Solving Optimal Linear Time-Variant Systems via Chebyshev Wavelet

[1,]Hatem Elaydi , [2,]Atya A. Abu Haya
Faculty of Engineering, Islamic University, Gaza, Palestine

Abstract

Over the last four decades, optimal control problem are solved using direct and indirect methods. Direct methods are based on using polynomials to represent the optimal problem. Direct methods can be implemented using either discretization or parameterization. The proposed method here is considered as a direct method in which the optimal control problem is directly converted into a mathematical programming problem. A wavelet-based method is presented to solve the linear quadratic optimal control problem. The Chebyshev wavelets functions are used as the basis functions. Numerical examples are presented to show the effectiveness of the method, several optimal control problems were solved, and the simulation results show that the proposed method gives good and comparable results with some other methods.

***Keywords**: Chebyshev wavelet, optimal control problem, time-variant systems*

I. INTRODUCTION

The goal of an optimal controller is the determination of the control signal such that a specified performance index is optimized, while at the same time keeping the system equations, initial condition, and any other constraints are satisfied. Many different methods have been introduced to solve optimal control problem for a system with given state equations. Examples of optimal control applications include environment, engineering, economics etc. The most popular method to solve the optimal control problem is the Riccati method for quadratic cost functions however this method results in a set of usually complicated differential equations [1]. In the last few decades orthogonal functions have been extensively used in obtaining an approximate solution of problems described by differential equations [2], which is based on converting the differential equations into an integral equation through integration. The state and/or control involved in the equation are approximated by finite terms of orthogonal series and using an operational matrix of integration to eliminate the integral operations. The form of the operational matrix of integration depends on the choice of the orthogonal functions like Walsh functions, block pulse functions, Laguerre series, Jacobi series, Fourier series, Bessel series, Taylor series, shifted Legendry, Chebyshev polynomials, Hermit polynomials and Wavelet functions [3].

This paper proposes a solution to solve the general optimal control problem using the parameterization direct method. The Chebyshev wavelets are used as new orthogonal polynomials to parameterize the states and control of the time-varying linear problem. Then, the cost function can be casted using the parameterized states and control.This paper is organized as follow: section 2 talks about the wavelets and scaling functions, section3 discusses using Chebyshev wavelets to approximate functions, section 4 presents the formulation of problems, section 5 gives numerical examples, and section 6 conclude this study.

II. SCALING FUNCTIONS AND WAVELETS

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously [4], the following family of continuous

$$\Psi_{a,b}(t) = |a|^{-\frac{1}{2}}\Psi\left(\frac{t-b}{a}\right), \qquad a,b \in \mathbb{R}, a \neq 0 \qquad (1)$$

Chebyshev wavelets $\psi_{nm}(t) = \psi(k,m,n,t)$ have four arguments; $k = 1,2,3,\ldots,$ $n = 1,2,3,\ldots,2^k$, m is the order for Chebyshev polynomials and t is the normalized time. They are defined on the interval [0,1) by:

$$\Psi_{nm}(t) = \begin{cases} \frac{\alpha_m 2^{\frac{k}{2}}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & \frac{n-1}{2^k} \le t \le \frac{n}{2^k} \\ 0 & elsewhere \end{cases} \qquad (2)$$

where

$$\alpha_m = \begin{cases} \sqrt{2} & m = 0 \\ 2\ , & m = 1,2,\dots \end{cases}$$

Here, $T_m(t)$ are the well−known Chebyshev polynomials of order m, which are orthogonal with respect to the weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$ and satisfy the following recursive formula [5]:

$$T_0(t) = 1,\ T_1(t) = t,$$
$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1,2,3,\dots \qquad (3)$$

The set of Chebyshev wavelets are an orthogonal set with respect to the weight function

$$\omega_n(t) = \omega(2^{k+1}\, t - 2n + 1) \qquad (4)$$

3. Statement of the optimal control of linear time-varying systems

Find the optimal control that minimizes the quadratic performance index

$$J = \int_0^{t_f} (x^T Q x + u^T Ru)dt \qquad (5)$$

subject to the time-varying system given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(0) = x_0 \qquad (6)$$

where

$x \in R^s$ *is the state variables vector*, $u \in R^r$ *is the control vector*, $x_0 \in R^s$ *is the vector of initial conditions*, $A(t)$ *and* $B(t)$ are time-varying matrices, Q is a positive semidefinite matrix , and R is a positive definite matrix .

III. OPTIMAL CONTROL PROBLEM

3.1 Control state parameterization

Approximating the state variables and the control variables by Chebyshev scaling functions, we get [5]

$$x_i(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} a^i{}_{nm}\phi_{nm}(t) \qquad i = 1,2,\dots,s \qquad (7)$$

$$u_i(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} b^i{}_{nm}\phi_{nm}(t) \qquad i = 1,2,\dots,r \qquad (8)$$

We can write these two equations in compact form as :

$$x(t) = (\Phi^T(t) \otimes I_s)a$$

$$u(t) = (\Phi^T(t) \otimes I_r)b \qquad (10)$$

Where I_s, I_r *are* $s\ x\ s$ *and* $r\ x\ r$ *identity matrices respectively* , $\Phi^T(t)$ *is* $N\ x\ 1$, $(N = 2^k(M))$, *vector of Chebyshev scaling function given by* :

$$\Phi(t) = [\Phi_{1m-1}(t), \Phi_{2m-1}(t), \Phi_{3m-1}(t), \dots, \phi_{2^k m-1}(t)]^T \qquad (11)$$
$$\Phi_{im-1}(t) = [\phi_{i0}(t), \phi_{i1}(t), \dots, \phi_{iM-1}(t)] \qquad (12)$$

and

$$a = [\alpha^1\ \alpha^2 \dots \alpha^s]^T \qquad (13)$$
$$\alpha^i = [a^i_{10}\ a^i_{11} \dots a^i_{1M-1}\ a^i_{20} \dots a^i_{2M-1} \dots a^i_{2^k 0} \dots\ a^i_{2^k M-1}] \quad i = 1,2,\dots,s \qquad (14)$$
$$b = [\beta^1\ \beta^2 \dots \beta^r]^T \qquad (15)$$
$$\beta^i = [b^i_{10}\ b^i_{11} \dots b^i_{1M-1}\ b^i_{1M-1} \dots b^i_{2M-1} \dots b^i_{2^k 0} \dots\ b^i_{2^k M-1}] \quad i = 1,2,\dots,r \qquad (16)$$

a *and* b are vectors of unknown parameters have dimensions $sN\ x\ 1$ *and* $rN\ x\ 1$ respectively.

3. 2 The product operational matrix of chebyshev wavelets

The following property of the product of two Chebyshev wavelets vectors [5] will also be used. Let

$$\Psi(t)\Psi^T(t)F = \tilde{F}\Psi(t), \qquad (17)$$

Where $\tilde{F}$ *is* $(2^kM)x\ (2^kM)$ *matrix. To illustrate the calculation procedure we choose*

$M = 3$ *and* $k = 2.$

Thus we have: $F = [f_{10}, f_{11}, f_{12}, \dots, f_{40}, f_{41}, f_{42}]^T$

$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \psi_{12}(t), \dots, \psi_{4,0}(t), \psi_{41}(t), \psi_{42}(t)]^T$

Then

$$\tilde{F} = \begin{bmatrix} \tilde{F}_1 & 0 & 0 & 0 \\ 0 & \tilde{F}_2 & 0 & 0 \\ 0 & 0 & \tilde{F}_3 & 0 \\ 0 & 0 & 0 & \tilde{F}_4 \end{bmatrix}$$

In general case $\tilde{F}$ *is a* $(\ 2^kM)\ X\ (\ 2^kM)$

$$\tilde{F} = \begin{bmatrix} \tilde{F}_1 & 0 & \cdots & 0 \\ 0 & \tilde{F}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{F}_{2^k} \end{bmatrix} \qquad (18)$$

Where

$$\tilde{F}_i = \frac{2^{\frac{k}{2}}}{\sqrt{\pi}} \begin{bmatrix} f_{i0} & f_{i1} & f_{i2} & f_{i3} & \cdots & f_{i,M-2} & f_{i,M-1} \\ f_{i1} & f_{i0}+\frac{1}{\sqrt{2}}f_{i2} & \frac{1}{\sqrt{2}}(f_{i1}+f_{i3}) & \frac{1}{\sqrt{2}}(f_{i2}+f_{i4}) & \cdots & \frac{1}{\sqrt{2}}(f_{i,M-3}+f_{i,M-1}) & \frac{1}{\sqrt{2}}f_{i,M-2} \\ f_{i2} & \frac{1}{\sqrt{2}}(f_{i1}+f_{i3}) & f_{i0}+\frac{1}{\sqrt{2}}f_{i4} & \frac{1}{\sqrt{2}}(f_{i1}+f_{i5}) & \cdots & \frac{1}{\sqrt{2}}f_{i,M-4} & \frac{1}{\sqrt{2}}f_{i,M-3} \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & f_{i0}+\frac{1}{\sqrt{2}}f_{i,\mu} & f_{i1}+\frac{1}{\sqrt{2}}f_{i,\mu+1} & \cdots & \frac{1}{\sqrt{2}}f_{i,\nu} \\ \cdots & \cdots & \cdots & f_{i1}+\frac{1}{\sqrt{2}}f_{i,\mu+1} & f_{i0} & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & f_{i0} & \frac{1}{\sqrt{2}}f_{i1} \\ f_{i,M-1} & \frac{1}{\sqrt{2}}f_{i,M-2} & \cdots & \cdots & \cdots & \frac{1}{\sqrt{2}}f_{i1} & f_{i0} \end{bmatrix}$$

$$\mu = \begin{cases} M-2 & M\ even \\ M-1 & M\ odd \end{cases}$$

$$\nu = \begin{cases} M/2 & M\ even \\ \dfrac{M-1}{2} & M\ odd \end{cases} \qquad (19)$$

3.3 Performance index approximation

To approximate the performance index, we substitute $Eq.(\ 9\ )$ and $(\ 10\ )$ into $(\ 5\ )$ to get [6]

$$J = \int_0^1 (a^T\ (\Phi(t) \otimes I_s)Q(\Phi^T(t) \otimes I_s)a + b^T(\Phi(t) \otimes I_r)R(\Phi^T(t) \otimes I_r)b)dt \quad (20)$$

It can be simplified as

$$J = \int_0^1 (a^T\ (\Phi(t)\Phi^T(t) \otimes Q)a + b^T(\Phi(t)\Phi^T(t) \otimes R)b)dt \qquad (21)$$

Because of orthogonality of Chebyshev scaling function and using Lemma1 in chapter three

$$\int_0^1 \Phi(t)\Phi^T(t)dt = RR \qquad (22)$$

Then $\quad J = a^T(RR \otimes Q)a + b^T(RR \otimes R)b \qquad (23)$

It can be wrote as

$$J = [\ a^T \quad b^T\ ]\begin{bmatrix} RR \otimes Q & 0_{N_s\ x\ N_r} \\ 0_{N_r\ x\ N_s} & RR \otimes R \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \qquad (24)$$

To approximate the state equations we write equation (9) as

$$x = \sum_{i=1}^{2^k} \sum_{j=0}^{M-1} \phi_{ij}(t)\ \alpha_{ij} \qquad (25)$$

Or

$$x = \Phi^T(t)\left[\alpha_{10}\alpha_{11} \dots \alpha_{1M-1}\alpha_{20} \ \dots\ \alpha_{2M-1}\alpha_{2^k 0} \ \dots\ \alpha_{2^k M-1}\ \right]^T$$
$$= \Phi^T(t)\alpha \qquad (26)$$

Where $\alpha_{ij} = [a_{ij}{}^1\ a_{ij}{}^2 \dots\ a_{ij}{}^s]$

The control variables (10) can be rewritten as

$$u = \sum_{i=1}^{2^k} \sum_{j=0}^{M-1} \phi_{ij}(t)\ \beta_{ij} \qquad (27)$$

Or

$$u = \Phi^T(t)\left[\beta_{10}\beta_{11} \dots \beta_{1M-1}\beta_{20} \ \dots\ \beta_{2M-1}\beta_{2^k 0} \ \dots\ \beta_{2^k M-1}\ \right]^T$$
$$= \Phi^T(t)\beta \qquad (28)$$

Where $\beta_{ij} = [b_{ij}{}^1\ b_{ij}{}^2 \dots\ b_{ij}{}^r]$

3.4 Time varying elements approximation

Then we need to express $A(t)$ *and* $B(t)$ in terms of Chebyshev scaling functions. The approximation of $A(t)$ can be given by [6] :

$$A(t) = \sum_{i=1}^{2^k} \sum_{j=0}^{M-1} A_{ij}\phi_{ij}(t) \qquad (29)$$

$$A(t) = \left[A_{10}\ \ A_{11} \dots\ A_{1M-1}\ \ A_{20} \dots\ A_{2M-1} \dots A_{2^k 0} \dots\ A_{2^k M-1}\right]\Phi(t) \qquad (30)$$

Where A_{ij} *is an* $s\ x\ s$ *constant matrix of the coefficents of Chebyshev scaling function* $\phi_{ij}(t)$. Theses constant matrices can be obtained as

$$A_{ij} = \int_{\frac{i-1}{2^k}}^{\frac{i}{2^k}} A(t)\phi_{ij}(t)\,dt \qquad (31)$$

Similarly , $B(t)$ can be expanded via Chebyshev scaling functions as follows

$$B(t) = \left[B_{10}\ \ B_{11} \dots\ B_{1M-1}\ \ B_{20} \dots\ B_{2M-1} \dots B_{2^k 0} \dots\ B_{2^k M-1}\right]\Phi(t) \qquad (32)$$

Where B_{ij} *is an* $s\ x\ r$ *constant matrix*

3.5 Initial condition

The initial condition vector x_o can be expressed via Chebyshev scaling function as

$$x_o = \frac{\sqrt{\pi/2}}{2^{k/2}}(\Phi^T(t))[\alpha_0^1\ \alpha_0^2 \dots .\ \alpha_0^s]$$
$$= \frac{\sqrt{\pi/2}}{2^{k/2}}(\Phi^T(t))g_o \qquad (33)$$

where $\qquad g_o = \left[\alpha_{10}^0\ \ 0 \dots\ 0\ \alpha_{20}^0\ 0 \dots 0\ \dots \alpha_{2^k 0}^0\ 0 \dots 0\right]^T$

and $\alpha_{i0}^0 = [x_i(0)\quad x_2(0) \dots\ x_s(0)\ ]$

We multiply Eq. (33) *by factor,*

$$\delta = \frac{\sqrt{\frac{\pi}{2}}}{2^{\frac{k}{2}}}$$

because from Eq. (2) *we can obtained*

$$\phi_{n0} = \frac{2^{k/2}}{\sqrt{\pi/2}}$$

To express the state equations in terms of the unknown parameters of the state variables and the control variables , *Eq.* (2) can be integrated as

$$x(t) - x_0 = \int_0^t A(t)\,x(\tau)\,d\tau + \int_0^t B(t)\,u(\tau)\,d\tau \qquad (34)$$

By substituting (26), (28), (30), (32) and (33) into (34), we get

$$\Phi^T(t)\alpha - \Phi^T(t)\delta g_o = \int_0^t [A_{10} \; \dots \; A_{2^kM-1}]\,\Phi(t)\Phi^T(t)\alpha dt + \int_0^t [B_{10} \; \dots \; B_{2^kM-1}]\,\Phi(t)\Phi^T(t)\beta dt \qquad (35)$$

But from (17) we have

$$[A_{10} \; \dots \; A_{2^kM-1}]\,\Phi(t)\Phi^T = \Phi^T\breve{A} \qquad (36)$$

$$[B_{10} \; \dots \; B_{2^kM-1}]\,\Phi(t)\Phi^T = \Phi^T\breve{B} \qquad (37)$$

where $\breve{A}$ and $\breve{B}$ are $sN \times sN$ and $sN \times rN$ constant matrices respectively. Substituting (36) and (37) into equation (35) gives

$$\Phi^T(t)\alpha - \Phi^T(t)\delta g_o = \int_0^t \Phi^T\breve{A}(t)\alpha dt + \int_0^t \Phi^T(t)\,\breve{B}\beta dt \qquad (38)$$

Using the integration operational matrix $\boldsymbol{P}$ of Chebyshev scaling function , we get

$$\Phi^T(t)\alpha - \Phi^T(t)\delta g_o = \Phi^T(t)P^T\breve{A}\alpha + \Phi^T(t)P^T\breve{B}\beta \qquad (39)$$

$$(\Phi^T(t)\otimes I_s)a - (\Phi^T(t)\otimes I_s)\delta g_o = (\Phi^T(t)P^T\otimes I_s)\breve{A}a + (\Phi^T(t)P^T\otimes I_s)\breve{B}b \qquad (40)$$

$$I_{Ns}a - \delta g_o = (P^T\otimes I_s)\breve{A}a + (P^T\otimes I_s)\breve{B}b \qquad (41)$$

3.6 Continuity of the state variables

To insure the continuity of the state variables between the different sections we must add constraints. There are $2^k - 1$ points at which the continuity of the state variables have to ensured [8].

Theses points are :

$$t_i = \frac{i}{2^k} \qquad i = 1,2,\dots,2^k-1 \qquad (42)$$

So there are $(2^k-1)s$ *equality constraints given by :*

$$(I_s\otimes\Phi'(t))a = 0_{(2^k-1)s\,x\,1} \qquad (43)$$

Where

$$\Phi' = \begin{bmatrix} \phi_{1m-1}(t_1) & -\phi_{2m-1}(t_1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & \phi_{2m-1}(t_2) & -\phi_{3m-1}(t_2) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \phi_{3m}(t_3) & -\phi_{4m}(t_3) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \phi_{(2^k-1)m}(t_{2^k-1}) & -\phi_{(2^k-1)m}(t_{2^k-1}) \end{bmatrix} \qquad (44)$$

Φ' *is* $(2^k-1)\;x\;(2^kM)$ matrix

IV. QUADRATIC PROGRAMMING PROBLEM TRANSFORMATION

Finally by combining the equality constraints (41) with (43) we get

$$\begin{bmatrix} (P^T\otimes I_s)\breve{A} - I_{Ns} & (P^T\otimes I_s)\breve{B} \\ (\Phi'\otimes I_s) & 0_{(2^k-1)s\,x\,Nr} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -g_0\delta \\ 0_{(2^k-1)s\,x\,1} \end{bmatrix} \qquad (45)$$

We saw that the optimal control problem is converted into a quadratic programming problem of minimizing the quadratic function (24) subject to the linear constraints (45) and solved it using MATLAB program.

4.1 Numerical Example

Find the optimal control $u(t)$ which minimizes

$$J = \frac{1}{2}\int_0^1 (x^2 + u^2)dt$$

subject to $\qquad \dot{x} = tx + u \qquad x(0) = 1$

We solved this problem for

$k = 2$ *and* $M = 3,4,5$, *the optimal value we get as in Table* (1) *as shown*

Table (1)

	$K = 2$, $M = 3$	$K = 2$, $M = 4$	$K = 2$, $M = 5$
J	0.4848235986044	0.4842684350618	0.4842678105389

The optimal state and control variables are shown in Figures (1-3), we noticed from Figures (1 – 3) and from Table (1) that when we increase M we obtained at a good trajectories plots and at good results of performance index (J).

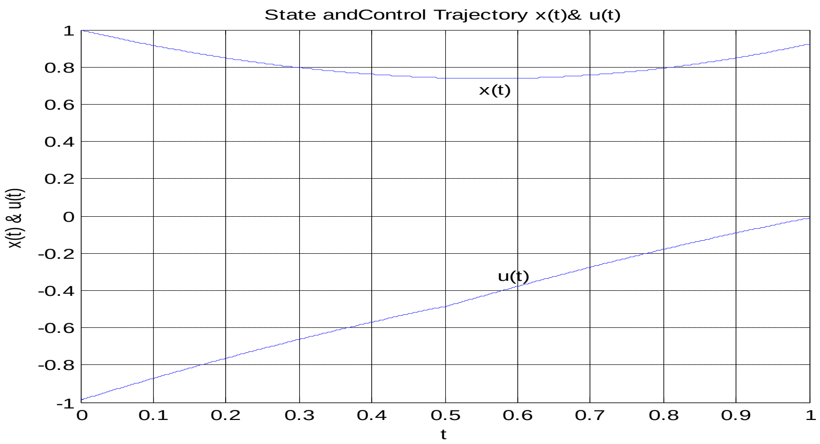

Figure (1) Optimal state and control K=2 M=3

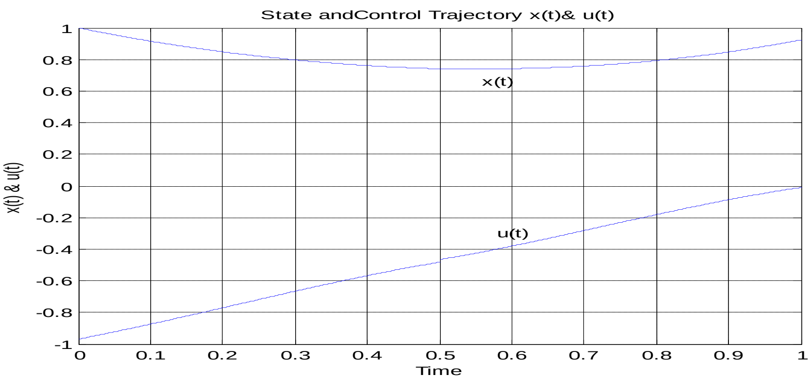

Figure (2) Optimal state and control K=2 M=4

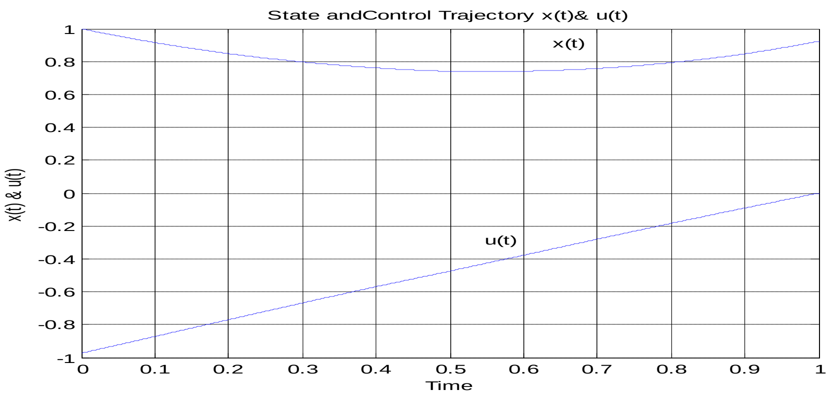

Figure (3) Optimal state and control K=2 M=5

Table (2)

Research name	$Jaddu[6]$	[7]	$This\ research$
J	0.48426760037684	0.48427022	0.484267810538982

Table (2) shows the comparison between our research and other researches to solve the previous problem , from the table we notice that our method is good compared with other methods. In this chapter we proposed a method to solve the optimal control problem time-varying systems using Chebyshev wavelet scaling function, we applied this method at a numerical example to see the effectiveness of the method and compared with other methods.We need to solve the optimal control problem time-varying systems because we must need it to solve the nonlinear optimal control problem in the next chapter.

V. CONCLUSION

In this paper, a numerical methods to solve optimal control problems for linear time-variant systems was proposed. This method was based on parameterizing the system state and control variables using a finite length Chebyshev wavelet. The aim of the proposed method is the determination of the optimal control and state vector by a direct method of solution based upon Chebyshev wavelet. An explicit formula for the performance index was presented. In addition, Chebyshev wavelet operational matrix of integration was presented and used to approximate the solution. A product operational matrix of Chebyshev wavelets was also presented and used to solve linear time-varying systems. Thus, the solution of the linear optimal control problem is reduced to a simple matrix-vector multiplication that can be solved easily using MATLAB. Numerical examples were solved to show the effectiveness and efficiency of the proposed method. The proposed method gave better or comparable results compared to other research. Future work can deal with using Chebyshev wavelet to solve nonlinear optimal control problems.

VI. REFERENCES

[1] Kirk, D. Optimal Control Theory: An Introduction. Prentice-Hall, Englewood Cliffs, NJ. 1970.

[2] Jaddu, H. Numerical methods for solving optimal control problems using Chebyshev polynomials. PHD Thesis. JAIST, Japan. 1998.

[3] Mohammad Ali Tavallaei and Behrouz Tousi. Closed form solution to an optimal control problem by orthogonal polynomial expansion. American J. of Engineering and Applied Sciences. 2008.

[4] E. Babolian, F. Fattahzadeh. Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration. Applied Mathematics and Computation. 188. 417–426. 2007.

[5] M. Ghasemi , M. Tavassoli Kajani. Numerical solution of time-varying delay systems by Chebyshev wavelets. Applied Mathematical Modelling. 35 5235–5244. 2011.

[6] H. Jaddu. Optimal control of time-varying linear systems using wavelets. School of Information Science, Japan Advanced Institute of Science and technology. 1998.

[7] G. Elnagar. State-control spectral Chebyshev parameterization for linearly constrained quadratic optimal control problems. Journal of Computational and Applied Mathematics. Vol. 79. 19-40. 1997.

[8] Hatem Elaydi, Atya A. Abu Haya. Solving Optimal Control Problem for Linear Time-invariant Systems via Chebyshev Wavele. International Journal of Electrical Engineering, 5:7.541-556. 2012.