SIAM J. CONTROL AND OPTIMIZATION
Vol. 29, No. 1, pp. 209-221, January 1991

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ON MINIMUM ENERGY PROBLEMS*

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Abstract. A stochastic system described by a semilinear equation with a small noise is considered. Under suitable hypotheses, the rate functionals for the family of distributions associated to the solution and the exit time and exit place of the solution are computed.

Key words. stochastic systems, optimization, semilinear equations, exit time

AMS(MOS) subject classifications. 93B05, 93C20, 93E20

1. Introduction. The paper is concerned with several deterministic optimization questions which arise in the theory of small noise distributed systems.

Let us assume that a stochastic system is described by a semilinear equation

$$dX^\varepsilon = (AX^\varepsilon + F(X^\varepsilon))dt + \sqrt{\varepsilon}\, dW, \quad X^\varepsilon(0) = a \in H, \qquad \varepsilon > 0, \tag{1.1}$$

where A and F are, respectively, linear and nonlinear parts of the drift term and W is a Wiener process with incremental covariance Q on a Hilbert space H.

The optimization problems considered in this paper are motivated by the problem of finding the rate functionals for the family of distributions $\mathscr{L}(X^\varepsilon(\cdot))$, $\varepsilon > 0$. They are also related to the problem of calculating (for a given domain) the exit time and exit place of the processes $X^\varepsilon(\cdot)$, $\varepsilon > 0$, (see [4], [5], [11], [12], [14], [15]). Here and in the sequel the distribution of a random variable ξ is denoted as $\mathscr{L}(\xi)$.

If $E_T(a,\cdot)$ is the rate functional for the family of measures $\mu_\varepsilon = \mathscr{L}(X^{x,\varepsilon}(T))$, $\varepsilon > 0$, then also of importance is the functional $E_\infty(a,b) = \inf_{T>0} E_T(a,b)$, $a, b \in H$, which is sometimes called the *quasipotential* ([4], [5]). For an appropriate choice of the initial condition, E_∞ is the rate functional for the invariant distributions (ν_ε) of the process X^ε.

Assume that a is a stable equilibrium point for the deterministic system $\dot z = Az + F(z)$, and let $\mathscr{D}$ be a set contained in H which is open with respect to the strong topology and contains the point a. Define

$$\tau^\varepsilon = \inf\{t \geqq 0;\, X^\varepsilon(t) \in \partial\mathscr{D}\}$$

then $\lim_{\varepsilon\downarrow 0} \ln \varepsilon\mathscr{E}(\tau^\varepsilon)$ is called the *exit rate.*

Now let $y^{a,\phi}(\cdot)$ be a solution to the following controlled equation:

$$\dot y = Ay + F(y) + Q^{1/2}\phi, \qquad y(0) = a$$

in which ϕ stands for a square integrable function from $[0, +\infty[$ into H.

Under fairly general conditions, (see [14]), we have

$$E_T(a,b) = \frac{1}{2}\inf\left\{\int_0^T \|\phi(s)\|^2\, ds;\, y^{a,\phi}(T) = b\right\}$$

and see [5], [12], and [14],

$$\lim_{\varepsilon\downarrow 0} \ln \varepsilon\mathscr{E}(\tau^\varepsilon) = \inf_{b\in\partial\mathscr{D}} E_\infty(a,b). \tag{1.2}$$

* Received by the editors September 8, 1989; accepted for publication February 7, 1990.
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The set of all those points where the infimum in (1.2) is attained is the *exit set*, and this also has an important probabilistic interpretation (see [4], [5], [13]).

The present paper is concerned with the problem of finding or estimating E_T and E_∞ and is also concerned with the problem of minimizing E_∞ over the boundary of a given set D. We will refer to them as the minimum energy problem and the exit problem, respectively. We show that in certain important cases explicit solutions are possible.

The paper is divided into three parts. The first part is devoted to the minimum energy problem for linear systems. Here we gather partially known results. Basic formulae and estimates are given in Theorem 2.2. The next section starts from an upper estimate for the energy E_∞ which, however, is only valid locally. The main result of the paper is formulated in Theorem 3.7 which gives explicit formulae for E_∞ for the so called *gradient systems*. The first part of the theorem is an extension of a result by Friedlin [5] which also allows for a much larger class of drift terms. The second part is concerned with systems of second order in time, which are not discussed in [5]. All the basic steps of the proof are the same as those for the related results in finite dimensional spaces (see [4]); they require more sophisticated control theoretic and analytical developments. The final part presents a complete solution of the exit problem when the dynamics are linear.

This paper is a shortened version of the report [3], to which we will refer for additional details.

2. Minimum energy problem for linear systems. Consider a linear control system

$$\dot{y} = Ay + Bu, \qquad y(0) = a \in H \tag{2.1}$$

on a Hilbert space H. The operator A generates a C_0-semigroup of linear operators $S(t)$, $t \geqq 0$ and B is a bounded linear operator from a Hilbert space U into H. We will always assume $u(\cdot) \in L^2[0, T; U]$ for arbitrary $T > 0$.

The mild solution of (2.1) is given by

$$y(t) = S(t)a + \int_0^t S(t-s)Bu(s)\, ds, \qquad t \geqq 0. \tag{2.2}$$

Let us fix $T > 0$ and consider the following linear operator L_T acting from $L^2[0, T; U]$ into H:

$$L_T u = \int_0^T S(T-s)Bu(s)\, ds. \tag{2.3}$$

Thus

$$y(T) = S(T)a + L_T u.$$

Recall that if L is a bounded linear operator between Hilbert spaces H_1, H_2, then the value of its pseudoinverse operator L^{-1} at a point $y \in \operatorname{Im} L \subset H_2$ is characterized as the unique vector $x \in H_1$ such that

$$Lx = y, \qquad \langle x - z, x \rangle = 0 \quad \text{for all } z \in H_1, \qquad Lz = y.$$

Equivalently $x = L^{-1}y$ is the element with the smallest norm satisfying $Lx = y$.

It is clear that there exists a control $u(\cdot) \in L^2[0, T; U]$ transferring a to b in time T if and only if $b - S(T)a \in \operatorname{Im} L_T$, and it is clear that the control which achieves this and minimizes the functional $u \to \int_0^T \|u(s)\|^2\, ds$—called the *energy* functional—is

$$u = L_T^{-1}(b - S(T)a). \tag{2.4}$$

Let us recall that the system (2.1) is null controllable in time $T>0$ if an arbitrary state $b\in H$ can be transferred to 0 in time T. Moreover, the set $\mathcal{R}_T$ of all states which can be reached from 0 in time $T>0$ with controls $u(\cdot)\in L^2[0, T; H]$ is called the reachable space in time T. If $\mathcal{R}_T$ is the whole space then the system is said to be exactly controllable in time T. Finally a semigroup $S(t)$ is said to be stable if, for some positive constants M and ω we have $\|S(t)\|\leqq Me^{-\omega t}$ (see [1], [2]).

Define the linear operator

$$R_t=\int_0^t S(r)BB^*S^*(r)\,dr,\qquad t\geqq 0.$$

We have the following proposition (see [1], [2]).

PROPOSITION 2.1. (i) *The function R_t, $t\geqq 0$, is the unique solution of the equation*

$$\frac{d}{dt}\langle R_t x, x\rangle = 2\langle R_t A^* x, x\rangle + \|B^*x\|^2,\qquad x\in D(A^*),\qquad t\geq 0;\qquad R_0=I. \tag{2.5}$$

(ii) *If A generates a stable semigroup then $\lim_{t\to+\infty} R_t=R$ exists and is the unique solution of the equation*

$$2\langle RA^*x, x\rangle + \|B^*x\|^2=0,\qquad x\in D(A^*). \tag{2.6}$$

The following theorem gives general results for the functionals $E_T(a, b)$, the minimal energy of transferring a to b in time T, and $E_\infty(a, b)$, $a, b\in H$, $T>0$. In its formulation we will use the convection that if an element x is not in the domain of an unbounded operator C we set $\|Cx\|=+\infty$.

THEOREM 2.2. (i) *For arbitrary $T>0$ and $a, b\in H$:*

$$E_T(a, b)=\|R_T^{1/2})^{-1}(S(T)a-b)\|^2.$$

(ii) *If $S(t)$ is stable and the system* (2.1) *is null controllable in time $T_0>0$, then*

$$E_\infty(0, b)=\|(R^{1/2})^{-1}b\|^2\qquad b\in H.$$

Moreover, there exists $C>0$, such that

$$\|(R^{1/2})^{-1}b\|^2\leqq E_T(0, b)\leqq C\|(R^{1/2})^{-1}b\|^2,\qquad b\in H,\qquad T\geqq T_0. \tag{2.7}$$

Proof. The proof of (i) can be found, for instance, in [1]. To prove (ii) let us remark that the null controllability in time T_0 is equivalent to the fact that for a constant $C_1>0$ and all $x\in H$,

$$\int_0^{T_0}\|B^*S^*(r)x\|^2\,dr=\|R_{T_0}^{1/2}x\|^2\geqq C_1\|S^*(T_0)x\|^2.$$

But

$$\|R^{1/2}x\|^2=\sum_{k=0}^{\infty}\int_{kT_0}^{(k+1)T_0}\|B^*S^*(r)x\|^2\,dr$$

and for a constant $C_2>0$,

$$\int_0^{T_0}\|B^*S^*(r)x\|^2\,dr\leqq C_2\|x\|^2,\qquad x\in H.$$

Consequently, for some constants $M>0$, $\omega>0$, $C_3>0$,

$$\begin{aligned}\|R^{1/2}x\|^2 &= \int_0^{T_0} \|B^*S^*(r)x\|^2\,dr + C_2\sum_{k=1}^{\infty}\|S^*(kT_0)x\|^2\\ &\leqq \int_0^{T_0}\|B^*S^*(r)x\|^2\,dr + C_2M^2\sum_{k=0}^{\infty} e^{-\omega kT_0}\|S^*(T_0)x\|^2\\ &\leqq \left\{1+\frac{C_2}{C_1}M^2(1-e^{-\omega T_0})^{-1}\right\}\int_0^{T_0}\|B^*S^*(r)x\|^2\,dr\\ &\leqq C_3\|R_{T_0}^{1/2}x\|^2, \qquad x\in H.\end{aligned}$$

Hence $\operatorname{Im} R^{1/2}\subset \operatorname{Im} R_{T_0}^{1/2}$. Since the operator $(R_{T_0}^{1/2})^{-1}R^{1/2}$ is closed and thus bounded it follows that (2.14) holds. Now $\lim_{T\uparrow\infty}R_T=R$ so (2.13) must hold as well.

We now consider two special cases. Assume that $A: D(A)\subset H\to H$ is a negative definite operator on a Hilbert space H and that $C: H\to H$ is a bounded operator. The operators A and $\mathcal{A}$,

$$\mathcal{A}=\begin{bmatrix}0 & I\\ A & C\end{bmatrix},\ D(\mathcal{A})=D(A)\times D(-A)^{1/2}$$

define C_0-semigroups on H and $\mathcal{H}=D(-A)^{1/2}\times H$. The semigroups define mild solutions of the following Cauchy problems

$$\dot{x}=Ax, \qquad x(0)\in H \tag{2.8}$$

$$\ddot{x}=Ax+C\dot{x}, \qquad x(0)\in D(-A)^{1/2}, \quad \dot{x}(0)\in H. \tag{2.9}$$

The controlled version of (2.8)-(2.9) are

$$\dot{y}=Ay+u, \qquad y(0)=x\in H \tag{2.10}$$

$$\ddot{y}=Ay+C\dot{y}+u, \qquad y(0)\in D(-A)^{1/2}), \qquad \dot{y}(0)=v\in H. \tag{2.11}$$

We have the following theorem.

THEOREM 2.3. (i) *Assume that the operator A is negative definite, then the reachable set $\mathcal{R}_T$ for the system* (2.10) *is, for all $T>0$, exactly $D((-A)^{1/2})$ and*

$$E_\infty(0,b)=2\|(-A)^{1/2}b\|^2, \qquad b\in H.$$

(ii) *If in addition the operator C is negative definite, bounded and $(-C)^{1/2}$ commutes with $(-A)^{1/2}$ then the system* (2.11) *is exactly controllable and*

$$E_\infty\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}a\\b\end{bmatrix}\right)=2\|(-C)^{1/2}(-A)^{1/2}a\|^2+2\|(-C)^{1/2}b\|^2, \qquad \begin{bmatrix}a\\b\end{bmatrix}\in\mathcal{H}.$$

Proof. For the details of the proof see [3]. The proof that (2.11) is exactly controllable is similar to the one given for the one dimensional wave equation in [2], although, of course, a more general spectral decomposition is required.

This result does not generalize to arbitrary semigroups; however for analytic semigroups the first part of (i) can be generalized. To do this we must introduce the real interpolation space $D_A(1/2,2)$. We recall that $D_A(1/2,2)$ is the set of all x in H such that there exists a function $y(\cdot)\in W^{1,2}(0,\infty;H)\cap L^2(0,\infty;D(A))$ such that $y(0)=x$ (see [7]).

THEOREM 2.4. *Suppose that A generates an analytic semigroup on H, then the reachable set for system* (2.10) *does not depend on time and is equal to $D_A(1/2,2)$. Moreover the energy norm $(E_\infty(0,\cdot))^{1/2}$ is equivalent to the norm of $D_A(1/2,2)$.*

Proof. Let $T>0$, $u(\cdot)\in L^2[0,T;H]$; then the solution of (2.10) is given by

$$y(t)=\int_0^t S(t-s)u(s)\,ds$$

and we have (see [8]) that $y(\cdot)\in W^{1,2}(0,\infty; H)\cap L^2(0,\infty; D(A))$. Hence $y(T)\in D_A(1/2, 2)$.

Conversely let $x\in D_A(1/2,2)$; then we want to show that there exists a control $u(\cdot)\in L^2[0, T; H]$ such that $y(T)=x$. Now since $x\in D_A(1/2,2)$ there exists $z(\cdot)\in W^{1/2}(0,\infty; H)\cap L^2(0,\infty; D(A))$ such that $z(0)=x$. Choose a C^∞ real function $\phi(\cdot)$ such that $\phi(0)=0$, $\phi(T)=1$ and set

$$\xi(t)=\phi(t)z(T-t),\qquad u(t)=\dot{\xi}(t)-A\xi(t);\qquad t\in[0, T]$$

Then $u(\cdot)L^2[0, T; H]$, $\xi(t)=y(t)$ and $\xi(T)=y(T)=x$ as required.

We now consider

$$y(t)=S(t)x+\int_0^t S(t-s)u(s)\,ds$$

and first suppose that $x\in D_A(1/2,2)$. Then there exists $z(\cdot)\in W^{1,2}(0,\infty; H)\cap L^2(,\infty; D(A))$ such that $z(0)=x$. Let h be a real valued C^∞ function such that $h(0)=1$, $h(T)=0$ and set $y(t)=h(t)z(t)$, then $y(0)=x$, $y(T)=0$ and $\dot{y}(\cdot)-Ay(\cdot)\in L^2[0, T; H]$. Thus it suffices to choose $u(t)=\dot{y}(t)-Ay(t)$. For $x\in H$ we first choose $u(t)=0$ in $[0, T/2]$; thus $y(T/2)\in D_A(\frac{1}{2}, 2)$. Now by the previous argument we can find a control $u(\cdot)\in L^2[T/2, T; H]$ such that $y(T)=0$.

3. Minimum energy problems for nonlinear systems. Results like Theorems 2.2, 2.4 for linear systems do not have immediate generalizations to nonlinear ones. However local results can be obtained via linearization as we shall show in Theorem 3.1. This theorem will also play a role in proving Theorem 3.7, which is an extension of Theorem 2.3 and is the most important result of the paper.

Denote by V_T the space $\operatorname{Im} L_T=\operatorname{Im} R_T^{1/2}$ associated with the control system (2.1), equipped with the norm $\|.\|_T$:

$$\|x\|_T=\|(R_T^{1/2})^{-1}x\|=\|L_T^{-1}x\|.$$

It follows immediately from the control theoretic interpretation that if $t\leqq s$,

$$V_t\subset V_s\quad\text{and}\quad \|x\|_s\leqq\|x\|_t\qquad x\in V_t.$$

Let us assume that for all $T>0$ sufficiently small, $F: V_T\to U$ and for all $r>0$, there exists $N_{r,T}>0$ such that

$$\|F(a)-F(b)\|_U\leqq N_{r,T}\|a-b\|_T\qquad\text{provided } \|a\|_T\leqq r,\quad \|b\|_T\leqq r. \tag{3.1}$$

Consider the following equation:

$$\dot{y}=(Ay+BF(y))+Bu,\qquad y(0)=0, \tag{3.2}$$

which has the following mild form:

$$y(t)=\int_0^t S(t-r)BF(y(r))\,dr+\int_0^t S(t-r)Bu(r)\,dr. \tag{3.3}$$

THEOREM 3.1. *If* $2N_{r,T}\sqrt{T}<1$ *and* $\|b\|_T\leqq r((1-2N_{r,T}\sqrt{T})/2N_{r,T}\sqrt{T})$ *then*

$$E_T(0,b)\leqq(\|b\|_T+rN_{r,T}\sqrt{T})^2.$$

We will need the following result, also of independent interest.

PROPOSITION 3.2. *A mild solution* $y(\cdot)$ *of* (2.1) *with initial condition* 0 *is* V_T*-continuous on* $[0, T]$.

Proof. Fix $0 \leqq t \leqq s \leqq T$, then

$$y(s)-y(t)=\int_0^s S(s-r)Bu(r)\,dr-\int_0^t S(t-r)Bu(r)\,dr$$
$$=\int_0^s S(s-r)B\{u(r)-u(r-(s-t))I_{[s-t,s]}(r)\}\,dr.$$

Thus, from the definition of the norms in V_s and V_T,

$$\|y(s)-y(t)\|_T^2 \leqq \|y(s)-y(t)\|_s^2 \leqq \int_0^s \|u(r)-u(r-(s-t))I_{[s-t,s]}(r)\|^2\,dr \tag{3.4}$$
$$=\int_{s-t}^s \|u(r)-u(r-(s-t))\|^2\,dr+\int_{s-t}^s \|u(r)\|^2\,dr.$$

But the right-hand side of (3.4) tends to 0 as $s-t\to 0$ and so the result follows.

Proof. The equation (3.2) can be written as

$$y(t)=L_tF(y)+L_tu$$

where $F(y)$ denotes the function $F(y(s))$ $s\in[0, T]$. If there exists a control $u(\cdot)$ that transfer zero to b in time T, then

$$x=L_TF(y)+L_Tu. \tag{3.4}$$

Set

$$u=L_T^{-1}(b-L_TF(y)). \tag{3.5}$$

We will now show that the following equation

$$y(t)=L_tF(y)+L_tL_T^{-1}(b-L_TF(y)) \qquad t\in[0, T] \tag{3.6}$$

has a V_T-continuous solution. Note that then necessarily

$$y(T)=L_TF(y)+x-L_TF(y)=x$$

and the transferring control is given by (4.6).

For $z\in Z=C[0, T;\, V_T]$ define $\phi(z)$ by

$$\phi(z)(t)=L_tF(z)+L_tL_T^{-1}(x-L_TF(z)).$$

It follows from Theorem 4.1 that ϕ: $Z\to Z$. Note

$$\phi(0)(t)=L_tL_T^{-1}x \quad t\in[0, T]$$

and hence

$$\sup_{t\leqq T}\|L_tL_T^{-1}b\|_T^2 \leqq \int_0^T \|L_T^{-1}b(s)\|_U^2\,ds=\|b\|_T^2.$$

So $\|\phi(0)\|_Z \leqq \|b\|_T$. Let $w, z\in Z$, then

$$\phi(w)(t)-\phi(z)(t)=L_t[F(w)-F(z)]+L_tL_T^{-1}(L_T[F(z)-F(w)])$$

and hence

$$\|\phi(w)-\phi(z)\|_Z \leqq \|L.(F(w)-F(z))\|_Z-\|L.L_T^{-1}(L_T[F(z)-F(w)])\|_Z$$
$$\leqq 2\left\{\int_0^T \|F(w(s))-F(z(s))\|_U^2\,ds\right\}^{1/2}.$$

If $\|w\|_Z \leqq r$, $\|z\|_Z \leqq r$, then

$$\|\phi(w)-\phi(z)\|_Z \leqq 2N_{r,T}\left\{\int_0^T \|w(s)-z(s)\|_T^2\, ds\right\}^{1/2} \leqq 2N_{r,T}\sqrt{T}\|w-z\|_Z.$$

To show that the iterates $z_n=\phi^n(0)$, $n=1,2,\cdots$, are convergent it is enough to prove that $\|z_n\|_Z \leqq r$, for $n=1,2,\cdots$. Set $k=2N_{r,T}\sqrt{T}$, then

$$\|\phi^n(0)\|_Z \leqq \|\phi^n(0)-\phi^{n-1}(0)\|_Z+\|\phi^{n-1}(0)-\phi^{n-2}(0)\|_Z+\cdots+\|\phi^2(0)-\phi^1(0)\|_Z$$

$$\leqq (k^{n-1}+\cdots+k)\|\phi(0)\|_Z \leqq \frac{k}{1-k}\|b\|_T.$$

So by the induction argument if

$$\frac{2N_{r,T}\sqrt{T}}{(1-2N_{r,T}\sqrt{T})}\|xb\|_T \leqq r,$$

then $\|\phi^n(0)\|_Z \leqq r$ for all $n=1,2,\cdots$. The sequence $\{z_n\}$ is thus convergent in Z to a solution $y(\cdot)$ of the equation (3.6). Now

$$\left\{\int_0^T \|u(s)\|^2\, ds\right\}^{1/2} = \|b-L_TF(u)\|_T \leqq \|b\|_T+\left\{\int_0^T \|F(y(s))\|_U^2\, ds\right\}^{1/2}$$

$$\leqq \|b\|_T+N_{r,T}\left\{\int_0^T \|y(s)\|_T^2\, ds\right\}^{1/2} \leqq \|b\|_T+rN_{r,T}\sqrt{T}.$$

This complete the proof.

Remark 3.3. With a similar proof to the one above we can show that there exists a unique solution of equation (3.2) on the interval $[0, T]$ for any control satisfying

$$2N_{r,T}\sqrt{T}<1 \quad \text{and} \quad \sup_{t\leqq T}\|L_tu\|_T < r\frac{1-2N_{r,T}\sqrt{T}}{2N_{r,T}\sqrt{T}}.$$

Also, nonzero initial states can be taken into account.

COROLLARY 3.4. *Assume that for a given $T>0$ the transformation F satisfies* (3.1) *with $N_{r,T}\downarrow 0$ as $r\downarrow 0$. Then for arbitrary $\varepsilon>0$ there exists $\delta>0$ such that if $\|b\|_T<\delta$, then $E_T(0, b)\leqq \varepsilon$.*

Proof. The result follows immediately from Theorem 3.1.

We will show that Theorem 2.3 can be extended to nonlinear systems of the form

$$\dot{y}=Ay-U'(y)+u, \qquad y(0)=a_0\in H \tag{3.7}$$

$$\ddot{y}=Ay-U'(y)-\beta\dot{y}+u, \qquad y(0)=a_0\in D(-A)^{1/2}, \qquad \dot{y}(0)=b_0\in H. \tag{3.8}$$

We will make the following assumptions

(i) *A is a negative definite operator on the Hilbert space H.*

(ii) *U is a functional from $V=D((-A)^{1/2})$ into R_+ of class C^1, $U(0)=0$, $DU(0)=0$.*

(iii) *There exists a mapping $U'\colon V\to H$, Lipschitz on bounded sets such that*

$$DU(x;\, h)=\langle U'(x), h\rangle, \quad \text{for all } x, h\in V,$$

where $DU(x;\, h)$ denotes the value of the Fréchet derivatives at x in the direction h.

(iv) *β is a positive constant.*

Example 3.5. Let $A=(d^2/dx^2)$, $D(A)=W_0^1(0, L)\cap W^2(0, L)$. For any positive integer k, we shall denote by $W^k(0, L)$ the Sobolev space consisting of all the real functions on $[0, T]$ which have square integrable derivatives of any order less or equal to k. Moreover, we set $W_0^1(0, L) = \{u \in W^1(0, L); u(0) = u(L) = 0\}$. Then $V = D(-A)^{1/2} = W_0^1(0, L)$. Define

$$U(x) = \int_0^L \phi(x(s))\, ds, \qquad x \in V,$$

where ϕ is a real valued function of class C^1. It is easy to see that

$$U'(x)(s) = \phi'(x(s)), \qquad s \in [0, L], \qquad x \in V.$$

The assumptions (ii) and (iii) are satisfied in this case.

Example 3.6. The functional $U(x) = \|(-A)^{1/2}x\|^2$, $x \in V$ obviously satisfies the condition (ii) and $DU(x; h) = 2\langle(-A)^{1/2}x, (-A)^{1/2}h\rangle$, $x, h \in V$. But $U'(x)$ is defined only for $x \in D(A)$, so (iii) does not hold.

The minimal energy required to transfer a_0 to a_1 for the system (3.7) and from $\left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]$ for the system (3.8) will be denoted by $E_T(a_0, a_1)$ and $E_T(\left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right])$, respectively. Also $E_\infty(\cdot, \cdot) = \inf_{T>0} E_T(\cdot, \cdot)$.

We denote by $z^a(\cdot)$, $z[\begin{smallmatrix} a \\ b \end{smallmatrix}](\cdot)$ the solutions of the uncontrolled systems

$$\dot z = Az - U'(z), \qquad z(0) = a \in H \tag{3.9}$$

$$\ddot z = Az - U'(z) - \beta \dot z, \qquad z(0) = a \in D(-A)^{1/2}, \qquad \dot z(0) = b \in H. \tag{3.10}$$

THEOREM 3.7. *Assume that the assumptions* (i)-(iv) *hold.*

(1) *If* $a \notin D(-A)^{1/2}$ *then* $E(0, a) = +\infty$.

(2) *If* $a \in D(-A)^{1/2}$ *and* $(-A)^{1/2} z^a(t) \to 0$ *as* $t \to \infty$ *in* H, *then*

$$E_\infty(0, a) = \|(-A)^{1/2}a\|^2 + 2U(a). \tag{3.11}$$

(3) *If* $\left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right] \in \mathscr{H}$ *and* $z^{[\begin{smallmatrix} a \\ b \end{smallmatrix}]}(t) \to 0$ *as* $t \to \infty$ *in* $\mathscr{H}$, *then*

$$E_\infty\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}\right) = \beta[\|(-A)^{1/2}a\|^2 + 2U(a) + \|b\|^2]. \tag{3.12}$$

Proof. The proof is based on the following identities. For the system (3.7), with $y(0) \in D(-A)^{1/2}$

$$\begin{aligned}\frac{1}{2}\int_0^T \|u(s)\|^2\, ds = \frac{1}{2}\int_0^T \|u(s) + 2Ay(s) - 2U'(y(s))\|^2\, ds \\ + [\|(-A)^{1/2}y(T)\|^2 + 2U(y(T)) - \|(-A)^{1/2}y(0)\|^2 \\ - 2U(y(0))].\end{aligned} \tag{3.13}$$

For the system (3.8), with $y(0) \in D(-A)^{1/2}$, $\dot y(0) \in H$,

$$\begin{aligned}\frac{1}{2}\int_0^T \|u(s)\|^2\, ds = \frac{1}{2}\int_0^T \|u(s) - 2\beta \dot y(s)\|^2\, ds \\ + \beta[\|(-A)^{1/2}y(T)\|^2 + 2U(y(T)) + \|\dot y(T)\|^2 \\ - \|(-A)^{1/2}y(0)\|^2 - 2U(y(0)) - \|\dot y(0)\|^2].\end{aligned} \tag{3.14}$$

To show that (3.13) holds let us use the fact that the mild solution of (3.7) is in fact a strong solution. Elementary calculations give

$$\begin{aligned}\backslash f(1, 2)\backslash i(0, T, \|u(s)\|^2\, ds) = \frac{1}{2}\int_0^T \|u(s) + 2Ay(s) - 2U'(y(s))\|^2\, ds) \\ - 2\int_0^T \langle \dot y(s), Ay(s) - U'(y(s))\rangle\, ds\end{aligned} \tag{3.15}$$

It remains to show that

$$\int_0^T \langle \dot{y}(s), U'(y(s))\rangle\, ds = U(y(T)) - U(y(0)) \tag{3.16}$$

and

$$-2\int_0^T \langle \dot{y}(s), Ay(s)\rangle\, ds = \|(-A)^{1/2}y(T)\|^2 - \|(-A)^{1/2}y(0)\|^2. \tag{3.17}$$

In fact the identities (3.16) and (3.17) are true for arbitrary functions $y(\cdot)$ from $W^{1,2}(0, T; H)\cap L^2(0, T; D(A))$. To see this consider a sequence $\{y_n(\cdot)\}$ of functions from $C^1(0, T; D(A))$ converging to y both in $W^{1,2}(0, T; H)$ and in $L^2(0, T; D(A))$ topologies. Such a sequence exists since the domain $D(A)$ is dense in H. For each n and all $t\in[0, T]$

$$\frac{d}{dt}U(y_n(t)) = DU(y_n(t); \dot{y}_n(t)) = \langle U'(y_n(t)), \dot{y}_n(t)\rangle$$

and

$$\frac{d}{dt}\|(-A)^{1/2}y_n(t)\|^2 = \frac{d}{dt}\langle Ay_n(t), y_n(t)\rangle = 2\langle Ay_n(t), \dot{y}_n(t)\rangle.$$

So the identities (3.16) and (3.17) hold for each y_n, $n = 1, 2, \cdots$. However, we can pass to the limit in the above identities and therefore (3.16) and (3.17) hold for general

To prove (3.14) note that the functional U is defined on all state space and is of class C^1. Thus if the control $u(\cdot)$ is smooth and initial condition is in the domain of the generator then

$$\frac{1}{2}\int_0^T \|u(s)\|^2\, ds = \frac{1}{2}\int_0^T \|u(s)-2\beta\dot{y}(s)+2\beta\dot{y}(s)\|^2\, ds$$
$$= \frac{1}{2}\int_0^T \|u(s)-2\beta\dot{y}(s)\|^2\, ds + 2\int_0^T \langle \ddot{y}(s) - Ay(s) + U'(y(s)), \dot{y}\rangle\, ds$$

and consequently (3.14) holds in this case. The general case is obtained by a standard approximation argument.

Note that if $\dot{z}(t) = Az(t) - U'(z(t))$, $t\in[0, T]$, then for $y(t) = z(T-t)$ we have

$$\dot{y}(t) + Ay(t) - U'(y(t)) = 0, \qquad y(0) = z(T), y(T) = z(0), \qquad t\in(0, T).$$

Moreover, the function $y(\cdot)$ is a solution of (3.7) when $u(t) = -2Ay(t) + 2U'(y(t))$, $t\in[0, T]$, and for this control the first term on the right-hand side of (3.13) vanishes. Hence $u(\cdot)\in L^2[0, T; H]$ and

$$\begin{aligned} &E_T(0, a) \geqq \|(-A)^{1/2}a\|^2 + 2U(a)\\ &E_T(z^a(T), a) = \|(-A)^{1/2}a\|^2 + 2U(a) - \|(-A)^{1/2}z^a(T)\|^2 - 2U(z^a(T)). \end{aligned} \tag{3.18}$$

But

$$E_{T+1}(0, a) \leqq E_1(0, z^a(T)) + E_T(z^a(T), a).$$

Since $\|(-A)^{1/2}z^a(T)\| \to 0$ as $T\to\infty$ it follows from Theorem 3.1 but $E_1(0, z^a(T))\to 0$ as $T\to\infty$. Thus

$$\lim_{T\to\infty} E_T(z^a(T), a) = \|(-A)^{1/2}a\|^2 + 2U(a)$$

and

$$\inf_{T>0} E_T(0, a) \leqq \|(-A)^{1/2} a\|^2 + 2U(a)$$

and so formula (3.11) holds. Formula (3.12) can be proved in a similar way.

Remark 3.8. Assume that Q is a linear positive definite operator that commutes with A (more precisely with the spectral measure associated with A) and consider the following system:

$$\ddot{y} = Q^{-1}Ay - \tfrac{1}{2}Q^{-1}U'(y) - \tfrac{1}{2}Q\dot{y} + u$$

$$y(0) = x \in D(-A)^{1/2},\ \dot{y}(0) = v \in H.$$

Then the formula (3.12) can be generalized to the following

$$E\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}\right) = \|(-A)^{1/2} a\|^2 + U(a) + \langle Qb, b\rangle \tag{3.19}$$

with basically the same proof.

Remark 3.9. Let $\mathscr{E}$ be a separable Banach space containing H such that the inclusion operator i: $\mathscr{H} \to \mathscr{E}$ is radonifying. This means that if $\mathscr{E}$ is a Hilbert space then i is Hilbert–Schmidt. Then there exists an invariant measure on $\mathscr{E}$ for the process

$$dX = Y dt \qquad dY = (Q^{-1}AX + \tfrac{1}{2}Q^{-1}U'(X) - \tfrac{1}{2}QY)\, dt + dW$$

and up to a multiplicative constant is of the form

$$e^{-(1/2)U(x)} \mu \begin{bmatrix} dx \\ dy \end{bmatrix}$$

where μ is a Gaussian invariant measure for the linear system

$$dX = Y\, dt \qquad dY = (Q^{-1}AX - \tfrac{1}{2}\, QY)\, dt + dW.$$

The measure μ is cylindrical on $\mathscr{H}$ with mean vector o and covariance operator

$$R = \begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}. \tag{3.20}$$

The representation (3.20) is valid provided we introduce a new but equivalent inner product $\langle \cdot, \cdot \rangle_1$ on $\mathscr{H}$

$$\left\langle \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \right\rangle_1 = \langle (-Q^{-1}A)^{1/2} a_1, (-Q^{-1}A)^{1/2} \rangle + \langle b_1, b_2 \rangle.$$

The proof of this result follows from [15].

4. The exit problem. For details of the stochastic exit problem see [5], [12]. Here we discuss its deterministic analogue. To fix ideas we concentrate on the system (3.7) and assume that the conditions of Theorem 3.7 are satisfied. In addition let $\mathscr{E}$ be a Banach space containing $D(-A)^{1/2}$ and such that the inclusion operator i: $\mathscr{H} \to \mathscr{E}$ is randonifying, which means that the image $i(\gamma)$ of the cylindrical Gaussian measure $N(0, I)$ has an extension to a σ-additive measure on Borel subsets of $\mathscr{E}$.

Example 4.1 (Compare [5]). Let

$$A = \frac{d^2}{dx^2}, \qquad D(A) = W_0^1(0, L) \cap W^2(0, L).$$

Then

$$D(-A)^{1/2} = W_0^1(0, L).$$

If $E = C[0, L]$, then the inclusion $i: W_0^1(0, L) \to \mathscr{E}$ is radonifying (see [6]).

Let $\mathscr{D}$ be a bounded open set in $\mathscr{E}$ containing 0. For arbitrary $\varepsilon > 0$ define

$$(\partial\mathscr{D})_\varepsilon = \{x \in \mathscr{E}, \text{distance}_{\mathscr{E}}(x, \partial\mathscr{D}) < \varepsilon\} \tag{4.1}$$

and let

$$r_\varepsilon^- = \inf\{E_\infty(0, b);\ b \in (\partial\mathscr{D})_\varepsilon \cap \bar{\mathscr{D}}\},$$

$$r_\varepsilon^+ = \inf\{E_\infty(0, b);\ b \in (\partial\mathscr{D})_\varepsilon \cap \mathscr{D}^c\},$$

where $\bar{\mathscr{D}}$ and $\mathscr{D}^c$ denote, respectively, the closure and the complement of $\mathscr{D}$.

$$\hat{r} = \inf\{E_\infty(0, b);\ b \in \partial\mathscr{D}\}.$$

If

$$r^- = \lim_{\varepsilon \downarrow 0} r_\varepsilon^-,\ r^+ = \lim_{\varepsilon \downarrow 0} r_\varepsilon^+$$

then $r^- \leqq \hat{r}$, $r^+ \leqq \hat{r}$ and we expect that in fact $r^- = r^+ = \hat{r}$. The numbers r^-, r^+, and $\hat{r}$ will be called, respectively, the lower, the upper exit rates, and the exit rate.

The following problems are of interest for both deterministic and stochastic systems.

PROBLEM 4.2. Under what conditions $r^- = r^+ = \hat{r}$?

PROBLEM 4.3. Assume that $r^- = r^+ = \hat{r}$. Calculate $\hat{r}$ and describe as explicitly as possible the set

$$\hat{\mathscr{E}} = \{b \in \partial\mathscr{D});\ E_\infty(0, b) = \hat{r}\} \tag{4.2}$$

which will be called the *exit set.*

For linear systems some answers to the above questions are available assuming that $\mathscr{E} = H$(see [12]); here we consider a different situation and give rather specific answers to both the problems. Namely we consider the problem

$$\inf_{u \in \mathscr{D}^c} \|Au\|_H^2,$$

where A is a closed operator on $H = L^2(\Gamma)$ with the domain $D(A) \subset C(\bar{\Gamma}) = E$, $C(\bar{\Gamma})$ being the space of continuous functions on $\bar{\Gamma} \subset R^n$ and $\mathscr{D}$ is a bounded neighborhood of 0 in E. If $u \notin D(A)$, we set $\|Au\| = +\infty$. We will assume also that:

(i) The operator $G = A^{-1}$ is an integral operator with a continuous kernel $g(\cdot, \cdot)$:

$$Gv(x) = \int_\Gamma g(x, y)v(y)\, dy, \qquad x \in \Gamma, \quad v \in H.$$

(ii) The set $\mathscr{D}$ is of the following form:

$$\mathscr{D} = \{u \in E;\ -b(x) < u(x) < a(x), x \in \Gamma\}$$

where $a(\cdot)$ and $b(\cdot)$ are positive functions on $\bar{\Gamma}$.

The following result holds.

THEOREM 4.4. *Assume* (i) *and* (ii) *hold. Then* $r^- = r^+ = \hat{r}$ *and*

$$\hat{r} = \inf_{x \in \Gamma}\left\{(a(x) \wedge b(x)\left(\int_\Gamma g^2(x, y)\, dy\right)^{-1}\right\}. \tag{4.3}$$

Let Γ_0 be the set of all $x \in \Gamma$ for which the infimum in (4.3) *is attained. If Γ_0 is nonempty then*

$$\hat{\mathscr{E}} = \left\{ (Gv^x)(\cdot);\ v^x(\cdot) = \frac{\hat{r}}{a(x) \wedge b(x)} g(x, \cdot),\ x \in \Gamma_0 \right\}.$$

Proof. Let us fix $x \in \Gamma$ and a positive number c; first we will solve the problem: $\inf\{\|Au\|^2;\ u(x) \geqq c,\ u \in E\}$ (if $u \notin D(A)$, $\|Au\| = +\infty$), which is equivalent to

$$\inf_{Gv(x) \geqq c} \|v\|^2 = \inf_{Gv(x) = c} \|v\|^2 \tag{4.4}$$

where $Gv(x) = \langle g(x, \cdot). v \rangle_H$. The proof of the following lemma is straightforward.

LEMMA 4.5. *Let H be a Hilbert space, $v \in H$ and $c > 0$, then for the problem $\inf\{\|u\|_H;\ \langle u, v\rangle = c\}$ the infimum is attained at $u = cv\|v\|^{-2}$ and is equal to $c\|v\|^{-1}$.*

Therefore the problem (4.4) has a unique solution $v^x(\cdot) = cg(x, \cdot)(\int_\Gamma g^2(x, y)\, dy)^{-1}$ and the minimum value is $c^2\|v^x(\cdot)\|^{-2}$. The statement of the theorem now follows easily.

Example 4.6. Let $A_1 = (d^2/dx^2)$, $D(A_1) = W_0^1(0, 1) \cap W^2(0, 1)$, $\mathscr{E} = C_0(0, 1)$ the space of continuous functions vanishing at 0 and 1, $U' = 0$ and

$$\mathscr{D} = \{z \in \mathscr{E};\ |z(x)| < a,\ x \in [0, 1]\}.$$

PROPOSITION 4.7. *For the above example $r^- = r^+ = \hat{r} = 4a^2$ and the exit set consists of exactly two functions $\pm\hat{z}$*

$$\hat{z}(t) = \begin{cases} (a/2)t & \text{if } t \in [0, \frac{1}{2}] \\ (a/2)(1 - t) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. The proof follows from Theorem 4.4 and elementary calculations.

Example 4.8. Here we take $A_2 = -A_1^2$ where A_1 is the same as in Example 4.6. Note that then $D(-A_2)^{1/2} = D(A_1)$ and

$$\|(-A_2)^{1/2} z\| = \int_0^1 \left[\frac{d^2 z}{dx^2}(x)\right]^2 dx.$$

The set $\mathscr{D}$ is the same as that in Example 4.6.

PROPOSITION 4.9. *For the above example $r^- = r^+ = \hat{r} = 48a^2$ and the exit set consists of exactly two functions $\pm\hat{z}$*

$$\hat{z}(t) = \begin{cases} at(3 - 4t^2) & \text{if } t \in [0, \frac{1}{2}] \\ a(1 - t)(3 - 4(1 - t)^2) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Remark 4.10. It is clear that a similar result is true for the operator $A_n = (-1)^{n+1} A_1^n$. We could also start from the operator $A_1 = (d^2/dx^2)$ on $L^2(0, 1; \mathbf{R}^d)$ of square integrable vector functions. $\mathscr{E} = C(0, 1; \mathbf{R}^d)$ and the set $\mathscr{D}$ could be of more general character

$$\mathscr{D} = \{z;\ z(s) \in T(s),\ s \in [0, 1]\}$$

where T is a multifunction with values in $\mathbf{R}^d$. Some additional subtleties arise here.

Remark 4.11. It would be interesting to consider in detail the case $A_1 = \Delta$ on $L^2(\Gamma)$, Γ bounded in $\mathbf{R}^n$, $D(A_1) = W_0^1(\Gamma) \cap W^2(\Gamma)$ and $A_m = (-1)^{m+1} A_1^m$. Under well-known conditions, $D(A_m) \subset c(\bar{\Gamma})$ and the exit problem, as formulated in Problems 1 and 2 can be posed correctly. Some related comments can be found in [5].

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