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The classical solution for the Bosonic string in the presence of three D-branes rotated by arbitrary SO(4) elements

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Abstract

We consider the classical instantonic contribution to the open string configuration associated with three D-branes with relative rotation matrices in SO(4) which corresponds to the computation of the classical part of the correlator of three non Abelian twist fields. We write the classical solution as a sum of a product of two hypergeometric functions. Differently from all the previous cases with three D-branes, the solution is not holomorphic and suggests that the classical bosonic string knows when the configuration may be supersymmetric. We show how this configuration reduces to the standard Abelian twist field computation. From the phenomenological point of view, the Yukawa couplings between chiral matter at the intersection in this configuration are more suppressed with respect to the factorized case in the literature.
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1. Introduction and conclusions

The study of viable phenomenological models in the framework of String Theory often involves the analysis of the properties of systems of D-branes. Clearly the inclusion of the physical requirements needed for a consistent theory deeply constrains the possible scenarios. In particular the chiral spectrum of the Standard Model acts as a strong restriction on the possible D-brane

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setup. Intersecting branes represent a relevant class of such models with interacting chiral matter.

In this paper we focus on the development of technical tools for the computation of Yukawa interactions for D-branes at angles [1–10]. These couplings, as well as the study of flavor changing neutral currents [11], are crucial in determining the validity of the different models. Moreover, several similar computations heavily require the ability to compute correlation functions of twist fields and excited twist fields and Green functions in the presence of twists.

The computation of the correlation functions of Abelian twist fields is the subject of a vast and solid literature and play a prominent role in many scenarios, such as magnetic branes with commuting magnetic fluxes [12–16], strings propagating in a gravitational wave background [17–20], D-brane bound states [21–23] and tachyon condensation in Superstring Field Theory [23–26]. A similar investigation can be extended to the properties of excited twist fields even though they are slightly more subtle to treat and hide many more delicate aspects [27–32]. Nonetheless, many results were found starting from the old dual models up to more modern interpretations of String Theory [33,34]. The generalization of the correlation function of pure twists fields to an arbitrary number of plain and excited twist fields (in combination with the usual vertex operators) is however more recent [35–37] and blends the CFT techniques with the path integral approach and the study of the Reggeon vertex [38–42]. The same result has also been recovered in the framework of the canonical quantization [43] and shows a global picture behind the computation of the correlators instead of a case-by-case dependence.

In the framework of intersecting D6-branes at angles we study the case of the D-branes whose relative rotations are non Abelian and, as a consequence, present non Abelian twist fields at the intersections. We try to understand the subtleties and technical issues arising from such scenario which has been studied only in few cases: in older days in the formulation of non Abelian orbifolds [44–47] and more recently for a D-branes system whose relative rotations are in SU(2) [48].

The configuration for which we develop the technical tools needed to study the Yukawa couplings is three D6-branes inside $\mathbb{R}^{1,9}$ with an internal space of the form $\mathbb{R}^4 \times \mathbb{R}^2$, prior to compactification to a torus, where the branes are embedded as lines in $\mathbb{R}^2$ and as bi-dimensional surfaces inside $\mathbb{R}^4$. In particular we focus on the relative rotations which characterize each brane in $\mathbb{R}^4$ with respect to the others. They will generally be non commuting SO(4) matrices.

In this paper we study the classical solution of the bosonic string which governs the behavior of the correlator of twist field and consequently the Yukawa couplings. In fact, once we separate the classical contribution of the string from the quantum fluctuations, using the path integral approach, we can write the correlator of N_B twist fields as

$$\left\langle \prod_{t=1}^{N_B} \sigma_{M_{(t)}}(x_t) \right\rangle = \mathcal{N}\left(\{x_t, M_{(t)}\}_{1 \leq t \leq N_B}\right) e^{-S_E\left(\{x_t, M_{(t)}\}_{1 \leq t \leq N_B}\right)},$$

where $M_{(t)}$ $(1 \leq t \leq N_B)$ are the monodromies induced by the twist fields, N_B is the number of D-branes (and their intersections) and x_t are the interaction points on the string worldsheet. Even though the quantum corrections in $\mathcal{N}\left(\{x_t, M_{(t)}\}_{1 \leq t \leq N_B}\right)$ are crucial to the complete determination of the correlator, the classical contribution to the Euclidean action represents the leading term of the Yukawa couplings. In this paper we address only this point in order to better understand the differences from the usual factorized case and generalize the results of the previous analysis on non Abelian rotations of the branes. We will not consider the quantum corrections since they cannot be computed with the actual techniques and their determination requires the

computation of the 4 twists correlator which requires knowledge of the connection formula for Heun functions which is not known.

In the second section of this paper we study the boundary conditions for the open string describing the D-branes embedded in $\mathbb{R}^4$. We first define the embedding of a brane locally in a well adapted frame of reference where all branes have the same embedding conditions, then we connect all these local descriptions using a global coordinate system. In this reference frame each brane is rotated with respect to the others and this gives raise to monodromies of the doubled string coordinate fields.

In the third section we choose the monodromies in SO(4) and we rewrite the boundary conditions problem in spinor representation by means of the local isomorphism SO(4) $\cong$ SU(2) $\times$ SU(2). In doing so we recast the issue of finding the solution intended as 4 real vector in the search of two solutions in the fundamental of SU(2), one for each SU(2).

In the fourth section we solve the previous problem of finding two functions transforming as a vector of SU(2) by means of a basis of hypergeometric functions. In particular we show how to relate the parameters of the rotations and the parameters of the hypergeometric equation and the fact that a rescaling factor is needed with respect to the conventionally normalized basis of solutions of the hypergeometric equation. Given the infinite number of solutions representing the same rotations and labeled by the choice of integer factors, we isolate the correct and finite number of solutions, actually two, by looking for independent hypergeometric functions and a finite Euclidean action.

The fifth section is dedicated to recovering the previous results from the general case. We compute the Abelian limit of the monodromies and we connect the parameters of the SU(2) $\times$ SU(2) to the usual parameters used in the geometrical construction. We then show how the known result follows naturally and we encounter the same analyticity properties of the field which have been shown in the past. We check also that the case of SU(2) monodromies is smoothly recovered.

Eventually, in the last section we give a natural interpretation of the result highlighting the key differences between the case of Abelian twist fields and the general setup and showing the physical consequences on the Yukawa couplings. The final result shows a substantial difference between the Abelian and the non Abelian case and even between the SU(2) $\times$ SU(2) and SU(2) cases. In the Abelian formulation the contribution of the Euclidean action is exactly the area of the triangle formed by the intersecting branes in $\mathbb{R}^2$, that is the string worldsheet is completely contained inside the polygon and the action is indeed proportional to its area. In the non Abelian case, even though the three intersection points still define a 2-dimensional plane in $\mathbb{R}^4$, the string worldsheet is no longer flat and spans a larger area with respect to the previous case. Intuitively, because of the non Abelian nature of the D-brane rotation, the string has to bend in order to stretch between the branes and cannot entirely reside on a flat surface. The difference between the SO(4) and SU(2) cases is more subtle: in the SU(2) case there exist complex coordinates for $\mathbb{R}^4$ for which the classical string solution is holomorphic in the upper half plane while in SO(4) case this does not happen. The reason of this can probably be traced back to supersymmetry, even if we are dealing with the bosonic part only. In fact, for branes rotated by SU(2) elements, part of the spacetime supersymmetry is preserved. The further suppression with respect to the Abelian case of the Yukawa interactions represents the physical interpretation of the result.

2. D-brane configuration and boundary conditions

Even though we are ultimately interested to the framework of superstrings and D6-branes intersecting at angles in the internal space, we will focus on the bosonic string embedded in

$\mathbb{R}^{1,d+4}$. The branes are seen as 2-dimensional Euclidean planes in $\mathbb{R}^4$ times possible further dimensions in $\mathbb{R}^{1,d}$. We then specifically concentrate on the Euclidean explicit solution for the classical bosonic string in this scenario.

The mathematical analysis is however more general and can be applied to any Dp-brane embedded in a generic Euclidean space $\mathbb{R}^q$. The full classical solution can in principle be written also in this case provided one can find the explicit form of the basis of functions with the proper boundary and monodromy conditions. This is possible in the case of three intersecting branes but in general it is an open mathematical issue. In fact, in the case of three branes with generic embedding we can usually connect a local basis around one intersection point to a local basis around a second intersection point, the third depending on the first two intersections, by means of Mellin-Barnes integrals. This way the solution can be explicitly and globally constructed. However, with more than three D-branes (consequently, intersection points) the situation is by far more difficult since the explicit form of the connection formulas is not known and therefore we cannot write any local basis with respect to the others. Hence the global solution cannot be fully specified.

2.1. Intersecting D-branes at angles

First of all we describe more precisely the embedding of the D-branes in $\mathbb{R}^{1,d+4}$ associated to the Euclidean space $\mathbb{R}^4$ which is the main focus of this paper. Let N_B be the total number of D-branes and $t = 1, 2, \ldots, N_B$ be an index defined modulo N_B to label them, then we can describe one of those D-branes in a well adapted system of coordinates $X^I_{(t)}$, where $I = 1, 2, 3, 4$, as:

$$X^3_{(t)} = X^4_{(t)} = 0. \tag{2.1}$$

That is, we choose $X^1_{(t)}$ and $X^2_{(t)}$ to be the coordinates parallel to the brane labeled with $D_{(t)}$ while $X^3_{(t)}$ and $X^4_{(t)}$ are the coordinates orthogonal to it.

This well adapted reference coordinates system is connected to the global $\mathbb{R}^4$ coordinates X^I, which we use to study the entire set of D-branes, as:

$$X^I_{(t)} = \left(R_{(t)} \right)^I_{\ J} X^J - g^I_{(t)} \quad \text{for} \quad I, J = 1, 2, 3, 4, \tag{2.2}$$

where $R_{(t)}$ represents the rotation of the D-brane $D_{(t)}$ and $g_{(t)}$ its translation with respect to the origin of the global set of coordinates (see Fig. 1 for a 2-dimensional example). While we could naively consider $R_{(t)} \in SO(4)$, rotating separately the subset of coordinates parallel and orthogonal to the D-brane does not affect the embedding and it just amounts to a trivial redefinition of the initial well adapted coordinates. Therefore $R_{(t)}$ is actually defined in the Grassmannian:

$$R_{(t)} \in \frac{SO(4)}{S\left(O(2) \times O(2)\right)}, \tag{2.3}$$

that is we need only consider the left coset where $R_{(t)}$ is a representative of an equivalence relation of the form

$$R_{(t)} \sim \mathcal{O}_{(t)} R_{(t)},$$

where the $S\left(O(2) \times O(2)\right)$ element $\mathcal{O}_{(t)}$ is defined as

$$\mathcal{O}_{(t)} = \begin{pmatrix} O^{\|}_{(t)} & \\ & O^{\perp}_{(t)} \end{pmatrix}$$
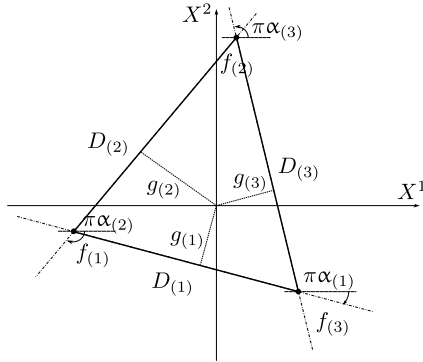
Fig. 1. Geometry of branes at angles in the factorized case: branes are embedded in the plane $\mathbb{R}^2$ as lines. Here, the geometrical interpretation of $g_{(t)}$ is straightforward and such that $g_{(t)}^\perp \geq 0$ and we describe the planar rotation $R_{(t)}(\alpha_{(t)}) \in$ SO(2)U(1) by means of a single parameter $-1 \leq \alpha_{(t)} < 1$.

with $O_{(t)}^\parallel \in O(2)$, $O_{(t)}^\perp \in O(2)$ and $\det(\mathcal{O}_{(t)}) = 1$. Here we adopted a shorthand notation which we will use again later: the superscript $\parallel$ represents any of the coordinates parallel to the brane, while $\perp$ any of the orthogonal directions. Notice that we write $S(O(2) \times O(2))$ and not $SO(2) \times SO(2)$ since the additional $\mathbb{Z}_2$ group can be used to set $g_{(t)}^\perp \geq 0$.

2.2. Boundary conditions for branes at angles

We now consider the implications of the embedding of the branes on the boundary conditions of the open strings. Let $\tau_E = i\tau$ be the Euclidean time direction, then we define the usual upper plane coordinates:

$$u = x + iy = e^{\tau_E + i\sigma} \in H \cup \{z \in \mathbb{C} \mid \text{Im}\, z = 0\},$$
$$\bar{u} = x - iy = e^{\tau_E - i\sigma} \in \overline{H} \cup \{z \in \mathbb{C} \mid \text{Im}\, z = 0\},$$

where $H = \{z \in \mathbb{C} \mid \text{Im}\, z > 0\}$ is the upper complex plane, $\overline{H} = \{z \in \mathbb{C} \mid \text{Im}\, z < 0\}$ is the lower complex plane and $\bar{u} = u^*$ by definition. In the conformal coordinates u and $\bar{u}$, D-branes are mapped to the real axis $\text{Im}\, z = 0$ and we use the symbol $D_{(t)}$ to specify both the brane and the interval representing it on the real axis of the upper half plane:

$$D_{(t)} = [x_t, x_{t-1}],$$

where $t = 2, 3, \ldots, N_B$ and $x_t < x_{t-1}$. The points x_t and x_{t-1} represent the worldsheet inter-section points of the brane $D_{(t)}$ with the branes $D_{(t+1)}$ and $D_{(t-1)}$ respectively. With this choice we have to consider carefully the interval $[x_1, x_{N_B}]$ representing the brane $D_{(1)}$: since the branes are defined modulo N_B, as shown in Fig. 2, it actually is:

$$D_{(1)} = [x_1, +\infty) \cup (-\infty, x_{N_B}].$$

In the global coordinates system X^I ($I = 1, 2, 3, 4$), associated to the subspace $\mathbb{R}^4 \subset \mathbb{R}^{1,d+4}$ where branes are generically rotated by a non Abelian rotation, the relevant part of the string action in conformal gauge is:
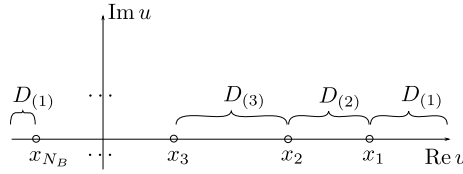
Fig. 2. Each interval $[x_t, x_{t-1}]$ for $t = 2, 3, \ldots, N_B$ defines the brane $D_{(t)}$. The brane $D_{(1)}$ is actually defined on the union of the intervals $(-\infty, x_{N_B}]$ and $[x_1, +\infty)$.

$$
\begin{aligned}
S_{\mathbb{R}^4} &= \frac{1}{2\pi\alpha'} \iint_{\mathrm{H}} \mathrm{d}^2 u\, \partial X^I \overline{\partial} X^I = \\
&= \frac{1}{4\pi\alpha'} \iint_{\mathbb{R}\times\mathbb{R}^+} \mathrm{d}x\, \mathrm{d}y \left(\left(\frac{\partial X^I}{\partial x} \right)^2 + \left(\frac{\partial X^I}{\partial y} \right)^2 \right),
\end{aligned}
\tag{2.4}
$$

where $\mathrm{d}^2 u = \mathrm{d}u\, \mathrm{d}\overline{u} = 2\,\mathrm{d}x\,\mathrm{d}y$ and

$$
\begin{aligned}
\partial &= \frac{\partial}{\partial u} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \\
\overline{\partial} &= \frac{\partial}{\partial \overline{u}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\end{aligned}
$$

Clearly, the equations of motion in these coordinates are:

$$
\partial \overline{\partial} X^I(u, \overline{u}) = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) X^I(x + iy, x - iy) = 0,
\tag{2.5}
$$

and their solution factorizes as usual in left and right moving parts:

$$
X^I(u, \overline{u}) = X^I_L(u) + X^I_R(\overline{u}).
$$

The information on the D-branes is in the boundary conditions which we now discuss.

In the well adapted coordinates, where the embedding is given by (2.1), we describe an open string with one of the endpoints on the brane $D_{(t)}$ through the relations:

$$
\partial_\sigma X^i_{(t)}(\tau, \sigma) \Big|_{\sigma=0} = \partial_y X^i_{(t)}(u, \overline{u}) \Big|_{y=0} = 0 \quad \text{for} \quad i = 1, 2,
\tag{2.6}
$$

$$
X^m_{(t)}(\tau, 0) = \quad X^m_{(t)}(x, x) \quad = 0 \quad \text{for} \quad m = 3, 4,
\tag{2.7}
$$

where $x \in D_{(t)} = [x_t, x_{t-1}]$ and the index i labels the Neumann boundary conditions associated with the parallel directions while m labels the Dirichlet coordinates associated to the normal ones. As argued in the previous section, this well adapted set of coordinates is connected to the global coordinates X^I as in (2.2).

In order to deal with the presence of $g^m_{(t)}$ in (2.2) and (2.7) and to get simpler boundary conditions, we consider the derivative along the boundary direction of (2.7) in such a way to remove the dependence on the translation $g^m_{(t)}$. This procedure produces simpler boundary conditions which are nevertheless not equivalent to the original ones: they will be recovered later by adding further constraints. The simpler boundary conditions for the global coordinates are:

$$\left(R_{(t)}\right)^i{}_J \, \partial_\sigma X^J(\tau,\sigma)\Big|_{\sigma=0} = 0 \quad \text{for} \quad i = 1, 2,$$

$$\left(R_{(t)}\right)^m{}_J \, \partial_\tau X^J(\tau,\sigma)\Big|_{\sigma=0} = 0 \quad \text{for} \quad m = 3, 4.$$

In the upper half plane coordinates and using the solution of the equations of motions they become:

$$\left(R_{(t)}\right)^i{}_J \left(\partial X^J_L(x+i0^+) - \overline{\partial} X^J_R(x-i0^+)\right) = 0 \quad \text{for} \quad i = 1, 2,$$

$$\left(R_{(t)}\right)^m{}_J \left(\partial X^J_L(x+i0^+) + \overline{\partial} X^J_R(x-i0^+)\right) = 0 \quad \text{for} \quad m = 3, 4,$$

where $x \in D_{(t)}$.

Introducing the matrix

$$S = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{2.8}$$

we can write the full boundary conditions (not just the simplified version we have just discussed) in terms of discontinuities along the branes and space time interactions points as:

$$\begin{cases} \partial X^I_L(x+i0^+) = \left(U_{(t)}\right)^I{}_J \, \overline{\partial} X^J_R(x-i0^+) & \text{for} \quad x_t \le x < x_{t-1} \\ X^I(x_t, x_t) = f^I_{(t)} \end{cases}, \tag{2.9}$$

where

$$U_{(t)} = R_{(t)}^{-1} S R_{(t)} \in \frac{\text{SO}(4)}{\text{S(O}(2) \times \text{O}(2))} \tag{2.10}$$

and $f_{(t)}$ is the target space embedding of the worldsheet interaction point between the brane $D_{(t)}$ and $D_{(t+1)}$. Given its definition, it is trivial but nonetheless critical to show that $U_{(t)}$ satisfies

$$U_{(t)} = U_{(t)}^{-1} = U_{(t)}^T.$$

On the other hand, another key point is the fact that $f_{(t)}$ recovers the apparent loss of information on the translation $g_{(t)}$. Consider for instance the embedding equations (2.7) for any two intersecting branes $D_{(t)}$ and $D_{(t+1)}$, then introducing the auxiliary quantities

$$\mathcal{R}_{(t,t+1)} = \begin{pmatrix} R^m_{(t)} \\ R^n_{(t+1)} \end{pmatrix} \in \text{GL}_4(\mathbb{R}) \quad \text{for} \quad m, n = 3, 4,$$

$$\mathcal{G}_{(t,t+1)} = \begin{pmatrix} g^m_{(t)} \\ g^n_{(t+1)} \end{pmatrix} \in \mathbb{R}^4 \quad \text{for} \quad m, n = 3, 4,$$

we compute the intersection point as:

$$f_{(t)} = \left(\mathcal{R}_{(t,t+1)}\right)^{-1} \mathcal{G}_{(t,t+1)}.$$

The result shows that information on the parameter $g_{(t)}$ is recovered through the global boundary conditions in the second equation of (2.9).

2.3. Doubling trick and branch cut structure

In going from the boundary conditions (2.6) and (2.7) to (2.9) we introduced discontinuities across each D-brane thus defining a non trivial cut structure on the complex plane. We introduce the doubling trick to deal with fields which take values on the whole complex plane by gluing the relations along an arbitrary but fixed D-brane $D_{(\bar{t})}$:

$$\partial \mathcal{X}(z) = \begin{cases} \partial X_L(u) & \text{if} \quad z = u \quad \text{and} \quad \mathrm{Im}\, z > 0 \quad \text{or} \quad z \in D_{(\bar{t})} \\ U_{(\bar{t})} \overline{\partial} X_R(\overline{u}) & \text{if} \quad z = \overline{u} \quad \text{and} \quad \mathrm{Im}\, z < 0 \quad \text{or} \quad z \in D_{(\bar{t})} \end{cases}. \tag{2.11}$$

Let

$$\mathcal{U}_{(t,t+1)} = U_{(t+1)} U_{(t)},$$
$$\widetilde{\mathcal{U}}_{(t,t+1)} = U_{(\bar{t})} U_{(t)} U_{(t+1)} U_{(\bar{t})},$$

then we restate the boundary conditions in terms of the doubling field:

$$\partial \mathcal{X}(x_t + e^{2\pi i}(\eta + i0^+)) = \mathcal{U}_{(t,t+1)} \partial \mathcal{X}(x_t + \eta + i0^+), \tag{2.12}$$
$$\partial \mathcal{X}(x_t + e^{2\pi i}(\eta - i0^+)) = \widetilde{\mathcal{U}}_{(t,t+1)} \partial \mathcal{X}(x_t + \eta - i0^+), \tag{2.13}$$

for $0 < \eta < \min(|x_{t-1} - x_t|, |x_t - x_{t+1}|)$ in order to consider only the two intersecting D-branes $D_{(t)}$ and $D_{(t+1)}$. The matrices $\mathcal{U}_{(t,t+1)}$ and $\widetilde{\mathcal{U}}_{(t,t+1)}$ represent the non trivial monodromies, consequence of the rotation of the branes and their boundary conditions. Notice however that they are somewhat special SO(4) matrices and in section 3.3 we give the general parametrization in term of $\mathrm{SU}_L(2) \times \mathrm{SU}_R(2)$ parameters. Given the non Abelian characteristic of the rotations, there are two different monodromies depending on the base point: one for paths starting in the upper plane H and one for paths starting in $\overline{\mathrm{H}}$. As a consequence of the nature of the rotations of the D-branes, a path on the complex plane enclosing all the branes simultaneously does not show any monodromy:

$$\prod_{t=1}^{N_B} \mathcal{U}_{(\bar{t}-t,\bar{t}+1-t)} = \prod_{t=1}^{N_B} \widetilde{\mathcal{U}}_{(\bar{t}+t,\bar{t}+1+t)} = \mathbb{1}_4,$$

where t is, as always, defined modulo N_B. These relations reveal that the complex plane has branch cuts running between the branes, at finite, as shown in Fig. 3. We therefore translated the rotations of the D-branes in terms of $\mathcal{U}_{(t,t+1)}$ and $\widetilde{\mathcal{U}}_{(t,t+1)}$ which are the matrix representation of the homotopy group of the complex plane with the described branch cut structure.

As a consistency check of the procedure, the string action (2.4) can be computed in terms of the new doubling field: the map

$$x_t + \eta \pm i0^+ \mapsto x_t + e^{2\pi i}(\eta \pm i0^+)$$

must leave the action invariant since it does not depend on the branch cut structure in the first place. In fact, it is easy to show that

$$S = \frac{1}{4\pi\alpha'} \iint\limits_{\mathbb{C}} \mathrm{d}z\, \mathrm{d}\overline{z}\, \partial \mathcal{X}^T(z) U_{(\bar{t})} \overline{\partial} \mathcal{X}(\overline{z}),$$

where $\mathrm{d}^2 z = \mathrm{d}z\, \mathrm{d}\overline{z} = 2\, \mathrm{d}x\, \mathrm{d}y$, is left untouched by the map.
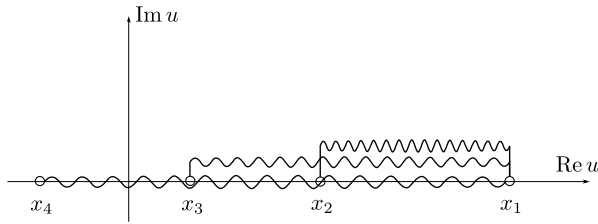
Fig. 3. The appearance of non trivial discontinuities or monodromies on the branes shows that the complex plane has a branch cut structure. The particular nature of the rotations of the D-branes is such that the branch cuts run along the boundary between finite points. Here we show the case of four D-branes, i.e. $N_B = 4$.

3. D-branes at angles in spinor representation

In the previous section we showed how to encode the rotations of the D-branes in matrices representing the non trivial monodromies of the doubling field. In order to find a solution to the equations of motion with the boundary conditions determined by the brane rotations, we should now find an explicit solution $\partial \mathcal{X}(z)$ such the non trivial monodromies in (2.12) and (2.13) can be reproduced.

At first analysis $\partial \mathcal{X}(z)$ is a 4-dimensional *real* vector which has N_B non trivial monodromies factors represented by 4×4 real matrices, one for each interaction point x_t. The solution to the string equations of motion is therefore represented by four linearly independent functions with N_B branch points. We can try to look for them among the solutions to fourth order differential equations with N_B finite Fuchsian points. This is however an open mathematical problem in its general statement: the basis of such functions around each branch point are usually complicated and defined up to several free parameters. Moreover, and more importantly, the connection between any two of these basis is an unsolved mathematical problem. Using contour integrals and representing the functions as Mellin-Barnes integrals it might be possible to solve the issue in the very special case $N_B = 3$ but it is certainly not the best course of action.

On the other hand our main interest is to find a solution precisely for $N_B = 3$. We then use the isomorphism

$$SO(4) \cong \frac{SU(2) \times SU(2)}{\mathbb{Z}_2}$$

in order to restate the problem of finding a 4-dimensional *real* solution to the equations of motion to a quest for a 2×2 complex matrix. This matrix can be seen as a linear superposition of tensor products of two (complex) vectors in the fundamental representation of two different SU(2). We can think of these vectors as a solution to a second order differential equation with three Fuchsian points, possibly the hypergeometric equation. Our task is then to map the original SO(4) monodromies into two sets of SU(2) monodromies and then to find the corresponding parameters of the hypergeometric functions.

3.1. Review of the isomorphism

In order to carry out the computations we consider the isomorphism between SO(4) and two different copies of SU(2). Here we sketch how operatively the isomorphism works in order to fix our notations while in Appendix A we review it in more details.

We first consider a basis

$$\tau = (i\mathbb{1}_2, \sigma)$$

where σ is a vector containing the usual Pauli matrices. We then choose to parameterize any matrix of SU(2) with a 3-dimensional vector

$$\mathbf{n} \in \left\{ \left(n^1, n^2, n^3\right) \in \mathbb{R}^3 \mid 0 \le n \le \frac{1}{2} \quad \text{and} \quad \mathbf{n} \equiv \mathbf{n}' \quad \text{when} \quad n = n' = \frac{1}{2} \right\}, \tag{3.1}$$

such that:

$$U(\mathbf{n}) = \cos(2\pi n)\mathbb{1}_2 + i\frac{\mathbf{n} \cdot \sigma}{n} \sin(2\pi n) \in \mathrm{SU}(2), \tag{3.2}$$

where $n = \|\mathbf{n}\|$ so that the following properties hold:

$$(U(\mathbf{n}))^* = \sigma^2 U(\mathbf{n})\sigma^2 = U(\widetilde{\mathbf{n}}), \tag{3.3}$$

$$(U(\mathbf{n}))^\dagger = (U(\widetilde{\mathbf{n}}))^T = U(-\mathbf{n}), \tag{3.4}$$

$$-U(\mathbf{n}) = U(\widehat{\mathbf{n}}), \tag{3.5}$$

where $\widetilde{\mathbf{n}} = \left(-n^1, +n^2, -n^3\right)$ and $\widehat{\mathbf{n}} = -\left(\frac{1}{2} - n\right)\mathbf{n}/n$.

We then define a new set of coordinates $X_{(s)}$ in this representation:

$$X_{(s)}(u, \overline{u}) = X^I(u, \overline{u})\tau_I, \tag{3.6}$$

where a rotation of $\mathrm{SU}_L(2) \times \mathrm{SU}_R(2)$ acts as[1]

$$X'_{(s)}(u, \overline{u}) = U_L(\mathbf{n})X_{(s)}(u, \overline{u})U_R^\dagger(\mathbf{m})$$

and it is equivalent to a 4-dimensional rotation

$$\left(X'(u, \overline{u})\right)^I = R^I_{\ J}(\mathbf{n}, \mathbf{m})X^J(u, \overline{u}),$$

where

$$R_{IJ}(\mathbf{n}, \mathbf{m}) = \frac{1}{2} \mathrm{tr}(\tau_I^\dagger U_L(\mathbf{n})\tau_J U_R^\dagger(\mathbf{m})) \in \mathrm{SO}(4).$$

We can therefore work directly on a representation of $\mathrm{SU}_L(2) \times \mathrm{SU}_R(2)$ since we have built an isomorphism which maps the two sets of three real parameters of SU(2) matrix (encoded in the vectors $\mathbf{n}$ and $\mathbf{m}$) to a SO(4) matrix which is in fact described by six real parameters. Notice that there is a residual $\mathbb{Z}_2$ symmetry acting as $\{U_L, U_R\} \leftrightarrow \{-U_L, -U_R\}$ so that we can fix the first non vanishing component of $\mathbf{n}$ to be positive. The correct isomorphism is therefore:

$$\mathrm{SO}(4) \cong \frac{\mathrm{SU}_L(2) \times \mathrm{SU}_R(2)}{\mathbb{Z}_2}.$$

---

[1] In the following we write $U_L(\mathbf{n})$ and $U_R(\mathbf{m})$ even if it is not necessary to specify whether the group element is in the left or right SU(2) since the parameters are explicitly given.

3.2. Doubling trick and rotations in spinor representation

In the light of the possibility to use the spinor representation of the rotations to find the solutions to the equations of motion of the classical bosonic string, we need to reproduce (2.11) as two separate SU(2) rotations. Consider then:

$$\partial \mathcal{X}_{(s)}(z) = \begin{cases} \partial X_{(s),\,L}(u) & \text{if} \quad z \in \mathrm{H} \quad \text{or} \quad z \in D_{(\bar{t})} \\ U_L(\mathbf{n}_{(\bar{t})}) \overline{\partial} X_{(s),\,R}(\overline{u}) U_R^\dagger(\mathbf{m}_{(\bar{t})}) & \text{if} \quad z \in \overline{\mathrm{H}} \quad \text{or} \quad z \in D_{(\bar{t})} \end{cases}, \tag{3.7}$$

where $\partial \mathcal{X}_{(s)}(z)$, $\partial X_{(s),\,L}(u)$ and $\overline{\partial} X_{(s),\,R}(\overline{u})$ are 2-dimensional square matrices in the sense of (3.6).

As in the real representation, we read the discontinuities on the branes with respect to the brane $D_{(\bar{t})}$ in terms of monodromies of $\partial \mathcal{X}(z)$, leaving the branch cut structure and the homotopy group considerations unchanged as long as we consider both left and right sectors of $\mathrm{SU}_L(2) \times \mathrm{SU}_R(2)$ at the same time. In particular, let $0 < \eta < \min(|x_t - x_{t-1}|, |x_{t+1} - x_t|)$, then we find:

$$\partial \mathcal{X}_{(s)}(x_t + e^{2\pi i}(\eta + i0^+)) = \mathcal{L}_{(t,t+1)} \partial \mathcal{X}_{(s)}(x_t + \eta + i0^+) \mathcal{R}_{(t,t+1)}^\dagger, \tag{3.8}$$

$$\partial \mathcal{X}_{(s)}(x_t + e^{2\pi i}(\eta - i0^+)) = \widetilde{\mathcal{L}}_{(t,t+1)} \partial \mathcal{X}_{(s)}(x_t + \eta - i0^+) \widetilde{\mathcal{R}}_{(t,t+1)}^\dagger, \tag{3.9}$$

where:

$$\mathcal{L}_{(t,t+1)} = U_L(\mathbf{n}_{(t+1)}) U_L^\dagger(\mathbf{n}_{(t)}),$$
$$\widetilde{\mathcal{L}}_{(t,t+1)} = U_L(\mathbf{n}_{(\bar{t})}) U_L^\dagger(\mathbf{n}_{(t)}) U_L(\mathbf{n}_{(t+1)}) U_L^\dagger(\mathbf{n}_{(\bar{t})}),$$
$$\mathcal{R}_{(t,t+1)} = U_R(\mathbf{m}_{(t+1)}) U_R^\dagger(\mathbf{m}_{(t)}),$$
$$\widetilde{\mathcal{R}}_{(t,t+1)} = U_R(\mathbf{m}_{(\bar{t})}) U_R^\dagger(\mathbf{m}_{(t)}) U_R(\mathbf{m}_{(t+1)}) U_R^\dagger(\mathbf{m}_{(\bar{t})}).$$

In the spinor representation the action (2.4) becomes

$$S = \frac{1}{4\pi\alpha'} \iint\limits_{\mathrm{H}} \mathrm{d}^2 u \, \mathrm{tr}(\partial X_{(s)}(u) \cdot \overline{\partial} X_{(s)}^\dagger(\overline{u}))$$

or, in terms of the doubling fields:

$$S = \frac{1}{8\pi\alpha'} \iint\limits_{\mathbb{C}} \mathrm{d}^2 z \, \mathrm{tr}(U_L(\mathbf{n}_{(\bar{t})}) \partial \mathcal{X}_{(s)}(z) U_R^\dagger(\mathbf{m}_{(\bar{t})}) \overline{\partial} \mathcal{X}_{(s)}^\dagger(\overline{z})). \tag{3.10}$$

Even in this case, the map $x_t + \eta \pm i0^+ \mapsto x_t + e^{2\pi i}(\eta \pm i0^+)$ does not generate additional contributions, leaving the action unchanged.

3.3. Special form of SU(2) matrices for branes at angles

We now show that the SU(2) involved in the branes at angles are of a very special form. For the left SU(2) sector we have:

$$\mathcal{L}_{(t,t+1)} = U_L(\mathbf{n}_{(t,t+1)}) = -\mathbf{v}_{(t+1)} \cdot \mathbf{v}_{(t)} + i(\mathbf{v}_{(t+1)} \times \mathbf{v}_{(t)}) \cdot \sigma,$$

with $\mathbf{v}_{(t)}^2 = 1$, and similarly for the right sector. This follows from the fact that the SO(4) matrix $U_{(t)}$ defined in (2.10) has also special properties and hence the corresponding the $\mathrm{SU}_L(2) \times \mathrm{SU}_R(2)$ element $(U_L(\mathbf{n}_{(t)}), U_L(\mathbf{m}_{(t)}))$ is special. In particular for the left part we have

$$U_L(\mathbf{n}_{(t)}) = i\mathbf{v}_{(t)} \cdot \sigma, \quad \mathbf{v}_{(t)}^2 = 1, \tag{3.11}$$

since $U_{(t)}^2 = \mathbb{1}_4$ implies $U_L^2 = \pm\mathbb{1}_2$ and similarly for the right part. In fact the matrix $\mathcal{S}$ in (2.8) can be represented as $U_L = U_R = i\sigma_1$, then any matrix $U_L(\mathbf{n}_{(t)})$ is of the form $U_L(\mathbf{n}_{(t)}) = U(\mathbf{r}_{(t)}) \cdot (i\sigma_1) \cdot U^\dagger(\mathbf{r}_{(t)})$, for some $\mathbf{r}_{(t)}$ as follows from (2.10). Now this matrix has vanishing trace and squares to $-\mathbb{1}_2$ hence the term proportional to $\mathbb{1}_2$ in the expression of the generic SU(2) element given in (3.2) vanishes and therefore $n_{(t)} = \frac{1}{4}$ so that (3.11) follows.

4. The classical solution

4.1. The choice of hypergeometric functions

As anticipated in the previous section, the spinorial representation entails using SU(2) matrices: it greatly simplifies the search for the basis of functions satisfying the desired boundary conditions. Fixing the $\mathrm{SL}_2(\mathbb{R})$ invariance naturally leads to consider a basis of hypergeometric functions in order to reproduce the monodromy matrices in (3.8) and (3.9). Specifically, since we are interested in a solution with $N_B = 3$, we fix the three intersection points $x_{\bar{i}-1}$, $x_{\bar{i}+1}$ and $x_{\bar{i}}$ to $\omega_{\bar{i}-1} = 0$, $\omega_{\bar{i}+1} = 1$ and $\omega_{\bar{i}} = \infty$ respectively through the map:

$$\omega_u = \frac{u - x_{\bar{i}-1}}{u - x_{\bar{i}}} \cdot \frac{x_{\bar{i}+1} - x_{\bar{i}-1}}{x_{\bar{i}+1} - x_{\bar{i}}} \tag{4.1}$$

The new cut structure for this choice is presented in Fig. 4 and fixes $\arg(\omega_t - \omega_z) \in [0, 2\pi)$ for $t = \bar{i} - 1, \bar{i} + 1$. We then choose for example $\bar{i} = 1$.

The natural choice of the functions to reproduce the given monodromies is a basis of hypergeometric functions. In particular we define:

$$F(a, b; c; z) = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{\Gamma(c+k)} \cdot \frac{z^k}{k!} = \frac{1}{\Gamma(c)} {}_2F_1(a, b; c; z),$$

where ${}_2F_1(a, b; c; z)$ is the usual Gauss hypergeometric function and $\Gamma(s)$ is the Euler Gamma function. With this choice, $F(a, b; c; z)$ is well defined for any value of a, b and c (not simply when c is a strictly positive integer, as in the definition of the Gauss hypergeometric function). We then choose:

$$B_0(z) = \begin{pmatrix} F(a, b; c; z) \\ (-z)^{1-c} F(a+1-c, b+1-c; 2-c; z) \end{pmatrix} \tag{4.2}$$

as a basis of hypergeometric functions around $z = 0$, with a branch cut on the interval $[0, +\infty)$. The choice of the branch cuts follows from the cut on $[1, +\infty)$ coming from $F(a, b; c; z)$ which has a singularity at $z = 1$ and the cut on $[0, +\infty)$ descending from $(-z)^{1-c}$, given the usual cut for z^α on the negative axis.

As we previously observed, the homotopy group of the branch cut plane, a sphere with three marked points, is such that a path enclosing all the singularities is homotopically trivial. Therefore the corresponding product of the monodromy matrices (in the inverse order with respect to the products of paths) is the unit matrix. That is, let $\mathcal{M}_{\omega_t}^\pm$ be the monodromy matrix which represents the homotopy loop around ω_t ($\mathcal{M}^+$ represents a path starting in H and $\mathcal{M}^-$ a path with base point in $\overline{\mathrm{H}}$). Then they satisfy:

$$\mathcal{M}_0^+ \mathcal{M}_1^+ \mathcal{M}_\infty^+ = \mathcal{M}_\infty^- \mathcal{M}_1^- \mathcal{M}_0^- = \mathbb{1}_2 \tag{4.3}$$
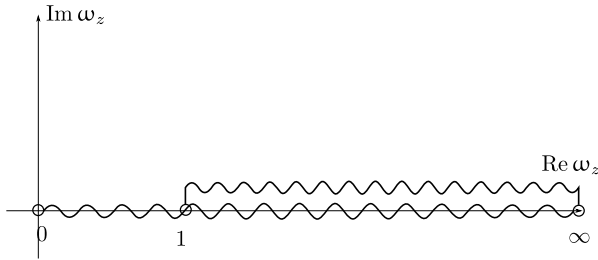
Fig. 4. Fixing the $SL_2(\mathbb{R})$ invariance for $N_B = 3$ and $\bar{i} = 1$ leads to a cut structure with all the cuts defined on the real axis towards $\omega_{\bar{i}} = \infty$.

which shows that we can recover the matrix of the monodromy factors in $\omega_{\bar{i}+1} = 1$ as a product of monodromies around 0 and ∞ given the properties

$$\mathcal{M}_0^+ = \mathcal{M}_0^- = \mathcal{M}_0,$$
$$\mathcal{M}_\infty^+ = \mathcal{M}_\infty^- = \mathcal{M}_\infty.$$

These matrices are an abstract representation of the monodromy group since they are in an arbitrary basis: to have an explicit representation we need to fix an explicit basis.

Using the basis around $z = 0$ in (4.2), it is straightforward to find the explicit representation M_0 of the abstract monodromy $\mathcal{M}_0$:

$$M_0 = \begin{pmatrix} 1 & \\ & e^{-2\pi i c} \end{pmatrix}. \tag{4.4}$$

In order to compute the monodromy matrix M_∞ at ∞ in the basis (4.2), it is best to first compute the explicit monodromy representation $\widetilde{M}_\infty$ of the abstract monodromy $\mathcal{M}_\infty$ in the canonical basis of hypergeometric functions around $z = \infty$:

$$B_\infty(z) = \begin{pmatrix} (-z)^{-a} F(a, a+1-c; a+1-b; z^{-1}) \\ (-z)^{-b} F(b, b+1-c; b+1-a; z^{-1}) \end{pmatrix}.$$

Then we use how this basis is connected by the matrix

$$\mathcal{C} = \frac{\pi}{\sin(\pi(a-b))} \begin{pmatrix} \frac{1}{\Gamma(b)\Gamma(c-a)} & -\frac{1}{\Gamma(a)\Gamma(c-b)} \\ \frac{1}{\Gamma(1-a)\Gamma(b+1-c)} & -\frac{1}{\Gamma(1-b)\Gamma(a+1-c)} \end{pmatrix}, \tag{4.5}$$

to $B_0(z) = \mathcal{C} B_\infty(z)$ in order to compute the explicit monodromy representation M_∞. In fact, performing the loop around the infinity as $z \to z e^{-i2\pi}$, we find

$$\widetilde{M}_\infty = \begin{pmatrix} e^{2\pi i a} & \\ & e^{2\pi i b} \end{pmatrix},$$

and finally

$$M_\infty = \mathcal{C} \widetilde{M}_\infty \mathcal{C}^{-1}. \tag{4.6}$$

4.2. The monodromy factors

The stage being set, our task is now to reproduce the monodromies of the doubling field in spinor representation (3.8) (we do not need to consider (3.9) since they are the same monodromies) by taking tensor products of two basis of hypergeometric functions: the first basis reproduces the monodromies defined as $\mathcal{L}$ and the second one those defined as $\mathcal{R}$.

In principle there can be several combinations of parameters of the hypergeometric function yielding the same monodromies, therefore we consider the full solution[2] to be a linear superposition of all possible contributions:

$$\partial \mathcal{X}(z) = \frac{\partial \omega_z}{\partial z} \sum_{l,r} c_{lr} \partial \mathcal{X}_{l,r}(\omega_z). \tag{4.7}$$

Explicitly we write any possible solution in a factorized form as

$$\partial \mathcal{X}_{l,r}(\omega_z) = (-\omega_z)^{A_{lr}} (1 - \omega_z)^{B_{lr}} \mathcal{B}_{\mathbf{0},l}^{(L)}(\omega_z) \mathcal{B}_{\mathbf{0},r}^{(R)}(\omega_z)^T, \tag{4.8}$$

where l and r label the possible parameters associate with the left and right hypergeometric. We have also introduced the left basis element

$$
\begin{aligned}
\mathcal{B}_{\mathbf{0},l}^{(L)}(\omega_z) &= D_l^{(L)} B_{\mathbf{0},l}^{(L)}(\omega_z) = \\
&= \begin{pmatrix} F(a_l, b_l; c_l; \omega_z) \\ K_l^{(L)}(-z)^{(1-c_l)} F(a_l + 1 - c_l, b_l + 1 - c_l; 2 - c_l; \omega_z) \end{pmatrix}
\end{aligned} \tag{4.9}
$$

where

$$D_l^{(L)} = \begin{pmatrix} 1 & \\ & K_l^{(L)} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \tag{4.10}$$

is a relative normalization of the two components of each basis. The right sector follows in a similar way. These may be different for each solution. Notice that the matrices $D_l^{(L)}$ do not fix an absolute factor which is contained in c_{lr} but only the normalization of one component of the basis with respect to the other.

After the determination of the possible solutions, we need to select the truly independent ones and among them those with a finite action. It will turn out actually to be easier to determine the solutions with finite action and then verify that they are independent.

4.2.1. Fixing the parameters in the most obvious case

We now determine the possible $\partial \mathcal{X}_{l,r}(\omega_z)$ which have the right monodromies. We will do this for the most general SU(2) matrices despite the fact the ones involved in our problem are of a very special form.

In order to reproduce the monodromies, consider the matrices in (4.4) and (4.6). We impose:

$$
\begin{cases}
D^{(L)} M_{\mathbf{0}}^{(L)} \left(D^{(L)}\right)^{-1} = e^{-2\pi i \delta_{\mathbf{0}}^{(L)}} \mathcal{L}(\mathbf{n_0}) \\
D^{(R)} M_{\mathbf{0}}^{(R)} \left(D^{(R)}\right)^{-1} = e^{-2\pi i \delta_{\mathbf{0}}^{(R)}} \mathcal{R}^*(\mathbf{m_0}) = e^{-2\pi i \delta_{\mathbf{0}}^{(R)}} \mathcal{R}(\widetilde{\mathbf{m}}_0) \\
e^{2\pi i (A_{lr} - \delta_{\mathbf{0}}^{(L)} - \delta_{\mathbf{0}}^{(R)})} = 1
\end{cases} \tag{4.11}
$$

---

[2] In the following we use only the spinor representation and for simplicity we write $\partial \mathcal{X}(z)$ instead of $\partial \mathcal{X}_{(s)}(z)$.

$$\begin{cases} D^{(L)} M_\infty^{(L)} \left(D^{(L)} \right)^{-1} = e^{-2\pi i \delta_\infty^{(L)}} \mathcal{L}(\mathbf{n}_\infty) \\ D^{(R)} M_\infty^{(R)} \left(D^{(R)} \right)^{-1} = e^{-2\pi i \delta_\infty^{(R)}} \mathcal{R}^*(\mathbf{m}_\infty) = e^{-2\pi i \delta_\infty^{(R)}} \mathcal{R}(\widetilde{\mathbf{m}}_\infty) \\ e^{2\pi i (A_{lr} + B_{lr} - \delta_\infty^{(L)} - \delta_\infty^{(R)})} = 1 \end{cases} \quad , \tag{4.12}$$

where we defined

$$\begin{aligned} \mathcal{L}(\mathbf{n_0}) &= \mathcal{L}_{(\bar{t}-1,\bar{t})} = U_L(\mathbf{n}_{(\bar{t})}) U_L^\dagger(\mathbf{n}_{(\bar{t}-1)}), \\ \mathcal{L}(\mathbf{n}_\infty) &= \mathcal{L}_{(\bar{t},\bar{t}+1)} = U_L(\mathbf{n}_{(\bar{t}+1)}) U_L^\dagger(\mathbf{n}_{(\bar{t})}), \\ \mathcal{R}(\mathbf{m_0}) &= \mathcal{R}_{(\bar{t}-1,\bar{t})} = U_R(\mathbf{n}_{(\bar{t})}) U_R^\dagger(\mathbf{n}_{(\bar{t}-1)}), \\ \mathcal{R}(\mathbf{m}_\infty) &= \mathcal{R}_{(\bar{t},\bar{t}+1)} = U_R(\mathbf{n}_{(\bar{t}+1)}) U_R^\dagger(\mathbf{n}_{(\bar{t})}). \end{aligned}$$

That is we highlighted the dependence on the parameters of the SU(2) matrices on the ω_t interaction point instead of the branes between which the interaction develops.

Notice that the range of definition of $\delta_{\mathbf{0}}^{(L)}$ is

$$\alpha \le \delta_{\mathbf{0}}^{(L)} \le \alpha + \frac{1}{2},$$

i.e. the width of the range is only $\frac{1}{2}$ and not 1 as one would naively expect since $e^{i4\pi\delta_{\mathbf{0}}^{(L)}}$ is the determinant of the right hand side of the first equation in (4.11). We will choose $\alpha = 0$ for simplicity. The same is true for all the other additional parameters $\delta_{\mathbf{0}}^{(R)}$ and $\delta_\infty^{(L, R)}$.

As we are interested in relative rotations of the branes, we can fix the rotation in $\omega_{\bar{t}-1} = 0$ to be in the maximal torus of $\mathrm{SU}_L(2) \times \mathrm{SU}_R(2)$ without loss of generality. Stated otherwise, since we have two independent groups we can choose the orientation of both the vectors $\mathbf{n_0}$ and $\mathbf{m_0}$. In particular we set:

$$\mathbf{n_0} = (0, 0, n_{\mathbf{0}}^3) \in \mathbb{R}^3 \quad \text{where} \quad 0 < n_{\mathbf{0}}^3 < \frac{1}{2}, \tag{4.13}$$

$$\widetilde{\mathbf{m}}_{\mathbf{0}} = (0, 0, -m_{\mathbf{0}}^3) \in \mathbb{R}^3 \quad \text{where} \quad 0 < m_{\mathbf{0}}^3 < \frac{1}{2}, \tag{4.14}$$

where the case $n_{\mathbf{0}}^3 = 0$ is excluded because we consider a non trivial rotation. We take the parameters of the rotation in $\omega = \infty$ to be the most general

$$\begin{aligned} \mathbf{n}_\infty &= (n_\infty^1, n_\infty^2, n_\infty^3), \\ \widetilde{\mathbf{m}}_\infty &= (-m_\infty^1, m_\infty^2, -m_\infty^3), \end{aligned}$$

even if we can set $n_\infty^2 = 0$ (and also $m_\infty^2 = 0$) because, after fixing $\mathbf{n_0}$, we can still perform $U(1)$ rotations which leave it invariant but mix n_∞^1, n_∞^2. We nevertheless keep the general expression in order to check our computations.

As we show in Appendix B, solving (4.11) and (4.12) links the parameters of the hypergeometric function to the parameter of the rotations, thus reproducing the boundary conditions of the intersecting D-branes through the non trivial monodromies of the basis. We find:

$$\begin{aligned} a_l^{(L)} &= n_{\mathbf{0}} + (-1)^{f^{(L)}} n_{\mathbf{1}} + n_\infty + \mathfrak{a}_l^{(L)} \quad \text{where} \quad \mathfrak{a}_l^{(L)} \in \mathbb{Z}, \\ b_l^{(L)} &= n_{\mathbf{0}} + (-1)^{f^{(L)}} n_{\mathbf{1}} - n_\infty + \mathfrak{b}_l^{(L)} \quad \text{where} \quad \mathfrak{b}_l^{(L)} \in \mathbb{Z}, \\ c_l^{(L)} &= 2n_{\mathbf{0}} + \mathfrak{c}_l^{(L)} \quad \text{where} \quad \mathfrak{c}_l^{(L)} \in \mathbb{Z}, \\ \delta_{\mathbf{0}}^{(L)} &= n_{\mathbf{0}}, \end{aligned}$$

$$\delta_\infty^{(L)} = -n_0 - (-1)^{f^{(L)}} n_1,$$

$$K_l^{(L)} = -\frac{1}{2\pi^2} \Gamma(1 - a_l^{(L)}) \Gamma(1 - b_l^{(L)}) \Gamma(a_l^{(L)} + 1 - c_l^{(L)}) \Gamma(b_l^{(L)} + 1 - c_l^{(L)}) \times$$

$$\times \sin(\pi c_l^{(L)}) \sin(\pi(a_l^{(L)} - b_l^{(L)})) \frac{n_\infty^1 + i\, n_\infty^2}{n_\infty},$$

where $f^{(L)} \in \{0, 1\}$ and we also introduced the norm n_1 of parameters of the rotation around $\omega_{\bar{l}+1} = 1$, that is $\mathbf{n_1}$, which depends on the other parameters through:

$$\cos(2\pi n_1) = \cos(2\pi n_0) \cos(2\pi n_\infty) - \sin(2\pi n_0) \sin(2\pi n_\infty) \frac{n_\infty^3}{n_\infty}. \tag{4.15}$$

This relation follows from (4.3) for the monodromy $M_1^+ = M_0^{-1} M_\infty^{-1}$ and the standard composition rule for the SU(2) parameters given in (A.5). The same relations for the right sector follow under the exchange $(L) \leftrightarrow (R)$ and $\mathbf{n} \leftrightarrow \widetilde{\mathbf{m}}$.

The other parameters A_{lr} and B_{lr} are then a consequence of the previous results and the equations (4.11) and (4.12):

$$A_{lr} = n_0 + m_0 + \mathfrak{A}_{lr},$$

$$B_{lr} = (-1)^{f^{(L)}} n_1 + (-1)^{f^{(R)}} m_1 + \mathfrak{B}_{lr}$$

where $\mathfrak{A}_{lr} \in \mathbb{Z}$ and $\mathfrak{B}_{lr} \in \mathbb{Z}$.

4.2.2. Solutions with different $f^{(L)}$ and $f^{(R)}$ are the same

As we see from the equations above the parameters of the hypergeometric function are still affected by some ambiguities: the choice of $f^{(L)}$ and $f^{(R)}$ seems an arbitrary decision leading to an undefined solution. However we can use the properties of the hypergeometric functions to show that any choice of their values does not affect the final result. Specifically, we could choose to start with certain values but we can recover the others through:

$$P \left\{ \begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix} \right\} = (1-z)^{c-a-b} P \left\{ \begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & c-b & z \\ 1-c & a+b-c & c-a & \end{matrix} \right\},$$

where P is the Papperitz-Riemann symbol for the hypergeometric functions. This way we can assign any of the possible values to $f^{(L)}$ and $f^{(R)}$ and then recover the other identifying:

$$f^{(L)\prime} = 1 + f^{(L)} \quad \mathrm{mod} \quad 2$$

$$\mathfrak{a}_l' = \mathfrak{c}_l - \mathfrak{b}_l,$$

$$\mathfrak{b}_l' = \mathfrak{c}_l - \mathfrak{a}_l,$$

$$\mathfrak{c}_l' = \mathfrak{c}_l,$$

$$\mathfrak{A}_{lr}' = \mathfrak{A}_{lr},$$

$$\mathfrak{B}_{lr}' = \mathfrak{B}_{lr} - \mathfrak{a}_l^{(L)} - \mathfrak{a}_r^{(R)} - \mathfrak{b}_l^{(L)} - \mathfrak{b}_r^{(R)} + \mathfrak{c}_l^{(L)} + \mathfrak{c}_r^{(R)},$$

and similarly the parameters of the right sector. This means that the choice of $f^{(L,R)}$ is simply a convenient relabeling of parameters. In what follows we choose $f^{(L)} = f^{(R)} = 0$ for simplicity.

As previously stated, in order to get a well defined solution we must impose some constraints on the hypergeometric parameters. Specifically we require:

$$c_l^{(L)} \notin \mathbb{Z},$$

$$a_l^{(L)} + b_l^{(L)} \notin \mathbb{Z} + \frac{1}{2}.$$

All these relations link the parameters of the hypergeometric function to the monodromies associated to the boundary conditions of the intersecting D-branes. They are however more general than actually needed: the number of parameters necessary to fix our configuration is 6 (i.e. $n_0^3, n_\infty^1, n_\infty^3$ and $m_0^3, m_\infty^1, m_\infty^3$), since as noticed before we can always fix $n_\infty^2 = m_\infty^2 = 0$. This is a consequence of the fact that all parameters depend on the moduli, exception made for $K^{(L)}$ and $K^{(R)}$ which depend on $n_\infty^1 + i n_\infty^2$ and $m_\infty^1 + i m_\infty^2$. Performing a $SU_L(2)$ and $SU_R(2)$ rotation around the third axis and a shift of the parameters δ_∞, the phases of K can then be made to vanish.

4.2.3. The importance of the normalization factors K

Using the Papperitz-Riemann symbol the solutions found can be symbolically written as

$$(-\omega)^{\mathfrak{A}}(1-\omega)^{\mathfrak{B}} \times$$

$$\times P \left\{ \begin{matrix} 0 & 1 & \infty \\ n_0 & n_1 & n_\infty + \mathfrak{a}^{(L)} & \omega \\ -n_0 + 1 - \mathfrak{c}^{(L)} & -n_1 - \mathfrak{a}^{(L)} - \mathfrak{b}^{(L)} + \mathfrak{c}^{(L)} & -n_\infty + \mathfrak{b}^{(L)} \end{matrix} \right\} \times \tag{4.16}$$

$$\times P \left\{ \begin{matrix} 0 & 1 & \infty \\ m_0 & m_1 & m_\infty + \mathfrak{a}^{(R)} & \omega \\ -m_0 + 1 - \mathfrak{c}^{(R)} & -m_1 - \mathfrak{a}^{(R)} - \mathfrak{b}^{(R)} + \mathfrak{c}^{(R)} & -m_\infty + \mathfrak{b}^{(R)} \end{matrix} \right\}.$$

This is exactly what one would have expected but the parameters K cannot be guessed from the P symbol. They nevertheless play a very important role for the consistency of the solution.

Hypergeometric functions can be connected by relations between contiguous functions. It is indeed possible to show that any hypergeometric function $F(a + \mathfrak{a}, b + \mathfrak{b}; c + \mathfrak{c}; z)$ can be written as a combination of $F(a, b; c; z)$ and any of its contiguous functions [49]. For example we could consider:

$$F(a + \mathfrak{a}, b + \mathfrak{b}; c + \mathfrak{c}; z) = h_1(a, b, c; z) F(a + 1, b; c; z) + h_2(a, b, c; z) F(a, b; c; z), \tag{4.17}$$

where $h_1(a, b, c; z)$ and $h_2(a, b, c; z)$ are in $C[\frac{1}{1-z}, \frac{1}{z}, z]$, i.e. they are finite sums of integer (both positive and negative) powers of z and negative powers of $1 - z$. For simplicity let:

$$F = F(a, b; c; z),$$

$$F(a + k) = F(a + k, b; c; z),$$

$$F(b + k) = F(a, b + k; c; z),$$

$$\dots$$

Similarly we use the shorthand notation[3] for the vector:

---

[3] Here we are a sloppy in writing $K_{a,b,c}$ since it depends on a phase which is not a function of a, b, c. See (4.15) and (B.16).

$$\mathcal{B}_\mathbf{0}(a, b, c; z) = \begin{pmatrix} F(a, b; c; z) \\ K_{a,b,c}(-z)^{(1-c)} F(a + 1 - c, b + 1 - c; 2 - c; z) \end{pmatrix}. \tag{4.18}$$

Then we can algorithmically use relations such as

$$(c - a)F(a - 1) + (2a - c + (b - a)z)F - a(1 - z)F(a + 1) = 0.$$
$$(b - a)F + aF(a + 1) - bF(b + 1) = 0,$$
$$(c - a - b)F + a(1 - z)F(a + 1) - (c - b)F(b - 1) = 0, \tag{4.19}$$
$$(a + (b - c)z)F - a(1 - z)F(a + 1) + (c - a)(c - b)zF(c + 1) = 0,$$
$$(c - a - 1)F + aF(a + 1) - F(c - 1) = 0,$$

in order to eliminate unwanted integer factors from each parameter and to keep only F and any of its contiguous functions.

Now $\mathcal{B}_\mathbf{0}$, considered as whole and made of two independent hypergeometric functions, is a basis element for the possible solutions of the classical and quantum string e.o.m. Using any relation in (4.19) we can change a, b or c by ± 1 in a coherent way in both hypergeometric functions, then the result must still be a linear combination of solutions $\mathcal{B}_\mathbf{0}$. For example from the first equation in (4.19) we expect:

$$(c - a)\mathcal{B}_\mathbf{0}(a - 1) + (2a - c + (b - a)z)\mathcal{B}_\mathbf{0} - a(1 - z)\mathcal{B}_\mathbf{0}(a + 1) = 0, \tag{4.20}$$

which can be used to lower and rise a. This relation holds only because of the presence of K. In fact the coefficients in this equation are exactly equal to those in the relation for F for the first component of $\mathcal{B}_\mathbf{0}$ but this is a non trivial fact for the second component where the factor K plays a fundamental role.

In a similar way, even if more complicated to prove, the relation which is needed to lower c which reads:

$$(a - c)(b - c)\mathcal{B}_\mathbf{0}(c + 1) + (a + (b - c)z)\mathcal{B}_\mathbf{0} - a(1 - z)\mathcal{B}_\mathbf{0}(a + 1) = 0. \tag{4.21}$$

4.3. Constraints from the finite euclidean action

In the previous section we found all the most obvious possible solutions to the classical string e.o.m. However we should look for a solution with finite action, thus restricting our attention to such property.

In principle it would be obvious to use (4.19) to restrict the possible arbitrary integers entering the solution to

$$\mathfrak{a}^{(L)} \in \{-1, 0\},$$
$$\mathfrak{b}^{(L)} = 0,$$
$$\mathfrak{c}^{(L)} = 0,$$

and analogously for the right counterparts and then to use (4.17) to write the possible solution as

$$\partial \mathcal{X}(z) = \frac{\partial \omega_z}{\partial z}(-\omega_z)^{n_0 + m_0}(1 - \omega_z)^{n_1 + m_1} \times$$
$$\times \sum_{\mathfrak{a}^{(L,R)} \in \{-1, 0\}} h(\omega_z, \mathfrak{a}^{(L,R)}) \times \tag{4.22}$$
$$\times \mathcal{B}_\mathbf{0}^{(L)}(a^{(L)} + \mathfrak{a}^{(L)}, b, c; \omega_z) \left(\mathcal{B}_\mathbf{0}^{(R)}(a^{(R)} + \mathfrak{a}^{(R)}, b, c; \omega_z)\right)^T.$$

We should finally find an explicit form for $h(\omega_z, \mathfrak{a}^{(L,R)})$ which yield a finite action.

It turns however out to be by far simpler to use the symbolic solution (4.16) to find the possible basis elements with finite action. Actually finding the possible solutions with finite action can be recast to the issue of finding finite solution, i.e. such that the field $\partial \mathcal{X}(z)$ is finite by itself. The latter formulation is by far simpler than the former since it is linear while the former is quadratic. From (3.10) it is clear that the action can be expressed as the sum of the product of any possible couple of elements of the over-complete expansion of the solution (4.7). Therefore we must exam all the possible behaviors of any couple $\partial \mathcal{X}_{l_1 r_1}(z) \overline{\partial} \mathcal{X}_{l_2 r_2}(\overline{z})$. In proximity of any singular point the behavior of any element of solution (4.7) can be easily read from its symbolic representation given by (4.16) and it is of the form:

$$\partial \mathcal{X}(z) \sim \omega_t^{C_t} \begin{pmatrix} \omega_t^{k_{t1}} \\ \omega_t^{k_{t2}} \end{pmatrix} \begin{pmatrix} \omega_t^{h_{t1}} & \omega_t^{h_{t2}} \end{pmatrix} \quad \text{for} \quad \omega_z \to \omega_t.$$

It is then easy to verify that imposing the convergence of the action both at finite and infinite intersection points yields the same constraints as imposing the convergence at any point of the classical solution (in spinor representation as follows from (3.7))

$$X_{(s)}(u, \overline{u}) = f_{(\overline{t}-1)\,(s)} + \int_{x_{\overline{t}-1}}^{u} du' \partial \mathcal{X}(u') + U_L^{\dagger}(\mathbf{n}_{(\overline{t})}) \int_{x_{\overline{t}-1}}^{\overline{u}} d\overline{u}' \partial \mathcal{X}(\overline{u}') \, U_R(\mathbf{m}_{(\overline{t})}), \tag{4.23}$$

where $f_{\overline{t}-1\,(s)} = f_{\overline{t}-1}^I \tau_I$. They are:

$$\begin{aligned} C_t + k_{ti} + h_{tj} > -1, \quad i, j \in \{1, 2\}, \ \omega_t \in \{0, 1\}, \\ C_t + k_{ti} + h_{tj} < -1, \quad i, j \in \{1, 2\}, \ \omega_t = \infty. \end{aligned} \tag{4.24}$$

To explain the approach in the easiest setup let us consider the case where the right rotation is trivial. In this case (4.16) becomes

$$(-\omega)^{\mathfrak{A}} (1 - \omega)^{\mathfrak{B}} \times$$

$$\times \mathrm{P} \left\{ \begin{matrix} 0 & 1 & \infty & \\ n_{\mathbf{0}} & n_{\mathbf{1}} & n_{\infty} + \mathfrak{a}^{(L)} & \omega \\ -n_{\mathbf{0}} + 1 - \mathfrak{c}^{(L)} & -n_{\mathbf{1}} - \mathfrak{a}^{(L)} - \mathfrak{b}^{(L)} + \mathfrak{c}^{(L)} & -n_{\infty} + \mathfrak{b}^{(L)} \end{matrix} \right\}. \tag{4.25}$$

Then it is easy to see that the only possible solution compatible with (4.24) is

$$\mathrm{P} \left\{ \begin{matrix} 0 & 1 & \infty & \\ n_{\mathbf{0}} - 1 & n_{\mathbf{1}} - 1 & n_{\infty} + 1 & \omega \\ -n_{\mathbf{0}} & -n_{\mathbf{1}} & -n_{\infty} + 2 \end{matrix} \right\}, \tag{4.26}$$

i.e. $\mathfrak{a}^{(L)} = -1$, $\mathfrak{b}^{(L)} = 0$, $\mathfrak{c}^{(L)} = 0$, $\mathfrak{A} = -1$ and $\mathfrak{B} = -1$. In the general case the situation is more complicated one could think that taking the product (4.26) and the corresponding solution for the right sector would yield the answer. Unfortunately it is not the case since for $\omega = 0$ we get $C_0 + h_{01} + k_{01} = n_{\mathbf{0}} + m_{\mathbf{0}} - 2 < -1$. To find the solution however we start from such product and we try to move integer factors between indices and between the left and right solutions. For each possible case the solution is unique and it is given by

1. $n_0 > m_0$ and $n_1 > m_1$

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ n_0 - 1 & n_1 - 1 & n_\infty + 1 & \omega \\ -n_0 & -n_1 & -n_\infty + 2 & \end{matrix}\right\} P\left\{\begin{matrix} 0 & 1 & \infty & \\ m_0 & m_1 & m_\infty & \omega \\ -m_0 + 1 & -m_1 & -m_\infty + 1 & \end{matrix}\right\}.$$

(4.27)

2. $n_0 > m_0$, $n_1 < m_1$ and $n_\infty > m_\infty$

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ n_0 - 1 & n_1 & n_\infty & \omega \\ -n_0 & -n_1 & -n_\infty + 2 & \end{matrix}\right\} P\left\{\begin{matrix} 0 & 1 & \infty & \\ m_0 & m_1 - 1 & m_\infty + 1 & \omega \\ -m_0 & -m_1 & -m_\infty + 1 & \end{matrix}\right\}.$$

(4.28)

3. $n_0 > m_0$, $n_1 < m_1$ and $n_\infty < m_\infty$

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ n_0 - 1 & n_1 & n_\infty + 1 & \omega \\ -n_0 & -n_1 & -n_\infty + 1 & \end{matrix}\right\} P\left\{\begin{matrix} 0 & 1 & \infty & \\ m_0 & m_1 - 1 & m_\infty & \omega \\ -m_0 & -m_1 & -m_\infty + 2 & \end{matrix}\right\}.$$

(4.29)

4. $n_0 < m_0$, $n_1 > m_1$ and $n_\infty > m_\infty$

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ n_0 & n_1 - 1 & n_\infty & \omega \\ -n_0 & -n_1 & -n_\infty + 2 & \end{matrix}\right\} P\left\{\begin{matrix} 0 & 1 & \infty & \\ m_0 - 1 & m_1 & m_\infty + 1 & \omega \\ -m_0 & -m_1 & -m_\infty + 1 & \end{matrix}\right\}.$$

(4.30)

5. $n_0 < m_0$, $n_1 > m_1$ and $n_\infty < m_\infty$

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ n_0 & n_1 - 1 & n_\infty + 1 & \omega \\ -n_0 & -n_1 & -n_\infty + 1 & \end{matrix}\right\} P\left\{\begin{matrix} 0 & 1 & \infty & \\ m_0 - 1 & m_1 & m_\infty & \omega \\ -m_0 & -m_1 & -m_\infty + 2 & \end{matrix}\right\}.$$

(4.31)

6. $n_0 < m_0$, $n_1 < m_1$

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ n_0 & n_1 & n_\infty & \omega \\ -n_0 & -n_1 & -n_\infty + 1 & \end{matrix}\right\} P\left\{\begin{matrix} 0 & 1 & \infty & \\ m_0 - 1 & m_1 - 1 & m_\infty + 1 & \omega \\ -m_0 & -m_1 & -m_\infty + 2 & \end{matrix}\right\}.$$

(4.32)

We can summarize the parameters which follows from the previous list and enter the solution in Table 1, where the obvious symmetry in the exchange of n and m becomes manifest.

4.4. The basis of solutions

In the previous section we have produced one solution for each possible ordering of the n_t with respect to m_t. This seems the end of the story but there are actually other solutions and they are connected to the $\mathbb{Z}_2$ in the isomorphism between SO(4) and SU(2) × SU(2)/$\mathbb{Z}_2$. Given any solution which is fixed by $(n_0, n_1, n_\infty) \oplus (m_0, m_1, m_\infty)$ we can replace any couple of $\mathbf{n}$ and $\mathbf{m}$ by $\widehat{\mathbf{n}}$ and $\widehat{\mathbf{m}}$ and produce an apparently new solution. For example we could consider $(\widehat{n}_0, \widehat{n}_1, n_\infty) \oplus (m_0, \widehat{m}_1, \widehat{m}_\infty)$. The necessity of changing a couple is because the monodromies are constrained by (4.3). On the other hand the previous substitution would change the SO(4) in both $\omega = 0$ and $\omega = \infty$: it does not represent a new solution. We are left therefore with three possibilities besides the original one:

Table 1

Integer shifts entering the hypergeometric parameters.

			$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{a}^{(L)}$	$\mathfrak{b}^{(L)}$	$\mathfrak{c}^{(L)}$	$\mathfrak{a}^{(R)}$	$\mathfrak{b}^{(R)}$	$\mathfrak{c}^{(R)}$
$n_0 > m_0$	$n_1 > m_1$	$n_\infty \lessgtr m_\infty$	-1	-1	-1	0	0	0	$+1$	$+1$
$n_0 > m_0$	$n_1 < m_1$	$n_\infty > m_\infty$	-1	-1	-1	$+1$	0	0	0	$+1$
$n_0 > m_0$	$n_1 < m_1$	$n_\infty < m_\infty$	-1	-1	0	0	0	-1	$+1$	$+1$
$n_0 < m_0$	$n_1 > m_1$	$n_\infty > m_\infty$	-1	-1	-1	$+1$	$+1$	0	0	0
$n_0 < m_0$	$n_1 > m_1$	$n_\infty < m_\infty$	-1	-1	0	0	$+1$	-1	$+1$	0
$n_0 < m_0$	$n_1 < m_1$	$n_\infty \lessgtr m_\infty$	-1	-1	0	$+1$	$+1$	-1	0	0

$$(\widehat{\mathbf{n}}_0, \widehat{\mathbf{n}}_1, \mathbf{n}_\infty) \oplus (\widehat{\mathbf{m}}_0, \widehat{\mathbf{m}}_1, \mathbf{m}_\infty),$$

$$(\widehat{\mathbf{n}}_0, \mathbf{n}_1, \widehat{\mathbf{n}}_\infty) \oplus (\mathbf{m}_0, \widehat{\mathbf{m}}_1, \widehat{\mathbf{m}}_\infty),$$

$$(\widehat{\mathbf{n}}_0, \mathbf{n}_1, \widehat{\mathbf{n}}_\infty) \oplus (\mathbf{m}_0, \widehat{\mathbf{m}}_1, \widehat{\mathbf{m}}_\infty). \tag{4.33}$$

Finally we want to gauge fix the $\mathbb{Z}_2$ by letting $\mathbf{n}_0^3, \mathbf{m}_0^3 > 0$ as required by (4.13) and (4.14). This eliminates the first two possibilities. We are therefore left with two possible solutions

$$(\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_\infty) \oplus (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_\infty),$$

$$(\mathbf{n}_0, \widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_\infty) \oplus (\mathbf{m}_0, \widehat{\mathbf{m}}_1, \widehat{\mathbf{m}}_\infty), \tag{4.34}$$

that is the original one and one which is obtained by acting with a parity-like operator P_2 on the rotation parameters at $\omega = 1, \infty$ on both left and right sector at the same time. In order to accept it as a further possible solution, we should now verify its independence with respect to the first one.

Actually looking to Table 1 we see that there are only two different cases up to left-right symmetry. The first case is

$$\left\{ (n_0 > m_0, n_1 > m_1, n_\infty > m_\infty) \,,\, (n_0 > m_0, \hat{n}_1 < \hat{m}_1, \hat{n}_\infty < \hat{m}_\infty) \right\}, \tag{4.35}$$

which is mapped to

$$\left\{ (n_0 < m_0, n_1 < m_1, n_\infty < m_\infty) \,,\, (n_0 < m_0, \hat{n}_1 > \hat{m}_1, \hat{n}_\infty > \hat{m}_\infty) \right\}, \tag{4.36}$$

by the left-right symmetry. The second one is

$$\left\{ (n_0 > m_0, n_1 > m_1, n_\infty < m_\infty) \,,\, (n_0 > m_0, \hat{n}_1 < \hat{m}_1, \hat{n}_\infty > \hat{m}_\infty) \right\}, \tag{4.37}$$

which is mapped to

$$\left\{ (n_0 < m_0, n_1 < m_1, n_\infty > m_\infty) \,,\, (n_0 < m_0, \hat{n}_1 > \hat{m}_1, \hat{n}_\infty < \hat{m}_\infty) \right\}, \tag{4.38}$$

by the left-right symmetry.

Let us now exam the two solutions in the two cases. We first perform a generic computation which is common to the two cases and then we explicitly specialize it. Computing the hypergeometric parameters for the first solution leads to:

$$\begin{cases} a^{(L)} = n_0 + n_1 + n_\infty + \mathfrak{a}^{(L)} \\ b^{(L)} = n_0 + n_1 - n_\infty + \mathfrak{b}^{(L)} \\ c^{(L)} = 2n_0 + \mathfrak{c}^{(L)} \end{cases} , \qquad \begin{cases} a^{(R)} = m_0 + m_1 + m_\infty + \mathfrak{a}^{(R)} \\ b^{(R)} = m_0 + m_1 - m_\infty + \mathfrak{b}^{(R)} \\ c^{(R)} = 2m_0 + 1 + \mathfrak{c}^{(R)} \end{cases} , \tag{4.39}$$

where the values of the constants can be read from Table 1. Then we compute the $K^{(L)}$ and $K^{(R)}$ factors using (4.15). Therefore the first solution is:

$$\partial_\omega \chi_1 = (-\omega)^{n_0 + m_0 - 1}(1 - \omega)^{n_1 + m_1 - 1} \times$$

$$\times \begin{pmatrix} F(a^{(L)}, b^{(L)}; c^{(L)}; \omega) \\ K^{(L)}(-\omega)^{1 - c^{(L)}} F(a^{(L)} + 1 - c^{(L)}, b^{(L)} + 1 - c^{(L)}; 2 - c^{(L)}; \omega) \end{pmatrix}$$

$$\times \begin{pmatrix} F(a^{(R)}, b^{(R)}; c^{(R)}; \omega) \\ K^{(R)}(-\omega)^{1 - c^{(R)}} F(a^{(R)} + 1 - c^{(R)}, b^{(R)} + 1 - c^{(R)}; 2 - c^{(R)}; \omega) \end{pmatrix}^T . \quad (4.40)$$

The parameters of the second solution read

$$\begin{cases} \hat{a}^{(L)} = n_0 + \hat{n}_1 + \hat{n}_\infty + \hat{\mathfrak{a}}^{(L)} &= c^{(L)} - a^{(L)} + \mathfrak{a}^{(L)} - \mathfrak{c}^{(L)} + \hat{\mathfrak{a}}^{(L)} + 1 \\ \hat{b}^{(L)} = n_0 + \hat{n}_1 - \hat{n}_\infty + \hat{\mathfrak{b}}^{(L)} &= c^{(L)} - b^{(L)} + \mathfrak{b}^{(L)} - \mathfrak{c}^{(L)} + \hat{\mathfrak{b}}^{(L)} \\ \hat{c}^{(L)} = 2n_0 + \hat{\mathfrak{c}}^{(L)} &= c^{(L)} - \mathfrak{c}^{(L)} + \hat{\mathfrak{c}}^{(L)} \end{cases} ,$$

$$\begin{cases} \hat{a}^{(R)} = m_0 + \hat{m}_1 + \hat{m}_\infty + \hat{\mathfrak{a}}^{(R)} &= c^{(R)} - a^{(R)} + \mathfrak{a}^{(R)} - \mathfrak{c}^{(R)} + \hat{\mathfrak{a}}^{(R)} + 1 \\ \hat{b}^{(R)} = m_0 + \hat{m}_1 - \hat{m}_\infty + \hat{\mathfrak{b}}^{(R)} &= c^{(R)} - b^{(R)} + \mathfrak{b}^{(R)} - \mathfrak{c}^{(R)} + \hat{\mathfrak{b}}^{(R)} \\ \hat{c}^{(R)} = 2m_0 + \hat{\mathfrak{c}}^{(R)} &= c^{(R)} - \mathfrak{c}^{(R)} + \hat{\mathfrak{c}}^{(R)} \end{cases} . \quad (4.41)$$

We see that the two cases differ only for the constants and not for the structure.

4.4.1. Case 1

We start with the case $n_0 > m_0$, $n_1 > m_1$ and $n_\infty > m_\infty$ for which the second solution is $n_0 > m_0$, $\hat{n}_1 < \hat{m}_1$ and $\hat{n}_\infty < \hat{m}_\infty$ The parameters for the second are explicitly

$$\begin{cases} \hat{a}^{(L)} = c^{(L)} - a^{(L)} \\ \hat{b}^{(L)} = c^{(L)} - b^{(L)} \\ \hat{c}^{(L)} = c^{(L)} \end{cases} , \qquad \begin{cases} \hat{a}^{(R)} = c^{(R)} - a^{(R)} \\ \hat{b}^{(R)} = c^{(R)} - b^{(R)} + 1 \\ \hat{c}^{(R)} = c^{(R)} + 1 \end{cases} . \quad (4.42)$$

The K factors are

$$\hat{K}^{(L)} = K^{(L)}, \qquad \hat{K}^{(R)} = \frac{K^{(R)}}{a^{(R)}(c^{(R)} - b^{(R)})}. \quad (4.43)$$

Using Euler relation

$$F(a, b; c; \omega) = (1 - \omega)^{c - a - b} F(c - a, c - b; c; \omega), \quad (4.44)$$

we can finally write the second solution as

$$\partial_\omega \chi_2 = (-\omega)^{n_0 + m_0 - 1}(1 - \omega)^{n_1 + m_1} \times$$

$$\times \begin{pmatrix} F(a^{(L)}, b^{(L)}; c^{(L)}; \omega) \\ K^{(L)}(-\omega)^{1 - c^{(L)}} F(a^{(L)} + 1 - c^{(L)}, b^{(L)} + 1 - c^{(L)}; 2 - c^{(L)}; \omega) \end{pmatrix}$$

$$\times \begin{pmatrix} F(a^{(R)} + 1, b^{(R)}; c^{(R)} + 1; \omega) \\ \hat{K}^{(R)}(-\omega)^{-c^{(R)}} F(a^{(R)} + 1 - c^{(R)}, b^{(R)} - c^{(R)}; 1 - c^{(R)}; \omega) \end{pmatrix}^T , \quad (4.45)$$

in which the left basis is exactly equal to the first solution while the right basis differs for $a^{(R)} \to a^{(R)} + 1$ and $c^{(R)} \to c^{(R)} + 1$.

4.4.2. Case 2

Consider now the second case $n_0 > m_0$, $n_1 > m_1$ and $n_\infty < m_\infty$. For the second solution we have $n_0 > m_0$, $\hat{n}_1 < \hat{m}_1$ and $\hat{n}_\infty > \hat{m}_\infty$ and the parameters are explicitly

$$
\begin{cases}
\hat{a}^{(L)} = c^{(L)} - a^{(L)} - 1 \\
\hat{b}^{(L)} = c^{(L)} - b^{(L)} + 1 \\
\hat{c}^{(L)} = c^{(L)}
\end{cases} ,
\quad
\begin{cases}
\hat{a}^{(R)} = c^{(R)} - a^{(R)} \\
\hat{b}^{(R)} = c^{(R)} - b^{(R)} \\
\hat{c}^{(R)} = c^{(R)}
\end{cases} .
\tag{4.46}
$$

The K factors are

$$
\hat{K}^{(L)} = K^{(L)} \frac{(b^{(L)} - 1)(c^{(L)} - a^{(L)} - 1)}{a^{(L)}(c^{(L)} - b^{(L)})}, \quad \hat{K}^{(R)} = K^{(R)}.
\tag{4.47}
$$

Using Euler relation we can finally write the second solution for the second case as

$$
\partial_\omega \chi_2 = (-\omega)^{n_0 + m_0 - 1} (1 - \omega)^{n_1 + m_1} \times
$$
$$
\times \begin{pmatrix} F(a^{(L)} + 1, b^{(L)} - 1; c^{(L)}; \omega) \\ \hat{K}^{(L)}(-\omega)^{1-c^{(L)}} F(a^{(L)} + 2 - c^{(L)}, b^{(L)} - c^{(L)}; 2 - c^{(L)}; \omega) \end{pmatrix}
$$
$$
\times \begin{pmatrix} F(a^{(R)}, b^{(R)}; c^{(R)}; \omega) \\ K^{(R)}(-\omega)^{1-c^{(R)}} F(a^{(R)} + 1 - c^{(R)}, b^{(R)} + 1 - c^{(R)}; 2 - c^{(R)}; \omega) \end{pmatrix}^T ,
\tag{4.48}
$$

in which the right basis is exactly equal to the first solution while the left basis differs for $a^{(L)} \to a^{(L)} + 1$ and $b^{(L)} \to b^{(L)} - 1$.

4.5. The solution

In the previous section we have shown that there are two independent solutions, therefore the general solution for $\partial_\omega \chi$ obviously reads

$$
\partial_\omega \chi = C_1 \partial_\omega \chi_1 + C_2 \partial_\omega \chi_2.
\tag{4.49}
$$

Therefore the final solution depends now only on two complex constants, C_1 and C_2 which we can fix imposing the global conditions in (2.9), i.e. the second equation for all t's in the solution (4.23). Since the three target space intersection points always define a triangle on a 2-dimensional plane, we can impose the boundary conditions knowing two angles formed by the sides (i.e. the branes between two intersections) and the length of one of them. We already fixed the parameters of the rotations, then we need to compute the length of one of the sides, and consider, for instance, the length of the side $X(x_{\bar{t}+1}, x_{\bar{t}+1}) - X(x_{\bar{t}-1}, x_{\bar{t}-1})$: Explicitly we impose the four real equations in spinorial formalism

$$
\int_0^1 d\omega \partial_\omega \mathcal{X}(\omega) + U_L^\dagger(\mathbf{n}_{(\bar{t})}) \int_0^1 d\bar{\omega} \partial_{\bar{\omega}} \mathcal{X}(\bar{\omega}) \, U_R(\mathbf{m}_{(\bar{t})}) = f_{(\bar{t}+1)(s)} - f_{(\bar{t}-1)(s)},
\tag{4.50}
$$

where we have used the mapping (4.1) to write the integrals directly in ω variables. This equation has then enough degrees of freedom to fix completely the two complex parameters C_1 and C_2, thus completing the determination of the full solution in its general form.

5. Recovering the SU(2) and the Abelian solution

Before analyzing further the result, we first show how this general procedure automatically includes the solution with both pure SU(2) and Abelian rotations of the D-branes. The Abelian solution emerges from the general construction as a limit and replicates the known result for Abelian SO(2) × SO(2) ⊂ SO(4) rotations in the case of a factorized space $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$.

5.1. Abelian Limit of the SU(2) monodromies

We want now to compute the parameter $\mathbf{n_1}$ when we are given two Abelian rotation in $\omega = 0$ and $\omega = \infty$ using the standard expression for two SU(2) element multiplication given in (A.5). We can summarize the results in Table 2. Notice that under the parity P_2 the previous four cases are grouped into two sets $\{n_1 = n_0 + n_\infty, \hat{n}_1 = -n_0 + \hat{n}_\infty\}$ and $\{n_1 = 1 - (n_0 + n_\infty), \hat{n}_1 = +n_0 - \hat{n}_\infty\}$. This can be also seen geometrically since the first group corresponds to the same geometry which is depicted in Fig. 5 while the second in Fig. 6. Arbitrarily fixing the orientation of $D_{(3)}$ we can in fact obtain these geometrical interpretations and since $n_0^3 > 0$ we can fix the orientation of $D_{(1)}$. The orientation of $D_{(2)}$ is then fixed relatively to $D_{(1)}$ by the sign of n_∞^3. The sign of n_1^3 then follows.

The usual Abelian convention is more geometrical and visual therefore it does not distinguish between the possible orientations of the branes while this group approach does. In fact comparing all possible brane orientations and the ensuing group parameter n^3 with the usual angles used in the Abelian configuration depicted in Fig. 7 we see that relation between the usual Abelian parameter $\epsilon_{(t)}$ and the group one n_t^3 is given by

$$\varepsilon_{\mathbf{t}} = n_{\mathbf{t}}^3 + \theta(-n_{\mathbf{t}}^3), \tag{5.1}$$

when all $m = 0$.

5.2. Abelian limit of the left solutions

Then we can compute the basis element for any entry of the Table 1 for any possible value of n_1 as given in Table 2. Here we consider for simplicity the left sector of the solution: everything can be stated in the same way for the right sector.

It turns out that either $K = 0$ or $K = \infty$. In the latter case we can absorb the infinite divergence in a constant term in front of the solution and effectively use:

$$D\mid_{K=0} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \tag{5.2}$$

$$D\mid_{K=\infty} = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}. \tag{5.3}$$

The result is then given in Table 3. In this table we have left some hypergeometrics in their symbolic form. However all of them are elementary functions since either a or $c - b$ is equal to -1.

5.3. The $SU_L(2)$ limit

We can recover the previously computed non Abelian SU(2) solution by considering $m_{\mathbf{t}} \sim 0$: this is case 1 of section 4.4. The first thing we notice is that the left solution $\mathcal{B}^{(L)}$ is always the

Table 2
Abelian limit of SU(2) monodromies.

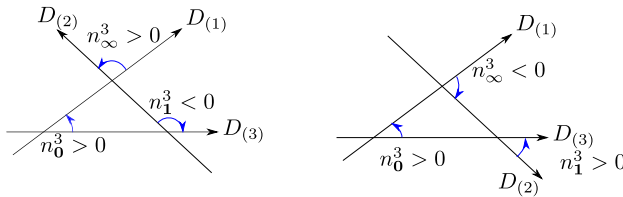| n_0	n_∞			n_1	n_1	$\sum_t n_{(t)}$
$n_0 k$	$n_\infty k$	$n_0 + n_\infty < \frac{1}{2}$	$n_0 \lesseqgtr n_\infty$	$n_0 + n_\infty$	$-n_1 k$	0
$n_0 k$	$n_\infty k$	$n_0 + n_\infty > \frac{1}{2}$	$n_0 \lesseqgtr n_\infty$	$1 - (n_0 + n_\infty)$	$+n_1 k$	k
$n_0 k$	$-n_\infty k$	$n_0 + n_\infty \lesseqgtr \frac{1}{2}$	$n_0 > n_\infty$	$n_0 - n_\infty$	$-n_1 k$	0
$n_0 k$	$-n_\infty k$	$n_0 + n_\infty \lesseqgtr \frac{1}{2}$	$n_0 < n_\infty$	$-n_0 + n_\infty$	$+n_1 k$	0

Fig. 5. The Abelian limit when the triangle has all acute angles. This corresponds to the cases $n_0 + n_\infty < \frac{1}{2}$ and $n_0 < n_\infty$ which are exchanged under the parity P_2. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)
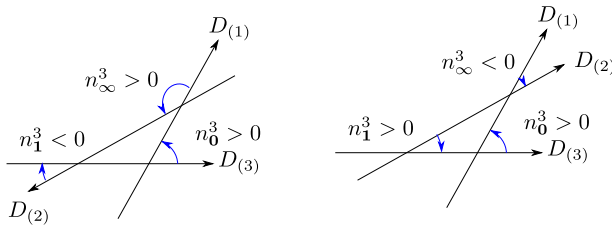

Fig. 6. The Abelian limit when the triangle has one obtuse angle. This corresponds to the cases $n_0 + n_\infty > \frac{1}{2}$ and $n_0 > n_\infty$ which are exchanged under the parity P_2.
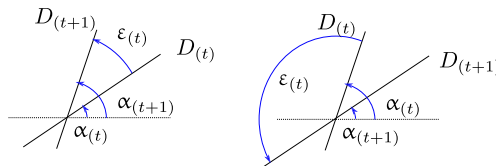

Fig. 7. The geometrical angles used in the usual geometrical approach to the Abelian configuration do not distinguish among the possible branes orientations. In fact we have $0 \leq \alpha < 1$ and $0 < \varepsilon < 1$.

same and matches the previous computation. Despite so, the right sector seems to give different solutions when different Abelian limits are taken. Actually, examining all the possible solutions,[4] we get that all of them give the same answer in the limit $m_t \to 0$, i.e. both $\mathcal{B}^{(R)} = (1, 0)^T$ and $\mathcal{B}^{(R)} = (0, 1)^T$. The only difference is which solution is obtained from the case $n_0 > m_0, n_1 > m_1$

---

[4] We write possible because the $m_1 = 1 - (m_0 + m_\infty)$ case is not.

Table 3
Abelian limit of the solutions.

$(\mathfrak{a}^{(L)}, \mathfrak{b}^{(L)}, \mathfrak{c}^{(L)})$	n_1	$\mathcal{B}^{(L)\,T}(z)$
$(-1, 0, 0)$	$n_0 + n_\infty$	$\left((1-z)^{-2n_\infty - 2n_0 + 1}, 0 \right)$
	$1 - (n_0 + n_\infty)$	$(1, 0)$
	$n_0 - n_\infty$	$\left(0, (-z)^{1-2n_0} \right)$
	$-n_0 + n_\infty$	$(1, 0)$
$(-1, 1, 0)$	$n_0 + n_\infty$	$\left(F\left(2n_\infty + 2n_0 - 1, 2n_0 + 1, 2n_0, z \right), 0 \right)$
	$1 - (n_0 + n_\infty)$	$(1, 0)$
	$n_0 - n_\infty$	$\left(0, (-z)^{1-2n_0} \right)$
	$-n_0 + n_\infty$	$\left(0, (1-z)^{2n_0 - 2n_\infty} (-z)^{1-2n_0} \right)$
$(0, 0, 0)$	$n_0 + n_\infty$	$\left((1-z)^{-2n_\infty - 2n_0}, 0 \right)$
	$1 - (n_0 + n_\infty)$	$\left(0, (1-z)^{2n_\infty + 2n_0 - 2} (-z)^{1-2n_0} \right)$
	$n_0 - n_\infty$	$\left((1-z)^{2n_\infty - 2n_0}, 0 \right)$
	$-n_0 + n_\infty$	$(1, 0)$
$(-1, 1, 1)$	$n_0 + n_\infty$	$\left((1-z)^{-2n_\infty - 2n_0 + 1}, 0 \right)$
	$1 - (n_0 + n_\infty)$	$(1, 0)$
	$n_0 - n_\infty$	$\left(0, F\left(-1, 1 - 2n_\infty, 1 - 2n_0, z \right) (-z)^{-2n_0} \right)$
	$-n_0 + n_\infty$	$\left(0, (1-z)^{-2n_\infty + 2n_0 + 1} (-z)^{-2n_0} \right)$
$(0, 0, 1)$	$n_0 + n_\infty$	$\left(0, (-z)^{-2n_0} \right)$
	$1 - (n_0 + n_\infty)$	$\left(0, (1-z)^{2n_\infty + 2n_0 - 1} (-z)^{-2n_0} \right)$
	$n_0 - n_\infty$	$\left(0, (-z)^{-2n_0} \right)$
	$-n_0 + n_\infty$	$(1, 0)$
$(0, 1, 1)$	$n_0 + n_\infty$	$\left((1-z)^{-2n_\infty - 2n_0}, 0 \right)$
	$1 - (n_0 + n_\infty)$	$\left(0, (1-z)^{2n_\infty + 2n_0 - 2} (-z)^{-2n_0} \right)$
	$n_0 - n_\infty$	$\left(0, (-z)^{-2n_0} \right)$
	$-n_0 + n_\infty$	$\left(0, (1-z)^{2n_0 - 2n_\infty} (-z)^{-2n_0} \right)$

and $n_\infty > m_\infty$ or from the $n_0 > m_0$, $\hat{n}_1 < \hat{m}_1$ and $\hat{n}_\infty < \hat{m}_\infty$. In any case we get a factorized solution of the form $\mathcal{B}^{(L)}(C, C')^T$ which is what expected since the right sector plays no role.

5.4. Relating the Abelian angles with the group parameters

Using the explicit expression for the SO(4) and SU(2) × SU(2) it is easy to verify that when the left and right SU(2) parameters are $\vec{n} = n^3 \vec{k}$ and $\vec{m} = m^3 \vec{k}$ the rotation in plane 14 is a SO(2) element $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ with angle $\theta = n^3 - m^3$ and the one in plane 23 is with angle $\theta = n^3 + m^3$.

Comparing with the case with $m = 0$ given in (5.1) we can then guess that the general relation between the group parameters and the usual Abelian angles is given by

$$\varepsilon_{\mathbf{t}} = n_{\mathbf{t}}^3 - m_{\mathbf{t}}^3 + \theta(-(n_{\mathbf{t}}^3 - m_{\mathbf{t}}^3)),$$

$$\varphi_{\mathbf{t}} = n_{\mathbf{t}}^3 + m_{\mathbf{t}}^3 + \theta(-(n_{\mathbf{t}}^3 + m_{\mathbf{t}}^3)). \tag{5.4}$$

5.5. Recovering the Abelian result: an example

In order to show how the Abelian limit works we consider the following example. We take case 1 as in section 4.4.1 with $n_{\mathbf{1}} = 1 - (n_{\mathbf{0}} + n_{\infty})$ and $m_{\mathbf{1}} = -m_{\mathbf{0}} + m_{\infty}$, which leads to two independent rational functions of ω_z:

$$
\begin{aligned}
\partial \mathcal{X}(\omega_z) &= \begin{pmatrix} i\partial\bar{\mathcal{Z}}^{\bar{1}}(\omega_z) & \partial\mathcal{Z}^2(\omega_z) \\ \partial\bar{\mathcal{Z}}^{\bar{2}}(\omega_z) & i\partial\mathcal{Z}^1(\omega_z) \end{pmatrix} = \\
&= \begin{pmatrix} i\partial(\mathcal{X}^1(\omega_z) - i\mathcal{X}^4(\omega_z)) & \partial(\mathcal{X}^2(\omega_z) + i\mathcal{X}^3(\omega_z)) \\ \partial(\mathcal{X}^2(\omega_z) - i\mathcal{X}^3(\omega_z)) & i\partial(\mathcal{X}^1(\omega_z) + i\mathcal{X}^4(\omega_z)) \end{pmatrix} = \\
&= \begin{pmatrix} 0 & C_1 (-\omega_z)^{\varepsilon_0 - 1}(1 - \omega_z)^{\varepsilon_1 - 1} \\ 0 & C_2 (-\omega_z)^{-\varphi_0}(1 - \omega_z)^{-\varphi_1} \end{pmatrix},
\end{aligned}
\tag{5.5}
$$

where C_1, C_2 are constants as in (4.49). This is the known result for the Abelian case, where we have two different U(1) sectors undergoing two different rotations $U_1(1) \times U_2(1) \subset SU_L(2) \times SU_R(2)$. In the previous expression we have used (5.4) to write the relation between the usual Abelian angles and the group parameters as

$$\varepsilon_0 = n_{\mathbf{0}} - m_{\mathbf{0}}, \qquad \varepsilon_1 = n_{\mathbf{1}} - m_{\mathbf{1}}, \qquad \varepsilon_{\infty} = n_{\infty} + m_{\infty},$$
$$\sum_t \varepsilon_{\mathbf{t}} = 1 \tag{5.6}$$

and

$$\varphi_0 = n_{\mathbf{0}} + m_{\mathbf{0}}, \qquad \varphi_1 = n_{\mathbf{1}} + m_{\mathbf{1}}, \qquad \varphi_{\infty} = n_{\infty} - m_{\infty},$$
$$\sum_t \varphi_{\mathbf{t}} = 2, \tag{5.7}$$

in order to approach the usual notation in the literature. As usual $\partial \mathcal{Z}^1(\omega_z) \neq [\partial\overline{\mathcal{Z}}^{\bar{1}}(\omega_z)]^*$. We can now build the Abelian solution to show the characteristic analytical structure of the Abelian limit. Explicitly we get

$$
\begin{pmatrix} i\bar{Z}^{\bar{1}}(u,\bar{u}) & Z^2(u,\bar{u}) \\ \bar{Z}^{\bar{2}}(u,\bar{u}) & iZ^1(u,\bar{u}) \end{pmatrix} = \begin{pmatrix} i f_{(\bar{t}-1)}^{\bar{1}} + i \int_0^{\bar{\omega}_{\bar{u}}} d\omega \partial\mathcal{Z}^1 & f_{(\bar{t}-1)}^2 + \int_0^{\omega_u} d\omega \partial\mathcal{Z}^2 \\ \bar{f}_{(\bar{t}-1)}^{\bar{2}} + \int_0^{\bar{\omega}_{\bar{u}}} d\omega \partial\mathcal{Z}^2 & i f_{(\bar{t}-1)}^1 + i \int_0^{\omega_u} d\omega \partial\mathcal{Z}^1 \end{pmatrix}, \tag{5.8}
$$

where for simplicity we have chosen $R_{(\bar{t})} = \mathbb{1}_4$ so that $U_{(\bar{t})}$ in (2.10) is mapped to the SU(2) $\times$ SU(2) element $(i\sigma_1, i\sigma_1)$. Notice however that $\mathbf{n_t} = n_{\mathbf{t}}^3 \mathbf{k}$ implies that $v_{(t)}^3 = 0$ in (3.11) and hence that U_L and U_R are always off diagonal and therefore their action on (5.5) is to fill the first column. From the previous relations we see the usual holomorphicity $\overline{Z}^{\bar{1}}(\bar{u}) = (Z^1(u))^*$ of the sector with $\sum_t \varepsilon_{\mathbf{t}} = 1$ and $\overline{Z}^{\bar{2}}(\bar{u}) = (Z^2(u))^*$ of the sector with $\sum_t \varphi_{\mathbf{t}} = 2$.

5.6. Abelian limits

Following the example of the previous section it is possible to consider both cases given in Section 4.4.1 and Section 4.4.2 for all possible combinations of the expression of $n_{\mathbf{1}}$ and $m_{\mathbf{1}}$ for

a total of $2 \cdot 4 \cdot 4$ possible combinations. In all cases but 6 the solution in spinorial formalism is a 2×2 matrix which has two non vanishing entries and hence two independent Abelian solutions. In the remaining 6 cases the matrix has only one non vanishing entry but the constraints on n and m are incompatible and therefore they should not be considered. The 6 inconsistent combinations are for case 1 when $\{n_1 = n_0 + n_\infty, \ m_1 = 1 - (m_0 + m_\infty)\}$ and $\{n_1 = 1 - (n_0 + n_\infty), \ m_1 = 1 - (m_0 + m_\infty)\}$ and for case 2 when $n_1 = -n_0 + n_\infty$.

6. The physical interpretation

In this section we would like to show some simple consequences of the explicit classical solution for the phenomenology of the branes at angles models. In particular we will focus on the value of the action which plays a fundamental role in the hierarchy of the Yukawa couplings.

6.1. Rewriting the action

Once the solution to the boundary conditions has been found, it is possible to compute the classical action to show its contribution to the correlation functions of twist fields and Yukawa couplings. We use the equations of motion (2.5) to simplify as much as possible the computation of the action (2.4) and get:

$$4\pi\alpha' \, S\,|_{\text{on-shell}} = i \sum_{t=1}^{3} \sum_{m \in \{3,4\}} g_{(t),\,m} \int_{x_t}^{x_{t-1}} dx \, \left(R_{(t)}\right)_{m\,I} \left(X'_L(x) - X'_R(x)\right)^I \Big|_{y=0^+}, \tag{6.1}$$

where $I = 1, 2, 3, 4$ and $m = 3, 4$ are the transverse directions in the well adapted frame with respect to the brane. Moreover, since the total number of D-branes is defined modulo $N_B = 3$, the interval defining $D_{(1)}$ is split on two separate intervals, namely:

$$[x_1, x_3] = [x_1, +\infty) \cup (-\infty, x_3],$$

as it is visually shown in Fig. 3. To proceed further we notice that for $x_t < x < x_{t-1}$ we have:

$$X(x + iy, x - iy) = X^*(x + iy, x - iy) \Rightarrow X^*_L(x - iy) = X_R(x - iy) + Y_{(t)},$$

where $Y_{(t)}$ is a constant factor which cannot depend on the particular brane $D_{(t)}$ and must be a real. From the continuity of $X_L(u)$ and $X_R(\overline{u})$ on the worldsheet intersection point we get

$$\lim_{x \to x_t^+} X(x, x) = \lim_{x \to x_t^-} X(x, x)$$

which does not allow $Y_{(t)}$ to depend on the brane while the reality of $X(u, \overline{u})$ implies that $\text{Im}\, Y = 0$. Then (6.1) becomes:

$$4\pi\alpha' \, S|_{\text{on-shell}} = -2 \sum_{t=1}^{3} \sum_{m \in \{3,4\}} g_{(t),\,m} \, \text{Im} \left(R_{(t)}\right)_{m\,I} X^I_L(x + i0^+) \Big|_{x=x_t}^{x=x_{t-1}}$$

$$= -2 \sum_{t=1}^{3} g^{(\perp)}_{(t),\,I} \, \text{Im}\, X^I_L(x + i0^+) \Big|_{x=x_t}^{x=x_{t-1}} \in \mathbb{R}, \tag{6.2}$$

where $g^{(\perp)}_{(t),\,I} = \sum_{m \in \{3,4\}} \left(R^{-1}_{(t)}\right)_{I\,m} g_{(t),\,m}$ is the transverse shift of $D_{(t)}$ in global coordinates and because of this is perpendicular to $f_{(t-1)} - f_{(t)}$ which is tangent to $D_{(t)}$, i.e.

$$g^{(\perp)}_{(t),\,I}(f_{(t-1)} - f_{(t)})^I = 0. \tag{6.3}$$

6.2. Holomorphic case

In this case there exist global complex coordinates for which the string solution is holomorphic, i.e.

$$Z^i(u, \bar{u}) = Z^i_L(u), \quad \bar{Z}^{\bar{i}}(u, \bar{u}) = \bar{Z}^{\bar{i}}(\bar{u}) = \left(Z^i_L(u)\right)^*, \tag{6.4}$$

where $i = 1$ in the Abelian case and $i = 1, 2$ in the SU(2) case. In these cases we have $f^i_{(t)} = Z^i_L(x_t + i0^+)$ and because of this the previous equations (6.3) and (6.2) become

$$\text{Re}\left(g^{(\perp)}_{(t),\,i}(f_{(t-1)} - f_{(t)})^i\right) = 0$$

$$4\pi\alpha'\,S|_{\text{on-shell}} = -2\sum_{t=1}^{3} \text{Im}\left(g^{(\perp)}_{(t),\,i}\,(f_{(t-1)} - f_{(t)})^i\right), \tag{6.5}$$

where the last equation shows that the action can be expressed only using the global data.

In the Abelian case where $i = 1$ we can further simplify the action and give a clear geometrical meaning. We notice that given to complex numbers $a, b \in \mathbb{C}$ such that $\text{Re}(a^*b) = 0$ then $\text{Im}(a^*b) = \pm|a||b|$. This can be seen either by direct computation or by using a U(1) rotation to set b equal to $|b|$. Since the action is positive then we can write

$$S|_{\text{on-shell}} = \frac{1}{2\pi\alpha'}\sum_{t=1}^{3}\frac{1}{2}|g^{(\perp)}_{(t)}|\,|f_{(t-1)} - f_{(t)}|, \tag{6.6}$$

where a factor $\frac{1}{2}$ comes from raising the $g^{(\perp)}_{(t)\,i=1}$ complex index. We now see that the right hand side is the sum of the areas of the triangles having as base the interval between two intersection points on a given brane $D_{(t)}$ and as height the distance between the brane and the origin as shown in Fig. 1.

For the SU(2) case we can use a SU(2) rotation to bring $(f_{(t-1)} - f_{(t)})^i$ to the form $\| f_{(t-1)} - f_{(t)} \| \delta^i_1$, then each term of the action can be interpreted again as an area of a triangle where the distance between the interaction points is the base. Also in this case a kind of flatness is playing a role to give the value of the action.

6.3. The general non Abelian case

In the general case there does not seem to be any possible way of computing the action (6.2) in term of the global data. It does not seem to be any kind of flatness involved and probably the action is bigger than in the holomorphic case since the string is no longer confined to a plane and, given the nature of the rotation, its worldsheet has to bend in order to be attached to the brane as pictorially shown in Fig. 8 in the case of a 3-dimensional space. The general case we considered then differs from the known factorized case by an additional contribution in the on-shell action which can be intuitively understood as a small "bump" of the string worldsheet in proximity of the boundary.

The physical consequence is an exponential suppression of the contribution of the classical action to the correlators of twist fields and to the Yukawa coupling with respect to the holomorphic case.
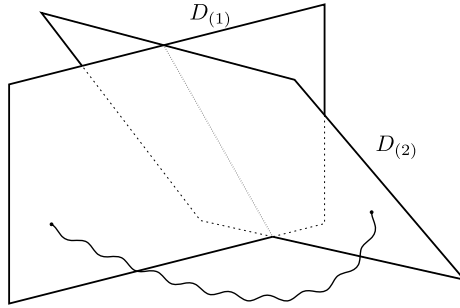
Fig. 8. This is a pictorical 3-dimensional representation of two D2-branes intersecting in the Euclidean space $\mathbb{R}^3$ along a line (in $\mathbb{R}^4$ the intersection is a point since the co-dimension of each brane is 2): since it is no longer constrained on a bi-dimensional plane, the string must be deformed in order to stretch between two consecutive branes. Its action will therefore be larger than the planar area.

Acknowledgements

Appendix A. The isomorphism in details

In this appendix we discuss our conventions for SU(2) and show the details on the constructions of the isomorphism between SO(4) and a class of equivalence of SU(2) × SU(2).

A.1. SU(2) *conventions*

We choose to parameterize any SU(2) matrix with a vector $\mathbf{n} \in \mathbb{R}^3$ such that:

$$U(\mathbf{n}) = \cos(2\pi n)\mathbb{1}_2 + i\frac{\mathbf{n} \cdot \sigma}{n} \sin(2\pi n), \tag{A.1}$$

where $n = \|\mathbf{n}\|$ and $0 \leq n \leq \frac{1}{2}$ with the identification of all $\mathbf{n}$ when $n = \frac{1}{2}$ since in this case $U(\mathbf{n}) = -\mathbb{1}_2$. The parametrization is such that:

$$(U(\mathbf{n}))^* = \sigma^2 U(\mathbf{n})\sigma^2 = U(\widetilde{\mathbf{n}}), \tag{A.2}$$

$$(U(\mathbf{n}))^\dagger = (U(\widetilde{\mathbf{n}}))^T = U(-\mathbf{n}), \tag{A.3}$$

$$-U(\mathbf{n}) = U(\widehat{\mathbf{n}}) \tag{A.4}$$

where $\widetilde{\mathbf{n}} = \left(-n^1, +n^2, -n^3\right)$ and $\widehat{\mathbf{n}} = -\left(\frac{1}{2} - n\right)\mathbf{n}/n$.

The product of two elements is given by $U(\mathbf{n} \circ \mathbf{m}) = U(\mathbf{n})U(\mathbf{m})$ or more explicitly by:

$$\cos(2\pi\|\mathbf{n} \circ \mathbf{m}\|) = \cos(2\pi n)\cos(2\pi m) - \sin(2\pi n)\sin(2\pi m)\frac{\mathbf{n}}{n} \cdot \frac{\mathbf{m}}{m},$$

$$\sin(2\pi\|\mathbf{n} \circ \mathbf{m}\|)\frac{\mathbf{n} \circ \mathbf{m}}{\|\mathbf{n} \circ \mathbf{m}\|} = \cos(2\pi n)\sin(2\pi m)\frac{\mathbf{m}}{m} + \sin(2\pi n)\cos(2\pi m)\frac{\mathbf{n}}{n}. \tag{A.5}$$

A.2. The isomorphism

Let $I = 1, 2, 3, 4$ as in the main text and define:

$$\tau_I = (i \mathbb{1}_2, \sigma),$$

where $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ are the usual Pauli matrices. It is then easy to show that:

$$(\tau_I)^\dagger = \eta_{IJ} \tau^J,$$
$$\left(\tau^I\right)^* = -\sigma_2 \tau_I \sigma_2, \tag{A.6}$$

where $\eta_{IJ} = \mathrm{diag}(-1, 1, 1, 1)$. We have the following useful relations:

$$\mathrm{tr}(\tau_I) = 2i \delta_{I1},$$
$$\mathrm{tr}(\tau_I \tau_J) = 2 \eta_{IJ},$$
$$\mathrm{tr}(\tau_I (\tau_J)^\dagger) = 2 \delta_{IJ}.$$

Now consider a vector in this spinor representation:

$$X_{(s)} = X^I \tau_I.$$

We can recover its components using the previous properties:

$$X^I = \frac{1}{2} \delta^{IJ} \, \mathrm{tr}(X_{(s)} (\tau_J)^\dagger) = \frac{1}{2} \eta^{IJ} \, \mathrm{tr}(X_{(s)} \tau_J).$$

If the vector X^I is real, using the properties in (A.6), then we have

$$X_{(s)}^\dagger = X^I \eta_{IJ} \tau^J = \frac{1}{2} \mathrm{tr}(X_{(s)} \tau_I) \tau^I,$$
$$X_{(s)}^* = -\sigma_2 X_{(s)} \sigma_2. \tag{A.7}$$

A rotation in the spinor representation is defined as:

$$X'_{(s)} = U_L(\mathbf{n}) X_{(s)} U_R^\dagger(\mathbf{m}) \tag{A.8}$$

and it is equivalent to:

$$(X')^I = R^I_{\ J} X^J \tag{A.9}$$

through

$$R_{IJ} = \frac{1}{2} \mathrm{tr}((\tau_I)^\dagger \, U_L(\mathbf{n}) \tau_J U_R^\dagger(\mathbf{m})). \tag{A.10}$$

R is indeed the matrix we are looking for since

$$\mathrm{tr}(X'_{(s)} X'^\dagger_{(s)}) = \mathrm{tr}(X_{(s)} X_{(s)}^\dagger) \Rightarrow R_{I\,K} R^*_{J\,K} = \delta_{I\,J}.$$

It is then necessary to show that R is a real matrix. From the second equation in (A.6) and the first equation in (A.4) we get:

$$R_{NM} = \frac{1}{2} \eta_{NI} \eta_{MJ} \, \mathrm{tr}(\tau_I^\dagger U_R \tau_J U_L^\dagger) = \frac{1}{2} \mathrm{tr}(\tau_N U_R \tau_M^\dagger U_L^\dagger) = (R_{NM})^*.$$

The property $R \in SO(4, \mathbb{R})$ can also be shown by a direct computation of the determinant using the parametrization of the SU(2) matrices: we find

$$\det(R) = 1.$$

Moreover the explicit choice of the basis τ ensures R to be a real matrix.

Since $\{U_L, U_R\}$ and $\{-U_L, -U_R\}$ generate the same SO(4) matrix then the correct isomorphism takes the form:

$$SO(4) \cong \frac{SU(2) \times SU(2)}{\mathbb{Z}_2}.$$

Appendix B. The parameters of the hypergeometric function

B.1. Consistency conditions for U(2) *and* U(1, 1) *monodromies*

In the main text we have set

$$DM_\infty (D)^{-1} = e^{-2\pi i \delta_\infty} \mathcal{L}(\mathbf{n}_\infty), \tag{B.1}$$

where $\mathcal{L}(\mathbf{n}_\infty)$ is a SU(2) matrix. This is a somewhat strong statement which may imply and implies some consistency conditions. The previous equation implies

$$[DM_\infty (D)^{-1}]^\dagger = [DM_\infty (D)^{-1}]^{-1}, \tag{B.2}$$

which can be rewritten as

$$\widetilde{M}_\infty^{-1} \, \mathcal{C}^\dagger D^\dagger D\mathcal{C} = \mathcal{C}^\dagger D^\dagger D\mathcal{C} \, \widetilde{M}_\infty^{-1}. \tag{B.3}$$

Since $\widetilde{M}_\infty$ is a generic diagonal matrix the previous equation implies that the off-diagonal elements of $\mathcal{C}^\dagger D^\dagger D\mathcal{C}$ must vanish. This means that

$$
\begin{aligned}
|K|^{-2} &= -\frac{\mathcal{C}_{21}\mathcal{C}_{22}^*}{\mathcal{C}_{11}\mathcal{C}_{12}^*} = \\
&= -\frac{\Gamma^*(a)\Gamma(b)\Gamma(c-a)\Gamma^*(c-b)}{\Gamma(1-a)\Gamma^*(1-b)\Gamma^*(1-c+a)\Gamma(1-c+b)} = \\
&= -\frac{1}{\pi^4}|\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)|^2 \times \\
&\quad \times \; \sin(\pi a)\sin^*(\pi(c-a)) \, (\sin(\pi b)\sin^*(\pi(c-b)))^*.
\end{aligned}
\tag{B.4}
$$

For real a, b and c this means that

$$\sin(\pi a)\sin(\pi(c-a)) \; \sin(\pi b)\sin(\pi(c-b)) < 0. \tag{B.5}$$

The previous equation is invariant under integer shift of any of the three parameters therefore we can limit to consider what happens to the fractional parts $0 \le \{a\}, \{b\}, \{c\} < 1$. Finally we get that the previous equation, i.e. the original position of having $U(2)$ monodromies requires

$$\text{either} \quad 0 \le \{b\} < \{c\} < \{a\} < 1 \quad \text{or} \quad 0 \le \{a\} < \{c\} < \{b\} < 1. \tag{B.6}$$

Should we require the monodromies be in $U(1, 1)$, as required by moving rotated branes, then we would get

$$|K|^{-2} = +\frac{\mathcal{C}_{21}\mathcal{C}_{22}^*}{\mathcal{C}_{11}\mathcal{C}_{12}^*}, \tag{B.7}$$

which would imply

$$\text{either} \quad 0 \leq \{c\} < \{a\}, \{b\} < 1 \quad \text{or} \quad 0 \leq \{a\}, \{b\} < \{c\} < 1. \tag{B.8}$$

B.2. Fixing the parameters

In this appendix we show in a detailed way how to compute the parameters of the basis of hypergeometric functions we used in the main text. The relation between such parameters and the SU(2) matrices are computed requiring that the monodromies induced by the choice of the parameters equal the monodromies of the rotations of the D-branes.

The monodromy in $\omega_{\bar{t}-1} = 0$ is simpler to compute since we chose $\mathcal{L}(\mathbf{n_0})$ and $\mathcal{R}(\widetilde{\mathbf{m}}_0)$ to be diagonal. We impose:

$$\begin{pmatrix} 1 & \\ & e^{-2\pi i c^{(L)}} \end{pmatrix} = e^{-2\pi i \delta_0^{(L)}} \begin{pmatrix} e^{2\pi i n_0} & \\ & e^{-2\pi i n_0} \end{pmatrix},$$

$$\begin{pmatrix} 1 & \\ & e^{-2\pi i c^{(R)}} \end{pmatrix} = e^{-2\pi i \delta_0^{(R)}} \begin{pmatrix} e^{-2\pi i m_0} & \\ & e^{2\pi i m_0} \end{pmatrix},$$

where $n_0^3 = n_0$ and $m_0^3 = m_0$ with $0 \leq n_0, m_0 < 1$ with the conventions of (4.13) and (4.14). In the left sector we therefore find:

$$\begin{aligned} \delta_0^{(L)} &= n_0 + k_{\delta_0^{(L)}}, \quad \text{where} \quad k_{\delta_0^{(L)}} \in \mathbb{Z}, \\ c^{(L)} &= 2n_0 + k_c, \quad \text{where} \quad k_c \in \mathbb{Z}. \end{aligned} \tag{B.9}$$

Since $e^{i4\pi \delta_0^{(L)}}$ is the determinant of the right hand side the range of definition of $\delta_0^{(L)}$ is $\alpha \leq \delta_0^{(L)} \leq \alpha + \frac{1}{2}$ since $0 \leq n_0 < \frac{1}{2}$ we can simply take $\alpha = 0$ and set

$$\delta_0^{(L)} = n_0.$$

Analogous results hold in the right sector. From the third equation in (4.11) and from the first equation in (B.9) we find:

$$-A + n_0 + m_0 \in \mathbb{Z}.$$

We now need to fix the 6 parameters $a^{(L)}, b^{(L)}, \delta_\infty^{(L)}, B$ and $|K^{(L)}|, \mathcal{T}^{(L)}$. Our strategy is first to find 3 equations to determine $a^{(L)}, b^{(L)}, \delta_\infty^{(L)}$ and then fix the remaining ones. Clearly everything holds true also for the right sector. All these equations follow from imposing the requests (4.12). The first two equations for $a^{(L)}, b^{(L)}, \delta_\infty^{(L)}$ follow easily by considering the trace of (4.12):

$$e^{i\pi(a^{(L)}+b^{(L)})} \cos(\pi(a^{(L)} - b^{(L)})) = e^{-2\pi i \delta_\infty^{(L)}} \cos(2\pi n_\infty),$$

which translates into:

$$\delta_\infty^{(L)} = -\frac{1}{2}(a^{(L)} + b^{(L)}) + \frac{1}{2}k_{\delta_\infty^{(L)}}, \quad \text{where} \quad k_{\delta_\infty} \in \mathbb{Z},$$

$$a^{(L)} - b^{(L)} = 2(-1)^{p^{(L)}} n_\infty + k_{\delta_\infty^{(L)}}(-1)^{q^{(L)}} + 2k'_{ab}, \quad \text{where} \quad k'_{ab} \in \mathbb{Z},$$

with $p^{(L)}, q^{(L)} \in \{0, 1\}$. A change of value of $p^{(L)}$ corresponds to an exchange between a and b: since the hypergeometric function is symmetric in a and b we can fix $p^{(L)} = 0$. Redefining k' we can always set $q^{(L)} = 0$, hence we can write

$$a^{(L)} - b^{(L)} = 2n_\infty + k_{\delta_\infty^{(L)}} + 2k_{ab}, \quad \text{where} \quad k_{ab} \in \mathbb{Z}. \tag{B.10}$$

A discussion of the possible values of $k_{\delta_\infty^{(L)}}$ is analogous to what done for the monodromy around $\omega_{\bar{\imath}-1} = 0$ but with an important difference: $\frac{1}{2}(a^{(L)} + b^{(L)})$ may a priori take values in an interval of width 1. Because of this since also in this case we have $\alpha \leq \delta_\infty^{(L)} \leq \alpha + \frac{1}{2}$ with α *a priori* arbitrary we cannot choose $k_{\delta_\infty^{(L)}} = 0$ but we have to consider $k_{\delta_\infty^{(L)}} = 0, 1$.

We then find a third relation by considering:

$$\text{Im}\left(e^{+2\pi i \delta_\infty^{(L)}} D^{(L)} M_\infty^{(L)} \left(D^{(L)}\right)^{-1}\right)_{11} = \text{Im}\left(\mathcal{L}(n_\infty)\right)_{11}.$$

With the help of

$$\det \mathcal{C} = \frac{\sin(\pi c^{(L)})}{\sin(\pi (a^{(L)} - b^{(L)}))},$$

and the second equation in (B.9) and (B.10), it leads to:

$$\cos\left(\pi (a^{(L)} + b^{(L)} - c^{(L)})\right) = (-1)^{k_c + k_{\delta_\infty^{(L)}}} \cos\left(2\pi \mathcal{A}^{(L)}\right),$$

where

$$\cos\left(2\pi \mathcal{A}^{(L)}\right) = \cos(2\pi n_0) \cos(2\pi n_\infty) - \sin(2\pi n_0) \sin(2\pi n_\infty) \frac{n_\infty^3}{n_\infty}. \tag{B.11}$$

The rotation parameter in the third interaction point $\omega_{\bar{\imath}+1} = 1$ is connected with the previous expression as

$$\cos\left(2\pi \mathcal{A}^{(L)}\right) = \cos(2\pi n_1).$$

Then we can write:

$$a^{(L)} + b^{(L)} - c^{(L)} = 2(-1)^{f^{(L)}} n_1 + k_c + k_{\delta_\infty^{(L)}} + 2k_{abc}, \quad \text{where} \quad k_{abc} \in \mathbb{Z},$$

with $f^{(L)} \in \{0, 1\}$.

We then fix the B parameter in the third equation of (4.12) requiring:

$$A + B - n_0 - m_0 - (-1)^{f^{(L)}} n_1 - (-1)^{f^{(R)}} m_1 \in \mathbb{Z}.$$

We can summarize the results so far as

$$a = n_0 + (-1)^{f^{(L)}} n_1 + n_\infty + m_a,$$
$$b = n_0 + (-1)^{f^{(L)}} n_1 - n_\infty + m_b,$$
$$c = 2n_0 + m_c,$$
$$\delta_0^{(L)} = n_0,$$
$$\delta_\infty^{(L)} = -n_0 - (-1)^{f^{(L)}} n_1 + m_c + 2m_\delta$$
$$A = n_0 + m_0 + m_A,$$
$$B = (-1)^{f^{(L)}} n_1 + (-1)^{f^{(R)}} m_1 + m_B,$$

where all the factors m are integers.

Finally we determine $K^{(L)}$. To this purpose we consider:

$$\left(D^{(L)} M_\infty \left(D^{(L)}\right)^{-1}\right)_{21} = e^{-2\pi i \delta_\infty^{(L)}} \left(\mathcal{L}(n_\infty)\right)_{21}, \tag{B.12}$$

and get:

$$
\begin{aligned}
K^{(L)} &= -\frac{1}{2\pi^2} \Gamma(1 - a^{(L)}) \Gamma(1 - b^{(L)}) \Gamma(a^{(L)} + 1 - c^{(L)}) \Gamma(b^{(L)} + 1 - c^{(L)}) \times \\
&\quad \times \sin(\pi c) \sin(\pi(a - b)) \frac{n_\infty^1 + i n_\infty^2}{n_\infty} \\
&= -\frac{(-1)^{m_a + m_b + m_c}}{2\pi^2} \Gamma(1 - a^{(L)}) \Gamma(1 - b^{(L)}) \Gamma(a^{(L)} + 1 - c^{(L)}) \Gamma(b^{(L)} + 1 - c^{(L)}) \times \\
&\quad \times \sin(2\pi n_{\mathbf{0}}) \sin(2\pi n_\infty) \frac{n_\infty^1 + i n_\infty^2}{n_\infty}.
\end{aligned}
\tag{B.13}
$$

B.3. Checking the consistency of the solution

Given the previous solution we can now check the consistency condition (B.6) with the help of (A.5). Another way of performing this check is to compute $K^{(L)}$ from

$$\left(D^{(L)} M_\infty \left(D^{(L)}\right)^{-1}\right)_{12} = e^{-2\pi i \delta_\infty^{(L)}} \left(\mathcal{L}(n_\infty)\right)_{12}, \tag{B.14}$$

instead of (B.12). The result is

$$
\begin{aligned}
\frac{1}{K^{(L)}} &= \frac{(-1)^{m_a + m_b + m_c}}{2\pi^2} \Gamma(a^{(L)}) \Gamma(b^{(L)}) \Gamma(-a^{(L)} + c^{(L)}) \Gamma(-b^{(L)} + c^{(L)}) \times \\
&\quad \times \sin(2\pi n_{\mathbf{0}}) \sin(2\pi n_\infty) \frac{n_\infty^1 - i n_\infty^2}{n_\infty}.
\end{aligned}
\tag{B.15}
$$

This expression and (B.13) are compatible only if

$$\frac{(n_\infty^1)^2 + (n_\infty^2)^2}{n_\infty^2} = -4 \frac{\sin(\pi a) \sin(\pi(c - a)) \sin(\pi b) \sin(\pi(c - b))}{\sin^2(\pi c) \sin^2(\pi(a - b))}. \tag{B.16}$$

Notice that this equation may be true only if the constraint found before and expressed in (B.5) is true. To proof it we rewrite (B.11) as

$$\frac{(n_\infty^3)^2}{n_\infty^2} = \frac{(\cos(\pi(a - b)) \cos(\pi c) - \cos(\pi(a + b - c)))^2}{\sin^2(\pi c) \sin^2(\pi(a - b))}, \tag{B.17}$$

and then we verify that the sum of the right hand side of this equation and the right hand side of (B.16) is equal to 1.

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