Comparison of Ising magnet on directed versus undirected Erdös-Rényi and scale-free networks

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Abstract

Scale-free networks are a recently developed approach to model the interactions found in complex natural and man-made systems. Such networks exhibit a power-law distribution of node link (degree) frequencies $n(k)$ in which a small number of highly connected nodes predominate over a much greater number of sparsely connected ones. In contrast, in an Erdös-Rényi network each of N sites is connected to every site with a low probability p (of the order of $1/N$). Then the number k of neighbors will fluctuate according to a Poisson distribution. One can instead assume that each site selects exactly k neighbors among the other sites. Here we compare in both cases the usual network with the directed network, when site A selects site B as a neighbor, and then B influences A but A does not influence B. As we change from undirected to directed scale-free networks, the spontaneous magnetization vanishes after an equilibration time following an Arrhenius law, while the directed ER networks have a positive Curie temperature.

Keywords: Ising Model, Directed and Undirected Erdös-Rényi network, Barabási-Albert network

Introduction

This paper deals with Ising spin on (mostly) directed Erdös-Rényi (ER) random graphs [1, 2, 3] and Barabási-Albert (BA) scale free networks [4]. Sumour and Shabat [5, 6] investigated Ising models with spin $S = 1/2$ on

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directed BA networks [4] with the usual Glauber dynamics. No spontaneous magnetisation was found, in contrast to the case of undirected BA networks [7, 8, 9] where a spontaneous magnetisation was found below a critical temperature which increases logarithmically with system size. In $S = 1/2$ systems on undirected, scale-free hierarchical-lattice small-world networks [10], conventional and algebraic (Berezinskii-Kosterlitz-Thouless) ordering, with finite transition temperatures, have been found. Lima and Stauffer [11] simulated directed square, cubic and hypercubic lattices in two to five dimensions with heat bath dynamics in order to separate the network effects form the effects of directedness. They also compared different spin flip algorithms, including cluster flips [12], for Ising-BA networks. They found a freezing-in of the magnetisation similar to [5, 6], following an Arrhenius law at least in low dimensions. This lack of a spontaneous magnetisation (in the usual sense) is consistent with the fact that if on a directed lattice a spin S_j influences spin S_i, then spin S_i in turn does not influence S_j, and there may be no well-defined total energy. Thus, they show that for the same scale-free networks, different algorithms give different results. The q-state Potts model has been studied in scale-free networks by Igloi and Turban [13] and depending on the value of q and the degree-exponent γ first- and second-order phase transitions are found, and also by Lima [15] on directed BA network, where only first-order phase transitions have being obtained independent of values of q for values of connectivity $z = 2$ and $z = 7$ of the directed BA network. More recently, Lima [14] simulated the Ising model for spin $S = 1$ on directed BA network and different from the Ising model for spin $S = 1/2$, an *unusual* order-disorder phase transition of order parameter was seen; this effect needs to be re-evaluated in the light of the time dependence presented below.

In previous work [5, 6] we also studied the Ising model on directed BA networks, by checking the magnetization on it. Analogously in this work we check the modified ER network (or Wilf graph) by taking an exact number of neighbors like in BA network, then we work in directed ER network with low probability, and finally go to undirected ER network. We also study the Ising model for spin $S = 1/2$, 1, 3/2 and 2 on directed BA network. In all cases we check whether or not a spontaneous magnetization exisst in equilibrium. The Ising model with spin 1/2 on the directed ER graphs and that with spin $S = 1/2$, 1, 3/2 and 2 on BA networks was seen not to show a *usual* spontaneous magnetisation and this decay time for flipping of the magnetisation followed an Arrhenius law for HeatBath algorithms that agrees with the results of the Ising model for spin $S = 1/2$ [5, 6] on directed
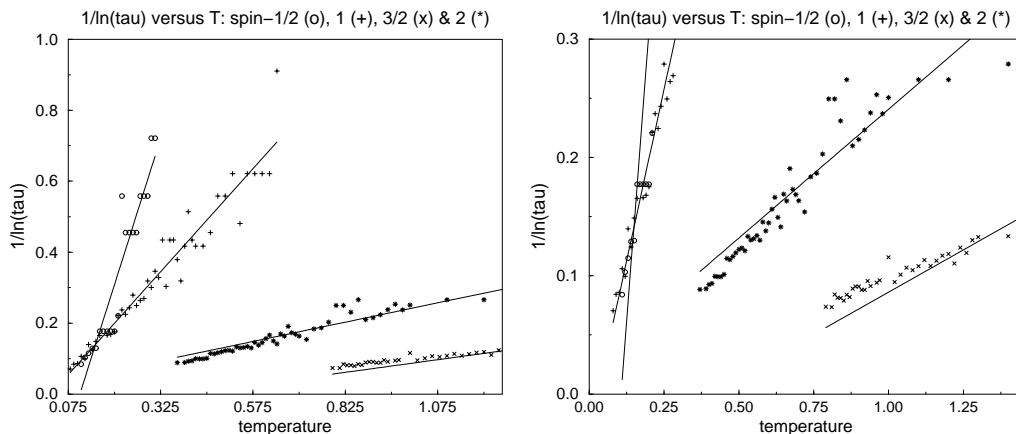
BA network.

Model and Simulation

Figure 1: Reciprocal logarithm of the relaxation times on directed BA networks for $S = 1/2$ to $S = 2$. The right part is a zoom of the left part for the longest times.

Ising model on directed Barabási-Albert Networks

We consider the spins $S = 1/2$, 1, 3/2 and 2 Ising model on directed Barabási-Albert (BA) networks, defined by a set of spin variables taking the values ± 1 for $S = 1/2$, ± 1 and 0 for $S = 1$, $\pm 3/2$ and $\pm 1/2$ for $S = 3/2$, and ± 2, ± 1 and 0 for 2, respectively, situated on every site of a directed BA networks with N sites.

The probability for spin S_i to change its state in these directed networks is

$$p_i = 1/[1 + \exp(2E_i)/k_B T], \quad E_i = -J \sum_j S_i S_j \qquad (1)$$

where the j-sum runs over all selected neighbors of S_i. In this network, each new site added to the network selects (preferential attachment proportional to the number of previous selections) with connectivity $z = 2$ already existing sites as neighbours influencing it; the newly added spin does not influence these neighbours.

To study the spins $S = 1/2$, 1, 3/2 and 2 Ising model we start with all spins up, a number of spins equal to $N = 500000$, and time up $2,000,000$ (in
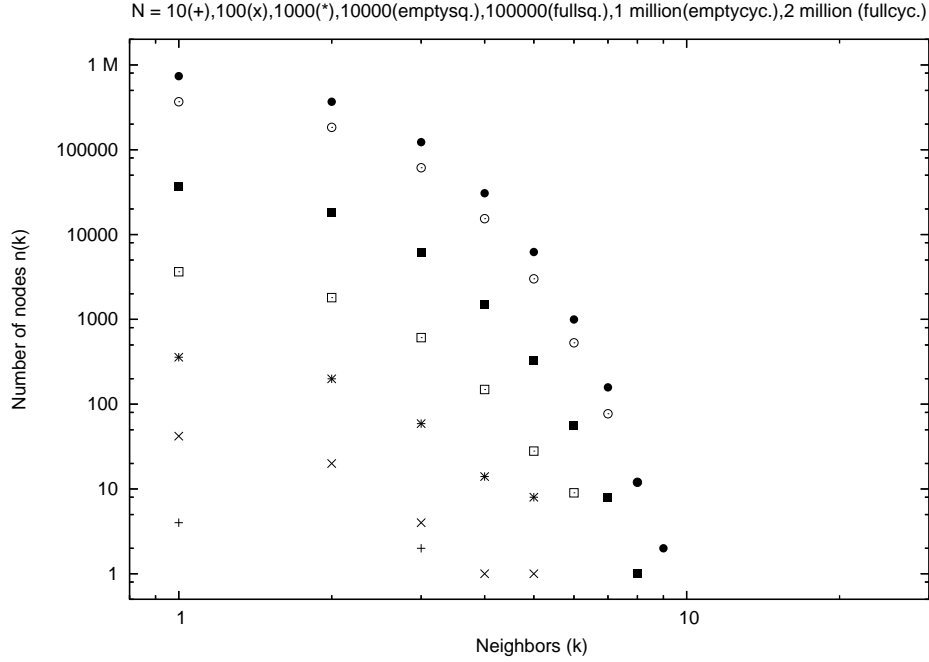
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N = 10(+),100(x),1000(*),10000(emptysq.),100000(fullsq.),1 million(emptycyc.),2 million (fullcyc.)

Figure 2: $n(k)$ versus k with different N (10, 100, 1000, 10000, 100000), for exactly 4 selected neighbors.

units of Monte Carlo steps per spin), with HeatBath Monte Carlo algorithm. Then we vary the temperature T and at each T study the time dependence for 9 samples. The temperature is measured in units of critical temperature of the square-lattice Ising model. We determine the time τ after which the magnetisation has flipped its sign for the first time, and then take the median values of our nine samples. So we get different values τ_1 for different temperatures.

In this study of the critical behavior this Ising model (with spins $S = 1/2$, 1, $3/2$ and 2) we define the variable $m = \sum_{i=1}^{N} S_i/N$ as normalized magnetization.

Our BA simulations, using the HeatBath algorithm, indicate that the spin $S = 1/2$, 1, $3/2$ and 2 Ising model does not display a phase transition and the plot of the time $1/\ln \tau$ versus temperature in Fig. 1 shows that our BA results for all spins agree with the modified Arrhenius law for relaxation time, defined as the first time when the sign of the magnetisation flips: $1/\ln(\tau) \propto T + \ldots$.

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Modified ER network

In the classical ER model all edges are equally probable and independent. We take a modification where each node connects with an exact number of neighbors like in the BA network; we take number of neighbors as 4. So we plot the number $n(k)$ of nodes versus the number k of neighbors in Fig. 2.

Fig.2 does not have the shape of the corresponding Poisson distribution for $< k >= 4$. The maximum number of nodes which select a site as neighbor is seen in Fig.3 to vary roughly logarithmically with the size N of the network.
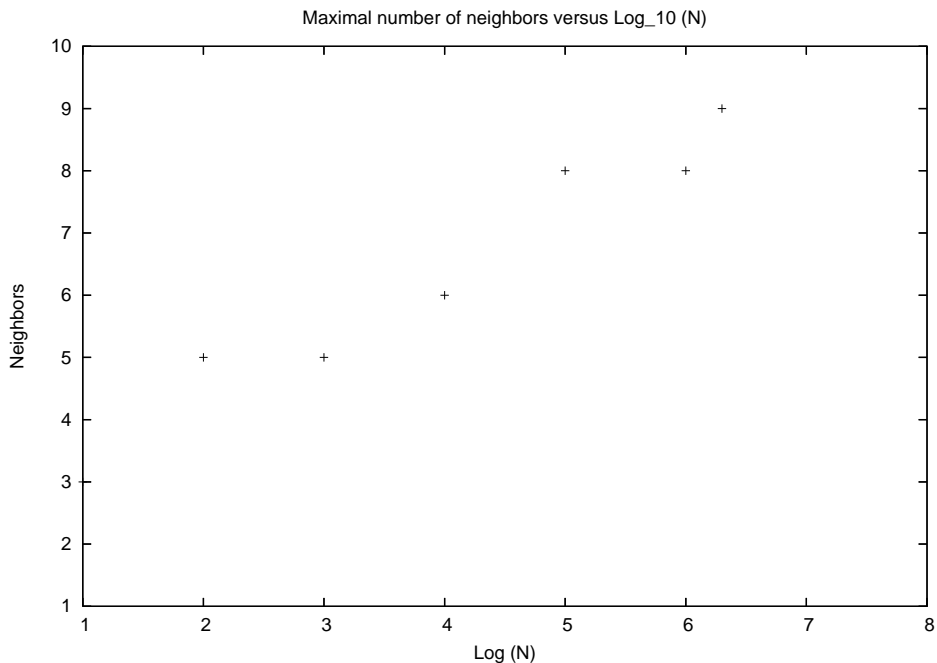

Maximal number of neighbors versus Log_10 (N)

Figure 3: Maximum number of neighbors versus decadic logarithm of the number N of nodes, for 4 neighbor selections per node.

Ising model on modified Erdös-Rényi network

When we put the Ising model on the Erdös-Rényi network with size of network = 2 million , temperatures 0.184 to 0.310, number of neighbors = 2 and 4, and time = 20000, and observe the time when the magnetization starts to change it's sign, we get Fig.4.
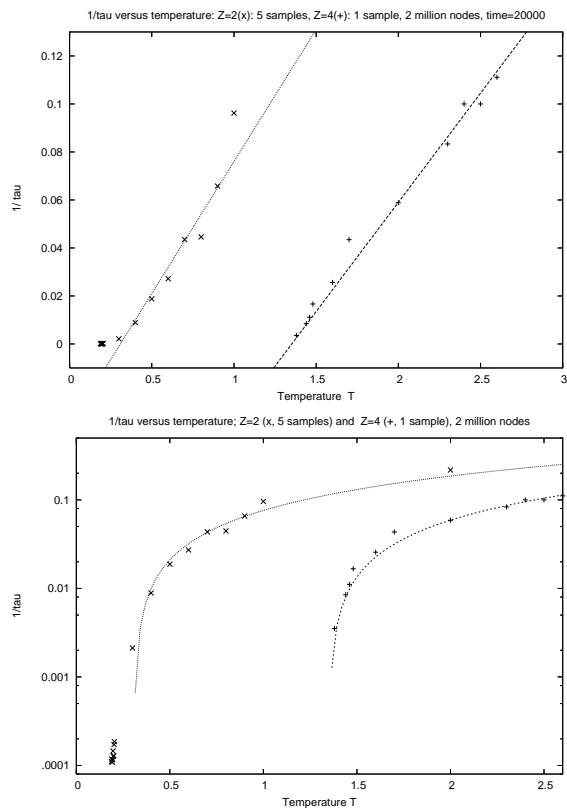
Figure 4: Reciprocal relaxation time $1/\tau$ versus temperature with network size = 2 and 4 million, time = 20000, for modified ER with exactly 2 or 4 neighbors. The upper part uses a linear, the lower a logarithmic scale for the reciprocal relaxation time.

The reciprocal relaxation time varies linearly with temperature for $z = 2$ and 4 neighbors selected by each new node, and can be extrapolated to vanish at some positive Curie temperature T_c: The relaxation time goes to infinity there.

Erdös-Rényi (ER) Network

Now we move from the modified to the classical ER graphs. In the normal ER model all edges are equally probable and appear independently. To get the proper number of nodes of normal ER network we need $N \gg 1$ and probability $p \ll 1$ with pN of order unity; each edge is chosen to appear

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with probability p. We used $p = 1/N$, $p = 2/N$, and $p = 3/N$. On average, each vertex will have a *small* number of neighbors. We found no significant difference between the two networks for large size in Poisson distribution and observed degree distribution $n(k)$.

Ising Model on Directed ER Network

We take different probabilities for different number of nodes N (1000, 10000, 50000, 100000), with different temperatures in Fig.5. There we check again the first time after which the magnetization changes sign, take the median from nine samples, and plot from the reciprocal of the time for three probabilities ($p = 3/N$, $p = 2/N$, $p = 1/N$) in Fig.5.
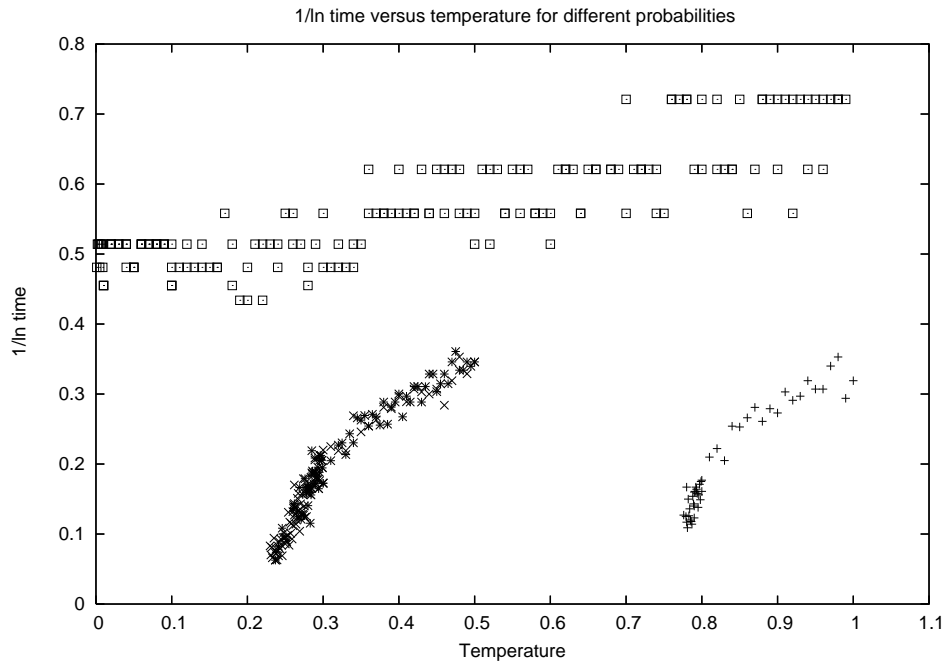

Figure 5: 1/ln(time) versus temperature for different probabilities $1/N$ (sq.), $2/N$ (x), $3/N$ (+).

The figure shows nicely the difference between probability $1/N$ (= percolation threshold) and larger probabilities. This figure shows that there is a spontaneous magnetization at $p = 2/N$ for the left curve and the right curve
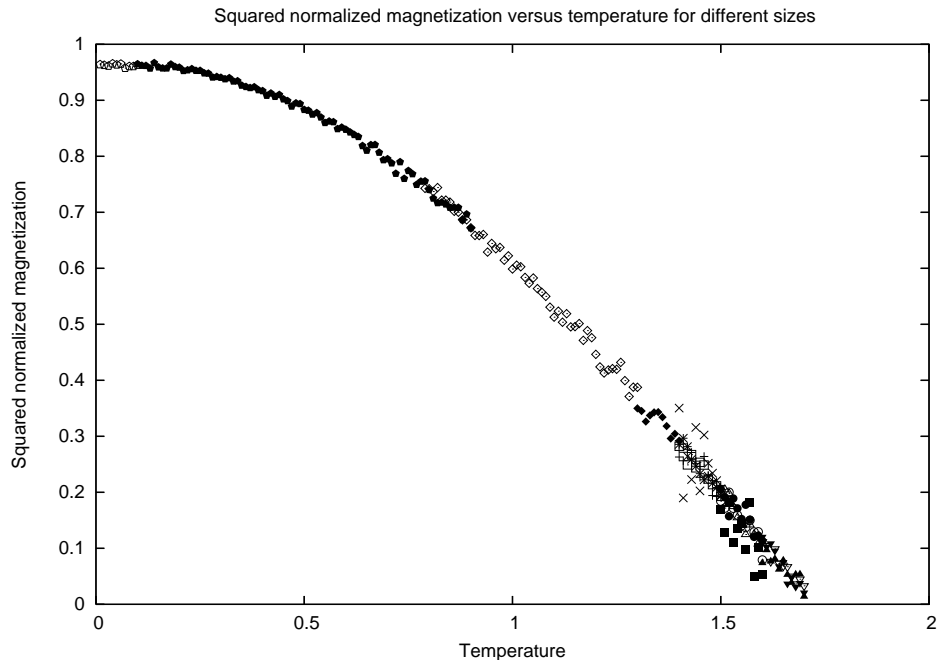
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Figure 6: Squared normalized magnetization versus temperatures for different sizes N of the undirected ER graph.

for $p = 3/N$, but not a spontaneous magnetization at $p = 1/N$ which is the percolation threshold.

Ising model on undirected ER network

For the undirected network we use only one probability equal $p = 2/N$, because it gives a clear answer compatible with the mean-field universality class, as expected because of the infinite range of the symmetric interaction. Before, for directed networks or graphs, when a new site A selects on old site B as a neighbor, then we had only one direction. Now, for undirected network, not only A is a neighbors of B but also B is a neighbor of A. From our simulation we see that the undirected version has a spontaneous magnetization, to which the system relaxes similarly to the standard Ising square lattice, Then we plot the square of normalized magnetization versus temperature in Fig.6. In equilibrium there is a Curie temperature T_c: below T_c we have a spontaneous magnetization and above T_c we do not have one as we see in Fig 6. The squared magnetization vanishes at this $T_c \simeq 1.7J/K_B$

linearly in temperature. This behavior corresponds, not unexpectedly, to a mean field critical exponent.

Discussion

For the directed BA networks we found an Arrhenius law. This means that for each positive temperature there is a finite relaxation time after which the magnetisation decay towards zero. Similarly to the one-dimensional Ising model there is no ferromagnetism on this directed Barabási-Albert network.

A directed (normal or modified) ER network has a phase transition temperature below which a spontaneous magnetization exists, while the directed BA network has no such phase transition. The undirected ER network has a spontaneous magnetization in the universality class of mean field theory.

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