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Distributions of Generalized Order Statistics and Parameters Estimation of Pareto Distribution in Statistical Explicit Forms

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Abstract

We study some distributions of generalized order statistics (GOS) for Pareto distribution. In particular, we have derived the joint probability density function (pdf) for GOS from Pareto distribution in statistical explicit form. In addition, the joint pdf of both range and midrange for GOS from Pareto distribution is obtained. The estimation of Pareto distribution parameters based on GOS using the method of Maximum Likelihood Estimators (MLE) have been derived in explicit forms. Furthermore, some special cases have been discussed.

Keywords Pareto Distribution, Generalized Order Statistics, Ordinary Order Statistics, Maximum Likelihood Estimation.

توزيعات الإحصاءات المرتبة المعممة وتقدير معالم توزيع باريتو في صيغ إحصائية محددة

ملخص

في هذه البحث تم دراسة توزيعات الإحصاءات المرتبة المعممة لتوزيع باريتو. حيث تم اشتقاق دالة الكثافة الاحتمالية المشتركة لتلك الإحصاءات لتوزيع باريتو وذلك في صيغة إحصائية محددة. بالإضافة تم اشتقاق دالة الكثافة الاحتمالية المشتركة للمدى وللمدى المتوسط في تلك الحالة. كذلك تم تقدير معالم توزيع باريتو للإحصاءات المرتبة المعممة باستخدام طريقة الإمكان الأعظم، وتم كذلك مناقشة بعض الحالات الخاصة.

كلمات مفتاحية: . توزيع باريتو، الإحصاءات المرتبة المعممة، الإحصاءات المرتبة العادية، التقدير باستخدام الإمكان الأعظم.

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1. Introduction

The Pareto distribution is originally found by the Italian economist Vilfredo Pareto. The Pareto distribution is a power law probability distribution found in a large number of real-world situations.

The family of Pareto distributions is parameterized by two parameters, υ>0 and θ >0. This distribution was originally used to describe the allocation of wealth among individuals since it seems to show rather well the way that a larger portion of the wealth of any society is owned by a smaller percentage of the people in that society.

Many authors have been studied the Generalized Order Statistics (GOS) and its distributions, see for example, Güngör et al. (2009), Kamps (1995), Kamps, and Cramer (1999).

GOS concept was introduced by Kamps (1995) as a unified approach to several models of ordered random variables such as upper order statistics, upper record values, sequential order statistics, ordering via truncated distributions, censoring schemes, among others. Ateya & Ahmad (2011), Jaheen (2005), Habibullah & Ahsanullah (2000), Raqab & Ahsanullah (2011) among others, utilized the GOS in their works.

Abu El-Fotouh & Nassar (2011) have investigated the estimation problem for the unknown parameters of Weibull extension model based on GOS by Maximum Likelihood Estimators (MLE). Alkasasbeh & Raqab (2009) considered the MLE of the different parameters of a generalized logistic distribution and compared the performances of these procedures through an extensive numerical simulation.

Adler, A. (2006) considered independent and identically distributed random variables $X_{nk}, 1 \le k \le m, n \ge 1$ from the Pareto distribution. He randomly selected a pair of order statistics from each row, $X_{n(i)}$ and $X_{n(j)}$, where $1 \le i < j \le m$. He tested whether or not Strong and Weak Laws of Large Numbers with nonzero limits for weighted sums of the random variables $X_{n(j)}/X_{n(i)}$ exist where he placed a prior distribution on the selection of each of these possible pairs of order statistics. He showed that there are some similarities and differences when dealing with ratios of order statistics from the Pareto distribution as to observing just one order statistic from the Pareto. In addition, he showed that if a weak law fails to hold then the corresponding strong law also cannot hold.

Mahmoud, M. et al. (2005) derived the exact explicit expressions for the single, double, triple, and quadruple moments of order statistics from the generalized Pareto distribution (GPD). They obtained the best linear unbiased estimates of the location and scale parameters (BLUE's) of the GPD. In addition, they developed approximate confidence intervals for the generalized Pareto parameters using Edgeworth approximation and compare them with those based on Monte Carlo simulations.

GOS contains a variety of models of ordered random variables with different interpretations. Let *F(x)* denotes continuous distribution function (*cdf*) with density function *f(x)*. The *cdf* or just distribution function, describes the probability that a real-valued random variable *X* with a given probability distribution will be found at a value less than or equal to *x*.

Definition 1.1. The Pareto distribution is frequently used a mode in study of incomes. A random variable X is said to have a Pareto distribution with two parameters ν and θ, if it's *pdf* and *cdf* are given, respectively, by

$$f(x;\theta,\nu) = \frac{\theta \nu^{\theta}}{x^{\theta+1}}, \quad \nu \le x < \infty, \ \theta > 0 \quad , \nu > 0 \qquad (1.1)$$

and

$$F(x;\theta,\nu)=\begin{cases}1-\left(\frac{\nu}{x}\right)^{\theta} & \nu\le x\\ 0 & \text{elsewhere}\end{cases} \tag{1.2}$$

This paper is structured as follows: Sections 2 presents the joint distribution for all GOS from Pareto distribution. The distribution of the sample range and midrange are derived in section 3. Section 4 demonstrates the estimation of Pareto distribution parameters based on GOS using MLE, the main results of this paper are stated and proved; and Section 5 summarizes the important results.

2. Joint Distribution of all Generalized Order Statistics

Definition 2.1. Let *F(x)* denote an absolutely continuous distribution function with density function *f(x)* and $X\left(1,n,\tilde{m},k\right),\ldots,X\left(n,n,\tilde{m},k\right)(k\ge 1,\ m$ is a real number) be '*n*' generalized order statistics. Then the joint probability density function (*pdf*) $f\left(x_1,\ldots,x_n\right)$ can be written as (See for example, Kamps, 1995)

$$f\left(x_1,\ldots,x_n\right)=\begin{cases}k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}\left(1-F\left(x_i\right)\right)^{m_i}f\left(x_i\right)\right]\left(1-F\left(x_n\right)\right)^{k-1}f\left(x_n\right),\\ \qquad\qquad \text{for } F^{-1}(0)<X_1\le X_2\le\ldots\le X_n<F^{-1}(1)\\ 0,\ \text{otherwise}\end{cases} \tag{2.1}$$

with parameters $n\in\mathbb{N}$, $n\ge 2$, $\tilde{m}=\left(m_1,\ldots,m_{n-1}\right)\in\mathbb{R}^{n-1}$, $M_r=\sum_{j=r}^{n-1}m_j$, such that $\gamma_r=k+n-r+M_r>0$ for all $r\in\{1,\ldots,n-1\}$, let $c_{r-1}=\prod_{j=1}^{r}\gamma_j$, $r=1,2,\ldots,n-1$ and $\gamma_n=k$.

GOS based on the standard uniform distribution are denoted by $U\left(r,n,\tilde{m},k\right)$. Choosing the parameters appropriately, models such as ordinary order statistics $\left(\gamma_i=n-i+1,\ i=1,\ldots,n,\ i.e.,m_1=\cdots=m_{n-1}=0\right)$ sequential order statistics $\left(\gamma_i=(n-i+1)\alpha_i;\ \alpha_1,\ldots,\alpha_n>0\right)$, progressive type II censored order statistics $\left(m_i\in\mathbb{N}_0,k\in\mathbb{N}\right)$, and record values $\left(\gamma_i=\beta_i;\beta_1,\ldots,\beta_n>0\right)$ are seen to be particular cases.

Definition 2.2. In Definition 2.1, if $k=1$ and $m_1=m_2=\ldots=m_{n-1}=zero$, $M_j=\sum_{r=j}^{n-1}m_r=zero$, then

$$f\left(x_1,\ldots,x_n\right)=\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n}f\left(x_i\right)\right).$$

$$\left(\prod_{j=1}^{n-1}\gamma_j\right)=\prod_{j=1}^{n-1}(1+n-j)$$
$$=(1+n-1)(1+n-2)(1+n-3)\ldots(1+n-n+1)$$
$$=n(n-1)\ldots 3.2.1=n!.$$

Therefore, the joint pdf of all the ordinary order statistics $f\left(x_1,\ldots,x_n\right)=n!\prod_{i=1}^{n}f\left(x_i\right)$, *which is the well known pdf of all ordinary order statistics*.

Theorem 2.3. The joint pdf of $X\left(1,n,\tilde{m},k\right),\ldots,X\left(n,n,\tilde{m},k\right)$ for Pareto distribution is

$$f\left(x_1,\ldots,x_n\right)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}\frac{\theta}{x_i}\left(\frac{\nu}{x_i}\right)^{\theta(m_i+1)}\right]\left[\frac{\theta}{x_n}\left(\frac{\nu}{x_n}\right)^{\theta k}\right] \tag{2.2}$$

Proof.

Using the *pdf* and *cdf* given in (1.1) and (1.2) in (2.1) we get

$$f\left(x_1,\ldots,x_n\right)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}\left(1-\left[1-\left(\frac{\upsilon}{x_i}\right)^{\theta}\right]\right)^{m_i}\frac{\theta\upsilon^{\theta}}{x_i^{\theta+1}}\right]\left(1-\left[1-\left(\frac{\upsilon}{x_n}\right)^{\theta}\right]\right)^{k-1}\frac{\theta\upsilon^{\theta}}{x_n^{\theta+1}}$$

$$=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}\frac{\theta\nu^{\theta(m_i+1)}}{x_i^{\theta(m_i+1)+1}}\right]\left[\frac{\theta\nu^{\theta k}}{x_n^{\theta k+1}}\right]$$

Collecting terms we get (2.2) and that completes the proof.

We discuss a special case in Corollary 2.4.

Corollary 2.4 (The joint pdf of all ordinary order statistics for Pareto Distribution)

In equation (2.2), let $k=1$ and $m=0,$ then the joint *pdf* of all ordinary order statistics $X\left(1,n,0,1\right),\ldots,X\left(n,n,0,1\right)$ for Pareto distribution is

$$f\left(x_1,\ldots,x_n\right)=\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}\frac{\theta\nu^{\theta}}{x_i^{\theta+1}}\right]\left[\frac{\theta\nu^{\theta}}{x_n^{\theta+1}}\right]$$

$$=\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n}\frac{\theta\nu^{\theta}}{x_i^{\theta+1}}\right]$$

Given (2.1), let $k=1$ and $m=0,$ then $\prod_{j=1}^{n-1}\gamma_j=\prod_{j=1}^{n-1}\left(1+n-j\right)=n!$. Therefore the joint *pdf* of all ordinary order statistics for Pareto Distribution is given by

$$f\left(x_1,\ldots,x_n\right)=n!\prod_{i=1}^{n}\frac{\theta\nu^{\theta}}{x_i^{\theta+1}} \tag{2.3}$$

3. The Sample Range and Midrange Distribution

An important quantity related to $X_{(1)}$ and $X_{(n)}$ is the sample range $R=X_{(n)}-X_{(1)}$, which provides information of how spread out the underlying distribution might be, Gut (1995). The range should reflect the dispersion in the population. The other important quantity related to $X_{(1)}$ and $X_{(n)}$ is the sample midrange $V=\frac{1}{2}\left(X_{(1)}+X_{(n)}\right),$ which is a measure of location like the sample median or the sample mean, Casella and Berger (2002).

Definition 3.1. The joint *pdf* of i^{th} j^{th} generalized order statistics, $X(i,n,\tilde{m},k)$ and $X(j,n,\tilde{m},k)$ is given by Garg (2009),

$$f_{i,j,n,\tilde{m},k}(x_i,x_j)=\frac{c_j}{(i-1)!(j-i-1)!}\left[1-F(x_i)\right]^m\left[1-F(x_j)\right]^{\gamma_j-1}\left[g_m(F(x_i))\right]^{i-1}$$
$$\times\left[g_m(F(x_j))-g_m(F(x_i))\right]^{j-i-1}f(x_i)f(x_j), \tag{3.1}$$

for $0<x_i<x_j<\infty, 1\le i<j\le n$,

where, $c_r=\prod_{j=1}^{r}\gamma_j$,

$\gamma_j=k+(n-j)(m+1)$

and

$$g_m(x)=\begin{cases}\dfrac{1-(1-x)^{m+1}}{m+1} & ,m\neq -1.\\ -\ln(1-x) & ,m=-1, x\in(0,1)\end{cases}$$

Since $\lim_{m\to -1}\frac{1-(1-x)^{m+1}}{m+1}=-\ln(1-x)$, we shall write $g_m(x)=\frac{1-(1-x)^{m+1}}{m+1}$ for all $x\in(0,1)$ and for all *m* with $g_{-1}(x)=\lim_{m\to -1}g_m(x)$.

Corollary 3.2. (The joint *pdf* of two ordinary order statistics)

In Definition (3.1) , if $k=1$ and $m=0$, then

$$f_{i,j,n,0,1}(x_i,x_j)=\frac{c_j}{(i-1)!(j-i-1)!}\left[1-F(x_j)\right]^{\gamma_j-1}\left[g_0(F(x_i))\right]^{i-1}\times$$
$$\left[g_0(F(x_j))-g_0(F(x_i))\right]^{j-i-1}f(x_i)f(x_j).$$

$g_0(F(x_i))=F(x_i), g_0(F(x_j))=F(x_j).$

$\gamma_j=n-j+1,\ c_j=\prod_{t=1}^{j}\gamma_t=\left[1+(n-1)\right]\left[1+(n-2)\right]...\left[1+(n-j)\right]=\frac{n!}{(n-j)!}.$

Then

$$f_{i,j,n,0,1}(x_i,x_j)=\frac{c_j}{(i-1)!(j-i-1)!}\left[1-F(x_j)\right]^{n-j}\left[F(x_i)\right]^{i-1}\times$$
$$\left[F(x_j)-F(x_i)\right]^{j-i-1}f(x_i)f(x_j),$$

for $-\infty<x_i<x_j<\infty,\ 1\le i<j\le n,$

which is the well known joint pdf of two ordinary order statistics x_i *and* x_j.

Theorem 3.3. Let $X(1,n,\tilde{m},k),X(2,n,\tilde{m},k),...,X(n,n,\tilde{m},k)$ denote the GOS of a random sample, $X_1,X_2,...,X_n$, from a continuous population with *cdf* $F_X(x)$ and pdf $f_X(x)$. Then the joint *pdf* of the sample range R and midrange V is

$$f_{R,V,n,m,k}(r,v)=\frac{c_n}{(n-2)!}\left[1-F\left(v-\tfrac{r}{2}\right)\right]^m\left[1-F\left(v+\tfrac{r}{2}\right)\right]^{k-1}$$
$$\times\left[g_m\left(F\left(v+\tfrac{r}{2}\right)\right)-g_m\left(F\left(v-\tfrac{r}{2}\right)\right)\right]^{n-2}f\left(v-\tfrac{r}{2}\right)f\left(v+\tfrac{r}{2}\right),\quad v\le x_{(1)}<x_{(n)}<\infty$$

where, $\left[g_m\left(F\left(v+\tfrac{r}{2}\right)\right)-g_m\left(F\left(v-\tfrac{r}{2}\right)\right)\right]^{n-2}=\left[\dfrac{\left(1-F\left(v-\tfrac{r}{2}\right)\right)^{m+1}-\left(1-F\left(v+\tfrac{r}{2}\right)\right)^{m+1}}{m+1}\right]^{n-2}$

and $c_n=\prod_{j=1}^{n}\gamma_j$.

We discuss a special case in Corollary 3.4.

Corollary 3.4. (The joint pdf of R and V of the ordinary order statistics)

For *k=1* and *m=0,* then $c_n=\prod_{j=1}^{n}\gamma_j$, $\gamma_n=k=1$, $c_n=n.(n-1)\ldots 3.2.1=n!$

$$\left[g_0\left(F\left(v+\tfrac{r}{2}\right)\right)-g_0\left(F\left(v-\tfrac{r}{2}\right)\right)\right]^{n-2}=\left[F\left(v+\tfrac{r}{2}\right)-F\left(v+\tfrac{r}{2}\right)\right]^{n-2}$$

Then,

$$f_{R,V,n,0,1}(r,v)=n(n-1)\left[F\left(v+\tfrac{r}{2}\right)-F\left(v-\tfrac{r}{2}\right)\right]^{n-2}f\left(v-\tfrac{r}{2}\right)f\left(v+\tfrac{r}{2}\right)$$

Which is the joint *pdf* of the sample range R and midrange V of the ordinary order statistics $X_{(1)},X_{(2)},\ldots,X_{(n)}$.

Theorem 3.5. The joint *pdf* of the sample range and midrange of Pareto distribution

$$f_{R,V,n,m,k}(r,v)=\frac{c_n}{(n-2)!}\left(\frac{\upsilon}{v-\frac{r}{2}}\right)^{\theta m}\left(\frac{\upsilon}{v+\frac{r}{2}}\right)^{\theta(k-1)}\left(\frac{\left(\theta\upsilon^{\theta}\right)^2}{\left[\left(v-\frac{r}{2}\right)\left(v+\frac{r}{2}\right)\right]^{\theta+1}}\right)$$
$$\times\left[g_m\left(F\left(v+\tfrac{r}{2}\right)\right)-g_m\left(F\left(v-\tfrac{r}{2}\right)\right)\right]^{n-2},$$

where,

$$\left[g_m\left(F\left(v+\tfrac{r}{2}\right)\right)-g_m\left(F\left(v-\tfrac{r}{2}\right)\right)\right]^{n-2}=\frac{1}{m+1}\left[\left(\frac{v}{v-\frac{r}{2}}\right)^{\theta(m+1)}-\left(\frac{v}{v+\frac{r}{2}}\right)^{\theta(m+1)}\right]^{n-2}$$

We discuss a special case in Corollary 3.6.

Corollary 3.6. (The joint pdf of R and V of the ordinary order statistics for Pareto Distribution)

For *k=1* and *m=0,* then $c_n=\prod_{j=1}^{n}\gamma_j$, $\gamma_n=k=1$, $c_n=n.(n-1)\ldots 3.2.1=n!$

$$\left[g_0\left(F\left(\nu+\tfrac{r}{2}\right)\right)-g_0\left(F\left(\nu-\tfrac{r}{2}\right)\right)\right]^{n-2}=\left[\left(\frac{\nu}{\nu-\frac{r}{2}}\right)^{\theta}-\left(\frac{\nu}{\nu+\frac{r}{2}}\right)^{\theta}\right]^{n-2}$$

Then,

$$f_{R,V,n,0,1}(r,\nu)=n(n-1)\left(\frac{\left(\theta\upsilon^{\theta}\right)^{2}}{\left[\left(\nu-\frac{r}{2}\right)\left(\nu+\frac{r}{2}\right)\right]^{\theta+1}}\right)\left[\left(\frac{\nu}{\nu-\frac{r}{2}}\right)^{\theta}-\left(\frac{\nu}{\nu+\frac{r}{2}}\right)^{\theta}\right]^{n-2}$$

Which is the joint *pdf* of the sample range R and midrange V of the ordinary order statistics $X_{(1)},X_{(2)},\dots,X_{(n)}$.

4. Estimation of Pareto Distribution Parameters Based on Generalized Order Statistics Using MLE

Theorem 4.1. Suppose $X(1,n,\tilde{m},k),X(2,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)$ denote the GOS of a Pareto distribution with pdf is given in (1.1). Then the MLE of ν and θ are given by

$$\hat{\nu}=x_{(1)}=\min x_{(i)}, \tag{4.1}$$

and

$$\hat{\theta}=n\left[\sum_{i=1}^{n-1}(m_i+1)\ln\left(\frac{x_i}{x_{(1)}}\right)+k\ln\left(\frac{x_n}{x_{(1)}}\right)\right]^{-1}, \tag{4.2}$$

respectively.

Proof. *The MLE for the parameter ν:* For fixed θ, $L(\theta,\nu|\text{x})$ is an increasing function of ν so it attains its maximum for large value of ν, $\nu>0$. We have $\nu\le x<\infty$, $\nu\le x_{(1)}\le x_{(2)}\le\dots\le x_{(n)}<\infty$. So both restricted and unrestricted MLEs of ν are $\hat{\nu}=x_{(1)}$. Therefore, MLE for ν is $\hat{\nu}=x_{(1)}=\min x_{(i)}$.

MLE for the parameter θ: For fixed ν,

$$ln\left[L(\theta,\nu;\mathbf{x})\right]=\log L=\ln k+\sum_{j=1}^{n-1}\ln\gamma_j+\left[\sum_{i=1}^{n-1}\ln\theta-\ln x_i+\theta(m_i+1)\ln\left(\frac{\nu}{x_i}\right)\right]+$$
$$\theta k\ln\left(\frac{\nu}{x_n}\right)-\ln(x_n)+\ln\theta$$

$$\frac{\partial ln\left[L(\theta,\nu;\mathbf{x})\right]}{\partial\theta}=\sum_{i=1}^{n-1}\left[\frac{1}{\theta}-(m_i+1)\ln\left(\frac{\nu}{x_i}\right)\right]+k\ln\left(\frac{\nu}{x_n}\right)+\frac{1}{\theta}$$

To find the MLE of θ, set

$$\sum_{i=1}^{n-1}\left[\frac{1}{\theta}-(m_i+1)\ln\left(\frac{\nu}{x_i}\right)\right]+k\ln\left(\frac{\nu}{x_n}\right)+\frac{1}{\theta}=0,$$

and solve for θ yielding

$$\theta = n\left[\sum_{i=1}^{n-1}(m_i+1)\ln\left(\frac{x_i}{\nu}\right)+k\ln\left(\frac{x_n}{\nu}\right)\right]^{-1} \qquad (4.3)$$

Set $\hat{\nu} = x_{(1)}$ in (4.3), we get the MLE of θ in (4.2).

Since $\frac{\partial^2 ln\left[L(\theta,\nu;\mathbf{x})\right]}{\partial\theta^2} = -\frac{n-1}{\theta^2} - \frac{1}{\theta^2} = -\frac{n}{\theta^2} < 0$ for all θ, then θ is the local maximum, and since it is the only value obtained when $\frac{\partial ln\left[L(\theta,\nu;\mathbf{x})\right]}{\partial\theta} = 0$, then θ is the global maximum for likelihood function. Therefore $\hat{\theta}$ in (4.2) is the MLE for θ.

We discuss a special case in Corollary 4.2.

Corollary 4.2. (The MLE for θ based on ordinary order statistics for a Pareto population)
For ordinary order statistics, let $m = 0$ and $k = 1$ in (4.2) and collecting terms we get,

$$\hat{\theta} = n\left[\ln\left(\frac{\prod_{i=1}^{n} x_i}{\left(\min_i x_i\right)^n}\right)\right]^{-1}, \qquad (4.4)$$

which is the well known MLE for θ based on ordinary order statistics for a Pareto population. See for example, Casella and Berger (2002).

5. Conclusion

In this paper, we have derived the joint *pdfs* of generalized order statistics for different cases from Pareto distribution in closed-forms. In addition, the *pdf* of range and midrange of GOS from Pareto distribution is given. Furthermore, some special cases have been discussed. Furthermore, we have derived the MLE of the parameters for Pareto distribution in closed forms and a special case is proven.

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