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Centralizing in a Complex Banach Algebra

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Abstract In this paper we show that for any subset B of a complex normed algebra A, (i) the quasi centralizer of B is the intersection of ρ and σ-quasi centralizer of B, and (ii) quasi, ρ and σ-quasi center of A are subsets of quasi, ρ and σ -quasi centralizer of B, respectively. We give an example to show that the subsets above may be proper subsets. But if B is a dense subset of A, then the equality in (ii) holds for the corresponding sets. The example also shows that QC(1,B), a part of the quasi centralizer of B (with k =1) need not equal the centralizer of B, but we prove that equality holds under certain conditions. Also we generalize some of the results in [5] and [1].

In this paper we study centralizing in a complex Banach algebra and we generalize some results related to centrality in a complex Banach algebra that was obtained by Le Page in [5] and As'ad and Sarsour in [1].

Introduction

Throughout this paper all linear spaces and algebras are assumed to be defined over ¢, the field of complex numbers.

Let A be any complex normed algebra, then we denote the center of A by

$Z(A) = \{ a \in A : ax = xa \text{ for all } x \in A \}$, and the centralizer of a subset B of A by

$C(B) = \{ a \in A : ax = xa \text{ for all } x \in B \}$. For $a \in A$, the spectrum in A of a will be

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written $\sigma_A(a)$ and the resolvent set, its complement, will be denoted by $\rho_A(a)$.

In [6] Rennison defined the set of all quasi central elements in a complex Banach algebra A by $Q(A) = \bigcup_{k\geq 1} Q(k, A)$,

where $Q(k, A) = \{ a \in A : \| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in A$ and all $\lambda \in ¢ \}$.

Also he defined the set of all σ-quasi central elements in A by $Q_\sigma (A) = \bigcup_{k\geq 1} Q\sigma(k, A)$,

where $Q_\sigma(k,A) = \{ a \in A: \| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in A$ and all $\lambda \in \rho_A (a)\}$.

In [3] we defined the set of all ρ-quasi central elements in A by $Q_\rho(A) = \bigcup_{k\geq 1} Q_\rho(k, A)$, where $Q_\rho(k,A) = \{ a \in A: \| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in A$ and all $\lambda \in \sigma_A(a)\}$.

Similarly we define the following three concepts, let B be a subset of a complex nomed algebra A, then

1) The quasi centralizer (quasi-commutant) of B is $QC(B) = \bigcup_{k\geq 1} QC(k, B)$, where $QC(k, B) = \{ a \in A : \| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in B$ and all $\lambda \in ¢ \}$.

2) The σ-quasi centralizer (σ -quasi-commutant) of B is $QC_\sigma(B) = \bigcup_{k\geq 1} QC_\sigma(k, B)$, where $QC_\sigma (k, B) = \{ a \in A: \| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in B$ and all $\lambda \in \rho_A(a) \}$.

3) The ρ-quasi centralizer (ρ-quasi commutant) of B is $QC_\rho(B) = \bigcup_{k\geq 1} QC_\rho(k, B)$, where $QC_\rho (k, B) = \{ a \in A: \| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in B$ and all $\lambda \in \sigma_A(a) \}$.

2. The Relation Between Centralizing and Centrality in a Complex Banach Algebra

We start by a theorem that is an elementary consequence of our definitions of quasi, σ and ρ-quasi centralizer in a complex normed Algebra.

<u>2.1 Theorem:-</u> *If A is a complex normed algebra and $D \subseteq B \subseteq A$. Then for $k \geq 1$,*

(i) $C(B) \subseteq QC(k, B) = QC_\sigma(k, B) \cap QC_\rho(k, B)$.

(ii) $Q(k, A) = QC(k, A) \subseteq QC(k, B) \subseteq QC(k, D)$.

(iii) $Q_\sigma(k, A) = QC_\sigma(k, A) \subseteq QC_\sigma(k, B) \subseteq QC_\sigma(k, D)$.

(iv) $Q_\rho(k, A) = QC_\rho(k, A) \subseteq QC_\rho(k, B) \subseteq QC_\rho(k, D)$.

Proof :

Left to the reader

<u>2.2 Proposition :-</u> *If A is a complex normed algebra such that $A = \bigcup_{i=1}^{n} B_i$, then*

(i) $\bigcap_{i=1}^{n} QC(B_i) = Q(A)$.

(ii) $\bigcap_{i=1}^{n} QC_\sigma(B_i) = Q_\sigma(A)$.

(iii) $\bigcap_{i=1}^{n} QC_\rho(B_i) = Q_\rho(A)$.

Proof:

We prove (iii) and omit the similar proofs of (i) and (ii).

Let $a \in \bigcap_{i=1}^{n} QC_\rho(B_i)$, then $a \in QC_\rho(B_i)$ for all i. However, $a \in QC_\rho(B_i)$ means

that there exists $k_i \geq 1$ such that $\|x(\lambda - a)\| \leq k_i \|(\lambda - a)x\|$ for all $x \in B_i$ and all $\lambda \in \sigma_A(a)$.

Then there exists $k = \sup\{ k_i : 1 \leq i \leq n \} \geq 1$ such that for all B_i, we have,

$\| x(\lambda - a) \| \leq k\| (\lambda - a) x \|$ for all $x \in B_i$ and all $\lambda \in \sigma_A(a)$. So that

$\| x(\lambda - a) \| \leq k\| (\lambda - a) x \|$ for all $x \in \bigcup_{i=1}^{n} B_i$, and all $\lambda \in \sigma_A(a)$, then $a \in Q_\rho(A)$.

Hence $\bigcap_{i=1}^{n} QC_\rho(B_i) \subseteq Q_\rho(A)$. By Theorem 2.1 (iv), $Q_\rho(A) \subseteq QC_\rho(B_i)$ for all i.

Hence $Q_\rho(A) \subseteq \bigcap_{i=1}^{n} QC_\rho(B_i)$. Therefore, $Q_\rho(A) = \bigcap_{i=1}^{n} QC_\rho(B_i)$

<u>**2.3 Proposition :-**</u> if A *is a complex normed algebra. Then for* $k \geq 1$,

(i) $\bigcap_{x \in A} QC(k, \{x\}) = Q(k, A)$.

(ii) $\bigcap_{x \in A} QC_\sigma(k, \{x\}) = Q_\sigma(k, A)$.

(iii) $\bigcap_{x \in A} QC_\rho(k, \{x\}) = Q_\rho(k, A)$.

Proof:

We prove (iii) and omit the similar proofs of (i) and (ii).

Let $a \in \bigcap_{x \in A} QC_{\rho}(k, \{x\})$, then $a \in QC_{\rho}(k, \{x\})$ for all $x \in A$.

However,

$a \in QC_{\rho}(k, \{x\})$ means that $\| x(\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $\lambda \in \sigma_A(a)$.

Hence $\| x(\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in A$ and all $\lambda \in \sigma_A(a)$, then $a \in Q_{\rho}(k, A)$. Hence $\bigcap_{x \in A} QC_{\rho}(k, \{x\}) \subseteq Q_{\rho}(k, A)$.

By Theorem 2. l(iv), $Q_{\rho}(k, A) \subseteq QC_{\rho}(k, \{x\})$ for all $x \in A$, and so $Q_{\rho}(k, A) \subseteq \bigcap_{x \in A} QC_{\rho}(k, \{x\})$. Therefore, $Q_{\rho}(k, A) = \bigcap_{x \in A} QC_{\rho}(k, \{x\})$

Similarly one can prove the following remark.

<u>Remark :-</u> *If A is a complex normed algebra and $\Im$ is a collection of subsets of A such that $A = \bigcup_{B \in \Im} B$. Then for $k \geq 1$,*

(i) $\bigcap_{B \in \Im} QC(k, B) = Q(k, A)$.

(ii) $\bigcap_{B \in \Im} QC_{\sigma}(k, B) = Q_{\sigma}(k, A)$.

(iii) $\bigcap_{B \in \Im} QC_{\rho}(k, B) = Q_{\rho}(k, A)$.

In Example 2.5 below we show that QC(B) need not be a subset of Q(A)
(the same can be said about σ and ρ-quasi center and centralizer). But the following Theorem shows that equality holds under certain conditions.

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2.4 Theorem :- *If A is a complex normed algebra and B is a dense subset of A, then*

for $k \geq 1$,

(i) $Z(A) = C(B)$.

(ii) $Q(k, A) = QC(k, B)$.

(iii) $Q_\sigma(k, A) = QC_\sigma(k, B)$.

(iv) $Q_\rho(k, A) = QC_\rho(k, B)$.

Proof :-

We prove (iii) and omit the similar proofs of (i), (ii) and (iv).

First of all note that $Q_\sigma(k, A) \subseteq QC_\sigma(k, B)$, see Theorem 2.1 (iii). Conversely let

$a \in QC_\sigma(k, B)$, then $\| x(\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in B$ and all $\lambda \in \rho_A(a)$. However, for any fixed $y \in A$, there exists a sequence (y_n) of elements of B such that $\text{Lim}\,(y_n) = y$, then $\| y_n (\lambda - a) \| \leq k \| (\lambda - a) y_n \|$ for all $n \in N$ and all $\lambda \in \rho_A(a)$. Then by the continuity of the norm we have, $\| y(\lambda - a) \| \leq k \| (\lambda - a) y \|$ for all $\lambda \in \rho_A(a)$. However, y is arbitrary in A, then $a \in Q_\sigma(k, A)$. Hence $QC_\sigma(k, B) \subseteq Q_\sigma(k, A)$. Therefore, $Q_\sigma(k, A) = QC_\sigma(k, B)$

In [1] we have shown that $Q(1,A) = Q_\sigma(1,A) = Z(A)$, where A is a complex Banach algebra with unity. But the following example shows that it is not the case for the quasi centralizer, where $QC(1,B)$ need not equal

C(B) and $QC_\sigma(1,B)$ need not equal C(B), but under certain conditions all of these three sets are equal, as we shall see in corollary 1 of Theorem 2.6.

The inclusions in Theorem 2.1 may be proper, where the following example explains this idea. Also our example shows that the quasi centralizer element need not be quasi central. The same things can be said a bout (the ρ and the σ)-quasi (center and centrelizer).

<u>2.5 Example :-</u> There is *a complex Banach algebra A and a closed subalgebra B of A with*

(i) $a \in QC(1, B)$, *but* $a \notin C(B)$.

(ii) $a \in QC(B)$, *but* $a \notin Q(A)$.

(iii) $a \in QC_\rho(B)$, *but* $a \notin Q_\rho(A)$.

(iv) $a \in QC_\sigma(B)$, *but* $a \notin Q_\sigma(A)$.

Construction :-

Let $A = \{ \beta = \begin{pmatrix} x & y \\ z & w \end{pmatrix} : x, y, z, w \in ¢ \}$ and define $\|\beta\| = \max\{ |x| + |y|, |z| + |w| \}$, which makes A a unital complex Banach algebra. Let $B = \{\alpha = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in ¢\}$ and let $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $a \notin C(B)$.

Now for any $\lambda \in ¢$, and any $\alpha \in B$, we have $\| \alpha (\lambda - a) \| = \| \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ -1 & \lambda \end{pmatrix} \| =$

$\| \begin{pmatrix} x\lambda & 0 \\ 0 & 0 \end{pmatrix} \| = | x\lambda |$, and $\| (\lambda - a) \alpha \| = \| \begin{pmatrix} \lambda & 0 \\ -1 & \lambda \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \| = \| \begin{pmatrix} \lambda x & 0 \\ -x & 0 \end{pmatrix} \| =$

$\max \{ | x\lambda |, | x | \}$. Hence $a \in QC(1,B)$, then $a \in QC(B)$ and by theorem 2.1(i)

$a \in QC_\sigma(1, B)$.

It is easy to see that $\| \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \| = 1$, and $\| \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \| = 0$.

Hence there exist $\beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A$ and $\lambda = 0$ such that $\| \beta (\lambda - a) \| > k \| (\lambda - a) \beta \|$ for all $k \geq 1$. Therefore, $a \notin Q(A)$

Since $a \in QC(B)$, then by Theorem 2.1(i), $a \in QC_\rho(B)$. Note that, $\sigma_A(a) = \{ \lambda \in ¢ : \begin{pmatrix} \lambda & 0 \\ -1 & \lambda \end{pmatrix}^{-1}$ does not exist $\} = \{0\}$. Hence as above there exist $\beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A$, and $\lambda = 0 \in \sigma_A(a)$ such that, $\|\beta (\lambda - a)\| > k\|(\lambda - a) \beta\|$ for all $k \geq 1$. Therefore, $a \notin Q_\rho(A)$. Again since $a \in QC(B)$, then by theorem 2.1(i) $a \in QC_\sigma(B)$. By the countablity of $\sigma_A(a)$ and [3, Corollary], $a \notin Q_\sigma(A)$

Now we prove our theorem that is stronger than both Theorem 5.1 in [1] and Le Page's Proposition in [5], where the theorem and the proposition appear in this paper as corollaries 3 and 4, respectively. And it would be noted here that the proof of Theorem 2.6 below is similar to the proof of Le Page's Proposition.

2.6 Theorem :- *Let J be an ideal of a unital complex Banach algebra A and let $a \in A$ be such that $\| x(\lambda - a)\| \le \|(\lambda - a)x\|$ for all $x \in J$ and all $\lambda \in ¢$ satisfying $|\lambda| > \|a\|$. Then $a \in C(J)$*

Proof :-

Fix any $u \in ¢$ with $u \neq 0$ and let n be any positive integer such that $n|u|^{-1} > \|a\|$. Then by [4, pp.400], $nu^{-1} \in \rho_A(a)$ and so by the definition of $\rho_A(a)$ we have $(nu^{-1} - a)^{-1} \in A$. Then $x = (nu^{-1} - a)^{-1} y \in J$ for all $y \in J$. But by assumption $\| x(\lambda - a)\| \le \|(\lambda - a)x\|$ for all $x \in J$ and all $\lambda \in ¢$ that satisfies $|\lambda| > \|a\|$. Hence for any $y \in J$ (take $\lambda = nu^{-1}$ and $x = (nu^{-1} - a)^{-1}y$), we have
$\|(nu^{-1} - a)^{-1} y (nu^{-1} - a)\| \le \|(nu^{-1} - a)(nu^{-1} - a)^{-1} y\| = \|y\|$, hence
$\|(e - un^{-1}a)^{-1} y (e - un^{-1}a)\| \le \|y\|$, where e is the unity of A.

Now, by induction we have that $\|(e - un^{-1}a)^{-m} y (e - un^{-1}a)^{m}\| \le \|y\|$ for all natural numbers m. In particular for $m = n$, we have,
$\|(e - un^{-1}a)^{-n} y (e - un^{-1}a)^{n}\| \le \|y\|$. Take the limit as $n \to \infty$, and use the continuity of the norm to get that $\|\exp(ua)\, y \exp(-ua)\| \le \|y\|$ .... (1).
Since $u \neq 0$ was arbitrary fixed complex number and (1) is true for $u = 0$, then we can define $f : ¢ \to A$ by $f(u) = \exp(ua)y\exp(-ua)$, where y is any fixed element in J. Then f is a bounded entire function (see(l)).

Now by Liouville's theorem we have f as a constant function, so that f(u) = y. But y was arbitrary fixed element in J, so that $\exp(ua)y\exp(-ua) = y$ for all $y \in J$, then $\exp(ua)y = y\exp(ua)$ for all $y \in J$. Hence, $\sum_{n=0}^{\infty} \frac{(ua)^n}{n!} y = y \sum_{n=0}^{\infty} \frac{(ua)^n}{n!}$ for all $y \in J$ and all $u \in ¢$. So that $ay = ya$ for all $y \in J$. Hence $a \in C(J)$

Corollary 1 *Let J be an ideal of a unital complex Banach algebra A. Then, $C(J) = QC(1, J) = QC_\sigma(1, J)$.*

Proof:-

First note that $C(J) \subseteq QC(1, J) \subseteq QC_\sigma(1, J)$, by Theorem 2.1(i).

Now let $a \in QC_\sigma(1,J)$, then $\|x(\lambda - a)\| \le \|(\lambda - a)x\|$ for all $x \in J$ and all $\lambda \in \rho_A(a)$. But by [4, pp.400], $\{\lambda \in ¢ : |\lambda| > \|a\|\} \subseteq \rho_A(a)$. Then

$\| x(\lambda - a) \| \le \| (\lambda - a) x \|$ for all $x \in J$ and all $\lambda \in ¢$ that satisfies $|\lambda| > \| a \|$. Then by Theorem 2.6, $a \in C(J)$, hence $QC_\sigma(1, J) \subseteq C(J)$. Therefore, $C(J) = QC(1, J) = QC_\sigma(1, J)$

Let A be a complex Banach algebra with unity and a.b denote the reversed product on A, that is a.b = ba for all $a,b \in A$. With the reversed product and the given norm on A, A becomes a complex Banach algebra with unity called the reversed algebra of A and is denoted by rev(A) [2, pp6].

<u>Corollary 2</u> *Let J be an ideal of a complex Banach algebra A with unity. Then,* $QC(l, J) = QC(l, rev(J)) = QC_\sigma(1, rev(J)) = QC_\sigma(1, J)$.

Proof: -

By Corollary 1, $C(J) = QC(1, J) = QC_\sigma(1, J)$. Since rev(J) is an ideal of rev(A), then by Corollary 1, $C(rev(J)) = QC(1, rev(J)) = QC_\sigma(1, rev(J))$. However, $C(rev(J)) = C(J)$. Hence $QC(l, J) = QC(l, rev(J)) = QC_\sigma(1, rev(J)) = QC_\sigma(1, J)$

<u>Corollary 3</u> [1, Theorem 5.1]

Let A be a Banach algebra with unity over the complex field ¢ and let $a \in A$ *be such that* $\| x(\lambda - a) \| \le \| (\lambda - a) x \|$ *for all* $x \in A$ *and all* $\lambda \in ¢$ *that satisfies* $|\lambda| > \| a \|$. *Then* $a \in Z(A)$.

Proof:-

Take J = A in Theorem 2.6 to get the result

<u>Corollary 4</u> [5, Proposition 1.1]

Let A be a complex Banach algebra with unity and let $a \in A$ *such that* $\| x(\lambda - a) \| \le \| (\lambda - a) x \|$ *for all* $x \in A$ *and all* $\lambda \in ¢$. *Then* $a \in Z(A)$.

Proof: -

Follows directly from corollary 3

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