universality principle and the law of corresponding states. In order to find the critical temperatures, for large n we no longer start with all spins parallel but instead with a random configuration, in order to facilitate spin flips. However, in all cases we have a Curie temperature with phase separation or long-range segregation only below this Curie temperature, and it is approximated by a simple formula: T_c is proportional to m*exp(n/constant)

Keywords: Ising Model, Modified Ising model, Statistical Physics, and Urban Segregation.

Introduction:
The Ising model is the simplest and most famous spin system model to study phase transitions. It was introduced in 1925 by Ernst Ising in his Ph.D. thesis[1]. He solved the model completely for one dimension, and found that no phase transition occurs. He concluded that this should be the case for all dimensions. After that, it was a mistake.

In the Schelling – Ising model of urban segregation [1-2] all lattice sites are equivalent. In reality, some houses are cheap and others are expensive. And usually of two groups in a population, one is poorer than the other.

We study the urban segregation by these models which use two groups A and B of people distributed on a square lattice, with group A corresponds to up spins $\uparrow$ (+1) and group B to down spins $\downarrow$ (-1), in a random magnetic field on finite samples, to check cheap and expensive residences.

Urban segregation in this article will be studied by using Statistical Physics, when the usual interaction energy of the random field Ising model is modified by adding to the random field an odd power of the sum of the neighbor spins. Within the Schelling model of urban segregation, this modification corresponds to housing prices depending on the immediate neighborhood.

The general equation of the random-field Ising model is given by :

$$E = -J \sum_{i,k} s_i s_k - \sum_i h_i s_i$$

(1)

where s_i is the spin (±1), J the interaction constant, h_i is a random field.
This random field Ising energy can be rewritten as:

$$E = -\sum_j s_i \left(J S_j + h_i \right)$$

(2)

Where $S_j = \sum_k s_k$ with the sum over all nearest neighbors of j.

In the Ising model without the random field in two and more dimensions, two neighboring spins have due to their interaction $-J \, s_i \, s_k$ a higher probability to belong to the same group than to belong to two different groups. If the difference between these two probabilities is large enough, $T < T_c$, domain sizes can grow to infinity in an infinite lattice, while only small clusters are formed for smaller differences in the probabilities, $T > T_c$ [2] . These probabilities - controlled through $-J/k_B T$ - lead to these different regimes, separated by a sharp phase transition at $T = Tc$, not obvious from

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the definition of the interaction $J\, s_i\, s_k$, physicists took many years to find it, and it is typical of complex systems.

The earlier standard Ising model gave results similar to the properly modified Schelling model [3-9].

The magnetic field is considered to refer to the price of residence: Group A prefers to go to the cheap residences and group B to the expensive residences. This can be simulated by a random magnetic field which is $+h$ on half of the places (attracting people of group A = up spins) and is $-h$ on the other half of the places (attracting group B = down spins). The signs of the field are distributed randomly, and the model is called the random-field Ising model.

In the spin 1/2 Ising model with Glauber kinetics on the square lattice, we interpret the two spin orientations as representing two groups of people. Nearest neighbors are coupled ferromagnetically, i.e. people prefer to be surrounded by others of the same group and not of the other group. Starting with random initial distribution of zero magnetization (= number of one group minus number of the other group), we check if "infinitely" large domain are formed. It is well known that they do so for $0 < T < T_c$ where T_c = 2.269 is the critical temperature in units of the interaction energy for the usual Ising model.

The Glauber kinetics is simulated on the computer by flipping a spin if and only if a random number between 0 and 1 is smaller than the probability which is equal $exp(-\Delta E/k_BT)/[1 + exp(-\Delta E/ k_BT)]$, where ΔE is the energy change produced by this spin flip.

Ising model in Sociophysics:

If two groups of people A and B, rich and poor, are distributed on a square lattice, we use the Ising model with random magnetic field at small and large temperatures for times up to 9000. The social meaning of temperature T is not what we hear in weather reports but an overall approximation for all the more or less random events which influence our decisions.

Thus the temperature here can have two meanings: tolerance and noise. Tolerance means that for high T one is willing to live among neighbors from a different group, and low T means that people strongly prefer to live among neighbors of their own group. The alternative interpretation is T = noise; T then measures all those facts of life outside the model which force people to move to another residence even though they like their old residence better.

Within the Ising model of urban segregation, our modification corresponds to housing prices depending on the immediate neighborhood. Now we add

to the J-term in equation (2) the energy $J'(S_j)^n$, where J' is another interaction constant and $n = 1,3,5,7,9,11,\ldots$ some odd integer.

For $n=1$ we recover the usual Ising model with $J+J'$ instead of only J as interaction constant. Thus with $n=1$ we can test whether our program gives correct results, and with $n = 3,5,7,9,11$, we can get results which to our knowledge are new. By $n=0$ we denote the standard Ising model with J only, setting J' to zero, while otherwise at first $J'=J$, later we use also $m= J'/J$ up to 5. With even n ($n=2,4,6...$) the new term S_j^n is always positive which seems less interesting.

Thus the energy of the new modified Ising model becomes

$$E = -\sum_j s_i \left(J \ S_j + J' \ S_j^n + h_j \right), \ S_j = \sum_k s_k \tag{3}$$

$k = $ neighbor of site j .

In physics we can imagine that the strength of the magnetic dipole moment of an atom is influenced in a nonlinear way by its neighbor atoms.

Discussion and results:

In the present work, we use the simple standard two-dimensional Ising model with Glauber dynamics instead of the complicated Schelling model. We initially carry out our simulation at temperature of $T = 2.0$, with size of square lattice of 500×500, at time = 4000, where time is the number of iterations, and a field value = 0.1.We noticed large domains at this temperature and this field as in figure 1, but when the field is changed from 0.1 to 0.9 with the same parameters the domains decrease and small domains can be noticed, as in figure 2. Simulations at high temperature $T = 17$ with the same parameters showed that the two populations mix without any large clusters as in figure 3.
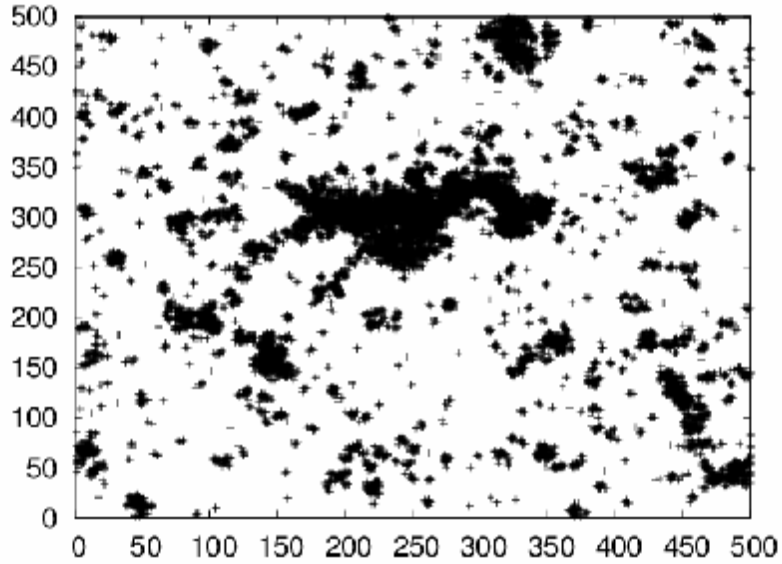
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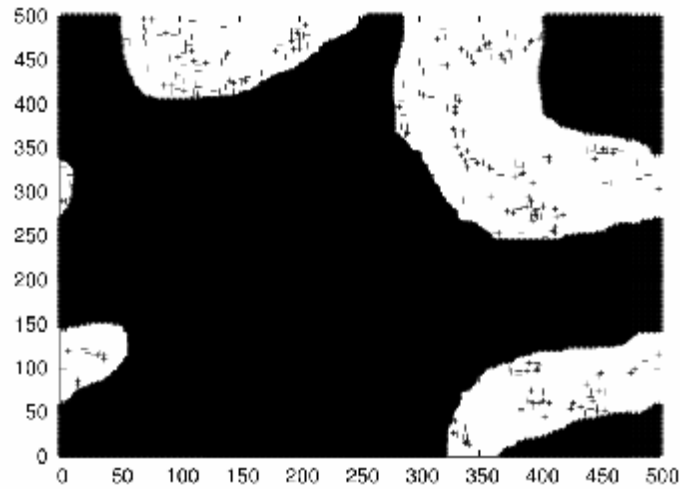
Figure(1): Shows the large domains of groups in small random field of + 0.1, and low temperature of 2.0, at time equal 4000, L=500 and n=3.
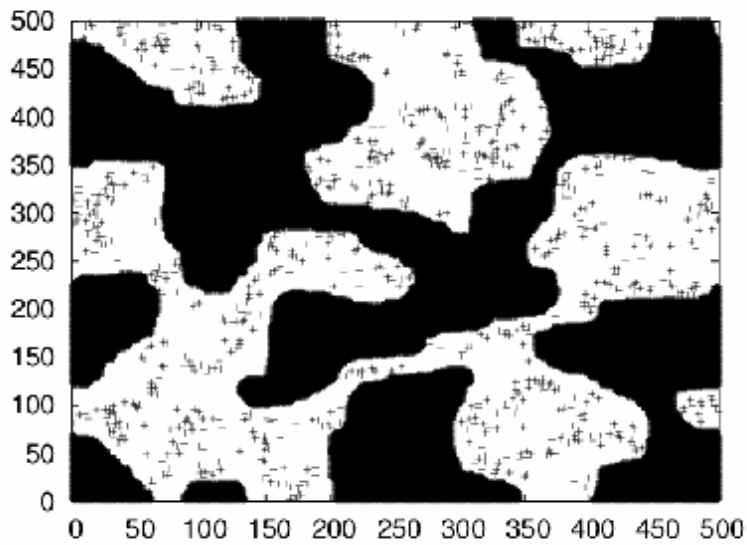
Figure(2): Shows the decrease of domains of groups as *h*=+ 0.9, and low temperature of 2.0, at time equal 4000, L=500 and *n*=3.

Figure(3): Shows the two populations mix without any large clusters. At *h*= + 0.9, and high temperature of 17, at time equal 4000, L=500 and *n*=3.

It was also demonstrated that there are no growing domains obtained at high field, while it was observed that the magnetization of the model changes from ferromagnetic to paramagnetic at the random field values higher than 0.256 for *n*=1,3,5,7,9,11.

In our simulation at different temperatures we check the critical temperatures T_c for different *n*=0,1,3,5,7,9,11. First we set for *n*=0, size of lattice 2000, time =9000, and put the value of the field=0.001. We find the critical temperature = 2.26 which agrees with the known critical temperature for the normal Ising model *n*=0.

Figure 4 shows the magnetization versus time where $T=T_c$ in the middle line, the lower line belongs to $T > T_c$, and the higher line to $T > T_c$.

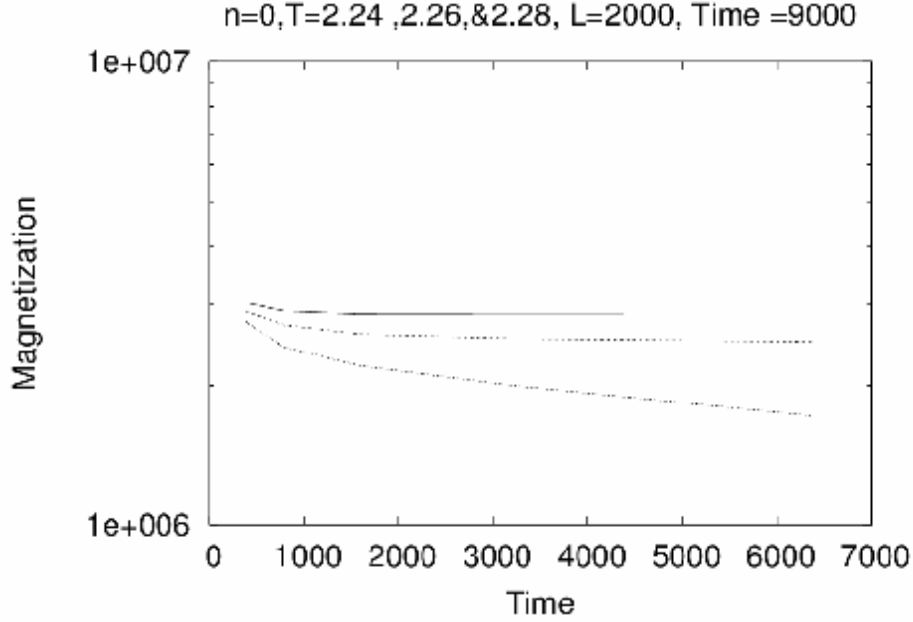**Highly Nonlinear Ising Model and Social Segregation**

Figure (4) : Magnetization versus time with temperature = 2.24(up), 2.26(middle) critical temperature, and 2.28 (down).

Similarly we go to n=1,3,5,7,9,11 for the same size of lattice L=2000,field=0,and time=9000, with different m and we take initially 1% spins down and 99% spins up at different temperatures, in this way the system can more easily move away from the initial configuration and find it's equilibrium, changing the initial fraction of down spins from 0.01 to 0.001 and 0.1 makes no difference in the calculated magnetization, critical temperatures for all n with different $m= J'/J$ listed in table 1:

Table(1) :T_c for different n and m, at h=0

n	1	3	5	7	9	11
m				T_c		
1	4.5	17.6	38.5	148	587	2350
2	6.9	32.5	76	290	1180	4710
3	9.2	47	112	442	1770	7090
4	11.5	61	149	590	2360	9415
5	13.8	76	186	737	2950	11750

Then we plot figures for each n to see the behavior of the magnetization versus time for *n*=1,3,5,7,9,11, we show for *T=T_c* the middle line, the lower line belongs *T>T_c*, and the higher line to *T < T_c* as figures(5-8) listed below for *m*=1 and selected *n*:
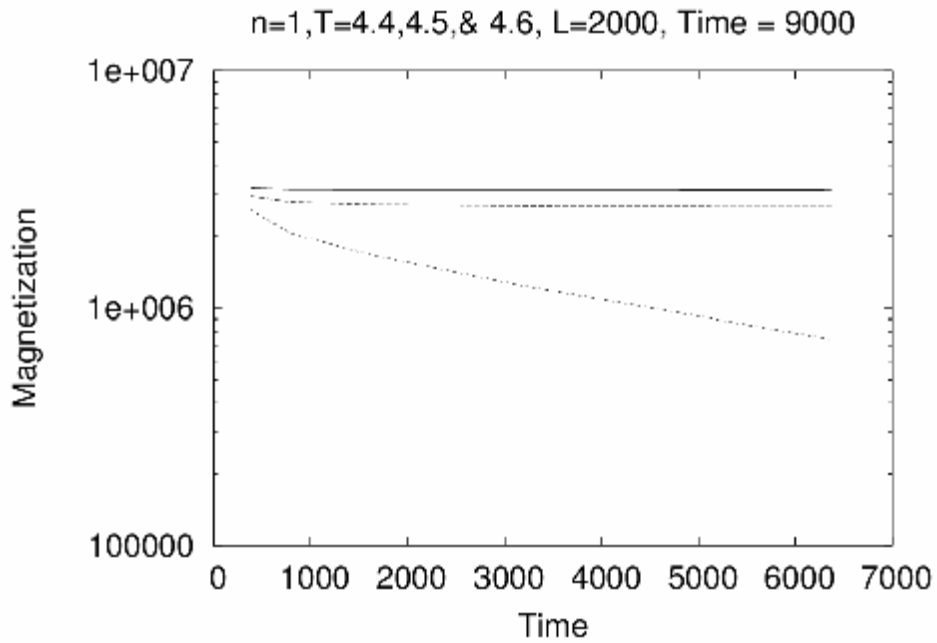

Figure (5) : Magnetization versus time   with temperature = 4.4(up), 4.5(middle) critical temperature, and 4.6 (down)

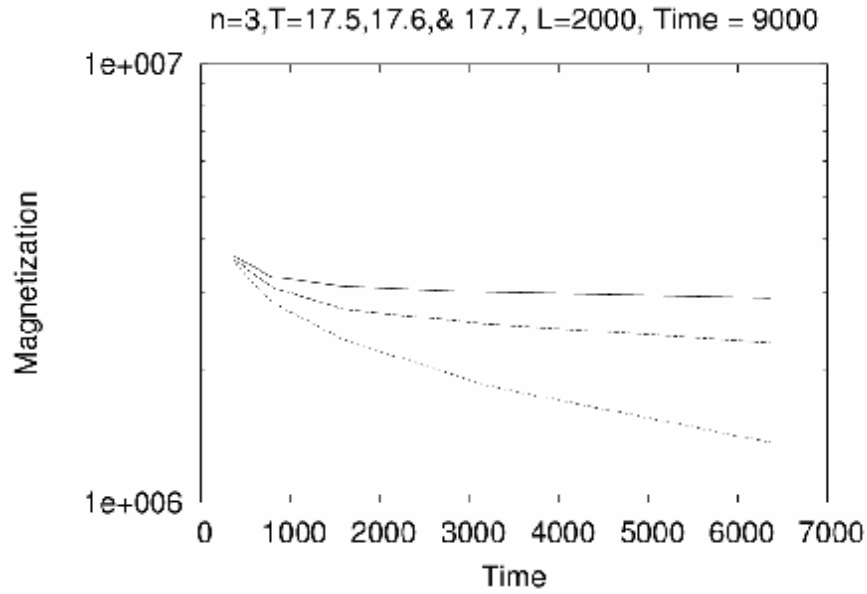**Highly Nonlinear Ising Model and Social Segregation**

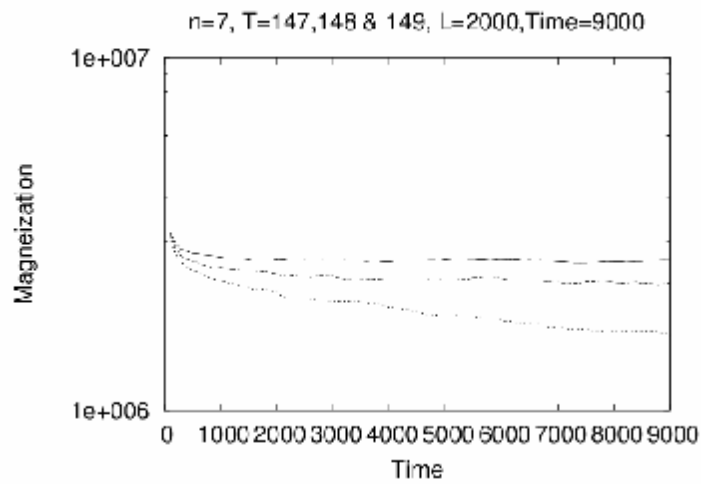**Figure (6)** : Magnetization versus time with temperature = 17.5(up), 17.6(middle) critical temperature, and 17.7(below).

Figure (7) : Magnetization versus time with temperature = 147(up), 148(middle) critical temperature, and 149(below).
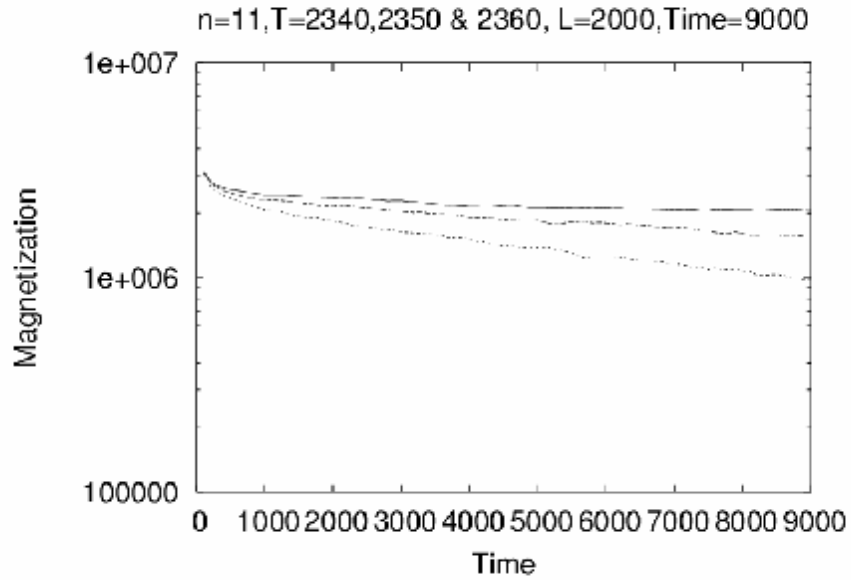
Figure (8) : Magnetization versus time with temperature = 2340(up), 2350(middle) critical temperature, and 2360(below).

Finally we had shown numerically that for n=1, T_c is exactly proportional to $1+m$, which is in accordance with the energy equation (eqn. 3) since the energy is proportional to ($S+mS$), while for $n \gg 1$ T_c is roughly proportional to m, since the term S^n dominates the energy, so the term S can be neglected leading to T_c roughly proportional to m. or we explain it as:

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we know that the interaction energy

$$E \propto \left(S_j + mS_j{}^n \right) \text{---------(4)}$$

So at $n = 1$ equation (1) becomes $E \propto \left(S_j + mS_j \right) \Rightarrow E \propto \left[S_j (1+m) \right]$

Thus $E \propto (1+m)$,

and we know that $E \propto T_c$

$\therefore T_c \propto (1+m)$.

And for $n \mathbf{f} 1$ the term of energy is :

$$E \propto \left(S_j + mS_j{}^n \right), \text{so the term} S_j{}^n \text{ is very high comparing with the term } S_j$$

Thus we can neglect the term S_j.

$$\therefore E \propto \left(mS_j{}^n \right), E \propto (m) \Rightarrow T_c \propto (m).$$

And if we take the logarithm for the equation $E \propto \left(mS_j{}^n \right) \Rightarrow \text{E=constant} \left(mS_i{}^n \right)$,

$E = \text{constant} \left(S_j{}^n \right) \text{at fixed } m, \text{ we get}$

$$\log(E) = \log(\text{constant}) + n \log S_j \Rightarrow \log(E) = (\text{constant}) + n \log S_j$$

Since the temperature enters for large n only through the ratio mS^n/T, it scales exponentially with n: T_c proportional to $m*\exp(n/\text{const})$ for large n.

And at a fixed m, for large n we found T_c varies exponentially with n.
From our simulation we get the magnetization, normalized by the number of sites, and plot it versus T/T_c, at 0.5 T_c to T_c for each n , size of lattice 2000, time =9000, and put the value of the field=0.001; thus we get figure 9.
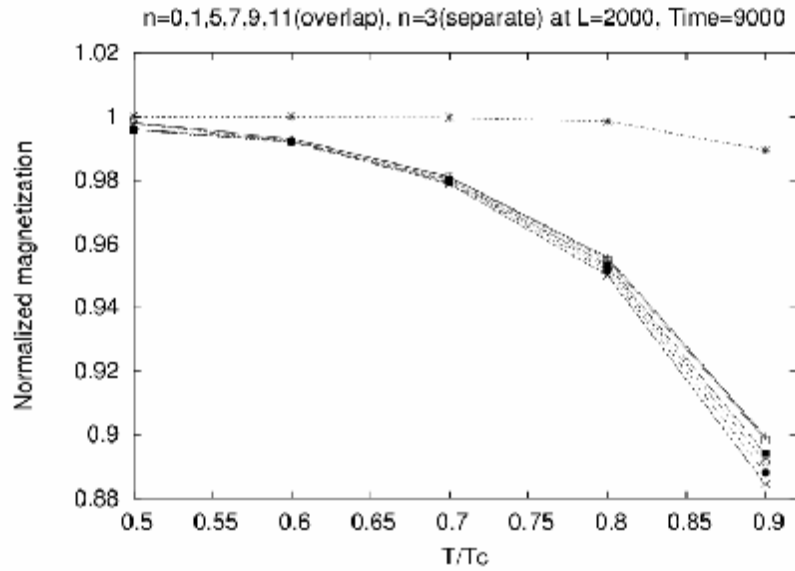
Figure (9) : Normalized magnetization versus *T/T$_c$* for 0.5*T$_c$* to *T$_c$* for each *n*.

But we see that *n*= 3 only does not agree with all values of *n*=0,1,5,7,9,11, and it's deviation from the others which is not understood.
 Finally we plot *T$_c$* at different strengths *J'/J=m* for each *n* as in figure10.

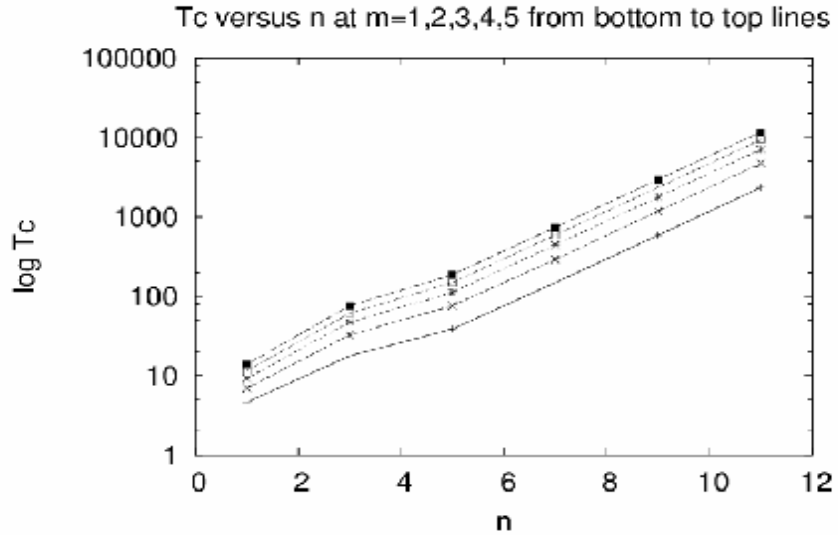**Highly Nonlinear Ising Model and Social Segregation**

Figure (10) : Critical temperatures versus n for m=1,2,3,4,5.

From this figure we find that for a fixed m, for large n, the critical temperature T_c varies exponentially with n, i.e. :
log $T_c = a\ n$ + constant, where a is the slope, which agrees with an Arrhenius law.

Conclusions:

- We assumed that residences are either cheap or expensive, randomly distributed over the square lattice, and that two groups of people, rich and poor, make up the population. We found that for small fields after a long time the domains are larger than for large fields, in this random-field Ising model of urban segregation. Housing price differences do not prevent segregation if they are not very large.
- We conclude that n=0 and n=1,5,7,9,11 are similar while n=3 differs, in violation of the universality principle and the law of corresponding states. However, in all cases we have a Curie temperature with phase separation or long-range segregation only below this Curie temperature.
- We found numerically that   for n=1 T_c is proportional to 1+m, while for $n \gg 1$ T_c is roughly proportional to m. And at a fixed m, for large n , T_c varies exponentially with n.

Acknowledgment

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