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Applications of integrand reduction to two-loop five-point scattering amplitudes in QCD

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We review the current state-of-the-art in integrand level reduction for five-point scattering amplitudes at two loops in QCD. We present some benchmark results for the evaluation of the leading colour two-loop five-gluon amplitudes in the physical region as well as the partonic channels for two quarks and three gluons and four quarks and one gluon.

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1. Introduction

In these proceedings we review some recent developments in integrand reduction methods for two-loop amplitudes and their application to five particle amplitudes in massless QCD. NNLO QCD corrections to processes such as $pp \to 3j$, $pp \to H+2j$ and $pp \to W+2j$ are currently a high priority within the community [1] where many measurements are already dominated by theoretical uncertainties. New methods to overcome the complexity of these amplitudes have been an on-going effort for many years and useful techniques have been proposed from many directions. High multiplicity multi-loop computations in maximally supersymmetric theories like $\mathcal{N}=4$ super Yang-Mills (sYM) are now common place where the current state of the art is planar 6-point amplitudes at 5-loops [2] with integrand representations known in principle to any loop order [3]. Understanding $\mathcal{N}=4$ sYM amplitudes has often provided valuable information when attempting to construct efficient computational strategies for more complicated and phenomenologically relevant gauge theories like QCD. On-shell methods such as unitarity [4, 5], generalised unitarity [6] and BCFW recursion [7] have been essential tools for deriving the compact five-point two-loop results in $\mathcal{N}=4$ sYM [8] and $\mathcal{N}=8$ supergravity [9].

Hadron collider phenomenology has gained significant precision thanks to automated codes that employ numerical reduction procedures to avoid the traditional bottleneck of large intermediate expressions. The integrand reduction procedure of Ossola, Papadopoulos and Pittau (OPP) [10] was an important development in this story and has been used in combination with generalised unitarity or a more traditional Feynman diagram approach to make phenomenological predictions for a wide range of complex final states with mass effects.

While extensions of the OPP method to two-loops (or indeed the multi-loop case) are now known [11, 12, 13, 14, 15, 16] applications had been previously restricted to specific 'all-plus' helicity amplitudes [17, 18, 19] where additional simplicity led to compact analytic representations. In the general case the integrand level expressions misses a large number of additional relations between basis integrals which can be identified from integration-by-parts (IBP) identities [20]. The standard method for approaching the computation of multi-loop amplitudes has been to construct a basis of master integrals using Laporta's algorithm [21] for solving systems of IBPs. This technique quickly becomes complicated for systems with many scales which has prompted new developments both in optimising current automated codes [22, 23, 24] and new approaches suitable for direct use in unitarity based approaches [25, 26, 27, 28]. There have been successful attempts to include IBP relations directly into the construction of the amplitudes from unitarity cuts using both the maximal unitarity method [29, 30] and more recently through numerical unitarity [31, 32].

Of course the reduction to a basis of integral functions is only part of the problem. The evaluation of multi-scale integral functions also presents a serious technical challenge. The first contributions to be completed were the planar topologies with all particles massless computed using (canonical) differential equation techniques [33, 34, 35]. Due to bottlenecks in the IBP reduction the non-planar topologies are still unfinished and are a high priority though many groups are working actively on the problem from different angles [36, 37].

In the last few months there has been increasing activity on the topic of five particle scattering amplitudes. The numerical unitarity approach has been used to reduce the planar five-gluon amplitudes to master integrals using finite field evaluations. Directly solving the system of IBP equations

in the planar case has been possible by optimising the route through the Laporta algorithm to produce some analytic, though quite lengthy, expressions [38, 39].

In these proceedings we report on an approach using integrand reduction to obtain analytic formulae for two-loop five-parton integrands and validate our expressions by providing numerical benchmarks of the integrated amplitudes for the set independent helicity configurations.

2. Integrand parametrisations and reconstruction over finite fields

In a recent paper by one of the authors [40] it was demonstrated that numerical sampling of generalised unitarity cuts over finite fields [41, 42, 43, 44] could be used together with integrand reduction to extract analytic representation of the integrands of complex amplitudes in QCD.

We start by parametrising the partial amplitudes of a standard colour decomposition in terms of irreducible numerators, Δ_T,

$$A^{(2)}(1,2,3,4,5) = \int \frac{d^d k_1}{i\pi^{d/2} e^{-\varepsilon\gamma_E}} \frac{d^d k_2}{i\pi^{d/2} e^{-\varepsilon\gamma_E}} \sum_T \frac{\Delta_T(\{k\},\{p\})}{\prod_{\alpha\in T} D_\alpha}, \tag{2.1}$$

where $\{k\} = \{k_1, k_2\}$ are the $(d = 4-2\varepsilon)$-dimensional loop momenta, T is the set of independent topologies and $\{p\} = \{1,2,3,4,5\}$ are the ordered external momenta. The index α runs over the set of propagators associated with the topology T. Our planar five-parton amplitudes are built from 57 distinct topologies, giving 425 irreducible numerators when including permutations of the external legs. Each topology, T, has an associated irreducible numerator Δ_T which depends on Lorentz invariants of the loop momenta and external momenta.

We construct a basis of monomials in these invariants by performing a transverse decomposition of the loop momentum for each topology along the lines of the construction of van Neerven and Vermaseren [45]. Specifically we keep track of both 4-dimensional and (-2ε)-dimensional components in the transverse space: $k_i^\mu = k_{\|,i}^\mu + k_{\perp,i}^{[4]} + k_{\perp,i}^{[-2\varepsilon]}$. Further details are given in references [40, 46]. After this decomposition has been performed the numerator can be parametrised using three classes of irreducible scalar products (ISPs). ISPs in the parallel space can be written in terms dot products between loop momenta and external momenta $k_i.p_j$ while ISPs in the transverse space are either in the 4-d 'spurious' space $k_i.\omega_j$ or extra dimensional space $\mu_{ij} = -k_{\perp,i}^{[-2\varepsilon]}.k_{\perp,j}^{[-2\varepsilon]}$. We take all external momenta to live in exactly four dimensions.

A basis for $\Delta_T(k_i.p_j, k_i.\omega_j, \mu_{ij})$ is then computed by first finding all possible monomials according to the gauge theory power counting and then solving a linear system to relate these to an independent basis that does not involve the μ_{ij} variables [46]. The choice of ordering within the over-complete set of monomials affects the basis and the final form of the integrand. This method avoids the polynomial division approach taken in previous integrand reduction methods [13, 14] and the linear system can be analysed efficiently by sampling over finite fields.

Once the basis of monomials is determined the integrand can be constructed by solving a system of generalised unitarity cuts. On each cut either a product of on-shell tree-amplitudes or a set of ordered Feynman diagrams can be used as input to an integrand fit using finite field evaluations. In our first applications of this procedure we have considered both tree-amplitude input using Berends-Giele recursion relations [47] in the six-dimensional spinor-helicity approach [48] and

Feynman diagrams in which the 't Hooft algebra has been used[1] to evaluate the extra-dimensional spinor strings. In the six-dimensional case we perform a dimensional reduction with scalar propagators to find integrands with including dependence on the spin dimension $g^{\mu}{}_{\mu} = d_s$.

In order to control kinematic complexity and to allow for numerical evaluations over finite fields, we use momentum twistor variables [52] to obtain a rational parametrisation of the multi-particle kinematics. Analytical representation of our integrand is obtained by performing functional reconstruction from finite fields evaluations. In this way, we avoid processing large intermediate algebraic expressions that normally appear in the analytical computation. After the integrand is reconstructed, the transverse space must be integrated [53] to obtain a form compatible with traditional integration-by-parts (IBP) relations. In this work, we integrate only over spurious space and keep μ_{ij} dependence, that can further be removed through dimension shifting identities.

3. Evaluation of two-loop five-parton amplitudes at benchmark phase-space points

In this section we provide some benchmark results for five-parton scattering amplitudes at two loops in QCD. Results are obtained in the leading colour approximation where only planar diagrams, and integrals, appear. These results have been obtained using the analytic expressions of the master integrals in [54], by-passing the time consuming step of integral evaluations with sector decomposition [55, 56] used in our previous publication.

3.1 Colour decompositions of five-parton amplitudes

The amplitudes in this section are defined with the normalisation

$$n = m_\varepsilon N_c \alpha_s/(4\pi), \quad \alpha_s = g_s^2/(4\pi), \quad m_\varepsilon = i(4\pi)^\varepsilon e^{-\varepsilon\gamma_E}, \tag{3.1}$$

where the dimensional regulator $\varepsilon = \frac{d-4}{2}$, N_C is the number of colours, and γ_E is the Euler–Mascheroni constant. Their colour decompositions are given by

$$\begin{aligned}\mathscr{A}^{(L)}(1_g,2_g,3_g,4_g,5_g) = n^L g_s^3 \sum_{\sigma\in S_5/Z_5} &\mathrm{tr}\left(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}}\right)\\ &\times A^{(L)}\left(\sigma(1)_g,\sigma(2)_g,\sigma(3)_g,\sigma(4)_g,\sigma(5)_g\right)\end{aligned} \tag{3.2}$$

for five gluons,

$$\mathscr{A}^{(L)}(1_q,2_g,3_g,4_g,5_{\bar{q}}) = n^L g_s^3 \sum_{\sigma\in S_3} \left(T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}\right)_{i_1}^{\bar{i}_5} A^{(L)}(1_q,\sigma(2)_g,\sigma(3)_g,\sigma(4)_g,5_{\bar{q}}) \tag{3.3}$$

for a quark pair and three gluons channel and,

$$\mathscr{A}^{(L)}(1_q,2_{\bar{q}},3_g,4_Q,5_{\bar{Q}}) = n^L g_s^3 \left[\left(T^{a_3}\right)_{i_4}^{\bar{i}_2}\delta_{i_1}^{\bar{i}_5} A^{(L)}(1_q,2_{\bar{q}},3_g,4_Q,5_{\bar{Q}}) + \left(1\leftrightarrow 4, 2\leftrightarrow 5\right)\right] \tag{3.4}$$

for the case of two distinct quark pairs and one gluon. In addition we normalise all amplitudes to the leading order amplitudes which removes any complex phase,

$$\widehat{A}^{(2)}_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} = \frac{A^{(2)}(1^{\lambda_1},2^{\lambda_2},3^{\lambda_3},4^{\lambda_4},5^{\lambda_5})}{A^{(0)}(1^{\lambda_1},2^{\lambda_2},3^{\lambda_3},4^{\lambda_4},5^{\lambda_5})}. \tag{3.5}$$

For helicity configurations that vanish at tree-level the leading term in the expansion around $d = 4-2\varepsilon$ and $d_s = 4$ at one-loop is used.

[1] We use QGRAF [49] to generate Feynman diagrams and FORM [50, 51] to perform algebraic manipulations

	ε^{-4}	ε^{-3}	ε^{-2}	ε^{-1}	ε^{0}
$\widehat{A}^{(2),[0]}_{--+++}$	12.5	27.7526	-23.7728	-168.1162	-175.2103
$\widehat{A}^{(2),[0]}_{-+-++}$	12.5	27.7526	2.5028	-35.8084	69.6695

Table 1: The (non-zero) leading colour primitive two-loop helicity amplitudes for the $d_s = 2$ component of $\widehat{A}^{(2)}(1_g, 2_g, 3_g, 4_g, 5_g)$ at the Euclidean phase space point given in the text.

	ε^{-4}	ε^{-3}	ε^{-2}	ε^{-1}	ε^{0}
$\widehat{A}^{(2),[1]}_{+++++}$	0	0	-2.5	-6.4324	-5.3107
$\widehat{A}^{(2),[1]}_{-++++}$	0	0	-2.5	-12.7492	-22.0981
$\widehat{A}^{(2),[1]}_{--+++}$	0	-0.625	-1.8175	-0.4869	3.1270
$\widehat{A}^{(2),[1]}_{-+-++}$	0	-0.625	-2.7759	-5.0018	0.1807

Table 2: The leading colour primitive two-loop helicity amplitudes for the $(d_s - 2)$ component of $\widehat{A}^{(2)}(1_g, 2_g, 3_g, 4_g, 5_g)$ at the Euclidean phase space point given in the text.

3.2 Evaluation of the master integrals

The master integrals were computed in [54] using first-order differential equations. All functions needed are expressed in terms of iterated integrals, where the integration kernels are taken from a set that was identified in [57]. The boundary conditions for the differential equations were determined by constraints such as the absence of unphysical branch cuts. We determined such boundary points for each of the physical regions, as well as for the Euclidean region.

Up to weight two, all master integrals are expressed in terms of logarithms and dilogarithms. Weight-three contributions are expressed in terms of Li_3 functions and in terms of one-dimensional integrals of logarithms and dilogarithms. At weight four, we use a representation proposed in [58] that allows to write the functions as a one-fold integral of known functions, leading to a fast and reliable numerical evaluation, for all kinematic regions.

As a validation of these formulas, we have performed numerical comparisons with [59] and, for the four-point subtopologies, with [60], finding perfect agreement.

3.3 Evaluation in the Euclidean region

We use the phase-space point defined by the invariants

$$s_{12} = -1, \quad s_{23} = -\frac{37}{78}, \quad s_{34} = -\frac{2023381}{3194997}, \quad s_{45} = -\frac{83}{102}, \quad s_{15} = -\frac{193672}{606645}, \tag{3.6}$$

which corresponds to the following values of momentum twistor variables[2],

$$x_1 = -1, \quad x_2 = \frac{79}{90}, \quad x_3 = \frac{16}{61}, \quad x_4 = \frac{37}{78}, \quad x_5 = \frac{83}{102}. \tag{3.7}$$

[2]The form of the momentum twistor parametrisation is given explicitly in Reference [46]

	$\widehat{A}^{(2),[2]}_{+++++}$	$\widehat{A}^{(2),[2]}_{-++++}$	$\widehat{A}^{(2),[2]}_{--+++}$	$\widehat{A}^{(2),[2]}_{-+-++}$
ε^0	3.6255	-0.0664	0.2056	0.0269

Table 3: The leading colour primitive two-loop helicity amplitudes for the $(d_s-2)^2$ component of $\widehat{A}^{(2)}(1_g,2_g,3_g,4_g,5_g)$ at the Euclidean phase space point given in the text.

	ε^{-4}	ε^{-3}	ε^{-2}	ε^{-1}	ε^0
$\widehat{A}^{(2)}_{++++-}$	0	0	-4	-13.53227	6.04865
$\widehat{A}^{(2)}_{+++--}$	8	7.96829	-52.39270	-140.15637	47.56872
$\widehat{A}^{(2)}_{++-+-}$	8	7.96829	-32.22135	-47.92349	145.97201
$\widehat{A}^{(2)}_{+-++-}$	8	7.96829	-40.88511	-87.02993	101.23299

Table 4: The leading colour primitive two-loop helicity amplitudes for $\widehat{A}^{(2)}(1_q,2_g,3_g,4_g,5_{\bar{q}})$ in the HV scheme at the Euclidean phase space point given in the text.

	ε^{-4}	ε^{-3}	ε^{-2}	ε^{-1}	ε^0
$\widehat{A}^{(2)}_{+-++-}$	4.5	2.28315	-32.09848	-41.39350	149.33050
$\widehat{A}^{(2)}_{+--+-}$	4.5	2.28315	-4.61657	-6.32369	-32.03278
$\widehat{A}^{(2)}_{+-+-+}$	4.5	2.28315	-38.29478	-43.52329	-56.71968
$\widehat{A}^{(2)}_{+---+}$	4.5	2.28315	-26.71316	-69.75805	22.23653

Table 5: The leading colour primitive two-loop helicity amplitudes for $\widehat{A}^{(2)}(1_q,2_{\bar{q}},3_g,4_Q,5_{\bar{Q}})$ in the HV scheme at the Euclidean phase space point given in the text.

The numerical results are shown in Tables 1 - 3, 4 and 5 for $ggggg$, $qggg\bar{q}$ and $q\bar{q}gQ\bar{Q}$ partonic channels, respectively. We have compared the poles of our results against the known universal IR structure [61, 62, 63, 64], and the d_s dependence of the IR pole formula in the 5-gluon case is extracted from the FDH results in [65].

3.4 Evaluation in the physical region

For numerical evaluation in the physical region, we use a phase space point defined by the invariants

$$s_{12}=\frac{113}{7},\quad s_{23}=-\frac{152679950}{96934257},\quad s_{34}=\frac{1023105842}{138882415},\quad s_{45}=\frac{10392723}{3968069},\quad s_{15}=-\frac{8362}{32585}, \tag{3.8}$$

which corresponds to the following values of our momentum twistor variables

$$x_1=\frac{113}{7},\quad x_2=-\frac{2}{9}-\frac{i}{19},\quad x_3=-\frac{1}{7}-\frac{i}{5},\quad x_4=\frac{1351150}{13847751},\quad x_5=-\frac{91971}{566867}. \tag{3.9}$$

The numerical results for the $ggggg$ partonic channel are shown in Tables 6 - 8.

	ε^{-4}	ε^{-3}	ε^{-2}	ε^{-1}	ε^{0}
$\widehat{A}^{(2),[0]}_{--+++}$	12.5	-9.17716 + 47.12389 i	-107.40046 - 25.96698 i	17.24014 - 221.41370 i	388.44694 - 167.45494 i
$\widehat{A}^{(2),[0]}_{-+-++}$	12.5	-9.17716 + 47.12389 i	-111.02853 - 12.85282 i	-39.80016 - 216.36601 i	342.75366 - 309.25531 i

Table 6: The leading colour primitive two-loop helicity amplitudes for the $d_s = 2$ component of $\widehat{A}^{(2)}(1_g, 2_g, 3_g, 4_g, 5_g)$ at the physical phase space point given in the text.

	ε^{-4}	ε^{-3}	ε^{-2}	ε^{-1}	ε^{0}
$\widehat{A}^{(2),[1]}_{+++++}$	0	0	-2.5	0.60532 - 12.48936 i	35.03354 + 9.27449 i
$\widehat{A}^{(2),[1]}_{-++++}$	0	0	-2.5	-7.59409 - 2.99885 i	-0.44360 - 20.85875 i
$\widehat{A}^{(2),[1]}_{--+++}$	0	-0.625	-0.65676 - 0.42849 i	-1.02853 + 0.30760 i	-0.55509 - 6.22641 i
$\widehat{A}^{(2),[1]}_{-+-++}$	0	-0.625	-0.45984 - 0.97559 i	1.44962 + 0.53917 i	-0.62978 + 2.07080 i

Table 7: The leading colour primitive two-loop helicity amplitudes for the $d_s - 2$ component of $\widehat{A}^{(2)}(1_g, 2_g, 3_g, 4_g, 5_g)$ at the physical phase space point given in the text.

	$\widehat{A}^{(2),[2]}_{+++++}$	$\widehat{A}^{(2),[2]}_{-++++}$	$\widehat{A}^{(2),[2]}_{--+++}$	$\widehat{A}^{(2),[2]}_{-+-++}$
ε^0	0.60217 - 0.01985 i	-0.10910 - 0.01807 i	-0.06306 - 0.01305 i	-0.03481 - 0.00699 i

Table 8: The leading colour primitive two-loop helicity amplitudes for the $(d_s - 2)^2$ component of $\widehat{A}^{(2)}(1_g, 2_g, 3_g, 4_g, 5_g)$ at the physical phase space point given in the text.

4. Outlook

The last few months have seen rapid progress in our ability to compute some of the missing two-loop amplitudes needed to improve the precision of theoretical predictions at hadron colliders. While benchmark numerical evaluations have been completed, the analytic representations of the integrand are extremely large. Further study of the analytic structure and direct reduction with a complete set of IBPs will be important to obtain representations suitable for flexible phenomenological applications. Continuing studies into the structure of amplitudes in maximally supersymmetric gauge theory such as local integrand structures in planar and non-planar sectors [66, 67, 68] may prove to give useful insights when persuing this direction.

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