

LEAF SUPERPOSITION PROPERTY FOR INTEGER RECTIFIABLE CURRENTS

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ABSTRACT. We consider the class of integer rectifiable currents without boundary in $\mathbb{R}^n \times \mathbb{R}$ satisfying a positivity condition. We establish that these currents can be written as a linear superposition of graphs of finitely many functions with bounded variation.

1. Introduction and statement of the main result. It is well known that a locally integrable function in \mathbb{R}^n belongs to BV_{loc} (the space of functions of locally bounded variation) if and only if its subgraph has locally finite perimeter in $\mathbb{R}^n \times \mathbb{R}$. The connections between the analytic properties of u and the geometric properties of its (sub)graph are well described, using the more powerful language of currents, in [7, 4.5.9] or [8, 4.1.5]. Recall that currents provide a very natural setting to discuss analytic problems with a geometrical content, and have been successfully used in many areas. In particular, Giaquinta, Modica, and Souček introduced the notion of Cartesian current and used it to attack many problems in the calculus of variations (see the extensive monograph [8]) including non-linear elasticity, harmonic maps between manifolds, relaxed energies, etc.

The aim of this paper is to show the representation of a suitable class of integer rectifiable currents in $\mathbb{R}^n \times \mathbb{R}$ as the superposition of finitely many graphs (referred to as “leaves”) of functions with bounded variation. In some sense this result has some connections with Almgren’s theory [1], [2] (developed in arbitrary dimension and codimension) of approximation, up to sets of small measure, of (minimal) currents

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by multi-valued Lipschitz graphs: here the regularity condition is weakened to BV , and this allows a complete description of the current, at least in codimension one, as a multi-valued graph. We rely here on techniques of geometric measure theory, especially the concept of BV maps and currents in metric spaces developed in Ambrosio [3] and Ambrosio and Kirchheim [6].

We refer to the following section for the notation and state here the main result of this paper. If $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is a locally BV function, we denote by $\mathbf{i}(u)$ the n -dimensional boundary-free current canonically associated with the graph of u in $\mathbb{R}^n \times \mathbb{R}$, obtained (roughly speaking) by completion of the discontinuities of u with vertical segments.

Theorem 1. *Let $T \in \mathcal{I}_n(\mathbb{R}^{n+1})$ be an n -dimensional integer rectifiable current in $\mathbb{R}^{n+1} = \mathbb{R}_x^n \times \mathbb{R}_y$ satisfying the zero-boundary condition $\partial T = 0$, the positivity condition $T \llcorner dx \geq 0$ and the cylindrical mass condition*

$$\mathbb{M}_{B_R(0) \times \mathbb{R}}(T) < \infty \quad \text{for every } R > 0. \quad (1)$$

Then, there exist a unique integer N and a unique family of functions $u_j \in BV_{\text{loc}}(\mathbb{R}^n; \mathbb{R})$, $1 \leq j \leq N$, satisfying

$$u_1 \leq u_2 \leq \dots \leq u_N, \quad (2)$$

such that the given current T is the superposition of the canonical Cartesian currents $\mathbf{i}(u_j)$ associated with the functions u_j , that is,

$$T = \sum_{j=1}^N \mathbf{i}(u_j). \quad (3)$$

In addition, the following additivity property holds:

$$\|T\| = \sum_{j=1}^N \|\mathbf{i}(u_j)\|. \quad (4)$$

We call each function u_j a leaf of the decomposition of T , and we refer to (3) as the *canonical leaf decomposition* of T . Heuristically (4) follows from (3) because all graphs have a common orientation in their intersection, so that no cancellations occur; notice that the additivity property (4) does not hold for more general decompositions which satisfy condition (3), but not the monotonicity assumption (2).

For an application of this result we refer to [5], where a geometric approach to tackle multi-dimensional scalar conservation laws is developed. Therein, solutions are defined geometrically as currents, rather than as functions satisfying entropy inequalities. The leaf decomposition is used to show the existence of entropy solutions in this setting, as the superposition of graphs of classical entropy solutions. (See [5] for details.)

2. Preliminaries and notation.

2.1. Currents. We denote by $\mathcal{D}_m(\mathbb{R}^k)$ the space of m -dimensional currents in \mathbb{R}^k , that is the dual space of all linear and continuous functionals defined on the space $\mathcal{D}^m(\mathbb{R}^k)$ of all smooth and compactly supported differential m -forms. The space $\mathcal{D}_m(\mathbb{R}^k)$ is equipped with the usual weak-star topology induced by this duality. The duality bracket between a current $T \in \mathcal{D}_m(\mathbb{R}^k)$ and a form $\omega \in \mathcal{D}^m(\mathbb{R}^k)$ is denoted by $\langle T, \omega \rangle$.

The boundary of a current $T \in \mathcal{D}_m(\mathbb{R}^k)$ is the current $\partial T \in \mathcal{D}_{m-1}(\mathbb{R}^k)$ defined by

$$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle, \quad \omega \in \mathcal{D}^{m-1}(\mathbb{R}^k),$$

where $d\omega \in \mathcal{D}^m(\mathbb{R}^k)$ denotes the differential of the form $\omega \in \mathcal{D}^{m-1}(\mathbb{R}^k)$. If $T \in \mathcal{D}_m(\mathbb{R}^k)$ and $\alpha \in \mathcal{D}^h(\mathbb{R}^k)$ for some $h \leq m$, we denote by $T \llcorner \alpha \in \mathcal{D}_{m-h}(\mathbb{R}^k)$ the saturation of the current T with the form α , which is defined by

$$\langle T \llcorner \alpha, \omega \rangle = \langle T, \alpha \wedge \omega \rangle, \quad \omega \in \mathcal{D}^{m-h}(\mathbb{R}^k).$$

The (local) mass of a current $T \in \mathcal{D}_m(\mathbb{R}^k)$ is defined for every open set $\Omega \subset \mathbb{R}^k$ as

$$\mathbb{M}_\Omega(T) = \sup \{ \langle T, \omega \rangle : \omega \in \mathcal{D}^m(\mathbb{R}^k), \text{supp } \omega \subset \Omega, \|\omega\| \leq 1 \}.$$

If T has locally finite mass, the set function $\Omega \mapsto \mathbb{M}_\Omega(T)$ is the restriction to bounded open sets of a nonnegative Radon measure that we shall denote by $\|T\|$, so that $\|T\|(\Omega) = \mathbb{M}_\Omega(T)$ for all bounded open sets $\Omega \subset \mathbb{R}^k$. Given a current $T \in \mathcal{D}_m(\mathbb{R}^k)$ with locally finite mass, there exists a unique (up to $\|T\|$ -negligible sets) $\|T\|$ -measurable map \vec{T} defined on \mathbb{R}^k and with values in the set of m -vectors such that \vec{T} is a unit m -vector $\|T\|$ -almost everywhere and (here $\langle \cdot, \cdot \rangle$ is the standard duality between m -vectors and m -covectors)

$$\langle T, \omega \rangle = \int_{\mathbb{R}^k} \langle \vec{T}(x), \omega(x) \rangle d\|T\|(x), \quad \omega \in \mathcal{D}^m(\mathbb{R}^k). \quad (5)$$

Whenever (5) holds, we shall write $T = \vec{T} \llcorner \|T\|$.

We will be especially interested in the subclass $\mathcal{I}_m(\mathbb{R}^k) \subset \mathcal{D}_m(\mathbb{R}^k)$ of all m -dimensional integer rectifiable currents T for which, by definition, there exists a triple (M, θ, τ) , where $M \subset \mathbb{R}^k$ is a countably \mathcal{H}^m -rectifiable set, $\theta : M \rightarrow \mathbb{N} \setminus \{0\}$ is a locally integrable function and τ is a Borel orientation of M (i.e. a Borel map $x \mapsto \tau(x) = \xi_1(x) \wedge \dots \wedge \xi_m(x)$ with values in unit and simple m -vectors whose span is the approximate tangent space to M at x) such that $T = \tau \theta \mathcal{H}^m \llcorner M$, or equivalently $\vec{T} = \tau$ and $\|T\| = \theta \mathcal{H}^m \llcorner M$. We shall also write $T = (M, \theta, \tau)$, and we refer to M as the support of T and to θ as the multiplicity of T (both are uniquely determined up to \mathcal{H}^m -negligible sets).

2.2. 0-dimensional integer rectifiable currents with finite mass. In this section we consider a very special class of integer rectifiable currents, the 0-dimensional ones with finite mass on the real line \mathbb{R} . We denote by $\tilde{\mathcal{I}}_0(\mathbb{R})$ the set of these currents and we notice that it consists of those currents that can be expressed as a finite sum of Dirac masses with weight ± 1 . This means that every $S \in \tilde{\mathcal{I}}_0(\mathbb{R})$ can be written as

$$S = \sum_{j=1}^l \sigma_j \delta_{A_j},$$

where the A_j are (not necessarily distinct) points of \mathbb{R} and $\sigma_j = \pm 1$. We will call *average* of the current $S \in \tilde{\mathcal{I}}_0(\mathbb{R})$ the integer $\sum_j \sigma_j$. For every $h \in \mathbb{N}$ we denote by $\mathcal{I}_0^h(\mathbb{R}) \subset \tilde{\mathcal{I}}_0(\mathbb{R})$ the set consisting of all nonnegative 0-dimensional integer rectifiable currents in \mathbb{R} with average h :

$$\mathcal{I}_0^h(\mathbb{R}) := \left\{ S \in \tilde{\mathcal{I}}_0(\mathbb{R}) : S = \sum_{j=1}^h \delta_{A_j} \right\}; \quad (6)$$

notice again that the points $A_j \in \mathbb{R}$ need not be distinct.

On the set $\bar{\mathcal{I}}_0(\mathbb{R})$ we define

$$\mathbb{F}(S) := \sup \{ \langle S, \phi \rangle : \phi \in \text{Lip}_{b,1}(\mathbb{R}) \}, \quad S \in \bar{\mathcal{I}}_0(\mathbb{R}),$$

where $\text{Lip}_{b,1}(\mathbb{R})$ denotes the set of bounded real-valued Lipschitz functions defined on \mathbb{R} with Lipschitz constant less or equal than one. Notice that, if $S \in \bar{\mathcal{I}}_0(\mathbb{R})$ has non-zero average, then obviously $\mathbb{F}(S) = +\infty$; on the other hand

$$\mathbb{F}(S) \leq \mathbb{M}_{\mathbb{R}}(S) \text{diam}(\text{supp } S) < +\infty$$

for all $S \in \bar{\mathcal{I}}_0(\mathbb{R})$ with zero average. It is also immediate to check that, for $S = \delta_A - \delta_B$, we have $\mathbb{F}(S) = |A - B|$. A generalization of this fact is given by the following well-known lemma.

Lemma 1. *If S and $S' \in \bar{\mathcal{I}}_0(\mathbb{R})$ are of the form*

$$S = \sum_{j=1}^h \delta_{A_j}, \quad S' = \sum_{j=1}^h \delta_{B_j},$$

with $A_1 \leq A_2 \leq \dots \leq A_h$ and $B_1 \leq B_2 \leq \dots \leq B_h$, then

$$\sum_{j=1}^h |A_j - B_j| = \mathbb{F}(S - S'). \quad (7)$$

Proof. We give an elementary proof, which uses ideas from the theory of optimal transportation (see [11]). We notice first that the inequality \geq in (7) is an obvious consequence of the inequality $|A_j - B_j| \geq |\phi(A_j) - \phi(B_j)|$ for all $\phi \in \text{Lip}_{1,b}(\mathbb{R})$, so we need only to build $\phi \in \text{Lip}_{1,b}(\mathbb{R})$ such that

$$\sum_{j=1}^h |A_j - B_j| \leq \langle S - S', \phi \rangle. \quad (8)$$

By the compactness of the support of $S - S'$, it suffices to construct a 1-Lipschitz function ϕ with this property. To this aim, we first notice that the fact that the list of the A_j 's and of the B_j 's are ordered implies

$$\sum_{j=1}^h |A_j - B_j| \leq \sum_{j=1}^h |A_j - B_{\sigma(j)}| \quad (9)$$

for any permutation σ of $\{1, \dots, h\}$ (this can be seen by showing that the right hand side does not increase if a permutation σ with $B_{\sigma(i)} > B_{\sigma(j)}$ for some $i < j$ is replaced by another one $\tilde{\sigma}$ with $\tilde{\sigma}(i) = \sigma(j)$, $\tilde{\sigma}(j) = \sigma(i)$ and $\tilde{\sigma}(k) = \sigma(k)$ for $k \neq i, j$). More generally, one can use (9) and the fact that permutation matrices are extremal points in the class of bi-stochastic matrices to obtain (the so-called Birkhoff theorem, see [11])

$$\sum_{j=1}^h |A_j - B_j| \leq \sum_{i,j=1}^h m_{ij} |A_j - B_i| \quad (10)$$

for any nonnegative m_{ij} with $\sum_i m_{ij} = \sum_i m_{ji} = 1$ for all $j = 1, \dots, h$.

The minimization of the functional $m \mapsto \sum_{i,j} m_{ij} |A_j - B_i|$ subject to the above constraints on m is a (very) particular case of Monge-Kantorovich optimal transport problem of finding an optimal coupling between S and S' with cost function $c(x, y) = |x - y|$. Kantorovich's duality theory gives that the infimum of this problem, namely

$\sum_j |A_j - B_j|$, is (see [11] again, where an explicit construction of the maximizing ϕ is given)

$$\max_{\phi \in \text{Lip}_1(\mathbb{R})} \langle S - S', \phi \rangle.$$

□

For every fixed $h \in \mathbb{N}$ we define

$$\mathbf{d}(S, S') := \mathbb{F}(S - S') = \sup \{ \langle S, \phi \rangle - \langle S', \phi \rangle : \phi \in \text{Lip}_{b,1}(\mathbb{R}) \}, \quad S, S' \in \mathcal{I}_0^h(\mathbb{R}),$$

which is easily seen to be a finite distance in $\mathcal{I}_0^h(\mathbb{R})$ (indeed, since S and S' belong to the same set $\mathcal{I}_0^h(\mathbb{R})$, the difference $S - S'$ has zero average).

2.3. Slices of a current. Given $T \in \mathcal{I}_n(\mathbb{R}^n \times \mathbb{R})$ we consider the vertical slices of T at $x \in \mathbb{R}^n$,

$$T_x := \langle T, dx, x \rangle \in \mathcal{I}_0(\mathbb{R}),$$

see for instance [6], [10]. This family of currents is uniquely determined, up to \mathcal{L}^n -negligible sets, by the identity $\int_{\mathbb{R}^n} T_x dx = T \llcorner dx$, i.e.

$$\int_{\mathbb{R}^n} \langle T_x, \varphi(x, \cdot) \rangle dx = \langle T \llcorner dx, \varphi \rangle \quad (11)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$. Furthermore, the masses of T_x are related to the mass of T by

$$\int_{\Omega} \mathbb{M}_{\mathbb{R}}(T_x) dx \leq \mathbb{M}_{\Omega \times \mathbb{R}}(T) \quad (12)$$

for all bounded open sets $\Omega \subset \mathbb{R}^n$. As a consequence, $T_x \in \bar{\mathcal{I}}_0(\mathbb{R})$ for \mathcal{L}^n -a.e. $x \in \Omega$ whenever $\mathbb{M}_{\Omega \times \mathbb{R}}(T) < +\infty$.

2.4. The current associated to the graph of a BV function. Recall that $u \in L_{\text{loc}}^1(\mathbb{R}^n)$ is said to be a locally BV function if its distributional derivative $Du = (D_1 u, \dots, D_n u)$ is an \mathbb{R}^n -valued measure with locally finite total variation in \mathbb{R}^n , and we shall denote by $\|Du\|$ this total variation.

In this section we are going to describe how we can canonically associate to $u \in BV_{\text{loc}}(\mathbb{R}^n)$ a current $\mathbf{i}(u) \in \mathcal{I}_n(\mathbb{R}^n \times \mathbb{R})$ with no boundary, finite mass on cylinders $\Omega \times \mathbb{R}$ with Ω bounded, and satisfying

$$\langle \mathbf{i}(u), \varphi dx \rangle := \int_{\Omega} \varphi(x, u(x)) dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}). \quad (13)$$

These two conditions are actually sufficient to characterize a unique current, see step 5 of the proof of Theorem 1.

Geometrically, this current corresponds to the integration on the graph of u , with the orientation induced by the map $x \mapsto (x, u(x))$, and this description works perfectly well when $u \in C^1$. In order to define $\mathbf{i}(u)$ in the general case when $u \in BV_{\text{loc}}$, we first define the subgraph $E(u)$ of u by

$$E(u) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y \leq u(x)\}.$$

It is well known that $E(u)$ has locally finite perimeter in $\mathbb{R}^n \times \mathbb{R}$ (i.e. $\chi_{E(u)} \in BV_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$), so it has a measure-theoretic boundary (the set of points where the density of $E(u)$ is neither 0 nor 1), that we shall denote by $\Gamma(u)$. De Giorgi's theorem on sets of finite perimeter ensures that $\Gamma(u)$ is countably \mathcal{H}^n -rectifiable, and that

$$D\chi_{E(u)} = -\nu_{E(u)} \mathcal{H}^n \llcorner \Gamma(u) \quad (14)$$

(the unit vector $\nu_{E(u)}$ is the so-called approximate outer normal to $E(u)$). Then, we define

$$\mathbf{i}(u) := (\Gamma(u), 1, \tau_u), \quad (15)$$

where τ_u is the unit n -vector spanning $\nu_{E(u)}^\perp$ (the approximate tangent space to $\Gamma(u)$), characterized by

$$\langle dx_1 \wedge \cdots \wedge dx_n \wedge dy, \tau_u \wedge \nu_{E(u)} \rangle \geq 0.$$

Equivalently, invoking the relation (14), we can define

$$\begin{aligned} \langle \mathbf{i}(u), \varphi dx \rangle &:= - \int_{\mathbb{R}^n \times \mathbb{R}} \varphi dD_y \chi_{E(u)}, \\ \langle \mathbf{i}(u), \varphi \widehat{dx}_j \wedge dy \rangle &:= \int_{\mathbb{R}^n \times \mathbb{R}} \varphi dD_j \chi_{E(u)}, \quad j = 1, \dots, n \end{aligned}$$

(here $\widehat{dx}_j := (-1)^{n-j} dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$). In the case $u \in C^1(\mathbb{R}^n)$, using the area formula, it is easy to check that this definition coincides with the geometric picture, and in particular that $\partial(\mathbf{i}(u)) = 0$ and (13) hold. In the general case both can be obtained, for instance, by approximation (notice that $u_i \rightarrow u$ in L^1_{loc} implies $E(u_i) \rightarrow E(u)$ in L^1_{loc} and therefore weak convergence of the associated currents).

We will need the following strong locality property of τ_u .

Lemma 2. *Let $u, v \in BV_{\text{loc}}(\mathbb{R}^n)$ with $u \geq v$. Then $\tau_u = \tau_v$ \mathcal{H}^n -a.e. on $\Gamma(u) \cap \Gamma(v)$.*

Proof. It suffices to show that $\nu_{E(u)} = \nu_{E(v)}$ \mathcal{H}^n -a.e. on $\Gamma(u) \cap \Gamma(v)$. It is a general property of sets of finite perimeter $E \subset \mathbb{R}^{n+1}$ that, for \mathcal{H}^n -a.e. $w \in \partial^* E$, the rescaled sets $(E - w)/r$ converge in L^1_{loc} as $r \downarrow 0$ to the halfspace having $\nu_E(w)$ as outer normal. In our case, $E(u) \supset E(v)$ because $u \geq v$, so that all points w where both $(E(u) - w)/r$ and $(E(v) - w)/r$ converge to a halfspace, the halfspace has to be the same. This implies the stated equality \mathcal{H}^n -a.e. of the outer normals. \square

2.5. Metric spaces valued BV functions. We now recall the main features of the theory of BV functions with values in a metric space, developed in Ambrosio [3] and Ambrosio and Kirchheim [6]. Let (E, d) be a metric space such that there exists a countable family $\mathcal{F} \subset \text{Lip}_{b,1}(E)$ which generates the distance, in the sense that

$$d(x, y) = \sup_{\Phi \in \mathcal{F}} |\Phi(x) - \Phi(y)|, \quad x, y \in E.$$

We say that a function $f : \mathbb{R}^n \rightarrow E$ is a function of metric locally bounded variation, and we write $f \in MBV_{\text{loc}}(\mathbb{R}^n; E)$, if $\Phi \circ f \in BV_{\text{loc}}(\mathbb{R}^n)$ for every $\Phi \in \mathcal{F}$ and if there exists a positive locally finite measure ν in \mathbb{R}^n such that

$$\nu \geq \|D(\Phi \circ f)\|, \quad \Phi \in \mathcal{F}.$$

The minimal ν such that the previous condition holds will still be denoted by $\|Df\|$. It is possible to check that the class $MBV_{\text{loc}}(\mathbb{R}^n; E)$ and the measure $\|Df\|$ are independent of the choice of the family \mathcal{F} .

We now consider the metric space $(\mathcal{I}_0^h(\mathbb{R}), \mathbf{d})$ previously defined. To every $\phi \in \text{Lip}_{b,1}(\mathbb{R})$ we associate the map $\Phi_\phi \in \text{Lip}_1(\mathcal{I}_0^h(\mathbb{R}))$ defined by

$$\Phi_\phi(S) := \langle S, \phi \rangle, \quad S \in \mathcal{I}_0^h(\mathbb{R}).$$

Indeed, it is immediate to check the Lipschitz continuity

$$|\Phi_\phi(S) - \Phi_\phi(S')| = |\langle S, \phi \rangle - \langle S', \phi \rangle| \leq \mathbf{d}(S, S'), \quad S, S' \in \mathcal{I}_0^h(\mathbb{R}).$$

By a standard density argument, it is possible to select a countable family $\mathcal{F} \subset \text{Lip}_{b,1}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ with the property that

$$\mathbf{d}(S, S') = \sup_{\phi \in \mathcal{F}} \{ \langle S, \phi \rangle - \langle S', \phi \rangle \}, \quad S, S' \in \mathcal{I}_0^h(\mathbb{R}). \quad (16)$$

Lemma 3. *Let E and F be metric spaces. Then $M \circ f \in MBV_{\text{loc}}(\mathbb{R}^k; F)$ whenever $f \in MBV_{\text{loc}}(\mathbb{R}^k; E)$ and $M : E \rightarrow F$ is an L -Lipschitz function, and $\|D(M \circ f)\| \leq L\|Df\|$. Furthermore, $MBV_{\text{loc}}(\mathbb{R}^k; \mathbb{R})$ coincides with $BV_{\text{loc}}(\mathbb{R}^k)$.*

Proof. Let $\phi \in \text{Lip}_{b,1}(F)$, $g = M \circ f$ and $\psi = \phi \circ M$; then $\psi \in \text{Lip}_b(E)$ and its Lipschitz constant is less than L ; as a consequence, $\|D(\psi \circ f)\| \leq L\|Df\|$. Since $\psi \circ f = \phi \circ g$ we obtain that $g \in MBV_{\text{loc}}(\mathbb{R}^k; F)$ and $\|Dg\| \leq L\|Df\|$.

The inclusion $BV_{\text{loc}}(\mathbb{R}^k) \subset MBV_{\text{loc}}(\mathbb{R}^k; \mathbb{R})$ is a simple consequence of the stability of BV functions under left composition with Lipschitz maps; to prove the opposite inclusion, let $f \in MBV_{\text{loc}}(\mathbb{R}^k; \mathbb{R})$ and fix an open ball $B \subset \mathbb{R}^k$; by definition all truncated functions $f_a := -a \vee (f \wedge a)$ belong to $BV(B)$ and $\|Df_a\| \leq \|Df\|$, since we can see f_a as the composition of f with the map $\eta_a \in \text{Lip}_{b,1}(\mathbb{R})$ defined as the identity for $x \in [-a, a]$, as the constant a for $x > a$ and as the constant $-a$ for $x < -a$. Therefore, denoting by \bar{f}_a their averages in B , by Poincaré inequality we obtain that $f_a - \bar{f}_a$ is bounded in $L^1(B)$. Thanks to the compactness of the embedding of BV in L^1 , we can find a sequence $a_i \rightarrow +\infty$ such that \bar{f}_{a_i} converges to some $m \in \overline{\mathbb{R}}$ and $f_{a_i} - \bar{f}_{a_i}$ converge in $L^1(B)$ and \mathcal{L}^n -almost everywhere to $g \in BV(B)$: if $m \in \mathbb{R}$ we immediately obtain that $f = m + g \in BV(B)$. If not, we obtain that $|f| = +\infty$ \mathcal{L}^n -almost everywhere, contradicting the assumption that f is real valued. \square

3. Proof of the main theorem. This section is entirely devoted to the proof of Theorem 1. We address separately the existence of the decomposition, its uniqueness and the equality of the total variations. During the proof we will occasionally use forms ω in $\mathbb{R}^n \times \mathbb{R}$ whose supports are not compact, but have a compact projection on \mathbb{R}^n . Their use can be easily justified by a truncation argument, based on the fact that the currents under consideration have finite mass on cylinders $\Omega \times \mathbb{R}$ with $\Omega \subset \mathbb{R}^n$ bounded.

3.1. Existence of a decomposition. We proceed in 5 steps.

Step 1. We begin by proving that there exists an integer N (depending on T only) such that, for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$, the slice $T_x \in \bar{\mathcal{I}}_0(\mathbb{R})$ is the sum of N Dirac masses with unit weight: more precisely, for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$ there exist N real values

$$u_1(x) \leq u_2(x) \leq \dots \leq u_N(x) \quad (17)$$

satisfying

$$T_x = \sum_{j=1}^N \delta_{u_j(x)}. \quad (18)$$

We first show that $T_x \geq 0$. Fix two nonnegative functions $\phi \in C_c^\infty(\mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R}^n)$, and apply (11) with $\varphi(x, y) = \psi(x)\phi(y)$ to get

$$\int_{\mathbb{R}^n} \langle T_x, \phi \rangle \psi(x) dx = \langle T \llcorner dx, \varphi \rangle \geq 0,$$

since we assumed $T \llcorner dx \geq 0$. Hence, by the arbitrariness of ψ , we deduce that $\langle T_x, \phi \rangle \geq 0$ for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$. By a simple density argument we can obtain an

\mathcal{L}^n -negligible set E independent of ϕ such that $\langle T_x, \phi \rangle \geq 0$ for all $\phi \in C_c^\infty(\mathbb{R})$ and $x \in \mathbb{R}^n \setminus E$. This proves that $T_x \geq 0$ for all $x \in \mathbb{R}^n \setminus E$.

Knowing that $T_x \geq 0$, the mass of T_x is simply given by $\langle T_x, 1 \rangle$ (notice that this function is locally integrable by (12) and assumption (1), and takes \mathcal{L}^n -almost everywhere its values in \mathbb{N} because \mathcal{L}^n -almost all the slices are integer rectifiable).

We want to show that the map $x \mapsto \langle T_x, 1 \rangle$ is \mathcal{L}^n -equivalent to a constant in \mathbb{R}^n . Indeed, for every function $\psi \in C_c^\infty(\mathbb{R}^n)$ we can compute (applying again (11))

$$\int_{\mathbb{R}^n} \langle T_x, 1 \rangle \frac{\partial \psi}{\partial x_i}(x) dx = \langle T \lrcorner dx, \frac{\partial \psi}{\partial x_i} \rangle = \langle T, d(\psi \widehat{dx}_i) \rangle = 0,$$

since $\partial T = 0$. Hence we denote by $N \in \mathbb{N}$ the \mathcal{L}^n -a.e. constant value of $\langle T_x, 1 \rangle$, and we can obviously assume that $N \geq 1$. In view of the representation (6), this means that $T_x \in \mathcal{I}_0^N(\mathbb{R})$ for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$. This leads us to the decomposition (17)–(18).

Step 2. Next, we claim that the map

$$\begin{aligned} \mathbb{R}^n &\rightarrow (\mathcal{I}_0^N(\mathbb{R}), \mathbf{d}), \\ x &\mapsto T_x, \end{aligned}$$

belongs to $MBV_{\text{loc}}(\mathbb{R}^n; \mathcal{I}_0^N(\mathbb{R}))$.

We proceed as in the proof of Theorem 8.1 of [6]. Recalling the definitions and the discussion in Subsection 2.5, we only need to show that for every $\phi \in \text{Lip}_{b,1}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ the map

$$x \mapsto \langle T_x, \phi \rangle$$

belongs to $BV_{\text{loc}}(\mathbb{R}^n)$, with a uniform (with respect to ϕ) control of the derivative.

For every $\psi \in C_c^\infty(\mathbb{R}^n)$, applying once more (11) we compute

$$\int_{\mathbb{R}^n} \langle T_x, \phi \rangle \frac{\partial \psi}{\partial x_i}(x) dx = \langle T \lrcorner dx, \frac{\partial \psi}{\partial x_i} \phi \rangle = \langle T, \frac{\partial \psi}{\partial x_i} \phi dx \rangle = (-1)^n \langle T, \phi' \psi \widehat{dx}_i \wedge dy \rangle,$$

using in the last equality the fact that $\partial T = 0$. Therefore, taking the modulus of both sides, we obtain

$$\left| \int_{\mathbb{R}^n} \langle T_x, \phi \rangle \frac{\partial \psi}{\partial x_i}(x) dx \right| \leq \int_{\mathbb{R}^n} |\psi| d\pi_{\#} \|T\|,$$

where $\pi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is the projection on the x variable. This implies that the total variation of the distributional derivative of $x \mapsto \langle T_x, \phi \rangle$ satisfies

$$\|D\langle T_x, \phi \rangle\| \leq n\pi_{\#} \|T\|.$$

Step 3. Given $S \in \mathcal{I}_0^N(\mathbb{R})$ of the form

$$S = \sum_{j=1}^N \delta_{A_j}, \quad \text{with } A_1 \leq A_2 \leq \dots \leq A_N,$$

let us prove that the map

$$\begin{aligned} (\mathcal{I}_0^N(\mathbb{R}), \mathbf{d}) &\rightarrow \mathbb{R}, \\ S &\mapsto A_N, \end{aligned}$$

is 1-Lipschitz continuous.

Let $S \in \mathcal{I}_0^N(\mathbb{R})$ be of the form above and $S' \in \mathcal{I}_0^N(\mathbb{R})$ be of the same form

$$S' = \sum_{j=1}^N \delta_{A'_j}, \quad \text{with } A'_1 \leq A'_2 \leq \dots \leq A'_N.$$

Then

$$|A_N - A'_N| \leq \sum_{j=1}^N |A_j - A'_j| = \mathbb{F} \left(\sum_{j=1}^N \delta_{A_j} - \sum_{j=1}^N \delta_{A'_j} \right) = \mathbf{d}(S, S'),$$

where we have used Lemma 1.

Step 4. Finally we claim that the map

$$x \mapsto u_N(x), \quad \mathbb{R}^n \rightarrow \mathbb{R}$$

belongs to $BV_{\text{loc}}(\mathbb{R}^n)$.

We have already seen in Step 2 that the map

$$x \mapsto T_x, \quad \mathbb{R}^n \rightarrow \mathcal{I}_0^N(\mathbb{R})$$

is MBV_{loc} and in Step 3 that the map defined by

$$\sum_{j=1}^N \delta_{z_j} \mapsto \max_{1 \leq i \leq n} z_i, \quad \mathcal{I}_0^N(\mathbb{R}) \rightarrow \mathbb{R}$$

is Lipschitz. Then, Lemma 3 yields that their composition, namely u_N , belongs to $MBV_{\text{loc}}(\mathbb{R}^n; \mathbb{R})$, which is nothing but $BV_{\text{loc}}(\mathbb{R}^n)$.

Step 5. Induction and conclusion of the proof.

Up to now we have selected the top leaf of the decomposition. Now define

$$\hat{T} = T - \mathbf{i}(u_N).$$

It is readily checked that \hat{T} is an n -dimensional integer rectifiable current in \mathbb{R}^{n+1} , satisfying the zero-boundary condition, the positivity condition and the cylindrical mass condition as in the statement of the theorem, and that for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$ we have

$$\hat{T}_x = T_x - \delta_{u_N(x)} = \sum_{j=1}^{N-1} \delta_{u_j(x)}.$$

Then, it suffices to apply again $N-1$ times the construction described in the previous steps to deduce that all functions u_j belong to $BV_{\text{loc}}(\mathbb{R}^n)$ and, by construction, $(T - \sum_{j=1}^N \mathbf{i}(u_j)) \llcorner dx = 0$. Let now $R := T - \sum_{j=1}^N \mathbf{i}(u_j)$ and let us prove that $\partial R = 0$ and $R \llcorner dx = 0$ imply $R = 0$. Indeed, given $\psi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$, let $\varphi(x, y) := \int_{-\infty}^y \psi(x, s) ds$; then

$$0 = \langle \partial R, \widehat{\varphi dx_j} \rangle = \langle R, \frac{\partial \varphi}{\partial x_j} dx + \psi dy \wedge \widehat{dx_j} \rangle = \langle R, \psi dy \wedge \widehat{dx_j} \rangle \quad \forall j = 1, \dots, n.$$

Finally, property (2) is a consequence of the choice we have done in (17).

3.2. Uniqueness of the decomposition. The uniqueness of this decomposition is immediate. Assume that we have two decompositions

$$T = \sum_{j=1}^N \mathbf{i}(u_j) = \sum_{j=1}^M \mathbf{i}(v_j),$$

with $u_j \in BV_{\text{loc}}(\mathbb{R}^n)$ for $j = 1, \dots, N$ and $v_j \in BV_{\text{loc}}(\mathbb{R}^n)$ for $j = 1, \dots, M$ satisfying

$$u_1 \leq u_2 \leq \dots \leq u_N \quad \text{and} \quad v_1 \leq v_2 \leq \dots \leq v_M. \quad (19)$$

For \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$ the slice T_x satisfies

$$T_x = \sum_{j=1}^N \delta_{u_j(x)} = \sum_{j=1}^M \delta_{v_j(x)}.$$

This immediately implies that $N = M$ and, together with (19), that $u_j(x) = v_j(x)$ for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$ for every $j = 1, \dots, N$.

3.3. Equality of the total variations. We know that $\mathbf{i}(u_j) = (\Gamma(u_j), 1, \tau_{u_j})$, and the locality property stated in Lemma 2 allows to find a Borel orientation τ of $\Gamma := \cup_j \Gamma(u_j)$ with the property

$$\tau = \tau_{u_j} \quad \mathcal{H}^n\text{-a.e. on } \Gamma(u_j), \text{ for } j = 1, \dots, N, \quad (20)$$

since by construction the functions u_j satisfy (2). Let us define $\theta(w)$ as the cardinality of the set $\{j \in \{1, \dots, N\} : w \in \Gamma(u_j)\}$; taking (20) into account, we have then

$$\langle T, \omega \rangle = \sum_{j=1}^N \langle \mathbf{i}(u_j), \omega \rangle = \sum_{j=1}^N \int_{\Gamma(u_j)} \langle \tau_{u_j}, \omega \rangle d\mathcal{H}^n = \int_{\Gamma} \theta \langle \tau, \omega \rangle d\mathcal{H}^n.$$

This proves that $T = (\Gamma, \theta, \tau)$. As a consequence

$$\|T\| = \theta \mathcal{H}^n \llcorner \Gamma = \sum_{j=1}^N \mathcal{H}^n \llcorner \Gamma(u_j) = \sum_{j=1}^N \|\mathbf{i}(u_j)\|.$$

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