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# Path Integral Quantization of Brink-Schwarz Superparticle

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**Abstract:** The quantization of the Brink-Schwarz superparticle is performed by canonical phase-space path integral.The supersymmetric particle is treated as a constrained system using the Hamilton-Jacobi approach. Since the equations of motion are obtained as total differential equations in many variables, we obtained the canonical phase space coordinates and the phase space Hamiltonian with out introducing Lagrange multipliers and with out any additional gauge fixing condition.



*Keywords: Field Theory; Path Integral, Constrained Systems, Hamilton-Jacobi Formalism, Supersymmetric, Superparticle*

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## 1. Introduction

Systems described by singular Lagrangians are called singular systems and this kind of systems contain inherent constraints [1, 2]. In a lot of physical domains, there extensively exist different singular systems, such as gauge field theories, gravitational field theory, supersymmetric theory, supergravity, superstring theory. A standard consistent way of dealing with singular systems was first formulated by Dirac [3]. In Dirac's method, when a singular Lagrangian in configuration space is transformed into a singular Lagrangian in phase space, the set of constraints would be generated, which are called primary constraints [4, 5]. Through the consistency conditions, using these primary constraints may generate more new constraints, which are called secondary constraints. Following Dirac, one classifies the constraints as being first or second class constraints. According to Dirac's conjecture each first class constraint generates a corresponding gauge symmetry,

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while second class constraints require for their implementation the replacement of Poisson brackets by Dirac brackets [6, 7, 8]. The quantization scheme for constrained systems is the path integral quantization. It is important because it serves as a basis to develop perturbation theory and to find out the Feynman rules. The path integral quantization of singular theories with first class constrains in canonical gauge was given by Faddeev and Popov [9, 10]. The generalization of the method to theories with second class constraints is given by Senjanovic [11]. Moreover, Fradkin and Vilkovisky [12, 13] considered quantization to bosonic theories with first class constraints and it is extension to include fermions in the canonical gauge. When the constrained dynamical systems possesses some second class constraints there exists another method given by Batalain and Fradkin [14]: the BFV- BRST operator quantization method. Which implies to extend the initial phase space by auxiliary variables to convert the original second class constraints into effective first class ones in the extended manifold. Recently, a new scheme of path integral quantization [19]-[22], depend on the Hamilton-Jacobi treatment of constrained systems [17]-[24]. According to Hamilton-Jacobi formalism the equations of motion are obtained as total differential equations in many variables which require to investigate the integrability conditions. The canonical path integral quantization is obtained directly as an integration over the canonical phase-space coordinates without any need to enlarge the initial phase-space by introducing extra-unphysical variables. The advantage of the Hamilton-Jacobi formalism is that we have no difference between first and second class constraints and we do not need gauge-fixing term to reduce or enlarge the physical phase-space. The better understanding of this features aries by applying the Hamilton-Jacobi formalism for supersymmetric constraint systems[25], which are subject to mixed fermionic first and second class constraints in an arbitrary space-time dimension. The main aim of this paper is to apply the Hamilton-Jacobi technique to discuss the classical dynamics of the Brink-Schwarz superparticle, then we try to quantize it by using the canonical path integral method.

The material presented in this paper is divided as follows: In the next section the Hamilton-Jacobi formulation is presented. Section *3*, is devoted to analyze the massive Brink-Schwarz superparticle model [26] by using Hamilton-Jacobi formalism. The conclusion is given in section *4*.

## 2. Hamilton-Jacobi Formalism Of Constrained Systems

The system that is described by singular Lagrangian $L(q_i, \dot{q}_i, t)$ with $i = 1, ..., N$, has a rank of Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \qquad\qquad i, j = 1, \dots, N, \tag{1}$$

equal to $(N - p)$ , $p < N$. In this case we have $p$ momenta which are dependent on each other. The generalized momenta $P_i$ corresponding to the generalized coordinates $q_i$ are defined as,

$$P_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, \ldots, N-p, \tag{2}$$

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = N-p+1, \ldots, N. \tag{3}$$

Since, the rank of the Hess matrix is $(N-p)$, one may solve (2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a\left(q_i, \dot{q}_\mu, P_b\right) \equiv \omega_a. \tag{4}$$

Substituting (4) into (3), we obtain relations in $q_i$, $P_a$, $\dot{q}_\nu$ and $t$ in the form

$$P_\mu = \left.\frac{\partial L}{\partial \dot{q}_\mu}\right|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, \dot{q}_\nu, \dot{q}_a = \omega_a, P_a, t), \qquad \nu = N-p+1, \ldots, N. \tag{5}$$

By mean of (4) and (5) the canonical Hamiltonian $H_0$ is defined as

$$H_0 = -L\big(q_i, \dot{q}_\mu, \dot{q}_a = \omega_a, t\big) + P_a\omega_a + \left.\dot{q}_\mu P_\mu\right|_{P_\nu = -H_\nu}. \tag{6}$$

The set of Hamilton-Jacobi partial differential equations (HJPDE) is expressed as

$$H'_\alpha\left(q_\beta; q_a; P_a = \frac{\partial S}{\partial q_a}; P_\mu = \frac{\partial S}{\partial q_\mu}\right) = 0, \qquad \alpha, \beta = 0, 1, \ldots, p. \tag{7}$$

where

$$H'_0 = P_0 + H_0; \tag{8}$$

and

$$H'_\mu = P_\mu + H_\mu. \tag{9}$$

with $q_0 \equiv t$ and $S$ being the action. The equations of motion are obtained as total differential equations in many variables such as,

$$dq_a = \frac{\partial H'_\alpha}{\partial P_a} dt_\alpha, \tag{10}$$

$$dP_r = -(-1)^{n_r n_\alpha} \frac{\partial H'_\alpha}{\partial q_r} dt_\alpha, \qquad r = 0, 1, \ldots, N, \tag{11}$$

$$dZ = \left(-H_\alpha + P_a \frac{\partial H'_\alpha}{\partial P_a}\right) dt_\alpha, \tag{12}$$

where $n_i = 0\,, 1,\ (i = r\,, \alpha)$ define the Grassmann parity of the corresponding quantity, and $Z = S\left(t_\alpha, q_a\right)$. These equations are integrable if and only if [27, 28]

$$dH'_0 = 0, \tag{13}$$

and

$$dH'_\mu = 0, \qquad \mu = N-p+1, \ldots, N. \tag{14}$$

If the conditions (13) and (14) are not satisfied identically, we consider them as new constraints and we examine their variations. Thus repeating this procedure, one may obtain a set of constraints such that all the variations vanish, then we may solve the equations of motion (10) and (11) to get the canonical phase-space coordinates as

$$q_a \equiv q_a(t, t_\mu), \qquad p_a \equiv p_a(t, t_\mu), \qquad \mu = 1, \dots, p. \tag{15}$$

In this case the path integral representation may be written as

$$\langle Out \mid S \mid In \rangle = \int \prod_{a=1}^{n-r} dq^a dp^a \exp\left[ i \int_{t_\alpha}^{t'_\alpha} \left( -H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right], \tag{16}$$

$a = 1, \dots, n-p, \qquad \alpha = 0, n-p+1, \dots, n.$

We should notice that the integral (16) is an integration over the canonical phase space coordinates $(q_a, p_a)$.

## 3. Hamilton-Jacobi Formulation of Brink-Schwarz Superparticle

One may write an action for a particle moving in a superspace; which is an extension of ordinary $4\,D$ spacetime to include extra anticommuting coordinates in the form of $N$ two-components Weyl spinors $\theta$, $\bar{\theta}$ , where $\bar{\theta}$ is the conjugate of $\theta$. Such action, firstly is written by Brink-Schwarz with simple supersymmetry $N = 1$ [26] by the Lagrangian

$$L = \frac{1}{2}[e^{-1}(\dot{x}^\mu - i\bar{\theta}\gamma^\mu\dot{\theta})^2 + em^2]. \tag{17}$$

The singularity of the the Lagrangian follows from the fact that the rank of the Hessian matrix $A_{ij}$ is one.

The canonical momenta defined in (2) and (3) read as

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = e^{-1}\left( \dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} \right), \tag{18}$$

$$\pi_\theta = \frac{\partial_r L}{\partial \dot{\theta}} = -i\bar{\theta}P_\mu\gamma^\mu = -H_\theta, \tag{19}$$

$$\bar{\pi}_{\bar{\theta}} = \frac{\partial_r L}{\partial \dot{\bar{\theta}}} = 0 = -H_{\bar{\theta}}, \tag{20}$$

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0 = -H_e. \tag{21}$$

Since the rank of the Hessian matrix is one, we can solve (18) for $\dot{x}^\mu$ in terms of $P_\mu$ and other coordinates, in the form

$$\dot{x}_\mu = eP_\mu + i\bar{\theta}\gamma_\mu\dot{\theta}. \tag{22}$$

The canonical Hamiltonian $H_0$ is

$$H_0 = \frac{1}{2}e[P^2 - m^2]. \tag{23}$$

The set of HJPDE's are

$$H_0' = P_0 + \frac{1}{2}e[P^2 - m^2], \tag{24}$$

$$H_\theta' = P_\theta + i\bar{\theta}P_\mu\gamma^\mu, \tag{25}$$

$$H_{\bar{\theta}}' = P_{\bar{\theta}}, \tag{26}$$

$$H_e' = P_e. \tag{27}$$

Therefore, the total differential equations for the characteristics read as

$$dx_\mu = eP_\mu d\tau + i\bar{\theta}\gamma_\mu d\theta, , \tag{28}$$

$$dP_\mu = 0, \tag{29}$$

$$dP_\theta = 0, \tag{30}$$

$$dP_{\bar{\theta}} = (-iP_\mu\gamma^\mu)d\theta, \tag{31}$$

$$dP_e = -\frac{1}{2}[P^2 - m^2]dt. \tag{32}$$

To check whether the set of (28) to (32) are integrable or not, let us consider the total variations of the set of (HJPDE)'s. The variation of

$$dH_0' = 0, \tag{33}$$

$$dH_\theta' = 0, \tag{34}$$

$$dH_{\bar{\theta}}' = 0, \tag{35}$$

are identically zero, whereas

$$dH_e' = -(\frac{1}{2}[P^2 - m^2])dt = H_e'' dt. \tag{36}$$

where

$$H_e'' = \frac{1}{2}[P^2 - m^2] = 0. \tag{37}$$

is a new constraint. We notice that the total differential of $H_e''$ vanish identically, i.e.

$$dH_e'' = 0. \tag{38}$$

Thus the set of equations (28)-(32) with (41) are integrable.According to (12) the action can be written as

$$\begin{aligned} dZ &= -H_0 d\tau - H_\theta d\theta - H_{\bar{\theta}} d\bar{\theta} - H_e de + P_\mu dx^\mu \\ &= \left\{ -\frac{1}{2}\, e\Big( P^2 - m^2 \Big) + P_\mu \Big( \dot{x} - i\bar{\theta}\gamma^\mu\dot{\theta} \Big) \right\} d\tau, \end{aligned} \tag{39}$$

and the canonical action integral becomes

$$S = \int \left\{ \frac{1}{2}\, e\Big( P^2 + m^2 \Big) \right\} d\tau. \tag{40}$$

By using (40) and (16) the canonical path integral quantization of Brink-Schwarz superparticle is expressed as

$$\left\langle x_\mu, \tau\, ;\, x'_\mu, \tau' \right\rangle = \int dx_\mu\, dp_\mu exp\Big[ i \int \Big\{ \frac{1}{2} e\big(P^2 + m^2\big) \Big\} d\tau \Big] \tag{41}$$

This path integral representation as an integration over the canonical phase-space with no need to introduce any gauge fixing to reduce or enlarge the phase-space as in covariant quantization of Brink-Schwarz superparticle described in references [29, 30, 31].

## Conclusion

In this work we presented Brink-Schwarz Superparticle as a singular system, and its Hamiltonian treatment contains all kinds of constraints (primary and secondary, first and second class ones). This model is very illustrative, since it allows a comparison between all features of Diracs and Hamilton-Jacobi formalisms. In Dirac's formalism, we must reduce any constrained singular system to one with first-class constraints only, we must call attention to the presence of arbitrary variables in some of the Hamiltonian equations of motion due to the fact that we have gauge dependent variables and we have made a gauge fixing. This does not occur in Hamilton-Jacobi formalism since it provides a gauge-independent description of the systems evolution due to the fact that the Hamilton-Jacobi function $S$ contains all the solutions that are related by gauge transformations. The canonical path integral quantization of Brink-Schwarz Superparticle is done, since the system is integrable, and the integration is taken over the canonical phase space.

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