Received: Added at production | Revised: Added at production | Accepted: Added at production
DOI: xxx/xxxx

ARTICLE TYPE

On the design and analysis of high order Weerakoon-Fernando methods based on weight functions

Prem B. Chand[1] | Francisco I. Chicharro[2] | Pankaj Jain[1]

[1]Department of Mathematics, South Asian University, Akbar Bhawan, Chanakya Puri, 110021 New Delhi, India
[2]Escuela Superior de Ingeniería y Tecnología, Universidad Internacional de La Rioja, Avenida de La Paz 137, 26006 Logroño, Spain

Correspondence
*F. I. Chicharro. Email: francisco.chicharro@unir.net

Abstract

In this paper, using the idea of weight functions on Weerakoon-Fernando's method, an optimal fourth order method and some higher order multipoint methods for solving nonlinear equations are proposed. These methods are tested in some real applications and numerical examples and the results are compared with some existing methods. Their dynamical behavior on complex polynomials is analyzed and basins of attraction of these methods are presented.

KEYWORDS:
weight function; nonlinear equation; Weerakoon-Fernando method; complex dynamics; applications

1 | INTRODUCTION

Nonlinear equations arise in almost all branches of Science and Engineering. The solution of such equations is rarely obtained by analytical methods, so iterative methods are employed. The most commonly used iterative method to solve the nonlinear equation $f(x) = 0$ is Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which is known to be quadratically convergent for simple zeros. Many mathematicians continue to contribute to Newton's method of improving the order of convergence in several ways.

Weerakoon[29] suggests an improvement in the iteration of Newton's method at the expense of an additional first derivative evaluation. The convergent cubic method of Weerakoon and Fernando method is given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x^*_{n+1})}, \tag{2}$$

where $x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is the Newton's component.

There are various techniques (see[20,22,23,27]) for developing new methods and improving the convergence of any iterative method. One of the most effective techniques is the use weight functions. This technique can be applied both on solving nonlinear equations[6,7,16,25] and systems of nonlinear equations[3,12,14].

In[7], authors proposed the following method:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(y_n)} \cdot H(t_n), \end{aligned} \tag{3}$$

where $t_n = \frac{f'(y_n)}{f'(x_n)}$ and H is a weight function. Method (3) was shown to be of order 4 if $H(1) = 1$, $H'(1) = \frac{1}{4}$, $H''(1) = \frac{3}{4}$ and $|H^{(3)}(1)| < \infty$.

In[25], authors proposed the following fourth-order iterative method using the weight function in Newton's method,

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left(-\frac{1}{2} + \frac{9}{8}\frac{f'(x_n)}{f'(y_n)} + \frac{3}{8}\frac{f'(y_n)}{f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}. \tag{4}$$

In this document, we propose new methods of order 4, 5, 6 and 8 using weight functions and based on the Weerakoon and Fernando method (2). An important ingredient of this paper is the dynamical behavior of the introduced methods. It is well known that the dynamical properties of the rational operator give important information about the convergence, efficiency, and stability of the iterative methods. In the last decades, the study of the dynamical behavior of the rational operator associated to an iterative method has become a fast growing area of research (see[4,13,17,21,24]). Further, there is an extensive literature[1,5,8,9] on the dynamics of rational functions. We discuss the dynamics of the proposed methods.

The paper is organized as follows: Section 2 presents the development of the methods and their corresponding error equations. In Section 3, these methods are tested in some Engineering applications and the results are compared with other known methods. Section 4 covers the dynamics of the methods for analyzing their stability. Finally, Section 5 presents the main conclusions.

2 | DEVELOPMENT OF METHODS AND THEIR CONVERGENCES

This section is focused on the generation of iterative methods based on (2) A fourth order method is generated using a weight function in the second step of (2).

Then a third step is introduced from which different iterative schemes are obtained. On the one hand, we apply a structure similar to (2) and introduce only a new functional evaluation, which results in a method of order five. On the other hand, we use a Newton-type expression with a frozen derivative and a weight function, obtaining two order six methods. Finally, if we directly include Newton's method as a third step we obtain an eighth order method.

2.1 | Fourth Order Method

We begin with the two step weighted Weerakoon-type method

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left(a_1 + a_2\frac{f'(x_n)}{f'(y_n)} + a_3\frac{f'(y_n)}{f'(x_n)}\right)\frac{2f(x_n)}{f'(x_n)+f'(y_n)}. \tag{5}$$

Below we prove the convergence of method (5).

Theorem 1. Let f be a complex valued function defined on some interval I having sufficient number of smooth derivatives. Let α be a simple root of the equation $f(x) = 0$. Then method (5) is of order four if $a_1 = -\frac{1}{4}$, $a_2 = \frac{3}{4}$ and $a_3 = \frac{1}{2}$.

Proof. Let e_n be the error in x_n. Denote $c_j = \frac{f^{(j)}(\alpha)}{j!\, f'(\alpha)}$. Then in view of Taylor's series expansion, we have

$$f(x_n) = f'(\alpha)\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)\right). \tag{6}$$

and

$$f'(x_n) = f'(\alpha)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3) + O(e_n^4). \tag{7}$$

so that,

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} = \alpha + \frac{e_n}{3} + \frac{2}{3}c_2e_n^2 - \frac{4}{3}(c_2^2 - c_3)e_n^3 + \frac{2}{3}(4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + O(e_n^5). \tag{8}$$

Now, using Taylor's series expansion of $f'(y_n)$ about α, (8) gives

$$f'(y_n) = f'(\alpha)\big(1 + \frac{2}{3}c_2e_n + \frac{1}{3}(4c_2^2 + c_3)e_n^2 - \frac{4}{27}(-18c_2^3 + 27c_2c_3 + c_4)e_n^3 + \frac{4}{9}(12c_2^4 - 24c_2^2c_3 + 6c_3^2 + 11c_2c_4)e_n^4 + O(e_n^5)\big). \tag{9}$$

Now, from (6), (7) and (9), we get

$$\frac{f(x_n)}{f'(x_n) + f'(y_n)} = \frac{1}{2}e_n - \frac{1}{6}c_2e_n^2 - \frac{1}{9}(c_2^2 + 3c_3)e_n^3 + \frac{1}{54}(50c_2^3 - 15c_2c_3 - 29c_4)e_n^4 + O(e_n^5). \tag{10}$$

$$\frac{f'(x_n)}{f'(y_n)} = 1 + \frac{4}{3}e_n - \frac{4}{9}(5c_2^2 - 6c_3)e_n^2 + \frac{8}{27}(8c_2^3 - 21c_2c_3 + 13c_4)e_n^3 - \frac{1}{81}(32c_2^4 - 540c_2^2c_3 + 288c_3^2 + 620c_2c_4 - 405c_5)e_n^4 + O(e_n^5). \tag{11}$$

and

$$\frac{f'(y_n)}{f'(x_n)} = 1 - \frac{4}{3}e_n + \frac{4}{3}(3c_2^2 - 2c_3)e_n^2 - \frac{8}{27}(36c_2^3 - 45c_2c_3 + 13c_4)e_n^3 + \frac{1}{27}(720c_2^4 - 1332c_2^2c_3 + 288c_3^2 + 484c_2c_4 - 135c_5)e_n^4 + O(e_n^5). \tag{12}$$

Using the results (10), (11) and (12) in (5), we obtain the error equation as:

$$\begin{aligned} e_{n+1} = {} & (1 - a_1 - a_2 - a_3)e_n + \frac{1}{3}(a_1 - 3a_2 + 5a_3)c_2e_n^2 \\ & + \frac{2}{9}(13a_2c_2^2 - 19a_3c_2^2 - 9a_2c_3 + 15a_3c_3 + a_1(c_2^2 + 3c_3))e_n^3 \\ & + \frac{1}{27}(-3a_2(42c_2^3 - 77c_2c_3 + 25c_4) + a_1(-50c_2^3 + 15c_2c_3 + 29c_4) \\ & + a_3(266c_2^3 - 393c_2c_3 + 133c_4))e_n^4 + O(e_n^5). \end{aligned}$$

In order to cancel the terms of order less than four, we solve the system

$$\left.\begin{aligned} 1 - a_1 - a_2 - a_3 &= 0, \\ a_1 - 3a_2 + 5a_3 &= 0, \\ 13a_2c_2^2 - 19a_3c_2^2 - 9a_2c_3 + 15a_3c_3 + a_1(c_2^2 + 3c_3) &= 0, \end{aligned}\right\}$$

which gives, $a_1 = -\frac{1}{4}$, $a_2 = \frac{3}{4}$ and $a_3 = \frac{1}{2}$. Consequently, the error equation becomes

$$e_{n+1} = \tfrac{1}{9}(17c_2^3 - 9c_2c_3 + c_4)e_n^4 + O(e_n^5).$$

and the assertion is proved. □

2.2 | Fifth Order Method

In view of Theorem 1, we obtain the following fourth order method:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \left(-\frac{1}{4} + \frac{3}{4}\frac{f'(x_n)}{f'(y_n)} + \frac{1}{2}\frac{f'(y_n)}{f'(x_n)}\right)\frac{2f(x_n)}{f'(x_n) + f'(y_n)}. \end{aligned} \tag{13}$$

We modify method (13) to increase the order of convergence including a third step. In this regard, we propose the following method:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(-\frac{1}{4} + \frac{3}{4}\frac{f'(x_n)}{f'(y_n)} + \frac{1}{2}\frac{f'(y_n)}{f'(x_n)}\right)\cdot\frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \\ x_{n+1} &= z_n - \frac{2f(z_n)}{f'(x_n)+f'(y_n)}. \end{aligned} \tag{14}$$

Below we prove the convergence of method (14).

Theorem 2. Let f be a complex valued function defined on some interval I having sufficient number of smooth derivatives. Let α be a simple root of the equation $f(x) = 0$. Then method (14) has fifth order of convergence.

Proof. Following the proof of Theorem 1 and using terms with more powers of e_n, we obtain

$$z_n = \alpha + \frac{1}{9}(17c_2^3 - 9c_2c_3 + c_4)e_n^4 + \frac{1}{36}(-336c_2^4 + 480c_2^2c_3 - 80c_2c_4 + 9(-8c_3^2 + c_5))e_n^5 + O(e_n^6). \tag{15}$$

Now, expanding $f(z_n)$ about α we get

$$\begin{aligned} f(z_n) = f'(\alpha)\Big(&\frac{1}{9}(17c_2^3 - 9c_2c_3 + c_4)e_n^4 \\ &+ \frac{1}{36}(-336c_2^4 + 480c_2^2c_3 - 80c_2c_4 + 9(-8c_3^2 + c_5))e_n^5 + O(e_n^6)\Big). \end{aligned} \tag{16}$$

and consequently, we have

$$\begin{aligned} \frac{f(z_n)}{f'(x_n)+f'(y_n)} = &\frac{1}{18}(17c_2^3 - 9c_2c_3 + c_4)e_n^4 \\ &+ \frac{1}{216}(-1280c_2^4 + 1584c_2^2c_3 - 256c_2c_4 - 216c_3^2 + 27c_5)e_n^5 + O(e_n^6). \end{aligned} \tag{17}$$

Using (15) and (17) in (14), we obtain the error equation as

$$e_{n+1} = \frac{4}{27}c_2(17c_2^3 - 9c_2c_3 + c_4)e_n^5 + O(e_n^6),$$

which shows that method (14) has order of convergence five. □

2.3 | Sixth Order Method

Applying some weight functions on the variant of method (14) of order five, higher order methods can be obtained. For instance, we propose the following methods:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(-\frac{1}{4} + \frac{3}{4}\frac{f'(x_n)}{f'(y_n)} + \frac{1}{2}\frac{f'(y_n)}{f'(x_n)}\right)\cdot\frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\cdot\left(a_1 + a_2\frac{f'(y_n)}{f'(x_n)}\right). \end{aligned} \tag{18}$$

and

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(-\frac{1}{4} + \frac{3}{4}\frac{f'(x_n)}{f'(y_n)} + \frac{1}{2}\frac{f'(y_n)}{f'(x_n)}\right)\cdot\frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(y_n)}\cdot\left(b_1 + b_2\frac{f'(x_n)}{f'(y_n)}\right). \end{aligned} \tag{19}$$

Below, we present the theorem of the convergence of methods (18) and (19).

Theorem 3. Let f be a real or complex valued function defined on some interval I having sufficient number of smooth derivatives. Let α be a simple root of the equation $f(x) = 0$. Then, method (18) is of order six, if $a_1 = \frac{5}{2}$ and $a_2 = -\frac{3}{2}$. Also, method (19) is of order six, if $b_1 = b_2 = \frac{1}{2}$.

Proof. It is easy to calculate from (7), (9) and (16), that

$$\frac{f(z_n)}{f'(x_n)} = \frac{1}{9}(17c_2^3 - 9c_2c_3 + c_4)e_n^4 + \frac{1}{36}(-472c_2^4 + 552c_2^2c_3 - 88c_2c_4 + 9(-8c_3^2 + c_5))e_n^5 + O(e_n^6), \tag{20}$$

and

$$\frac{f(z_n)}{f'(y_n)} = \frac{1}{9}(17c_2^3 - 9c_2c_3 + c_4)e_n^4 + \big(\frac{-286}{27}c_2^4 + 14c_2^2c_3 - 2c_3^2 - \frac{62}{27}c_2c_4 + \frac{1}{4}c_5\big)e_n^5 + O(e_n^6). \tag{21}$$

Now, using (12) and (20) in (18), we get the error equation of method (18) as

$$\begin{aligned} e_{n+1} = & -\frac{1}{9}(a_1 + a_2 - 1)(17c_2^3 - 9c_2c_3 + c_4)e_n^4 + \frac{1}{108}(8(177a_1 + 211a_2 - 126)c_2^4 \\ & - 72(23a_1 + 25a_2 - 20)c_2^2c_3 + 8(33a_1 + 35a_2 - 30)c_2c_4 + 27(a_1 + a_2 - 1)(8c_3^2 - c_5))e_n^5 \\ & + \frac{1}{324}(-8c_2^5(2227a_1 + 3241a_2 - 1165) + 12c2^3c_3(2799a_1 + 3595a_2 - 1818) \\ & - 24c_2^2c_4(335a_1 + 385a_2 - 269) - 27c_2(4c_3^2(107a_1 + 123a_2 - 86) \\ & + c_5(-51a_1 - 55a_2 + 45)) + 12c_3c_4(207a_1 + 215a_2 - 198))e_n^6 + O(e_n^7). \end{aligned}$$

Therefore, method (18) is of order 6 if

$$\left.\begin{aligned} a_1 + a_2 - 1 &= 0, \\ 8(177a_1 + 211a_2 - 126)c_2^4 - 72(23a_1 + 25a_2 - 20)c_2^2c_3 + 8(33a_1 + 35a_2 - 30)c_2c_4 + 27(a_1 + a_2 - 1)(8c_3^2 - c_5) &= 0, \end{aligned}\right\}$$

whose solution is $a_1 = \frac{5}{2}$ and $a_2 = -\frac{3}{2}$. Hence, we get a sixth order method of the form:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(-\frac{1}{4} + \frac{3}{4}\frac{f'(x_n)}{f'(y_n)} + \frac{1}{2}\frac{f'(y_n)}{f'(x_n)}\right) \cdot \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \cdot \left(\frac{5}{2} - \frac{3}{2}\frac{f'(y_n)}{f'(x_n)}\right). \end{aligned} \tag{22}$$

Again, using (11) and (21) in equation (19), we get the error equation of the form:

$$\begin{aligned} e_{n+1} = & -\frac{1}{9}((-1 + b_1 + b_2)(17c_2^3 - 9c_2c_3 + c_4)e_n^4 + \frac{1}{108}(8(-126 + 143b_1 + 109b_2)c_2^4 \\ & - 72(-20 + 21b_1 + 19b_2)c_2^2c_3 + 8(-30 + 31b_1 + 29b_2)c_2c_4 + 27(-1 + b_1 + b_2)(8c_3^2 - c_5))e_n^5 \\ & + \frac{1}{324}(-8(-1165 + 1349b_1 + 607b_2)c_2^5 + 12(-1818 + 2051b_1 + 1351b_2)c_2^3c_3 \\ & - 8(-807 + 863b_1 + 729b_2)c_2^2c_4 + 12(-198 + 199b_1 + 191b_2)c_3c_4 \\ & - 27c_2(4(-86 + 91b_1 + 75b_2)c3^2 + (45 - 47b_1 - 43b_2)c_5))e_n^6 + O(e_n^7) \end{aligned}$$

so that method (19) is of order 6 if

$$\left.\begin{aligned} & -1 + b_1 + b_2 = 0 \\ & 8(-126 + 143b_1 + 109b_2)c_2^4 - 72(-20 + 21b_1 + 19b_2)c_2^2c_3 + +8(-30 + 31b_1 + 29b_2)c_2c_4 + 27(-1 + b_1 + b_2)(8c_3^2 - c_5) = 0, \end{aligned}\right\}$$

which on solving gives $b_1 = b_2 = \frac{1}{2}$. Hence, we get a sixth order method of the form

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(-\frac{1}{4} + \frac{3}{4}\frac{f'(x_n)}{f'(y_n)} + \frac{1}{2}\frac{f'(y_n)}{f'(x_n)}\right) \cdot \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(y_n)} \cdot \left(\frac{1}{2} + \frac{1}{2}\frac{f'(x_n)}{f'(y_n)}\right). \end{aligned} \tag{23}$$

□

TABLE 1 Comparison of different methods according to their order of convergence and efficiency.

Methods	N	S	M_1	M_2	M_3	M_4	M_5
n_f	2	3	3	4	4	4	5
p	2	4	4	5	6	6	8
E	1.414	1.587	1.587	1.495	1.565	1.565	1.516

2.4 | Eighth Order Method

When the fourth order method is composed with Newton's method, we get

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(-\frac{1}{4} + \frac{3}{4}\frac{f'(x_n)}{f'(y_n)} + \frac{1}{2}\frac{f'(y_n)}{f'(x_n)}\right)\cdot\frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}. \end{aligned} \tag{24}$$

Theorem 4. Let f be a real or complex valued function defined on some interval I having sufficient number of smooth derivatives. Let α be a simple root of the equation $f(x) = 0$. Then method (24) has eighth order of convergence.

Proof. Taylor's series expansion of $f'(z_n)$ about α by using (15) gives

$$\begin{aligned} f'(z_n) = f'(\alpha)[1 &+ \frac{2}{9}c_2(17c_2^3 - 9c_2c_3 + c_4)e_n^4 - \frac{1}{18}(c_2(336c_2^4 - 480c_2^2c_3 + 72c_3^2 + 80c_2c_4 - 9c_5))e_n^5 \\ &+ \frac{1}{162}(c_2(9320c_2^5 - 21816c_2^3c_3 + +6456c_2^2c_4 - 2376c_3c_4 + 27c_2(344c_3^2 - 45c_5))e_n^6 \\ &- \frac{1}{162}(c_2(22352c_2^6 - 74048c_2^4c_3 - 6192c_3^3 + 29600c_2^3c_4 - 27120c_2c_3c_4 + 2096c_4^2 \\ &+ 324c_2^2(176c_3^2 - 25c_5) + 3645c_3c_5))e_n^7 + O(e_n^8)] \end{aligned} \tag{25}$$

so that, from (16) and (25), we obtain

$$\begin{aligned} \frac{f(z_n)}{f'(z_n)} = &\frac{1}{9}(17c_2^3 - 9c_2c_3 + c_4)e_n^4 + \frac{1}{36}(-336c_2^4 + 480c_2^2c_3 - 80c_2c_4 + 9(-8c_3^2 + c_5)e_n^5 \\ &+ (\frac{2330}{81}c_2^5 - \frac{202}{3}c_2^3c_3 + \frac{86}{3}c_2c_3^2 + \frac{538}{27}c_2^2c_4 - \frac{22}{3}c_3c_4 - \frac{15}{4}c_2c_5)e_n^6 \\ &+ \frac{1}{324}(-22352c_2^6 + 74048c_2^4c_3 + 6192c_3^3 - 29600c_2^3c_4 + 27120c_2c_3c_4 - 2096c_4^2 \\ &- 324c_2^2(176c_3^2 - 25c_5) - 3645c_3c_5)e_n^7 + \frac{1}{2916}(392972c_2^7 - 1757952c_2^5c_3 + 870752c_2^4c_4 \\ &- 1376244c_2^2c_3c_4 + 36c_2^3(57299c_3^2 - 8928c_5) + 27c_4(8836c_3^2 - 2097c_5) \\ &- 6c_2(96066c_3^3 - 29500c_4^2 - 50301c_3c_5))e_n^8 + O(e_n^9). \end{aligned} \tag{26}$$

Applying (15) and (26) in (24), the error equation of method (24) is

$$e_{n+1} = \tfrac{1}{81}c_2(17c_2^3 - 9c_2c_3 + c_4)^2e_n^8 + O(e_n^9).$$

□

2.5 | Comparison of the methods

In this section, we compare the methods introduced previously in terms of order of convergence (p) and efficiency index ($E = p^{1/n_f}$), where n_f is the number of functional evaluations per iteration. Newton's (1) and Sharma's (4) methods are also included for the sake of comparison, denoted by N and S, respectively. The introduced methods given by (13), (14), (22), (23) and (24) are denoted by M_1, M_2, M_3, M_4 and M_5, respectively. The results are presented in Table 1.

TABLE 2 Results for drag coefficient c in the parachutist problem.

#Iteration	Results	M_1	M_2	M_3	M_4	M_5
	c	12.2715	12.4266	12.4616	12.4955	12.5310
1	$f(c)$	0.4859	0.1969	0.1321	0.0695	0.0042
	c	12.5333	12.5333	12.5333	12.5333	12.5333
2	$f(c)$	3.8857 $E(-7)$	5.2061 $E(-11)$	5.6843 $E(-14)$	0.0	0.0
	c	12.5333	12.5333	12.53337	12.5333	12.5333
3	$f(c)$	0.0	0.0	0.0	0.0	0.0

3 | APPLICATIONS AND EXAMPLES

In this section, numerical tests on the introduced methods are performed. In Subsections 3.1, 3.2 and 3.3 the methods are applied on standard engineering examples[11]. Subsection 3.4 covers a set of analytical examples. In all tables, $BE(-A)$ stands for $B \times 10^{-A}$ and M_i, $i = 1, 2, 3, 4, 5$ represent the methods of order 4, 5, 6, 6 and 8, respectively.

3.1 | Parachutist's problem

The total force F acting on a falling parachutist is the sum of two opposite forces, namely the downward force due to gravity F_d and the upward force due to air resistance F_u, so that $F = F_d + F_u$.

The force due to gravity is given by $F_d = mg$, where $g \approx 9.8\, m/s^2$ is the acceleration due to gravity and m is the mass of the parachutist. Air resistance can be assumed to be linearly proportional to the velocity v and acts in an upward direction, so $F_u = -cv$, where c is the proportionality constant called the drag coefficient (kg/s) and the negative sign indicates the upward direction. The parameter c is responsible for the properties of the falling object, such as the shape or surface roughness, which affect air resistance. In the case of parachutist, c may be the type of jumpsuit or the orientation used by the parachutist during free-fall.

The total force is obtained as

$$F = mg - cv.$$

On the other hand, if a denotes the acceleration, then the second law of motion gives $F = ma = m\frac{dv}{dt}$. Therefore, $\frac{dv}{dt} = g - \frac{c}{m}v$ is a differential equation. Assuming that the parachutist is initially at rest, $v(t = 0) = 0$, its solution is

$$v(t) = \frac{gm}{c}\left(1 - e^{-\frac{c}{m}t}\right). \tag{27}$$

Suppose that it is required to determine the drag coefficient c for a parachutist of a given mass m to attain a prescribed velocity v in a give time period t. Rearranging (27), we get the nonlinear equation

$$f(c) = \frac{gm}{c}\left(1 - e^{-\frac{c}{m}t}\right) - v. \tag{28}$$

We assume the values of the parameters as $g = 9.8\, m/s^2$, $m = 68\, kg$, $t = 8\, s$, and $v = 41\, m/s$. Below, we are applying the introduced iterative methods for solving (28). Initial guess for the drag coefficient is $c_0 = 3.0\, kg/s$ The results from first three iterations of each of the methods are presented in Table 2.

3.2 | Open-channel flow

An open problem in civil engineering is to relate the flow of water with other factors affecting the flow in open channels such as rivers or canals. The flow rate is determined as the volume of water passing a particular point in a channel per unit time. A further concern is related to what happens when the channel is slopping.

Under uniform flow conditions, the flow of water in an open channel is given by Manning's equation

$$Q = \frac{\sqrt{S}}{n} A R^{2/3}, \tag{29}$$

TABLE 3 Results for the depth water y in the open channel problem

#Iteration	Results	M_1	M_2	M_3	M_4	M_5
1	y	1.4701	1.4656	1.4653	1.4653	1.4650
	$f(y)$	0.0690	0.0080	0.0038	0.0032	0.0000
2	y	1.4650	1.4650	1.4650	1.4650	1.4650
	$f(y)$	5.5427 $E(-11)$	-3.5527 $E(-15)$	-3.5527 $E(-15)$	1.7763 $E(-15)$	1.7763 $E(-15)$
3	y	1.4650	1.4650	1.46506	1.4650	1.4650
	$f(y)$	-3.5527 $E(-15)$	-3.5527 $E(-15)$	-3.5527 $E(-15)$	1.7763 $E(-15)$	1.7763 $E(-15)$

where S is the slope of the channel, A is the cross-sectional area of the channel, R is the hydraulic radius of the channel and n is the Manning's roughness coefficient. For a rectangular channel of width B and water depth in the channel y, it is known that $A = By$ and $R = \dfrac{By}{B+2y}$. With these values, (29) becomes

$$Q = \frac{\sqrt{S}}{n} By \left(\frac{By}{B+2y} \right)^{2/3}. \tag{30}$$

Now, if it is required to determine the depth of water in the channel for a given quantity of water, (29) can be rearranged as

$$f(y) = \frac{\sqrt{S}}{n} By \left(\frac{By}{B+2y} \right)^{2/3} - Q. \tag{31}$$

In our work, we estimate y when remaining parameters are assumed to be given as $Q = 14.15\, m^3/s$, $B = 4.572\, m$, $n = 0.017$ and $S = 0.0015$. We choose the initial guess $y_0 = 3.0\, m$. The results obtained from first three iterations by using the methods $M_i, i = 1, ..., 5$, are presented in Table 3.

3.3 | Design of an electric circuit

A common problem in electrical engineering is the study of the steady state behavior of electric circuits. The voltages V_R, V_L and V_C are the potentials across the resistor R, the inductor L and the capacitor C, respectively. It is known that $V_R = iR$, $V_L = L\dfrac{di}{dt}$ and $V_C = \dfrac{q}{C}$, where q is the charge and $i = \frac{dq}{dt}$ is the flow of current.

According to Kirchhoff's second law, the algebraic sum of all the voltages around any closed circuit is zero which leads to $L\dfrac{di}{dt} + iR + \dfrac{q}{C} = 0$. Therefore,

$$L\frac{d^2q}{dt^2} + \frac{dq}{dt}R + \frac{q}{C} = 0$$

which is a differential equation whose solution can be obtained as

$$q(t) = q_0 e^{-\frac{Rt}{2L}} \cos\left(\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} t \right), \tag{32}$$

assuming that $q(t = 0) = q_0 = V_0 C$, being V_0 the voltage source.

Let us assume that the appropriate resistor R is required to be determined with known values of $L = 5\ H$ and $C = 10^{-4}\ F$ and C. Moreover, we take $\frac{q}{q_0} = 0.01$, i.e., the charge must be dissipated to 1 percent of its original value in time $t = 0.05\ s$. We choose the initial guess for the resistance $R_0 = 30\ \Omega$. The results obtained from first three iterations by using the methods M_i are presented in Table 4.

3.4 | Further examples

In this subsection, we study and compare the performance of some of the known methods as well as those obtained in the previous sections on some numerical examples. The methods have been tested on the following functions:

- $f_4(x) = \sqrt{x^2 + 2x + 5} - 2\sin x - x^2 + 3$,

TABLE 4 Results for the resistor R in the design of an electric circuit.

#Iteration	Results	M_1	M_2	M_3	M_4	M_5
1	R	304.7563	314.9646	317.5493	320.8943	327.1815
	$f(R)$	-0.0243	-0.0134	-0.0107	-0.0073	-0.0009
2	R	328.1494	328.1514	328.1514	328.1514	328.1514
	$f(R)$	-1.9369 E(-6)	-6.3306 $E(-9)$	-1.5278 $E(-10)$	-5.2303 $E(-12)$	-5.0306 $E(-17)$
3	R	328.1514	328.1514	328.1514	328.1514	328.15142908514815
	$f(R)$	3.2959 $E(-17)$	3.2959 $E(-17)$	3.2959 $E(-17)$	3.2959 $E(-17)$	-5.0306 $E(-17)$

- $f_5(x) = x - 3\ln x$,
- $f_6(x) = e^{-x}\sin x + \ln(1+x^2) - 2$,
- $f_7(x) = (x-1)^3 - 1$.

The stopping criteria for the iterative process has been set in $\Delta x = |x_{n+1} - x_n| \leq 10^{-12}$. The details of the work are presented in Table 5. The values displayed are the initial guess x_0, the number of iterations n for achieving the stopping criteria, and the approximated computational order of convergence ACOC[18]. The value NA in ACOC stands for Not Available, because the number of iterates is not enough for its calculation.

4 | DYNAMICS OF THE METHODS

In this section, we discuss the dynamics of the methods presented in Section 2 and compare them with some of the existing methods.

4.1 | Basics on complex dynamics

Below, some preliminaries of complex dynamics are presented. For further information, see[9,19].

For any rational function $f(z)$, a point z_0 is called a fixed point if $f(z_0) = z_0$. The critical points of $f(z)$ are those points that satisfy $f'(z) = 0$. The critical points may or may not coincide with the fixed points. Fixed and critical points are not necessarily the roots of $f(z) = 0$. In this case, they are called strange fixed and free critical points, respectively. A fixed point z_0 of $f(z)$ is superattracting, attracting, repelling or neutral if $f'(z_0) = 0$. $|f'(z_0)| < 1$. $|f'(z_0)| > 1$ or $|f'(z_0)| = 1$, respectively.

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function, where $\hat{\mathbb{C}}$ is the Riemann sphere. Successive iterations of R over z are called orbits and are given by the sequence $\{z, R(z), R^2(z), R^3(z), \cdots\}$. When iterative methods for solving nonlinear equations are applied on polynomials, they result in a rational function, commonly known as the rational operator. The above considerations for fixed and critical points can be directly extended to rational functions.

4.2 | Fixed and critical points

The strange fixed points of any operator may complicate the root finding procedure. If they have an attracting behavior, they may can trap an iteration sequence, giving incorrect results for a z^* root of the $p(z)$ polynomial. Even as the repulsive or neutral fixed points, however, they may alter the structure of the basin of attraction for the roots[28]. Generally, increasing the order of convergence of any method increases the number of strange fixed points, which may adversely affect the basin of attraction of the method[21].

Corresponding to the Newton's method, define the operator

$$N(z) = z - \frac{p(z)}{p'(z)},$$

where $p(z)$ is a polynomial. Similarly, we consider the operators S and $M_i(z)$, $i = 1,2,3,4,5$ corresponding to the methods M_i, $i = 1,2,3,4,5$ as defined in Subsection 2.5.

TABLE 5 Numerical performance of the introduced methods.

Methods		$f_4(x)$ ($x_0 = 3.0$)	$f_5(x)$ ($x_0 = 1.0$)	$f_6(x)$ ($x_0 = 1.5$)	$f_7(x)$ ($x_0 = 3.0$)
N	n	5	7	5	7
	Δx	$0.444E-15$	0.0	$0.444E-15$	0.0
	x_{n+1}	2.331967655883964	1.857183860207835	2.447748286452425	2.0
	$\lvert f(x_{n+1})\rvert$	$0.888E-15$	0.0	0.0	0.0
	ACOC	2.046848	2.000437	1.853573	2.001497
S	n	3	4	3	4
	Δx	$0.888E-15$	$0.444E-15$	$0.444E-15$	$0.768E-13$
	x_{n+1}	2.331967655883964	1.857183860207835	2.447748286452425	2.0
	$\lvert f(x_{n+1})\rvert$	$0.888E-15$	0.0	0.0	0.0
	ACOC	3.848687	3.202382	3.776679	3.914934
M_1	n	3	4	3	4
	Δx	$0.888E-14$	0.0	$0.222E-14$	$0.111E-13$
	x_{n+1}	2.331967655883964	1.857183860207835	2.447748286452425	2.0
	$\lvert f(x_{n+1})\rvert$	$0.888E-14$	0.0	$0.222E-15$	0.0
	ACOC	4.018510	3.702951	4.026750	3.932947
M_2	n	3	4	3	4
	Δx	0.0	0.0	0.0	0.0
	x_{n+1}	2.331967655883964	1.857183860207835	2.447748286452425	2.0
	$\lvert f(x_{n+1})\rvert$	$0.888E-15$	$0.222E-15$	0.0	0.0
	ACOC	NA	4.601259	NA	3.975725
M_3	n	3	4	3	4
	Δx	0.0	0.0	0.0	0.0
	x_{n+1}	2.331967655883964	1.857183860207835	2.447748286452425	2.0
	$\lvert f(x_{n+1})\rvert$	$0.888E-15$	$0.222E-15$	0.0	0.0
	ACOC	NA	5.280387	NA	4.486354
M_4	n	3	4	3	4
	Δx	0.0	0.0	$0.888E-15$	0.0
	x_{n+1}	2.331967655883964	1.857183860207835	2.447748286452425	2.0
	$\lvert f(x_{n+1})\rvert$	$0.888E-15$	0.0	$0.222E-15$	0.0
	ACOC	NA	5.495705	NA	4.777322
M_5	n	3	3	3	3
	Δx	0.0	$0.444E-15$	0.0	$0.4E-14$
	x_{n+1}	2.331967655883964	1.857183860207835	2.447748286452425	2.0
	$\lvert f(x_{n+1})\rvert$	$0.888E-15$	0.0	0.0	0.0
	ACOC	NA	5.657281	NA	6.827861

4.3 | Conjugacy Classes

The methods presented in this paper verify the Scaling Theorem. By M_{ip}, we denote the method M_i when applied on the polynomial $p(z)$.

Theorem 5. (**Scaling theorem**): Let $T(z) = az + b$, $a \neq 0$ be an affine map in the Riemann sphere $\hat{\mathbb{C}}$, and let $\lambda \in \mathbb{C}$ be a non zero constant. Let $p(z)$ be a polynomial defined in $\hat{\mathbb{C}}$. Define $q(z) = \lambda(p \circ T)(z)$. Then

$$T \circ M_{iq} \circ T^{-1} = M_{ip},$$

i.e. T is a conjugacy between M_{ip} and M_{iq}, where $i = 1, 2, 3, 4, 5$.

Proof. We shall prove the assertion for M_1 only. For other M_i, it can be proved similarly. We have

$$M_{1q}(z) = z - \left[-\frac{1}{4} + \frac{3}{4} \frac{q'(z)}{q'(z - \frac{2}{3}\frac{q(z)}{q'(z)})} + \frac{1}{2} \frac{q'(z - \frac{2}{3}\frac{q(z)}{q'(z)})}{q'(z)} \right] \frac{2q(z)}{q'(z) + q'(z - \frac{2}{3}\frac{q(z)}{q'(z)})}.$$

Since $T^{-1}(z) = \frac{z-b}{a}$, and $T'(z) = a$, we obtain

$$\begin{aligned} T \circ M_{1q} \circ T^{-1}(z) &= T \circ M_{1q}(\frac{z-b}{a}) = a\left(M_{1q}(\frac{z-b}{a})\right) + b \\ &= a\left(\frac{z-b}{a} - \frac{2p(z)\left(\frac{3p'(z)}{4p'\left(a\left(\frac{z-b}{a} - \frac{2p(z)}{3ap'(z)}\right)+b\right)} + \frac{p'\left(a\left(\frac{z-b}{a} - \frac{2p(z)}{3ap'(z)}\right)+b\right)}{2p'(z)} - \frac{1}{4} \right)}{ap'\left(a\left(\frac{z-b}{a} - \frac{2p(z)}{3ap'(z)}\right)+b\right) + ap'(z)} \right) + b \\ &= M_{1p}. \end{aligned}$$

□

Verification of the scaling theorem involves how the methods for a given polynomial relate to the methods for a rescaled polynomial. In other words, we can transform the roots into a related map without qualitatively changing the dynamics of the corresponding methods[9,10].

4.3.1 | Corresponding Conjugacy Class for Quadratic Polynomials

Newton's method $N_p(z)$ on any quadratic polynomial $p(z)$ is conjugate to the quadratic polynomial z^2[9]. Moreover, method $S_p(z)$ is conjugate to $z^4\frac{7+8z+3z^2}{3+8z+7z^2}$. In this way, $R_N(z)$ and $R_S(z)$ refer to the application of the Scaling Theorem to $N_p(z)$ and $S_p(z)$, respectively. We show that methods $M_{ip}(z)$ on quadratic polynomials have more complicated expression for the conjugacy classes. The results are proved in the following theorem.

Theorem 6. Let $p(z) = (z - a)(z - b)$ with $a \neq b$, be a quadratic polynomial. Then, the operators $M_{ip}(z)$ with $i = 1, 2, 3, 4, 5$ are conjugated to the rational maps $R_i(z)$ with $i = 1, 2, 3, 4, 5$, respectively. Their expressions are

$$R_1(z) = z^4\frac{9z^2+18z+17}{17z^2+18z+9},$$

$$R_2(z) = z^5\frac{243z^7+1296z^6+3672z^5+6624z^4+7959z^3+6196z^2+2790z+612}{612z^7+2790z^6+6196z^5+7959z^4+6624z^3+3672z^2+1296z+243},$$

$$R_3(z) = z^6\frac{918+4032z+8843z^2+11560z^3+10081z^4+6048z^5+2493z^6+648z^7+81z^8}{(9+18z+17z^2)(9+54z+152z^2+266z^3+301z^4+180z^5+54z^6)},$$

$$R_4(z) = z^6\frac{243z^6+972z^5+2484z^4+4068z^3+4671z^2+3160z+1122}{(17z^2+18z+9)(66z^4+116z^3+117z^2+54z+27)},$$

$$R_5(z) = z^8\frac{(9z^2+18z+17)^2}{(17z^2+18z+9)^2}.$$

Proof. We prove the result for $M_1(z)$ only. The rest of cases can be proved similarly.

Consider the Möbius transformation $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by $h(z) = \frac{z-a}{z-b}$. Note that $h^{-1}(z) = \frac{bz-a}{z-1}$. Then

$$(h \circ M_1 \circ h^{-1})(z) = z^4\frac{9z^2+18z+17}{17z^2+18z+9} = R_1(z).$$

□

TABLE 6 Number of extraneous fixed points (#EFP) and free critical points (#FCP) for different rational functions when a quadratic polynomial is applied.

Method	$N(z)$,	$S(z)$	$R_2(z)$	$R_3(z)$	$R_4(z)$	$R_5(z)$	$R_6(z)$
#EFP	1	5	5	11	13	11	11
#FCP	0	3	4	14	13	8	6

4.4 | Basins of Attraction

The basins of attraction of any rational operator $R(z)$ explain the dynamical behavior of the methods. If z^* is an attracting fixed point of the rational operator $R(z)$, then the basin of attraction of z^* is the set

$$\mathcal{B}(z^*) = \left\{ z \in \hat{\mathbb{C}} \,:\, R^n(z) \to z^* \text{ as } n \to \infty \right\}.$$

Since the Möbius transform can be performed for quadratic polynomials, we are studying their behaviour. Regarding cubic polynomials, the analysis is performed in terms of different cubic polynomials.

In our work, we use Mathematica 9.0 for the symbolic calculations, while the representations of the basins of attraction, that follows the guideline of [15], are performed with Matlab R2017b. We divide the complex plane into 500×500 initial points in the domain $[-3, 3] \times [-3, 3]$ to determine the basins of attraction of the roots of the polynomials. Each root of the polynomial is mapped to a different color. If an initial guess tends to a root of the polynomial, this point is represented with its corresponding color. Indeed, the dynamical planes include information about the number of iterations required to converge to the root: the more iterations required, the darker color.

4.4.1 | Dynamical analysis applying the methods on quadratic polynomials

The rational maps of Theorem 6 correspond to the rational functions of the introduced methods when they are applied to quadratic polynomials. In every case, $z_1^* = 0$ is a superattracting fixed point. As introduced in [13], $z_2^* = \infty$ is also a superattracting fixed point. The rest of fixed points are repelling, improving the stability of the method. The number of fixed and critical points is displayed in Table 6.

Since every strange fixed point is repelling, their presence does not affect negatively to the stability of the corresponding methods. In order to verify this fact, and also to analyze the shape and width of the basins of attraction, Figure 1 represents the dynamical planes of the introduced methods. Note that the orange color is devoted to the root $z_1^* = 0$ and the blue one corresponds to the superattracting point $z_2^* = \infty$.

4.4.2 | Dynamical analysis applying the methods on cubic polynomials

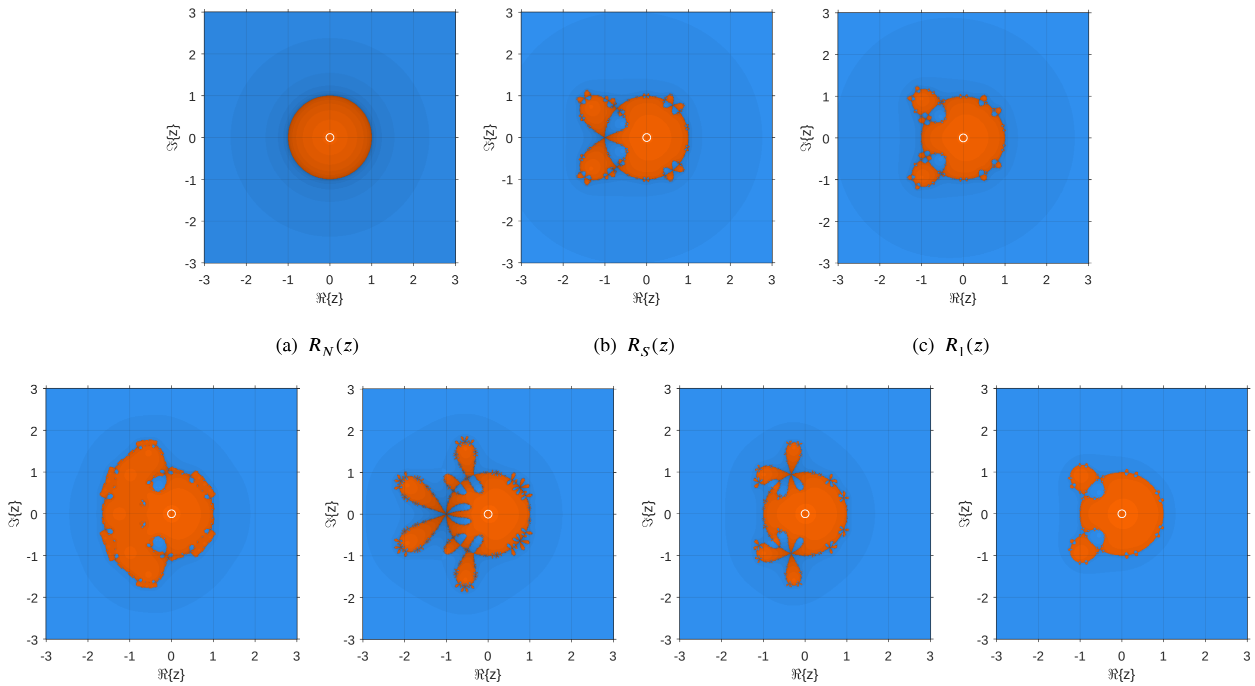
When the rational functions are applied on cubic polynomials, the resulting expressions are in terms of the original method instead of using their Möbius transform. In order to analyze a complete behavior over the cubic polynomials, the iterative methods are applied on $p_\bullet(z) = z^3$, $p_+(z) = z^3 + z$ and $p_-(z) = z^3 - z$, as performed in [2].

In the $p_\bullet(z)$ case, there is no presence of fixed points, but the root $z_1^* = 0$, that is superattracting, and the $z_2 = \infty$, whose behavior is repelling.

When the polynomial $p_+(z)$ is applied, the roots z_0^* and $z_{3-4}^* = \pm i$ are superattracting points. As in the previous polynomial, the point $z_2 = \infty$ is a repelling point. In this case, there is a big presence of extraneous fixed points, but they behave in a repelling way.

Finally, the application of the methods on the polynomial $p_-(z)$ gives a similar performance than in the $p_+(z)$ case. The roots $z_0^* = 0$ and $z_{5-6}^* = \pm 1$ are superattracting, the point $z_2 = \infty$ is repelling, and the rest of strange fixed points are also repelling. Table 7 gathers the information about the extraneous fixed points and free critical points when the three polynomials are applied on the introduced methods.

Figures 2-4 represent the dynamical planes of the introduced methods. In Figure 2 a unique basin of attraction can be observed, due to the polynomial $p_\bullet(z) = z^3$ has only a superattracting point in $z_1^* = 0$. Therefore, every initial guess whose orbit tends to this root is represented in orange.



(a) $R_N(z)$ (b) $R_S(z)$ (c) $R_1(z)$

(d) $R_2(z)$ (e) $R_3(z)$ (f) $R_4(z)$ (g) $R_5(z)$

FIGURE 1 Dynamical planes of several methods for quadratic polynomials.

TABLE 7 Number of extraneous fixed points (#EFP) and free critical points (#FCP) for different rational functions when a cubic polynomial is applied.

$p(z)$	Method	$N(z)$,	$S(z)$	$R_2(z)$	$R_3(z)$	$R_4(z)$	$R_5(z)$	$R_6(z)$
$p_\bullet(z)$	#EFP	0	0	0	0	0	0	0
	#FCP	0	0	0	0	0	0	0
$p_+(z)$	#EFP	0	12	12	42	48	42	42
	#FCP	0	12	18	44	48	42	42
$p_-(z)$	#EFP	0	12	12	42	48	42	42
	#FCP	0	12	18	44	48	44	22

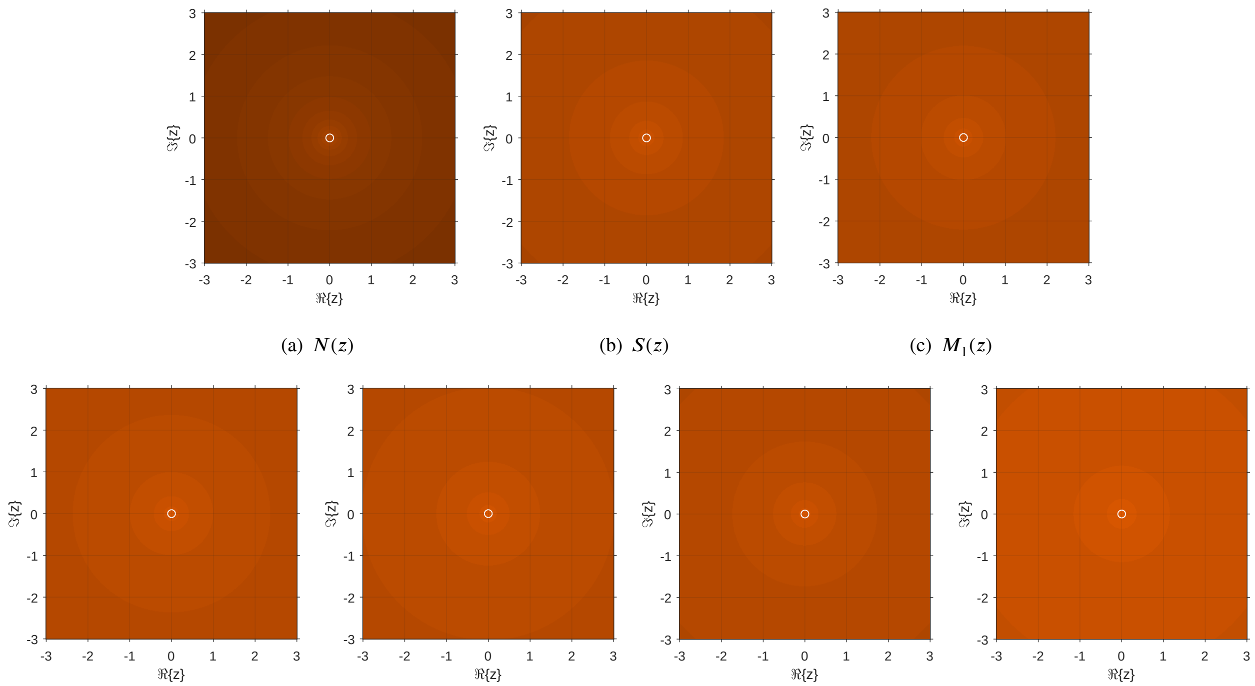
The stability of every method, when they are applied on $p_\bullet(z)$, is verified in the complete plane, as pictured in Figure 2. Note that every initial guess in the complex plane tends to the superattracting point $z_1^* = 0$.

Figure 3 illustrates the dynamical planes of methods $M_{1-5}(z)$ when they are applied on $p_+(z) = z^3 + z$. In this case, there are three superattracting points: z_0^* is mapped to color orange, $z_3^* = i$ is mapped to color blue and $z_4^* = -i$ is mapped to color green.

The dynamical planes of Figure 3 confirm the theoretical analysis. There is not any fixed point different to the roots that attracts any orbit. All initial estimates tend to one of the three superattracting points, showing the wide region of stability of every method. The methods of order 6 show a more intricated Julia set than the other ones. However, when the order is increased, this behavior is not observed. The composition with Newton's method makes the borderlines of the dynamical planes smoother.

Finally, Figure 4 represents the dynamical planes of the methods when they are applied on $p_-(z) = z^3 - z$. In this case, the orange basin remains with $z_1^* = 0$, while blue and green basins are referred to $z_5^* = 1$ and $z_6^* = -1$, respectively.

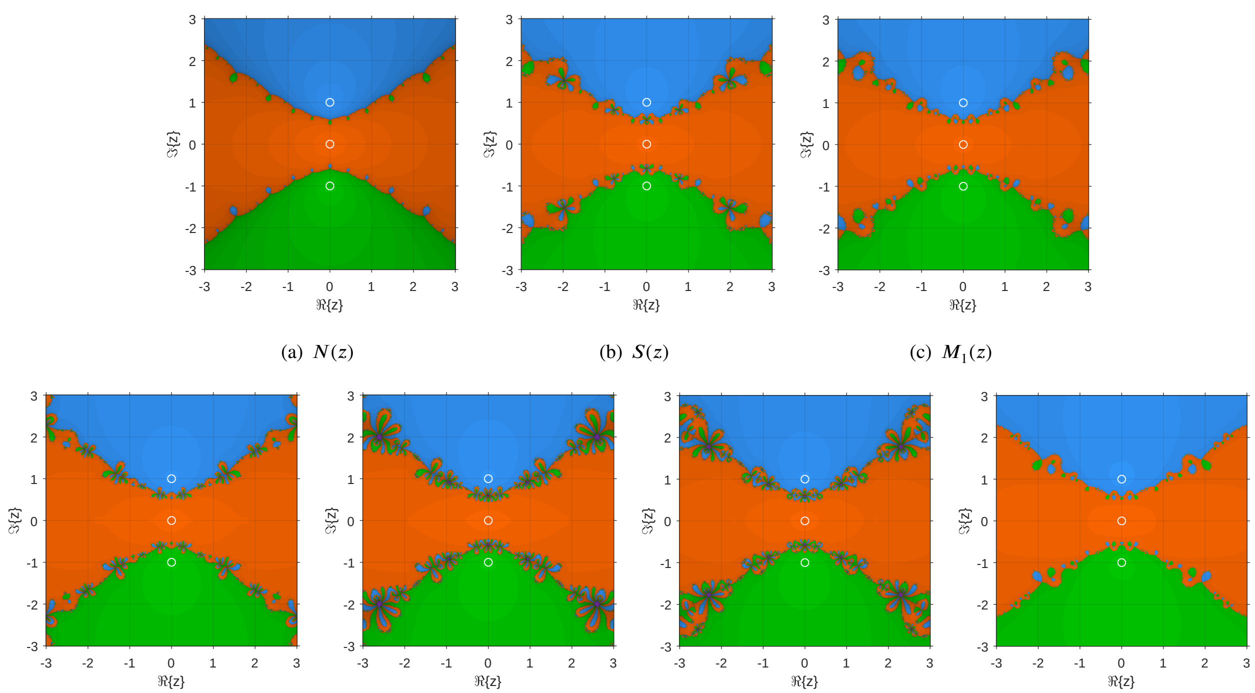
The features of Figure 4 coincide with the Figure 3 corresponding ones. Note that the planes of Figure 4 are just a rotation of the planes of Figure 3.



(a) $N(z)$ (b) $S(z)$ (c) $M_1(z)$

(d) $M_2(z)$ (e) $M_3(z)$ (f) $M_4(z)$ (g) $M_5(z)$

FIGURE 2 Dynamical planes of several methods for the cubic polynomial $p_\bullet(z)$.



(a) $N(z)$ (b) $S(z)$ (c) $M_1(z)$

(d) $M_2(z)$ (e) $M_3(z)$ (f) $M_4(z)$ (g) $M_5(z)$

FIGURE 3 Dynamical planes of several methods for the cubic polynomial $p_+(z)$.

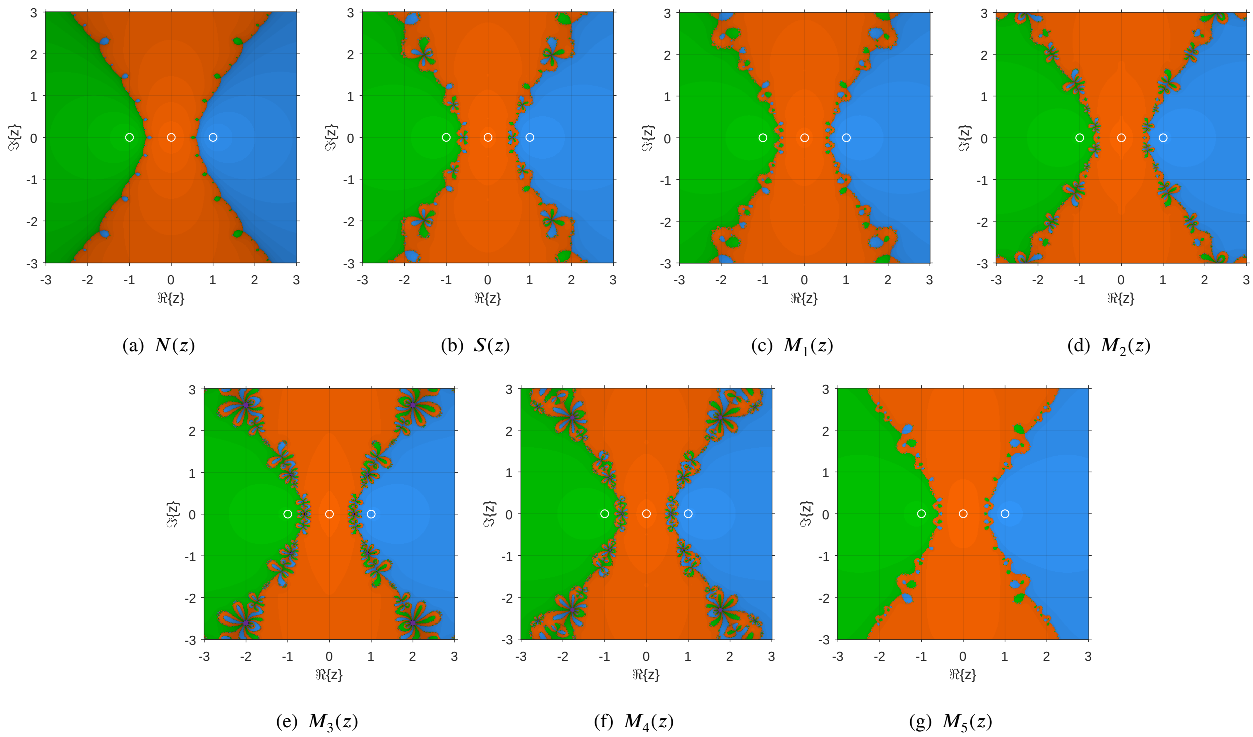


FIGURE 4 Dynamical planes of several methods for the cubic polynomial $p\_(z)$.

5 | CONCLUSIONS

A new set of iterative methods, based on Weerakoon and Fernando method, have been introduced. These methods, of different orders of convergence, have been checked by numerical and stability tests. On the one hand, the numerical performance makes evident the power of these methods, as they all converge to the expected root and reduce the number or iterations compared to Newton's method. On the other hand, the stability analysis, carried out in terms of complex dynamics, shows that every initial guess tends to a superattracting point that matches with the root of the corresponding polynomial. This fact is due to the absence of strange fixed points that behave attracting in both quadratic and cubic polynomials.

ACKNOWLEDGEMENTS

This research was supported by PGC2018-095896-B-C22 (MCIU/AEI/FEDER/UE).

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

References

1. Alexander, D. S. *A History of Complex Dynamics: From Schröder to Fatou and Julia*. Friedr & Sohn Veriage; 1994.

2. Amat, S., Busquier, S., Plaza, S. Chaotic dynamics of a third-order Newton-type method. *J. Math. Anal. Appl.* 2005; 366: 24–32.

3. Amiri, A., Cordero, A., Darvishi, M. T., Torregrosa, J. R. Stability analysis of a parametric family of seventh-order iterative methods for solving nonlinear systems. *Appl. Math. Comput.* 2018; 323: 43–57.

4. Argyros, I. K., Cordero, A., Magreñán, Á. A., Torregrosa, J. R. Third-degree anomalies of Traub's method. *J. Comput. Appl. Math.* 2017; 309: 511–521.

5. Argyros, I. K., Magreñán, Á. A. *Iterative Methods and Their Dynamics with Applications: A Contemporary Study*. CRC Press; 2016.

6. Artidiello, S., Cordero, A., Torregrosa, J. R., Vassileva, M. P. Design and multidimensional extension of iterative methods for solving nonlinear problems. *Appl. Math. Comput.* 2017; 293: 194–203.

7. Babajee, D. K. R. Several improvements of the 2-point third order midpoint iterative method using weight functions. *Appl. Math. Comput.* 2012; 218: 7958–7966.

8. Beardon, A. F. *Iteration of Rational Functions*, Springer-Verlag; 1991.

9. Blanchard, P. The dynamics of Newton's method. *Proc. Sympos. Appl. Math.* 1994; 49: 139–154.

10. Carleson, L., Gamelin, T. W. *Complex Dynamics*. Springer-Verlag; 1992.

11. Chapra, S. C., Canale, R. C. *Numerical Methods for Engineers*. McGraw Hill; 2015.

12. Chicharro, F. I., Cordero, A., Garrido, N., Torregrosa, J. R. Generalized high-order classes for solving nonlinear systems and their applications. *Mathematics* 2019; 1194: 1–14.

13. Chicharro, F. I., Cordero, A., Gutiérrez, J. M., Torregrosa, J. R. Complex dynamics of derivative free methods for nonlinear equations. *Appl. Math. Comput.* 2013; 219: 7023–7035.

14. Chicharro, F. I., Cordero, A., Torregrosa, J. R. Dynamics of iterative families with memory based on weight functions procedure. *J. Comput. Appl. Math.* 2019; 354: 286–298.

15. Chicharro, F. I., Cordero, A., Torregrosa, J. R. Drawing dynamical and parameters planes of iterative families and methods. *Sci. World J.* 2013; 780153: 1–11.

16. Chun, C., Neta, B., Kozdon, J., Scott, M. Choosing weight functions in iterative methods for simple roots. *Appl. Math. Comput.* 2014; 227: 788–800.

17. Cordero, A., Feng, L., Magreñán, Á. A., Torregrosa, J. R. A new fourth order family for solving nonlinear problems and its dynamics. *J. Math. Chem.* 2015; 53: 893–910.

18. Cordero, A., Torregrosa, J. R. Variants of Newton's method using fifth order quadrature formulas. *App. Math. Comput.* 2007; 190: 686–698.

19. Devaney, R. L. *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley; 1989.

20. Kung, H. T., Traub, J. F. Optimal order of one-point and multipoint iterations. *J. Assoc. Comput. Mach.* 1984; 21: 643–651.

21. Neta, B., Chun, C. Basins of attractions for several methods to find the simple roots of nonlinear equations. *Appl. Math. Comput.* 2012; 218: 10548–10556.

22. Ostrowski, A. M. *Solutions of Equations and System of Equations*. Academic Press; 1960.

23. Petković, M. S., Neta, B, Petković, L. D., Džunić, J. *Multipoint Methods for Solving Nonlinear Equations*. Elsevier; 2013.

24. Sharma, J. R., Arora, H. A new family of optimal eighth order methods with dynamics for nonlinear equations. *Appl. Math. Comput.* 2016; 273: 924–933.

25. Sharma, R., Bahl, A. An optimal fourth order iterative method for solving nonlinear equations and its dynamics. *J. Complex Anal.* 2015; 259167: 1–9.

26. Scott, M., Neta, B., Chun, C. Basin attractors for various methods. *Appl. Math. Comput.* 2011; 218: 2584–2599.

27. Traub, J. F. *Iterative Methods for the Solution of Equations*. Chelsea; 1977.

28. Vrscay, E. R., Gilbert, W. J. Extraneous fixed points, basin boundaries and chaotic dynamics for Schröder and König rational iteration functions. *Numer. Math.* 1988; 52: 1–16.

29. Weerakoon, S., Fernando, T. G. I. A variant of Newton's method with accelerated third-order convergence. *Appl. Math. Lett.* 2000; 13: 87–93.

How to cite this article: P. B. Chand, F. I. Chicharro, and P. Jain (2020), On the design and analysis of high order Weerakoon-Fernando methods based on weight functions, *Computational and Mathematical Methods*, .