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"Matei Demetrescu, Iliyan Georgiev, Paulo M.M. Rodrigues, A. M. Robert Taylor"

Essex Business School, University of Essex, Wivenhoe Park, Colchester, CO4 3SQ
Web site: http://www.essex.ac.uk/ebs/

Extensions to IVX Methods of Inference for Return Predictability*

Matei Demetrescu[a], Iliyan Georgiev[b], Paulo M.M. Rodrigues[c] and A.M. Robert Taylor[d]

[a] Institute for Statistics and Econometrics, Christian-Albrechts-University of Kiel

[b] Department of Economics, University of Bologna, and Institut d'Anàlisi Econòmica, CSIC

[c] Banco de Portugal and NOVA School of Business and Economics

[d] Essex Business School, University of Essex

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Abstract

Predictive regression methods are widely used to examine the predictability of (excess) returns on stocks and other equities by lagged macroeconomic and financial variables. Extended IV [IVX] estimation and inference has proved a particularly valuable tool in this endeavour as it allows for possibly strongly persistent and endogenous regressors. This paper makes three distinct contributions to the literature. First we demonstrate that, provided either a suitable bootstrap implementation is employed or heteroskedasticity-consistent standard errors are used, the IVX-based predictability tests of Kostakis *et al.* (2015) retain asymptotically pivotal inference, regardless of the degree of persistence or endogeneity of the (putative) predictor, under considerably weaker assumptions on the innovations than are required by Kostakis *et al.* (2015) in their analysis. In particular, we allow for quite general forms of conditional and unconditional heteroskedasticity in the innovations, neither of which are tied to a parametric model. Second, and associatedly, we develop asymptotically valid bootstrap implementations of the IVX tests under these conditions. Monte Carlo simulations show that the bootstrap methods we propose can deliver considerably more accurate finite sample inference than the asymptotic implementation of these tests used in Kostakis *et al.* (2015) under certain problematic parameter constellations, most notably for their implementation against one-sided alternatives, and where multiple predictors are included. Third, under the same conditions as we consider for the full-sample tests, we show how sub-sample implementations of the IVX approach, coupled with a suitable bootstrap, can be used to develop asymptotically valid one-sided and two-sided tests for the presence of temporary windows of predictability.

Keywords: predictive regression; IVX estimation; (un)conditional heteroskedasticity; subsample tests; unknown regressor persistence; endogeneity; residual wild bootstrap.

JEL classification: C12, C22, G17

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1 Motivation

There exists a large body of empirical research investigating whether stock returns can be predicted using publicly available data. A wide range of lagged financial and macroeconomic variables has been considered as putative predictors for returns, including: valuation ratios such as the dividend-price ratio, dividend yield, earnings-price ratio, and book-to-market ratio; various interest rates and interest rate spreads, and macroeconomic variables including inflation and industrial production.

Empirical evidence on the predictability of returns largely derives from inference obtained from predictive regressions and, as such, the size and power properties of predictability tests from these regressions are of fundamental importance. These depend on the time series properties of the predictor, in particular its degree of persistence and endogeneity. Campbell and Yogo (2006) and Welch and Goyal (2008), among others, find that many of the variables used in predictive regressions are highly persistent and that a strong negative correlation often exists between returns and the predictors' innovations. Nelson and Kim (1993) and Stambaugh (1999) show that the estimated slope coefficient in such cases will be heavily biased.

In the context of a formulation of strong persistence where the predictor, x_{t-1} say, is assumed to follow a first-order autoregression with a local-to-unity coefficient $\rho = 1 - c/T$, where c is a finite constant and T is the sample size, standard likelihood-based statistics from the predictive regression have limiting distributions which depend on c and on the correlation between the innovations driving the predictor and returns; see, for example, Cavanagh *et al.* (1995). In particular, the standard regression t statistic may severely over-reject under the null of no predictability when the predictor is endogenous. As a result, a number of likelihood-based predictability tests have been developed in the literature designed to be asymptotically valid (by which we mean asymptotically correctly sized under the null hypothesis) under the assumption that the predictor is endogenous and displays strong persistence in the local-to-unity class of processes; see, in particular, Cavanagh *et al.* (1995), Campbell and Yogo (2006) and Jansson and Moreira (2006).

A major practical drawback with these likelihood-based approaches is that they are invalid if the predictor is stationary or near-stationary; the theoretical validity of the methods requires each predictor to be at least as persistent as a local-to-unity process. An alternative approach which has been developed in the literature is to base predictability tests on methods of estimating the predictive regression which are robust to the properties of the regressor. Various approaches have been considered, but by far the most successful is proposed in Kostakis *et al.* (2015) who estimate the predictive regression using the extended instrumental variable [IVX] procedure of Phillips and Magdalinos (2009); see also, Gonzalo and Pitarakis (2012), Phillips and Lee (2013), Breitung and Demetrescu (2015), Lee (2016), Demetrescu and Hillmann (2020) and Demetrescu *et al.* (2020). In the IVX approach each predictor in the predictive regression has an associated stochastic instrument formed by constructing a mildly integrated variable from the first differences of the predictor. The IVX instrument, by construction, has lower persistence than a near-integrated variable and, as a consequence, delivers an asymptotically pivotal predictability statistic.

Kostakis *et al.* (2015) demonstrate that, under certain regularity conditions on the system innovations, IVX-based predictability statistics possess standard pivotal limiting null distributions regardless of whether the predictor is local-to-unity or weakly dependent (stationary). The asymptotic theory for IVX predictability statistics can, however, provide a very poor approximation to their finite sample behaviour, particularly for highly persistent and endogenous predictors which, as noted above, is arguably the case of most practical relevance. To ameliorate these finite sample distortions from the asymptotic theory, Kostakis *et al.* (2015) (see also Chevillon *et al.*, 2020) suggest a finite sample modification to the standard errors used in computing the IVX statistics. While this finite sample correction appears to work well for tests against two-sided alternatives reported in the simulation study for the case of a single regressor in Kostakis *et al.* (2015), as we will show in this paper, tests against one-sided alternatives remain very badly size-distorted for highly persistent and endogenous regressors. Moreover, Xu and Guo (2020) present simulation evidence which suggests that the quality of the prediction from the asymptotic theory, even with the finite sample correction employed, also markedly deteriorates as the number of regressors specified in the predictive regression is increased.

The regularity conditions required by Kostakis *et al.* (2015) to establish asymptotic mixed normality for their IVX estimator, which delivers the result that the associated IVX predictability statistics have standard pivotal limiting null distributions, include an assumption of unconditional homoskedasticity in the vector of innovations driving the predictive model. Although the conditions imposed in Kostakis *et al.* (2015) do allow for conditional heteroskedasticity in the innovation vector (provided heteroskedasticity-consistent standard errors are used in constructing their IVX test statistics) these conditions are rather restrictive in practice. In particular, even though a relatively weak martingale difference assumption is placed on the innovations driving the regressors, the errors in the predictive regression equations are assumed to follow a finite-order parametric GARCH model. This has the unfortunate consequence that it imposes the absence of any dependence of the conditional variance of the regression errors on lagged values of the innovations driving the predictors. This assumption is likely to be unrealistic for many predictors used to predict stock returns. Moreover, while GARCH models are very widely used in empirical finance, their usefulness for returns data is not uncontentious; see, for example, Carnero *et al.* (2004), who argue that the class of autoregressive stochastic volatility [ARSV] models is much better suited to capturing the main empirical properties of the volatility of financial returns series.

A major contribution of this paper is to address the foregoing issues with practical implementation of the IVX tests. First regarding the regularity conditions needed, we show that the IVX predictability tests of Kostakis *et al.* (2015) continue to deliver asymptotically pivotal inference, again regardless of the degree of persistence or endogeneity of the regressors, in cases where unconditional heteroskedasticity and/or conditional heteroskedasticity are allowed in the innovations, provided either a suitable bootstrap implementation of the test is employed or heteroskedasticity-consistent standard errors are used in the construction of the IVX test statistics. In particular, we establish the conditions required for asymptotic validity to hold for both of these approaches. These permit quite general patterns of unconditional time heteroskedasticity in the innovations, allowing not only for time-varying innovation variances but also the possibility of time-varying correlations between the innovations. Similarly we show that asymptotic validity holds for a much larger martingale difference class of innovations than considered in Kostakis *et al.* (2015) with no need to exclude interdependence between the conditional variances of the innovations in the model. Moreover, the practitioner is not required to assume a parametric model for either the conditional or unconditional time-variation in the innovations.

Second, and associatedly, in order to improve on their finite sample performance we also discuss bootstrap implementations of the IVX tests which are asymptotically valid under these conditions. Although there are papers already in the literature that consider the problem of bootstrapping mildly integrated variables, see Fan and Lee (2019) and Smeekes and Westerlund (2019), neither of these are capable of allowing for the generality of time-variation in the variance matrix of the vector of innovations that we consider here. Moreover, neither of these approaches is concerned with partial-sums based statistics. More relevant to the IVX tests of Kostakis *et al.* (2015) considered in this paper, Demetrescu *et al.* (2020), develop subsample implementations of the two-stage least squares (2SLS)-based predictability tests of Breitung and Demetrescu (2015) and base inference on a fixed regressor wild bootstrap [FRWB] resampling scheme. In this approach the regressor (and instrument in the case of Breitung and Demetrescu, 2015) is treated as fixed in the resampling exercise, while the returns series is resampled using a wild bootstrap scheme. Demetrescu *et al.* (2020) demonstrate that the FRWB approach correctly replicates the first-order limiting null distributions of the temporary predictability statistics they propose under conditional and unconditional heteroskedasticity of a similar form to that considered in this paper. The FRWB is also used by Georgiev *et al.* (2018, 2019) who develop tests for structural change in the predictive regression model.

The FRWB can also be used to successfully replicate the first-order limiting null distribution of the full sample IVX statistics under the conditions on the innovations considered in this paper. However, in Monte Carlo simulations we find that it does not address the finite sample distortions with the asymptotic IVX tests discussed above, most notably the distortions that occur when the regressor is highly persistent and endogenous. This is perhaps unsurprising given that the FRWB does not replicate in the bootstrap data the contemporaneous correlation present between the model's innovations. We therefore also discuss an alternative residual wild bootstrap [RWB] resampling scheme which is designed to replicate this correlation. Here we jointly wild resample the residuals from the fitted predictive regression model and a parametric autoregressive model fitted to the predictor. We also investigate the conditions under which the RWB-based IVX predictability tests are first-order asymptotically valid, and show that these deliver substantial improvements in finite sample behaviour relative to the asymptotic IVX tests.

Although the main application of the IVX methodology has been to predictive regressions for forecasting stock returns, it has also recently been applied to Fama regressions in the context of detecting episodic bubble-type behaviour in foreign exchange markets by Pavlidis *et al.* (2017). In their empirical analysis, Pavlidis *et al.* (2017) consider a rolling subsample-based implementation of one-sided IVX tests of Kostakis *et al.* (2015) and consider a test which rejects the null hypothesis of no bubble if any of the subsample statistics in the rolling sequence exceeds a given critical value. To avoid the inherent multiple testing bias, Pavlidis *et al.* (2017) base their approach on a conservative critical value obtained using a Bonferroni correction (i.e. adjusting the nominal significance level by the number of statistics in the rolling sequence). Pavlidis *et al.* (2017) note that this approach is likely to deliver a highly conservative test and suggest that a bootstrap implementation might deliver more powerful size controlled tests.

Tests based on the suprema of rolling and recursive subsample sequences of the 2SLS predictability tests of Breitung and Demetrescu (2015) have also been implemented recently in the context of detecting temporary periods of stock return predictability (so-called *pockets of predictability*) by Demetrescu *et al.* (2020). As noted above, Demetrescu *et al.* use a FRWB to implement these tests. The final contribution of this paper is to show that both the RWB and FRWB approaches can also be implemented in the context of the corresponding tests from sequences of subsample IVX statistics and that these are asymptotically valid under the same regularity conditions on the innovations as are required for the corresponding bootstrap implementations of the full sample tests. Moreover, unlike the 2SLS-based tests of Demetrescu *et al.* (2020) which can only be implemented as two-sided tests, these tests can be implemented as either one-sided or two-sided tests for the presence of temporary windows of predictability, so that more powerful tests can be obtained in cases where the direction of predictability under the alternative is know.

The remainder of the paper is organised as follows. Section 2, introduces the time-varying predictive regression model we consider together with the assumptions needed for our analysis. Section 3 reviews the standard full sample IV-based predictability tests of Kostakis *et al.* (2015) and details the subsample implementations of these statistics. Representations for the limiting distributions of these statistics under both the null and local alternatives are provided. These are shown to depend in general on any heteroskedasticity present, regardless of whether the putative predictor follows a strongly persistent process (modeled as near-integrated) or a weakly persistent process (modeled as a stable autoregression). Moreover, the form of these limiting distributions depends on whether the predictor is near-integrated or weakly dependent, even under homoskedasticity. In the context of the full sample IVX statistic, however, the use of Eicker-White standard errors is shown to deliver a standard pivotal limiting null distribution regardless of the predictor's persistence. Section 4 discusses bootstrap implementations of the IVX tests and demonstrates the first-order asymptotic validity of these. Section 5 presents the results from a Monte Carlo analysis into the finite sample behaviour of the tests. Concluding comments including some suggestions for further research are provided in Section 6. Detailed proofs of the technical results given in the paper along with other supporting material appear in a supplementary appendix.

In terms of notation, we use L to denote the lag operator, $Lw_t = w_{t-1}$, $\forall t$, and $\mathbb{I}(\cdot)$ to denote the indicator function, taking value one when its argument is true and zero otherwise. We furthermore denote by $\mathcal{D}^k$ the space of càdlàg real functions on $[0,1]^k$ equipped with the Skorokhod topology, and abbreviate $\mathcal{D}^1$ to $\mathcal{D}$. The weak convergence of probability measures on $\mathcal{D}^k$ and on $\mathbb{R}^k$ is denoted by $\Rightarrow$. We use the notation P, E etc. for probability, expectation etc. with respect to the distribution of the original data and use P*, E* etc. for probability, expectation etc. induced by the data and the wild bootstrap multipliers (which we shall denote $\{R_t\}$) conditionally on the data. If w_T, w ($T \in \mathbb{N}$) are random elements of metric spaces, the weak-in-probability convergence $w_T \overset{w}{\Rightarrow}_p w$ means that $\mathrm{E}^* f(w_T) \overset{p}{\to} \mathrm{E} f(w)$ for all continuous bounded real functions with matching domain. Finally, the probabilistic Landau symbols O_p and o_p have their usual meaning.

2 The Episodic Predictive Regression Model

Consider the predictive regression model for stock returns,[1] y_t, allowing for time-variation in the slope coefficient on a lagged predictor, x_{t-1}, of the form

$$y_t = \alpha + \beta_t x_{t-1} + u_t, \qquad t = 1, \dots, T, \tag{1}$$

where x_t satisfies the additive component model

$$x_t = \mu_x + \xi_t, \qquad t = 0, \dots, T, \tag{2}$$

$$\xi_t = \rho \xi_{t-1} + w_t, \qquad t = 1, \dots, T, \tag{3}$$

in which w_t is assumed to follow a pth order stable autoregression; that is, $A(L)\, w_t = v_t$ where $A(z) := (1 - a_1 z - a_2 z^2 - \cdots - a_p z^p)$. For future reference, we define $\omega := 1/A(1)$ and, for the case where x_t does follow a stable autoregression, we let κ^2 denote the sum of the squared coefficients of the filter $((1-\rho L)A(L))^{-1}$. In our exposition and technical analysis we follow the bulk of this literature and focus attention on the case of a single predictor; that is, where x_{t-1} in (1) is a scalar variable. Extensions to the case where the predictive regression contains multiple predictors will be discussed at various points in the text, although we leave a detailed treatment of this case for future research.

The DGP in (1) generalises the constant parameter predictive regression model considered in Kostakis *et al.* (2015) by allowing for the possibility that the slope coefficient on x_{t-1} varies over time, allowing for changes over time in the predictive content of the regressor x_{t-1}. The constant parameter predictive regression model obtains by setting a constant slope parameter such that $\beta_t = \beta$, for all $t = 1, \dots, T$. The tests we consider in this paper are all for the null hypothesis, H_0, that $(y_t - \alpha)$ is a MD sequence and, hence, that y_t is not predictable by x_{t-1}, which entails that $\beta_t = 0$, for all $t = 1, \dots, T$, in (1).[2] The full-sample

[1] This framework can also be applied to Fama regressions as is done in Pavlidis *et al.* (2017). Here $y_t = s_t - f_{t-1,1}$ and $x_t = f_{t,1} - s_t$ (the forward premium), where s_t is (the log of) the spot exchange rate at time t and $f_{t,1}$ is (the log of) the forward rate at time t for maturity at time $t+1$. The efficient market hypothesis then corresponds to $\beta_t = 0$, $t = 1, ..., T$, in (1), while an exchange rate bubble is present in any time periods where $\beta_t > 0$.

[2] The methods which we outline in this paper could equally well be used to test the null hypothesis that $\beta_t = \beta_0$ for all $t = 1, \dots T$, but as the focus in both equity forecasting and Fama regressions is on testing the null hypothesis of a zero coefficient on the lagged predictor we will restrict our discussion to $\beta_0 = 0$.

IVX tests of Kostakis *et al.* (2015) test the same null hypothesis, H_0, against the alternative that y_t is predictable by x_{t-1} with a constant slope parameter holding across the whole sample; that is, $\beta_t = \beta \neq 0$ for all $t = 1, \ldots, T$. The subsample implementations of IVX we discuss will be used to test against alternatives such that $\beta_t \neq 0$ for some t but without imposing constancy on β_t. In any case, some structure needs to be placed on the class of alternative hypotheses we may consider and this will be formalised below.

The degree of persistence of the regressor, x_t, is controlled via the parameter ρ. We allow x_t to be either weakly or strongly persistent through the following assumption.

Assumption 1 *Let the pth order lag polynomial $A(L)$ be invertible with characteristic roots bounded away from the complex unit circle and ξ_0 be a mean zero $O_p(1)$ variate. Moreover, exactly one of the two following conditions holds on ρ:*

1. ***Weakly persistent regressor:*** *The autoregressive parameter ρ in (3) is fixed and bounded away from unity, $|\rho| < 1$.*

2. ***Strongly persistent regressor:*** *The autoregressive parameter ρ in (3) is local-to-unity with $\rho := 1 - cT^{-1}$ where c is a fixed constant.*

Remark 1. Assumption 1 imposes the condition that the errors w_t in (3) follow a finite-order autoregression. This parametric assumption is imposed for the purposes of facilitating the RWB implementations of the full sample and subsample IVX tests proposed in section 4. Asymptotic versions of these tests (i.e. tests based on critical values from the limiting null distributions of the statistics) could equally well be based on a linear process assumption for w_t of the form considered in Assumption INNOV of Kostakis *et al.* (2015, p. 1512) or the slightly weaker Assumption M of Magdalinos (2020); in particular, Proposition 1 of this paper would remain valid in such cases. The FRWB implementations of the IVX tests discussed in section 4 would also be asymptotically valid under a linear process assumption of this form. Moreover, we conjecture that the RWB bootstrap tests would also be asymptotically valid in this case provided a sieve device is adopted in Step 2 of Algorithm 1 below, whereby the truncation lag for the fitted autoregression is allowed to increase at a suitable rate with the sample size, T. ◇

Remark 2. We follow the bulk of the literature on predictive regressions in considering regressors that follow either stable (weakly dependent) processes, see Amihud and Hurvich (2004), or are near-integrated, see Campbell and Yogo (2006), without assuming knowing of which of these is satisfied in the data. As we shall see, the limiting behavior of the IVX statistics can differ under the two types of persistence, but this can be consistently replicated (to asymptotic first order) by the bootstrap procedures we propose. ◇

The basic idea underlying the IVX procedure of Phillips and Magdalinos (2009) is to instrument the regressor x_{t-1} by a variable of controlled persistence, constructed as

$$z_0 = 0 \quad \text{and} \qquad z_t = (1 - \varrho L)_+^{-1} \Delta x_t := \sum_{j=0}^{t-1} \varrho^j \Delta x_{t-j}, t = 1, \ldots, T, \tag{4}$$

and where $\varrho := 1 - aT^{-\eta}$ with $0 < \eta < 1$. Where x_t is near-integrated satisfying Assumption 1.2, the instrument z_t is approximately a mildly integrated process and therefore of lower persistence than x_t. Moreover, where x_t is weakly dependent satisfying Assumption 1.1, we have that $z_t \approx x_t$. As a result, Kostakis *et al.* (2015) demonstrate that the IVX full-sample estimator of the slope parameter in (1) is asymptotically (mixed) Gaussian under H_0 regardless of whether Assumption 1.1 or Assumption 1.2 holds and that, consequently, the full-sample instrumental variable tests for H_0 they propose have standard limiting null distributions regardless of the degree of persistence or endogeneity of x_t.

For the purposes of this paper we follow Demetrescu *et al.* (2020) and conduct our theoretical analysis of the large sample properties of both the full-sample and sub-sample IVX predictability statistics under local alternatives such that the slope parameter β_t is local-to-zero for an asymptotically non-vanishing set of the sample observations. This is an important generalisation of the large sample results presented for the full sample IVX-based tests in Kostakis *et al.* (2015) and Magdalinos (2020) which only apply under H_0. The localisation rate (or Pitman drift) will need to be such that β_t is specified to lie in a neighbourhood of zero which shrinks with the sample size, T. The appropriate Pitman drift is dictated by which of Assumption 1.1 and Assumption 1.2 holds in (3); see also Demetrescu and Rodrigues (2020). Where x_t is near-integrated the appropriate rate is $T^{-1/2-\eta/2}$, while for weakly dependent x_{t-1}, the rate is $T^{-1/2}$. Formally, we specify β_t to satisfy the following assumption.

Assumption 2 *In the context of* (1)–(3)*, let* $\beta_t := n_T^{-1} b\,(t/T)$*, where* $b(\cdot)$ *is a piecewise Lipschitz-continuous real function on* $[0,1]$*, with* $n_T = \sqrt{T}$ *under Assumption 1.1, and* $n_T = T^{1/2+\eta/2}$ *under Assumption 1.2.*

Under the structure of Assumption 2, the null hypothesis H_0 that $\beta_t = 0$, for all $t = 1, \ldots, T$, can be expressed as

$$H_0 : \textit{The function}\, b(\cdot)\ \textit{is identically zero on}\ [0,1], \tag{5}$$

while the alternative hypothesis can be written as

$$H_{1,b(\cdot)} : \textit{The function}\, b(\cdot)\ \textit{is non-zero over at least one non-empty open subinterval of}\ [0,1]. \tag{6}$$

The latter entails that at least one subset of the sample observations (this need not be a strict subset, so it could contain all of the sample observations) comprising contiguous observations exists for which $\beta_t \neq 0$, and where the size of this subset is proportional to the sample size T. One-sided alternatives that $\beta_t > 0$ ($\beta_t < 0$) in some subset(s) of the data can be considered simply by defining $b\,(\cdot)$ to be a non-negative (non-positive) function.

We conclude this section by detailing in Assumption 3 the conditions that we will place on the disturbances u_t and v_t in (1) and (3), respectively. Subsequently we will provide some discussion of these conditions before providing the key (multivariate) invariance principles that hold under these conditions.

Assumption 3 *Let*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} := \mathbf{H}\left(\frac{t}{T}\right)\begin{pmatrix} a_t \\ e_t \end{pmatrix},$$

where:

1. $\mathbf{H}(\cdot) := \begin{pmatrix} h_{11}(\cdot) & h_{12}(\cdot) \\ h_{21}(\cdot) & h_{22}(\cdot) \end{pmatrix}$ *is a matrix of piecewise Lipschitz-continuous bounded functions on* $(-\infty, 1]$*, which is of full rank at all but a finite number of points;*

2. $\boldsymbol{\psi}_t := (a_t, e_t)'$ *is a* L_4*-bounded stationary and ergodic martingale difference sequence satisfying* $\mathrm{E}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t') = \mathbf{I}_2$ *and* $\mathrm{E}\|\mathrm{E}_0 \sum_{t=1}^{T}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2)\|^2 = O(T^{2\epsilon})$ *for some* $\epsilon < \frac{1}{2}$*, with* $\mathrm{E}_0(\cdot)$ *denoting expectation conditional on* $\{\boldsymbol{\psi}_{-i}\}_{i=0}^{\infty}$ *and* $\mathbf{I}_k$ *denoting the* $k \times k$ *identity matrix.*

Remark 3. Assumption 3 is similar to Assumption 3 of Demetrescu *et al.* (2020) and we refer the reader to Demetrescu *et al.* (2020) for a detailed discussion of these conditions. Briefly, Assumption 3.1 allows for unconditional time heteroskedasticity of quite general form in the innovations through the function $\mathbf{H}$, whereby the unconditional covariance matrix of $(u_t, v_t)'$ is given by $\mathbf{H}(t/T)\mathbf{H}'(t/T)$. This structure allows both u_t and v_t to display time-varying unconditional variances and for both contemporaneous and time-varying (unconditional) correlation between u_t and v_t. Empirically plausible models of single or multiple (co-) variance shifts, (co-)variances which follow a broken trend, and smooth transition (co-) variance shifts are all permitted under this assumption. In contrast, Assumption INNOV of Kostakis *et al.* (2015, p. 1512) and Assumption M of Magdalinos (2020) impose a constant unconditional variance matrix on $(u_t, v_t)'$. Assumption 3.2 imposes a martingale difference [MD] structure on $\boldsymbol{\psi}_t$ thereby allowing for conditional heteroskedasticity. In common with Assumption INNOV of Kostakis *et al.* (2015) and Assumption M of Magdalinos (2020), Assumption 3.2 imposes finite fourth-order moments on $\boldsymbol{\psi}_t$. ◇

Remark 4. As we will see below, in order to establish the large sample properties of the IVX tests of Kostakis *et al.* (2015) in the strong persistence case we rely on a weak convergence result for $\frac{1}{\sqrt{T^{1+\eta}}}\sum_{t=1}^{[\tau T]} z_{t-1}u_t$. For the case of full-sample sums, Kostakis *et al.* (2015) and Magdalinos (2020) make the parametric assumption that u_t is generated by a stationary finite-order GARCH(p, q) model with finite fourth moments. This assumption therefore has the consequence that it imposes the absence of any dependence of the conditional variance of u_t on lags of v_t which is likely to be unrealistic for many predictors used to predict stock returns; see Example 1 in the supplementary appendix for further discussion on this point. Moreover, a number of authors, including Carnero *et al.* (2004) and Johannes *et al.* (2014) argue that ARSV models capture the main empirical properties of the volatility of financial returns series better than GARCH models. To eliminate the need to choose a specific parametric volatility model, Assumption 3.2 instead adopts an explicit assumption of martingale approximability whereby $\mathrm{E}\|\mathrm{E}_0 \sum_{t=1}^{T}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2)\|^2 = O(T^{2\epsilon})$ for some $\epsilon < \frac{1}{2}$, see Merlevède *et al.* (2006). The exponent ϵ controls the degree of persistence permitted in the conditional variances of the innovations. Stationary vector GARCH processes with finite fourth-order moments satisfy Assumption 3.2 with $\epsilon = 0$, but the assumption is considerably more general as it also allows for asymmetric effects in the conditional variance. Stationary ARSV processes as, for example, are assumed in Johannes *et al.* (2014) also satisfy Assumption 3.2. ♢

Under Assumption 1.1 (weak persistence), $\xi_t = (1-\rho L)_+^{-1} A(L)^{-1} v_t + \rho^t \xi_0$, which, given the exponential decay of the coefficients under weak persistence, is asymptotically equivalent to the process $(1-\rho L)^{-1} A(L)^{-1} v_t$, and with a slight abuse of notation, we will write $\xi_t = (1-\rho L)^{-1} A(L)^{-1} v_t$ in what follows, ignoring the asymptotically negligible term. Under Assumption 3, the normalised partial sums of $(u_t, v_t, \xi_{t-1} u_t)$ in the weak persistence case satisfy the multivariate invariance principle,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor \tau T \rfloor} \begin{pmatrix} u_t \\ v_t \\ \xi_{t-1} u_t \end{pmatrix} \Rightarrow \int_0^\tau \mathbf{G}(s) \mathrm{d}\boldsymbol{B}(s) := \begin{pmatrix} M_u(\tau) \\ M_v(\tau) \\ M_{\xi u}(\tau) \end{pmatrix} \tag{7}$$

on $\mathcal{D}^3$, where $\mathbf{G}(\tau)$ is a 3×6 matrix of piecewise Lipschitz functions whose elements are formed from the elements of $\mathbf{H}(\tau)$, and where $\boldsymbol{B}(\tau)$ is a 6-dimensional Brownian motion. Explicit expressions for the covariance matrix of $\boldsymbol{B}(\tau)$ and for $\mathbf{G}(\tau)$ are provided in Lemma 4 in the supplementary appendix, where the convergence result in (7) is also formally established. Using the well-known Phillips-Solo device, it is straightforwardly obtained from (7) that the suitably normalised partial sums of ξ_t weakly converge to $\omega/(1-\rho) M_v$.

Remark 5. The limiting processes M_u, M_v and $M_{\xi u}$ in (7) are individually variance-transformed Brownian motions; cf. Davidson (1994, section 29.4). These three processes are, in general, correlated under Assumption 3, and indeed this correlation can be time-varying; see the supplementary appendix for precise expressions. Under conditional homoskedasticity, $M_{\xi u}$ can be seen to be uncorrelated with either M_u or M_v. Under conditional heteroskedasticity, however, M_v and $M_{\xi u}$ are in general dependent (as are M_u and $M_{\xi u}$), even where $\mathbf{H}(\tau)$ is constant, because $\mathrm{Cov}(\xi_{t-1} u_t, v_t)$ is not necessarily zero if the conditional correlation between u_t and v_t is nonzero. Where $\mathbf{H}(\tau)$ is constant, such that $(u_t, v_t)'$ is unconditionally homoskedastic, $\int_0^\tau \mathbf{G}(s) \mathrm{d}\boldsymbol{B}(s)$ reduces to a standard Brownian motion process. Where $\mathbf{H}(\tau)$ is non-constant the variance profiles of M_u, M_v and $M_{\xi u}$ will, in general, differ (we define the variance profile of a generic stochastic process $W(s)$ as $[W](s)/[W](1)$ where $[W](s)$ denotes the quadratic variation process of $W(s)$). Even in the special case where $\mathbf{H}(\tau)$ is a scalar multiple of the identity matrix, although M_u and M_v will share the same variance profile, this will not in general coincide with variance profile of $M_{\xi u}$ because the variance of its increments is a polynomial of degree four in the elements of $\mathbf{H}(\tau)$, while those of M_u and M_v are both polynomials of degree two (see the proof of Lemma 4 in the supplementary appendix). ♢

Under Assumption 1.2 (strong persistence), the normalized partial sums of (u_t, v_t) converge as previously to (M_u, M_v), where M_u and M_v are the same limiting processes as in

(7). Moreover, the normalized partial sums of $(v_t, \frac{1}{\sqrt{T^\eta}} z_{t-1} u_t)$ converge weakly as well,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[\tau T]} \begin{pmatrix} v_t \\ \frac{1}{\sqrt{T^\eta}} z_{t-1} u_t \end{pmatrix} \Rightarrow \begin{pmatrix} M_v(\tau) \\ M_{zu}(\tau) \end{pmatrix} \tag{8}$$

on $\mathcal{D}^2$, with $M_{zu}(\tau) := \frac{\omega}{\sqrt{2a}} \int_0^\tau \sqrt{[M_v]'(s)[M_u]'(s)} \mathrm{d}B(s)$, where B is a standard Brownian motion independent of M_v, and where $[M_v]'(s)$ and $[M_u]'(s)$ denote the derivatives (with respect to s) of $[M_v](s)$ and $[M_u](s)$, respectively. These derivatives are well-defined at all but finitely many $s \in [0, 1]$, see Lemma 3 in the Supplementary Appendix. Convergence (8) is established in Lemma 5 in the Supplementary Appendix. Under strong persistence, the levels of ξ_t satisfy the weak convergence result $T^{-1/2}\xi_{\lfloor \tau T \rfloor} \Rightarrow \omega J_{c,\mathrm{H}}(\tau)$, where $J_{c,\mathrm{H}}(\tau)$ is an Ornstein-Uhlenbeck-type process driven by $M_v(\tau)$; that is, $J_{c,\mathrm{H}}(\tau) := \int_0^\tau e^{-c(\tau-s)} \mathrm{d}M_v(s)$.

Remark 6. The limiting process M_{zu} in (8) is a variance-transformed Brownian motion as well. An important difference between the invariance principles in (7) and (8) is that M_{zu} is independent of M_v irrespective of any conditional heteroskedasticity while, as discussed in Remark 5, $M_{\xi u}$ and M_v are in general dependent. Another important difference between the invariance principles under weak and strong persistence is that the processes $M_{\xi u}$ and M_{zu}, despite being driven by the same innovations, can have quite different behaviour depending on the pattern of conditional and unconditional heteroskedasticity present in $\boldsymbol{\psi}_t$. To illustrate, under unconditional heteroskedasticity the variance profiles of $M_{\xi u}$ and M_{zu} will in general differ where conditional heteroskedasticity is also present; see Example 2 in the Supplementary Appendix. ◇

3 IVX-based Predictability Tests

In section 3.1 we first outline the full sample IVX-based predictability tests of Kostakis *et al.* (2015). In section 3.2 we then discuss subsample-based implementations of these tests. The limiting distributions of the full sample and subsample IVX statistics are established under the local alternative in section 3.3. Here we show that in the case of the full-sample IVX statistics basing these on Eicker-White standard errors yields standard normal limiting null distributions. For the subsample based statistics these will still depend, in general, on any heteroskedasticity present in the innovations.

3.1 Full-sample IVX tests

The full-sample IVX-based t-ratio, proposed in Kostakis *et al.* (2015), for testing the null hypothesis $H_0 : \beta_t = 0$ for all $t = 1, ..., T$ in (1) is given by

$$t_{zx} := \frac{\hat{\beta}_{zx}}{s.e.(\hat{\beta}_{zx})} \tag{9}$$

where $\hat{\beta}_{zx}$ is the IVX estimator of β,

$$\hat{\beta}_{zx} := \frac{\sum_{t=1}^{T} z_{t-1}(y_t - \bar{y})}{\sum_{t=1}^{T} z_{t-1}(x_{t-1} - \bar{x}_{-1})} \tag{10}$$

with $\bar{y} := T^{-1}\sum_{t=1}^{T} y_t$ and $\bar{x}_{-1} := T^{-1}\sum_{t=1}^{T} x_{t-1}$, and[3]

$$s.e.(\hat{\beta}_{zx}) := \frac{\sqrt{\hat{\sigma}_u^2 \sum_{t=1}^{T} z_{t-1}^2}}{\sum_{t=1}^{T} z_{t-1}(x_{t-1} - \bar{x}_{-1})} \tag{11}$$

with $\hat{\sigma}_u^2 = T^{-1}\sum_{t=1}^{T} \hat{u}_t^2$. A variety of choices for the residuals $\hat{u}_t$ is possible. Both Breitung and Demetrescu (2015) and Kostakis *et al.* (2015) recommend the use of the OLS residuals from estimating (1) on the grounds that they come from the best linear projection of y_t on x_{t-1} regardless of the persistence of the putative predictor, and that their finite-sample behaviour appears to be more stable than that of the corresponding IV residuals. One could also use residuals computed under the null; that is, $\hat{u}_t := y_t - \frac{1}{T}\sum_{s=1}^{T} y_s$. Under the local alternatives considered in Assumption 2, these two possible choices can be shown to be asymptotically equivalent to one another in so far as the behaviour of the resulting IVX statistic is concerned. Given that the IV residuals have reduced convergence rates compared to the two possible choices above, we shall not consider them in the following.

One-sided tests based on t_{zx} can be formed by rejecting against the right-sided alternative that $\beta_t = \beta > 0$, for all $t = 1, ..., T$, for large positive values of the statistics and against the left-sided alternative that $\beta_t = \beta < 0$, for all $t = 1, ..., T$, for large negative values of the statistics. The latter can be equivalently implemented as right-sided tests simply by replacing the predictor x_{t-1} by $-x_{t-1}$. Two-sided tests can be formed by rejecting against the alternative that $\beta_t = \beta \neq 0$, for all $t = 1, ..., T$, for large positive values of $(t_{zx})^2$.

Remark 7. In order to correct for the finite sample effects of estimating the intercept term in (1), which are most pronounced for highly persistent regressors that are strongly correlated with the predictive model's innovations, Kostakis *et al.* (2015, p. 1516) recommend the use of a finite-sample correction factor; see also the discussion in Demetrescu and Hosseinkouchack (2020). This entails replacing the numerator of (11) by $\sqrt{\hat{\sigma}_u^2 \sum_{t=1}^{T} z_{t-1}^2 - \Xi}$ where Ξ is the finite-sample correction factor given by $\Xi := T\bar{z}_{-1}^2(\hat{\sigma}_u^2 - \hat{\sigma}_{uw}^2\hat{\sigma}_w^{-2})$, with $\bar{z}_{-1} := T^{-1}\sum_{t=1}^{T} z_{t-1}$, and where $\hat{\sigma}_w^2$ and $\hat{\sigma}_{uw}$ are estimates of the long-run variance of w_t, and of the long-run covariance between u_t and w_t, respectively; a discussion on the practical choice of these estimators is provided in Kostakis *et al.* (2015, pp. 1513 and 1524). The inclusion of this correction factor does not alter any of the large sample results that follow. $\diamondsuit$

Remark 8. Kostakis *et al.* (2015) also consider a variant of the t_{zx} statistic based on the use of heteroskedasticity-robust standard errors. Replacing the conventional standard error,

[3]Notice that, as discussed in Kostakis *et al.* (2015, p. 1514), z_{t-1} does not need to be demeaned in (10) because the IV estimator, $\hat{\beta}_{zx}$, is invariant to whether z_{t-1} is demeaned or not.

$s.e.(\hat{\beta}_{zx})$, in (9) by the corresponding Eicker-White standard error,

$$s.e.^{EW}(\hat{\beta}_{zx}) := \frac{\sqrt{\sum_{t=1}^{T} z_{t-1}^2 \hat{u}_t^2}}{\sum_{t=1}^{T} z_{t-1} (x_{t-1} - \bar{x}_{-1})} \tag{12}$$

the Eicker-White form of the IVX t-ratio is then defined as

$$t_{zx}^{EW} := \frac{\hat{\beta}_{zx}}{s.e.^{EW}(\hat{\beta}_{zx})}. \tag{13}$$

As we will show in section 3.3, the t_{zx}^{EW} statistic has a standard normal limiting null distribution even under unconditional and/or conditional heteroskedasticity of the form specified in Assumption 3, regardless of whether x_t is strongly or weakly persistent. Kostakis *et al.* (2015) and Magdalinos (2020) have previously shown that this result holds under unconditional homoskedasticity and for the form of conditional heteroskedasticity they assume which as discussed in section 2 is a special case of our Assumption 3.2. The same result is also true for the t_{zx} statistic based on conventional standard errors in the strongly persistent case when the innovations are unconditionally homoskedastic, but does not hold in general otherwise. The finite sample correction factor Ξ discussed in Remark 7 can also be applied to the numerator of (12). ◇

Remark 9. Kostakis *et al.* (2015) consider the more general set-up of multiple predictive regressions of the form $y_t = \alpha + \boldsymbol{\beta}'\boldsymbol{x}_{t-1} + u_t$, $t = 1, \ldots, T$, where $\boldsymbol{\beta} := (\beta_1, ..., \beta_k)'$ and where $\boldsymbol{x}_t := (x_{1,t}, ..., x_{k,t})'$ is such that $\boldsymbol{x}_t = \boldsymbol{\mu}_{\boldsymbol{x}} + \boldsymbol{\xi}_t$ where $\boldsymbol{\xi}_t$ satisfies the k-dimensional generalisation of (3), $\boldsymbol{\xi}_t = \boldsymbol{\Gamma}\boldsymbol{\xi}_{t-1} + \boldsymbol{v}_t$, $t = 1, \ldots, T$, and where $\boldsymbol{\mu}_{\boldsymbol{x}}$ is a k-vector of constants. Kostakis *et al.* (2015) specify the matrix $\boldsymbol{\Gamma}$ to be diagonal with ith diagonal element ρ_i, $i = 1, ..., k$, and assume that the predictors all lie within the same persistence class; that is, the $x_{i,t}$, $i = 1, ..., k$, either all satisfy Assumption 1.1, or they all satisfy Assumption 1.2. Generating the set of k instruments, $\boldsymbol{z}_t := (z_{1,t}, ..., z_{k,t})'$, from the predictors $x_{i,t}$, $i = 1, ..., k$, each generated according to (4), a two-sided Wald-type IVX based test rejects the null $\mathbf{R}\boldsymbol{\beta} = \mathbf{0}$, where $\mathbf{R}$ is a known $q \times k$ matrix of full row rank, for large values of $W_{zx}^{\mathbf{R}} := \hat{\boldsymbol{\beta}}_{zx}'\mathbf{R}'(\mathbf{R}\widehat{\text{Cov}}(\hat{\boldsymbol{\beta}}_{zx})\mathbf{R}')^{-1}\mathbf{R}\hat{\boldsymbol{\beta}}_{zx}$ where $\hat{\boldsymbol{\beta}}_{zx} := \boldsymbol{A}_T^{-1}\boldsymbol{C}_T$ with $\boldsymbol{A}_T := \sum_{t=1}^{T} \boldsymbol{z}_{t-1}(\boldsymbol{x}_{t-1} - \bar{\boldsymbol{x}}_{-1})'$, $\boldsymbol{C}_T := \sum_{t=1}^{T} \boldsymbol{z}_{t-1}(y_t - \bar{y})$, $\bar{\boldsymbol{x}}_{-1} := T^{-1}\sum_{t=1}^{T} \boldsymbol{x}_{t-1}$, and where $\widehat{\text{Cov}}(\hat{\boldsymbol{\beta}}_{zx}) := \hat{\sigma}_u^2 \boldsymbol{A}_T^{-1}\boldsymbol{B}_T(\boldsymbol{A}_T^{-1})'$ with $\boldsymbol{B}_T := \sum_{t=1}^{T} \boldsymbol{z}_{t-1}\boldsymbol{z}_{t-1}'$, $\hat{\sigma}_u^2 := T^{-1}\sum_{t=1}^{2} \hat{u}_t^2$ and $\hat{u}_t$ being the residuals of the estimated predictive regression. An Eicker-White version of $W_{zx}^{\mathbf{R}}$ can be formed by replacing $\hat{\sigma}_u^2 \mathbf{B}_T$ in the expression of $\widehat{\text{Cov}}(\hat{\boldsymbol{\beta}}_{zx})$ with $\mathbf{D}_T := \sum_{t=1}^{T} \boldsymbol{z}_{t-1}\boldsymbol{z}_{t-1}'\hat{u}_t^2$. A finite sample correction factor can again be used; see Kostakis *et al.* (2015, p. 1515) for precise details. IVX (partial) t-type tests of the null hypothesis $\beta_i = 0$, $i \in \{1, ..., k\}$, can also be considered. ◇

3.2 Subsample IVX Tests

As we will subsequently show in Proposition 1, the full-sample test based on t_{zx} has non-trivial asymptotic local power against $H_{1,b(\cdot)}$ of (6) for both weakly and strongly persistent regressors. However, these tests are clearly designed for the case where the function $b(\cdot)$ of Assumption 2 is such that $b(t/T) = b$, $t = 1, ..., T$. If it were known that a pocket of predictability might occur only over the particular subsample $t = \lfloor \tau_1 T \rfloor + 1, \ldots, \lfloor \tau_2 T \rfloor$, such that $b(t/T) = b$ for $t = \lfloor \tau_1 T \rfloor + 1, \ldots, \lfloor \tau_2 T \rfloor$ but was zero elsewhere, then it would be more logical to base a test for this on the IVX statistic computed only on the subsample $t = \lfloor \tau_1 T \rfloor + 1, \ldots, \lfloor \tau_2 T \rfloor$, viz,

$$t_{zx}(\tau_1, \tau_2) := \frac{\hat{\beta}_{zx}(\tau_1, \tau_2)}{s.e.(\hat{\beta}_{zx}(\tau_1, \tau_2))} \tag{14}$$

where

$$\hat{\beta}_{zx}(\tau_1, \tau_2) := \frac{\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} z_{t-1} \left(y_t - \bar{y}(\tau_1, \tau_2) \right)}{\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} z_{t-1} \left(x_{t-1} - \bar{x}_{-1}(\tau_1, \tau_2) \right)} \tag{15}$$

$$s.e.(\hat{\beta}_{zx}(\tau_1, \tau_2)) := \frac{\hat{\sigma}_u(\tau_1, \tau_2) \sqrt{\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} z_{t-1}^2}}{\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} z_{t-1} \left(x_{t-1} - \bar{x}_{-1}(\tau_1, \tau_2) \right)} \tag{16}$$

with $\bar{y}(\tau_1, \tau_2) := (T^*)^{-1} \sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} y_t$, $\bar{x}_{-1}(\tau_1, \tau_2) := (T^*)^{-1} \sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} x_{t-1}$, where $T^* := (\lfloor \tau_2 T \rfloor - \lfloor \tau_1 T \rfloor)$, and where $\hat{\sigma}_u(\tau_1, \tau_2)^2$ is the analogue of $\hat{\sigma}_u^2$ in (11) computed for the subsample $t = \lfloor \tau_1 T \rfloor + 1, \ldots, \lfloor \tau_2 T \rfloor$. The corresponding subsample analogue of the full sample Eicker-White t_{zx}^{EW} statistic in (13) can be defined similarly and will be denoted $t_{zx}^{EW}(\tau_1, \tau_2)$.

In practice it is unlikely the practitioner will know which specific subsample(s) of the data might admit predictive regimes. As discussed in Demetrescu *et al.* (2020), a conventional approach in such cases is to base tests on certain functionals of sequences of subsample predictability statistics. These sequences need to be agnostic of the data to avoid any endogenous selection bias and any test formed from them must be such that multiple testing issues are also avoided. Given we are testing the null of no predictability against the alternative of predictability in at least one subsample of the data, an approach based on the maximum (in the case of two-sided and right-tailed tests) or minimum (in the case of left-sided tests) of the sequence of subsample predictability statistics would seem appropriate.

Common choices of such agnostic sequences of statistics include forward and reverse recursive sequences and rolling sequences, and we will use those here. Tests based on the forward recursive sequence of statistics are designed to detect pockets of predictability which begin at or near the start of the full sample period, while those based on the reverse recursive sequence are designed to detect end-of-sample pockets of predictability. For a given window width, tests based on a rolling sequence of statistics are designed to pick up a window of predictability, of (roughly) the same length, within the data.

The subsample IVX tests we propose based on these sequences of subsample statistics are then formally defined as follows. We will outline these for the case of IVX statistics computed with conventional standard errors, but these can also be implemented with Eicker-White standard errors as in Remark 8 by replacing $t_{zx}(\cdot, \cdot)$ with $t_{zx}^{EW}(\cdot, \cdot)$ throughout.

• The sequence of *forward recursive* statistics is given by $\{t_{zx}(0,\tau)\}_{\tau_L \le \tau \le 1}$, where the parameter $\tau_L \in (0,1)$ is chosen by the user. The forward recursive regression approach uses $\lfloor T\tau_L \rfloor$ start-up observations, where τ_L is the *warm-in* fraction, and then calculates the sequence of subsample predictive regression statistics $t_{zx}(0,\tau)$ for $t = 1, ..., \lfloor \tau T \rfloor$, with τ travelling across the interval $[\tau_L, 1]$. An upper-tailed test can then be based on the maximum taken across this sequence, *viz*,

$$\mathcal{T}_U^F := \max_{\tau_L \le \tau \le 1} \{t_{zx}(0,\tau)\}. \tag{17}$$

The corresponding left-tailed test can be based on the minimum across this sequence, denoted $\mathcal{T}_L^F$, and a two-tailed test can be based on the corresponding maximum taken over the sequence of $(t_{zx}(0,\tau))^2$ statistics, denoted $\mathcal{T}_2^F$.

• The sequence of *backward recursive* statistics is given by $\{t_{zx}(\tau,1)\}_{0 \le \tau \le \tau_U}$ with $\tau_U \in (0,1)$ again chosen by the user. Here one calculates the sequence of subsample predictive regression statistics $t_{zx}(\tau,1)$ for $t = \lfloor \tau T \rfloor + 1, ..., T$, with τ travelling across the interval $[0, \tau_U]$. Analogously to the forward recursive case, an upper-tailed test can again be based on the maximum from this sequence,

$$\mathcal{T}_U^B := \max_{0 \le \tau \le \tau_U} \{t_{zx}(\tau,1)\} \tag{18}$$

while corresponding lower-tailed tests and two-sided tests can be formed from the statistics $\mathcal{T}_L^B$ and $\mathcal{T}_2^B$, defined analogously to the forward recursive case.

• The sequence of *rolling* statistics is given by $\{t_{zx}(\tau, \tau + \Delta\tau)\}_{0 \le \tau \le 1-\Delta\tau}$ where the user-defined parameter $\Delta\tau \in (0,1)$. Here one calculates the sequence of subsample statistics $t_{zx}(\tau, \tau+\Delta\tau)$ for $t = \lfloor \tau T \rfloor + 1, ..., \lfloor \tau T \rfloor + \lfloor T\Delta\tau \rfloor$, where $\Delta\tau$ is the window fraction with $\lfloor T\Delta\tau \rfloor$ the window width, with τ travelling across the interval $[0, 1-\Delta\tau]$. An upper-tailed test can again be based on the maximum from this rolling sequence,

$$\mathcal{T}_U^R := \max_{0 \le \tau \le 1-\Delta\tau} \{t_{zx}(\tau, \tau+\Delta\tau)\} \tag{19}$$

while corresponding lower-tailed tests and two-sided tests can again be formed from the statistics $\mathcal{T}_L^R$ and $\mathcal{T}_2^R$, defined analogously to the recursive cases.

Remark 10. Notice that the full sample IVX statistic t_{zx} of (9) is contained within the forward recursive, backward recursive, and rolling sequences of statistics and obtains by setting $\tau = 1$, $\tau = 0$, and $\Delta\tau = 1$, respectively, in those sequences. ♢

Remark 11. Subsample implementations of the multiple predictor IVX Wald tests discussed in Remark 9 can also be defined in an analogous fashion to $\mathcal{T}_U^F$, $\mathcal{T}_U^B$ and $\mathcal{T}_U^R$ of (17), (18) and (19), respectively. Here, defining the subsample analogue of the IVX Wald statistic $W_{zx}^{\mathbf{R}}$ computed over the data subsample $t = \lfloor \tau_1 T \rfloor + 1, \ldots, \lfloor \tau_2 T \rfloor$, as $W_{zx}^{\mathbf{R}}(\tau_1, \tau_2)$, we can consider tests which reject for large values of the maxima from analogous forward recursive, backward recursive and rolling sequences of such subsample statistics, which we will denote $\mathcal{W}_F^{\mathbf{R}}$, $\mathcal{W}_B^{\mathbf{R}}$ and $\mathcal{W}_R^{\mathbf{R}}$, respectively. ♢

Tests based on recursive and rolling sequences of subsample statistics have also been proposed in the literature on testing for episodic bubbles; see, in particular, Phillips *et al.* (2011) and Homm and Breitung (2012). Pavlidis *et al.* (2017) propose tests for detecting episodic bubbles in foreign exchange markets by considering a rolling subsample-based implementation of the right-sided IVX t-ratios of Kostakis *et al.* (2015) applied to Fama regressions (see footnote 1) estimated over a rolling sequence of subsamples of the data. They consider a test which rejects the null hypothesis of no bubble if any of the subsample statistics in the rolling sequence exceeds a given critical value. In order to deliver size-controlled inference, they base their test on a conservative critical value obtained using a Bonferroni correction, adjusting the nominal significance level by the number of statistics in the sequence. Given that the number of statistics in the rolling sequence will generally be quite large (for a given sample size, T, the number of statistics in the sequence will be larger the smaller the rolling window width, $\lfloor T\Delta\tau \rfloor$), Pavlidis *et al.* (2017) acknowledge that this approach will deliver a very conservative test. The methods developed in this paper therefore provide an alternative test to that in Pavlidis *et al.* (2017), likely to be considerably more powerful, based on the maximum from the rolling sequence of statistics.

Demetrescu *et al.* (2020) also consider tests for episodic predictability based on the maxima from corresponding sequences of rolling and recursive subsample implementations of a 2SLS predictability statistic as discussed by Breitung and Demetrescu (2015). As a necessary consequence of overidentified IV inference with strictly exogenous instruments, the approach proposed in Demetrescu *et al.* (2020) can only be used to test against two-sided alternatives, while as we have seen the subsample IVX-based tests considered in this paper can be used to test against either one-sided or two-sided alternatives. Where, as is often the case, theory predicts the sign of the slope parameter on x_{t-1} under predictability, being able to consider one-sided tests will clearly deliver tests with greater power relative to two-sided testing.

3.3 Asymptotic Theory

In this section we provide limiting distribution theory for the IVX statistics from sections 3.1 and 3.2 in Proposition 1. In Proposition 2 we then provide the limiting null distribution of the Eicker-White form of the IVX statistic, t_{zx}^{EW} in (13). Some remarks follow both Propositions including a comparison with existing large sample results available in the literature.

Proposition 1 *Consider the model in* (1)–(3) *and let Assumptions 2 and 3 hold. Then under the local alternative* $H_{1,b(\cdot)}$ *of* (6)*:*

(i) *Under Assumption 1.1, as* $T \to \infty$

$$
\begin{aligned}
t_{zx}(\tau_1,\tau_2) &\Rightarrow \frac{M_{\xi u}(\tau_2)-M_{\xi u}(\tau_1)+\kappa^2\int_{\tau_1}^{\tau_2}[M_v]'(s)b(s)\mathrm{d}s}{\sqrt{\frac{\kappa^2}{\tau_2-\tau_1}\left([M_u](\tau_2)-[M_u](\tau_1)\right)\left([M_v](\tau_2)-[M_v](\tau_1)\right)}} := G_1(b,\tau_1,\tau_2);\\
\mathcal{T}_U^F &\Rightarrow \sup_{\tau\in[\tau_L,1]}\{G_1(b,0,\tau)\} := G_{1,U}^F(b);\\
\mathcal{T}_U^B &\Rightarrow \sup_{\tau\in[0,\tau_U]}\{G_1(b,\tau,1)\} := G_{1,U}^B(b);\\
\mathcal{T}_U^R &\Rightarrow \sup_{\tau\in[0,1-\Delta\tau]}\{G_1(b,\tau,\tau+\Delta\tau)\} := G_{1,U}^R(b).
\end{aligned}
$$

(ii) *Under Assumption 1.2, and with* $\epsilon < \min\{1-\eta, \frac{1}{2}\eta\}$ *in Assumption 3,*

$$
\begin{aligned}
t_{zx}(\tau_1,\tau_2) &\Rightarrow \frac{M_{zu}(\tau_2)-M_{zu}(\tau_1)}{\sqrt{\frac{1}{(\tau_2-\tau_1)}\left([M_u](\tau_2)-[M_u](\tau_1)\right)\left([M_v](\tau_2)-[M_v](\tau_1)\right)}}\\
&\quad+\sqrt{\frac{2\omega^2}{a}}\frac{J_{c,\mathrm{H}}Z_b|_{\tau_1}^{\tau_2}-\int_{\tau_1}^{\tau_2}Z_b(s)\mathrm{d}J_{c,\mathrm{H}}(s)-\frac{1}{\tau_2-\tau_1}Z_b|_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2}J_{c,\mathrm{H}}(s)\mathrm{d}s}{\sqrt{\frac{1}{\tau_2-\tau_1}\left([M_u](\tau_2)-[M_u](\tau_1)\right)\left([M_v](\tau_2)-[M_v](\tau_1)\right)}} := G_2(b,\tau_1,\tau_2);\\
\mathcal{T}_U^F &\Rightarrow \sup_{\tau\in[\tau_L,1]}\{G_2(b,0,\tau)\} := G_{2,U}^F(b);\\
\mathcal{T}_U^B &\Rightarrow \sup_{\tau\in[0,\tau_U]}\{G_2(b,\tau,1)\} := G_{2,U}^B(b);\\
\mathcal{T}_U^R &\Rightarrow \sup_{\tau\in[0,1-\Delta\tau]}\{G_2(b,\tau,\tau+\Delta\tau)\} := G_{2,U}^R(b),
\end{aligned}
$$

where a *and* η *are the parameters defining the IVX filter in (4),* ω *and* κ^2 *are as defined in section 2,* $Z_b(\tau) := b(\tau)J_{c,\mathrm{H}}(\tau)-\int_0^\tau J_{c,\mathrm{H}}(s)\mathrm{d}b(s)$*, and for a generic stochastic process* $W(r)$*,* $W|_{r_1}^{r_2} := W(r_2)-W(r_1)$*. The results for* $t_{zx}(\tau_1,\tau_2)$ *hold for any given fixed values of* τ_1 *and* τ_2*,* $0 \le \tau_1 < \tau_2 \le 1$*.*

Remark 12. Corresponding representations for the limiting distributions of the left-sided $\mathcal{T}_L^F$, $\mathcal{T}_L^B$ and $\mathcal{T}_L^R$ statistics under the conditions of Proposition 1 can be obtained simply by replacing the sup operator by the inf operator in the representations given in Proposition 1, and with an obvious notation we denote these limiting distributions as $G_{j,L}^F(b)$, $G_{j,L}^B(b)$ and $G_{j,L}^R(b)$, $j = 1, 2$, respectively. Similarly, representations for the limiting distributions of the two-sided statistics $\mathcal{T}_2^F$, $\mathcal{T}_2^B$ and $\mathcal{T}_2^R$, denoted $G_{j,2}^F(b)$, $G_{j,2}^B(b)$ and $G_{j,2}^R(b)$, $j = 1, 2$, respectively, can be obtained by squaring the limiting quantities over which the supremum is taken in the expressions in Proposition 1. ♢

Remark 13. Part (ii) of Proposition 1, which relates to the case where x_t is strongly dependent, imposes a further restriction on the degree of persistence permitted in the conditional variances via the additional requirement that $\epsilon < \min\{1-\eta, \frac{1}{2}\eta\}$. This restriction therefore entails that $\epsilon < 1/3$ (with this maximum upper bound for ϵ corresponding to the use of an IVX filter with $\eta = 2/3$). Recalling, for example, that parametric GARCH models are such that $\epsilon = 0$, it seems likely that this additional restriction would not be restrictive in practice. ♢

Remark 14. The results in Proposition 1 establish the asymptotic local power functions of the tests based on the subsample and full sample IVX-based statistics (the latter obtained by setting $\tau_2 = 1$ and $\tau_1 = 0$ in the limiting representations for $t_{zx}(\tau_1, \tau_2)$) from sections 3.1 and 3.2, respectively, under the local alternative $H_{1,b(\cdot)}$. These local power functions depend, in general, on any heteroskedasticity and/or weak autocorrelation (short-run dynamics) present in the errors and differ according to whether x_t is weakly or strongly persistent. In the strongly persistent case they also depend on the parameter a used in the IVX filter and on the local-to-unity parameter, c. For the full sample t_{zx} test these results therefore complement those provided in Kostakis *et al.* (2015) and Magdalinos (2020) which apply only under the null hypothesis. From Proposition 1 it can be seen that the full sample t_{zx} test exhibits non-trivial power against the class of time-varying local alternatives we consider in this paper; that is, it has power to detect predictive episodes. In the case where $b(s) = b$, for some constant b, the results in Proposition 1 provide the asymptotic local power functions of the tests in the case where (local) predictability holds across the full sample; in this case the limiting process $Z_b(\tau)$ in part (ii) of Proposition 1 simplifies to $bJ_{c,\mathrm{H}}(\tau)$. ♢

Remark 15. The limiting null distributions of the statistics obtain from the results in Proposition 1 on setting $b(s) = 0$ for all s (whereby $Z_b(\tau)$ collapses to zero). Doing so, the limiting null distributions of the individual statistics $t_{zx}(\tau_1, \tau_2)$ can be seen to be (pointwise) normal. For example, under strong persistence, we have for the full-sample statistic that

$$t_{zx} \Rightarrow \frac{M_{zu}(1)}{\sqrt{[M_u](1)[M_v](1)}} = \frac{\int_0^1 \sqrt{[M_u]'(s)[M_v]'(s)}\mathrm{d}B(s)}{\sqrt{[M_u](1)[M_v](1)}} \overset{d}{=} N\left(0, \frac{\int_0^1 [M_u]'(s)[M_v]'(s)\mathrm{d}s}{\int_0^1 [M_u]'(s)\mathrm{d}s \int_0^1 [M_v]'(s)\mathrm{d}s}\right).$$

It can then be seen that in the unconditionally homoskedastic case where $\mathbf{H}$ is constant, the limiting null distribution of t_{zx} is standard normal under strong persistence, and hence that of $(t_{zx})^2$ is χ_1^2. This holds regardless of any conditional heteroskedasticity present in the innovations. In the weakly persistent case, however, we have that

$$t_{zx} \Rightarrow \frac{M_{\xi u}(1)}{\sqrt{\kappa^2[M_u](1)[M_v](1)}} \overset{d}{=} N\left(0, \frac{[M_{\xi u}](1)}{\kappa^2[M_u](1)[M_v](1)}\right)$$

whereby it follows that the variance of the limiting distribution of t_{zx} will in general depend on any conditional heteroskedasticity and/or short-run dynamics (the latter through the parameter κ^2) present, even where $\mathbf{H}$ is constant. On the other hand, κ^2 drops out of this expression under conditional homoskedasticity of $\boldsymbol{\psi}_t$, even if $\mathbf{H}$ is time-varying. For further details see the proof of Lemma 4 in the supplementary appendix. ♢

Remark 16. The limiting null distributions of the subsample-based statistics, $\mathcal{T}_j^F$, $\mathcal{T}_j^B$ and $\mathcal{T}_j^R$, $j \in \{U, L, 2\}$, all depend, in general, in a highly complicated way on nuisance parameters arising from any heteroskedasticity and (in the weakly dependent case) serial correlation present in $(u_t, v_t)'$ and on whether x_t is strongly or weakly persistent. While, as we show below in Proposition 2, these dependencies can be be removed from the limiting null distribution of the full sample t_{zx} statistic by basing the statistic on Eicker-White standard errors, this is not true of the subsample-based statistics. ♢

As discussed in Remark 15, the standard t_{zx} statistic, while having a limiting null distribution that is free of nuisance parameters when x_t is strongly persistent and the innovations are unconditionally homoskedastic, does not in general have a pivotal limiting null distribution when x_t is weakly persistent. The non-pivotal nature of the limiting null distribution of t_{zx} under conditional heteroskedasticity in the case of a weakly persistent predictor motivated Kostakis *et al.* (2015) to also consider the Eicker-White statistic t_{zx}^{EW} in (13). In Proposition 2 we demonstrate that the limiting (marginal) null distribution of the subsample Eicker-White $t_{zx}^{EW}(\tau_1, \tau_2)$ statistic has a standard normal limiting null distribution under the conditions of Proposition 1 and regardless of whether x_t is weakly dependent or near-integrated.

Proposition 2 *Under the conditions of Proposition 1, and for any given fixed values of* τ_1 *and* τ_2, $t_{zx}^{EW}(\tau_1, \tau_2) \Rightarrow N(0, 1)$, *and hence* $(t_{zx}^{EW}(\tau_1, \tau_2))^2 \Rightarrow \chi_1^2$, *under the null hypothesis,* H_0, *regardless of whether Assumption 1.1 or Assumption 1.2 holds.*

Remark 17. As a consequence of Proposition 2 the full-sample t_{zx}^{EW} statistic of (13) is seen to have a standard normal limiting null distribution under H_0 regardless of whether x_t is weakly or strongly persistent. The standard normality of the limiting null distribution of t_{zx}^{EW} has previously been shown to hold by Kostakis *et al.* (2015) under their Assumption INNOV and by Magdalinos (2020) under his Assumption M, both of which assume unconditional homoskedasticity. The result in Proposition 2 therefore establishes that this result holds under the much more general conditions of Assumption 3, which includes: (i) the case where $\mathbf{H}$ is non-constant such that the innovations are unconditionally heteroskedastic, and (ii) the case where the sequence $\boldsymbol{\psi}_t$ exhibits conditional heteroskedasticity of very general form; see again the discussion in Remarks 3 and 4. $\diamondsuit$

Remark 18. Provided the vector $(u_t, \boldsymbol{v}_t')'$ satisfies an obvious $(k+1)$-dimensional generalisation of Assumption 3, then the multiple predictor full sample Wald statistic, $W_{zx}^{\mathbf{R}}$ of Remark 9, when implemented with Eicker-White standard errors, can be shown to have a χ_q^2 limiting null distribution regardless of whether $\boldsymbol{x}_t$ is strongly or weakly persistent. The limiting null distributions of the corresponding subsample-based statistics, $\mathcal{W}_F^{\mathbf{R}}$, $\mathcal{W}_B^{\mathbf{R}}$ and $\mathcal{W}_R^{\mathbf{R}}$, of Remark 11 will, like the corresponding subsample-based tests for a scalar predictor, x_t, discussed in this section, have limiting null distributions which will, in general, depend in a highly complicated way on nuisance parameters arising from any heteroskedasticity and (in the weakly dependent case) serial correlation present in $(u_t, \boldsymbol{v}_t')'$ and on whether $\boldsymbol{x}_t$ is strongly or weakly persistent. $\diamondsuit$

As discussed above the subsample IVX statistics proposed in this paper, even when based on sequences of Eicker-White $t_{zx}^{EW}(\cdot, \cdot)$ statistics, have non-pivotal limiting null distributions whose form depends on whether the putative predictor x_t is a near-integrated or a weakly dependent process. The same is true of the corresponding 2SLS subsample-based supremum statistics of Demetrescu *et al.* (2020). This poses significant problems for conducting inference not encountered with the test based on the full sample t_{zx}^{EW} statistic which has a standard normal limiting null regardless of whether x_t is weakly or strongly persistent. In the next section we discuss how these issues can be solved by using bootstrap methods.

4 Bootstrap IVX Tests

As the results in section 3.3 show, implementing tests based on either the full sample t_{zx} statistic from section 3.1 or the subsample-based $\mathcal{T}_j^F$, $\mathcal{T}_j^B$ and $\mathcal{T}_j^R$, $j = U, L, 2$, statistics from section 3.2 will require us to address the fact that their limiting null distributions will, in general, depend on nuisance parameters arising from heteroskedasticity and/or serial correlation present in the data, and on whether the predictor x_{t-1} is weakly dependent or near-integrated.

We will consider two bootstrap resampling schemes in this section. The first, a residual wild bootstrap [RWB], is outlined in Algorithm 1. In Algorithm 2 we then outline how the fixed regressor wild bootstrap [FRWB] employed by Demetrescu *et al.* (2020) can also be used with the full sample and subsample IVX statistics discussed in this paper.[4]

Algorithm 1 (Residual Wild Bootstrap)

Step 1: Fit the predictive regression to the sample data $(y_t, x_{t-1})'$ to obtain the residuals $\hat{u}_t$, $t = 1, ..., T$, using any of the two choices outlined below (11).

Step 2: Fit by OLS an autoregression of order $p+1$ to x_t; viz,

$$x_t = \hat{m} + \sum_{j=1}^{p+1} \hat{a}_j x_{t-j} + \hat{v}_t$$

and compute the OLS residuals $\hat{v}_t$, $t = p+1, \ldots, T$. Set $\hat{v}_t = 0$ for $t = 1, \ldots, p$.

Step 3: Generate bootstrap innovations $(u_t^*, v_t^*)' := (R_t \hat{u}_t,\ R_t \hat{v}_t)'$, $t = 1 \ldots, T$, where R_t, $t = 1, ..., T$, is a scalar i.i.d.$(0,1)$ sequence with $\mathrm{E}(R_t^4) < \infty$, which is independent of the sample data.

Step 4: Define the bootstrap data $(y_t^*, x_{t-1}^*)'$ where $y_t^* = u_t^*$ (so that the null hypothesis is imposed on the bootstrap y_t^*) and where x_t^* is generated according to the recursion

$$x_t^* = \sum_{j=1}^{p+1} \hat{a}_j x_{t-j}^* + v_t^*, \quad t = 1, ..., T$$

with initial conditions $x_0^* = \ldots = x_{-p}^* = 0$. Create the associated bootstrap IVX instrument, z_t^*, as:

$$z_0^* = 0 \quad \textit{and} \quad z_t^* = \sum_{j=0}^{t-1} \varrho^j \Delta x_{t-j}^*, \ t = 1, \ldots, T,$$

[4] In what follows to save space we outline our proposed bootstrap procedures only for the case where conventional standard errors are used and where the finite sample correction factor of Kostakis *et al.* (2015) is not employed; cf. Remarks 7 and 8. Bootstrap implementations of the tests with the finite sample correction factor can instead be used without altering any of the large sample properties given in this section. Moreover, bootstrap implementations of the IVX tests based around Eicker-White standard errors may also be considered and again share the same asymptotic validity properties as the bootstrap tests based on conventional standard errors.

where ϱ is the same value as used in constructing the original IVX instrument, z_t.

Step 5: Using the bootstrap sample data, $(y_t^*, x_{t-1}^*, z_{t-1}^*)'$, in place of the original sample data, $(y_t, x_{t-1}, z_{t-1})'$, construct the bootstrap analogues of the $t_{zx}(\tau_1, \tau_2)$, $\mathcal{T}_j^F$, $\mathcal{T}_j^B$ and $\mathcal{T}_j^R$, $j = U, L, 2$, statistics from section 3.2. Denote these bootstrap statistics as $t_{zx}^*(\tau_1, \tau_2)$, $\mathcal{T}_j^{*,F}$, $\mathcal{T}_j^{*,B}$ and $\mathcal{T}_j^{*,R}$, $j = U, L, 2$.

Step 6: Taking the test based on $\mathcal{T}_U^F$ to illustrate, a bootstrap p-value is then computed as $p_{1,T}^* := 1 - G_{1,T}^*(\mathcal{T}_U^F)$, where $G_{1,T}^*(\cdot)$ denotes the conditional (on the original sample data) cumulative distribution function (cdf) of $\mathcal{T}_U^{*,F}$. Notice, therefore, that the bootstrap test, run at the λ significance level, based on $\mathcal{T}_U^F$ is then defined such that it rejects H_0 if $p_{1,T}^* < \lambda$. Bootstrap p-values for the other tests are similarly obtained.

Algorithm 2 (Fixed Regressor Wild Bootstrap)

Step 1: As Step 1 in Algorithm 1.

Step 2: Generate bootstrap innovations $u_t^* := R_t \hat{u}_t$, $t = 1 \ldots, T$, where R_t satisfies the same conditions as given in Step 3 of Algorithm 1

Step 3: For $t = 1, \ldots, T$, define the bootstrap data $y_t^* = u_t^*$ (so that the null hypothesis is imposed on the bootstrap y_t^*).

Step 4: As detailed in Step 5 of Algorithm 1, but where the original sample data, $(y_t, x_{t-1}, z_{t-1})'$ are instead replaced by the fixed regressor bootstrap sample data, $(y_t^*, x_{t-1}, z_{t-1})'$.

Step 5: As Step 6 of Algorithm 1

Remark 19. The key difference between the RWB outlined in Algorithm 1 and the FRWB outlined in Algorithm 2 surrounds the generation of the bootstrap analogue data for x_t and, hence, z_t. In the FRWB scheme one calculates the bootstrap statistics in Step 4 using the data $(y_t^*, x_{t-1}, z_{t-1})'$; that is, y_t^* is generated exactly as in Algorithm 1, but the observed outcomes on $\mathbf{x} := [x_0, x_1, ..., x_T]'$ and $\mathbf{z} := [z_0, z_1, ..., z_T]'$ are treated as a fixed regressor and fixed instrument vector, respectively, when implementing the bootstrap procedure. As such, while the RWB rebuilds into the bootstrap data (an estimate of) the correlation between the innovations u_t and v_t through Step 3 of Algorithm 1 (it is crucial in doing so that the same R_t is used to multiply both $\hat{u}_t$ and $\hat{v}_t$), the FRWB does not. This is an important distinction because, as the simulation results we report in section 5 will show, the finite sample behaviour of the IVX statistics is heavily dependent on the correlation between u_t and v_t in the case where x_t is strongly persistent. As a result we find that the RWB delivers considerably better finite sample performance than the FRWB in the case where x_t is strongly persistent. ◊

Remark 20. A further difference between the RWB and the FRWB is that because one creates bootstrap analogues of x_t and z_t, x_t^* and z_t^* respectively, one implicitly has to use an estimate of ρ in doing so. Under Assumption 1.2 (strong persistence) it is well known that the associated local-to-unity parameter, c, cannot be consistently estimated. Consequently, when x_t is strongly persistent the bootstrap data on x_t^* will not be generated with the same local-to-unity parameter as the original data x_t. In the case of the FRWB this issue does not arise because the original data on x_t is used in calculating the bootstrap statistics. However, the IVX statistics instrument x_{t-1} by z_{t-1}, and their bootstrap analogue statistics instrument x_{t-1}^* by z_{t-1}^*, where z_t and z_t^* are, by construction, both mildly integrated processes regardless of the value of c under Assumption 1.2. There is therefore no necessity for the estimate of c from Step 2 to be consistent in order to validly implement the RWB in Algorithm 1. Notice that this would not be true under Assumption 1.2 if we were bootstrapping the standard OLS t-statistic from (1) because this statistic does not instrument x_t by a variable of lower persistence and, as result, has a limiting null distribution which depends on c. $\diamond$

Remark 21. It could also be possible to implement a moving block bootstrap [MBB] based scheme, similar to that used in Fan and Lee (2019), for the IVX-based tests considered here. An outline of this algorithm can be found in the Supplementary Appendix. We conjecture that this MBB procedure is asymptotically valid provided $\mathbf{H}$ were constant such that the innovations were unconditionally homoskedastic. To account for unconditional heteroskedasticity a block wild adaptation of this bootstrap could be employed and again this is outlined in the Supplementary Appendix. We will not pursue either of these MBB-based methods further here as in unreported simulations we found them to perform poorly in finite samples relative to the RWB-based tests. $\diamond$

Remark 22. With simple modifications, the RWB of Algorithm 1 can be implemented for the multiple regressor full sample Wald statistic, $W_{zx}^{\mathbf{R}}$ of Remark 9, and the corresponding subsample-based statistics, $\mathcal{W}_F^{\mathbf{R}}$, $\mathcal{W}_B^{\mathbf{R}}$ and $\mathcal{W}_R^{\mathbf{R}}$, of Remark 11. In Step 2 of Algorithm 1 a vector autoregression of order $p+1$ is fitted to $\boldsymbol{x}_t$ to obtain the residuals $\hat{\boldsymbol{v}}_t$ with the residuals from these collected together into $\hat{\boldsymbol{v}}_t$. In Step 3 one then calculates the bootstrap innovations $(u_t^*, \boldsymbol{v}_t^{*\prime})' = \left(R_t\hat{u}_t, R_t\hat{\boldsymbol{v}}_t'\right)'$, $t = 1, \ldots, T$. In Step 4 one generates the bootstrap data $y_t^* = u_t^*$ imposing the null hypothesis, together with the bootstrap predictor vector, $\boldsymbol{x}_t^*$, by the recursion based on the coefficient estimates obtained in Step 2. The bootstrap instruments, $\boldsymbol{z}_t^*$, are derived from $\boldsymbol{x}_t^*$ according to the same IVX filter used to obtain $\boldsymbol{z}_t$ from $\boldsymbol{x}_t$. The RWB statistics are then computed from the bootstrap sample data, $\left(y_t^*, \boldsymbol{x}_{t-1}^*, \boldsymbol{z}_{t-1}^*\right)'$. The FRWB of Algorithm 2 can also be modified to allow for multiple regressors by using the bootstrap sample data, $(y_t^*, \boldsymbol{x}_{t-1}, \boldsymbol{z}_{t-1})'$ in Step 4. Provided the conditions outlined in Remark 18 hold, both the FRWB and RWB bootstrap tests for multiple regressors will share analogous asymptotic validity properties to the bootstrap tests in the case of a single regressor established below. $\diamond$

Remark 23. In practice the autoregressive lag truncation order used in Step 2 of Algorithm 1 will be unknown. This can be selected in the usual way using a consistent information criterion such as the Bayes Information Criterion (BIC) or Hannan-Quinn [HQ] information criterion. A less parsimonious information criterion, such as the Akaike Information Criterion [AIC] could also be used, or even a deterministic truncation lag chosen according to, for example, the popular Schwert (1989) rule where the lag truncation is set equal to $\lfloor \kappa (T/100)^{1/4} \rfloor$, for some positive constant κ. The lag length fitted in Step 2 actually has rather little bearing on the power of the resulting bootstrap tests, as is also shown in the context of bootstrap augmented Dickey-Fuller unit root tests in Palm *et al.* (2008). Notice that no choice of p is required in connection with the FRWB outlined in Algorithm 2. $\diamondsuit$

In Proposition 3 we now demonstrate the large sample validity of the RWB and FRWB bootstrap implementations of the IVX tests from Algorithms 1 and 2, respectively. In particular, we show that these correctly replicate the first order asymptotic null distributions of the IVX statistics under both the null hypothesis and local alternatives. However, for the RWB-based tests this result requires a further restriction to hold on the fourth moments of the innovations in the case where x_t is weakly persistent. This additional restriction is not required for the asymptotic validity of the FRWB tests.

Proposition 3 *Consider the model in* (1)–(3) *and let Assumptions 2 and 3 hold. Then under the local alternative* $H_{1,b(\cdot)}$ *of* (6)*:*

(i) *Under Assumption 1.1,*

- **(a)** *For the bootstrap statistics generated according to the RWB scheme in Algorithm 1, provided* $\mathrm{E}[(\boldsymbol{\psi}_1 \boldsymbol{\psi}_1') \otimes (\boldsymbol{\psi}_{-i} \boldsymbol{\psi}_{-j}')] = 0$ *for all natural* $i \neq j$*, it holds that* $t^*_{zx}(\tau_1, \tau_2) \overset{w}{\Rightarrow}_p G_1(0, \tau_1, \tau_2)$, $\mathcal{T}_j^{*,F} \overset{w}{\Rightarrow}_p G^F_{1,j}(0)$, $\mathcal{T}_j^{*,B} \overset{w}{\Rightarrow}_p G^B_{1,j}(0)$*, and* $\mathcal{T}_j^{*,R} \overset{w}{\Rightarrow}_p G^R_{1,j}(0)$*, in each case for* $j = U, L, 2$.
- **(b)** *For the bootstrap statistics generated according to the FRWB scheme in Algorithm 2,* $t^*_{zx}(\tau_1, \tau_2) \overset{w}{\Rightarrow}_p G_1(0, \tau_1, \tau_2)$, $\mathcal{T}_j^{*,F} \overset{w}{\Rightarrow}_p G^F_{1,j}(0)$, $\mathcal{T}_j^{*,B} \overset{w}{\Rightarrow}_p G^B_{1,j}(0)$*, and* $\mathcal{T}_j^{*,R} \overset{w}{\Rightarrow}_p G^R_{1,j}(0)$*, in each case for* $j = U, L, 2$.

(ii) *Under Assumption 1.2, and with* $\epsilon < \min\{\eta, \frac{1}{2}\}$ *in Assumption 3, and regardless of whether the bootstrap statistics are generated according to the RWB scheme in Algorithm 1 or the FRWB scheme in Algorithm 2,* $t^*_{zx}(\tau_1, \tau_2) \overset{w}{\Rightarrow}_p G_2(0, \tau_1, \tau_2)$, $\mathcal{T}_j^{*,F} \overset{w}{\Rightarrow}_p G^F_{2,j}(0)$, $\mathcal{T}_j^{*,B} \overset{w}{\Rightarrow}_p G^B_{2,j}(0)$*, and* $\mathcal{T}_j^{*,R} \overset{w}{\Rightarrow}_p G^R_{2,j}(0)$*, in each case for* $j = U, L, 2$.

Remark 24. A comparison of the limiting results for the bootstrap statistics in Proposition 3 with those given for the corresponding statistics in Proposition 1 demonstrates the usefulness of the RWB and FRWB procedures from Algorithms 1 and 2, respectively; as the number of observations increases, the bootstrapped statistics have the same first-order limiting null distributions as the corresponding original test statistic.[5] For this result to hold for the RWB statistics, however, it is seen that fourth moments of the form $\mathrm{E}[(\boldsymbol{\psi}_1 \boldsymbol{\psi}_1') \otimes (\boldsymbol{\psi}_{-i} \boldsymbol{\psi}_{-j}')]$ for $i \neq j$ should not contribute to the quadratic variation of the process $M_{\xi u}$. The reason is

[5]Observe that the condition placed on ϵ in part (ii) of Proposition 3 is less restrictive than that imposed for part (ii) of Proposition 1 regardless of the value of η used in the IVX filter and therefore this result holds for all DGPs such that Proposition 1 holds.

that in the RWB world the mixed fourth moments $\mathrm{E}^*[(R_t^2\boldsymbol{\psi}_t\boldsymbol{\psi}_t')\otimes(R_{t-i}R_{t-j}\boldsymbol{\psi}_{t-i}\boldsymbol{\psi}_{t-j}')] = 0$ by construction for all natural $i \neq j$, and hence, these do not contribute to the quadratic variation of the RWB analogue of $M_{\xi u}$. As with the conditions placed on $\{\boldsymbol{\psi}_t\}$ by Assumption 3.2, this assumption is not tied to any specific parametric model. Even where this condition is violated, the impact on the (asymptotic) size of the resulting RWB test might still be relatively small, given that the quantities $\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\otimes(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')]$, for all natural $i \neq j$, only constitute part of the quadratic variation of $M_{\xi u}$ and it is this latter quantity which the bootstrap limit needs to reproduce. A well known class of models which violate this condition are GARCH models with non-zero leverage effects. We will explore the impact of such a model on the finite sample size behaviour of the RWB tests in section 5. ◇

Remark 25. A consequence of the results in Proposition 3, using the same arguments as in the proof of Theorem 5 in Hansen (2000), is that for each of the tests the bootstrap p-values are (asymptotically) uniformly distributed under the unit root null hypothesis, H_0, leading to tests with (asymptotically) correct size, thereby establishing the asymptotic validity of the bootstrap tests. In the case of the FRWB, this validity result is achieved without the practitioner needing to have knowledge of whether x_t is weakly or strongly persistent and holds regardless of any autocorrelation or heteroskedasticity present in u_t and v_t satisfying Assumption 3. For the RWB this is also true, provided the condition $\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\otimes(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')] = 0$ for all natural $i \neq j$ holds. A further consequence of the result in Proposition 3 for $t_{zx}^*(\tau_1, \tau_2)$, setting $\tau_1 = 0$ and $\tau_2 = 1$, is therefore that under the null the RWB and FRWB bootstrap implementations of the full sample t_{zx} test deliver asymptotically pivotal inference under Assumption 3 (or the restricted version thereof in the case of the RWB scheme) without the need for Eicker-White standard errors. ◇

Remark 26. An additional implication of the results in Proposition 3 is that each of the bootstrap IVX-based tests proposed in Algorithms 1 and 2 will admit the same asymptotic local power functions under the local alternative $H_{1,b(\cdot)}$ of (6) as the corresponding (infeasible) size-adjusted tests based on the corresponding original IVX statistic. ◇

Remark 27. As discussed in Remark 24, a key difference between the large sample properties of the RWB and FRWB is that the former can only be validly applied in the case where x_t is weakly persistent if the mixed fourth moments $\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\otimes(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')]$ with $i \neq j$ do not contribute to the quadratic variation of the process $M_{\xi u}$. However, as we will see in the simulations in section 5, the RWB delivers considerably better finite sample performance than the FRWB when x_t is strongly persistent, while the two display similar performance when the degree of persistence in x_t is weaker. In principle then one might use the sample data on x_t to decide which of the RWB and FRWB to use. In particular, one could adopt the RWB of Algorithm 1 unless the sample data on x_t suggested the persistence in x_t was relatively weak. This idea has previously been advocated in the predictability testing literature by Elliott *et al.* (2015) who propose a testing procedure which switches between a weighted average power test where x_t is strongly persistent and the standard OLS t-test from (1) when x_t is weakly persistent. The switching mechanism they adopt is to use the

OLS t-test when $\hat{c} \geq 130$ and the weighted average power test otherwise, where $\hat{c}$ is an estimate of the local-to-unity parameter, c. A similar rule could be used here, whereby we use the RWB unless $\hat{c}$ exceeds some specified value. An obvious estimate of c, based on the autoregressive estimates from Step 2 of Algorithm 1, is $\hat{c} := T(1 - \sum_{j=1}^{p} \hat{a}_j)$. This rule ensures that, with probability approaching one, the RWB would not be chosen in large samples when x_t was weakly dependent, and therefore this hybrid bootstrap will share the asymptotic validity result enjoyed by the FRWB in the weak persistence case. ♢

Remark 28. In practice the cdf $G^*_{1,T}(\cdot)$ of the bootstrap $\mathcal{T}^{*,F}_U$ statistic, and the corresponding cdfs for the other statistics, required in Step 6 of Algorithm 1 and Step 5 of Algorithm 2 will be unknown but can be approximated in the usual way through numerical simulation. To illustrate, again for the case of the $\mathcal{T}^F_U$ statistic, this is achieved by generating B bootstrap (conditionally) independent statistics, say $\mathcal{T}^{*,F}_{U,b}$, $b = 1, ..., B$, each computed as in Algorithm 1 above. The simulated bootstrap p-value for the test is then computed as $\tilde{p}^*_{1,T} = B^{-1} \sum_{b=1}^{B} \mathbb{I}\left(\mathcal{T}^{*,F}_{U,b} > \mathcal{T}^F_U\right)$ and is such that $\tilde{p}^*_{1,T} \overset{a.s.}{\to} p^*_{1,T}$ as $B \to \infty$, where $\overset{a.s.}{\to}$ denotes almost sure convergence. An approximate standard error for $\tilde{p}^*_{1,T}$ is given by $(\tilde{p}^*_{1,T}(1 - \tilde{p}^*_{1,T})/B)^{1/2}$; see Hansen (1996, p. 419). For a discussion on the choice of B see, *inter alia*, Davidson and MacKinnon (2000). Simulated bootstrap critical values can also be obtained for the tests. Again illustrating for the case of a test based on the $\mathcal{T}^F_U$ statistic, a λ level empirical bootstrap critical value, $cv_{\lambda,B}$ say, can be calculated as the upper tail λ percentile from the order statistic formed from the B bootstrap statistics, $\mathcal{T}^{*,F}_{U,b}$, $b = 1, ..., B$. The resulting bootstrap test, which rejects H_0 if $\mathcal{T}^F_U > cv_{\lambda,B}$, will have asymptotic size that for sufficiently large B will be as close as desired to the given nominal level, λ. ♢

5 Finite Sample Results

In this section we present results from a detailed Monte Carlo study into the finite sample properties of the IVX tests of Kostakis *et al.* (2015) based on the use of asymptotic critical values. We will consider versions of these tests implemented both with and without Eicker-White corrected standard errors. We will compare the finite sample behaviour of these asymptotic tests with their RWB and FRWB bootstrap implementations developed in this paper. In section 5.1 we report finite sample size and power results for the leading case of a single predictor. Then in section 5.2 we report results for the case where multiple predictors are considered. In order to present results from as a wide a range of empirically plausible DGPs as possible, tabulations of results will only be reported in the main text for a subset of the cases we discuss in the text. Tables pertaining to the other cases appear in the supplementary appendix. All of the results we present pertain to the case of full sample statistics. For all of the statistics considered, OLS residuals are used in computing the standard errors.

5.1 Single Predictor Regressions

We first consider the case where a single predictor, x_{t-1}, is included in the predictive regression. Results are reported for the IVX test of Kostakis *et al.* (2015) both with and without Eicker-White corrected standard errors, t_{zx}^{EW} and t_{zx}, respectively; these statistics were computed exactly as detailed in section 3.1 with the finite sample correction factor, Ξ, included; see Remarks 7 and 8. We will compare these with their RWB and FRWB bootstrap analogues, $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$, described in Algorithms 1 and 2 in section 4, respectively. In the context of the RWB the autoregressive lag length used in Step 2 of Algorithm 1 was chosen applying the BIC over $p \in \{0, ..., \lfloor 4(T/100)^{0.25} \rfloor\}$. The bootstrap statistics are all based on conventional standard errors and all include the finite sample correction factor. Our analysis consists of testing the null hypothesis of no predictability, $H_0 : \beta = 0$, in (1) in the context of a constant parameter prediction model, so that $\beta_t = \beta$, for all $t = 1, ..., T$. We will consider tests directed against both one-sided alternatives, left-tailed tests for $H_1 : \beta < 0$, and right-tailed tests for $H_1 : \beta > 0$, together with two-sided tests for $H_1 : \beta \neq 0$. Results are reported for tests run at the 1%, 5% and 10% nominal significance levels. For the bootstrap implementations we use 999 replications and all results are based on 10000 Monte Carlo replications. All simulations are preformed in MATLAB, versions R2018b and R2020a, using the Mersenne Twister random number generator.

5.1.1 Empirical Size

To investigate the finite sample size properties of t_{zx}, t_{zx}^{EW}, $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ under the null hypothesis of no predictability, we generate data according to (1)-(3) with $\beta_t = \beta = 0$ for all $t = 1, ..., T$. In generating the data we set the intercepts α and μ_x in (1) and (2), respectively, to zero with no loss of generality. We initialised the autoregressive process characterising the dynamics of the putative predictor, x_t, in (3) at $\xi_0 = 0$, and considered a wide range of values for the autoregressive parameter ρ in (3) covering stationary, near-integrated and mildly explosive predictors; in particular, we set $\rho = 1 - c/T$ with $c \in \{-5, -2.5, 0, 2.5, 5, 10, 25, 50, 75, 100, 125, 150, 200, 250\}$. All results reported, both in the main text and in the supplementary appendix, are for sample sizes $T = 250$ and $T = 1000$. In total, for the single predictor case, we consider 11 distinct classes of DGP. For the sake of space we will present Tables of results for two of these DGPs in this section. A summary of the results for the other 9 DGPs will also be given, with the full details of these DGPs and the associated tables of results for these cases relegated to the accompanying supplementary appendix.

Main Results

The first DGP (DGP1) we will consider corresponds to (1)-(3) with the innovation vector $(u_t, v_t)'$ drawn from an i.i.d. bivariate Gaussian distribution with mean vector zero and covariance matrix $\mathbf{\Sigma} = \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}$, where ϕ corresponds to the correlation between u_t and

v_t. Results from DGP1 for $\phi = -0.95$, -0.90, -0.50 and 0 are reported in Tables 1–4.[6]

The second DGP (DGP2) we will consider is one designed to be such that the regularity conditions needed for the validity of the RWB when x_t is weakly persistent are violated. The DGP we consider is a well known model where the conditional variance of the innovations $(u_t, v_t)'$ follows a stationary ARCH model with leverage effects and is of the form

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \psi_t \tag{20}$$

with

$$\psi_t = \begin{pmatrix} a_t \\ e_t \end{pmatrix} = \begin{pmatrix} \varepsilon_{1t}\sqrt{1 + \frac{1}{2}a_{t-1}^2 \mathbb{I}_{\{a_{t-1}<0\}}} \\ \varepsilon_{2t} \end{pmatrix}$$

and $(\varepsilon_{1t}, \varepsilon_{2t})' \sim NIID(\mathbf{0}, \mathbf{I_2})$. The AR parameter ρ is again set equal to $1 - c/T$.

DGP2 satisfies our assumptions of finite fourth moments of ψ_t and martingale approximability of $\psi_t\psi_t'$ (with $\epsilon = 0$). However, and crucially, the quadratic variation of $M_{\xi u}$ depends on,

$$\begin{aligned} h_{11}^2 h_{21}^2 b_1 b_2 \,\mathrm{E}(a_t^2 a_{t-1} a_{t-2}) &= \rho^3 \,\mathrm{E}(a_t^2 a_{t-1} a_{t-2}) \\ &= \frac{\rho^3}{8} \,\mathrm{E}\,|\varepsilon_1|^3 \,\mathrm{E}\left\{ |a_1| \left[\sqrt{\left(1 + \frac{1}{2}a_1^2\right)^3} - 1 \right] \right\} > 0; \end{aligned} \tag{21}$$

see the proof of Lemma 4. This model therefore violates the limiting condition that $M_{\xi u}^* \stackrel{d}{=} M_{\xi u}$ which is necessary and sufficient for the validity of the RWB in the case where x_t is weakly persistent. Specifically, the non-zero term in (21) is absent from the quadratic variation of $M_{\xi u}^*$ in the limiting distribution of the RWB bootstrap statistic when x_t is weakly persistent; cf. Remark 24. Because the focus is therefore on the weakly persistent case results will be reported only for $c \in \{5, 10, 25, 50, 75, 100, 125, 150, 200, 250\}$. Recall, however, that this limiting condition is not required for the asymptotic validity of the FRWB statistic.

Consider first the results pertaining to the homoskedastic DGP1. A comparison of the results in Tables 1–4 for $\phi = -0.95$, -0.90, -0.50 and 0, respectively, show that when the innovations are homoskedastic the endogeneity correlation parameter, ϕ, has relatively little impact on the size properties of the two-sided tests, regardless of the significance level considered, at least for cases where the autoregressive parameter c is positive and not close to zero. Here there is relatively little difference between the tests based on asymptotic critical values and the corresponding RWB and FRWB bootstrap tests. For all of these cases the two-sided tests display finite sample size close to the nominal levels considered. However, where x_t is mildly explosive with $c = -5$ there is a tendency to undersize in t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ for both $\phi = -0.95$ and $\phi = -0.90$ which is largely redressed by $t_{zx}^{*,RWB}$. For $0 \le c \le 10$ slight oversizing is also seen for both $\phi = -0.95$ and $\phi = -0.90$ with t_{zx},

[6] Notice that, because we report results for both left-tailed, right-tailed and two-tailed tests, it is not necessary to report results for positive values of ϕ; cf. Campbell and Yogo (2006, p. 30)

t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ which is again largely eliminated by $t_{zx}^{*,RWB}$.

A rather different picture emerges when considering one-sided implementations of the tests. The one-sided t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ tests display severe size distortions for $c < 50$ when $\phi = -0.95$. Specifically, for $\phi = -0.95$ the left-tailed t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ tests display very significant undersizing, while their right-tailed counterparts are severely oversized (for instance when $c < 10$ empirical size is in most cases more than double the nominal size considered). The size distortions observed for these one-sided tests decrease, other things equal, as $|\phi|$ decreases, but significant size distortions are still observed even for $\phi = -0.5$. We also observe that the empirical rejection frequencies of the one-sided t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ tests under DGP1 are all very similar to each other for given values of ϕ and c. Consequently, the FRWB based implementations of the one-sided IVX tests do not appear to offer any tangible improvement on the finite sample size properties of the asymptotic tests, as might be expected in the light of Remark 19. In contrast, both the left-sided and right-sided tests implemented with the RWB offer empirical size properties which are close to the nominal level throughout.

Consider next the results in in Table 5 for DGP2 where the conditional variance of $(u_t, v_t)'$ follows an ARCH model with leverage effects. The results show that in general the two-sided versions of the t_{zx}^{EW}, $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ tests all display reasonable size control throughout. In contrast, significant size distortions are seen for the two-sided t_{zx} test regardless of the significance level considered. The latter finding is consistent with our discussion in Remark 15 on the non-pivotal nature of the limiting null distribution of t_{zx} under conditional heteroskedasticity when x_t is weakly dependent. Large size distortions are also seen for the one-sided t_{zx} tests. Moreover, and as observed with DGP1, although the two-sided t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ tests shows decent finite sample size control the same is not true of the one-sided versions of these tests. In contrast the one-sided $t_{zx}^{*,RWB}$ tests deliver decent finite sample size control for all values of c and regardless of the sample size. Consequently, although the limiting condition $M_{\xi u}^* \stackrel{d}{=} M_{\xi u}$ formally required for the asymptotic validity of the RWB tests is not met by DGP2, the results in Table 5 suggest that $t_{zx}^{*,RWB}$ nonetheless displays arguably the most reliable finite sample size control among the tests considered for data generated according to DGP2.

Summary of Additional Results

In addition to the results discussed above for DGP1 and DGP2 we have also investigated the impact on the finite sample performance of the IVX statistics and their bootstrap implementations from a variety of additional empirically relevant models which allow for serial correlation and heteroskedasticity. Full details of the simulation DGPs considered and the tabulated results (which appear in Tables D.1 - D.36) are given in the supplementary appendix. In what follows we provide a summary of those results.

- The results in Tables 1–4 relate to the case where the error process, v_t, driving the predictor in DGP1 is serially uncorrelated. We have also repeated these experiments for the case where v_t in DGP1 admits short-run dependence following either a positively autocorrelated (DGP3) or negatively autocorrelated (DGP4) stationary AR(1) process. These results, which can be found in Tables D.1 - D.8, were qualitatively very similar to those reported above for serially uncorrelated v_t.

- We consider two DGPs which include a contemporaneous one-time break of equal magnitude in the unconditional variances of u_t and v_t, as in Georgiev *et al.* (2018) and Demetrescu *et al.* (2020). The first, labelled DGP5, contains an upward change in the unconditional variances of u_t and v_t at the sample midpoint (Tables D.9 - D.12), while the second, labelled DGP6, contains a corresponding downward change in the unconditional variances of u_t and v_t (Tables D.13–D.16).

  The results reported in Tables D.9 to D.16 reveal that, as expected, the two-sided IVX test with conventional standard errors, t_{zx}, displays significant size distortions. For example, for a 5% significance level and $\phi = -0.95$ the rejection frequencies observed across all values of c considered, when an upward change in variance occurs (Table D.9) are in the range $[0.064, 0.095]$ for $T = 250$ and $[0.066, 0.097]$ for $T = 1000$. For a downward change in variance (Table D.13) results are similar ($[0.017, 0.098]$ for $T = 250$ and $[0.018, 0.091]$ for $T = 1000$), except for cases where $c < 0$ (mildly explosive predictors) in which case some undersizing is observed. The magnitude of these size distortions are relatively stable across the values of ϕ considered.

  In contrast, for the one-sided versions of t_{zx} the empirical size distortions for the former worsen, other things equal, as $|\phi|$ increases. For example, for DGP5 with $T = 250$ and $\phi = -0.95$ the range of empirical rejection frequencies for the left-sided tests is $[0.003, 0.075]$ and for the right-sided tests $[0.085, 0.151]$; see Table D.9. On the other hand, for $\phi = 0$ the left and right-sided tests rejection frequencies' range is $[0.064, 0.081]$; see Table D.12.

  The size distortions observed with the two-sided t_{zx} test for both DGP5 and DGP6 are significantly ameliorated by the use of Eicker-White standard errors (t_{zx}^{EW}) when $c \geq -2.5$. However, the one-sided (left and right-sided) t_{zx}^{EW} tests do not seem to improve much relative to t_{zx} when $c \leq 25$; see Tables D.9 to D.16.

  The RWB and FRWB bootstrap implementations of the two-sided t_{zx} test are both seen to do a very good job at controlling finite sample size in the presence of unconditional heteroskedasticity. For the one-sided tests, $t_{zx}^{*,RWB}$ displays empirical rejection frequencies which are again in general close to the nominal significance level considered, regardless of the values of c and ϕ. In contrast, the one-sided $t_{zx}^{*,FRWB}$ test displays significant size distortions for values of $c \leq 25$; these improve as $|\phi|$ decreases, as anticipated by the discussion in Remark 19.

- To further evaluate the impact of conditional heteroskedasticity we considered three further volatility specifications: i) a GARCH(1,1) model coupled with either Gaussian (DGP7) or Student-t distributed innovations with 5 degrees of freedom (DGP8), thereby allowing for unconditionally heteroskedastic and fat-tailed innovations (Bollerslev, 1986); ii) a GoGARCH(1,1) model [see Van der Weide (2002) and Boswijk and Weiden (2011)] also allowing for either Gaussian (DGP9) or Student-t distributed innovations with 5 degrees of freedom (DGP10); and iii) an autoregressive stochastic volatility process (DGP11), as used in Gonçalves and Kilian (2004) and Cavaliere and Taylor (2008).

As observed earlier in relation to the results from DGP2, the non-pivotal nature of the t_{zx} statistic's limiting null distribution under GARCH type conditional heteroskedasticity is also apparent in the results in Tables D.17 to D.20 and D.21 to D.24 corresponding to DGP7 and DGP8, respectively. These results highlight that the size distortion of the two-sided t_{zx} statistic increases as $|\phi|$ increases regardless of whether $N(0,1)$ (Tables D.17 to D.20) or Student-t innovations (Tables D.21 to D.24) are used in generating the data. The magnitude of the size distortions is, however, considerably exacerbated when the innovations are heavy tailed (DGP8). For instance, for $N(0,1)$ innovations, $T = 250$, $\phi = -0.95$ and for a 5% significance level the range of the empirical rejection frequencies for t_{zx} is $[0.042, 0.082]$, whereas for Student-t distributed innovations the range is $[0.081, 0.167]$. The Eicker-White correction does a good job in correcting the size distortion of the two-sided t_{zx} test regardless of whether the innovations are $N(0,1)$ or Student-t distributed. In the previous example, the ranges of the rejection frequencies of t_{zx}^{EW} when the innovations are $N(0,1)$ and Student-t distributed is $[0.047, 0.066]$ and $[0.062, 0.068]$, respectively. The results also show that the RWB and FRWB both display good empirical size properties in a two-sided hypothesis testing context. However, for one-sided testing $t_{zx}^{*,RWB}$ delivers significantly better finite sample size control than $t_{zx}^{*,FRWB}$ when x_t is strongly persistent, while they display similar performance for weaker levels of persistence in x_t. Overall $t_{zx}^{*,RWB}$ is the best performing test regardless of the nominal significance levels used and regardless of the underlying distribution of the innovations. All of the other one-sided tests display serious size distortions when the predictor is strongly persistent ($c < 25$), for both $N(0,1)$ or Student-t distributed innovations.

For the GoGARCH models (DGP9 and DGP10 in Tables D.25 to D.28 and Tables D.29 to D.32, respectively), qualitatively similar conclusions can be drawn to those discussed above for the GARCH(1,1) case albeit the magnitude of the size distortions observed for the $t_{zx}^{*,FRWB}$, t_{zx}^{EW} and t_{zx} tests are generally smaller.

Finally, regarding the impact of stochastic volatility (DGP11), the results in Tables D.33 to D.36 suggest that all of the two-sided tests display adequate finite sample size control, with the exception of t_{zx}^{EW} which is oversized for $T = 250$, although its size properties are improved for $T = 1000$. For the one-sided tests, similar conclusions are drawn as for the GARCH and GoGARCH specifications. Specifically, $t_{zx}^{*,FRWB}$, t_{zx}^{EW} and t_{zx} are considerably oversized when the predictor is strongly persistent and $\phi = -0.95$, but $t_{zx}^{*,RWB}$ consistently displays reliable empirical rejection frequencies close to the nominal level across the range of values of c considered.

5.1.2 Finite Sample Local Power

We next provide a brief analysis of the relative finite sample local power properties of the IVX tests and their bootstrap analogues. To that end, we again generate simulation data from DGP1, but now for a variety of local alternatives. For the sake of space, we only report results for $\phi = -0.95$, for a sample of size $T = 250$ and for four values of the persistence parameter, c, associated with x_t; specifically, $c = \{-5, 0, 10, 20\}$. The slope parameter β is parameterised in (1) as $\beta = b/T$, with the following values considered for the Pitman drift parameter, $b \in \{-20, -19, ..., 19, 20\}$.

Because of the large finite sample size distortions associated with the one-sided t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ tests discussed in section 5.1.1 for these combinations of c and ϕ, we only report local power results results for the two-sided t_{zx}, t_{zx}^{EW}, $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ tests all of which have well controlled empirical size properties under DGP1. The finite sample local power curves of these tests are graphed in Figure 1. Recalling from Remark 26 that the RWB and FRWB tests share the same asymptotic local power functions as the corresponding (size-adjusted) asymptotic IVX test, Figure 1 shows that this prediction from the limiting theory is borne out well even for a sample of size $T = 250$ with the power curves of the bootstrap and asymptotic tests being almost indistinguishable from each other for all of the values of c considered.

5.2 Multiple Predictors

In our final set of experiments, we investigate the finite sample behaviour of the asymptotic IVX test and its RWB and FRWB bootstrap counterparts in cases where multiple predictors are included in the predictive regression. For our analysis we use the same DGP as is considered in Xu and Guo (2020); that is,

$$y_t = \alpha + \mathbf{x}_{t-1}'\boldsymbol{\beta} + u_t, \qquad t = 1, \dots, T, \tag{22}$$

$$\mathbf{x}_t = \boldsymbol{\rho}\mathbf{x}_{t-1} + \mathbf{v}_t, \qquad t = 0, \dots, T, \tag{23}$$

where $\mathbf{x}_t := (x_{1,t}, ..., x_{K,t})'$ is a $K \times 1$ vector of predictor variables, $\boldsymbol{\beta}$ is a $K \times 1$ vector of parameters, $\alpha = 0.25$, $\boldsymbol{\rho}$ is a $K \times K$ diagonal matrix with common diagonal element ρ, i.e., $\boldsymbol{\rho} := \mathsf{diag}(\rho, ..., \rho)$, and $(u_t, \mathbf{v}_t')' \sim NIID(\mathbf{0}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_u^2 & \sigma_{u,v_1} & 0 & \cdots & 0 \\ \sigma_{u,v_1} & \sigma_{v_1}^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{v_2}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{v_K}^2 \end{pmatrix} \tag{24}$$

with $\sigma_u^2 = 0.037$, $\sigma_{u,v_1} = -0.035$, $\sigma_{v_1}^2 = ... = \sigma_{v_K}^2 = 0.045$. Notice, therefore, that the first predictor, $x_{1,t}$ is endogenous (with an endogeneity correlation parameter $\phi_1 = -0.83$), while the remaining predictors $x_{2,t}, ..., x_{K,t}$ are exogenous. For the autoregressive parameter we again consider $\rho = 1 - c/T$ with $c \in \{-5, 2.5, 0, 2.5, 5, 10, 25, 50, 75, 100, 125, 150, 200, 250\}$.

Table 6 reports the empirical rejection frequencies, for $T = 250$ and $T = 1000$ and for $K \in \{1, 3, 5, 10\}$, for the Wald-type IVX tests W_{zx} and W_{zx}^{EW} discussed in Remark 9, together with the RWB and FRWB bootstrap implementations of W_{zx}, denoted $W_{zx}^{*,RWB}$ and $W_{zx}^{*,FRWB}$, respectively, computed as described in Remark 22. In the context of $W_{zx}^{*,RWB}$, in Step 2 of the multivariate version of Algorithm 1 autoregressions of length $p+1$ were fitted to each element of $\boldsymbol{x}_t$ with p selected in each case by BIC using the same range of values of p as were used in the simulations for a single predictor.

For $K = 1$ (the single predictor case), and in line with what was observed in section 5.1.1 for the two-sided tests based under DGP1, all of the Wald-based IVX statistics display empirical rejection frequencies close to the nominal level. Again, $W_{zx}^{*,RWB}$ displays the smallest size distortions among the tests considered. For instance, for a 5% significance level the rejection frequencies of $W_{zx}^{*,RWB}$ are in the range [0.042, 0.056] for $T = 250$ and [0.038, 0.056] for $T = 1000$, whereas for $W_{zx}^{*,FRWB}$, W_{zx}^{EW} and W_{zx} these are [0.037, 0.058], [0.045, 0.064], and [0.040, 0.060], respectively, when $T = 250$ and [0.034, 0.060], [0.036, 0.060] and [0.035, 0.059], respectively, when $T = 1000$.

However, it is as K increases that the significant advantage of the RWB becomes clear, particularly in the case where the predictors are strongly persistent. It is clear from the results that the $W_{zx}^{*,FRWB}$, W_{zx}^{EW} and W_{zx} tests are not reliable when the predictors are strongly persistent. The rejection frequencies we observe for W_{zx} are in line with those reported in Xu and Guo (2020) who also show that the quality of the prediction from the asymptotic theory deteriorates as the number of regressors, K, specified in the predictive regression increases. For instance, for $K = 3$ and $c < 0$; for $K = 5$ and $c < 2.5$; and for $K = 10$ and $c < 25$, even for $T = 1000$ all three of these tests display rejection frequencies larger than 15% at a 5% nominal level. For the smaller sample, $T = 250$, qualitatively similar size behaviour is observed (but with distortions of larger magnitude) for $W_{zx}^{*,FRWB}$ and W_{zx}. However, W_{zx}^{EW} becomes severely oversized as K increases, for all values of c. For instance, for $K = 10$, $T = 250$ and a 5% significance level, the smallest empirical rejection frequencies seen for this statistic is more than double the significance level considered. To illustrate the severity of the size distortions, observe from Table 6 that, for $K = 10$ unit root predictors ($c = 0$) and a 5% significance level, the empirical rejection frequencies of $W_{zx}^{*,FRWB}$, W_{zx}^{EW} and W_{zx} are 30.6%, 40.6% and 32.4%, respectively, for $T = 250$, and 29.5%, 30.0% and 28.0%, respectively for $T = 1000$. For mildly explosive predictors, the situation is even worse with empirical size in the region of 70% for each of $W_{zx}^{*,FRWB}$, W_{zx}^{EW} and W_{zx} when $K = 10$ and $c = -5$.

In contrast, the residual wild bootstrap based test, $W_{zx}^{*,RWB}$, controls empirical size much better than the other tests with empirical rejection frequencies acceptably close to the nominal level for all of the values of K considered. Some size distortions remain for values of $c \leq 5$, albeit unlike with the other tests these do not get appreciably worse as K increases. Moreover, in those cases where size distortions are seen with the $W_{zx}^{*,RWB}$ test, these are very much smaller than those seen for those cases with the other tests. For example, for tests run at the 5% nominal level, there are no entries in Table 6 where $W_{zx}^{*,RWB}$

displays an empirical size in excess of 10%, which compares very favourably with the other tests.

Finally, although not reported here we also investigated the finite sample behaviour of the partial IVX t-type tests discussed in Remark 9. To summarise our findings, we found that, for both one-sided and two-sided implementations, the t-type tests associated with the exogenous predictors, $x_{2,t}, ..., x_{K,t}$, all displayed qualitatively similar finite sample size properties to those which were observed in section 5.1.1 for the single predictive regression case for DGP1 with $\phi = 0$ (see Table 4). For the t-type tests associated with the endogenous predictor, $x_{1,t}$, both one-sided and two-sided versions of the RWB implementation of the tests continued to display good finite sample size control, regardless of the number of predictors, K, and the value of c. In contrast, however, the empirical sizes of the other implementations of the tests, including those based on the FRWB, deteriorated very badly as K increased, rendering these tests highly unreliable in practice.

6 Conclusions

In this paper we have extended the IVX-based predictability tests of Kostakis *et al.* (2015) in three distinct ways. First, we have shown that provided either a suitable bootstrap implementation is employed or Eicker-White standard errors are used, these tests still deliver asymptotically pivotal inference, regardless of the degree of persistence or endogeneity of the predictor, under considerably weaker assumptions on the innovations, including quite general forms of conditional and unconditional heteroskedasticity, than are required by Kostakis *et al.* (2015) in their analysis. Second, we have developed asymptotically valid residual and fixed regressor wild bootstrap implementations of the IVX tests and established the conditions required for their asymptotic validity. Simulation evidence has been provided which demonstrates that tests based around a residual wild bootstrap resampling scheme perform well in finite samples, largely correcting the finite sample size distortions seen with the asymptotic tests of Kostakis *et al.* (2015) in some scenarios. Third, we have shown how sub-sample implementations of the IVX approach, again based on the residual wild bootstrap, can be used to develop asymptotically valid one-sided and two-sided tests for the presence of temporary windows of predictability.

We finish with two suggestions for further research. First, our exposition in the paper has focused, like the bulk of this literature, on the case where the predictive regression contains a single predictive regressor. As we have discussed in the text, the methods discussed in this paper readily extend to the case of multiple regressors, provided these satisfy the condition imposed by Kostakis *et al.* (2015) that all of the regressors belong to the same persistence class; that is, they are all either strongly persistent or are all weakly persistent. However, based on the results in this paper, we conjecture that the bootstrap IVX-based tests considered in this paper would also retain asymptotic validity in the considerably more general scenario where some of the regressors were weakly persistent and others were strongly persistent, and where the strongly persistent regressors could be allowed to be cointegrated with each other. The practitioner would not need to know which of the regressors were weakly persistent and which were strongly persistent, and would not need to know the form of any cointegrating relations holding among the latter. A formal proof of this conjecture is likely to be very involved and is certainly beyond the remit of this paper, but constitutes an important next step in this research agenda. The technical material in this paper provides important groundwork for this endeavour. Second, the finite sample efficacy of the residual wild bootstrap IVX tests proposed in this paper will depend, in part, on the finite sample properties of the autoregressive parameter estimates obtained in Step 2 of Algorithm 1. The OLS estimates we have employed are known to suffer from non-negligible finite sample biases. It might be useful to explore a refinement of Algorithm 1 based on the bootstrap-after-bootstrap approach of Kilian (1998) (in this approach the bootstrap data in Step 5 are generated not using the original point estimates from the fitted autoregressive model but using bias-corrected estimates which are themselves obtained by bootstrap methods) to investigate if this further improves on the finite sample properties of our proposed bootstrap tests.

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Table 1: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.95; \quad -0.95 \quad 1]$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				**5%**				**10%**			
Left-sided tests - T=250												
-5	0.010	0.000	0.001	0.000	0.046	0.004	0.004	0.003	0.097	0.011	0.013	0.012
-2.5	0.006	0.000	0.000	0.000	0.045	0.000	0.000	0.001	0.111	0.002	0.002	0.002
0	0.014	0.000	0.000	0.000	0.041	0.001	0.001	0.001	0.065	0.002	0.002	0.003
2.5	0.021	0.001	0.001	0.001	0.062	0.005	0.005	0.005	0.099	0.012	0.012	0.011
5	0.023	0.002	0.001	0.001	0.068	0.010	0.011	0.010	0.113	0.026	0.026	0.025
10	0.020	0.003	0.003	0.002	0.064	0.019	0.019	0.018	0.115	0.043	0.044	0.042
25	0.017	0.006	0.006	0.005	0.057	0.029	0.030	0.028	0.108	0.059	0.059	0.058
50	0.012	0.007	0.006	0.006	0.056	0.034	0.036	0.035	0.105	0.072	0.074	0.071
75	0.011	0.007	0.007	0.007	0.056	0.037	0.038	0.037	0.105	0.078	0.082	0.080
100	0.011	0.007	0.008	0.008	0.054	0.038	0.040	0.038	0.108	0.083	0.087	0.084
125	0.011	0.007	0.008	0.007	0.054	0.039	0.042	0.041	0.109	0.089	0.091	0.087
150	0.011	0.007	0.008	0.008	0.055	0.043	0.046	0.042	0.107	0.090	0.094	0.090
200	0.010	0.008	0.009	0.009	0.054	0.046	0.048	0.045	0.108	0.092	0.097	0.094
250	0.011	0.010	0.011	0.009	0.054	0.048	0.051	0.048	0.110	0.099	0.101	0.098
Right-sided tests - $T = 250$												
-5	0.011	0.016	0.020	0.017	0.046	0.074	0.080	0.073	0.092	0.151	0.155	0.150
-2.5	0.010	0.016	0.018	0.017	0.041	0.094	0.097	0.093	0.088	0.240	0.241	0.238
0	0.011	0.022	0.025	0.023	0.053	0.105	0.114	0.110	0.112	0.225	0.231	0.228
2.5	0.014	0.022	0.027	0.023	0.064	0.112	0.116	0.115	0.124	0.226	0.233	0.228
5	0.013	0.023	0.026	0.023	0.062	0.107	0.116	0.112	0.128	0.208	0.215	0.211
10	0.014	0.022	0.025	0.024	0.062	0.097	0.102	0.099	0.120	0.181	0.186	0.184
25	0.012	0.017	0.019	0.017	0.057	0.078	0.084	0.080	0.110	0.147	0.150	0.148
50	0.011	0.013	0.016	0.015	0.052	0.067	0.072	0.067	0.108	0.135	0.139	0.136
75	0.011	0.014	0.015	0.014	0.053	0.064	0.068	0.065	0.105	0.125	0.129	0.126
100	0.011	0.012	0.016	0.014	0.053	0.061	0.065	0.062	0.105	0.119	0.124	0.119
125	0.011	0.012	0.014	0.013	0.052	0.060	0.063	0.060	0.103	0.116	0.120	0.116
150	0.011	0.012	0.013	0.013	0.053	0.056	0.060	0.059	0.103	0.111	0.115	0.111
200	0.010	0.012	0.013	0.011	0.050	0.054	0.056	0.053	0.103	0.109	0.112	0.109
250	0.010	0.011	0.013	0.010	0.051	0.051	0.055	0.053	0.103	0.103	0.107	0.102
Two-sided tests - $T = 250$												
-5	0.010	0.008	0.012	0.011	0.048	0.038	0.044	0.039	0.095	0.075	0.083	0.077
-2.5	0.009	0.008	0.010	0.009	0.038	0.040	0.048	0.044	0.083	0.094	0.098	0.094
0	0.010	0.011	0.013	0.011	0.047	0.051	0.057	0.053	0.095	0.105	0.115	0.110
2.5	0.012	0.012	0.015	0.013	0.053	0.058	0.062	0.060	0.107	0.116	0.121	0.120
5	0.012	0.012	0.014	0.012	0.054	0.058	0.063	0.060	0.111	0.118	0.127	0.121
10	0.012	0.012	0.015	0.013	0.055	0.060	0.066	0.060	0.109	0.115	0.122	0.118
25	0.011	0.010	0.014	0.012	0.056	0.056	0.060	0.058	0.105	0.109	0.114	0.109
50	0.011	0.010	0.012	0.011	0.051	0.051	0.054	0.052	0.102	0.101	0.108	0.102
75	0.011	0.010	0.012	0.011	0.049	0.047	0.052	0.049	0.102	0.100	0.106	0.102
100	0.009	0.010	0.011	0.010	0.049	0.048	0.052	0.050	0.101	0.099	0.105	0.100
125	0.010	0.010	0.011	0.011	0.050	0.049	0.053	0.051	0.101	0.100	0.105	0.101
150	0.009	0.009	0.011	0.010	0.051	0.049	0.054	0.052	0.102	0.099	0.106	0.101
200	0.009	0.010	0.012	0.010	0.050	0.048	0.054	0.050	0.101	0.099	0.104	0.098
250	0.010	0.010	0.012	0.011	0.049	0.048	0.053	0.050	0.101	0.100	0.105	0.101

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				**5%**				**10%**			
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.045	0.002	0.003	0.003	0.094	0.010	0.011	0.010
-2.5	0.005	0.000	0.000	0.000	0.046	0.000	0.000	0.000	0.109	0.001	0.001	0.001
0	0.015	0.000	0.000	0.000	0.042	0.001	0.001	0.001	0.068	0.003	0.003	0.003
2.5	0.024	0.001	0.001	0.001	0.060	0.006	0.006	0.006	0.097	0.015	0.016	0.016
5	0.023	0.002	0.002	0.002	0.068	0.013	0.014	0.013	0.109	0.028	0.028	0.028
10	0.021	0.004	0.004	0.004	0.064	0.022	0.021	0.021	0.113	0.044	0.044	0.044
25	0.016	0.007	0.007	0.008	0.061	0.030	0.030	0.031	0.107	0.064	0.064	0.063
50	0.014	0.008	0.008	0.008	0.057	0.035	0.035	0.035	0.108	0.072	0.072	0.071
75	0.012	0.008	0.008	0.008	0.055	0.039	0.039	0.038	0.107	0.078	0.080	0.078
100	0.013	0.008	0.007	0.008	0.055	0.040	0.040	0.040	0.105	0.082	0.081	0.081
125	0.013	0.008	0.008	0.008	0.054	0.041	0.042	0.041	0.104	0.082	0.084	0.082
150	0.011	0.009	0.008	0.008	0.053	0.043	0.043	0.043	0.105	0.085	0.085	0.083
200	0.012	0.009	0.008	0.008	0.053	0.044	0.044	0.042	0.105	0.088	0.088	0.088
250	0.012	0.009	0.009	0.009	0.053	0.044	0.044	0.044	0.106	0.090	0.091	0.089
Right-sided tests - $T = 1000$												
-5	0.007	0.012	0.014	0.013	0.039	0.064	0.065	0.064	0.086	0.139	0.140	0.137
-2.5	0.008	0.018	0.017	0.016	0.041	0.092	0.092	0.091	0.086	0.229	0.226	0.225
0	0.010	0.019	0.020	0.019	0.050	0.103	0.104	0.102	0.105	0.223	0.222	0.223
2.5	0.010	0.020	0.021	0.020	0.059	0.107	0.108	0.107	0.117	0.218	0.218	0.218
5	0.010	0.020	0.020	0.019	0.059	0.105	0.106	0.106	0.121	0.205	0.207	0.205
10	0.010	0.020	0.020	0.018	0.059	0.097	0.098	0.098	0.116	0.181	0.182	0.181
25	0.011	0.016	0.017	0.016	0.055	0.081	0.082	0.081	0.108	0.154	0.154	0.151
50	0.010	0.015	0.015	0.015	0.051	0.070	0.071	0.070	0.104	0.133	0.133	0.133
75	0.011	0.013	0.013	0.013	0.051	0.067	0.068	0.067	0.102	0.126	0.127	0.126
100	0.010	0.013	0.013	0.012	0.051	0.065	0.066	0.065	0.104	0.123	0.124	0.123
125	0.009	0.012	0.012	0.011	0.051	0.063	0.063	0.063	0.103	0.121	0.122	0.121
150	0.009	0.012	0.011	0.011	0.051	0.061	0.061	0.061	0.103	0.121	0.121	0.120
200	0.008	0.012	0.011	0.010	0.050	0.059	0.060	0.059	0.106	0.119	0.120	0.119
250	0.010	0.011	0.011	0.010	0.050	0.059	0.059	0.059	0.105	0.116	0.116	0.116
Two-sided tests - $T = 1000$												
-5	0.007	0.006	0.007	0.006	0.040	0.030	0.032	0.031	0.087	0.066	0.067	0.066
-2.5	0.007	0.008	0.008	0.008	0.037	0.042	0.043	0.042	0.080	0.091	0.092	0.091
0	0.008	0.009	0.009	0.009	0.041	0.050	0.050	0.049	0.090	0.103	0.105	0.103
2.5	0.008	0.011	0.011	0.009	0.050	0.056	0.057	0.058	0.101	0.112	0.114	0.113
5	0.009	0.011	0.010	0.009	0.052	0.056	0.058	0.058	0.108	0.118	0.120	0.119
10	0.009	0.011	0.011	0.011	0.055	0.061	0.063	0.062	0.108	0.117	0.119	0.119
25	0.011	0.013	0.012	0.012	0.052	0.056	0.057	0.056	0.105	0.111	0.112	0.112
50	0.011	0.012	0.012	0.011	0.051	0.053	0.054	0.053	0.102	0.104	0.106	0.105
75	0.012	0.011	0.011	0.011	0.051	0.052	0.054	0.052	0.105	0.106	0.107	0.105
100	0.010	0.010	0.010	0.010	0.051	0.051	0.053	0.052	0.103	0.104	0.106	0.105
125	0.010	0.009	0.010	0.009	0.050	0.051	0.052	0.052	0.106	0.104	0.105	0.104
150	0.009	0.010	0.010	0.009	0.050	0.052	0.053	0.051	0.103	0.102	0.105	0.103
200	0.009	0.010	0.009	0.009	0.052	0.051	0.052	0.051	0.103	0.102	0.104	0.102
250	0.011	0.010	0.011	0.009	0.050	0.051	0.051	0.051	0.102	0.102	0.103	0.103

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4.

Table 2: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.90; \quad -0.90 \quad 1]$.

Left-sided tests - $T = 250$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
		1%				**5%**				**10%**		
-5	0.009	0.001	0.001	0.000	0.049	0.004	0.005	0.004	0.097	0.014	0.016	0.013
-2.5	0.008	0.000	0.000	0.000	0.047	0.000	0.001	0.001	0.111	0.002	0.002	0.002
0	0.012	0.000	0.000	0.000	0.039	0.001	0.001	0.001	0.063	0.004	0.004	0.004
2.5	0.019	0.001	0.001	0.001	0.059	0.005	0.005	0.005	0.096	0.014	0.014	0.013
5	0.022	0.002	0.001	0.002	0.066	0.010	0.011	0.010	0.112	0.027	0.027	0.027
10	0.018	0.003	0.003	0.003	0.063	0.019	0.020	0.020	0.111	0.044	0.045	0.044
25	0.015	0.006	0.006	0.006	0.055	0.030	0.032	0.030	0.107	0.061	0.061	0.060
50	0.011	0.006	0.007	0.006	0.054	0.033	0.036	0.034	0.105	0.072	0.074	0.072
75	0.009	0.007	0.007	0.007	0.054	0.038	0.038	0.037	0.105	0.078	0.081	0.080
100	0.010	0.008	0.009	0.008	0.050	0.037	0.040	0.039	0.106	0.084	0.086	0.084
125	0.011	0.008	0.008	0.007	0.053	0.041	0.043	0.041	0.107	0.087	0.090	0.088
150	0.011	0.008	0.009	0.009	0.054	0.043	0.045	0.044	0.106	0.089	0.092	0.089
200	0.011	0.009	0.010	0.009	0.054	0.046	0.049	0.047	0.107	0.093	0.095	0.092
250	0.011	0.009	0.011	0.010	0.055	0.048	0.051	0.048	0.106	0.096	0.099	0.097

Left-sided tests - $T = 1000$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
		1%				**5%**				**10%**		
-5	0.008	0.000	0.000	0.000	0.047	0.003	0.003	0.003	0.095	0.013	0.013	0.013
-2.5	0.006	0.000	0.000	0.000	0.048	0.000	0.000	0.000	0.109	0.001	0.001	0.001
0	0.013	0.000	0.000	0.000	0.040	0.002	0.002	0.002	0.066	0.004	0.004	0.004
2.5	0.021	0.001	0.001	0.001	0.058	0.007	0.007	0.007	0.095	0.018	0.017	0.018
5	0.022	0.003	0.003	0.003	0.065	0.014	0.014	0.014	0.106	0.030	0.031	0.030
10	0.019	0.004	0.004	0.004	0.063	0.023	0.022	0.022	0.110	0.047	0.045	0.045
25	0.016	0.008	0.008	0.008	0.059	0.031	0.031	0.031	0.110	0.063	0.063	0.063
50	0.014	0.008	0.008	0.008	0.056	0.036	0.036	0.036	0.108	0.076	0.076	0.075
75	0.012	0.008	0.008	0.008	0.054	0.039	0.040	0.039	0.108	0.080	0.081	0.080
100	0.013	0.008	0.008	0.008	0.054	0.040	0.041	0.040	0.107	0.083	0.083	0.082
125	0.013	0.008	0.008	0.008	0.055	0.044	0.043	0.043	0.105	0.085	0.084	0.083
150	0.012	0.008	0.008	0.008	0.055	0.045	0.044	0.044	0.103	0.085	0.086	0.084
200	0.012	0.008	0.008	0.008	0.054	0.045	0.046	0.046	0.104	0.088	0.089	0.088
250	0.011	0.009	0.009	0.009	0.052	0.046	0.046	0.046	0.105	0.090	0.091	0.091

Right-sided tests - $T = 250$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.010	0.015	0.019	0.017	0.044	0.074	0.079	0.073	0.093	0.150	0.155	0.149
-2.5	0.010	0.016	0.019	0.017	0.042	0.093	0.100	0.094	0.091	0.234	0.238	0.234
0	0.011	0.021	0.025	0.022	0.054	0.104	0.112	0.108	0.113	0.225	0.231	0.226
2.5	0.013	0.023	0.026	0.024	0.063	0.110	0.114	0.112	0.125	0.218	0.227	0.221
5	0.013	0.024	0.026	0.024	0.062	0.106	0.114	0.109	0.128	0.202	0.208	0.204
10	0.014	0.022	0.026	0.023	0.061	0.094	0.100	0.096	0.121	0.178	0.183	0.180
25	0.011	0.017	0.019	0.017	0.058	0.078	0.082	0.079	0.111	0.146	0.150	0.148
50	0.011	0.014	0.016	0.014	0.053	0.067	0.071	0.068	0.107	0.132	0.136	0.132
75	0.010	0.014	0.017	0.014	0.052	0.064	0.067	0.064	0.104	0.125	0.130	0.127
100	0.010	0.013	0.014	0.014	0.052	0.062	0.065	0.062	0.104	0.119	0.124	0.121
125	0.010	0.012	0.014	0.012	0.053	0.059	0.062	0.059	0.103	0.114	0.118	0.113
150	0.010	0.012	0.014	0.012	0.052	0.056	0.059	0.058	0.101	0.110	0.113	0.111
200	0.010	0.012	0.013	0.011	0.051	0.055	0.058	0.055	0.104	0.109	0.111	0.107
250	0.011	0.011	0.012	0.011	0.048	0.049	0.053	0.051	0.104	0.105	0.109	0.106

Right-sided tests - $T = 1000$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.008	0.012	0.013	0.013	0.041	0.066	0.067	0.064	0.085	0.140	0.141	0.140
-2.5	0.009	0.017	0.017	0.016	0.040	0.092	0.091	0.090	0.087	0.225	0.225	0.224
0	0.010	0.020	0.020	0.019	0.051	0.098	0.101	0.100	0.103	0.219	0.216	0.217
2.5	0.010	0.020	0.021	0.020	0.058	0.103	0.106	0.105	0.118	0.212	0.213	0.212
5	0.011	0.020	0.020	0.019	0.059	0.100	0.102	0.102	0.118	0.198	0.198	0.197
10	0.010	0.018	0.018	0.017	0.059	0.094	0.095	0.094	0.116	0.173	0.175	0.175
25	0.010	0.015	0.015	0.015	0.055	0.080	0.080	0.078	0.110	0.149	0.151	0.149
50	0.011	0.014	0.014	0.014	0.051	0.069	0.070	0.069	0.103	0.133	0.133	0.132
75	0.010	0.013	0.013	0.013	0.052	0.066	0.066	0.066	0.101	0.126	0.125	0.123
100	0.009	0.012	0.012	0.012	0.053	0.065	0.064	0.064	0.102	0.121	0.122	0.122
125	0.009	0.011	0.012	0.011	0.051	0.060	0.062	0.062	0.102	0.122	0.122	0.120
150	0.009	0.011	0.011	0.011	0.051	0.061	0.060	0.060	0.103	0.119	0.120	0.119
200	0.009	0.011	0.011	0.010	0.053	0.059	0.061	0.061	0.103	0.118	0.118	0.119
250	0.009	0.011	0.011	0.010	0.051	0.058	0.059	0.060	0.103	0.115	0.115	0.115

Two-sided tests - $T = 250$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.010	0.008	0.011	0.010	0.045	0.036	0.044	0.038	0.096	0.077	0.084	0.077
-2.5	0.009	0.008	0.010	0.010	0.037	0.042	0.047	0.044	0.084	0.095	0.100	0.095
0	0.010	0.011	0.013	0.011	0.048	0.052	0.057	0.053	0.095	0.104	0.113	0.109
2.5	0.011	0.011	0.015	0.014	0.055	0.059	0.062	0.061	0.107	0.113	0.119	0.117
5	0.012	0.012	0.015	0.013	0.054	0.059	0.064	0.060	0.107	0.115	0.125	0.119
10	0.012	0.012	0.016	0.013	0.055	0.057	0.065	0.061	0.108	0.112	0.121	0.116
25	0.010	0.011	0.013	0.012	0.054	0.055	0.060	0.057	0.105	0.107	0.114	0.109
50	0.010	0.010	0.011	0.011	0.049	0.050	0.054	0.052	0.102	0.103	0.107	0.102
75	0.010	0.010	0.011	0.011	0.048	0.049	0.053	0.049	0.102	0.099	0.105	0.101
100	0.010	0.010	0.011	0.010	0.049	0.048	0.052	0.050	0.100	0.099	0.105	0.101
125	0.009	0.008	0.011	0.010	0.052	0.050	0.054	0.052	0.101	0.098	0.105	0.101
150	0.008	0.009	0.010	0.010	0.049	0.050	0.054	0.052	0.102	0.099	0.104	0.102
200	0.009	0.009	0.011	0.010	0.049	0.048	0.052	0.050	0.104	0.102	0.107	0.102
250	0.009	0.010	0.012	0.010	0.049	0.049	0.054	0.049	0.101	0.097	0.103	0.099

Two-sided tests - $T = 1000$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.008	0.006	0.007	0.006	0.041	0.032	0.034	0.033	0.087	0.068	0.070	0.067
-2.5	0.007	0.008	0.008	0.008	0.037	0.040	0.043	0.041	0.081	0.090	0.091	0.090
0	0.009	0.011	0.010	0.010	0.043	0.048	0.049	0.048	0.087	0.100	0.102	0.101
2.5	0.008	0.011	0.010	0.010	0.050	0.056	0.057	0.057	0.098	0.109	0.113	0.112
5	0.010	0.011	0.011	0.010	0.051	0.058	0.059	0.058	0.104	0.117	0.116	0.116
10	0.009	0.010	0.010	0.010	0.054	0.059	0.061	0.060	0.105	0.115	0.117	0.116
25	0.010	0.011	0.011	0.010	0.052	0.056	0.056	0.055	0.105	0.110	0.111	0.109
50	0.010	0.011	0.012	0.011	0.051	0.053	0.054	0.052	0.102	0.106	0.106	0.105
75	0.010	0.011	0.011	0.011	0.051	0.053	0.053	0.053	0.102	0.105	0.106	0.104
100	0.010	0.011	0.010	0.010	0.052	0.052	0.052	0.053	0.103	0.103	0.105	0.104
125	0.010	0.010	0.010	0.010	0.051	0.052	0.053	0.053	0.103	0.103	0.105	0.104
150	0.009	0.010	0.010	0.009	0.051	0.052	0.054	0.052	0.104	0.105	0.105	0.104
200	0.009	0.010	0.009	0.009	0.049	0.050	0.053	0.051	0.106	0.105	0.107	0.107
250	0.010	0.009	0.009	0.009	0.050	0.050	0.051	0.050	0.105	0.104	0.105	0.106

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4.

Table 3: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.50; \quad -0.50 \quad 1]$.

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
	1%				**5%**				**10%**			
Left-sided tests - T=250												
-5	0.010	0.002	0.005	0.002	0.053	0.019	0.024	0.019	0.105	0.046	0.052	0.047
-2.5	0.010	0.000	0.001	0.000	0.050	0.006	0.007	0.005	0.102	0.016	0.017	0.016
0	0.006	0.000	0.001	0.001	0.030	0.005	0.006	0.006	0.059	0.017	0.017	0.017
2.5	0.009	0.002	0.002	0.002	0.045	0.016	0.017	0.016	0.086	0.037	0.039	0.037
5	0.012	0.004	0.004	0.004	0.050	0.023	0.024	0.023	0.095	0.051	0.052	0.050
10	0.012	0.006	0.006	0.006	0.052	0.030	0.031	0.030	0.099	0.062	0.064	0.063
25	0.011	0.007	0.007	0.006	0.050	0.036	0.038	0.036	0.101	0.076	0.079	0.077
50	0.010	0.006	0.007	0.007	0.048	0.039	0.040	0.039	0.100	0.083	0.087	0.085
75	0.008	0.007	0.007	0.006	0.050	0.042	0.045	0.043	0.097	0.082	0.085	0.083
100	0.009	0.007	0.008	0.007	0.049	0.043	0.045	0.043	0.097	0.085	0.088	0.087
125	0.010	0.008	0.009	0.008	0.051	0.044	0.046	0.045	0.096	0.086	0.088	0.089
150	0.010	0.008	0.009	0.009	0.051	0.046	0.048	0.047	0.097	0.089	0.091	0.089
200	0.010	0.019	0.010	0.010	0.052	0.047	0.050	0.048	0.102	0.093	0.096	0.095
250	0.010	0.010	0.012	0.010	0.053	0.049	0.052	0.051	0.103	0.098	0.100	0.097
Right-sided tests - $T = 250$												
-5	0.009	0.016	0.020	0.015	0.046	0.072	0.079	0.072	0.097	0.144	0.152	0.143
-2.5	0.012	0.020	0.026	0.020	0.053	0.101	0.107	0.101	0.107	0.196	0.203	0.197
0	0.013	0.020	0.022	0.019	0.061	0.097	0.102	0.096	0.121	0.191	0.197	0.190
2.5	0.014	0.019	0.022	0.020	0.061	0.090	0.096	0.090	0.119	0.171	0.175	0.174
5	0.013	0.018	0.020	0.019	0.061	0.081	0.087	0.085	0.113	0.158	0.162	0.158
10	0.012	0.017	0.018	0.018	0.056	0.074	0.078	0.076	0.111	0.142	0.147	0.145
25	0.012	0.014	0.016	0.016	0.053	0.065	0.069	0.067	0.110	0.131	0.134	0.131
50	0.010	0.013	0.014	0.013	0.054	0.063	0.066	0.064	0.108	0.120	0.124	0.122
75	0.009	0.012	0.013	0.011	0.055	0.060	0.065	0.062	0.107	0.116	0.121	0.119
100	0.009	0.011	0.012	0.011	0.054	0.059	0.063	0.061	0.109	0.116	0.119	0.117
125	0.010	0.010	0.012	0.011	0.055	0.059	0.061	0.059	0.109	0.112	0.118	0.114
150	0.010	0.011	0.012	0.010	0.055	0.057	0.061	0.058	0.107	0.111	0.115	0.112
200	0.009	0.010	0.010	0.010	0.053	0.054	0.057	0.054	0.105	0.107	0.111	0.108
250	0.009	0.009	0.010	0.009	0.051	0.052	0.055	0.051	0.105	0.105	0.110	0.107
Two-sided tests - $T = 250$												
-5	0.009	0.009	0.015	0.009	0.048	0.043	0.055	0.042	0.098	0.089	0.102	0.090
-2.5	0.010	0.011	0.014	0.011	0.049	0.050	0.059	0.051	0.101	0.106	0.113	0.106
0	0.011	0.010	0.012	0.012	0.048	0.051	0.057	0.053	0.098	0.100	0.108	0.102
2.5	0.012	0.011	0.013	0.013	0.052	0.054	0.059	0.055	0.101	0.105	0.113	0.106
5	0.011	0.012	0.014	0.013	0.052	0.055	0.059	0.057	0.103	0.105	0.110	0.107
10	0.013	0.011	0.013	0.013	0.052	0.053	0.057	0.055	0.104	0.104	0.110	0.106
25	0.011	0.011	0.013	0.012	0.049	0.050	0.054	0.052	0.101	0.102	0.107	0.103
50	0.009	0.008	0.011	0.008	0.049	0.050	0.054	0.050	0.101	0.100	0.106	0.104
75	0.008	0.008	0.010	0.008	0.049	0.049	0.054	0.052	0.104	0.102	0.110	0.105
100	0.008	0.009	0.011	0.010	0.050	0.048	0.054	0.052	0.103	0.103	0.108	0.104
125	0.008	0.009	0.011	0.009	0.051	0.050	0.055	0.050	0.105	0.103	0.107	0.104
150	0.008	0.010	0.011	0.009	0.051	0.049	0.054	0.052	0.105	0.104	0.109	0.105
200	0.009	0.010	0.012	0.010	0.049	0.049	0.055	0.052	0.104	0.100	0.107	0.102
250	0.009	0.011	0.013	0.010	0.049	0.049	0.053	0.050	0.103	0.099	0.107	0.102

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
	1%				**5%**				**10%**			
Left-sided tests - $T = 1000$												
-5	0.010	0.002	0.002	0.002	0.050	0.018	0.018	0.018	0.097	0.045	0.045	0.044
-2.5	0.008	0.001	0.000	0.000	0.048	0.004	0.004	0.004	0.099	0.014	0.015	0.014
0	0.008	0.001	0.001	0.001	0.032	0.009	0.008	0.008	0.064	0.020	0.020	0.020
2.5	0.013	0.002	0.003	0.002	0.049	0.018	0.018	0.018	0.091	0.042	0.042	0.042
5	0.013	0.004	0.003	0.003	0.054	0.024	0.024	0.025	0.101	0.055	0.055	0.054
10	0.012	0.006	0.005	0.005	0.053	0.031	0.031	0.031	0.104	0.067	0.066	0.066
25	0.012	0.008	0.007	0.008	0.053	0.039	0.039	0.039	0.105	0.079	0.079	0.078
50	0.013	0.011	0.009	0.008	0.051	0.042	0.042	0.043	0.102	0.085	0.085	0.085
75	0.013	0.011	0.010	0.010	0.053	0.045	0.045	0.045	0.103	0.087	0.088	0.087
100	0.014	0.011	0.011	0.011	0.052	0.047	0.046	0.045	0.101	0.089	0.089	0.088
125	0.013	0.011	0.011	0.011	0.051	0.045	0.046	0.045	0.101	0.090	0.091	0.090
150	0.013	0.011	0.011	0.012	0.052	0.046	0.046	0.046	0.103	0.093	0.094	0.092
200	0.013	0.011	0.011	0.011	0.053	0.047	0.046	0.046	0.104	0.095	0.096	0.095
250	0.013	0.012	0.012	0.012	0.052	0.048	0.047	0.046	0.104	0.095	0.095	0.095
Right-sided tests - $T = 1000$												
-5	0.008	0.013	0.015	0.013	0.047	0.070	0.072	0.070	0.097	0.139	0.141	0.139
-2.5	0.009	0.018	0.016	0.016	0.047	0.095	0.096	0.094	0.102	0.191	0.193	0.191
0	0.011	0.018	0.019	0.018	0.056	0.091	0.090	0.091	0.116	0.189	0.185	0.185
2.5	0.012	0.019	0.020	0.019	0.058	0.087	0.087	0.086	0.114	0.167	0.168	0.167
5	0.011	0.018	0.018	0.018	0.057	0.082	0.081	0.080	0.111	0.154	0.155	0.155
10	0.011	0.017	0.015	0.016	0.053	0.073	0.074	0.074	0.106	0.139	0.140	0.139
25	0.009	0.013	0.013	0.012	0.052	0.064	0.065	0.063	0.101	0.122	0.125	0.124
50	0.011	0.012	0.012	0.012	0.049	0.057	0.059	0.058	0.097	0.115	0.115	0.114
75	0.010	0.012	0.012	0.012	0.049	0.056	0.057	0.057	0.096	0.112	0.112	0.111
100	0.011	0.012	0.012	0.011	0.049	0.056	0.059	0.057	0.101	0.110	0.111	0.111
125	0.010	0.012	0.012	0.011	0.049	0.054	0.056	0.055	0.101	0.109	0.110	0.111
150	0.011	0.012	0.012	0.012	0.050	0.055	0.055	0.054	0.100	0.109	0.109	0.110
200	0.011	0.012	0.012	0.012	0.050	0.054	0.054	0.054	0.098	0.110	0.109	0.109
250	0.011	0.012	0.013	0.012	0.050	0.053	0.054	0.055	0.102	0.108	0.108	0.108
Two-sided tests - $T = 1000$												
-5	0.008	0.008	0.008	0.007	0.047	0.042	0.043	0.041	0.098	0.088	0.090	0.089
-2.5	0.008	0.008	0.008	0.007	0.043	0.047	0.047	0.045	0.093	0.097	0.100	0.098
0	0.009	0.010	0.010	0.010	0.045	0.049	0.050	0.048	0.094	0.099	0.099	0.099
2.5	0.010	0.011	0.011	0.010	0.050	0.052	0.053	0.052	0.100	0.105	0.105	0.104
5	0.010	0.011	0.011	0.010	0.050	0.053	0.054	0.053	0.100	0.105	0.105	0.105
10	0.010	0.011	0.010	0.010	0.050	0.052	0.054	0.054	0.101	0.104	0.105	0.105
25	0.010	0.011	0.011	0.011	0.049	0.050	0.052	0.051	0.102	0.102	0.104	0.102
50	0.011	0.010	0.011	0.011	0.054	0.052	0.053	0.053	0.101	0.099	0.100	0.101
75	0.012	0.012	0.012	0.012	0.053	0.053	0.054	0.053	0.100	0.101	0.102	0.102
100	0.013	0.013	0.013	0.013	0.052	0.052	0.053	0.052	0.102	0.103	0.105	0.103
125	0.012	0.012	0.012	0.013	0.052	0.052	0.052	0.053	0.102	0.099	0.102	0.100
150	0.011	0.012	0.012	0.012	0.052	0.052	0.053	0.054	0.100	0.099	0.102	0.100
200	0.012	0.012	0.012	0.012	0.052	0.052	0.053	0.053	0.102	0.100	0.100	0.099
250	0.012	0.011	0.011	0.012	0.050	0.050	0.051	0.051	0.102	0.099	0.101	0.101

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4.

Table 4: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad 0; \quad 0 \quad 1]$.

	1%				5%				10%			
c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.011	0.011	0.017	0.010	0.052	0.051 1	0.060	0.052	0.102	0.102	0.111	0.102
-2.5	0.011	0.011	0.015	0.010	0.050	0.050	0.058	0.051	0.103	0.103	0.111	0.102
0	0.011	0.011	0.013	0.011	0.049	0.049	0.056	0.050	0.098	0.100	0.100	0.097
2.5	0.009	0.009	0.012	0.010	0.051	0.050	0.053	0.050	0.099	0.099	0.101	0.099
5	0.010	0.010	0.011	0.010	0.052	0.051	0.053	0.050	0.098	0.098	0.099	0.097
10	0.010	0.011	0.012	0.011	0.051	0.051	0.054	0.052	0.102	0.100	0.102	0.101
25	0.010	0.010	0.011	0.010	0.050	0.050	0.052	0.051	0.098	0.096	0.098	0.099
50	0.009	0.009	0.010	0.009	0.048	0.048	0.051	0.049	0.100	0.098	0.101	0.101
75	0.009	0.009	0.010	0.010	0.047	0.046	0.050	0.048	0.099	0.096	0.099	0.098
100	0.009	0.010	0.010	0.011	0.048	0.048	0.051	0.048	0.097	0.095	0.098	0.098
125	0.010	0.010	0.011	0.011	0.048	0.046	0.049	0.048	0.096	0.093	0.097	0.096
150	0.009	0.010	0.011	0.011	0.047	0.047	0.048	0.048	0.095	0.092	0.096	0.095
200	0.009	0.009	0.011	0.010	0.047	0.047	0.049	0.047	0.096	0.093	0.095	0.095
250	0.009	0.011	0.011	0.010	0.049	0.048	0.051	0.048	0.096	0.094	0.098	0.095
Right-sided tests - $T = 250$												
-5	0.012	0.011	0.017	0.0118	0.052	0.051	0.060	0.050	0.101	0.101	0.110	0.100
-2.5	0.010	0.011	0.015	0.0100	0.053	0.052	0.058	0.052	0.097	0.100	0.105	0.097
0	0.010	0.012	0.014	0.0111	0.051	0.051	0.055	0.049	0.098	0.100	0.103	0.098
2.5	0.010	0.011	0.012	0.0103	0.052	0.052	0.053	0.052	0.102	0.101	0.103	0.103
5	0.011	0.011	0.013	0.0106	0.052	0.051	0.053	0.051	0.104	0.101	0.105	0.102
10	0.011	0.010	0.011	0.0096	0.051	0.051	0.054	0.051	0.102	0.101	0.104	0.104
25	0.012	0.011	0.013	0.0116	0.051	0.052	0.054	0.052	0.104	0.103	0.105	0.104
50	0.011	0.010	0.012	0.0109	0.056	0.054	0.057	0.056	0.103	0.103	0.106	0.103
75	0.010	0.010	0.010	0.0100	0.054	0.055	0.058	0.054	0.104	0.102	0.108	0.104
100	0.010	0.010	0.012	0.0104	0.054	0.052	0.056	0.055	0.106	0.104	0.106	0.104
125	0.009	0.011	0.012	0.0114	0.053	0.052	0.056	0.053	0.104	0.103	0.106	0.103
150	0.010	0.010	0.012	0.0111	0.051	0.052	0.054	0.052	0.104	0.101	0.105	0.102
200	0.010	0.010	0.011	0.0103	0.051	0.051	0.055	0.052	0.102	0.099	0.103	0.102
250	0.009	0.010	0.011	0.011	0.053	0.051	0.054	0.054	0.100	0.098	0.102	0.100
Two-sided tests - $T = 250$												
-5	0.010	0.011	0.020	0.011	0.050	0.052	0.065	0.051	0.102	0.101	0.119	0.102
-2.5	0.011	0.011	0.019	0.011	0.052	0.052	0.066	0.052	0.101	0.101	0.116	0.102
0	0.009	0.010	0.015	0.011	0.050	0.051	0.058	0.051	0.098	0.100	0.111	0.099
2.5	0.010	0.010	0.012	0.010	0.050	0.050	0.057	0.052	0.100	0.101	0.106	0.102
5	0.010	0.010	0.012	0.011	0.049	0.051	0.055	0.052	0.102	0.100	0.106	0.101
10	0.012	0.012	0.013	0.012	0.049	0.048	0.052	0.050	0.100	0.101	0.108	0.103
25	0.011	0.011	0.013	0.013	0.053	0.053	0.057	0.055	0.102	0.100	0.106	0.103
50	0.010	0.010	0.011	0.010	0.052	0.051	0.056	0.052	0.103	0.101	0.108	0.104
75	0.009	0.010	0.011	0.009	0.052	0.051	0.056	0.053	0.102	0.101	0.108	0.102
100	0.008	0.009	0.011	0.010	0.050	0.049	0.053	0.052	0.101	0.100	0.107	0.102
125	0.009	0.010	0.012	0.011	0.050	0.049	0.054	0.050	0.100	0.097	0.105	0.101
150	0.009	0.010	0.012	0.010	0.049	0.051	0.056	0.052	0.099	0.097	0.102	0.100
200	0.009	0.009	0.012	0.010	0.052	0.053	0.056	0.054	0.098	0.099	0.104	0.099
250	0.010	0.010	0.012	0.010	0.049	0.051	0.054	0.051	0.101	0.100	0.105	0.101

	1%				5%				10%			
c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.009	0.009	0.009	0.009	0.051	0.050	0.051	0.051	0.097	0.096	0.098	0.096
-2.5	0.010	0.009	0.010	0.009	0.050	0.050	0.050	0.050	0.097	0.096	0.096	0.095
0	0.011	0.011	0.011	0.010	0.053	0.052	0.052	0.053	0.102	0.102	0.103	0.103
2.5	0.012	0.012	0.011	0.012	0.055	0.055	0.054	0.053	0.102	0.103	0.103	0.103
5	0.012	0.011	0.011	0.011	0.052	0.051	0.053	0.051	0.105	0.106	0.105	0.105
10	0.010	0.011	0.010	0.011	0.051	0.052	0.052	0.052	0.104	0.103	0.104	0.104
25	0.011	0.011	0.011	0.011	0.054	0.053	0.053	0.052	0.104	0.104	0.104	0.103
50	0.012	0.011	0.011	0.011	0.053	0.052	0.053	0.053	0.105	0.104	0.106	0.104
75	0.012	0.011	0.011	0.011	0.053	0.051	0.051	0.051	0.103	0.102	0.104	0.102
100	0.012	0.011	0.011	0.011	0.052	0.052	0.052	0.051	0.103	0.104	0.104	0.103
125	0.012	0.011	0.012	0.012	0.051	0.051	0.052	0.051	0.103	0.103	0.103	0.102
150	0.011	0.012	0.012	0.012	0.052	0.051	0.053	0.052	0.103	0.102	0.103	0.102
200	0.011	0.011	0.011	0.011	0.053	0.052	0.053	0.053	0.100	0.100	0.101	0.100
250	0.011	0.011	0.011	0.012	0.051	0.051	0.051	0.052	0.099	0.099	0.100	0.098
Right-sided tests - $T = 1000$												
-5	0.011	0.011	0.012	0.010	0.048	0.048	0.049	0.048	0.100	0.099	0.103	0.099
-2.5	0.010	0.010	0.010	0.009	0.048	0.048	0.049	0.046	0.096	0.096	0.097	0.095
0	0.009	0.009	0.009	0.009	0.050	0.049	0.048	0.048	0.099	0.100	0.101	0.100
2.5	0.009	0.010	0.010	0.009	0.051	0.051	0.052	0.050	0.102	0.103	0.102	0.101
5	0.009	0.009	0.009	0.009	0.052	0.053	0.053	0.053	0.102	0.103	0.103	0.103
10	0.010	0.010	0.010	0.009	0.052	0.052	0.053	0.052	0.100	0.101	0.102	0.100
25	0.010	0.010	0.010	0.010	0.050	0.049	0.050	0.050	0.096	0.097	0.098	0.097
50	0.010	0.010	0.010	0.009	0.049	0.048	0.050	0.049	0.096	0.096	0.096	0.096
75	0.011	0.010	0.011	0.010	0.050	0.049	0.049	0.049	0.098	0.098	0.098	0.097
100	0.011	0.011	0.012	0.011	0.049	0.047	0.049	0.048	0.100	0.099	0.099	0.099
125	0.011	0.010	0.011	0.010	0.049	0.048	0.050	0.049	0.100	0.098	0.100	0.101
150	0.011	0.010	0.011	0.011	0.052	0.050	0.051	0.050	0.100	0.100	0.101	0.101
200	0.010	0.010	0.010	0.009	0.053	0.051	0.052	0.052	0.102	0.101	0.102	0.101
250	0.009	0.009	0.009	0.010	0.052	0.051	0.053	0.053	0.103	0.102	0.102	0.102
Two-sided tests - $T = 1000$												
-5	0.009	0.010	0.011	0.009	0.048	0.049	0.052	0.049	0.098	0.097	0.099	0.099
-2.5	0.011	0.010	0.011	0.010	0.050	0.050	0.051	0.049	0.098	0.097	0.099	0.096
0	0.010	0.010	0.010	0.010	0.050	0.050	0.050	0.049	0.100	0.100	0.100	0.101
2.5	0.010	0.011	0.010	0.010	0.050	0.051	0.051	0.050	0.102	0.105	0.105	0.103
5	0.010	0.011	0.010	0.010	0.052	0.050	0.050	0.051	0.104	0.104	0.106	0.104
10	0.011	0.010	0.010	0.010	0.051	0.050	0.051	0.050	0.104	0.103	0.104	0.104
25	0.011	0.010	0.010	0.011	0.052	0.053	0.052	0.050	0.103	0.100	0.103	0.103
50	0.012	0.012	0.012	0.011	0.052	0.051	0.051	0.051	0.103	0.100	0.103	0.103
75	0.011	0.011	0.012	0.012	0.052	0.052	0.052	0.052	0.102	0.099	0.101	0.100
100	0.012	0.012	0.012	0.012	0.052	0.051	0.052	0.053	0.098	0.098	0.101	0.099
125	0.011	0.012	0.011	0.011	0.051	0.051	0.051	0.052	0.100	0.100	0.102	0.100
150	0.011	0.012	0.011	0.011	0.052	0.051	0.051	0.052	0.103	0.102	0.104	0.102
200	0.013	0.011	0.012	0.011	0.051	0.050	0.051	0.051	0.105	0.103	0.106	0.105
250	0.011	0.011	0.011	0.011	0.050	0.050	0.051	0.050	0.103	0.102	0.104	0.105

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4.

Table 5: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP2 (ARCH with Leverage Effects):** $\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \boldsymbol{\psi}_t$ with $\boldsymbol{\psi}_t = (a_t; e_t)' = (\varepsilon_{1t}\sqrt{1 + \frac{1}{2}a_{t-1}^2 \mathbb{I}_{\{a_{t-1}<0\}}}; \varepsilon_{2t})'$ and $(\varepsilon_{1t}, \varepsilon_{2t})' \sim NIID(0, \mathbf{I}_2)$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				**5%**				**10%**			
Left-sided tests - $T = 250$												
5	0.011	0.022	0.024	0.024	0.062	0.099	0.106	0.107	0.127	0.195	0.203	0.205
10	0.013	0.019	0.023	0.024	0.059	0.088	0.096	0.099	0.122	0.174	0.181	0.184
25	0.012	0.015	0.018	0.022	0.059	0.076	0.081	0.092	0.116	0.144	0.152	0.162
50	0.012	0.015	0.016	0.023	0.058	0.067	0.074	0.089	0.117	0.131	0.137	0.158
75	0.012	0.014	0.016	0.025	0.060	0.062	0.069	0.090	0.118	0.124	0.131	0.157
100	0.011	0.014	0.015	0.026	0.060	0.061	0.067	0.089	0.115	0.120	0.125	0.154
125	0.012	0.013	0.016	0.027	0.059	0.060	0.066	0.088	0.117	0.116	0.120	0.151
150	0.013	0.014	0.016	0.026	0.059	0.058	0.063	0.085	0.113	0.111	0.119	0.149
200	0.011	0.012	0.015	0.024	0.057	0.056	0.060	0.082	0.111	0.108	0.113	0.146
250	0.011	0.012	0.013	0.023	0.053	0.053	0.057	0.078	0.109	0.106	0.110	0.138
Right-sided tests - $T = 250$												
5	0.017	0.002	0.002	0.003	0.058	0.014	0.015	0.015	0.101	0.035	0.035	0.036
10	0.016	0.004	0.004	0.004	0.059	0.021	0.022	0.024	0.110	0.050	0.050	0.053
25	0.013	0.006	0.006	0.010	0.061	0.030	0.030	0.039	0.111	0.065	0.066	0.078
50	0.016	0.007	0.008	0.015	0.062	0.037	0.038	0.053	0.109	0.071	0.073	0.092
75	0.015	0.008	0.009	0.019	0.061	0.039	0.041	0.061	0.111	0.074	0.078	0.104
100	0.016	0.008	0.010	0.022	0.059	0.037	0.040	0.065	0.111	0.078	0.080	0.109
125	0.015	0.009	0.009	0.024	0.057	0.039	0.041	0.067	0.110	0.081	0.083	0.114
150	0.014	0.008	0.010	0.025	0.059	0.040	0.042	0.071	0.109	0.082	0.085	0.117
200	0.013	0.010	0.011	0.025	0.056	0.041	0.045	0.072	0.106	0.087	0.091	0.118
250	0.012	0.011	0.011	0.025	0.054	0.043	0.047	0.075	0.102	0.088	0.093	0.119
Two-sided tests - $T = 250$												
5	0.010	0.012	0.015	0.014	0.053	0.055	0.065	0.063	0.106	0.112	0.122	0.122
10	0.011	0.012	0.014	0.014	0.051	0.053	0.060	0.062	0.108	0.109	0.117	0.124
25	0.011	0.011	0.013	0.017	0.056	0.052	0.057	0.069	0.112	0.105	0.111	0.131
50	0.014	0.012	0.014	0.022	0.059	0.050	0.056	0.081	0.117	0.103	0.112	0.142
75	0.015	0.011	0.013	0.027	0.061	0.051	0.057	0.090	0.118	0.101	0.110	0.150
100	0.015	0.011	0.014	0.030	0.063	0.053	0.058	0.095	0.118	0.098	0.107	0.154
125	0.014	0.012	0.014	0.032	0.062	0.051	0.059	0.098	0.116	0.098	0.107	0.155
150	0.014	0.011	0.014	0.033	0.060	0.051	0.057	0.097	0.115	0.096	0.105	0.156
200	0.012	0.011	0.014	0.033	0.058	0.050	0.056	0.096	0.110	0.096	0.104	0.154
250	0.013	0.011	0.014	0.029	0.053	0.051	0.054	0.092	0.106	0.096	0.104	0.153

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				**5%**				**10%**			
Left-sided tests - $T = 1000$												
5	0.012	0.021	0.021	0.021	0.060	0.097	0.101	0.102	0.124	0.191	0.193	0.194
10	0.012	0.019	0.020	0.020	0.059	0.090	0.092	0.093	0.118	0.170	0.173	0.177
25	0.012	0.017	0.016	0.019	0.055	0.076	0.075	0.080	0.109	0.146	0.146	0.152
50	0.011	0.014	0.015	0.018	0.053	0.066	0.068	0.074	0.107	0.128	0.130	0.140
75	0.011	0.013	0.014	0.017	0.055	0.064	0.066	0.075	0.105	0.120	0.121	0.136
100	0.011	0.013	0.013	0.018	0.057	0.061	0.063	0.076	0.107	0.118	0.118	0.137
125	0.012	0.012	0.013	0.018	0.058	0.061	0.062	0.078	0.109	0.116	0.118	0.139
150	0.012	0.012	0.012	0.019	0.059	0.061	0.063	0.080	0.114	0.119	0.119	0.142
200	0.012	0.012	0.012	0.022	0.061	0.060	0.063	0.085	0.116	0.115	0.118	0.150
250	0.012	0.013	0.013	0.023	0.062	0.062	0.063	0.088	0.120	0.116	0.118	0.151
Right-sided tests - $T = 1000$												
5	0.015	0.002	0.002	0.002	0.057	0.013	0.013	0.013	0.107	0.034	0.034	0.035
10	0.014	0.003	0.003	0.004	0.057	0.021	0.020	0.021	0.109	0.047	0.047	0.049
25	0.012	0.006	0.006	0.007	0.057	0.030	0.030	0.034	0.110	0.069	0.068	0.073
50	0.014	0.008	0.007	0.011	0.058	0.037	0.037	0.045	0.110	0.075	0.075	0.084
75	0.015	0.008	0.008	0.013	0.061	0.041	0.040	0.052	0.108	0.079	0.079	0.094
100	0.016	0.009	0.009	0.015	0.060	0.041	0.041	0.055	0.109	0.082	0.081	0.099
125	0.016	0.010	0.009	0.018	0.061	0.042	0.042	0.058	0.111	0.081	0.082	0.105
150	0.017	0.009	0.009	0.021	0.060	0.042	0.042	0.062	0.111	0.083	0.084	0.109
200	0.017	0.009	0.010	0.023	0.062	0.042	0.042	0.067	0.112	0.083	0.084	0.112
250	0.018	0.010	0.010	0.026	0.062	0.042	0.044	0.071	0.111	0.085	0.086	0.119
Two-sided tests - $T = 1000$												
5	0.010	0.011	0.011	0.011	0.051	0.056	0.058	0.058	0.103	0.109	0.114	0.114
10	0.010	0.012	0.011	0.012	0.053	0.056	0.058	0.059	0.103	0.108	0.112	0.115
25	0.012	0.011	0.012	0.013	0.053	0.053	0.055	0.059	0.105	0.104	0.106	0.113
50	0.012	0.012	0.012	0.014	0.054	0.051	0.052	0.062	0.106	0.102	0.105	0.119
75	0.014	0.011	0.011	0.017	0.059	0.052	0.054	0.069	0.115	0.103	0.106	0.127
100	0.014	0.012	0.011	0.019	0.061	0.055	0.054	0.074	0.115	0.102	0.104	0.130
125	0.015	0.011	0.011	0.021	0.064	0.054	0.055	0.079	0.115	0.102	0.104	0.136
150	0.016	0.011	0.012	0.024	0.064	0.054	0.056	0.084	0.119	0.101	0.105	0.142
200	0.016	0.010	0.011	0.027	0.068	0.053	0.056	0.089	0.122	0.104	0.105	0.152
250	0.016	0.011	0.011	0.030	0.068	0.052	0.055	0.097	0.124	0.102	0.106	0.160

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4.

Table 6: Empirical rejection frequencies of Wald-type IVX based tests for predictability in a multiple predictive regression context with $K \in \{1, 3, 5, 10\}$ predictors, for sample sizes $T = 250$ and $T = 1000$.

$T = 250$

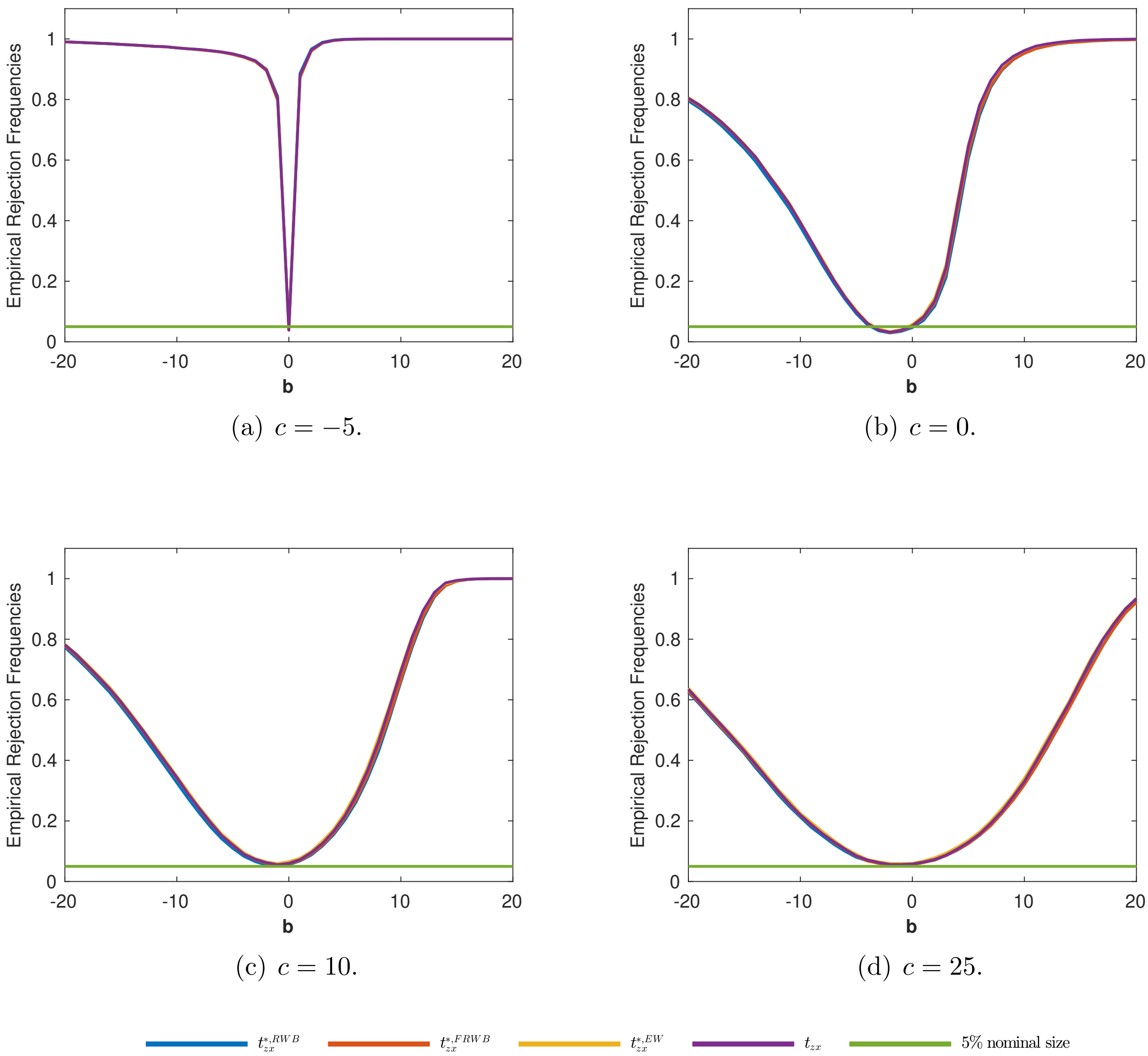
		1%				**5%**				**10%**			
K	c	$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}	$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}	$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}
1	-5	0.009	0.008	0.011	0.009	0.048	0.037	0.045	0.040	0.098	0.080	0.088	0.080
	-2.5	0.009	0.010	0.012	0.010	0.042	0.045	0.051	0.048	0.098	0.106	0.113	0.108
	0	0.011	0.011	0.013	0.012	0.050	0.053	0.059	0.056	0.102	0.108	0.115	0.114
	2.5	0.012	0.012	0.015	0.013	0.054	0.058	0.062	0.059	0.106	0.112	0.119	0.115
	5	0.013	0.012	0.015	0.013	0.056	0.058	0.064	0.060	0.108	0.113	0.119	0.115
	10	0.013	0.013	0.015	0.013	0.054	0.056	0.061	0.056	0.103	0.110	0.116	0.113
	25	0.011	0.012	0.014	0.012	0.055	0.057	0.060	0.057	0.105	0.104	0.112	0.108
	50	0.012	0.013	0.015	0.014	0.054	0.055	0.059	0.056	0.107	0.104	0.111	0.106
	75	0.013	0.012	0.015	0.014	0.055	0.055	0.060	0.056	0.105	0.103	0.108	0.103
	100	0.012	0.012	0.015	0.012	0.055	0.054	0.059	0.055	0.106	0.102	0.109	0.104
	125	0.012	0.011	0.014	0.013	0.055	0.054	0.058	0.054	0.104	0.103	0.109	0.104
	150	0.011	0.011	0.014	0.012	0.053	0.051	0.056	0.052	0.106	0.103	0.111	0.105
	200	0.010	0.010	0.013	0.012	0.052	0.051	0.056	0.052	0.105	0.102	0.108	0.103
	250	0.009	0.011	0.012	0.011	0.051	0.049	0.055	0.051	0.106	0.103	0.109	0.103
3	-5	0.020	0.135	0.171	0.148	0.085	0.352	0.385	0.366	0.158	0.494	0.521	0.507
	-2.5	0.023	0.052	0.067	0.054	0.097	0.176	0.193	0.177	0.177	0.284	0.301	0.283
	0	0.016	0.027	0.035	0.027	0.075	0.105	0.117	0.104	0.134	0.184	0.196	0.183
	2.5	0.014	0.020	0.028	0.022	0.067	0.086	0.103	0.090	0.122	0.157	0.174	0.161
	5	0.014	0.018	0.025	0.021	0.059	0.077	0.095	0.083	0.118	0.145	0.166	0.151
	10	0.013	0.016	0.024	0.018	0.054	0.066	0.083	0.071	0.109	0.129	0.152	0.137
	25	0.011	0.012	0.019	0.014	0.052	0.061	0.075	0.066	0.104	0.112	0.131	0.120
	50	0.011	0.012	0.018	0.014	0.053	0.057	0.070	0.061	0.104	0.110	0.131	0.115
	75	0.011	0.012	0.018	0.014	0.053	0.053	0.069	0.058	0.105	0.107	0.130	0.114
	100	0.010	0.012	0.018	0.014	0.051	0.053	0.069	0.057	0.107	0.106	0.129	0.113
	125	0.011	0.012	0.018	0.014	0.052	0.054	0.070	0.058	0.107	0.105	0.128	0.113
	150	0.011	0.013	0.018	0.013	0.052	0.054	0.069	0.058	0.107	0.107	0.128	0.114
	200	0.010	0.011	0.017	0.014	0.052	0.055	0.071	0.059	0.109	0.107	0.130	0.114
	250	0.009	0.011	0.018	0.013	0.053	0.055	0.071	0.060	0.107	0.108	0.127	0.114
5	-5	0.018	0.167	0.225	0.184	0.074	0.402	0.466	0.421	0.138	0.558	0.606	0.573
	-2.5	0.022	0.087	0.117	0.089	0.091	0.239	0.281	0.241	0.160	0.372	0.405	0.374
	0	0.020	0.050	0.067	0.050	0.082	0.157	0.186	0.156	0.152	0.258	0.289	0.254
	2.5	0.017	0.036	0.053	0.039	0.069	0.120	0.156	0.129	0.132	0.208	0.246	0.215
	5	0.014	0.028	0.046	0.033	0.063	0.105	0.138	0.116	0.124	0.183	0.223	0.195
	10	0.013	0.022	0.040	0.028	0.062	0.086	0.120	0.098	0.114	0.157	0.197	0.171
	25	0.012	0.017	0.029	0.021	0.053	0.067	0.100	0.080	0.110	0.129	0.167	0.141
	50	0.011	0.014	0.025	0.017	0.052	0.059	0.089	0.069	0.107	0.118	0.155	0.130
	75	0.011	0.013	0.024	0.017	0.051	0.055	0.085	0.063	0.104	0.110	0.149	0.122
	100	0.010	0.013	0.022	0.015	0.049	0.053	0.082	0.062	0.103	0.106	0.145	0.119
	125	0.009	0.011	0.022	0.013	0.049	0.053	0.080	0.062	0.102	0.105	0.142	0.118
	150	0.008	0.011	0.020	0.013	0.046	0.052	0.078	0.061	0.101	0.106	0.142	0.117
	200	0.006	0.009	0.020	0.012	0.047	0.051	0.079	0.060	0.098	0.103	0.139	0.114
	250	0.007	0.010	0.019	0.012	0.044	0.049	0.077	0.058	0.100	0.104	0.142	0.116
10	-5	0.011	0.243	0.396	0.298	0.058	0.513	0.635	0.559	0.114	0.658	0.754	0.691
	-2.5	0.016	0.169	0.280	0.195	0.072	0.398	0.505	0.425	0.144	0.542	0.631	0.558
	0	0.020	0.114	0.208	0.132	0.087	0.306	0.406	0.324	0.168	0.443	0.531	0.456
	2.5	0.016	0.078	0.162	0.098	0.075	0.238	0.342	0.262	0.144	0.360	0.459	0.384
	5	0.013	0.061	0.133	0.078	0.067	0.191	0.301	0.225	0.130	0.303	0.415	0.337
	10	0.012	0.041	0.100	0.058	0.060	0.141	0.244	0.175	0.115	0.238	0.356	0.278
	25	0.011	0.022	0.064	0.034	0.050	0.089	0.174	0.118	0.104	0.162	0.263	0.201
	50	0.010	0.016	0.048	0.025	0.048	0.067	0.142	0.091	0.099	0.129	0.223	0.164
	75	0.009	0.014	0.044	0.022	0.046	0.060	0.129	0.081	0.100	0.118	0.204	0.146
	100	0.009	0.013	0.043	0.020	0.046	0.056	0.120	0.077	0.097	0.110	0.197	0.139
	125	0.008	0.011	0.041	0.020	0.043	0.053	0.117	0.074	0.093	0.104	0.191	0.134
	150	0.007	0.012	0.039	0.019	0.042	0.052	0.116	0.071	0.092	0.101	0.191	0.131
	200	0.006	0.010	0.036	0.017	0.039	0.049	0.116	0.070	0.091	0.101	0.190	0.132
	250	0.005	0.010	0.035	0.016	0.036	0.050	0.116	0.072	0.094	0.104	0.193	0.133

$T = 1000$

		1%				**5%**				**10%**			
K	c	$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}	$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}	$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}
1	-5	0.009	0.008	0.008	0.008	0.043	0.034	0.036	0.035	0.094	0.075	0.077	0.074
	-2.5	0.008	0.008	0.009	0.009	0.038	0.042	0.043	0.043	0.088	0.098	0.098	0.097
	0	0.008	0.010	0.010	0.009	0.045	0.051	0.051	0.050	0.093	0.105	0.106	0.105
	2.5	0.009	0.010	0.011	0.011	0.051	0.054	0.056	0.056	0.103	0.114	0.114	0.113
	5	0.010	0.011	0.010	0.010	0.054	0.059	0.060	0.058	0.107	0.117	0.117	0.117
	10	0.012	0.012	0.014	0.013	0.055	0.060	0.060	0.059	0.108	0.118	0.119	0.117
	25	0.012	0.013	0.013	0.013	0.056	0.059	0.059	0.059	0.105	0.113	0.113	0.112
	50	0.011	0.011	0.011	0.011	0.055	0.059	0.059	0.058	0.105	0.108	0.109	0.107
	75	0.012	0.011	0.012	0.012	0.056	0.057	0.058	0.058	0.104	0.106	0.107	0.106
	100	0.011	0.012	0.012	0.012	0.056	0.056	0.057	0.058	0.103	0.104	0.105	0.105
	125	0.012	0.012	0.012	0.012	0.056	0.056	0.057	0.056	0.104	0.105	0.106	0.105
	150	0.011	0.012	0.012	0.011	0.055	0.053	0.055	0.055	0.104	0.105	0.106	0.104
	200	0.011	0.012	0.012	0.011	0.054	0.054	0.053	0.052	0.104	0.106	0.106	0.104
	250	0.011	0.011	0.011	0.011	0.053	0.052	0.053	0.053	0.104	0.105	0.106	0.104
3	-5	0.020	0.126	0.131	0.126	0.083	0.346	0.354	0.346	0.151	0.493	0.498	0.492
	-2.5	0.024	0.048	0.049	0.047	0.092	0.162	0.159	0.155	0.167	0.262	0.264	0.260
	0	0.016	0.026	0.027	0.025	0.071	0.096	0.096	0.095	0.130	0.168	0.168	0.166
	2.5	0.013	0.021	0.022	0.020	0.063	0.081	0.084	0.083	0.117	0.149	0.153	0.149
	5	0.013	0.019	0.020	0.018	0.060	0.076	0.079	0.077	0.114	0.144	0.149	0.145
	10	0.013	0.016	0.018	0.017	0.057	0.072	0.078	0.075	0.111	0.135	0.140	0.136
	25	0.011	0.014	0.015	0.014	0.056	0.064	0.067	0.065	0.107	0.120	0.126	0.123
	50	0.011	0.012	0.013	0.012	0.053	0.058	0.061	0.059	0.103	0.112	0.115	0.113
	75	0.010	0.011	0.012	0.010	0.050	0.055	0.058	0.055	0.103	0.107	0.112	0.109
	100	0.010	0.011	0.011	0.010	0.048	0.052	0.054	0.051	0.100	0.104	0.107	0.105
	125	0.010	0.010	0.011	0.010	0.048	0.049	0.054	0.050	0.098	0.100	0.106	0.103
	150	0.009	0.010	0.010	0.010	0.047	0.049	0.052	0.049	0.098	0.098	0.104	0.102
	200	0.009	0.009	0.009	0.010	0.046	0.048	0.051	0.048	0.097	0.099	0.103	0.098
	250	0.009	0.010	0.010	0.010	0.046	0.048	0.050	0.048	0.097	0.096	0.101	0.099
5	-5	0.015	0.162	0.176	0.164	0.074	0.398	0.408	0.403	0.143	0.549	0.558	0.548
	-2.5	0.021	0.080	0.082	0.075	0.091	0.237	0.238	0.230	0.170	0.360	0.360	0.352
	0	0.019	0.044	0.046	0.043	0.085	0.152	0.154	0.148	0.155	0.253	0.252	0.246
	2.5	0.017	0.032	0.034	0.031	0.069	0.118	0.126	0.117	0.131	0.206	0.209	0.203
	5	0.014	0.027	0.029	0.027	0.063	0.104	0.110	0.105	0.122	0.184	0.192	0.185
	10	0.012	0.021	0.024	0.021	0.058	0.089	0.096	0.092	0.115	0.161	0.171	0.164
	25	0.012	0.016	0.019	0.017	0.052	0.069	0.077	0.071	0.108	0.134	0.143	0.138
	50	0.011	0.014	0.016	0.015	0.049	0.057	0.064	0.059	0.103	0.118	0.126	0.120
	75	0.011	0.013	0.014	0.013	0.050	0.055	0.062	0.057	0.102	0.112	0.120	0.114
	100	0.009	0.010	0.013	0.011	0.051	0.056	0.060	0.057	0.101	0.107	0.115	0.109
	125	0.009	0.011	0.012	0.011	0.052	0.055	0.060	0.057	0.100	0.104	0.112	0.107
	150	0.009	0.010	0.012	0.010	0.051	0.055	0.059	0.056	0.100	0.104	0.111	0.107
	200	0.009	0.011	0.012	0.011	0.051	0.052	0.057	0.054	0.098	0.102	0.110	0.104
	250	0.010	0.010	0.012	0.010	0.051	0.052	0.058	0.054	0.097	0.100	0.108	0.103
10	-5	0.014	0.244	0.265	0.245	0.060	0.502	0.526	0.501	0.118	0.650	0.666	0.647
	-2.5	0.019	0.166	0.177	0.154	0.076	0.384	0.394	0.371	0.149	0.523	0.531	0.509
	0	0.023	0.112	0.120	0.103	0.091	0.295	0.300	0.280	0.169	0.426	0.431	0.407
	2.5	0.019	0.079	0.090	0.075	0.078	0.229	0.244	0.224	0.146	0.349	0.363	0.346
	5	0.016	0.059	0.071	0.061	0.068	0.188	0.211	0.191	0.133	0.301	0.318	0.304
	10	0.014	0.044	0.053	0.044	0.061	0.147	0.166	0.151	0.119	0.242	0.268	0.249
	25	0.013	0.027	0.033	0.027	0.057	0.101	0.119	0.108	0.111	0.175	0.201	0.184
	50	0.012	0.020	0.025	0.020	0.057	0.081	0.096	0.085	0.108	0.144	0.167	0.152
	75	0.012	0.016	0.022	0.017	0.055	0.071	0.085	0.077	0.108	0.129	0.152	0.136
	100	0.011	0.015	0.021	0.016	0.055	0.066	0.080	0.071	0.106	0.122	0.143	0.129
	125	0.011	0.014	0.018	0.016	0.055	0.061	0.077	0.067	0.105	0.120	0.140	0.123
	150	0.011	0.013	0.017	0.014	0.052	0.058	0.075	0.064	0.103	0.115	0.134	0.120
	200	0.010	0.012	0.016	0.013	0.053	0.057	0.070	0.063	0.104	0.111	0.130	0.119
	250	0.009	0.011	0.015	0.012	0.051	0.055	0.070	0.061	0.103	0.109	0.130	0.116

Note: W_{zx} and W_{zx}^{EW} are the Wald-type IVX based statistics discussed in Remark 9 of the main text, and $W_{zx}^{*,RWB}$ and $W_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) versions of W_{zx} computed as described in Algorithms 1 and 2 of Section 4 of the main text.

Figure 1: Power plots for two-sided tests for predictability. Data is generated from DGP1 with $\phi = -0.95$ and for $T = 250$. c is the noncentrality parameter which controls the persistence of the predictor used in the predictive regression and b are the values of the Pitman drift parameter.



(a) $c = -5$.

(b) $c = 0$.

(c) $c = 10$.

(d) $c = 25$.

On-Line Supplementary Appendix

to

"Extensions to IVX Methods of Inference for Return Predictability"

by

Matei Demetrescu, Iliyan Georgiev, Paulo Rodrigues and Robert Taylor

Summary of Contents

This supplement contains four sections. Section A contains Examples 1 and 2 referred in Remarks 4 and 6 of the main paper. Section B outlines how moving blocks bootstrap methods can be applied to the setting considered in this paper. Section C contains detailed proofs of Propositions 1-3. Section D reports additional supporting Monte Carlo results to those reported in section 5 of the paper.

A Additional material

Example 1 *Let*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_t \\ e_t \end{pmatrix} \tag{A.1}$$

where, with $a_0, b_0 > 0$*,*

$$\begin{aligned} a_t &= \sqrt{a_0}\nu_{t,1} \\ e_t &= \sqrt{b_0 + b_1 e_{t-1}^2}\nu_{t,2} \end{aligned}$$

with $\{\nu_{t,1}\}$ *and* $\{\nu_{t,2}\}$ *two mutually independent zero-mean unit-variance IID sequences. Assume* $\nu_{t,1}$, $\nu_{t,2}$ *to be uniformly* L_4 *bounded, and* $b_1^2 < 1/\operatorname{E}(\nu_{t,2}^4)$ *to ensure that* e_t *does itself have finite 4th moment. The process* $v_t = e_t$ *is therefore a stationary ARCH(1) process whenever* $0 \leq b_1 < 1$*, whereas* a_t *is conditionally homoskedastic (*a_t *is an IID sequence).*

The natural filtration is $\mathcal{F}_t = \{(\nu_{t1}; \nu_{t2}), (\nu_{t-1,1}; \nu_{t-1,2}), \ldots\}$*, and the conditional variance of* u_t *is easily seen to be*

$$\mathrm{E}\left(u_t^2 | \mathcal{F}_{t-1}\right) = a_0 + \gamma^2 \left(b_0 + b_1 v_{t-1}^2\right).$$

In this model, the conditional variance of u_t *obviously does not depend on the past innovations* v_t *when* $\gamma = 0$*; however, this restriction also implies the absence of any contemporaneous correlation between* u_t *and* v_t*, inconsistent with the conditions ordinarily expected to hold in a predictive regression model for financial variables.*

The model outlined above satisfies our Assumption 3.2, (see remark 4), but violates assumption INNOV of Kostakis et al. (2015) because u_t *from (A.1) cannot have a so-called strict finite-order GARCH representation (i.e. with IID shocks) in general:*

1. *If* u_t *did have such a strict GARCH representation, it would hold that* $u_t = \sqrt{h_t}\eta_t$ *where* η_t *is an IID sequence, and* $h_t = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1}$ *(we show below that* u_t^2 *has an ARMA(1,1) representation, such that* u_t *itself can only have a GARCH(1,1) representation). The ARMA(1,1) representation of this squared GARCH model equation is then*

   $$u_t^2 = \alpha_0 + (\alpha_1 + \beta_1) u_{t-1}^2 + \vartheta_t - \beta_1 \vartheta_{t-1}$$

   where $\vartheta_t = h_t(\eta_t^2 - 1)$*. The errors* ϑ_t *in the squared GARCH model equation must be conditionally heteroskedastic martingale differences with the particular conditional variance,* $\mathrm{E}(\vartheta_t^2|\mathcal{F}_t) = h_t^2\, \mathrm{E}\left((\eta_t^2 - 1)^2\right)$.

2. *The squared* u_t *implied by the model in (A.1) is given as*

   $$\begin{aligned} u_t^2 &= a_0 + \gamma^2 b_0 + \gamma^2 b_1 v_{t-1}^2 + \left(a_t^2 - a_0\right) + \gamma^2\left(v_t^2 - \left(b_0 + b_1 v_{t-1}^2\right)\right) + 2\gamma a_t v_t \\ &= a_0 + \gamma^2 b_0 + b_1 \gamma^2 v_{t-1}^2 + \xi_t \end{aligned}$$

   where

   $$\xi_t = \left(a_t^2 - a_0\right) + \gamma^2\left(b_0 + b_1 v_{t-1}^2\right)\left(\nu_{t,2}^2 - 1\right) + 2\gamma a_t v_t$$

   is a MD sequence w.r.t. $\mathcal{F}_t$*. Furthermore,*

   $$\gamma^2 v_{t-1}^2 = u_{t-1}^2 - a_{t-1}^2 - 2\gamma a_{t-1} v_{t-1}$$

   such that, plugging this in, we obtain

   $$\begin{aligned} u_t^2 &= \left(a_0 + \gamma^2 b_0\right) + b_1\left(u_{t-1}^2 - a_{t-1}^2 - 2\gamma a_{t-1} v_{t-1}\right) + \xi_t \\ &= \left(a_0 + \gamma^2 b_0 - b_1 a_0\right) + b_1 u_{t-1}^2 + \pi_t \end{aligned}$$

where

$$\begin{aligned}\pi_t &= \xi_t - b_1\left(a_{t-1}^2 - a_0\right) - 2b_1\gamma a_{t-1}v_{t-1} \\ &= \left(a_t^2 - a_0 + 2\gamma a_t v_t\right) - b_1\left(a_{t-1}^2 - a_0 + 2\gamma a_{t-1}v_{t-1}\right) \\ &\quad + \gamma^2\left(b_0 + b_1 v_{t-1}^2\right)\left(\nu_{t,2}^2 - 1\right)\end{aligned}$$

is a weakly stationarity process and therefore possesses a linear representation. The autocovariance function of π_t *is obtained as follows,*

$$\begin{pmatrix} \pi_t \\ \tilde{\pi}_t \end{pmatrix} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} L\right)\begin{pmatrix} a_t^2 - a_0 + 2\gamma a_t v_t \\ \gamma^2\left(b_0 + b_1 v_{t-1}\right)\left(\nu_{t,2}^2 - 1\right) \end{pmatrix}$$

where $\begin{pmatrix} a_t^2 - a_0 + 2\gamma a_t v_t \\ \gamma^2\left(b_0 + b_1 v_{t-1}\right)\left(\nu_{t,2}^2 - 1\right) \end{pmatrix}$ *is easily seen to be a zero-mean white noise sequence under our assumptions, such that* $\begin{pmatrix} \pi_t \\ \tilde{\pi}_t \end{pmatrix}$ *is a vector MA(1) process. Therefore,* π_t *does has a* marginal *MA(1) representation – but one where the innovations are uncorrelated, and not MD sequences in general. In turn, this does make* u_t^2 *an ARMA(1,1) process, but not necessarily one with MD innovations, so, in general, the model (A.1) does not have a GARCH representation where the driving shocks are IID.*

Example 2 *Consider the following particular case where* $A(L) = 1$ *but* $\rho \neq 0$ *is fixed and bounded away from unity and* $\boldsymbol{\psi}_t$ *is conditionally heteroskedastic. Assume also that* $h_{12}(\tau) = 0\ \forall\tau$*. Then,* $\xi_t = \sum_{j=0}^{\infty}\rho^j v_{t-j}$ *such that*

$$\operatorname{Var}\left(\xi_{t-1}u_t\right) = \operatorname{E}\left(h_{11}^2(t/T)a_t^2\left(\sum_{j=0}^{\infty}\rho^j[h_{21}((t-1-j)/T)a_{t-1-j} + h_{22}((t-1-j)/T)e_{t-1-j}]\right)^2\right),$$

where some algebra shows that

$$\operatorname{Var}\left(\xi_{t-1}u_t\right) = \operatorname{E}\left(h_{11}^2(t/T)a_t^2\left(h_{21}(t/T)\sum_{j=0}^{\infty}\rho^j a_{t-1-j} + h_{22}(t/T)\sum_{j=0}^{\infty}\rho^j e_{t-1-j}\right)^2\right) + o(1).$$

One therefore obtains

$$\operatorname{Var}\left(\xi_{t-1}u_t\right) = h_{11}^2(t/T)\left(C_1 h_{21}^2(t/T) + C_2 h_{22}^2(t/T) + C_3 h_{21}(t/T)h_{22}(t/T)\right) + o(1)$$

where $C_1 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\rho^j\rho^k\operatorname{E}\left(a_t^2 a_{t-1-j}a_{t-1-k}\right)$, $C_2 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\rho^j\rho^k\operatorname{E}\left(a_t^2 e_{t-1-j}e_{t-1-k}\right)$ *and* $C_3 = 2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\rho^j\rho^k\operatorname{E}\left(a_t^2 a_{t-1-j}e_{t-1-k}\right)$*. Therefore, we have at all differentiability points*

$$\left[M_{\xi u}\right]'(s) = h_{11}^2(s)\left(C_1 h_{21}^2(s) + C_2 h_{22}^2(s) + C_3 h_{21}(s)h_{22}(s)\right).$$

At the same time, it follows analogously that

$$[M_u]'(s) = h_{11}^2(s) \qquad [M_v]'(s) = h_{21}^2(s) + h_{22}^2(s)$$

such that

$$[M_{zu}]'(s) = h_{11}^2(s)\left(h_{21}^2(s) + h_{22}^2(s)\right).$$

Summing up, the quadratic variation (and thus the variance profile) of M_{zu} is in general different from that of $M_{\xi u}$.

B Moving blocks bootstrap

Following Fan and Lee (2019), one could employ a block-bootstrap scheme. This amounts, in their notation, to the following algorithm:

1. Let b be an integer block length and let $B(t) = (\boldsymbol{w}_t, \boldsymbol{w}_{t+1}, \ldots, \boldsymbol{w}_{t+b-1})$ denote a data block with starting point $t \in \{1, \ldots, T-b+1\}$, where the data to be resampled stacks $\boldsymbol{w}_t = (y_t, z_t)'$.

2. The total number of possible blocks and the number of blocks in one bootstrapped sample are denoted by q and m. The letter ℓ indicates the bootstrapped sample size, $T = q + b - 1$ and $\ell = mb$. (Intuitively, one should choose m such that $\ell \approx T$; Fan and Lee (2019) only require $m = O(T)$ and $T = O(m)$.)

3. Sample m blocks randomly with replacement from $\{B(t) : t = 1, \ldots, n-b+1\}$: the resulting bootstrap sample $\boldsymbol{w}_1^*, \ldots, \boldsymbol{w}_\ell^*$ is $(B(I_1), \ldots, B(I_m))$ with I_i are IID discrete uniform variables on $\{1, ..., n-b+1\}$.

4. Compute e.g. the full-sample bootstrap IVX t statistic,

$$t_{zx}^* = \frac{\sum_{t=1}^{\ell} \tilde{z}_{t-1}^* \tilde{y}_t^*}{\sqrt{\hat{\sigma}_{u^*}^2}\sqrt{\sum_{t=1}^{\ell}\left(\tilde{z}_{t-1}^*\right)^2}};$$

   this step is different from the corresponding step of Fan and Lee (2019), since they work in a quantile regression framework.

5. Use quantiles of distribution of t_{zx}^* for inference rather than quantiles of the standard normal.

The above procedure does not replicate the null hypothesis in the bootstrap data, so one would need to either construct confidence intervals and invert them to obtain a test, or replace y_t with the OLS residuals $\hat{u}_t := y_t - \hat{\alpha} - \hat{\beta} x_{t-1}$ in the definition of $\boldsymbol{w}_t$ in Step 1 to ensure that the null is imposed on the bootstrap data.

Block wild bootstrap

To account for unconditional heteroskedasticity, Step 3 of the above MBB scheme could be replaced with a block wild bootstrap. In this case, one needs to impose the null when resampling, i.e. replace y_t with the OLS residuals $\hat{u}_t := y_t - \hat{\alpha} - \hat{\beta}x_{t-1}$ in the definition of $\boldsymbol{w}_t$ in step 1.

We do not provide theoretical results for either moving block bootstrap.

C Technical appendix

We denote by P^*, E^* and Var^* respectively probability, expectation and variance conditional on the original data. Further, we use E^*_{t-1} for expectation conditional on the data and $\{R_s\}_{s=1}^{t-1}$. Weak in-probability convergence is denoted by $\overset{w}{\Rightarrow}_p$. If w is a degenerate (deterministic) element, an alternative notation to $w_T \overset{w}{\Rightarrow}_p w$ is $w_T \overset{p}{\Rightarrow}_p w$. If the metric space of interest is a linear space with zero element 0, we use $w_T \overset{w}{\Rightarrow}_p 0$ interchangeably with $w_T = o_p^*(1)$. For instance, $w_T \overset{p}{\Rightarrow}_p w$ is equivalent to $d(w_T, w) = o_p^*(1)$ for the metric d of the underlying space. We introduce $w_T = O_p^*(1)$ by the standard property that for every $\epsilon > 0$ there exists a $K_\epsilon \in \mathbb{R}$ such that $\mathrm{P}(\mathrm{P}^*(d(w_T, 0) > K_\epsilon) < \epsilon) > 1 - \epsilon$ for all $T \in \mathbb{N}$. As usual, $o_p^*(T^\alpha) := T^\alpha o_p^*(1)$ and $O_p^*(T^\alpha) := T^\alpha O_p^*(1)$. The o_p and O_p symbols retain their usual meaning. For r.v.'s w we write $\|w\|_r$ for $(E|w|^r)^{1/r}$, $r > 0$. Finally, C is an unspecified positive constant whose value may change across the expressions where it appears.

C.1 Toolbox

We start with some results that structure our approach to the derivation of the main theory.

Martingale approximation

Assumption 3.2 implies that the components of $\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2$ are well approximated by martingale differences. Specifically, let

$$\begin{pmatrix} \Psi_T^a & \Psi_T^{ae} \\ \Psi_T^{ae} & \Psi_T^e \end{pmatrix} := \sum_{t=1}^{T} (\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2).$$

Then the condition $\mathrm{E}\|\mathrm{E}_0 \sum_{t=1}^T (\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2)\|^2 = O(T^{2\epsilon})$ with $\epsilon \in (0, \frac{1}{2})$ ensures, by Jensen's inequality, that component-wise $\|\mathrm{E}(\Psi_T^a|\mathcal{F}_0^a)\|_2 = O(T^\epsilon)$, $\|\mathrm{E}(\Psi_T^e|\mathcal{F}_0^e)\|_2 = O(T^\epsilon)$ and $\|\mathrm{E}(\Psi_T^{ae}|\mathcal{F}_0^{ae})\|_2 = O(T^\epsilon)$ for $\mathcal{F}_0^c := \sigma(c_{-i} : i \in \mathbb{N} \cup \{0\})$, $c \in \{a, e, ae\}$ and for the same ϵ. Together with the stationarity of $\boldsymbol{\psi}_t$ and the finite fourth moment of its components, this implies that the martingale approximation results of Merlevède *et al.* (2006) are applicable to Ψ_T^a, Ψ_T^e and Ψ_T^{ae}. The Lipschitz-by-parts property of the function $\mathbf{H}$ transfers this behavior to the sequences $u_t^2 - \sigma_{ut}^2$ and $v_t^2 - \sigma_{vt}^2$, where $\sigma_{ut}^2 := \mathrm{E}u_t^2 = h_{11}^2(t/T) + h_{12}^2(t/T)$ and similarly for σ_{vt}^2. Some implications are collected in the next lemma.

Lemma 1 *Let* $S^u_{T(t+1,r)} := \sum_{s=t+1}^r (u_s^2 - \sigma_{us}^2)$ *and* $S^v_{T(t+1,r)} := \sum_{s=t+1}^r (v_s^2 - \sigma_{vs}^2)$ *for* $1 \le t < r \le T$. *Under Assumption 3 it holds that:*

(a) $\max_{1\le t\le T} |T^{-1/2} S^u_{T(1,t)}| = O_p(1)$ *and* $\max_{1\le t\le T} |T^{-1/2} S^v_{T(1,t)}| = O_p(1)$

(b) $\mathrm{E}\left[\max_{1\le t<r\le T} (S^u_{T(t+1,r)})^2\right] = O(T)$

(c) $\max_{1\le t<r\le T} |\mathrm{E}[(u_t^2 - \sigma_{ut}^2) S^u_{T(t+1,r)}]| = O(T^\epsilon)$, $\max_{1\le t<r\le T} |\mathrm{E}[(v_t^2 - \sigma_{vt}^2) S^v_{T(t+1,r)}]| = O(T^\epsilon)$ *and* $\max_{1\le s<t<r\le T} |\mathrm{E}(v_t v_s S^v_{T(t+1,r)})| = O(T^\epsilon)$.

Exponential averaging

For an arbitrary real sequence w_t, partial summation produces

$$\left|\sum_{t=1}^r \varrho^{t-1} w_t\right| = \left|\varrho^{r-1}\sum_{t=1}^r w_t + (1-\varrho)\sum_{s=1}^{r-1}\varrho^{s-1}\sum_{t=1}^s w_t\right| \le \max_{1\le s\le r}\left|\sum_{t=1}^s w_t\right|. \qquad \text{(A.2)}$$

Some implications of this estimate (and not only) are collected next. Here and in what follows, E_t denotes expectation conditional on $\sigma(\boldsymbol{\psi}_{-i} : i \in \mathbb{N} \cup \{0, -1, ..., -t\})$.

Lemma 2 *Let* w_{Tt} *be an array of r.v.'s.*

(a) If $T^{-\alpha}\sum_{t=1}^{\lfloor T\tau\rfloor} w_{Tt} \Rightarrow W(\tau)$ *in the sense of weak convergence of probability measures on* $\mathcal{D}$, *then* $\max_{1\le s\le T} |\sum_{t=1}^s \varrho^{t-1} w_{Tt}| = O_p(T^\alpha)$;

(b) $\max_{1\le s\le T} \mathrm{E}\,|\sum_{t=1}^s \varrho^{t-1} w_{Tt}| \le \max_{1\le s\le T} \mathrm{E}\,|\sum_{t=1}^s w_{Tt}|$;

(c) If w_{Tt} *is an MD array with* $\mathrm{E}|w_{Tt}|^p < \infty$ *for some* $p > 2$, *then*

$$\max_{1\le t\le T}\left\|\sum_{j=0}^{t-1}\varrho^j w_{T,t-j}\right\|_p = O(T^{\eta/2})\left(\max_{t\le T}\mathrm{E}|w_{Tt}|^p\right)^{1/p}$$
$$\max_{1\le t\le T}\left|\sum_{j=0}^{t-1}\varrho^j w_{T,t-j}\right| = o_p(T^{1/2})\left(\max_{t\le T}\mathrm{E}|w_{Tt}|^p\right)^{1/p}.$$

In the following parts, let Assumption 3 hold. Then:

(d) $\max_{1\le t\le T} |\sum_{r=t+1}^T \varrho^{2(r-t-1)}(u_r^2 - \sigma_{ur}^2)| = O_p(T^{1/2})$;

(e) $\max_{1\le t\le T} \|\mathrm{E}_t \sum_{r=t+1}^T \varrho^{2(r-t-1)}(u_r^2 - \sigma_{ur}^2)\|_2 = O(T^\epsilon)$ *and* $\max_{1\le t\le T}\left\|\mathrm{E}_t\sum_{r=t+1}^T \varrho^{r-t} u_r v_r\right\|_2 = O(T^\epsilon)$;

(f) $T^{-1-\eta}\sum_{s=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2\right)u_t^2 = T^{-1-\eta}\sum_{s=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2\right)\sigma_{ut}^2 + O_p(T^{(\epsilon-\eta)/2})$ *pointwise;*

(g) If $\epsilon < \eta$, *then* $T^{-1-\eta}\sum_{s=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2\right)\sigma_{ut}^2 \xrightarrow{p} \int_0^\tau [M_u(s)]'[M_v(s)]'\mathrm{d}s$ *pointwise and uniformly (the derivatives exist everywhere except at finitely many points and are continuous on the intervals where they exist).*

The space $\mathcal{D}(T_\Delta)$

Let $T_\Delta = [0,1]^2 \cap \{(\tau_1, \tau_2) \in \mathbb{R}^2 : \tau_2 - \tau_1 \geq \Delta_\tau\}$ for some $\Delta_\tau \in (0,1)$. Let $\mathcal{D}(T_\Delta)$ be the set of real functions on T_Δ which are continuous from the 'right' (i.e., $f(\tau_1^{(n)}, \tau_2^{(n)}) \to f(\tau_1, \tau_2)$ when $\tau_i^{(n)} \downarrow \tau_i$, $i = 1, 2$, for $(\tau_1^{(n)}, \tau_2^{(n)})$, $(\tau_1, \tau_2) \in T_\Delta$ and $f \in \mathcal{D}(T_\Delta)$) and have limits from within each of the four right angles $[A_1 \times A_2] \cap T_\Delta$, $A_i \in \{[0, \tau_i), [\tau_i, 1]\}$, $i = 1, 2$, when the angles are non-empty. For clarity, note that all bivariate cdf's with domain restricted to T_Δ belong to $\mathcal{D}(T_\Delta)$. It is well-known (e.g. Bickel and Wichura, 1971, p. 1662) that $\mathcal{D}(T_\Delta)$ can be equipped with a Skorokhod-like metric which makes it a separable and complete metric space such that stochastic process with values in $\mathcal{D}(T_\Delta)$ are measurable w.r.t. the resulting Borel σ-algebra. Moreover, the resulting topology relativised to $\mathcal{C}(T_\Delta) \subset \mathcal{D}(T_\Delta)$, the subspace of continuous real functions on T_Δ, coincides with the uniform topology. As we will only be interested in convergence to limits in $\mathcal{C}(T_\Delta)$, in what follows convergence and continuity issues involving elements of $\mathcal{D}(T_\Delta)$ are always discussed w.r.t. the uniform metric on $\mathcal{D}(T_\Delta)$. It is then straightforward to see that the function from $\mathcal{D}^2$ to $\mathcal{D}(T_\Delta)$ which associates to every $(f_1, f_2) \in \mathcal{D}^2$ the element $(\tau_1, \tau_2) \mapsto f_2(\tau_2) - f_1(\tau_1)$ of $\mathcal{D}(T_\Delta)$ is continuous on the subspace of continuous functions $\mathcal{C}^2$ of $\mathcal{D}^2$. Moreover, linearly combining functions in $\mathcal{D}(T_\Delta)$, multiplication of functions in $\mathcal{D}(T_\Delta)$ and division of functions in $\mathcal{D}(T_\Delta)$ (for denominators bounded away from zero) are continuous transformations of the product subspace $\mathcal{C}(T_\Delta) \times \mathcal{C}(T_\Delta)$ of $\mathcal{D}(T_\Delta) \times \mathcal{D}(T_\Delta)$. Finally, also the functionals $\sup_{A^s} |f|$, $s \in \{F, B, R\}$, are continuous on $\mathcal{C}(T_\Delta)$, where $A^F = \{0\} \times [\tau_L, 1]$, $A^B = [0, \tau_U] \times \{1\}$ and $A^R = \{(\tau, \tau + \Delta_\tau) : \tau \in [0, 1 - \Delta_\tau]\}$ with $\tau_L \geq \Delta_\tau$ and $1 - \tau_U \geq \Delta_\tau$.

C.2 Asymptotics on the space of the original data

The first result is independent of the persistence properties of x_t.

Lemma 3 *Under Assumption 3, it holds as $T \to \infty$ that*

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau \rfloor} \begin{pmatrix} u_t^2 \\ v_t^2 \end{pmatrix} = \frac{1}{T}\sum_{t=1}^{\lfloor T\tau \rfloor} \begin{pmatrix} \sigma_{ut}^2 \\ \sigma_{vt}^2 \end{pmatrix} + o_p(T^{-1/2}) \xrightarrow{p} \begin{pmatrix} [M_u] \\ [M_v] \end{pmatrix}(\tau) = \int_0^\tau \begin{pmatrix} h_{11}^2(s) + h_{12}^2(s) \\ h_{21}^2(s) + h_{22}^2(s) \end{pmatrix} \mathrm{d}s$$

uniformly over $\tau \in [0, 1]$.

We now turn to the weakly persistent case.

Lemma 4 *Under Assumptions 1.1 and 3, we have as $T \to \infty$:*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor \tau T \rfloor} \begin{pmatrix} u_t \\ v_t \\ u_t \xi_{t-1} \end{pmatrix} \Rightarrow \int_0^\tau \mathbf{G}(s) \mathrm{d}\boldsymbol{B}(s)$$

where

$$\mathbf{G}(\tau) = \begin{pmatrix} h_{11} & h_{12} & 0 & 0 & 0 & 0 \\ h_{21} & h_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{11}h_{21} & h_{11}h_{22} & h_{12}h_{21} & h_{12}h_{22} \end{pmatrix}(\tau)$$

and $\boldsymbol{B}(\tau)$ is a 6-variate Brownian motion of covariance matrix defined in the proof.

The next lemma collects some product-moment limits in the strongly persistent case.

Lemma 5 *Under Assumptions 1.2 and 3 with $\epsilon < \min\{1-\eta, \frac{1}{2}\eta\}$, the following hold jointly as $T \to \infty$:*

(a) $\frac{1}{T^{1/2+\eta}} \sum_{t=1}^{[\tau T]} z_{t-1} \Rightarrow \frac{\omega}{a} J_{c,\mathrm{H}}(\tau) = \frac{\omega}{a} \int_0^\tau e^{-c(\tau-s)} \mathrm{d}M_v(s)$

(b) $\frac{1}{T^{1+\eta}} \sum_{t=1}^{[\tau T]} z_{t-1}x_{t-1} \Rightarrow \frac{\omega^2}{a} \left(J_{c,\mathrm{H}}^2(\tau) - \int_0^\tau J_{c,\mathrm{H}}(s)\mathrm{d}J_{c,\mathrm{H}}(s)\right)$

(c) $\frac{1}{T^{1+\eta}} \sum_{t=1}^{[\tau T]} z_{t-1}^2 \xrightarrow{p} \frac{\omega^2}{2a}[M_v](\tau)$ *uniformly in $\tau \in [0,1]$*

(d) $\frac{1}{T^{1/2+\eta/2}} \sum_{t=1}^{[\tau T]} z_{t-1}u_t \Rightarrow \frac{\omega}{\sqrt{2a}} \int_0^\tau \sqrt{[M_u]'(s)[M_v]'(s)}\mathrm{d}B(s)$ *where B is a standard Brownian motion independent of M_v (and thus, of $J_{c,\mathrm{H}}$).*

(e) $\frac{1}{T^{1/2+\eta}} \sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1}b(t/T) \Rightarrow \frac{\omega}{a} \left(b(\tau)J_{c,\mathrm{H}}(\tau) - \int_0^\tau J_{c,\mathrm{H}}(s)\mathrm{d}b(s)\right) := \frac{\omega}{a} Z_b(\tau)$.

(f) $\frac{1}{T^{1+\eta}} \sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1}b(t/T)x_{t-1} \Rightarrow \frac{\omega^2}{a} \left(J_{c,\mathrm{H}}(\tau)Z_b(\tau) - \int_0^\tau Z_b(s)\mathrm{d}J_{c,\mathrm{H}}(s)\right)$.

Proof of Proposition 1. For the space $\mathcal{D}(T_\Delta)$ and our approach to the weak convergence of probability measures on it, see Section C.1.

We have

$$t_{zx}(\tau_1,\tau_2) = \frac{\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} z_{t-1}\left(u_t - \bar{u}(\tau_1,\tau_2)\right)}{\hat{\sigma}_u(\tau_1,\tau_2)\sqrt{\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} z_{t-1}^2}} + \frac{\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} z_{t-1}\beta_t\left(\xi_{t-1} - \bar{\xi}_{-1}(\tau_1,\tau_2)\right)}{\hat{\sigma}_u(\tau_1,\tau_2)\sqrt{\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} z_{t-1}^2}}.$$

Under Assumption 1.1, we notice that, given our moment restrictions and the absolute summability of the Wold coefficients of ξ_t, $\sup_t |\xi_{t-1}| = O_p\left(T^{1/4}\right)$; also, $\hat{\alpha}$ and $\hat{\beta}$ are easily shown to be $\sqrt{T}$-consistent, so

$$\hat{u}_t = u_t - (\hat{\alpha} - \alpha) - \left(\hat{\beta} - \beta\right) x_{t-1} = u_t + o_p(1)$$

uniformly in t. (The same is easily shown to hold for the residuals computed under the null and we omit the details.) Then,

$$\begin{aligned} \hat{\sigma}_u^2(\tau_1,\tau_2) &= \frac{1}{T}\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} \hat{u}_t^2 = \frac{1}{T}\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} u_t^2 + \frac{1}{T}\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} \left(\hat{u}_t^2 - u_t^2\right) \\ &= \frac{1}{T}\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} \sigma_{ut}^2 + \frac{1}{T}\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} \left(u_t^2 - \sigma_{ut}^2\right) + o_p(1) \end{aligned}$$

uniformly in τ_1 and τ_2 with $0 \le \tau_1 < \tau_2 \le 1$, such that, thanks e.g. to Lemma 3,

$$\hat{\sigma}_u^2(\tau_1,\tau_2) \Rightarrow \frac{1}{\tau_2 - \tau_1}\left([M_u](\tau_2) - [M_u](\tau_1)\right). \tag{A.3}$$

Moving on, we have like in the proof of Lemma 6 that $z_t = \xi_t + R_{t,T}$ where the rest term $R_{t,T}$ vanishes as $T \to \infty$ and can be controlled for in the relevant sums, such that we may conclude that, uniformly in τ_1 and τ_2 with $0 \le \tau_1 < \tau_2 \le 1$,

$$\frac{1}{T}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} z_{t-1}^2 = \frac{1}{T}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} \xi_{t-1}^2 + o_p(1) \Rightarrow \kappa^2\left([M_v](\tau_2) - [M_v](\tau_1)\right). \tag{A.4}$$

Similarly, we have uniformly in τ_1 and τ_2 with $0 \le \tau_1 < \tau_2 \le 1$ that

$$\frac{1}{\sqrt{T}}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} z_{t-1}\left(u_t - \bar{u}(\tau_1,\tau_2)\right) = \frac{1}{\sqrt{T}}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} \xi_{t-1}\left(u_t - \bar{u}(\tau_1,\tau_2)\right) + o_p(1)$$
$$= \frac{1}{\sqrt{T}}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} \xi_{t-1}u_t - \left(\frac{1}{T(\tau_2-\tau_1)}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} u_t\right)\left(\frac{1}{\sqrt{T}}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} \xi_{t-1}\right) + o_p(1),$$

where the weak convergence of the partial sums of ξ_t and u_t implies

$$\frac{1}{\sqrt{T}}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} \xi_{t-1} = O_p(1) \qquad \frac{1}{\sqrt{T}}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} u_t = O_p(1)$$

uniformly in τ_1 and τ_2 with $0 \le \tau_1 < \tau_2 \le 1$. The weak convergence of the partial sums of $\xi_{t-1}u_t$ therefore implies

$$\frac{1}{\sqrt{T}}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} z_{t-1}\left(u_t - \bar{u}(\tau_1,\tau_2)\right) \Rightarrow M_{\xi u}(\tau_2) - M_{\xi u}(\tau_1).$$

To assess the drift term under the local alternative $\beta_t = T^{-1/2}b(t/T)$, write like above

$$\frac{1}{\sqrt{T}}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} z_{t-1}\beta_t\left(\xi_{t-1} - \bar{\xi}_{-1}(\tau_1,\tau_2)\right)$$
$$= \frac{1}{T}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} b(t/T)\xi_{t-1}^2 - \bar{\xi}_{-1}(\tau_1,\tau_2)\frac{1}{\sqrt{T}}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} \xi_{t-1}b(t/T) + o_p(1)$$

uniformly in τ_1 and τ_2 with $0 \le \tau_1 < \tau_2 \le 1$. It is then not difficult to establish analogously to Lemma 3 that

$$\frac{1}{T}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} b(t/T)\xi_{t-1}^2 \Rightarrow \kappa^2\int_{\tau_1}^{\tau_2}[M_v]'(s)b(s)\mathrm{d}s$$

and we omit the details. Finally,

$$\frac{1}{\sqrt{T}}\sum_{t=\lfloor \tau_1 T\rfloor+1}^{\lfloor \tau_2 T\rfloor} \xi_{t-1}b(t/T) = O_p(1) \qquad \bar{\xi}_{-1}(\tau_1,\tau_2) = O_p\left(1/\sqrt{T}\right)$$

as required for the first part of the result.

Moving on to the part concerning Assumption 1.2, $\hat{\sigma}_u^2(\tau_1, \tau_2)$ is easily shown to have the same behavior as under the stable regressor case considering that the OLS residuals satisfy

$$\begin{aligned} \hat{u}_t &= u_t - (\hat{\alpha} - \alpha) - \left(\hat{\beta} - \beta\right) x_{t-1} \\ &= u_t + O_p\left(T^{-1/2}\right) \end{aligned}$$

uniformly in t since $\hat{\alpha} - \alpha = O_p\left(T^{-1/2}\right)$, $\hat{\beta} - \beta = O_p\left(T^{-1}\right)$ and $\sup_{1\leq t\leq T}|x_{t-1}| = O_p\left(\sqrt{T}\right)$ given the weak convergence of $T^{-1/2}x_{[\tau T]}$ to an a.s. continuous process. (An analogous argument applies for the residuals computed under the null). Then, under Assumption 1.2, Lemma 5 part (c) then leads to

$$\frac{1}{T^{1+\eta}} \sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2 \Rightarrow \frac{\omega^2}{2a}\left([M_v](\tau_2) - [M_v](\tau_1)\right). \tag{A.5}$$

Lemma 5 parts (a) and (d) furthermore imply

$$\frac{1}{T^{1/2+\eta/2}} \sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}\left(u_t - \bar{u}(\tau_1, \tau_2)\right) \Rightarrow \frac{\omega}{\sqrt{2a}}\left(M_{zu}(\tau_2) - M_{zu}(\tau_1)\right),$$

and, given the weak convergence of $\xi_{[\tau T]} = x_{[\tau T]} - \mu_x$ and also Lemma 5 part (e),

$$\frac{\bar{\xi}_{-1}(\tau_1, \tau_2)}{\sqrt{T}} \frac{1}{T^{1/2+\eta}} \sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1} b(t/T) \Rightarrow \frac{1}{\tau_2 - \tau_1} \frac{\omega^2}{a} \int_{\tau_1}^{\tau_2} J_{c,\mathrm{H}}(s)\mathrm{d}s\left(Z_b(\tau_2) - Z_b(\tau_1)\right),$$

with $Z_b(\tau)$ defined there. Finally, Lemma 5 part (f) leads to

$$\begin{aligned} \frac{1}{T^{1+\eta}} \sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1} b(t/T)\xi_{t-1} \;\Rightarrow\; & \frac{\omega^2}{a}\left(J_{c,\mathrm{H}}(\tau_2)Z_b(\tau_2) - \int_0^{\tau_2} Z_b(s)\mathrm{d}J_{c,\mathrm{H}}(s)\right) \\ & -\frac{\omega^2}{a}\left(J_{c,\mathrm{H}}(\tau_1)Z_b(\tau_1) - \int_0^{\tau_1} Z_b(s)\mathrm{d}J_{c,\mathrm{H}}(s)\right) \end{aligned}$$

such that the 2nd part of the result then follows by the continuous mapping theorem. □

Proof of Proposition 2.

Under the null hypothesis,

$$t_{zx}^{EW}(\tau_1, \tau_2) = \frac{\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}\left(u_t - \bar{u}(\tau_1, \tau_2)\right)}{\sqrt{\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2 \hat{u}_t^2}}$$

and we only need to tackle the limiting behavior of the denominator, for which we have that

$$\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2\hat{u}_t^2 = \sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2 u_t^2 + \sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2\left(\hat{u}_t^2 - u_t^2\right).$$

We recall from the proof of Proposition 1 that $\sup_{1\le t\le T}|\hat{u}_t^2 - u_t^2| = o_p(1)$ under both Assumptions 1.1 and 1.2.

Under Assumption 1.1, we have

$$\left|\frac{1}{T}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2\left(\hat{u}_t^2 - u_t^2\right)\right| \le \sup_{1\le t\le T}\left|\hat{u}_t^2 - u_t^2\right| \frac{1}{T}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2 = o_p(1)$$

see Equation (A.5), and, using the same argument leading to Equation (A.4), we obtain

$$\frac{1}{T}\sum_{t=+1}^{\lfloor\tau T\rfloor} z_{t-1}^2 u_t^2 = \frac{1}{T}\sum_{t=+1}^{\lfloor\tau T\rfloor} \xi_{t-1}^2 u_t^2 + o_p(1)$$

where $\frac{1}{T}\sum_{t=+1}^{\lfloor\tau T\rfloor}\xi_{t-1}^2 u_t^2 \Rightarrow [M_{\xi u}](\tau)$ is a byproduct of establishing the weak convergence of the partial sums of $\xi_{t-1}u_t$.

Under Assumption 1.2, we then immediately have thanks to Lemma 5 part (c) that

$$\left|\frac{1}{T^{1+\eta}}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2\left(\hat{u}_t^2 - u_t^2\right)\right| \le \sup_t\left|\hat{u}_t^2 - u_t^2\right| \frac{1}{T^{1+\eta}}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2 = o_p(1),$$

while the quadratic variation

$$\frac{1}{T^{1+\eta}}\sum_{t=+1}^{\lfloor\tau T\rfloor} z_{t-1}^2 u_t^2 \Rightarrow [M_{zu}](\tau)$$

is dealt with in the proof of Lemma 5 part (d). □

C.3 Bootstrap asymptotics

The next lemma establishes the asymptotics of the processes in the numerator and the denominator of the bootstrap statistic t_{zx}^* in the weakly persistent case.

Lemma 6 *Let Assumptions 1.1 and 3 hold. Let B be a standard Brownian motion on $[0,1]$ and $\mathbf{H}_{1\cdot}$,$\mathbf{H}_{2\cdot}$ denote the rows of $\mathbf{H}$. Then, as $T\to\infty$:*

(a) $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor} z_{t-1}u_t^* \overset{w}{\Rightarrow}_p M_{\xi u}(\tau) = \int_0^\tau \chi(s)^{1/2}\mathrm{d}B(s)$ *on $\mathcal{D}$, with*

$$\chi(s) = \sum_{i,j\ge 0} b_i b_j \mathrm{E}[\mathbf{H}_{1\cdot}(s)(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\mathbf{H}_{1\cdot}(s)'\mathbf{H}_{2\cdot}(s)(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')\mathbf{H}_{2\cdot}(s)'];$$

(b) $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor} z_{t-1}^{*}u_t^{*} \overset{w}{\Rightarrow}_p M_{\xi u}^{*}(\tau) := \int_0^{\tau} \chi^{*}(s)^{1/2}\mathrm{d}B(s)$ on $\mathcal{D}$, with

$$\chi^{*}(s) = \sum_{j\geq 0} b_j^2 \mathrm{E}[\mathbf{H}_{1\cdot}(s)(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\mathbf{H}_{1\cdot}(s)'\mathbf{H}_{2\cdot}(s)(\boldsymbol{\psi}_{-j}\boldsymbol{\psi}_{-j}')\mathbf{H}_{2\cdot}(s)'];$$

(c) $T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}^{*})^2 \overset{p}{\Rightarrow}_p \kappa^2[M_v](\tau)$ on $\mathcal{D}$;

(d) $\hat{\sigma}_u^{2*}(0,\tau) = T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor}(\hat{u}_t^{*})^2 \overset{p}{\Rightarrow}_p [M_u](\tau)$ on $\mathcal{D}$.

We now turn to the case of a strongly persistent posited predictor variable and discuss the process $N_T^{*}(\tau) := T^{-(1+\eta)/2}\sum_{t=1}^{\lfloor T\tau\rfloor} z_{t-1}^{*}u_t^{*}$ in steps similar to those of Magdalinos (2020). First, we approximate $N_T^{*}(\tau)$ by $\tilde{N}_T^{*}(\tau) := T^{-(1+\eta)/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^{*}\tilde{u}_t$ for $\zeta_t^{*} := \omega\sum_{j=0}^{t-1}\varrho^j v_{t-j}^{*}$ and $\tilde{u}_t := u_t R_t$. Second, we discuss the predictable quadratic variation of $\tilde{N}_T^{*}$ conditional on the data,

$$\tilde{V}_T^{*}(\tau) := T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}\mathrm{E}_{t-1}^{*}(\zeta_{t-1}^{*}\tilde{u}_t)^2 = T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}(\zeta_{t-1}^{*})^2 u_t^2,$$

whose asymptotics determine those of N_T^{*}.

Lemma 7 *Under Assumptions 1.2 and 3, it holds that*

(a) $\sup_{[0,1]}|N_T^{*} - \tilde{N}_T^{*}| = o_p^{*}(1)(1+\sup_{[0,1]}|\tilde{N}_T^{*}|)$;

(b) $\tilde{V}_T^{*}(\tau) = \tilde{V}(\tau) + o_p^{*}(1)(1+\tilde{V}(1))$ *pointwise for* $\tilde{V}(\tau) := T^{-1-\eta}\omega^2\sum_{s=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2\right)u_t^2$;

(c) *If* $\epsilon < \eta$ *in Assumption 3.2, then* $\tilde{V}_T^{*}(\tau) \overset{p}{\Rightarrow}_p \frac{\omega^2}{2a}\int_0^{\tau}[M_u(s)]'[M_v(s)]'\mathrm{d}s$ *on* $\mathcal{D}$.

We are now in a position to establish the asymptotic behaviour of the processes in the numerator and the denominator of the bootstrap statistic t_{zx}^{*} in the strongly persistent case.

Lemma 8 *Under Assumptions 1.2 and 3 with* $\epsilon < \eta$ *it holds that*

(a) $N_T^{*}(\tau) \overset{w}{\Rightarrow}_p N(\tau) = \frac{|\omega|}{\sqrt{2a}}\int_0^{\tau}\sqrt{[M_v(s)]'[M_u(s)]'}\mathrm{d}B(s)$ on $\mathcal{D}$;

(b) $T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}^{*})^2 \overset{p}{\Rightarrow}_p \frac{\omega^2}{2a}[M_v](\tau)$ on $\mathcal{D}$;

(c) $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor}(\hat{u}_t^{*})^2 = T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor}\hat{u}_t^2 + o_p^{*}(1) \overset{p}{\Rightarrow}_p [M_u](\tau)$ on 1.

Proof of Proposition 3. For the space $\mathcal{D}(T_\Delta)$, see Section C.1.

Using the limits in Lemma 6 and a CMT for weak convergence in probability (e.g., Theorem 10 of Sweeting, 1989), it follows that under Assumption 1.1,

$$t_{zx}^{*,FR}(\tau_1,\tau_2) \overset{w}{\Rightarrow}_p \frac{\sqrt{\tau_2-\tau_1}(M_{\xi u}(\tau_2)-M_{\xi u}(\tau_1))}{|\kappa|\sqrt{\{[M_u](\tau_2)-[M_u](\tau_1)\}\{[M_v](\tau_2)-[M_v](\tau_1)\}}},$$
$$t_{zx}^{*,RB}(\tau_1,\tau_2) \overset{w}{\Rightarrow}_p \frac{\sqrt{\tau_2-\tau_1}(M_{\xi u}^{*}(\tau_2)-M_{\xi u}^{*}(\tau_1))}{|\kappa|\sqrt{\{[M_u](\tau_2)-[M_u](\tau_1)\}\{[M_v](\tau_2)-[M_v](\tau_1)\}}}$$

on $\mathcal{D}(T_\Delta)$, respectively for the FRWB and the RWB t-processes. Similarly, using the limits in Lemma 8 and a CMT for weak convergence in probability, it follows that under Assumption 1.2,

$$t_{zx}^{*,RB}(\tau_1,\tau_2) \overset{w}{\Rightarrow}_p \frac{\sqrt{\tau_2-\tau_1}\int_0^\tau \sqrt{[M_u]'[M_v]'}\mathrm{d}B}{\sqrt{\{[M_u](\tau_2)-[M_u](\tau_1)\}\{[M_v](\tau_2)-[M_v](\tau_1)\}}}$$

on $\mathcal{D}(T_\Delta)$. From here the $\overset{w}{\Rightarrow}_p$- limits of $\mathcal{T}^x$, $x\in\{R,F,B\}$, follow by a further application of the same CMT, except for the fixed-regressor bootstrap statistics under Assumption 1.2. These latter limits follow from the theory of Demetrescu *et al.* (2020).

We notice that the condition $M_{\xi u}^* \overset{d}{=} M_{\xi u}$, which is necessary and sufficient for the validity of the residual-based fixed-regressor bootstrap under Assumption 1.1, is satisfied iff $\sum_{i,j\geq 0}\mathbb{I}_{\{i\neq j\}}b_ib_j\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\otimes(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')]=0$. For the latter to hold, it suffices that $\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\otimes(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')]=0$ for all natural $i\neq j$. □

C.4 Proofs of the auxiliary results

We observe for use throughout the proofs that $(u_t,v_t)'$ inherit the uniform L_4-boundedness of $(a_t,e_t)'$ inasmuch as $\sup_{t\leq T}\mathrm{E}u_t^4\leq C\|\mathbf{H}\|_\infty(\sup_t \mathrm{E}a_t^4+\sup_t\mathrm{E}e_t^4)$ with $\|\mathbf{H}\|_\infty := \sup_{r\in(-\infty,1]}\|\mathbf{H}(r)\| < \infty$, and similarly for $\sup_{t\leq T}\mathrm{E}v_t^4$.

Proof of Lemma 1. In parts (a)-(c) we provide a proof for the sequences constructed from u_t, as for those constructed from v_t the argument is analogous. It holds that

$$S_{T(1,t)}^u = \sum_{s=1}^t h_{11}^2(\frac{s}{T})\Delta\Psi_s^a + 2\sum_{s=1}^t h_{11}(\frac{s}{T})h_{12}(\frac{s}{T})\Delta\Psi_s^{ae} + \sum_{t=1}^T h_{12}^2(\frac{s}{T})\Delta\Psi_s^e.$$

In part (a) we find by partial summation that

$$\left|\sum_{s=1}^t h_{11}^2(\frac{s}{T})\Delta\Psi_s^a\right| = \left|\Psi_t^a h_{11}^2(\frac{t}{T}) - \sum_{s=2}^t \Psi_{s-1}^a\Delta h_{11}^2(\frac{s}{T})\right| \leq C\max_{1\leq s\leq t}|\Psi_s^a|,$$

and similarly for the other two summations in the decomposition of $S_{T(1,t)}^u$, with the constant C depending on the global Lipschitz constant of $\mathbf{H}$. Therefore,

$$\max_{1\leq t\leq T}|S_{T(1,t)}^u| \leq C\left(\max_{1\leq t\leq T}|\Psi_t^a| + 2\max_{1\leq t\leq T}|\Psi_t^{ae}| + \max_{1\leq t\leq T}|\Psi_t^e|\right). \tag{A.6}$$

The three maxima on the right-hand side are all $O_p(T^{1/2})$ by Theorem 11 of Merlevède *et al.* (2006). Hence, also $\max_{1\leq t\leq T}|S_{T(1,t)}^u| = O_p(T^{1/2})$.

In part (b), by writing $(S_{T(t+1,r)}^u)^2 = (S_{T(1,r)}^u - S_{T(1,t)}^u)^2 \leq 4\max_{1\leq t\leq T}(S_{T(1,t)}^u)^2$ and then using (A.6) we can conclude that

$$\mathrm{E}\left[\max_{1\leq t<r\leq T}(S_{T(t+1,r)}^u)^2\right] \leq C\left(\mathrm{E}\left[\max_{1\leq t\leq T}(\Psi_t^a)^2\right] + \mathrm{E}\left[\max_{1\leq t\leq T}(\Psi_t^{ae})^2\right] + \mathrm{E}\left[\max_{1\leq t\leq T}(\Psi_t^e)^2\right]\right).$$

Under Assumption 3 with $\epsilon < \frac{1}{2}$, the three expectations on the r.h.s. are $O(T)$ by Proposition 9 of Merlevède *et al.* (2006), and thus, so is the expectation on the l.h.s.

In part (c), $|\mathrm{E}[(u_t^2 - \sigma_{ut}^2) S_{T(t+1,r)}^u]| = |\mathrm{E}[(u_t^2 - \sigma_{ut}^2) \mathrm{E}_t S_{T(t+1,r)}^u]| \leq \|u_t^2 - \sigma_{ut}^2\|_2 \|\mathrm{E}_t S_{T(t+1,r)}^u\|_2$, where

$$\left\|\mathrm{E}_t S_{T(t+1,r)}^u\right\|_2 \leq \left\|\mathrm{E}_t \left(\sum_{s=t+1}^{r} h_{11}^2(\frac{s}{T})\Delta\Psi_s^a\right)\right\|_2 + 2\left\|\mathrm{E}_t \left(\sum_{s=t+1}^{r} h_{11}(\frac{s}{T})h_{12}(\frac{s}{T})\Delta\Psi_s^{ae}\right)\right\|_2 + \left\|\mathrm{E}_t \left(\sum_{s=t+1}^{r} h_{12}^2(\frac{s}{T})\Delta\Psi_s^e\right)\right\|_2,$$

and, using partial summation and the stationarity of a_t,

$$\begin{aligned}\left\|\mathrm{E}_t \left(\sum_{s=t+1}^{r} h_{11}^2(\frac{s}{T})\Delta\Psi_s^a\right)\right\|_2 &= \left\|\mathrm{E}_t \left[(\Psi_r^a - \Psi_t^a)h_{11}^2(\frac{r}{T}) - \sum_{s=t+2}^{r} (\Psi_{s-1}^a - \Psi_t^a)\Delta h_{11}^2(\frac{s}{T})\right]\right\|_2 \\ &= \left\|\mathrm{E}_0 \left(\Psi_{r-t}^a h_{11}^2(\frac{r}{T}) - \sum_{s=t+2}^{r} \Psi_{s-t-1}^a \Delta h_{11}^2(\frac{s}{T})\right)\right\|_2 \\ &\leq h_{11}^2(\frac{r}{T}) \left\|\mathrm{E}_0\Psi_{r-t}^a\right\|_2 + \sum_{s=t+2}^{r} \left\|\mathrm{E}_0\Psi_{s-t-1}^a\right\|_2 \left|\Delta h_{11}^2(\frac{s}{T})\right| \\ &\leq C \max_{1\leq t\leq T} \|\mathrm{E}_0\Psi_T^a\|_2 = O(T^\epsilon)\end{aligned}$$

uniformly in r, t, and similarly for the other two conditional expectations in the upper bound for $|\mathrm{E}[(u_t^2 - \sigma_{ut}^2) S_{T(t+1,r)}^u]|$, with the constant C depending on the global Lipschitz constant of $\mathbf{H}$. We conclude that $\|\mathrm{E}_t S_{T(t+1,r)}^u\|_2 = O(T^\epsilon)$ uniformly in r, t. As $\|u_t^2 - \sigma_{ut}^2\|_2$ is a bounded sequence, part (c) follows. □

Proof of Lemma 2. In part (a), by using (A.2), we find that

$$\max_{s\leq T}\left|T^{-\alpha}\sum_{t=1}^{s}\varrho^{t-1}w_t\right| \leq \max_{s\leq T}\left|T^{-\alpha}\sum_{t=1}^{s} w_t\right| \Rightarrow \sup_{\tau\in[0,1]}|W(\tau)|$$

by the CMT, from where the magnitude order of $\max_{s\leq T}|T^{-\alpha}\sum_{t=1}^{s}\varrho^{t-1}w_t|$ follows.

In part (b), for every $r \in \{1, ..., T\}$ (A.2) yields

$$\begin{aligned}\mathrm{E}\left|\sum_{t=1}^{r}\varrho^{t-1}w_t\right| &\leq \varrho^{r-1}\mathrm{E}\left|\sum_{t=1}^{r}w_t\right| + (1-\varrho)\sum_{s=1}^{r-1}\varrho^{s-1}\mathrm{E}\left|\sum_{t=1}^{s}w_t\right| \\ &\leq (\varrho^{r-1} + (1-\varrho)\sum_{s=1}^{r-1}\varrho^{s-1})\max_{1\leq s\leq T}\mathrm{E}\left|\sum_{t=1}^{s}w_t\right| = \max_{1\leq s\leq T}\mathrm{E}\left|\sum_{t=1}^{s}w_t\right|\end{aligned}$$

and the conclusion follows by taking maxima over r.

We turn to part (c) and discuss the nontrivial case $m_T := \max_{t\leq T}\mathrm{E}|w_{Tt}|^p > 0$. If w_{Tt} is an MD array with $\mathrm{E}|w_{Tt}|^p < \infty$ for some $p > 2$, then

$$\mathrm{E}\left|\sum_{j=0}^{t-1} \varrho^j w_{T,t-j}\right|^p \leq C \left(\sum_{j=0}^{t-1} \varrho^{2j} \left(\mathrm{E}|w_{T,t-j}|^p\right)^{2/p}\right)^{p/2}$$

by Lemma 2.5.2 of Giraitis et al. (2012). Further,

$$\mathrm{E}\left|\sum_{j=0}^{t-1} \varrho^j w_{T,t-j}\right|^p \leq C m_T \left(\sum_{j=0}^{T} \varrho^{2j}\right)^{p/2} \leq C m_T T^{\frac{\eta p}{2}},$$

such that $\sum_{j=0}^{t-1} \varrho^j w_{T,t-j} = O_p(m_T^{1/p} T^{\eta/2}) = o_p(m_T^{1/p} T^{1/2})$ for every fixed $t \leq T$. To obtain the same infinitesimality order uniformly, we apply Billingsley's (1968, Theorem 15.6) tightness criterion to $m_T^{-1/p} T^{-1/2} W_T(\tau)$ with $W_T(\tau) := \sum_{j=0}^{\lfloor T\tau \rfloor - 1} \varrho^j w_{T,\lfloor T\tau \rfloor - j}$. For $0 \leq \tau_1 < \tau < \tau_2 \leq 1$, it holds that

$$\mathrm{E}[|W_T(\tau_2) - W_T(\tau)|^{p/2} |W_T(\tau) - W_T(\tau_1)|^{p/2}] \leq \sqrt{\mathrm{E}|W_T(\tau_2) - W_T(\tau)|^p \mathrm{E}|W_T(\tau) - W_T(\tau_1)|^p}$$

where

$$\begin{aligned}\mathrm{E}|W_T(\tau_2) - W_T(\tau)|^p &= \mathrm{E}\left|\sum_{j=0}^{\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor - 1} \rho^j w_{T,\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor - j} + (\varrho^{\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor} - 1) \sum_{j=0}^{\lfloor \tau T \rfloor - 1} \rho^j w_{T,\lfloor \tau T \rfloor - j}\right|^p \\ &\leq \Bigg[\sum_{j=0}^{\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor - 1} \rho^{2j} (\mathrm{E}|w_{T,\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor - j}|^p)^{2/p} \\ &\qquad + (\varrho^{\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor} - 1)^2 \sum_{j=0}^{\lfloor \tau T \rfloor - 1} \rho^{2j} (\mathrm{E}|w_{T,\lfloor \tau T \rfloor - j}|^p)^{2/p}\Bigg]^{p/2}.\end{aligned}$$

by Lemma 2.5.2 of Giraitis et al. (2012), then

$$\begin{aligned}\mathrm{E}|W_T(\tau_2) - W_T(\tau)|^p &\leq m_T (1-\rho^2)^{-p/2} \left[1 - \rho^{2(\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor)} + (\varrho^{\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor} - 1)^2 (1 - \varrho^{2\lfloor \tau T \rfloor})\right]^{p/2} \\ &= m_T (1-\rho^2)^{-p/2} (1 - \rho^{\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor})^{p/2} \big[1 + \rho^{\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor} \\ &\qquad + (1 - \rho^{\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor})(1 - \varrho^{2\lfloor \tau T \rfloor})\big]^{p/2} \\ &\leq m_T (1-\rho^2)^{-p/2} (1 - \rho^{\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor})^{p/2} 3^{p/2}.\end{aligned}$$

and by Bernoulli's inequality,

$$\mathrm{E}|W_T(\tau_2) - W_T(\tau)|^p \leq m_T \frac{(3a)^{p/2} (\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor)^{p/2}}{T^{\eta p/2} (1-\rho^2)^{p/2}} \leq C m_T (\lfloor \tau_2 T \rfloor - \lfloor \tau T \rfloor)^{p/2},$$

such that, with a similar estimate for $\mathrm{E}|W_T(\tau) - W_T(\tau_1)|^p$, eventually

$$m_T^{-1}T^{-p/2}\mathrm{E}[|W_T(\tau_2) - W_T(\tau)|^{p/2}|W_T(\tau) - W_T(\tau_1)|^{p/2}] \leq CT^{-p/2}(\lfloor\tau_2 T\rfloor - \lfloor\tau T\rfloor)^{p/4}(\lfloor\tau_1 T\rfloor - \lfloor\tau_1 T\rfloor)^{p/4}$$
$$\leq C(\frac{\lfloor\tau_2 T\rfloor - \lfloor\tau_1 T\rfloor}{T})^{p/2} \leq C(\tau_2 - \tau_1)^{p/2}.$$

Since $p/2 > 1$, as required by Billingsley's criterion, it follows that $m_T^{-1/p}T^{-1/2}W_T(\tau)$ is tight.

In part (d), (A.2) yields $\max_{1\leq s\leq T} | \sum_{t=1}^{T-s} \varrho^{2(t-1)}(u_{s+t}^2 - \sigma_{u,s+t}^2)| \leq 2\max_{1\leq s\leq T} |S_{T(1,s)}^u| = O_p(T^{1/2})$ by Lemma 1(a). Similarly, in part (e),

$$\max_{1\leq t\leq T} \left\| \mathrm{E}_t \sum_{r=t+1}^{T} \varrho^{2(r-t-1)}(u_r^2 - \sigma_{ur}^2) \right\|_2 \leq \max_{1\leq t<r\leq T} \left\|\mathrm{E}_t S_{T(t+1,r)}\right\|_2 = O(T^{\epsilon})$$

by the proof of Lemma 1(c).

We turn to the proof of part (f). It holds that

$$\left[\sum_{s=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2\right)(u_t^2 - \sigma_{ut}^2)\right]^2 = \left[\sum_{s=1}^{\lfloor T\tau\rfloor-1} v_s^2 \sum_{t=s+1}^{T}\varrho^{2(t-s-1)}(u_t^2 - \sigma_{ut}^2)\right]^2$$
$$\leq \sum_{s=1}^{\lfloor T\tau\rfloor-1} v_s^4 \sum_{s=1}^{\lfloor T\tau\rfloor-1}\left[\sum_{t=s+1}^{T}\varrho^{2(t-s-1)}(u_t^2 - \sigma_{ut}^2)\right]^2.$$

As $\sum_{s=1}^{\lfloor T\tau\rfloor-1} v_s^4 = O_p(T)$ by Markov's inequality, part (c) will follow if

$$\sum_{s=1}^{\lfloor T\tau\rfloor-1}\left[\sum_{t=s+1}^{T}\varrho^{2(t-s-1)}(u_t^2 - \sigma_{ut}^2)\right]^2 = O_p(T^{1+\eta+\epsilon}). \tag{A.7}$$

In the decomposition

$$\mathrm{E}\left[\sum_{t=s+1}^{T}\varrho^{2(t-s-1)}(u_t^2 - \sigma_{ut}^2)\right]^2 = \sum_{t=s+1}^{T}\varrho^{4(t-s-1)}\mathrm{E}(u_t^2 - \sigma_{ut}^2)^2$$
$$+ 2\sum_{t=s+1}^{T}\varrho^{2(t-s-1)}\sum_{r=t+1}^{T}\varrho^{2(r-t-1)}\mathrm{E}[(u_t^2 - \sigma_{ut}^2)(u_r^2 - \sigma_{ur}^2)]$$

eq. (A.2) can be used to bound the mixed products as follows:

$$\left|\sum_{r=t+1}^{T}\varrho^{2(r-t-1)}\mathrm{E}[(u_t^2 - \sigma_{ut}^2)(u_r^2 - \sigma_{ur}^2)]\right| \leq \max_{t+1\leq q\leq T}\left|\mathrm{E}\left[(u_t^2 - \sigma_{ut}^2)\sum_{r=t+1}^{q}(u_r^2 - \sigma_{ur}^2)\right]\right|$$
$$\leq \max_{t+1\leq q\leq T}\left|\mathrm{E}[(u_t^2 - \sigma_{ut}^2)S_{T(t+1,q)}^u]\right|.$$

As $\max_{1\leq t\leq T} \|u_t^2-\sigma_{ut}^2\|_2 = O(1)$, it can be concluded that

$$\mathrm{E}\left[\sum_{t=s+1}^{T} \varrho^{2(t-s-1)}(u_t^2-\sigma_{ut}^2)\right]^2 = O(T^\eta) + 2\max_{1\leq t\leq q\leq T}\left|\mathrm{E}[(u_t^2-\sigma_{ut}^2)S_{T(t+1,q)}^u]\right| \sum_{t=s+1}^{T} \varrho^{2(t-s-1)}$$

uniformly in $s\leq T$, such that (A.7) follows by Markov's inequality and Lemma 1(c). This completes the proof of part (f).

Finally, to prove part (g), we first show that $\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\sigma_{ut}^2 = o_p(T^{1+\eta})$ pointwise. In fact,

$$\left[\sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right)\sigma_{ut}^2\right]^2 \leq \sum_{t=1}^{T}\left[\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right]^2\sum_{t=1}^{T}\sigma_{ut}^4,$$

where $\sum_{t=1}^{T}\sigma_{ut}^4 = O(T)$, whereas the other factor on the right-hand side is $O_p(T^{1+\eta+\epsilon})$ similarly to an analogous expression in the proof of part (f). Specifically,

$$\begin{aligned}\mathrm{E}\left[\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right]^2 &= \sum_{j=0}^{t-2}\varrho^{4j}\mathrm{E}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)^2\\ &\quad + 2\sum_{j=0}^{t-2}\sum_{i=j+1}^{t-2}\varrho^{2(j+i)}\mathrm{E}[v_{t-i-1}^2(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)],\end{aligned}$$

where $\sum_{j=0}^{t-2}\varrho^{4j}\mathrm{E}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)^2 \leq \max_{1\leq t\leq T}\|v_t^2-\sigma_{vt}^2\|_2^2\sum_{j=0}^{T}\varrho^{4j} = O(T^\eta)$ and

$$\sum_{j=0}^{t-2}\sum_{i=j+1}^{t-2}\varrho^{2(j+i)}\mathrm{E}[v_{t-i-1}^2(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)] = \sum_{s=1}^{t-1}\varrho^{4(t-s-1)}\sum_{r=s+1}^{t-1}\varrho^{2(s-r)}\mathrm{E}[v_s^2(v_r^2-\sigma_{vr}^2)]$$

with

$$\begin{aligned}\left|\sum_{r=s+1}^{t-1}\varrho^{2(s-r)}\mathrm{E}[v_s^2(v_r^2-\sigma_{vr}^2)]\right| &\leq \max_{s+1\leq q\leq t-1}\left|\sum_{r=s+1}^{q}\mathrm{E}[v_s^2(v_r^2-\sigma_{vr}^2)]\right|\\ &= \max_{s+1\leq q\leq t-1}\left|\mathrm{E}(v_s^2S_{T(s+1,q)}^v)\right|\\ &\leq \max_{1\leq s<q\leq T}\left|\mathrm{E}(v_s^2S_{T(s+1,q)}^v)\right| = O(T^\epsilon)\end{aligned}$$

using (A.2) and Lemma 1(c). As the upper bounds are uniform in $t=1,...,T$, it follows that

$$\mathrm{E}\sum_{t=1}^{T}\left[\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right]^2 = O(T^{1+\eta}) + O(T^\epsilon)\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varrho^{4(t-s-1)} = O(T^{1+\eta+\epsilon}).$$

This and Markov's inequality let us conclude that $\sum_{t=1}^{T}\left[\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right]^2 = O_p(T^{1+\eta+\epsilon})$ and hence, $\sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right)\sigma_{ut}^2 = O_p(T^{1+(\eta+\epsilon)/2}) = o_p(T^{1+\eta})$

for $\epsilon < \eta$. Equivalently, $\tilde{V}(\tau) = T^{-1-\eta}\omega^2 \sum_{t=1}^{\lfloor T\tau \rfloor} \left(\sum_{j=0}^{t-2} \varrho^{2j} \sigma_{v,t-j-1}^2 \right) \sigma_{ut}^2 + o_p(1)$ pointwise.

Second, we discuss the convergence of the deterministic $\sum_{t=1}^{\lfloor T\tau \rfloor} \left(\sum_{j=0}^{t-2} \varrho^{2j} \sigma_{v,t-j-1}^2 \right) \sigma_{ut}^2$ to an integral. Say for concreteness that the function $\mathbf{H}$ (determining the unconditional variance profile) of (u_t, v_t) is Lipschitz continuous on $[0, \lambda)$ and $(\lambda,1]$, the case of more than two (but finitely many) maximal intervals of Lipschitz continuity being analogous. Without loss of generality, let $\mathbf{H}$ be right-continuous at λ. Then, for $t < \lfloor T\lambda \rfloor$ it holds that

$$\sum_{j=0}^{t-2} \varrho^{2j} \left| \sigma_{v,t-j-1}^2 - \sigma_{v,t}^2 \right| \leq C \sum_{j=0}^{t-2} \varrho^{2j} \left(\frac{j-1}{T} \right) = O(T^{2\eta-1})$$

uniformly in t, where C depends on the Lipschitz constant of the function $\mathbf{H}$, whereas for $t = \lfloor T\lambda \rfloor, ..., T$ the analogous estimate is

$$\left| \sum_{j=0}^{t-2} \varrho^{2j} (\sigma_{v,t-j-1}^2 - \sigma_{v,t}^2) - \sum_{j=t-\lfloor T\lambda \rfloor}^{t-2} \varrho^{2j} (\sigma_{v,\lfloor T\lambda \rfloor -1}^2 - \sigma_{v,t}^2) \right| \leq \sum_{j=0}^{t-\lfloor T\lambda \rfloor -1} \varrho^{2j} \left| \sigma_{v,t-j-1}^2 - \sigma_{v,t}^2 \right|$$
$$+ \sum_{j=t-\lfloor T\lambda \rfloor}^{t-2} \varrho^{2j} \left| \sigma_{v,t-j-1}^2 - \sigma_{v,\lfloor T\lambda \rfloor -1}^2 \right| \leq C \sum_{j=0}^{t-\lfloor T\lambda \rfloor -1} \varrho^{2j} \left(\frac{j-1}{T} \right)$$
$$+C \sum_{j=t-\lfloor T\lambda \rfloor}^{t-2} \varrho^{2j} \left(\frac{j-\lfloor T\lambda \rfloor}{T} \right) = O(T^{2\eta-1}).$$

As a result,

$$\sum_{t=1}^{\lfloor T\tau \rfloor} \left(\sum_{j=0}^{t-2} \varrho^{2j} \sigma_{v,t-j-1}^2 \right) \sigma_{ut}^2 = \sum_{t=1}^{\lfloor T\tau \rfloor} \left(\sum_{j=0}^{t-2} \varrho^{2j} \right) \sigma_{vt}^2 \sigma_{ut}^2 + \sum_{t=\lfloor T\lambda \rfloor}^{\lfloor T\tau \rfloor} \left(\sum_{j=t-\lfloor T\lambda \rfloor}^{t-2} \varrho^{2j} \right) (\sigma_{v,\lfloor T\lambda \rfloor -1}^2 - \sigma_{v,t}^2) \sigma_{ut}^2$$
$$+ O(T^{2\eta-1}) \sum_{t=1}^{\lfloor T\tau \rfloor} \sigma_{ut}^2$$
$$= \frac{1}{2a} T^\eta \sum_{t=1}^{\lfloor T\tau \rfloor} \sigma_{vt}^2 \sigma_{ut}^2 + O(T^\eta) \sum_{t=\lfloor T\lambda \rfloor}^{\lfloor T\tau \rfloor} \varrho^{2(t-\lfloor T\lambda \rfloor)} (\sigma_{v,\lfloor T\lambda \rfloor -1}^2 - \sigma_{v,t}^2) \sigma_{ut}^2$$
$$+ O(T^{2\eta}) = \frac{T^{1+\eta}}{2a} \int_0^\tau [M_v(s)]'[M_u(s)]' \mathrm{d}s + o(T^{1+\eta}) + O(T^{2\eta})$$

using the boundedness of σ_{ut}^2 and σ_{vt}^2. This establishes the pointwise limit asserted in part (g). As the involved processes are increasing and the limiting function is also continuous, the limit is a uniform one as well. □

Proof of Lemma 3. It holds that

$$\frac{1}{T} \sum_{t=1}^{\lfloor T\tau \rfloor} \begin{pmatrix} u_t^2 - \sigma_{ut}^2 \\ v_t^2 - \sigma_{vt}^2 \end{pmatrix} = T^{-1} \begin{pmatrix} S_{T(1,\lfloor T\tau \rfloor)}^u \\ S_{T(1,\lfloor T\tau \rfloor)}^v \end{pmatrix} = O_p(T^{-1/2})$$

uniformly in τ, by Lemma 1(a). Further, $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor}(\sigma_{ut}^2,\sigma_{vt}^2)'$ are Riemann sums of the limiting integral, which exists by the Lipschitz-by-parts property of $\mathbf{H}$, and convergence follows from the definition of the integral. The convergence is uniform because the involved coordinate functions are increasing and the limiting coordinate functions are continuous. □

Proof of Lemma 4. With b_j the coefficients of $[A(L)(1-\rho L)]^{-1}$, where $|\rho|<1$ is bounded away from unity, let

$$\begin{aligned}\tilde{\xi}_{t-1} &= \sum_{j\geq 0} b_j\left(h_{21}(t/T)a_{t-1-j}+h_{22}(t/T)e_{t-1-j}\right)\\ &= h_{21}(t/T)\sum_{j\geq 0} b_j a_{t-1-j}+h_{22}(t/T)\sum_{j\geq 0} b_j e_{t-1-j}\end{aligned}$$

and note that

$$\xi_{t-1}-\tilde{\xi}_{t-1}=\sum_{j\geq 0} b_j\left(\left(h_{21}\left(\frac{t-1-j}{T}\right)-h_{21}\left(\frac{t}{T}\right)\right)a_{t-1-j}+\left(h_{22}\left(\frac{t-1-j}{T}\right)-h_{22}\left(\frac{t}{T}\right)\right)e_{t-1-j}\right).$$

Therefore,

$$\sum_{t=1}^{T}\mathrm{E}\left(\left|\xi_{t-1}-\tilde{\xi}_{t-1}\right|\right)\leq CT\sum_{j\geq 0}\frac{j+1}{T}b_j=O\left(1\right)$$

since the absolute moments are uniformly bounded, b_j are 1-summable (in fact they have exponential decay), and $h_{ij}(\cdot)$ are piecewise Lipschitz, where the discontinuities are accounted for along the lines of the proof of Lemma 2 (g). We may therefore write

$$\sup_{\tau\in[0,1]}\left|\sum_{t=1}^{\lfloor\tau T\rfloor}u_t\xi_{t-1}-\sum_{t=1}^{\lfloor\tau T\rfloor}u_t\tilde{\xi}_{t-1}\right|\leq\sum_{t=1}^{\lfloor\tau T\rfloor}|u_t|\left|\xi_{t-1}-\tilde{\xi}_{t-1}\right|\leq\sup_{1\leq t\leq T}|u_t|\sum_{t=1}^{T}\left|\xi_{t-1}-\tilde{\xi}_{t-1}\right|=o_p\left(\sqrt{T}\right)$$

thanks to Markov's inequality and the fact that uniformly bounded 4th order moments imply $\sup_{1\leq t\leq T}|u_t|=o_p\left(\sqrt{T}\right)$.

Then, uniformly in τ,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor\tau T\rfloor}\begin{pmatrix}u_t\\ v_t\\ u_t\xi_{t-1}\end{pmatrix}=\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor\tau T\rfloor}\begin{pmatrix}u_t\\ v_t\\ u_t\tilde{\xi}_{t-1}\end{pmatrix}+o_p\left(1\right).$$

Now,

$$\begin{aligned}u_t\tilde{\xi}_{t-1} &= h_{11}(t/T)h_{21}(t/T)a_t\sum_{j\geq 0} b_j a_{t-1-j}+h_{11}(t/T)h_{22}(t/T)a_t\sum_{j\geq 0} b_j e_{t-1-j}\\ &\quad +h_{12}(t/T)h_{21}(t/T)e_t\sum_{j\geq 0} b_j a_{t-1-j}+h_{12}(t/T)h_{22}(t/T)e_t\sum_{j\geq 0} b_j e_{t-1-j}\end{aligned}$$

and we note (with all functions h_{ij} evaluated at t/T) that

$$\begin{pmatrix} u_t \\ v_t \\ u_t\tilde{\xi}_{t-1} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & 0 & 0 & 0 & 0 \\ h_{21} & h_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{11}h_{21} & h_{11}h_{22} & h_{12}h_{21} & h_{12}h_{22} \end{pmatrix} \begin{pmatrix} a_t \\ e_t \\ a_t \sum_{j\geq 0} b_j a_{t-1-j} \\ a_t \sum_{j\geq 0} b_j e_{t-1-j} \\ e_t \sum_{j\geq 0} b_j a_{t-1-j} \\ e_t \sum_{j\geq 0} b_j e_{t-1-j} \end{pmatrix} = \mathbf{G}(t/T)\tilde{\boldsymbol{\psi}}_t.$$

Furthermore, the covariance matrix of $\tilde{\boldsymbol{\psi}}_t$ is constant and can be determined in a straightforward manner, e.g.

$$Cov\left(\tilde{\boldsymbol{\psi}}_{t,3}, \tilde{\boldsymbol{\psi}}_{t,4}\right) = \sum_{j\geq 0}\sum_{k\geq 0} b_j b_k \mathrm{E}\left(a_t^2 a_{t-1-j} e_{t-1-k}\right).$$

Finally, $\tilde{\boldsymbol{\psi}}_t$ is easily seen to obey an invariance principle for stationary and ergodic square-integrable MDs, such that, summing up,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor \tau T\rfloor} \tilde{\boldsymbol{\psi}}_t \Rightarrow \int_0^\tau \mathbf{G}(s)\mathrm{d}\boldsymbol{B}(s)$$

where $\boldsymbol{B}(\tau)$ is a 6-variate Brownian motion of covariance matrix $Cov\left(\tilde{\boldsymbol{\psi}}_t\right)$.

Of particular importance is the quadratic variation (and implicitly the variance profile) of $M_{\xi u}(\tau)$ (the third component of $\int_0^\tau \mathbf{G}(s)\mathrm{d}\boldsymbol{B}(s)$), we have at all differentiability points

$$\frac{\mathrm{d}\left[M_{\xi u}\right](\tau)}{\mathrm{d}\tau} = \mathrm{Var}\left(u_{\lfloor \tau T\rfloor}\tilde{\xi}_{\lfloor \tau T\rfloor -1}\right) + O\left(\frac{1}{T}\right)$$

where (again with all functions h_{ij} evaluated at t/T),

$$\mathrm{Var}\left(u_t\tilde{\xi}_{t-1}\right) = \mathrm{E}\left(\left(h_{11}a_t + h_{12}e_t\right)^2\left(h_{21}\sum_{j\geq 0} b_j a_{t-1-j} + h_{22}\sum_{j\geq 0} b_j e_{t-1-j}\right)^2\right)$$

or

$$
\begin{aligned}
\mathrm{Var}\left(u_t\tilde{\xi}_{t-1}\right) &= h_{11}^2 h_{21}^2 \sum_{j\geq 0}\sum_{k\geq 0} b_j b_k \mathrm{E}\left(a_t^2 a_{t-1-j} a_{t-1-k}\right) \\
&\quad +2h_{11}^2 h_{21} h_{22} \sum_{j\geq 0}\sum_{k\geq 0} b_j b_k \mathrm{E}\left(a_t^2 a_{t-1-j} e_{t-1-k}\right) \\
&\quad +h_{11}^2 h_{22}^2 \sum_{j\geq 0}\sum_{k\geq 0} b_j b_k \mathrm{E}\left(a_t^2 e_{t-1-j} e_{t-1-k}\right) \\
&\quad +2h_{11} h_{12} h_{21}^2 \sum_{j\geq 0}\sum_{k\geq 0} b_j b_k \mathrm{E}\left(a_t e_t a_{t-1-j} a_{t-1-k}\right) \\
&\quad +4h_{11} h_{12} h_{21} h_{22} \sum_{j\geq 0}\sum_{k\geq 0} b_j b_k \mathrm{E}\left(a_t e_t a_{t-1-j} e_{t-1-k}\right) \\
&\quad +2h_{11} h_{12} h_{22}^2 \sum_{j\geq 0}\sum_{k\geq 0} b_j b_k \mathrm{E}\left(a_t e_t e_{t-1-j} e_{t-1-k}\right) \\
&\quad +h_{12}^2 h_{21}^2 \sum_{j\geq 0}\sum_{k\geq 0} b_j b_k \mathrm{E}\left(e_t^2 a_{t-1-j} a_{t-1-k}\right) \\
&\quad +2h_{12}^2 h_{21} h_{22} \sum_{j\geq 0}\sum_{k\geq 0} b_j b_k \mathrm{E}\left(e_t^2 a_{t-1-j} e_{t-1-k}\right) \\
&\quad +h_{12}^2 h_{22}^2 \sum_{j\geq 0}\sum_{k\geq 0} b_j b_k \mathrm{E}\left(e_t^2 e_{t-1-j} e_{t-1-k}\right).
\end{aligned}
$$

The previous variance is precisely $\chi(t/T)$ as defined in Lemma 6(a). □

In the proof of Lemma 5, z_t is frequently approximated by $\omega\zeta_t$, where $\zeta_{t-1} = (1-\varrho L)_+^{-1} v_{t-1}$ and the approximation error can be controlled for in most sums, but not all (see the partial sums of z_t). □

Proof of Lemma 5(a). It holds that

$$
z_t = \sum_{j=0}^{t-1} \varrho^j w_{t-j} - (c/T) \sum_{j=0}^{t-1} \varrho^j \xi_{t-j-1} \tag{A.8}
$$

where, by using the Beveridge-Nelson decomposition $w_t = A^{-1}(L)v_t = \omega v_t - \Delta\tilde{v}_t$ (which defines $\tilde{v}_t$) and (A.2),

$$
\begin{aligned}
\sum_{j=0}^{t-1} \varrho^j w_{t-j} &= \omega \sum_{j=0}^{t-1} \varrho^j v_{t-j} - \sum_{j=0}^{t-1} \varrho^j \Delta\tilde{v}_{t-j} \\
&= \omega\zeta_t - \tilde{v}_t + \varrho^{t-1}\tilde{v}_0 + (1-\varrho)\sum_{s=1}^{t-1} \varrho^{s-1}\tilde{v}_{t-s}
\end{aligned}
$$

with $\zeta_t = \sum_{j=0}^{t-1} \varrho^j v_{t-j}$. Write $\sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1} = \omega Z_1(\tau) + Z_2(\tau) - (c/a)T^{\eta-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \xi_{t-2}$ with, first,

$$
\begin{aligned}
Z_1(\tau) := \sum_{t=1}^{\lfloor \tau T \rfloor} \zeta_{t-1} &= \left(\sum_{j=0}^{\infty} \varrho^j \right) \sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} - \sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} \left(\sum_{j=\lfloor \tau T \rfloor - t+1}^{\infty} \varrho^j \right) \\
&= a^{-1} T^{\eta} \left(\sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} - \sum_{t=2}^{\lfloor \tau T \rfloor} \varrho^{\lfloor \tau T \rfloor - t+1} v_{t-1} \right) \\
&= a^{-1} T^{\eta} \left(\sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} - \zeta_{\lfloor \tau T \rfloor - 1} \right) = a^{-1} T^{\eta} \sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} + o_p(T^{\eta+1/2})
\end{aligned}
$$

uniformly in $\tau \in [0,1]$ because $\max_{t \le T} |\zeta_t| = o_p(T^{1/2})$ by Lemma 2(c) with $w_{Tt} = v_t, p = 4$ and $\max_{1 \le t \le T} \mathrm{E} v_t^4 = O(1)$. Second,

$$
Z_2(\tau) := \sum_{t=1}^{\lfloor \tau T \rfloor} \sum_{j=0}^{t-2} \varrho^j \Delta \tilde{v}_{t-1-j} = \sum_{j=0}^{\lfloor \tau T \rfloor - 2} \varrho^j \tilde{v}_{\lfloor \tau T \rfloor - 1 - j} - \tilde{v}_0 \sum_{j=0}^{\lfloor \tau T \rfloor - 2} \varrho^j = o_p(T^{\eta+1/2})
$$

uniformly in $\tau \in [0,1]$ because because $\max_{t \le T} |\sum_{j=0}^{t-1} \varrho^j \tilde{v}_{t-j}| = o_p(T^{1/2})$ by Lemma 2(c) with $w_{Tt} = \tilde{v}_t, p = 4$ and $\max_{1 \le t \le T} \mathrm{E} \tilde{v}_t^4 = O(1)$. By collecting the previous results, it follows that, uniformly in $\tau \in [0,1]$,

$$
\frac{1}{T^{1/2+\eta}} \sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1} = \frac{\omega}{a} \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor \tau T \rfloor} v_{t-1} - \frac{c}{a} \frac{1}{T^{3/2}} \sum_{t=2}^{\lfloor \tau T \rfloor} \xi_{t-2} + o_p(1) \Rightarrow \frac{\omega}{a} \left(M_v(\tau) - c \int_0^{\tau} J_{c,H}(s) \mathrm{d}s \right),
$$

using in particular the continuity of the two summand processes. The latter limit is $\frac{\omega}{a} J_{c,H}(\tau)$ by the Ornstein-Uhlenbeck differential equation. □

Proof of Lemma 5(b). We first show that

$$
\max_{t \le T} \left| \sum_{j=0}^{t-1} \varrho^j \xi_{t-j} v_{t-j} \right| = o_p(T).
$$

For any fixed $K > 0$, the following decomposition holds:

$$
\begin{aligned}
\sum_{j=0}^{t-1} \varrho^j \xi_{t-j} v_{t-j} &= \sum_{j=0}^{t-1} \varrho^j \xi_{t-j-1} v_{t-j} + \sum_{j=0}^{t-1} \varrho^j \Delta \xi_{t-j} v_{t-j} \\
&= (1 - \frac{c}{T}) \sum_{j=0}^{t-1} \varrho^j \xi_{t-j-1} v_{t-j} + \sum_{j=0}^{t-1} \varrho^j w_{t-j} v_{t-j} \\
&= (1 - \frac{c}{T}) \left(\sum_{j=0}^{t-1} \varrho^j \mathbb{I}_{\{|\xi_{t-j-1}| \le T^{1/2} K\}} \xi_{t-j-1} v_{t-j} + \sum_{j=0}^{t-1} \varrho^j \mathbb{I}_{\{|\xi_{t-j-1}| > T^{1/2} K\}} \xi_{t-j-1} v_{t-j} \right) \\
&\quad + \sum_{j=0}^{t-1} \varrho^j (v_{t-j}^2 - \sigma_{v,t-j}^2) + \sum_{j=0}^{t-1} \varrho^j \sigma_{v,t-j}^2 + \sum_{j=0}^{t-1} \varrho^j v_{t-j} (w_{t-j} - v_{t-j}).
\end{aligned}
$$

Here $\sum_{j=0}^{t-1} \varrho^j \mathbb{I}_{\{|\xi_{t-j-1}|\le T^{1/2}K\}} \xi_{t-j-1} v_{t-j} = o_p(T)$ by Lemma 2(c) with $w_{Tt} = \mathbb{I}_{\{|\xi_{t-j-1}|\le T^{1/2}K\}} \xi_{t-j-1} v_{t-j}$, $p = 4$ and $\max_{1\le t\le T} \mathrm{E} w_{Tt}^4 = O(T^2)$. Since $\max_{t\le T} |\xi_t| = O_p(T^{1/2})$, it follows that, by choosing K sufficiently large, $\sum_{j=0}^{t-1} \varrho^j \mathbb{I}_{\{|\xi_{t-j-1}|> T^{1/2}K\}} \xi_{t-j-1} v_{t-j}$ can be made equal to zero with probability as close to one as desired. Next, by (A.2) and Lemma 1(a),

$$\max_{1\le t\le T} \left| \sum_{j=0}^{t-1} \varrho^j (v_{t-j}^2 - \sigma_{v,t-j}^2) \right| \le \max_{1\le t\le T} |S^v_{T(1,t)}| = O_p(T^{1/2}),$$

whereas $\sum_{j=0}^{t-1} \varrho^j \sigma_{v,t-j}^2 = O(T^\eta) = o(T)$ by the boundedness of σ_{vt}^2. Finally, for any fixed $L > 0$,

$$\sum_{j=0}^{t-1} \varrho^j v_{t-j}(w_{t-j} - v_{t-j}) = \sum_{j=0}^{t-1} \varrho^j \left[\mathbb{I}_{\{|w_{t-j}-v_{t-j}|\le T^{1/2}L\}} + \mathbb{I}_{\{|w_{t-j}-v_{t-j}|> T^{1/2}L\}} \right] v_{t-j}(w_{t-j} - v_{t-j}),$$

where $w_{t-j} - v_{t-j} = \sum_{i=1}^{\infty} b_i v_{t-j-i}$ is in the past of v_{t-j}. Thus, $\sum_{j=0}^{t-1} \mathbb{I}_{\{|w_{t-j}-v_{t-j}|\le T^{1/2}L\}} v_{t-j}(w_{t-j} - v_{t-j}) = o_p(T)$ by Lemma 2(c) with $w_{Tt} = \mathbb{I}_{\{|w_{t-j}-v_{t-j}|\le T^{1/2}L\}} v_{t-j}(w_{t-j} - v_{t-j})$,$p = 4$ and $\max_{1\le t\le T} \mathrm{E} w_{Tt}^4 = O(T^2)$. As $\max_{t\le T} |w_{t-j} - v_{t-j}| = o_p(T^{1/2})$ because $\mathrm{E}|w_t - v_t|^4$ is a bounded sequence, by choosing L sufficiently large $\sum_{j=0}^{t-1} \mathbb{I}_{\{|w_{t-j}-v_{t-j}|> T^{1/2}L\}} v_{t-j}(w_{t-j} - v_{t-j})$ can be made equal to zero with probability as close to one as desired. By combining the previous conclusions, it follows that $\max_{t\le T} \left| \sum_{j=0}^{t-1} \varrho^j \xi_{t-j} v_{t-j} \right| = o_p(T)$.

We turn to the process of main interest in part (b). Similarly to part (a), it holds that $\sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1} x_{t-1} = \omega Z X_1(\tau) + Z X_2(\tau) - (c/a) T^{\eta-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \xi_{t-2} \xi_{t-1} + o_p(T^{1/2+\eta})$ uniformly in $\tau \in [0,1]$, with the remainder $\mu_x \sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1}$ discussed in part (a). The summands $Z X_i(\tau) := \sum_{t=2}^{\lfloor \tau T \rfloor} \Delta Z_i(\frac{t}{T}) \xi_{t-1}$ $(i = 1,2)$ behave as follows. First,

$$\begin{aligned} Z X_1(\tau) &= \sum_{t=2}^{\lfloor \tau T \rfloor} \zeta_{t-1} \xi_{t-1} = a^{-1} T^\eta \left(\sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} \xi_{t-1} - \sum_{t=2}^{\lfloor \tau T \rfloor} \varrho^{\lfloor \tau T \rfloor - t + 1} v_{t-1} \xi_{t-1} \right) \\ &= a^{-1} T^\eta \sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} \xi_{t-1} + o_p(T^{1+\eta}) \end{aligned}$$

uniformly in $\tau \in [0,1]$ because $\max_{t\le T} \left| \sum_{j=0}^{t-1} \varrho^j \xi_{t-j} v_{t-j} \right| = o_p(T)$ as shown previously. Second,

$$\begin{aligned} Z X_2(\tau) &:= \sum_{t=2}^{\lfloor \tau T \rfloor} (\tilde{v}_{t-1} - (1-\varrho) \sum_{j=1}^{t-2} \varrho^{j-1} \tilde{v}_{t-1-j} - \varrho^{t-2} \tilde{v}_0) \xi_{t-1} \\ &= O_p(T) + (1-\varrho) O_p(T^{1+\eta}) + O_p(T^{1/2+\eta}) = o_p(T^{1+\eta}) \end{aligned}$$

uniformly in $\tau \in [0,1]$ because $T^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{v}_{t-1} \xi_{t-1}$ converges weakly in $\mathcal{D}$, $\sum_{t=2}^{\lfloor \tau T \rfloor} \sum_{j=1}^{t-2} \varrho^{j-1} \tilde{v}_{t-1-j} \xi_{t-1}$ is of the same form (and thus, uniform magnitude order) as $Z X_1(\tau)$, and $\max_{t\le T} |\xi_t| =$

$O_p(T^{1/2})$. Recollecting the results about $ZX_i(\tau)$ $(i = 1, 2)$, we find that

$$\frac{1}{T^{1+\eta}} \sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1} x_{t-1} = \frac{\omega}{a} \frac{1}{T} \sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} \xi_{t-1} - \frac{c}{a} \frac{1}{T^2} \sum_{t=2}^{\lfloor \tau T \rfloor} \xi_{t-2} \xi_{t-1} + o_p(1),$$

where the summations on the right-hand side are not affected by mild integration. It then follows by standard near-integration asymptotics that

$$\begin{aligned} \frac{1}{T^{1+\eta}} \sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1} x_{t-1} &\Rightarrow \frac{\omega^2}{a} \left(\int_0^\tau J_{c,H} dM_v + [M_v]_\tau - c \int_0^\tau J_{c,H}^2 \right) \\ &= \frac{\omega^2}{a} \left(\int_0^\tau J_{c,H} dJ_{c,H} + [M_v]_\tau \right), \end{aligned}$$

the equality by the Ornstein-Uhlenbeck differential equation. It remains to note that $[M_v]_\tau = [J_{c,H}]_\tau$ and $J_{c,H}^2(\tau) - \int_0^\tau J_{c,H} dJ_{c,H} = \int_0^\tau J_{c,H} dJ_{c,H} + [J_{c,H}]_\tau$, the latter by the semimartingale property of $J_{c,H}$. (Alternative functional representations of the limit are, thus, are given by $\frac{1}{2}(J_{c,H}^2(\tau) + [J_{c,H}]_\tau) = \frac{1}{2}(J_{c,H}^2(\tau) + [M_v]_\tau)$. □

Proof of Lemma 5(c). Since the involved processes are increasing and the function $[M_v](\tau)$ is continuous, with the interval $[0, 1]$ compact, it is sufficient to show that the asserted convergence holds pointwise in probability for $\tau \in [0, 1]$.

First, we argue that terms involving the local parameter c and v_{-i}, $i \in \mathbb{N} \cup \{0\}$, are asymptotically negligible. Recall (A.8). Since

$$\sum_{t=1}^{\lfloor \tau T \rfloor} \left(\sum_{j=0}^{t-1} \varrho^j \xi_{t-j-1} \right)^2 \leq \max_{t=0,\ldots,T} \xi_t^2 \sum_{t=1}^{T} \left(\sum_{j=0}^{t-1} \varrho^j \right)^2 = O_p(T^{2+2\eta}) = o_p(T^{3+\eta})$$

and $\theta_{\lfloor \tau T \rfloor} = \sum_{t=1}^{\lfloor \tau T \rfloor} \left\{ \sum_{i=t-1}^{\infty} v_{t-1-i} \left(\sum_{j=0}^{t-1} \varrho^j b_{i-j} \right) \right\}^2 \geq 0$ with

$$\begin{aligned} \mathrm{E}\theta_{\lfloor \tau T \rfloor} &= \sum_{t=1}^{\lfloor \tau T \rfloor} \sum_{i=t-1}^{\infty} \mathrm{E} v_{t-1-i}^2 \left(\sum_{j=0}^{t-1} \varrho^j b_{i-j} \right)^2 \leq C \sum_{t=1}^{T} \sum_{j,k=0}^{t-1} \varrho^j \varrho^k \sum_{i=t-1}^{\infty} b_{i-j} b_{i-k} \\ &\leq C \sum_{k=0}^{T-1} \sum_{j=0}^{k} \varrho^j \varrho^k \sum_{t=1}^{T-k} \sum_{i=t-1}^{\infty} |b_{i+k-j}||b_i| \leq C \sum_{k=0}^{T-1} \sum_{j=0}^{k} \varrho^j \varrho^k \sum_{i=0}^{\infty} (i+1)|b_i| = O(T^{2\eta}), \end{aligned}$$

it follows using Markov's inequality that

$$\frac{1}{T^{1+\eta}} \sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1}^2 = \frac{1}{T^{1+\eta}} \sum_{t=1}^{\lfloor \tau T \rfloor - 1} \tilde{z}_{t-1}^2 + o_p \left(\frac{1}{T^{1+\eta}} \sum_{t=1}^{T} z_{t-1}^2 \right).$$

for $\tilde{z}_{t-1} := \sum_{j=0}^{t-2} \varrho^j \sum_{i=0}^{t-j-2} b_i v_{t-j-i-1}$.

Second, we establish the pointwise expansion

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]}\tilde{z}_{t-1}^2 = \frac{\omega^2}{T^{1+\eta}}(1+o_p(1))\sum_{t=1}^{[\tau T]}\zeta_{t-1}^2. \tag{A.9}$$

The following Beveridge-Nelson decomposition holds:

$$\tilde{z}_{t-1}-\omega\zeta_{t-1} = -\sum_{i=0}^{t-2}\tilde{b}_i v_{t-i-1} + (1-\varrho)\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\tilde{b}_i v_{t-i-j-2} \tag{A.10}$$

with

$$\mathrm{E}\sum_{t=1}^{T}(\sum_{i=0}^{t-2}\tilde{b}_i v_{t-i-1})^2 = \sum_{t=1}^{T}\sum_{i=0}^{t-2}\tilde{b}_i^2\mathrm{E}(v_{t-i-1}^2) \le C\sum_{t=1}^{T}\sum_{i=0}^{\infty}\tilde{b}_i^2 = O(T)$$

and $\mathrm{E}\sum_{t=1}^{T}(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\tilde{b}_i v_{t-i-j-2})^2 = O(T^{1+\eta})$ as shown next:

$$\begin{aligned}\mathrm{E}\sum_{t=1}^{T}(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\tilde{b}_i v_{t-i-j-2})^2 &= \mathrm{E}\sum_{t=1}^{T}(\sum_{s=1}^{t-2}v_s\sum_{j=0}^{t-s-2}\varrho^j\tilde{b}_{t-s-j-2})^2\\ &= \sum_{t=1}^{T}\sum_{s=1}^{t-2}\mathrm{E}v_s^2(\sum_{j=0}^{t-s-2}\varrho^j\tilde{b}_{t-s-j-2})^2\\ &\le C\sum_{t=1}^{T}\sum_{s=1}^{t-2}\varrho^{2(t-s)} = O(T^{1+\eta})\end{aligned}$$

using the exponential decay of $\tilde{b}_i$. As $1-\varrho = aT^{-\eta}$, we can conclude that $\mathrm{E}\sum_{t=1}^{T}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2 = o(T^{1+\eta})$ and $\sum_{t=1}^{T}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2 = o_p(T^{1+\eta})$, by Markov's inequality. This estimate and the bound

$$\left|\sum_{t=1}^{[\tau T]}(\tilde{z}_{t-1}^2-\omega^2\zeta_{t-1}^2)\right| \le \sum_{t=1}^{T}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2 + 2|\omega|\sqrt{\sum_{t=1}^{T}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2\sum_{t=1}^{[\tau T]}\zeta_{t-1}^2}$$

establish (A.9).

Third, we consider

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]}\zeta_{t-1}^2 = \frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-1-j}^2 + \frac{2}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1}\sum_{s=t+1}^{[\tau T]-1}\varrho^{s-t}v_t v_s\sum_{j=0}^{[\tau T]-s-1}\varrho^{2j}$$

with the second addend on the r.h.s. being $o_p(1)$, as shown next. It holds that

$$\sum_{t=1}^{[\tau T]-1}\sum_{s=t+1}^{[\tau T]-1}\varrho^{s-t}v_t v_s\sum_{j=0}^{[\tau T]-s-1}\varrho^{2j} = (1-\varrho^2)\sum_{t=1}^{[\tau T]}\sum_{s=t+1}^{[\tau T]}\varrho^{s-t}v_t v_s(1-\varrho^{2([\tau T]-s)}) \tag{A.11}$$

with

$$\mathrm{E}\left(\sum_{t=1}^{[\tau T]-1}\sum_{s=t+1}^{[\tau T]-1}\varrho^{s-t}v_tv_s\right)^2 = \sum_{t=1}^{[\tau T]-1}\sum_{s=t+1}^{[\tau T]-1}\varrho^{2(s-t)}\mathrm{E}(v_t^2v_s^2)+2\sum_{t=1}^{[\tau T]-1}\sum_{r=t+1}^{[\tau T]-1}\sum_{s=r+1}^{[\tau T]-1}\varrho^{2s-t-r}\mathrm{E}(v_tv_rv_s^2)$$
$$\leq CT^{1+\eta}+2\sum_{t=1}^{[\tau T]-1}\sum_{r=t+1}^{[\tau T]-1}\varrho^{r-t}\sum_{s=r+1}^{[\tau T]-1}\varrho^{2(s-r)}\mathrm{E}(v_tv_r(v_s^2-\sigma_{vs}^2))$$
$$= O(T^{1+\eta})+O(T^{1+\eta+\epsilon})$$

because, by (A.2),

$$\left|\sum_{t=1}^{[\tau T]-1}\sum_{r=t+1}^{[\tau T]-1}\varrho^{r-t}\sum_{s=r+1}^{[\tau T]-1}\varrho^{2(s-r)}\mathrm{E}(v_tv_r(v_s^2-\sigma_{vs}^2))\right| \leq \max_{1\leq t<r\leq[\tau T]-2}|\mathrm{E}(v_tv_rS^v_{T(r+1,[\tau T]-1)})|\sum_{t=1}^{[\tau T]-1}\sum_{r=t+1}^{[\tau T]-1}\varrho^{r-t}$$
$$= O(T^{1+\eta+\epsilon})$$

using Lemma 1(c), such that $(1-\varrho^2)\sum_{t=1}^{[\tau T]}\sum_{s=t+1}^{[\tau T]}\varrho^{s-t}v_tv_s = O_p(T^{(1+3\eta+\epsilon)/2})$ by Chebyshev's inequality, and a similar estimate holds for the aggregare contribution of the terms in (A.11) involving $\varrho^{2([\tau T]-s)}$. Hence,

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]}\zeta_{t-1}^2 = \frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-1-j}^2+o_p(1)$$

under the assumption that $\eta+\epsilon<1$.

Fourth,

$$\sum_{t=1}^{\lfloor\tau T\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 = \sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}\sigma_{v,t-j-1}^2+o_p(T^{1+\eta}) = \sum_{t=1}^{\lfloor\tau T\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}\right)\sigma_{vt}^2+o_p(T^{1+\eta})$$
$$= \frac{T^\eta}{2a}\sum_{t=1}^{\lfloor\tau T\rfloor}\sigma_{vt}^2+o_p(T^{1+\eta})$$

by formally substituting σ_{ut}^2 with 1 in the proof of Lemma 2(f). The pointwise convergence $T^{-1-\eta}\sum_{t=1}^{\lfloor\tau T\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 \xrightarrow{p} \frac{1}{2a}[M_v](\tau)$ is now immediate. In conjunction with (A.9) this yields $T^{-1-\eta}\sum_{t=1}^{\lfloor\tau T\rfloor}z_{t-1}^2 \xrightarrow{p} \frac{\omega^2}{2a}[M_v](\tau)$. As the involved processes are increasing and the limit function is continuous, the convergence is in fact uniform. □

Proof of Lemma 5(d). First, we expand $\sum_{t=1}^{[\tau T]}z_{t-1}u_t = \sum_{t=1}^{[\tau T]}\tilde{z}_{t-1}u_t+o_p(T^{1/2+\eta/2})$ uniformly in τ, and second, we continue the expansion as $\sum_{t=1}^{[\tau T]}\tilde{z}_{t-1}u_t = \omega\sum_{t=1}^{[\tau T]}\zeta_{t-1}u_t+o_p(T^{1/2+\eta/2})$, with $\tilde{z}_t$ and ζ_{t-1} defined previously. Third, we show that

$$\frac{1}{T^{1/2}}\sum_{t=1}^{[\tau T]}\begin{pmatrix} v_t \\ \frac{1}{T^{\eta/2}}\zeta_{t-1}u_t\end{pmatrix} \Rightarrow \begin{pmatrix} M_v(\tau) \\ \frac{1}{\sqrt{2a}}\int_0^\tau[M_v]'(s)[M_u]'(s)\mathrm{d}B(s)\end{pmatrix} \tag{A.12}$$

by discussing its predictable variation and applying a Lindeberg-style martingale CLT. This is the most involved step of the proof and its structure will be detailed later.

First,

$$\sum_{t=1}^{[\tau T]}(z_{t-1}-\tilde{z}_{t-1})u_t = -(c/T)\sum_{t=1}^{[\tau T]}\left(\sum_{j=0}^{t-2}\varrho^j\xi_{t-j-2}\right)u_t + \sum_{t=1}^{[\tau T]}\left[\sum_{i=t-1}^{\infty}v_{t-1-i}\left(\sum_{j=0}^{t-1}\varrho^j b_{i-j}\right)\right]u_t$$
$$= -(c/T)ZU_1(\tau) + ZU_2(\tau).$$

Choose $\delta = \frac{1}{4}(1-\eta) > 0$. As $\max_{t\le T}|\xi_t| = O_p(T^{1/2})$, the process $ZU_1(\tau)$ equals with probability approaching one the martingale $Z\tilde{U}_1(\tau) = \sum_{t=1}^{[\tau T]}\left(\sum_{j=0}^{t-2}\varrho^j\mathbb{I}_{\{|\xi_{t-j-2}|\le T^{1/2+\delta}\}}\xi_{t-j-2}\right)u_t$ with

$$\mathrm{Var}(Z\tilde{U}_1(1)) = \sum_{t=1}^{T}\mathrm{E}\left[\left(\sum_{j=0}^{t-2}\varrho^j\mathbb{I}_{|\xi_{t-j-2}|\le T^{1/2+\delta}}\xi_{t-j-2}\right)^2 u_t^2\right] \le CT^{1+2\delta}\sum_{t=1}^{T}\left(\sum_{j=0}^{t-2}\varrho^j\right)^2\mathrm{E}\left[u_t^2\right]$$
$$= O(T^{2+2\eta+2\delta}) = o(T^{3+\eta}),$$

such that, by Doob's martingale inequality, $Z\tilde{U}_1(\tau) = o_p(T^{3/2+\eta/2})$ uniformly in $\tau\in[0,1]$, and the same magnitude order is inherited by $ZU_1(\tau)$. The process $ZU_2(\tau)$ is a martingale with

$$\mathrm{E}|ZU_1(1)| \le \sum_{t=1}^{T}\sum_{i=t-1}^{\infty}\mathrm{E}|v_{t-1-i}u_t|\sum_{j=0}^{t-1}\varrho^j|b_{i-j}| \le C\sum_{t=1}^{T}\sum_{i=t-1}^{\infty}\sum_{j=0}^{t-1}\varrho^j|b_{i-j}|$$
$$= C\sum_{j=0}^{T-1}\varrho^j\sum_{t=1}^{T-j}\sum_{i=t-1}^{\infty}|b_i| \le C\sum_{j=0}^{T-1}\varrho^j\sum_{i=0}^{\infty}(i+1)|b_i| = O(T^\eta) = o(T^{1/2+\eta/2})$$

and, again by Doob's martingale inequality, $ZU_1(\tau) = o_p(T^{1/2+\eta/2})$ uniformly in $\tau\in[0,1]$. As a result, $\sum_{t=1}^{[\tau T]}(z_{t-1}-\tilde{z}_{t-1})u_t = o_p(T^{1+\eta})$ uniformly in $\tau\in[0,1]$.

Second, $\sum_{t=1}^{[\tau T]}(\tilde{z}_{t-1}-\omega\zeta_{t-1})u_t$ is a martingale with variance at 1 given by

$$\sum_{t=1}^{T}\mathrm{E}[(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2u_t^2] \le \sum_{t=1}^{T}\sqrt{\mathrm{E}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^4\mathrm{E}u_t^4} \le C\sum_{t=1}^{T}\sqrt{\mathrm{E}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^4},$$

where, by using (A.10) and Lemma 2.5.2 of Giraitis et al. (2012),

$$\begin{aligned}
\mathrm{E}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^4 &\le C\left[\mathrm{E}\left(\sum_{i=0}^{t-2}\tilde{b}_i v_{t-i-1}\right)^4+(1-\varrho)^4\mathrm{E}\left(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\tilde{b}_i v_{t-i-j-2}\right)^4\right]\\
&\le C\left[\mathrm{E}\left(\sum_{i=0}^{t-2}\tilde{b}_i v_{t-i-1}\right)^4+T^{-4\eta}\mathrm{E}\left(\sum_{s=1}^{t-2}v_s\sum_{j=0}^{t-s-2}\varrho^j\tilde{b}_{t-s-j-2}\right)^4\right]\\
&\le C\left[\left(\sum_{i=0}^{t-2}\tilde{b}_i^2\sqrt{\mathrm{E}v_{t-i-1}^4}\right)^2+T^{-4\eta}\left(\sum_{s=1}^{t-2}\left(\sum_{j=0}^{t-s-2}\varrho^j\tilde{b}_{t-s-j-2}\right)^2\sqrt{\mathrm{E}v_s^4}\right)^2\right]\\
&\le C\left[\left(\sum_{i=0}^{\infty}\tilde{b}_i^2\right)^2+T^{-4\eta}\left(\sum_{s=1}^{t-2}\left(\sum_{j=0}^{t-s-2}\varrho^j\tilde{b}_{t-s-j-2}\right)^2\right)^2\right].
\end{aligned}$$

Further, in view of the exponential decay of $\tilde{b}_i$,

$$\mathrm{E}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^4 \le C+O(T^{-4\eta})\left(\sum_{s=1}^{t-2}\varrho^{2(t-s)}\right)^2=O(1)$$

uniformly in $t=1,\ldots,T$, such that $\sum_{t=1}^{T}\mathrm{E}[(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2u_t^2]=O(T)=o(T^{1+\eta})$ and, by Doob's martingale inequality, $\sum_{t=1}^{[\tau T]}(\tilde{z}_{t-1}-\omega\zeta_{t-1})u_t=o_p(T^{1+\eta})$ uniformly in $\tau\in[0,1]$.

Third, we establish that the predictable quadratic variation of the l.h.s. martingale in (A.12) satisfies

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\begin{pmatrix}\frac{1}{T}\mathrm{E}_{t-1}v_t^2 & \frac{1}{T^{1+\eta/2}}\zeta_{t-1}\mathrm{E}_{t-1}(u_tv_t)\\ \frac{1}{T^{1+\eta/2}}\zeta_{t-1}\mathrm{E}_{t-1}(u_tv_t) & \frac{1}{T^{1+\eta}}\zeta_{t-1}^2\mathrm{E}_{t-1}u_t^2\end{pmatrix}\xrightarrow{p}\begin{pmatrix}[M_v](\tau) & 0\\ 0 & \frac{1}{2a}\int_0^\tau[M_v]'(s)[M_u]'(s)\mathrm{d}s\end{pmatrix}. \tag{A.13}$$

Indeed, only the entries in the second row require detailed discussion. The analysis of the off-diagonal entry relies on the martingale approximability of $\sum_{t=1}^{\lfloor T\tau\rfloor}u_tv_t$. We write

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}\mathrm{E}_{t-1}(u_tv_t)=\sum_{t=1}^{\lfloor T\tau\rfloor-1}v_t\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}u_sv_s-\sum_{t=1}^{\lfloor T\tau\rfloor-1}v_t\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}[u_sv_s-\mathrm{E}_{s-1}(u_sv_s)] \tag{A.14}$$

where

$$\mathrm{E}\left|\sum_{t=1}^{\lfloor T\tau\rfloor-1}v_t\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}u_sv_s\right|\le\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sqrt{\mathrm{E}v_t^2}\sqrt{\mathrm{E}\left(\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}u_sv_s\right)^2}$$

with

$$\mathrm{E}\left(\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}u_sv_s\right)^2=\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{2(s-t-1)}\mathrm{E}(u_s^2v_s^2)+2\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{2(s-t-1)}\mathrm{E}\left(u_sv_s\mathrm{E}_s\sum_{r=s+1}^{\lfloor T\tau\rfloor-1}\varrho^{r-s}u_rv_r\right)$$

$$\leq CT^{\eta}+2\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{2(s-t-1)}\sqrt{\mathrm{E}(u_s^2v_s^2)}\sqrt{\mathrm{E}\left(\mathrm{E}_s\sum_{r=s+1}^{\lfloor T\tau\rfloor-1}\varrho^{r-s}u_rv_r\right)^2}$$

$$\leq CT^{\eta}+C\max_{1\leq s\leq T}\left\|\mathrm{E}_s\sum_{r=s+1}^{T}\varrho^{r-s}u_rv_r\right\|_2\sum_{s=0}^{T}\varrho^{2s}=O(T^{\eta+\epsilon})$$

by using Lemma 1(e) and the condition $\epsilon<\eta$. To deal with the possible nonexistence of a finite second moment of the second summation on the r.h.s. of (A.14), we notice first that it equals, with probability approaching one,

$$\sum_{t=1}^{\lfloor T\tau\rfloor-1}\mathbb{I}_{\{|v_t|\leq T^{1/3}\}}v_t\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}[u_sv_s-\mathrm{E}_{s-1}(u_sv_s)]$$

because $\max_{1\leq t\leq T}|v_t|=o(T^{1/3})$ under uniform L_4-boundedness of v_t. By MD considerations, the variance of the quantity in the previous display is

$$\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{2(s-t-1)}\mathrm{E}\{\mathbb{I}_{\{|v_t|\leq T^{1/3}\}}v_t^2[u_sv_s-\mathrm{E}_{s-1}(u_sv_s)]^2\}=O(T^{5/3+\eta})=o(T^{2+\eta}).$$

By combining the previous estimates and Markov's inequality, we can conclude that $\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}\mathrm{E}_{t-1}(u_tv_t)=o_p(T^{1+\eta/2})$ provided that $\epsilon<\eta$.

Based on the martingale approximability of $\sum_{t=1}^{\lfloor T\tau\rfloor}(u_t^2-\sigma_{ut}^2)$, we discuss next

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^2\mathrm{E}_{t-1}u_t^2=\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^2u_t^2+\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^2(\mathrm{E}_{t-1}u_t^2-u_t^2),\tag{A.15}$$

where

$$\begin{aligned}\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^2u_t^2-\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2u_t^2&=2\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\sum_{i=0}^{j-1}\varrho^{i+j}v_{t-j-1}v_{t-i-1}u_t^2\\&=2\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{s=1}^{t-1}\varrho^{t-s}v_sv_t\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}u_r^2\\&=2\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{s=1}^{t-1}\varrho^{t-s}v_sv_t\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}\sigma_{ur}^2\\&\quad+2\varrho\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}v_t\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}(u_r^2-\sigma_{ur}^2).\end{aligned}\tag{A.16}$$

The first term on the r.h.s. of (A.16) is $o_p(T^{1+\eta})$ by Chebyshev's inequality:

$$\begin{aligned}
\mathrm{E}\left(\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{s=1}^{t-1}\varrho^{t-s}v_s v_t\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}\sigma_{ur}^2\right)^2 &= \sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}\sigma_{ur}^2\right)^2 \mathrm{E}(\sum_{s=1}^{t-1}\varrho^{t-s}v_s v_t)^2 \\
&\le CT^{2\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}\left[\sum_{s=1}^{t-1}\varrho^{2(t-s)}\mathrm{E}(v_s^2 v_t^2)\right. \\
&\qquad \left. + \sum_{s=1}^{t-1}\varrho^{t-s}\sum_{q=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{2(q-t)}\mathrm{E}(v_t v_s v_q^2)\right] \\
&\le CT^{1+3\eta} + CT^{2\eta}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varrho^{t-s}\max_{1\le s<t<r\le T}\left|\sum_{q=t+1}^{r}\mathrm{E}(v_t v_s v_q^2)\right| \\
&= CT^{1+3\eta} + CT^{2\eta}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varrho^{t-s}\max_{1\le s<t<r\le T}\left|\mathrm{E}(v_t v_s S^v_{T(t+1,r)})\right| \\
&= O(T^{1+3\eta+\epsilon}) = o(T^{2+2\eta})
\end{aligned}$$

for $\epsilon < 1-\eta$, using (A.2) and Lemma 1(c) for the estimate involving a maximum. For the discussion of the second term on the r.h.s. of (A.16) we define $\check{\zeta}_t = \zeta_t \mathbb{I}_{\{|\zeta_t|\le T^{1/2}\}}$, $\check{v}_t = v_t \mathbb{I}_{\{|v_t|\le T^{1/3}\}}$ and $A^{u\tau}_{t+1} := \sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}(u_r^2-\sigma_{ur}^2)$ and notice that, with probability approaching one,

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}v_t\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}(u_r^2-\sigma_{ur}^2) = \sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t A^{u\tau}_{t+1}$$

because $\max_{1\le t\le T}|\zeta_t| = o(T^{1/2})$ and $\max_{1\le t\le T}|v_t| = o(T^{1/3})$; the purpose of truncation is to ensure square integrability. Furthermore, we use the decomposition

$$\begin{aligned}
\sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t A^{u\tau}_{t+1} &= \sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t\iota_t(A^{u\tau}_{t+1}-\mathrm{E}_t A^{u\tau}_{t+1}) \\
&\quad + \sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t\iota_t\mathrm{E}_t A^{u\tau}_{t+1} + \sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t(1-\iota_t)A^{u\tau}_{t+1},
\end{aligned} \tag{A.17}$$

where $\iota_t := \mathbb{I}\{\mathrm{E}_t[\max_{r\le T}(S^u_{T(t+1,r)})^2] \le T^{1+\delta}\}$ for some $\delta\in(0,\eta)$ (notice that $\mathrm{E}_t[(A^{u\tau}_{t+1} - \mathrm{E}_t A^{u\tau}_{t+1})^2] \le \mathrm{E}_t[(A^{u\tau}_{t+1})^2] \le \mathrm{E}_t[\max_{r\le T}(S^u_{T(t+1,r)})^2]$ a.s., by using eq. (A.2)). For the terms in the decomposition in (A.17), first, by MD considerations,

$$
\begin{aligned}
\mathrm{E}\left[\sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t\iota_t(A_{t+1}^{u\tau}-\mathrm{E}_tA_{t+1}^{u\tau})\right]^2 &= \sum_{t=1}^{\lfloor T\tau\rfloor}\mathrm{E}\left[\check{\zeta}_{t-1}\check{v}_t\iota_t(A_{t+1}^{u\tau}-\mathrm{E}_tA_{t+1}^{u\tau})\right]^2 \\
&\leq \sum_{t=1}^{\lfloor T\tau\rfloor}\mathrm{E}\left[\check{\zeta}_{t-1}^2\check{v}_t^2\iota_t\mathrm{E}_t[(A_{t+1}^{u\tau}-\mathrm{E}_tA_{t+1}^{u\tau})^2]\right] \\
&\leq \sum_{t=1}^{\lfloor T\tau\rfloor}\mathrm{E}\left[\check{\zeta}_{t-1}^2\check{v}_t^2\iota_t\mathrm{E}_t[\max_{r\leq T}(S_{T(t+1,r)}^u)^2]\right] \\
&\leq T^{1+\delta}\sum_{t=1}^{T}\|\zeta_{t-1}\|_4^2\|v_t\|_4^2 = O(T^{2+\eta+\delta}) \\
&= o(T^{2+2\eta})
\end{aligned}
$$

by Lemma 2(c) and given the choice of δ; second,

$$
\begin{aligned}
\mathrm{E}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t\iota_t\mathrm{E}_tA_{t+1}^{u\tau}\right| &\leq \max_{1\leq t\leq T}\left\|\mathrm{E}_tA_{t+1}^{u\tau}\right\|_2\sum_{t=1}^{T}\|\zeta_{t-1}\|_4\|v_t\|_4 \\
&= O(T^{1+\eta/2+\epsilon}) = o(T^{1+\eta})
\end{aligned}
$$

by Lemma 2(c) and given that $2\epsilon<\eta$, and third,

$$
\mathrm{P}\left(\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}v_t(1-\iota_t)A_{t+1}^{u\tau}=0\right)\geq \mathrm{P}\left(\min_{t\leq T}\iota_t=1\right)\to 1
$$

because

$$
\iota_t=\mathbb{I}\{\mathrm{E}_t[\max_{r\leq T}(S_{T(t+1,r)}^u)^2]\leq T^{1+\delta}\}\geq \mathbb{I}\{\max_{t\leq T}\mathrm{E}_t[\max_{1\leq s<r\leq T}(S_{T(s+1,r)}^u)^2]\leq T^{1+\delta}\}
$$

for all $t=1,...,T$,such that

$$
\begin{aligned}
\mathrm{P}\left(\min_{t\leq T}\iota_t=1\right) &\geq 1-\mathrm{P}\left(\max_{t\leq T}\mathrm{E}_t[\max_{1\leq s<r\leq T}(S_{T(s+1,r)}^u)^2]>T^{1+\delta}\right) \\
&\geq 1-T^{-1-\delta}\mathrm{E}\mathrm{E}_T[\max_{1\leq s<r\leq T}(S_{T(s+1,r)}^u)^2] \\
&= 1-T^{-1-\delta}\mathrm{E}[\max_{1\leq s<r\leq T}(S_{T(s+1,r)}^u)^2]=1-O(T^{-\delta})
\end{aligned}
$$

by Doob's martingale inequality applied to the martingale $\mathrm{E}_t[\max_{1\leq s<r\leq T}(S_{T(s+1,r)}^u)^2]$ ($t=1,...,T$) and by Lemma 1(b). By collecting the previous three results, we conclude that also the second term on the r.h.s. of (A.16) is $o_p(T^{1+\eta})$, such that

$$
T^{-1-\eta}\sum_{t=1}^{[\tau T]}\zeta_{t-1}^2u_t^2=T^{-1-\eta}\sum_{t=1}^{[\tau T]}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2u_t^2+o_p(1)\xrightarrow{p}\frac{1}{2a}\int_0^\tau[M_v]'(s)[M_u]'(s)\mathrm{d}s
$$

by Lemma 2(e,f).

In view of (A.15), to establish the convergence of the predictable quadratic variation as stated in (A.13), it remains to show that

$$\sum_{t=1}^{[\tau T]} \zeta_{t-1}^2 (u_t^2 - \mathrm{E}_{t-1} u_t^2) = \sum_{t=1}^{[\tau T]} v_t^2 B_{t+1}^{u\tau} + 2\varrho \sum_{t=1}^{\lfloor T\tau \rfloor} \zeta_{t-1} v_t B_{t+1}^{u\tau} = o_p(T^{1+\eta}),$$

where $B_{t+1}^{u\tau} := \sum_{r=t+1}^{\lfloor T\tau \rfloor} \varrho^{2(r-t-1)} (u_r^2 - \mathrm{E}_{r-1} u_r^2)$. This can be achieved similarly to the discussion of (A.17), though with some simplifications due to the martingale difference property $\mathrm{E}_t B_{t+1}^{u\tau} = 0$. We skip the details but mention that $S_{T(t+1,r)}^u$ could be replaced by $\tilde{S}_{T(t+1,r)}^u := \sum_{s=t+1}^{r} (u_s^2 - \mathrm{E}_{s-1} u_s^2)$, with $\mathrm{E}[\max_{1 \le s < r \le T} (\tilde{S}_{T(s+1,r)}^u)^2] = O(T)$ as a consequence of the martingale property of $\tilde{S}_{T(1,r)}^u$ and the uniform L_4-boundedness of u_t (e.g. Proposition 9 of Merlevède *et al.* (2006) asserts this under much weaker conditions).

Finally, to complete the proof of convergence (A.12), a conditional Lindeberg condition now suffices, by the function-space version of Corollary 3.1 of Hall and Heyde (1980). The following conditional Lindeberg condition can be established along the lines of Lemma 3.5(ii) of Magdalinos (2020):

$$\begin{aligned}\sum_{t=1}^{T} \mathrm{E}_{t-1} \left[\left(\frac{v_t^2}{T} + \frac{\zeta_{t-1}^2 u_t^2}{T^{1+\eta}} \right) \mathbb{I} \left\{ \sqrt{\frac{v_t^2}{T} + \frac{\zeta_{t-1}^2 u_t^2}{T^{1+\eta}}} > 2\delta \right\} \right] &\le \sum_{t=1}^{T} \mathbb{I}\{|\zeta_{t-1}| > T^{\frac{1}{2}}\} \mathrm{E}_{t-1} \left(\frac{v_t^2}{T} + \frac{\zeta_{t-1}^2 u_t^2}{T^{1+\eta}} \right) \\ &+ \sum_{t=1}^{T} \mathrm{E}_{t-1} \left[\left(\frac{v_t^2}{T} + \frac{\delta^2 u_t^2}{T^{\eta}} \right) \mathbb{I}\{|v_t| > T^{1/2}\delta\} \right] \\ &+ \sum_{t=1}^{T} \mathrm{E}_{t-1} \left[\left(\frac{v_t^2}{T} + \frac{\zeta_{t-1}^2 u_t^2}{T^{1+\eta}} \right) \mathbb{I}\{|u_t| > T^{\frac{\eta}{2}}\} \right] \\ &= o_p(1)\end{aligned}$$

for any $\delta > 0$. In fact, the first and the second term on the majorant side are zero with probability approaching one, respectively because $\max_{t \le T} |\zeta_{t-1}| = o_p(T^{1/2})$ (by Lemma 2(c) with $w_{Tt} = v_t, p = 4$ and $\max_{1 \le t \le T} \mathrm{E} v_t^4 = O(1)$) and $\max_{1 \le t \le T} |v_t| = o_p(T^{1/2})$ (because $\max_{1 \le t \le T} \mathrm{E} v_t^4 = O(1)$). The third term on the majorant side is $o_p(1)$ because it is non-negative and its expectation is bounded by

$$\begin{aligned}&2 \sum_{t=1}^{T} \left(\frac{\sqrt{\mathrm{E} v_t^4}}{T} \sqrt{\mathrm{P}(|u_t| > T^{\eta/2})} + \frac{\sqrt{\mathrm{E} \zeta_{t-1}^4}}{T^{1+\eta}} \sqrt{\mathrm{E}(u_t^4 \mathbb{I}\{|u_t| > T^{\eta/2}\})} \right) \\ &\quad \le 2 \sum_{t=1}^{T} \left(\frac{\sqrt{\mathrm{E} v_t^4 \mathrm{E} u_t^4}}{T^{1+\eta}} + \frac{\sqrt{\mathrm{E} \zeta_{t-1}^4}}{T^{1+\eta}} \sqrt{\mathrm{E}(u_t^4 \mathbb{I}\{|u_t| > T^{\eta/2}\})} \right) = o(1)\end{aligned}$$

by Markov's inequality, by Lemma 2(c) for $\max_{t \le T} \mathrm{E} \zeta_{t-1}^4 = O(T^{2\eta})$ and because u_t^4 are uniformly integrable (a property inherited from the uniformly L_4-bounded and stationary sequence $\boldsymbol{\psi}_t$ because $\mathbf{H}$ is bounded).

Convergence (A.15) and the conditional Lindeberg condition imply convergence (A.12).

In view of the first two steps of this proof, also the convergence

$$\frac{1}{T^{1/2}}\sum_{t=1}^{[\tau T]}\begin{pmatrix} v_t \\ \frac{1}{T^{\eta/2}}z_{t-1}u_t \end{pmatrix} \Rightarrow \begin{pmatrix} M_v(\tau) \\ \frac{1}{\sqrt{2a}}\int_0^\tau [M_v]'(s)[M_u]'(s)\mathrm{d}B(s) \end{pmatrix}$$

follows, making the convergence of $T^{-1/2-\eta/2}\sum_{t=1}^{[\tau T]} z_{t-1}u_t$ joint with the one established in part (c). □

Proof of Lemma 5(e).

Apply the partial summation formula to obtain

$$\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1}b(t/T) = b([\tau T]/T)\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]-2} z_t - \frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]-1}\left(\sum_{j=1}^{t} z_j\right)\left(b\left(\frac{t+1}{T}\right) - b\left(\frac{t}{T}\right)\right).$$

We know from part (a) that $\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1} \Rightarrow \frac{\omega}{a}J_{c,\mathrm{H}}(\tau)$, such that the first summand converges to $\frac{\omega}{a}b(\tau)J_{c,\mathrm{H}}(\tau)$. Moreover, since the Ornstein-Uhlenbeck process $J_{c,\mathrm{H}}(\tau)$ is pathwise Hölder-continuous of any order $\alpha < 1/2$, and b is Hölder-continuous of order 1, the second summand converges to the Stieltjes integral $\frac{\omega}{a}\int_0^\tau J_{c,\mathrm{H}}(s)\mathrm{d}b(s)$ as required. □

Proof of Lemma 5(f).

Apply the partial summation formula to obtain

$$\begin{aligned}\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1}b(t/T)x_{t-1} &= x_{[\tau T]-1}\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1} z_t b((t+1)/T) \\ &\quad - \frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1}\left(\sum_{j=1}^{t} z_{j-1}b(j/T)\right)\left(w_t - \frac{c}{T}\xi_{t-1}\right)\end{aligned}$$

where the limit of the first summand on the r.h.s. follows with with part (e) and the weak convergence of $T^{-1/2}\xi_{[\tau T]}$.

Then, following the arguments in the proof of part (b), it straightforward to show that, uniformly in τ,

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1}\left(\sum_{j=1}^{t} z_{j-1}b(j/T)\right)w_t = \omega\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1}\left(\sum_{j=1}^{t} z_{j-1}b(j/T)\right)v_t + o_p(1)$$

such that, with v_t orthogonal to $\sum_{j=1}^{t} z_{j-1}b(1/T)$, we obtain as required

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1}\left(\sum_{j=1}^{t} z_j b((j+1)/T)\right)\left(w_t - \frac{c}{T}\xi_{t-1}\right) \Rightarrow \frac{\omega^2}{a}\left(\int_0^\tau Z_b(s)\mathrm{d}M_v(s) - c\int_0^\tau Z_b(s)J_{c,\mathrm{H}}(s)\mathrm{d}s\right).$$

□

We now turn to the derivation of the limit bootstrap distributions.

Under Assumption 1.1, let $\hat{A}(z) := 1 - \sum_{i=1}^{p+1} \hat{a}_i z^i$, whereas under Assumption 1.2, let $\hat{A}(z) := 1 - \sum_{i=1}^{p} \tilde{a}_i z^i$ for $\tilde{a}_i$ as in $\Delta x_t^* = \hat{\phi} x_{t-1}^* + \sum_{i=1}^{p} \tilde{a}_i \Delta x_{t-i}^* + v_t^*$. Let further $\sum_{i=0}^{\infty} \hat{b}_i z^i = (\hat{A}(z))^{-1}$ with $\hat{b}_0 = 1$. As the coefficients of $\hat{A}(z)$ estimate consistently those of $(1-\rho z)A(z)$ and $A(z)$ respectively under Assumption 1.1 and Assumption 1.2, and since $(1-\rho z)A(z)$ and $A(z)$, under the respective assumptions, have their roots outside a complex disk of radius $1+2\delta'$ for some $\delta' > 0$, it follows that with probability approaching one $\hat{A}(z)$ has its roots outside the complex disk of radius $1+\delta'$, such that the coefficients of the power series $\sum_{i=0}^{\infty} \hat{b}_i z^i$ decrease exponentially ($|\hat{b}_i| \leq C\delta^i$ for some $\delta \in (0,1)$,with probability approaching one). Since we are interested in results 'in probability', in the proof of such results we proceed, without loss of generality, as if the roots of $\hat{A}(z)$ were a.s. outside the complex disk of radius $1+\delta'$. Thus, as x_t^* is initialized with zero initial values, under Assumption 1.1 we write $x_t^* = \sum_{i=0}^{t-1} \hat{b}_i v_{t-i}^*$, where $\hat{b}_i$ a.s. decay at an exponential rate which is uniform over T. Similarly, under Assumption 1.1, we write

$$\Delta x_t^* = \sum_{i=0}^{t-1} \hat{b}_i (\hat{\phi} x_{t-i-1}^* + v_{t-i}^*),$$
$$x_t^* = \sum_{i=0}^{t-1} ((\hat{A}(1))^{-1} + \hat{b}_i^*)(\hat{\phi} x_{t-i-1}^* + v_{t-i}^*),$$

where $\hat{b}_i$ and the Beveridge-Nelson coefficients $\hat{b}_i^*$ a.s. decay at an exponential rate which is uniform over T.

We often use the estimates $\max_{1\leq t\leq T} |\hat{v}_t - v_t| = O_p(T^{-1/4})$, $\max_{1\leq t\leq T} |\hat{v}_t^2 - v_t^2| = O_p(1)$, $\max_{1\leq t\leq T} |\hat{u}_t - u_t| = O_p(T^{-1/2})$ and $\max_{1\leq t\leq T} |\hat{u}_t^2 - u_t^2| = O_p(T^{-1/4})$ which hold as a result of consistent parameter estimation and the assumptions on (u_t, v_t). Their standard implications where $(\hat{u}_t, \hat{v}_t)$ are approximated by (u_t, v_t) are usually used without explicit justification, e.g., $\sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} \hat{v}_{t-j-1}^2 = \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 + o_p(T^{1+\eta})$. In this specific case, a possible justification would be

$$\left| \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} \hat{v}_{t-j-1}^2 - \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 \right| \leq 2 \max_{1\leq t\leq T} |\hat{v}_t - v_t| \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} |v_{t-j-1}| \qquad (A.18)$$
$$+ \max_{1\leq t\leq T} (\hat{v}_t - v_t)^2 \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} = o_p(T^{1+\eta})$$

because $\sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} |v_{t-j-1}| = O_p(T^{1+\eta})$ by Markov's inequality.

Proof of Lemma 6. As μ_x of the DGP of x_t cancels out in the definition of z_t,we assume that $\mu_x = 0$, without loss of generality. Also without loss of generality when distributional results are concerned, we regard the independent sequences $\boldsymbol{\psi}_t$ and R_t as defined on a product probability space with a generic outcome (ω, ω^*).

In part (a) it holds that

$$\sum_{t=1}^{\lfloor T\tau\rfloor} z_{t-1}u_t^* = \sum_{t=1}^{\lfloor T\tau\rfloor} x_{t-1}u_tR_t + \sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}-x_{t-1})u_tR_t + \sum_{t=1}^{\lfloor T\tau\rfloor} z_{t-1}(\hat{u}_t-u_t)R_t,$$

where $\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}-x_{t-1})u_tR_t$ and $\sum_{t=1}^{\lfloor T\tau\rfloor} z_{t-1}(\hat{u}_t-u_t)R_t$ conditionally on the data are martingales in τ with, first,

$$\mathrm{E}^*\left(\sum_{t=1}^{T}(z_{t-1}-x_{t-1})u_tR_t\right)^2 = \sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}-x_{t-1})^2u_t^2 \le \sqrt{\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}-x_{t-1})^4\sum_{t=1}^{T}u_t^4} = o_p(T)$$

because $\sum_{t=1}^{T}u_t^4 = O_p(T)$ and $\|z_t-x_t\|_4 = O_p(T^{-\eta/2}+\varrho^t)$ under Assumption 1.2 (see e.g. Lemma 4 (a) in Demetrescu and Hillmann, 2020), and second,

$$\mathrm{E}^*\left(\sum_{t=1}^{T} z_{t-1}(\hat{u}_t-u_t)R_t\right)^2 = \sum_{t=1}^{T} z_{t-1}^2(\hat{u}_t-u_t)^2 \le \max_{1\le t\le T}|\hat{u}_t-u_t|^2\sum_{t=1}^{T} z_{t-1}^2 = o_p(T)$$

because $\sum_{t=1}^{T} z_{t-1}^2 = O_p(T)$ by Markov's inequality and the uniform L_4-boundedness of z_t. By using Doob's martingale inequality, it follows that

$$\sum_{t=1}^{\lfloor T\tau\rfloor} z_{t-1}u_t^* = \sum_{t=1}^{\lfloor T\tau\rfloor} x_{t-1}u_tR_t + o_p^*(T^{1/2}) = \sum_{t=1}^{\lfloor T\tau\rfloor}\xi_{t-1}u_tR_t + o_p^*(T^{1/2})$$

uniformly over $\tau\in[0,1]$. Then, by using the Lipschitz-by-parts property of $\mathbf{H}$,

$$\max_{\tau\in[0,1]}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}\xi_{t-1}u_tR_t - \sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}u_tR_t\right| = o_p(T^{1/2})$$

by the same argument as in the proof of Lemma 4, with $\tilde{\xi}_t$ defined there. As convergence in probability to zero becomes $\overset{p}{\Rightarrow}_p$ convergence upon conditioning, the previous estimates holds weakly in probability conditionally on the data, such that $\sum_{t=1}^{\lfloor T\tau\rfloor} z_{t-1}u_t^* = \sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}u_tR_t + o_p^*(T^{1/2})$ uniformly over $\tau\in[0,1]$. The limit of $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor} z_{t-1}u_t^*$ asserted in part (a) will then follow if we show that this limit holds for $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}u_tR_t$, as we do next.

Similarly to the proof of Lemma 4, consider the representation

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}u_tR_t = \sum_{t=1}^{\lfloor T\tau\rfloor}\begin{pmatrix}\mathbf{H}_{1\cdot}(\tfrac{t}{T}) & \mathbf{H}_{2\cdot}(\tfrac{t}{T})\end{pmatrix}\left[(\boldsymbol{\psi}_tR_t)\otimes\sum_{j\ge0} b_j\boldsymbol{\psi}_{t-1-j}\right] \tag{A.19}$$

where $\boldsymbol{\psi}_t$ is as in Assumption 3. Notice that, by the ergodic theorem and the dominated convergence theorem,

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t')\otimes\sum_{i,j\geq 0}b_ib_j\boldsymbol{\psi}_{t-1-i}\boldsymbol{\psi}_{t-1-j}'\overset{a.s.}{\to}\tau\sum_{i,j\geq 0}b_ib_j\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\otimes(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')]:=\tau\Omega, \quad \text{(A.20)}$$

as it was already used in the proof of Lemma 4. Moreover, the convergence holds in the functional sense (on $\mathcal{D}$) given that the involved functions are increasing and the limit function is also continuous. For $\check{\psi}_t := (\boldsymbol{\psi}_t R_t)\otimes\sum_{j\geq 0}b_j\boldsymbol{\psi}_{t-1-j}$, let $\mathcal{B}$ be an almost certain event in the factor space of the data such that $\mathrm{E}_{\omega^*}\|\check{\psi}_t(\omega,\omega^*)\|^2<\infty$ for every fixed $\omega\in\mathcal{B}$, where the expectation is taken w.r.t. the probability measure on the factor space of the bootstrap multipliers; such a $\mathcal{B}$ exists by the L_4-boundedness of $\boldsymbol{\psi}_t$. Let g_{TN} be measurable functions from $\mathbb{R}^\infty$ to $\mathbb{R}$ such that $g_{TN}(\boldsymbol{\psi}_t,\boldsymbol{\psi}_{t-1},...)$ are versions of $\mathrm{E}^*[\|\check{\psi}_t\|^2\mathbb{I}_{\{\|\check{\psi}_t\|>N\}}]$ and the equalities

$$g_{TN}(\boldsymbol{\psi}_t(\omega),\boldsymbol{\psi}_{t-1}(\omega),...)=\mathrm{E}_{\omega^*}[\|\check{\psi}_t(\omega,\omega^*)\|^2\mathbb{I}_{\{\|\check{\psi}_t(\omega,\omega^*)\|>N\}}]$$

are satisfied for all $T,N\in\mathbb{N}$ and $\omega\in\mathcal{B}$; such g_{TN} exist by the ergodicity of $\boldsymbol{\psi}_t$ and the product structure of the underlying probability space. By the ergodic theorem, it holds that

$$\frac{1}{T}\sum_{t=1}^{T}g_{TN}(\boldsymbol{\psi}_t,\boldsymbol{\psi}_{t-1},...)\overset{a.s.}{\to}\mathrm{E}\left[\|\check{\psi}_1\|^2\mathbb{I}_{\{\|\check{\psi}_1\|>N\}}\right] \quad \text{(A.21)}$$

for every $N\in\mathbb{N}$. Let $\mathcal{A}\subset\mathcal{B}$ be an almost certain event in the factor space of the data such that the countably many convergence facts (A.20) and (A.21) (with (A.20) counted as a single functional convergence) hold simultaneously for every $\omega\in\mathcal{A}$. Then, for every fixed $\omega\in\mathcal{A}$, the process

$$T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}(\boldsymbol{\psi}_t(\omega)R_t(\omega^*))\otimes\sum_{j\geq 0}b_j\boldsymbol{\psi}_{t-1-j}(\omega) \quad \text{(A.22)}$$

is a martingale with variance function

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}(\boldsymbol{\psi}_t(\omega)\boldsymbol{\psi}_t'(\omega))\otimes\sum_{i,j\geq 0}b_ib_j\boldsymbol{\psi}_{t-1-i}(\omega)\boldsymbol{\psi}_{t-1-j}'(\omega)\overset{a.s.}{\to}\tau\Omega$$

and, moreover, for every $n,N\in\mathbb{N}$, it holds for large T that

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}_{\omega^*}\left[\|\check{\psi}_t(\omega,\omega^*)\|^2\mathbb{I}_{\{\|\check{\psi}_t(\omega,\omega^*)\|>\sqrt{T}/n\}}\right]&\leq\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}_{\omega^*}\left[\|\check{\psi}_t(\omega,\omega^*)\|^2\mathbb{I}_{\{\|\check{\psi}_t(\omega,\omega^*)\|>N\}}\right]\\&=\frac{1}{T}\sum_{t=1}^{T}g_{TN}(\boldsymbol{\psi}_t(\omega),\boldsymbol{\psi}_{t-1}(\omega),...)\\&\to\mathrm{E}\left[\|\check{\psi}_1\|^2\mathbb{I}_{\{\|\check{\psi}_1\|>N\}}\right].\end{aligned}$$

Since $\mathrm{E}[\|\check{\psi}_1\|^2 \mathbb{I}_{\{\|\check{\psi}_1\|>N\}}]$ can be made arbitrarily small by choosing N large, it follows that the Lindeberg condition

$$\frac{1}{T}\sum_{t=1}^{T} \mathrm{E}\left[\|\check{\psi}_t(\omega,\omega^*)\|^2 \mathbb{I}_{\{\|\check{\psi}_t(\omega,\omega^*)\|>\sqrt{T}/n\}}\right] \to 0$$

is satisfied for every $n \in \mathbb{N}$, and therefore, for every $\omega \in \mathcal{A}$, the process (A.22) weakly converges to a quadrivariate Brownian motion B_Ω with variance matrix at unity Ω. Then, using the Lipshits-by-parts property of $\mathbf{H}$ and representation (A.19), we can conclude that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor T\tau \rfloor} \tilde{\xi}_{t-1}(\omega)u_t(\omega)R_t(\omega^*) \Rightarrow \int_0^\tau [\mathbf{H}_{1\cdot}(s) \otimes \mathbf{H}_{2\cdot}(s)]\mathrm{d}B_\Omega(s) \overset{d}{=} \int_0^\tau \sqrt{\chi(s)}\mathrm{d}B(s)$$

for every fixed $\omega \in \mathcal{A}$, where B is a standard univariate Brownian motion. As the probability of $\mathcal{A}$ is one and the underlying probability space has a product structure, the previous convergence yields

$$T^{-1/2}\sum_{t=1}^{\lfloor T\tau \rfloor} \tilde{\xi}_{t-1}u_t R_t \overset{w}{\Rightarrow}_{a.s.} \int_0^\tau \sqrt{\chi(s)}\mathrm{d}B(s).$$

By the earlier discussion, the same limit is inherited by $T^{-1/2}\sum_{t=1}^{\lfloor T\tau \rfloor} \tilde{\xi}_{t-1}u_t R_t$ in the weak-in-probability mode.

We turn to part (b). Notice for further reference that, for $s \geq t$, it holds (a.s., without loss of generality) that

$$|\mathrm{E}^*(x_s^* x_t^*)| \leq \sum_{i=0}^{t-1} |\hat{b}_i||\hat{b}_{i+s-t}|\hat{v}_{t-i}^2 \leq C\delta^{s-t}\sum_{i=0}^{t-1} \delta^{2i}\hat{v}_{t-i}^2. \tag{A.23}$$

Let $\xi_t^* := \sum_{i=0}^{t-1} b_i v_{t-i} R_{t-i}$. Then

$$\sum_{t=1}^{\lfloor T\tau \rfloor} z_{t-1}^* u_t^* = \sum_{t=1}^{\lfloor T\tau \rfloor} \xi_{t-1}^* u_t R_t + \sum_{t=1}^{\lfloor T\tau \rfloor}(x_{t-1}^* - \xi_{t-1}^*)u_t R_t + \sum_{t=1}^{\lfloor T\tau \rfloor} x_{t-1}^*(\hat{u}_t - u_t)R_t - (1-\varrho)\sum_{t=1}^{\lfloor T\tau \rfloor}\sum_{j=0}^{t-1} \varrho^j x_{t-j-2}^* u_t^*$$

where, conditionally on the data, the processes $\sum_{t=1}^{\lfloor T\tau \rfloor}(x_{t-1}^* - \xi_{t-1}^*)u_t R_t$, $\sum_{t=1}^{\lfloor T\tau \rfloor} x_{t-1}^*(\hat{u}_t - u_t)R_t$

and $\sum_{t=1}^{\lfloor T\tau \rfloor}\sum_{j=0}^{t-1}\varrho^j x_{t-j-2}^* u_t^*$ are martingales in τ with, first,

$$\begin{aligned}
\mathrm{E}^*\left(\sum_{t=1}^{T}(x_{t-1}^*-\xi_{t-1}^*)u_tR_t\right)^2 &= \sum_{t=1}^{T}\mathrm{E}^*[(x_{t-1}^*-\xi_{t-1}^*)^2]u_t^2 = \sum_{t=1}^{T}\sum_{i=0}^{t-2}(\hat{b}_i\hat{v}_{t-i-1}-b_iv_{t-i-1})^2u_t^2 \\
&\leq 2\sum_{t=1}^{T}\sum_{i=0}^{t-2}\hat{b}_i^2(\hat{v}_{t-i-1}-v_{t-i-1})^2u_t^2 + 2\sum_{t=1}^{T}\sum_{i=0}^{t-2}(\hat{b}_i-b_i)^2v_{t-i-1}^2u_t^2 \\
&\leq 2\max_{1\leq t\leq T}(\hat{v}_t-v_t)^2\sum_{i=0}^{\infty}\hat{b}_i^2\sum_{t=1}^{T}u_t^2 + C\max_{1\leq t\leq T}|\hat{b}_i-b_i|\sum_{t=1}^{T}\sum_{i=0}^{t-2}\delta^iv_{t-i-1}^2u_t^2 \\
&= o_p(T)
\end{aligned}$$

by Markov's inequality for $\sum_{t=1}^{T}\sum_{i=0}^{t-2}\delta^iv_{t-i-1}^2u_t^2$; second,

$$\begin{aligned}
\mathrm{E}^*\left(\sum_{t=1}^{T}x_{t-1}^*(\hat{u}_t-u_t)R_t\right)^2 &= \sum_{t=1}^{T}\mathrm{E}^*[(x_{t-1}^*)^2](\hat{u}_t-u_t)^2 = \sum_{t=1}^{T}\sum_{i=0}^{t-2}\hat{b}_i^2\hat{v}_{t-i-1}^2(\hat{u}_t-u_t)^2 \\
&\leq \max_{1\leq t\leq T}(\hat{u}_t-u_t)^2\sum_{i=0}^{\infty}\hat{b}_i^2\sum_{t=1}^{T}\hat{v}_t^2 = o_p(1)\sum_{t=1}^{T}v_t^2 = o_p(T)
\end{aligned}$$

and third, using (A.23),

$$\begin{aligned}
\mathrm{E}^*\left(\sum_{t=1}^{T}\sum_{j=0}^{t-1}\varrho^jx_{t-j-2}^*u_t^*\right)^2 &= \sum_{t=1}^{T}\hat{u}_t^2\sum_{i,j=0}^{t-1}\varrho^{j+i}\mathrm{E}^*(x_{t-i-2}^*x_{t-j-2}^*) \\
&\leq C\sum_{t=1}^{T}\hat{u}_t^2\sum_{i=0}^{t-1}\sum_{j=0}^{i-1}\varrho^{j+i}\delta^{i-j}\sum_{k=i+2}^{t-1}\delta^{2(k-i-2)}\hat{v}_{t-k}^2 \\
&\leq C\sum_{t=1}^{T}\hat{u}_t^2\sum_{i=0}^{t-1}\varrho^i\sum_{k=i+2}^{t-1}\delta^{2(k-i-2)}\hat{v}_{t-k}^2 \\
&\leq C(1+o_p(1))\sum_{t=1}^{T}u_t^2\sum_{i=0}^{t-1}\varrho^i\sum_{k=i+2}^{t-1}\delta^{2(k-i-2)}v_{t-k}^2 = O_p(T^{1+\eta})
\end{aligned}$$

by Markov's inequality, as $\mathrm{E}(u_t^2v_{t-k}^2)$ are bounded uniformly in t,k and

$$\sum_{t=1}^{T}\sum_{i=0}^{t-1}\varrho^i\sum_{k=i+2}^{t-1}\delta^{2(k-i-2)} = O(T^{1+\eta}).$$

Hence, by Doob's martingale inequality,

$$\sum_{t=1}^{\lfloor T\tau\rfloor}z_{t-1}^*u_t^* = \sum_{t=1}^{\lfloor T\tau\rfloor}\xi_{t-1}^*u_tR_t + o_{p^*}(T^{1/2})$$

uniformly in τ. Further, similarly to the proof of Lemma 4, let

$$\tilde{\xi}^*_{t-1} \;=\; h_{21}(\frac{t}{T})\sum_{j\geq 0} b_j a_{t-1-j}R_{t-1-j} + h_{22}(\frac{t}{T})\sum_{j\geq 0} b_j e_{t-1-j}R_{t-1-j}.$$

Then, as in the proof of Lemma 4, the Lipschitz-by-parts property of h_{21}, h_{22} can be used to check that

$$\max_{\tau\in[0,1]}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}\xi^*_{t-1}u_tR_t - \sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}^*_{t-1}u_tR_t\right| = o_p(T^{1/2}).$$

As convergence to zero in probability becomes $\overset{p}{\Rightarrow}_p$ convergence upon conditioning, it follows that $\sum_{t=1}^{\lfloor T\tau\rfloor} z^*_{t-1}u^*_t = \sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}^*_{t-1}u_tR_t + o_{p^*}(T^{1/2})$ and part (b) will be proved if we show that $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}^*_{t-1}u_tR_t$ conditionally on the data converges weakly in probability to the asserted limit of $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor} z^*_{t-1}u^*_t$. We show this convergence next.

Consider the representation

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}^*_{t-1}u_tR_t = \sum_{t=1}^{\lfloor T\tau\rfloor}\begin{pmatrix}\mathbf{H}_{1\cdot}(\frac{t}{T}) & \mathbf{H}_{2\cdot}(\frac{t}{T})\end{pmatrix}\left[(\boldsymbol{\psi}_tR_t)\otimes\sum_{j\geq 0} b_j\boldsymbol{\psi}_{t-1-j}R_{t-1-j}\right]. \tag{A.24}$$

Notice that, by the ergodic theorem and the dominated convergence theorem,

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}(\boldsymbol{\psi}_t\boldsymbol{\psi}'_t)\otimes\sum_{j\geq 0} b_j^2\boldsymbol{\psi}_{t-1-j}\boldsymbol{\psi}'_{t-1-j}\overset{a.s.}{\to}\tau\sum_{j\geq 0} b_j^2\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}'_1)\otimes(\boldsymbol{\psi}_{-j}\boldsymbol{\psi}'_{-j})] := \tau\Omega^*$$

and

$$\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}^*\left[\|\boldsymbol{\psi}^*_t\|^2\mathbb{I}_{\{\|\boldsymbol{\psi}^*_t\|>N\}}\right]\overset{a.s.}{\to}\mathrm{E}\left[\|\boldsymbol{\psi}^*_1\|^2\mathbb{I}_{\{\|\boldsymbol{\psi}^*_1\|>N\}}\right]$$

where $\boldsymbol{\psi}_t$ is as in Assumption 3 and $\boldsymbol{\psi}^*_t := (\boldsymbol{\psi}_tR_t)\otimes\sum_{j\geq 0} b_j\boldsymbol{\psi}_{t-1-j}R_{t-1-j}$. As a result, similarly to the proof of part (a), in the factor space of $\boldsymbol{\psi}_t$ there exists an event $\mathcal{A}^*$ of probability one such that, for every fixed $\omega\in\mathcal{A}^*$, the process

$$T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}(\boldsymbol{\psi}_t(\omega)R_t(\omega^*))\otimes\sum_{j\geq 0} b_j\boldsymbol{\psi}_{t-1-j}(\omega)R_{t-1-j}(\omega^*) = T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\boldsymbol{\psi}^*_t(\omega,\omega^*), \tag{A.25}$$

with randomness originating from ω^* alone, is a martingale with variance function

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}(\boldsymbol{\psi}_t(\omega)\boldsymbol{\psi}'_t(\omega))\otimes\sum_{j\geq 0} b_j^2\boldsymbol{\psi}_{t-1-j}(\omega)\boldsymbol{\psi}'_{t-1-j}(\omega)\to\tau\Omega^*$$

and satisfies the Lindeberg condition

$$\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}_{\omega^*}\left[\|\boldsymbol{\psi}_t^*(\omega,\omega^*)\|^2\mathbb{I}_{\{\|\boldsymbol{\psi}_t^*(\omega,\omega^*)\|>\sqrt{T}/n\}}\right]\to 0$$

for all $n\in\mathbb{N}$. By a martingale FCLT it follows that the process (A.25) converges weakly to a quadrivariate Brownian motion B_Ω^* defined on $[0,1]$ and having variance matrix Ω^*. This fact and representation (A.24), together with the Lipschitz-by-parts property of $\mathbf{H}$, imply that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}^*(\omega,\omega^*)u_t(\omega)R_t(\omega^*)\Rightarrow\int_0^\tau[\mathbf{H}_{1\cdot}(s)\otimes\mathbf{H}_{2\cdot}(s)]\mathrm{d}\boldsymbol{B}_{\boldsymbol{\Omega}}^*(s)\overset{d}{=}\int_0^\tau\sqrt{\chi^*(s)}\mathrm{d}B(s)$$

for every fixed $\omega\in\mathcal{A}^*$, where B is a standard Brownian motion. As the probability of $\mathcal{A}^*$ is one, and given the product structure of the probability space, the previous convergence implies that

$$T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}^*u_tR_t\overset{w}{\Rightarrow}_{a.s.}\int_0^\tau\sqrt{\chi^*(s)}\mathrm{d}B(s)$$

conditionally on the data, and hence, the same convergence holds also weakly in probability. By the discussion earlier in this proof, the convergence is inherited by $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}z_{t-1}^*u_t^*$ as asserted in part (b).

In part (c), we first find that

$$\sum_{t=1}^{\lfloor T\tau\rfloor-1}|(z_t^*)^2-(x_t^*)^2|\le\sum_{t=1}^{\lfloor T\tau\rfloor-1}(z_t^*-x_t^*)^2+2\left[\sum_{t=1}^{\lfloor T\tau\rfloor-1}(z_t^*-x_t^*)^2\right]^{1/2}\left[\sum_{t=1}^{\lfloor T\tau\rfloor-1}(x_t^*)^2\right]^{1/2},$$

where, using (A.23),

$$\begin{aligned}\mathrm{E}^*\sum_{t=1}^{\lfloor T\tau\rfloor-1}(z_t^*-x_t^*)^2&=(1-\varrho)^2\sum_{t=1}^{\lfloor T\tau\rfloor-1}\mathrm{E}^*\left(\sum_{j=0}^{t-2}\varrho^jx_{t-j-1}^*\right)^2=a^2T^{-2\eta}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i,j=0}^{t-2}\varrho^{i+j}\mathrm{E}^*(x_{t-i-1}^*x_{t-j-1}^*)\\&=O_p(T^{-2\eta})\sum_{t=1}^{T}\sum_{i=0}^{t-2}\sum_{j=0}^{i}\varrho^{i+j}\delta^{i-j}\sum_{k=0}^{t-i-2}\delta^{2i}\hat{v}_{t-i-k-1}^2\\&=O_p(T^{-2\eta})\sum_{t=1}^{T}\sum_{i=0}^{t-2}\varrho^i\sum_{k=0}^{t-i-2}\delta^{2i}v_{t-i-k-1}^2+O_p(T^{1-\eta})=O_p(T^{1-\eta})\end{aligned}$$

by Markov's inequality, such that

$$T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}(z_t^*)^2=T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}(x_t^*)^2+o_{p^*}(1)\left[1+T^{-1}\sum_{t=1}^{T}(x_t^*)^2\right]^{1/2}\qquad\text{(A.26)}$$

again by Markov's inequality. Second,

$$\sum_{t=1}^{\lfloor T\tau\rfloor-1} |(x_t^*)^2 - (\xi_t^*)^2| \leq \sum_{t=1}^{\lfloor T\tau\rfloor-1} (x_t^* - \xi_t^*)^2 + 2\left[\sum_{t=1}^{\lfloor T\tau\rfloor-1} (x_t^* - \xi_t^*)^2\right]^{1/2} \left[\sum_{t=1}^{\lfloor T\tau\rfloor-1} (\xi_t^*)^2\right]^{1/2},$$

where

$$\begin{aligned}
\mathrm{E}^* \sum_{t=1}^{\lfloor T\tau\rfloor-1} (x_t^* - \xi_t^*)^2 &\leq \sum_{t=1}^{T} \mathrm{E}^*(x_{t-1}^* - \xi_{t-1}^*)^2 = \sum_{t=1}^{T} (\hat{b}_i\hat{v}_{t-i-1} - b_i v_{t-i-1})^2 \\
&\leq 2\sum_{t=1}^{T}\sum_{i=0}^{t-2} \hat{b}_i^2(\hat{v}_{t-i-1} - v_{t-i-1})^2 + 2\sum_{t=1}^{T}\sum_{i=0}^{t-2}(\hat{b}_i - b_i)^2 v_{t-i-1}^2 \\
&\leq 2T \max_{1\leq t\leq T}(\hat{v}_t - v_t)^2 \sum_{i=0}^{\infty}\hat{b}_i^2 + C\max_{1\leq t\leq T}|\hat{b}_i - b_i| \sum_{t=1}^{T}\sum_{i=0}^{t-2}\delta^i v_{t-i-1}^2 \\
&= o_p(T)
\end{aligned}$$

using Markov's inequality, such that

$$T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1} (x_t^*)^2 = T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1} (\xi_t^*)^2 + o_{p^*}(1)\left[1 + T^{-1}\sum_{t=1}^{T}(\xi_t^*)^2\right]^{1/2} \tag{A.27}$$

again by Markov's inequality. Third,

$$\sum_{t=1}^{\lfloor T\tau\rfloor-1} [(\xi_t^*)^2 - \sum_{i=0}^{t-1} b_i^2 v_{t-i}^2 R_{t-i}^2] = 2\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1}\sum_{j=i+1}^{t-1} b_i b_j v_{t-i}v_{t-j}R_{t-i}R_{t-j},$$

where the r.h.s., conditionally on the data, has expected square bounded by

$$4\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1}\sum_{j=i+1}^{t-1} |b_i||b_j|v_{t-i}^2 v_{t-j}^2 \sum_{s=1}^{\lfloor T\tau\rfloor-1} |b_{s-t+i}||b_{s-t+j}| \leq C\sum_{t=1}^{T}\sum_{i=0}^{t-1}\sum_{j=i+1}^{t-1} |b_i||b_j|v_{t-i}^2 v_{t-j}^2 = O_p(T)$$

by Markov's inequality, such that

$$T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1} [(\xi_t^*)^2 - \sum_{i=0}^{t-1} b_i^2 v_{t-i}^2 R_{t-i}^2] = T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1} b_i^2 v_{t-i}^2 R_{t-i}^2 + o_{p^*}(1) \tag{A.28}$$

again by Markov's inequality. From (A.26)-(A.28) it follows that $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}(z_t^*)^2$ will converge to the limit asserted in part (c) if $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1} b_i^2 v_{t-i}^2 R_{t-i}^2$ converges to that same limit. We establish the latter convergence next.

From the Beveridge-Nelson decomposition $\sum_{i=0}^{t-1} b_i^2 v_{t-i}^2 R_{t-i}^2 = \kappa^2 v_t^2 R_t^2 + \Delta\tilde{v}_t$, where $\tilde{v}_t =$

$\sum_{i=0}^{t-1} c_i v_{t-i}^2 R_{t-i}^2$ for an appropriate exponentially decreasing sequence c_i, it follows that

$$\begin{aligned}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1} b_i^2 v_{t-i}^2 R_{t-i}^2 &= \kappa^2 \sum_{t=1}^{\lfloor T\tau\rfloor-1} v_t^2 + \kappa^2 \sum_{t=1}^{\lfloor T\tau\rfloor-1} v_t^2(R_t^2-1) + \tilde{v}_{\lfloor T\tau\rfloor-1} - \tilde{v}_0 \\ &= \kappa^2 \sum_{t=1}^{\lfloor T\tau\rfloor-1} v_t^2 + o_p(T)\end{aligned}$$

by Chebyshev's inequality for $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1} v_t^2(R_t^2-1)$ and Markov's inequality for the quantity $\sum_{i=0}^{\lfloor T\tau\rfloor-1} |c_i| v_{\lfloor T\tau\rfloor-i-1}^2 R_{\lfloor T\tau\rfloor-i-1}^2$. As $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor} v_t^2 \xrightarrow{p} [M_v](\tau)$ by Lemma 3 and the limiting function is continuous, we can conclude that

$$T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1} b_i^2 v_{t-i}^2 R_{t-i}^2 \xrightarrow{p} [M_v](\tau).$$

As, in addition, the functions on the l.h.s. and the r.h.s. of the previous convergence are increasing, the convergence is uniform in τ:

$$\sup_{\tau\in[0,1]}\left|\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1} b_i^2 v_{t-i}^2 R_{t-i}^2 - [M_v](\tau)\right| \xrightarrow{p} 0.$$

Finally, since convergence in probability to zero implies $\overset{p}{\Rightarrow}_p$-convergence to zero upon conditioning on the data, it holds that $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1} b_i^2 v_{t-i}^2 R_{t-i}^2 \overset{p}{\Rightarrow}_p [M_v](\tau)$ in $\mathcal{D}$ conditionally on the data, which establishes part (c) by virtue also of the earlier discussion.

Finally, in part (d) the full-sample bootstrap residuals computed under the null hypothesis are $\hat{u}_t^* = u_t^* - T^{-1}\sum_{s=1}^T u_s^*$, such that

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor} |(\hat{u}_t^*)^2 - (u_t^*)^2| \le \frac{\lfloor T\tau\rfloor}{T^3}\left(\sum_{s=1}^T u_s^*\right)^2 + \frac{2\sqrt{\lfloor T\tau\rfloor}}{T^2}\left|\sum_{s=1}^T u_s^*\right|\left[\sum_{t=1}^{\lfloor T\tau\rfloor}(u_t^*)^2\right]^{1/2}. \quad \text{(A.29)}$$

Here, first,

$$\mathrm{E}^*\left(\sum_{s=1}^T u_s^*\right)^2 = \sum_{s=1}^T \hat{u}_s^2 = T\hat{\sigma}_u^2(0,1) = O_p(T)$$

by (A.3), such that $\sum_{s=1}^T u_s^* = O_p^*(T^{1/2})$ by Chebyshev's inequality. Second,

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}(u_t^*)^2 - \frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}\hat{u}_t^2 = \frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}\hat{u}_t^2(R_t^2-1) = o_p^*(1)$$

again by Chebyshev's inequality:

$$\mathrm{E}^*\left(\sum_{t=1}^{\lfloor T\tau\rfloor}\hat{u}_t^2(R_t^2-1)\right)^2 = C\sum_{t=1}^{T}\hat{u}_t^4 \le C\sum_{t=1}^{T}u_t^4 + C\sum_{t=1}^{T}(\hat{u}_t-u_t)^4 = O_p(T)$$

because u_t are uniformly L_4-bounded. Therefore,

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}(u_t^*)^2 = \frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}\hat{u}_t^2 + o_p(1) \xrightarrow{p} [M_u](\tau)$$

by (A.3). Returning to (3), it follows that $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor}(\hat{u}_t^*)^2 = [M_u](\tau) + o_p^*(1)$, where the infinitesimal term is uniform in τ because the involved processes are increasing and $[M_u](\tau)$ is moreover continuous. The discussion of bootstrap residuals $\hat{u}_t^*$ computed over subsamples is similar. □

The proof of Lemma 7 will make use of the following estimates.

Lemma 9 *Under Assumptions 1.2 and 3:*

(a) $\max_{1\le s\le t\le T}|\mathrm{E}^*(v_s^*x_t^*)| = O_p(1)\hat{v}_s^2$ *and* $\mathrm{E}^*(v_s^*x_t^*) = 0$ *for* $s > t$*;*

(b) $\max_{1\le s,t\le T}|\mathrm{E}^*(x_s^*x_t^*)| = O_p(1)\sum_{t=1}^{T}\hat{v}_t^2$.

Proof of Lemma 9. For $s > t$, the expectation in part (a) is zero by the conditional independence of v_s^* and x_t^*. For $s \le t$ it holds that

$$\begin{aligned}|\mathrm{E}^*(v_s^*x_t^*)| &= \left|\hat{\phi}\sum_{i=0}^{t-s-1}[(\hat{A}(1))^{-1}+\hat{b}_i^*]\mathrm{E}^*(v_s^*x_{t-i-1}^*) + [(\hat{A}(1))^{-1}+\hat{b}_{t-s}]\hat{v}_s^2\right| \\ &\le \hat{C}\left(|\hat{\phi}|\sum_{i=0}^{t-s-1}|\mathrm{E}^*(v_s^*x_{t-i-1}^*)| + \hat{v}_s^2\right)\end{aligned}$$

with $\hat{C} := |\hat{A}(1)|^{-1} + \sup_{i\ge 0}(|\hat{b}_i| + |\hat{b}_i^*|) = O_p(1)$. Thus, by recursive substitution, $|\mathrm{E}^*(v_s^*x_s^*)| \le \hat{C}\hat{v}_s^2$ and $|\mathrm{E}^*(v_s^*x_{s+i}^*)| \le \hat{C}(1+\hat{C}|\hat{\phi}|)^i\hat{v}_s^2$ for $i \ge 1$. These imply the estimate $|\mathrm{E}^*(v_s^*x_t^*)| \le \hat{C}(1+\hat{C}|\hat{\phi}|)^T\hat{v}_s^2$ uniformly in $t = 1, ..., T$. As $\hat{\phi} = O_p(T^{-1})$ and $(1+\hat{C}|\hat{\phi}|)^T = O_p(1)$, part (a) follows.

Using the previous estimate, in part (b) we find that

$$\begin{aligned}|\mathrm{E}^*(x_s^*x_t^*)| &= \left|\sum_{i=0}^{t-1}\{(\hat{A}(1))^{-1}+\hat{b}_i^*\}\{\hat{\phi}\mathrm{E}^*(x_s^*x_{t-i-1}^*) + \mathrm{E}^*(x_s^*v_{t-i}^*)\}\right| \\ &\le \hat{C}\left(|\hat{\phi}|\sum_{i=0}^{t-2}|\mathrm{E}^*(x_s^*x_{t-i-1}^*)| + \hat{C}(1+\hat{C}|\hat{\phi}|)^T\sum_{i=1}^{T}\hat{v}_i^2\right).\end{aligned}$$

Again by recursive substitution, $|\mathrm{E}^*(x_s^*x_1^*)| \le \hat{C}^2(1+\hat{C}|\hat{\phi}|)^T\sum_{i=1}^{T}\hat{v}_i^2$ and $|\mathrm{E}^*(v_s^*x_t^*)| \le \hat{C}^2(1+\hat{C}|\hat{\phi}|)^{T+t-1}\sum_{i=1}^{T}\hat{v}_i^2$. The estimate $\max_{1\le s,t\le T}|\mathrm{E}^*(x_s^*x_t^*)| \le \hat{C}^2(1+\hat{C}|\hat{\phi}|)^{2T}\sum_{i=1}^{T}\hat{v}_i^2$ follows, and since $\hat{C}|\hat{\phi}| = O_p(T^{-1})$, also part (b). □

Proof of Lemma 7. In part (a), we write

$$T^{(1+\eta)/2}\left[N_T^*(\tau)-\tilde{N}_T^*(\tau)\right]=T^{(1+\eta)/2}[DN_{T1}^*(\tau)+DN_{T2}^*(\tau)+DN_{T3}^*(\tau)]$$

with

$$\begin{aligned}
T^{(1+\eta)/2}DN_{T1}^*(\tau) &:= \hat{\phi}\sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^j\sum_{i=0}^{t-j-2}\hat{b}_i x_{t-i-j-2}^*\right)u_t^*\\
T^{(1+\eta)/2}DN_{T2}^*(\tau) &:= \sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^j\left(\sum_{i=0}^{t-j-2}\hat{b}_i v_{t-i-j-1}^*-\omega v_{t-j-1}^*\right)u_t^*\\
T^{(1+\eta)/2}DN_{T3}^*(\tau) &:= \sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^*(u_t^*-\tilde{u}_t).
\end{aligned}$$

and notice that $T^{(1+\eta)/2}DN_{T1}^*(\tau)$ is a martingale conditional on the data, with conditional variance at 1 given by

$$\begin{aligned}
&\hat{\phi}^2\sum_{t=1}^{T}\left(\sum_{j,k=0}^{t-2}\varrho^{j+k}\sum_{i=0}^{t-j-2}\sum_{m=0}^{t-k-2}\hat{b}_i\hat{b}_m \mathrm{E}^*(x_{t-i-j-2}^*x_{t-m-k-2}^*)\right)\hat{u}_t^2\\
&\qquad=O_p(1)\hat{\phi}^2\sum_{t=1}^{T}\hat{v}_t^2\sum_{t=1}^{T}\left(\sum_{j,k=0}^{t-2}\varrho^{j+k}\sum_{i=0}^{t-j-2}\sum_{m=0}^{t-k-2}|\hat{b}_i\hat{b}_m|\right)\hat{u}_t^2
\end{aligned}$$

by Lemma 9(b). Further, as $\hat{\phi}^2\sum_{t=1}^{T}\hat{v}_t^2=O_p(T^{-1})$, this conditional variance equals

$$O_p(T^{-1})\left(\sum_{i=0}^{T-2}|\hat{b}_i|\right)^2\left(\sum_{j=0}^{T-2}\varrho^j\right)^2\sum_{t=1}^{T}\hat{u}_t^2=O_p(T^{2\eta})=o_p(T^{1+\eta}).$$

Therefore, by Doob's martingale inequality, $\sup_{[0,1]}|DN_{T1}^*(\tau)|=o_p^*(1)$.

Since $\omega-\sum_{i=0}^{\infty}\hat{b}_i=A(1)^{-1}-(1-\sum_{i=1}^{p}\tilde{a}_i)^{-1}=o_p(1)$ and $\sup_{[0,1]}|\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^j v_{t-j-1}^*u_t^*|=O_p^*(T^{1/2+\eta/2})$ by Doob's martingale inequality, it will follow that $\sup_{[0,1]}|DN_{T2}^*(\tau)|=o_p^*(1)$ if we show that $\sup_{[0,1]}|D\tilde{N}_{T2}^*(\tau)|=o_p^*(1)$ for

$$T^{(1+\eta)/2}D\tilde{N}_{T2}^*(\tau):=\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^j\left(\sum_{i=0}^{t-j-2}\hat{b}_i v_{t-i-j-1}^*-\sum_{i=0}^{\infty}\hat{b}_i v_{t-j-1}^*\right)u_t^*.$$

Consider the decomposition

$$\sum_{j=0}^{t-2}\varrho^j\left(\sum_{i=0}^{t-j-2}\hat{b}_i v_{t-i-j-1}^*-\sum_{i=0}^{\infty}\hat{b}_i v_{t-j-1}^*\right)=-\sum_{i=0}^{t-2}\hat{b}_i^*v_{t-i-1}^*+(1-\varrho)\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\hat{b}_i^*v_{t-i-j-2}^*. \tag{A.30}$$

It holds that $\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{i=0}^{t-2}\hat{b}_i^*v_{t-i-1}^*u_t^*$ and $\sum_{t=1}^{\lfloor T\tau\rfloor}(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\hat{b}_i^*v_{t-i-j-2}^*)u_t^*$ are martingales conditional on the data with conditional variances at 1 equal respectively to

$$\sum_{t=1}^{T}\left[\mathrm{E}^*\left(\sum_{i=0}^{t-2}\hat{b}_i^* v_{t-i-1}^*\right)^2\right]\hat{u}_t^2 = \sum_{t=1}^{T}\sum_{i=0}^{t-2}(\hat{b}_i^*)^2\hat{v}_{t-i-1}^2\hat{u}_t^2$$

$$= (1+o_p(1))\sum_{t=1}^{T}\sum_{i=0}^{t-2}(\hat{b}_i^*)^2 v_{t-i-1}^2 u_t^2 = O_p(T)$$

and

$$\begin{aligned}\sum_{t=1}^{T}\left[\mathrm{E}^*\left(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\hat{b}_i^* v_{t-i-j-2}^*\right)^2\right]\hat{u}_t^2 &= \sum_{t=1}^{T}\left[\mathrm{E}^*\left(\sum_{s=1}^{t-2}v_s^*\sum_{j=0}^{t-s-2}\varrho^j\hat{b}_{t-s-j-2}^*\right)^2\right]\hat{u}_t^2 \qquad \text{(A.31)}\\ &= \sum_{t=1}^{T}\sum_{s=1}^{t-2}\hat{v}_s^2\left(\sum_{j=0}^{t-s-2}\varrho^j\hat{b}_{t-s-j-2}^*\right)^2\hat{u}_t^2\\ &= O_p(1)\sum_{t=1}^{T}\left(\sum_{s=1}^{t-2}\hat{v}_s^2\varrho^{2(t-s)}\right)\hat{u}_t^2\\ &= O_p(1)\sum_{t=1}^{T}\left(\sum_{s=1}^{t-2}v_s^2\varrho^{2(t-s)}\right)u_t^2\\ &= O_p(T^{1+\eta}),\end{aligned}$$

using the estimate $\left|\sum_{j=0}^{t-s-2}\varrho^j\hat{b}_{t-s-j-2}\right| \leq C\sum_{j=0}^{t-s-2}\varrho^j\delta^{t-s-j-2} = \frac{C}{\rho-\delta}(\varrho^{t-s-1}-\delta^{t-s-1}) = O(\varrho^{t-s})$ a.s. ($\delta\in(0,1)$), and Markov's inequality. Therefore, by Doob's martingale inequality, it holds that

$$\sup_{[0,1]}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{i=0}^{t-2}\hat{b}_i^* v_{t-i-1}^* u_t^*\right| = O_p^*(T^{1/2}) \text{ and } \sup_{[0,1]}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\hat{b}_i^* v_{t-i-j-2}^*)u_t^*\right| = O_p^*(T^{(1+\eta)/2})$$

. As $1-\varrho = O(T^{-\eta})$, by combining the previous conclusions it follows that indeed $\sup_{[0,1]}|D\tilde{N}_{T2}^*(\tau)| = o_p^*(1)$.

Finally, also $T^{(1+\eta)/2}D\tilde{N}_{T3}^*(\tau)$ conditionally on the data is a martingale and its conditional variance at 1 is given by

$$\begin{aligned}\sum_{t=1}^{T}[\mathrm{E}^*(\zeta_{t-1}^*)^2](\hat{u}_t-u_t)^2 &\leq \max_{1\leq t\leq T}(\hat{u}_t-u_t)^2\sum_{t=1}^{T}\mathrm{E}^*(\zeta_{t-1}^*)^2\\ &= o_p(1)\sum_{t=1}^{T}\sum_{j=0}^{t-2}\varrho^{2j}\hat{v}_{t-j-1}^2\\ &= o_p(1)\sum_{t=1}^{T}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 + o_p(T^{1+\eta}) = o_p(T^{1+\eta})\end{aligned}$$

by (A.18) and by Markov's inequality for $\sum_{t=1}^{T}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 = O_p(T^\eta)$. Therefore, by

Doob's martingale inequality, $\sup_{[0,1]}|D\tilde{N}^*_{T3}(\tau)| = o^*_p(1)$. Part (a) follows by combining the previous results.

In part (b), let $\tilde{\zeta}_t := \omega\sum_{j=0}^{t-1}\varrho^j\tilde{v}_{t-j}$, $(\tilde{u}_t, \tilde{v}_t) := (u_t, v_t)R_t$. Consider first

$$\begin{aligned}\mathrm{E}^*\sum_{t=1}^{T}(\zeta^*_{t-1}-\tilde{\zeta}_{t-1})^2u_t^2 &= \sum_{t=1}^{T}\sum_{j=0}^{t-2}\varrho^{2j}(\hat{v}_{t-j-1}-v_{t-j-1})^2u_t^2\\ &\leq \max_{1\leq t\leq T}|\hat{v}_t-v_t|^2\sum_{j=0}^{T-2}\varrho^{2j}\sum_{t=1}^{T}u_t^2 = o_p(T^{1+\eta}).\end{aligned}$$

Hence, by Markov's inequality, $T^{-1-\eta}\sum_{t=1}^{T}(\zeta^*_{t-1}-\tilde{\zeta}_{t-1})^2u_t^2 = o^*_p(1)$. As a result,

$$\begin{aligned}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}[(\zeta^*_{t-1})^2-(\tilde{\zeta}_{t-1})^2]u_t^2\right| &\leq \sum_{t=1}^{T}(\zeta^*_{t-1}-\tilde{\zeta}_{t-1})^2u_t^2 + 2\sqrt{\sum_{t=1}^{T}(\zeta^*_{t-1}-\tilde{\zeta}_{t-1})^2u_t^2}\sqrt{\sum_{t=1}^{T}\tilde{\zeta}^2_{t-1}u_t^2}\\ &= o^*_p(T^{1+\eta}) + 2o^*_p(T^{1+\eta})\sqrt{\check{V}_T(1)}\end{aligned}$$

with $\check{V}_T(\tau) := T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\zeta}^2_{t-1}u_t^2$, such that

$$\tilde{V}^*_T(\tau) = \omega^2\check{V}_T(\tau) + o^*_p(1)\left(1+\check{V}_T(1)\right) \tag{A.32}$$

pointwise.

Next, it holds that $\mathrm{P}^*(\max_{1\leq t\leq T}|v_t| \leq T^{1/3}) = \mathbb{I}\{\max_{1\leq t\leq T}|v_t| \leq T^{1/3}\} \xrightarrow{p} 0$ because $\max_{1\leq t\leq T}|v_t| = o_p(T^{1/3})$. Then, with $\check{v}_t = \mathbb{I}\{|v_t| \leq T^{1/3}\}v_t$, the decomposition

$$\begin{aligned}T^{1+\eta}\check{V}_T(\tau) &= \sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v^2_{t-j-1}u_t^2 + \sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j,k=0}^{t-2}\varrho^{j+k}\mathbb{I}_{j\neq k}\tilde{v}_{t-j-1}\tilde{v}_{t-k-1}u_t^2\\ &= \sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v^2_{t-j-1}u_t^2 + DV_{1T}(\tau) + 2DV_{2T}(\tau) + o^*_p(T^{1+\eta}) \quad (A.33)\end{aligned}$$

holds, where DV_{1T} and DV_{2T} are square-integrable under Assumption 3 and defined as follows. On the one hand, $DV_{1T}(\tau) := \sum_{s=1}^{\lfloor T\tau\rfloor-1}\check{v}_s^2(R_s^2-1)\sum_{t=s+1}^{\lfloor T\tau\rfloor}\varrho^{2(t-s-1)}u_t^2$ has

$$\begin{aligned}\mathrm{E}^*\left(DV_{1T}(\tau)\right)^2 &\leq \mathrm{Var}(R_1^2)\sum_{s=1}^{T-1}\check{v}_s^4\left[\sum_{t=s+1}^{T}\varrho^{2(t-s-1)}u_t^2\right]^2\\ &\leq C\sum_{s=1}^{T-1}v_s^4\left\{\left[\sum_{t=s+1}^{T}\varrho^{2(t-s-1)}(u_t^2-\sigma^2_{ut})\right]^2 + \max_{1\leq t\leq T}\sigma^4_{ut}\left[\sum_{t=s+1}^{T}\varrho^{2(t-s-1)}\right]^2\right\}\\ &\leq [O_p(T)+O_p(T^{2\eta})]\sum_{s=1}^{T-1}v_s^4\end{aligned}$$

by Lemma 2(d). Therefore, $\mathrm{E}^*\left(DV_{1T}(\tau)\right)^2 = O_p(T^2)+O_p(T^{2\eta+1}) = o_p(T^{2+2\eta})$ such that

$T^{-1-\eta} DV_{1T}(\tau) = o_p^*(1)$ by Chebyshev's inequality. On the other hand,

$$DV_{2T}(\tau) := \sum_{t=1}^{\lfloor T\tau \rfloor} \sum_{j=0}^{t-2} \sum_{k=j+1}^{t-2} \varrho^{j+k} R_{t-j-1} \check{v}_{t-j-1} R_{t-k-1} \check{v}_{t-k-1} u_t^2$$
$$= \sum_{s=1}^{\lfloor T\tau \rfloor -1} R_s \check{v}_s \sum_{r=s+1}^{\lfloor T\tau \rfloor -1} \varrho^{r-s} R_r \check{v}_r \sum_{t=r+1}^{\lfloor T\tau \rfloor} \varrho^{2(t-r-1)} u_t^2$$

has

$$\mathrm{E}^* \left(DV_{2T}(\tau)\right)^2 \leq \sum_{s=1}^{T-1} v_s^2 \sum_{r=s+1}^{T-1} \varrho^{2(r-s)} v_r^2 \left(\sum_{t=r+1}^{T} \varrho^{2(t-r-1)} u_t^2 \right)^2$$
$$= O_p(T) \sum_{s=1}^{T-1} v_s^2 \sum_{r=s+1}^{T-1} \varrho^{2(r-s)} v_r^2 + O(1) \sum_{s=1}^{T-1} v_s^2 \sum_{r=s+1}^{T-1} \varrho^{2(r-s)} v_r^2 \left(\sum_{t=1}^{T-r} \varrho^{2(t-1)} \right)^2$$

by Lemma 2(d) and further

$$\mathrm{E}^* \left(DV_{2T}(\tau)\right)^2 = O_p(T^{2+\eta}) + O(T^{2\eta}) \sum_{s=1}^{T-1} v_s^2 \sum_{r=s+1}^{T-1} \varrho^{2(r-s)} v_r^2$$
$$= O_p(T^{2+\eta}) + O_p(T^{1+3\eta}) = o_p(T^{2+2\eta})$$

using Markov's inequality, such that $T^{-1-\eta} DV_{2T}(\tau) = o_p^*(1)$ pointwise. Combining the previous results establishes the pointwise expansion $\check{V}_T(\tau) = T^{-1-\eta} \sum_{t=1}^{\lfloor T\tau \rfloor} \sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 u_t^2 + o_p^*(1)$ and, in view of (A.32), also point (b).

Part (c) follows by combining part (b) with Lemma 2(f,g). □

Proof of Lemma 8. In view of Lemma 7(a), to proof part (a) it is sufficient to show that $\tilde{N}_T^*(\tau) \overset{w}{\Rightarrow}_p \frac{\omega}{\sqrt{2a}} \int_0^\tau \sqrt{[M_v(s)]'[M_u(s)]'} \mathrm{d}B(s)$. We accomplish this by means of a Skorokhod representation on a probability space with a product structure.

Similarly to Lemma 3.5(ii) of Magdalinos (2020), the conditional Lindeberg condition $\sum_{t=2}^{T} \mathrm{E}_{t-1}^* \{\varsigma_t^2 \mathbb{I}_{|\varsigma_t|>1/n}\} = o_{p^*}(1)$ holds for $\varsigma_t := T^{-(1+\eta)/2} \zeta_{t-1}^* \tilde{u}_t$ and all $n \in \mathbb{N}$. In fact, as

$$\mathrm{E}\left[\mathrm{E}^*(\zeta_{t-1}^*)^4\right] = \mathrm{E}\left[\sum_{j=0}^{t-2} \varrho^{4j} v_{t-j}^4 \mathrm{E}(R_{t-j}^4) + 6 \sum_{j=0}^{t-2} \sum_{i=j+1}^{t-2} \varrho^{2(i+j)} v_{t-i-1}^2 v_{t-1-j}^2 \right]$$
$$\leq C \sup_{1 \leq t \leq T} \mathrm{E} v_t^4 \left[\sum_{j=0}^{T-2} \varrho^{4j} + 6 \sum_{j=0}^{T-2} \sum_{i=j+1}^{T-2} \varrho^{2(i+j)} \right] = O(T^{2\eta}),$$

it follows that $\max_{t \leq T} |\zeta_{t-1}^*| \leq [\sum_{t=1}^{T} (\zeta_{t-1}^*)^4]^{1/4} = O_p^*(T^{1/4+\eta/2})$ by Markov's inequality, such that

$$\sum_{t=2}^{T}\mathrm{E}^*_{t-1}(\varsigma_t^2\mathbb{I}\{|\varsigma_t|>1/n\})\leq\sum_{t=2}^{T}\mathrm{E}^*_{t-1}(\varsigma_t^2)\mathbb{I}\{|\zeta^*_{t-1}|>\frac{T^{-\frac{1}{8}}T^{\frac{1+\eta}{2}}}{n}\}$$
$$+\sum_{t=2}^{T}\mathrm{E}^*_{t-1}(\varsigma_t^2\mathbb{I}\{|R_t|>T^{1/16}\})$$
$$+\sum_{t=2}^{T}\mathrm{E}^*_{t-1}(\varsigma_t^2\mathbb{I}\{|u_t|>T^{1/8}\})=o_{p^*}(1)$$

because

$$\mathrm{P}^*\left(\sum_{t=2}^{T}\varsigma_t^2\mathbb{I}\{|\zeta^*_{t-1}|>\frac{T^{-\frac{1}{8}}T^{\frac{1+\eta}{2}}}{n}\}=0\right)\leq 1-\mathrm{P}^*\left(\max_{t\leq T}|\zeta^*_{t-1}|>\frac{T^{-\frac{1}{8}}T^{\frac{1+\eta}{2}}}{n}\right)$$
$$=1-o_p(1),$$

$$\mathrm{E}\left[\mathrm{E}^*\sum_{t=2}^{T}\mathrm{E}^*_{t-1}(\varsigma_t^2\mathbb{I}\{|R_t|>T^{1/16}\})\right]=T^{-1-\eta}\sum_{t=2}^{T}\mathrm{E}\left[\{\mathrm{E}^*(\zeta^*_{t-1})^2\}u_t^2R_t^2\mathbb{I}\{|R_t|>T^{1/16}\}\right]$$
$$=T^{-1-\eta}\sum_{t=2}^{T}\sum_{j=0}^{t-2}\varrho^{2j}\mathrm{E}(v_{t-j-1}^2u_t^2R_t^2\mathbb{I}\{|R_t|>T^{1/16}\})$$
$$\leq T^{-1-\eta}\sum_{t=2}^{T}\sum_{j=0}^{t-2}\varrho^{2j}\sqrt{\mathrm{E}v_{t-j-1}^4}\sqrt{\mathrm{E}u_t^4\mathrm{E}[R_1^4\mathbb{I}\{|R_1|>T^{1/8}\}]}$$
$$=o(T^{-1-\eta})\sum_{t=2}^{T}\sum_{j=0}^{t-2}\varrho^{2j}=o(1),$$

and similarly

$$\mathrm{E}\left[\mathrm{E}^*\sum_{t=2}^{T}\mathrm{E}^*_{t-1}(\varsigma_t^2\mathbb{I}\{|u_t|>T^{1/8}\})\right]\leq T^{-1-\eta}\sum_{t=2}^{T}\sum_{j=0}^{t-2}\varrho^{2j}\sqrt{\mathrm{E}v_{t-j-1}^4}\sqrt{\mathrm{E}R_1^4\mathrm{E}[u_t^4\mathbb{I}\{|u_t|>T^{1/8}\}]}$$
$$\leq o(T^{-1-\eta})\sum_{t=2}^{T}\sum_{j=0}^{t-2}\varrho^{2j}=o(1)$$

using the uniform integrability of u_t^4 (inherited from the uniformly L_4-bounded and stationary sequence $\boldsymbol{\psi}_t$ because $\mathbf{H}$ is bounded).

For $X_T=(x_0,...,x_{T-1})$, $Y_T:=(y_1,...,y_T)$ and $R_T^*:=(R_1,...,R_T)$, write $\varsigma_t=\varsigma_t(X_T,Y_T,R_T^*)$ for the measurable transformation defining ς_t, and similarly, $\tilde{V}_T^*=\tilde{V}_T^*(X_T,Y_T,R_T^*)$. Fix the measurable functions functions $g_{Tn}:\mathbb{R}^{3T}\to\mathbb{R}$ as $g_{Tn}(x,y,R_T^*)=\sum_{t=2}^{T}\mathrm{E}^*_{t-1}\{\varsigma_t^2(x,y,R_T^*)\mathbb{I}_{|\varsigma_t(x,y,R_T^*)|>1/n}\}$ such that, by the independence of (X_T,Y_T) and R_T^*, it holds that $g_{Tn}(X_T,Y_T,R_T^*)=\sum_{t=2}^{T}\mathrm{E}^*_{t-1}\{\varsigma_t^2\mathbb{I}_{|\varsigma_t|>1/n}\}$ a.s. Introduce also $\delta_T:\mathbb{R}^{3T}\to\mathbb{R}$ by $\delta_T(x,y,R_T^*):=\sup_{[0,1]}|\tilde{V}_T^*(x,y,R_T^*)(\tau)-\frac{\omega^2}{2a}\int_0^\tau[M_u(s)]'[M_v(s)]'\mathrm{d}s|$ such that $\delta_T(X_T,Y_T,R_T^*)=o_p^*(1)$ by Lemma 7(d). Then the $\mathbb{R}^\infty$-

valued function

$$\gamma_T(X_T, Y_T, R_T^*) := (\delta_T(X_T, Y_T, R_T^*), g_{T1}(X_T, Y_T, R_T^*), g_{T2}(X_T, Y_T, R_T^*), ...)$$

satisfies $\gamma_T(X_T, Y_T, R_T^*) = o_p^*(1)$ in the sense that $d(\gamma_T(X_T, Y_T, R_T^*), 0^\infty) = o_p^*(1)$ for the Frechet metric d and the zero sequence $0^\infty \in \mathbb{R}^\infty$. Equivalently,

$$\mathrm{E}^* f(d(\gamma_T(X_T, Y_T, R_T^*), 0^\infty)) \xrightarrow{p} f(0)$$

for every continuous and bounded $f : [0, \infty) \to \mathbb{R}$. Let $\{f_k\}_{k \in \mathbb{N}}$ be a countable collection of continuous and bounded functions $[0, \infty) \to \mathbb{R}$ such that for any $w_T = w_T(R_T^*)$ the convergence $\mathrm{E} f_k(w_T) \to f_k(0)$ for all functions in this collection is equivalent to $w_T \xrightarrow{p} 0$ (the expectation and the latter convergence are w.r.t. the distribution of R_T^*). Define on the support of (X_T, Y_T) the measurable deterministic functions $h_{Tk}(\cdot, \cdot) = \mathrm{E} f_k(d(\gamma_T(\cdot, \cdot, R_T^*), 0^\infty))$ (the expectation is w.r.t. the distribution of R_T^*), such that $h_{Tk}(X_T, Y_T) = \mathrm{E}^* f_k(d(\gamma_T(X_T, Y_T, R_T^*), 0^\infty))$ a.s., then

$$\chi_T(X_T, Y_T) := (h_{T1}(X_T, Y_T), h_{T2}(X_T, Y_T), ...) \xrightarrow{p} (f_1(0), f_2(0), ...)$$

in $\mathbb{R}^\infty$. By extended Skorokhod coupling (Corollary 5.12 of Kallenberg, 1997), there exist a probability space and random elements $(\tilde{X}_T, \tilde{Y}_T) \stackrel{d}{=} (X_T, Y_T)$ such that $\chi_T(\tilde{X}_T, \tilde{Y}_T) \xrightarrow{a.s.} (f_1(0), f_2(0), ...)$. On an extension of this probability space, define the i.i.d. $R_t^* \stackrel{d}{=} R_1$ and $\tilde{R}_T^* := (\tilde{R}_1, ..., \tilde{R}_T)$. Choose an almost certain event $\mathcal{A}$ in the factor space of $(\tilde{X}_T, \tilde{Y}_T)$ such that, for every fixed $\omega \in \mathcal{A}$ and every $k \in \mathbb{N}$,

$$\mathrm{E} f_k(d(\gamma_T(\tilde{X}_T(\omega), \tilde{Y}_T(\omega), \tilde{R}_T^*), 0^\infty)) = h_{Tk}(\tilde{X}_T(\omega), \tilde{Y}_T(\omega)) \to f_k(0),$$

where the expectation is w.r.t. the distribution of $\tilde{R}_T = R_T$. Then, due to the choice of f_k, it follows that $d(\gamma_T(\tilde{X}_T(\omega), \tilde{Y}_T(\omega), \tilde{R}_T^*), 0^\infty) \xrightarrow{p} 0$ for every fixed $\omega \in \mathcal{A}$. Equivalently, $\gamma_T(\tilde{X}_T(\omega), \tilde{Y}_T(\omega), \tilde{R}_T^*) \xrightarrow{p} 0^\infty$ for every $\omega \in \mathcal{A}$. A component-wise reading of this convergence shows that, for every fixed $\omega \in \mathcal{A}$, the conditions (predictable variance + Lindeberg) of a martingale invariance principle apply to $\tilde{N}_T^*(\tau)$ (redefined on the Skorokhod representation space) and regarded, upon fixing $\omega \in \mathcal{A}$, as a transformation of $\tilde{R}_T$ alone. Specifically, $\tilde{N}_T^*(\tau)$ on the Skorokhod representation space weakly converges to a continuous Gaussian martingale with variance $\frac{\omega^2}{2a}\int_0^\tau [M_u(s)]'[M_v(s)]'\mathrm{d}s$ for every $\omega \in \mathcal{A}$, and therefore, almost surely. Thus, on a general probability space $\tilde{N}_T^*(\tau)$ converges to the same (nonrandom) limit weakly in probability.

We now turn to the proof of part (b). The steps are analogous to those in Lemma 7. We show that, first, $\sum_{t=1}^{\lfloor T\tau \rfloor} (z_{t-1}^*)^2 = \hat{\omega}^2 \sum_{t=1}^{\lfloor T\tau \rfloor} (\zeta_{t-1}^*)^2 + o_p^*(T^{-1-\eta})$ pointwise, next, $\sum_{t=1}^{\lfloor T\tau \rfloor} (\zeta_{t-1}^*)^2 = \sum_{t=1}^{\lfloor T\tau \rfloor} \sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 + o_p^*(T^{-1-\eta})$ pointwise, and last, $T^{-1-\eta} \sum_{t=1}^{\lfloor T\tau \rfloor} \sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 \xrightarrow{p}_p \frac{1}{2a} M_v(\tau)$.

To accomplish the first step, for $\tilde{z}_t := \sum_{j=1}^{t-2} \varrho^{2j} \sum_{i=0}^{t-j-2} \hat{b}_i v^*_{t-j-i-1}$ we find that

$$\begin{aligned}
\mathrm{E}^* \sum_{t=1}^{T} (z^*_{t-1} - \tilde{z}_{t-1})^2 &= \hat{\phi}^2 \sum_{t=1}^{T} \mathrm{E}^* \left(\sum_{j=1}^{t-2} \varrho^{2j} \sum_{i=0}^{t-j-2} \hat{b}_i x^*_{t-j-i-2} \right)^2 \\
&= \hat{\phi}^2 \sum_{t=1}^{T} \sum_{j,k=1}^{t-2} \varrho^{2(j+k)} \sum_{i=0}^{t-j-2} \sum_{m=0}^{t-k-2} \hat{b}_i \hat{b}_m \mathrm{E}^*(x_{t-j-i-2} x^*_{t-k-m-2}) \\
&= O_p(T^{-2}) \sum_{t=1}^{T} \hat{v}_t^2 \left(\sum_{j=1}^{T} \varrho^{2j} \right)^2 \left(\sum_{i=0}^{\infty} |\hat{b}_i| \right)^2 = O_p(T^{2\eta-1})
\end{aligned}$$

using Lemma 9(b), such that $T^{-1-\eta} \sum_{t=1}^{\lfloor T\tau \rfloor} (z^*_{t-1} - \tilde{z}_t)^2 = o^*_p(1)$. Hence,

$$\begin{aligned}
\left| \sum_{t=1}^{\lfloor T\tau \rfloor} (z^*_{t-1})^2 - \sum_{t=1}^{\lfloor T\tau \rfloor} (\tilde{z}^*_{t-1})^2 \right| &\le \sum_{t=1}^{T} (z^*_{t-1} - \tilde{z}_t)^2 + 2 \sqrt{\sum_{t=1}^{T} (\tilde{z}^*_{t-1})^2} \sqrt{\sum_{t=1}^{T} (z^*_{t-1} - \tilde{z}_t)^2} \qquad \text{(A.34)} \\
&= o^*_p(T^{1+\eta}) \left(1 + T^{-1-\eta} \sum_{t=1}^{T} (\tilde{z}^*_{t-1})^2 \right).
\end{aligned}$$

Similarly, by using the decomposition

$$\tilde{z}^*_{t-1} - \hat{\omega} \zeta^*_{t-1} = - \sum_{i=0}^{t-2} \hat{b}^*_i v^*_{t-i-1} + (1 - \varrho) \sum_{j=0}^{t-3} \varrho^j \sum_{i=0}^{t-j-3} \hat{b}^*_i v^*_{t-i-j-2}$$

with

$$\mathrm{E}^* \sum_{t=1}^{T} (\sum_{i=0}^{t-2} \hat{b}^*_i v^*_{t-i-1})^2 = \sum_{t=1}^{T} \sum_{i=0}^{t-2} (\hat{b}^*_i)^2 \hat{v}^2_{t-i-1} \le \sum_{t=1}^{T} \hat{v}_t^2 \sum_{i=0}^{\infty} (\hat{b}^*_i)^2 = O_p(T)$$

and $\mathrm{E}^* \sum_{t=1}^{T} (\sum_{j=0}^{t-3} \varrho^j \sum_{i=0}^{t-j-3} \hat{b}^*_i v^*_{t-i-j-2})^2 = O_p(T^{1+\eta})$, the latter by formally substituting $\hat{u}_t^2$ with 1 in (A.31), we can conclude that $\mathrm{E}^* \sum_{t=1}^{T} (\tilde{z}^*_{t-1} - \hat{\omega} \zeta^*_{t-1})^2 = o_p(T^{1+\eta})$ and $\sum_{t=1}^{T} (\tilde{z}^*_{t-1} - \hat{\omega} \zeta^*_{t-1})^2 = o^*_p(T^{-1-\eta})$. Therefore,

$$\begin{aligned}
\left| \sum_{t=1}^{\lfloor T\tau \rfloor} (\tilde{z}^*_{t-1})^2 - \hat{\omega}^2 \sum_{t=1}^{\lfloor T\tau \rfloor} (\zeta^*_{t-1})^2 \right| &\le \sum_{t=1}^{T} (\tilde{z}^*_{t-1} - \hat{\omega} \zeta^*_{t-1})^2 + 2|\hat{\omega}| \sqrt{\sum_{t=1}^{T} (\zeta^*_{t-1})^2} \sqrt{\sum_{t=1}^{T} (\tilde{z}^*_{t-1} - \hat{\omega} \zeta^*_{t-1})^2}. \\
&= o^*_p(T^{1+\eta}) \left(1 + T^{-1-\eta} \sum_{t=1}^{T} (\zeta^*_{t-1})^2 \right).
\end{aligned}$$

As it will be shown next that $T^{-1-\eta} \sum_{t=1}^{T} (\zeta^*_{t-1})^2 = O_p(1)$, from the previous estimate and (A.34) it follows that $\sum_{t=1}^{\lfloor T\tau \rfloor} (z^*_{t-1})^2 = \hat{\omega}^2 \sum_{t=1}^{\lfloor T\tau \rfloor} (\zeta^*_{t-1})^2 + o^*_p(T^{1+\eta}) = \omega^2 \sum_{t=1}^{\lfloor T\tau \rfloor} (\zeta^*_{t-1})^2 + o^*_p(T^{1+\eta})$.

At the second step, by formally substituting $(T, \tilde{v}_t, u_t)$ with $(\lfloor T\tau \rfloor, v^*_t, 1)$ in the discussion of eq. (A.33), it can be concluded that $\sum_{t=1}^{\lfloor T\tau \rfloor} (\zeta^*_{t-1})^2 = \sum_{t=1}^{\lfloor T\tau \rfloor} \sum_{j=0}^{t-2} \varrho^{2j} \hat{v}^2_{t-j-1} + o^*_p(T^{1+\eta})$.

Then $\sum_{t=1}^{\lfloor T\tau \rfloor}(\zeta_{t-1}^*)^2 = \sum_{t=1}^{\lfloor T\tau \rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 + o_p^*(T^{1+\eta})$ by (A.18).

Finally, in the third step,

$$\begin{aligned}\sum_{t=1}^{\lfloor T\tau \rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 &= \sum_{t=1}^{\lfloor T\tau \rfloor}\sum_{j=0}^{t-2}\varrho^{2j}\sigma_{v,t-j-1}^2 + O(T^{2\eta}) \\ &= \sum_{t=1}^{\lfloor T\tau \rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}\right)\sigma_{vt}^2 + O(T^{2\eta}) = \frac{T^\eta}{2a}\sum_{t=1}^{\lfloor T\tau \rfloor}\sigma_{vt}^2 + O(T^{2\eta})\end{aligned}$$

by formally substituting (T, σ_{ut}^2) with $(\lfloor T\tau \rfloor, 1)$ in the proof of Lemma 2(d). The pointwise convergence $T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau \rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 \xrightarrow{p} \frac{1}{2a}[M_v](\tau)$ is now immediate. In conjunction with steps one and two it yields $T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau \rfloor}(z_{t-1}^*) \xrightarrow{p} \frac{\omega^2}{2a}[M_v](\tau)$. As the involved processes are increasing and the limit function is continuous, the convergence is in fact uniform.

The proof of part (c) is a matter of routine and we omit it for brevity. □

Additional References

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D Additional Monte Carlo Simulations

D.1 Single Predictor Case

Our baseline DGP for all simulation results below is the same as the one that was used in Section 5, i.e.,

$$y_t = \beta x_{t-1} + u_t, \tag{D.1}$$

where x_t satisfies the additive component model

$$x_t = \rho x_{t-1} + w_t, \tag{D.2}$$

$$w_t = \psi w_{t-1} + v_t. \tag{D.3}$$

The autoregressive process characterising the dynamics of the putative predictor, x_t, in (D.3) was initialised at $x_0 = 0$. Results are reported for a range of values of the autoregressive parameter ρ in (D.2) that cover stationary, persistent, and mildly explosive predictors; i.e., we consider $\rho = 1 - c/T$ with $c \in \{-5, -2.5, 0, 2.5, 5, 10, 25, 50, 75, 100, 125, 150, 200, 250\}$. The specific generation mechanism of the innovation vector $(u_t, v_t)'$ used to generated the artificial data for each specific DGP that will be considered is characterized in each Table, and four values for the innovations' correlation are considered, $\phi \in \{-0.95, -0.90, -0.50, 0\}$. For all cases results are reported for samples of length $T = 250$ and $T = 1000$.

Specifically, results based on the following DGPs will be reported:

- **DGP with positively and negatively autocorrelated predictor innovations:** To evaluate the impact on the test statistics when the autoregressive process of the predictor displays short-run dependence we generate data from (D.1) - (D.3) with $\psi \neq 0$. The two cases considered are:
  - **DGP3**: Positively autocorrelated predictor innovations ($\psi = 0.5$); see Tables D.1 - D.4.
  - **DGP4**: Negatively autocorrelated predictor innovations ($\psi = -0.5$); see Tables D.5 - D.8.

- **DGP with Unconditional Heteroskedasticity:** To evaluate the impact of unconditional heteroskedasticity a contemporaneous one-time break of equal magnitude in the variances of u_t and v_t is considered. Specifically, defining the variance of $(u_t, v_t)'$ as

$$\mathbf{\Sigma}_t = \begin{bmatrix} \sigma_{ut}^2 & \phi\sigma_{ut}\sigma_{vt} \\ \phi\sigma_{vt}\sigma_{ut} & \sigma_{vt}^2 \end{bmatrix}$$

  in DGP5 the simulation design considers an upward change in variance such that $\sigma_{ut}^2 = \sigma_{vt}^2 = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 4\mathbb{I}(t > \lfloor 0.5T \rfloor)$, and in DGP6 a downward change in variance is imposed, i.e., $\sigma_{ut}^2 = \sigma_{vt}^2 = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + \frac{1}{4}\mathbb{I}(t > \lfloor 0.5T \rfloor)$. Notice, therefore, that in both DGP5 and DGP6 the correlation between u_t and v_t does not display a break and is equal to ϕ throughout the sample. These experiments allow us to examine the impact of unconditional heteroskedasticity, both in isolation and in its interaction with ϕ, on the finite sample size of the tests. In both DGPs change in variance of a larger magnitude than we might expect to see in practice is imposed, but this serves to illustrate how the tests behave in the presence of a large change in unconditional volatility.

Hence, the two cases for which we provide results for are:

- **DGP5**: The innovations are characterised by an upward change in the unconditional variance; see Tables D.9 - D.12.
- **DGP6**: The innovations are characterised by a downward change in the unconditional variance; see Tables D.13 - D.16.

- **DGP with Conditional Heteroskedasticity - GARCH(1,1):** A further important feature of financial data is conditional heteroskedasticity. Hence, to evaluate the impact of this feature on the tests performance innovations $(u_t, v_t)'$ are generated to exhibit time-varying conditional second-order moments according to the design,

$$(u_t, v_t)' = \begin{bmatrix} \sigma_{1t} & 0 \\ 0 & \sigma_{2t} \end{bmatrix} \boldsymbol{\eta}_t; \quad E(\boldsymbol{\eta}_t) = \mathbf{0}, \quad E(\boldsymbol{\eta}_t \boldsymbol{\eta}_t') =: \boldsymbol{\Omega}_\phi = \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}$$

where $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})'$ is an i.i.d. vector drawn from either a bivariate Gaussian distribution or a (heavy-tailed) bivariate Student-t distribution with 5 degrees of freedom. The innovations' covariance matrix $\boldsymbol{\Omega}_\phi$ depends on the contemporaneous correlation coefficient ϕ, $\phi \in \{-0.95, -0.90, -0.50, 0\}$. The conditional variances $\{\sigma_{it}^2\}$ are driven by (normalised) stationary GARCH(1,1) processes $\sigma_{it}^2 = (1-\alpha-\beta)+\alpha e_{i,t-1}^2+\beta\sigma_{i,t-1}^2$, $i = 1, 2$ with $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$, such that $E(u_t^2) = E(v_t^2) = 1$. We consider $(\alpha, \beta) = (0.1, 0.85)$.

Hence, the two cases considered are:

- **DGP7**: GARCH(1,1) with Normal Innovations; see Tables D.17 - D.20.
- **DGP8**: GARCH(1,1) with Student-t distributed innovations with 5 degrees of freedom; see Tables D.21 - D.24.

- **DGP with Conditional Heteroskedasticity - GoGARCH(1,1):** In addition to the GARCH we also consider a GoGARCH characterisation of the conditional second moments of the innovations. Specifically, innovation vector $(u_t, v_t)'$ is generated as,

$$(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t, \tag{D.4}$$

where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z}$ is a 2×2 non-singular matrix, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2, i=1,2 are GARCH processes generated as $\sigma_{it}^2 = (1 - \alpha - \beta) + \alpha e_{i,t-1}^2 + \beta\sigma_{i,t-1}^2$, $i = 1, 2$ with $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$, such that $E(u_t^2) = E(v_t^2) = 1$. We consider $(\alpha, \beta) = (0.1, 0.85)$. Moreover, $\boldsymbol{\varepsilon}_t$ is either a vector of Gaussian innovations, $\boldsymbol{\varepsilon}_t \sim NIID(\mathbf{0}, diag(1, 1))$, or drawn from a bivariate Student-t distribution with 5 degrees of freedom, $\boldsymbol{\varepsilon}_t \sim iidt_5(\mathbf{0}, diag(1,1))$. The unconditional covariance matrix of $(u_t, v_t)'$, $\boldsymbol{\Sigma}$, is $\boldsymbol{\Sigma} = \mathbf{ZZ}' = \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}$; see, for instance, Boswijk and van der Weide (2011) for further details on the GoGARCH model.

Thus, also for the GoGARCH two cases are considered:

- **DGP9**: GoGARCH(1,1) with Normal innovations; see Tables D.25 - D.28.
- **DGP10**: GoGARCH(1,1) with Student-t distributed innovations with 5 degrees of freedom; see Tables D.29 - D.32.

- **DGP with Conditional Heteroskedasticity - Stochastic Volatility:** Finally, we also evaluate the tests when the innovations are generated from an autoregressive (AR) stochastic volatility process. The innovations $(u_t, v_t)'$ follow from a first-order AR stochastic volatility process as $(u_t = e_{1t}\texttt{exp}(h_{1t}), v_t = e_{2t}\texttt{exp}(h_{2t}))'$, and

$$h_{it} = \lambda h_{i,t-1} + 0.5\xi_{it} \tag{D.5}$$

with $(\xi_{it}, e_{it})' \sim NIID(0, diag(\sigma_\xi^2, 1))$, independent across i = 1,2. Results are reported for $(\lambda, \sigma_\xi)' = (0.951, 0.314)'$.

  - **DGP 11**: Stochastic Volatility; see Tables D.33 - D.36.

Table D.1: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP3 (Positive Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.95; \quad -0.95 \quad 1]$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				**5%**				**10%**			
Left-sided tests - $T = 250$												
-5	0.009	0.001	0.001	0.000	0.045	0.003	0.005	0.004	0.097	0.013	0.014	0.013
-2.5	0.006	0.000	0.000	0.000	0.042	0.000	0.001	0.001	0.105	0.002	0.002	0.002
0	0.013	0.000	0.000	0.000	0.040	0.001	0.001	0.001	0.064	0.002	0.002	0.002
2.5	0.021	0.000	0.001	0.001	0.059	0.005	0.005	0.004	0.094	0.012	0.012	0.011
5	0.023	0.001	0.001	0.001	0.068	0.010	0.011	0.010	0.110	0.025	0.026	0.024
10	0.019	0.003	0.003	0.002	0.065	0.019	0.019	0.018	0.113	0.041	0.042	0.041
25	0.016	0.006	0.006	0.005	0.056	0.027	0.029	0.027	0.107	0.056	0.056	0.057
50	0.015	0.007	0.007	0.007	0.055	0.032	0.033	0.032	0.105	0.068	0.070	0.068
75	0.012	0.007	0.007	0.007	0.055	0.036	0.038	0.036	0.106	0.073	0.076	0.073
100	0.011	0.006	0.007	0.007	0.056	0.038	0.040	0.038	0.105	0.077	0.079	0.076
125	0.011	0.007	0.007	0.007	0.055	0.039	0.040	0.038	0.103	0.079	0.080	0.079
150	0.011	0.008	0.008	0.007	0.055	0.038	0.040	0.038	0.104	0.080	0.083	0.082
200	0.011	0.007	0.009	0.008	0.054	0.038	0.040	0.040	0.108	0.085	0.087	0.085
250	0.011	0.007	0.008	0.007	0.053	0.040	0.042	0.041	0.107	0.087	0.091	0.087
Right-sided tests - $T = 250$												
-5	0.011	0.018	0.022	0.019	0.045	0.075	0.082	0.077	0.090	0.155	0.161	0.155
-2.5	0.009	0.017	0.019	0.018	0.044	0.102	0.109	0.104	0.093	0.256	0.257	0.258
0	0.012	0.021	0.027	0.024	0.056	0.113	0.122	0.117	0.115	0.242	0.250	0.244
2.5	0.013	0.023	0.030	0.026	0.064	0.118	0.124	0.120	0.128	0.234	0.244	0.239
5	0.013	0.024	0.029	0.026	0.065	0.114	0.121	0.115	0.128	0.217	0.225	0.219
10	0.013	0.024	0.027	0.024	0.063	0.101	0.110	0.105	0.123	0.189	0.194	0.192
25	0.012	0.020	0.022	0.021	0.058	0.083	0.089	0.086	0.109	0.153	0.160	0.156
50	0.010	0.016	0.018	0.016	0.055	0.072	0.076	0.074	0.106	0.136	0.140	0.138
75	0.010	0.015	0.018	0.015	0.053	0.068	0.071	0.070	0.104	0.132	0.137	0.135
100	0.010	0.015	0.016	0.015	0.053	0.066	0.069	0.067	0.105	0.129	0.131	0.129
125	0.012	0.014	0.016	0.014	0.052	0.064	0.069	0.066	0.103	0.126	0.129	0.127
150	0.011	0.014	0.015	0.014	0.053	0.064	0.068	0.065	0.103	0.122	0.127	0.123
200	0.011	0.014	0.016	0.014	0.051	0.060	0.066	0.062	0.102	0.120	0.123	0.120
250	0.011	0.012	0.014	0.013	0.054	0.060	0.063	0.060	0.103	0.115	0.120	0.116
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.013	0.012	0.046	0.040	0.046	0.041	0.092	0.078	0.087	0.081
-2.5	0.009	0.009	0.011	0.010	0.040	0.047	0.054	0.049	0.087	0.102	0.110	0.104
0	0.011	0.012	0.014	0.013	0.048	0.054	0.060	0.057	0.098	0.112	0.123	0.117
2.5	0.011	0.011	0.015	0.014	0.054	0.059	0.066	0.062	0.109	0.123	0.129	0.124
5	0.012	0.013	0.015	0.013	0.056	0.062	0.070	0.063	0.110	0.123	0.132	0.125
10	0.010	0.012	0.016	0.013	0.054	0.062	0.069	0.062	0.110	0.120	0.129	0.123
25	0.010	0.011	0.013	0.012	0.055	0.059	0.065	0.060	0.107	0.110	0.117	0.113
50	0.010	0.010	0.012	0.012	0.054	0.054	0.060	0.056	0.102	0.103	0.109	0.106
75	0.009	0.009	0.012	0.011	0.052	0.052	0.058	0.054	0.103	0.102	0.109	0.106
100	0.010	0.011	0.012	0.011	0.051	0.050	0.057	0.052	0.103	0.102	0.109	0.105
125	0.012	0.011	0.012	0.012	0.051	0.051	0.056	0.051	0.102	0.104	0.109	0.104
150	0.012	0.010	0.012	0.012	0.050	0.049	0.054	0.050	0.105	0.101	0.107	0.103
200	0.010	0.010	0.012	0.011	0.049	0.049	0.054	0.049	0.102	0.099	0.106	0.102
250	0.009	0.010	0.011	0.011	0.051	0.050	0.053	0.051	0.104	0.099	0.105	0.101

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				**5%**				**10%**			
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.045	0.003	0.003	0.003	0.093	0.011	0.011	0.011
-2.5	0.005	0.000	0.000	0.000	0.044	0.000	0.000	0.000	0.108	0.001	0.001	0.001
0	0.015	0.000	0.000	0.000	0.042	0.001	0.001	0.001	0.068	0.003	0.003	0.004
2.5	0.023	0.001	0.001	0.001	0.059	0.006	0.006	0.006	0.096	0.015	0.016	0.015
5	0.024	0.002	0.002	0.002	0.068	0.013	0.014	0.013	0.108	0.027	0.027	0.028
10	0.021	0.003	0.004	0.004	0.064	0.021	0.022	0.022	0.114	0.044	0.043	0.043
25	0.016	0.007	0.007	0.008	0.061	0.031	0.030	0.029	0.108	0.063	0.064	0.063
50	0.014	0.007	0.008	0.007	0.058	0.035	0.035	0.034	0.107	0.073	0.074	0.072
75	0.013	0.008	0.008	0.008	0.057	0.039	0.038	0.038	0.105	0.078	0.078	0.078
100	0.013	0.008	0.008	0.008	0.054	0.040	0.040	0.040	0.106	0.079	0.080	0.079
125	0.012	0.008	0.008	0.008	0.052	0.040	0.040	0.038	0.106	0.082	0.082	0.081
150	0.012	0.009	0.009	0.008	0.052	0.040	0.042	0.039	0.106	0.083	0.083	0.081
200	0.012	0.008	0.008	0.008	0.052	0.041	0.041	0.041	0.104	0.084	0.085	0.084
250	0.012	0.009	0.008	0.008	0.052	0.043	0.042	0.041	0.104	0.085	0.085	0.084
Right-sided tests - $T = 1000$												
-5	0.007	0.013	0.015	0.014	0.040	0.066	0.068	0.066	0.084	0.143	0.144	0.140
-2.5	0.008	0.018	0.017	0.016	0.041	0.099	0.099	0.098	0.087	0.241	0.238	0.238
0	0.009	0.020	0.021	0.019	0.052	0.108	0.109	0.107	0.109	0.230	0.232	0.231
2.5	0.011	0.022	0.022	0.021	0.060	0.111	0.113	0.112	0.119	0.222	0.225	0.225
5	0.011	0.020	0.022	0.021	0.062	0.112	0.111	0.111	0.123	0.209	0.209	0.209
10	0.011	0.020	0.019	0.018	0.062	0.099	0.101	0.100	0.118	0.184	0.185	0.184
25	0.011	0.017	0.017	0.017	0.055	0.083	0.083	0.083	0.110	0.152	0.155	0.154
50	0.010	0.015	0.014	0.014	0.052	0.071	0.072	0.070	0.105	0.139	0.139	0.140
75	0.010	0.014	0.014	0.014	0.050	0.066	0.066	0.065	0.103	0.129	0.130	0.129
100	0.010	0.014	0.014	0.013	0.051	0.063	0.062	0.062	0.102	0.126	0.125	0.125
125	0.009	0.013	0.013	0.013	0.048	0.061	0.062	0.061	0.098	0.121	0.121	0.120
150	0.009	0.013	0.013	0.012	0.050	0.063	0.063	0.063	0.099	0.118	0.116	0.117
200	0.010	0.011	0.012	0.011	0.053	0.062	0.062	0.062	0.099	0.116	0.117	0.116
250	0.009	0.012	0.011	0.010	0.053	0.061	0.061	0.061	0.101	0.115	0.116	0.116
Two-sided tests - $T = 1000$												
-5	0.006	0.007	0.007	0.007	0.040	0.033	0.033	0.033	0.086	0.069	0.071	0.069
-2.5	0.007	0.008	0.008	0.008	0.037	0.042	0.045	0.044	0.082	0.096	0.099	0.098
0	0.008	0.010	0.010	0.009	0.044	0.052	0.053	0.051	0.091	0.110	0.110	0.108
2.5	0.009	0.010	0.011	0.010	0.052	0.061	0.060	0.059	0.104	0.117	0.120	0.117
5	0.009	0.010	0.011	0.010	0.054	0.061	0.062	0.061	0.109	0.124	0.125	0.124
10	0.010	0.011	0.011	0.011	0.058	0.062	0.065	0.064	0.110	0.120	0.123	0.122
25	0.011	0.011	0.011	0.011	0.053	0.059	0.059	0.058	0.107	0.112	0.113	0.112
50	0.011	0.012	0.011	0.011	0.052	0.055	0.055	0.054	0.100	0.108	0.107	0.104
75	0.010	0.011	0.010	0.010	0.051	0.052	0.053	0.053	0.102	0.102	0.104	0.103
100	0.010	0.010	0.010	0.010	0.050	0.051	0.052	0.051	0.100	0.101	0.102	0.102
125	0.009	0.010	0.010	0.010	0.051	0.051	0.053	0.051	0.098	0.101	0.102	0.100
150	0.010	0.011	0.011	0.010	0.051	0.050	0.051	0.050	0.101	0.103	0.105	0.102
200	0.011	0.011	0.010	0.010	0.051	0.051	0.051	0.051	0.101	0.101	0.103	0.102
250	0.010	0.011	0.011	0.010	0.051	0.052	0.053	0.052	0.103	0.103	0.103	0.102

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.2: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP3 (Positive Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0,\ \rho = 1 - c/T, \psi = 0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.90; \quad -0.90 \quad 1]$.

Left-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
-5	0.009	0.000	0.001	0.000	0.046	0.005	0.006	0.005	0.097	0.015	0.017	0.014
-2.5	0.007	0.000	0.000	0.000	0.045	0.001	0.001	0.001	0.106	0.002	0.002	0.002
0	0.011	0.000	0.000	0.000	0.039	0.001	0.001	0.001	0.062	0.003	0.003	0.003
2.5	0.019	0.001	0.001	0.001	0.056	0.005	0.005	0.005	0.093	0.013	0.013	0.013
5	0.022	0.001	0.002	0.001	0.065	0.011	0.011	0.010	0.108	0.027	0.028	0.026
10	0.018	0.003	0.003	0.003	0.064	0.018	0.020	0.019	0.114	0.042	0.043	0.043
25	0.015	0.006	0.006	0.005	0.056	0.028	0.029	0.028	0.105	0.058	0.060	0.058
50	0.014	0.007	0.007	0.006	0.053	0.032	0.035	0.032	0.104	0.067	0.068	0.068
75	0.013	0.007	0.008	0.007	0.054	0.036	0.037	0.035	0.105	0.073	0.075	0.072
100	0.011	0.006	0.007	0.007	0.053	0.037	0.040	0.038	0.104	0.075	0.076	0.075
125	0.010	0.007	0.007	0.006	0.053	0.039	0.040	0.039	0.103	0.079	0.080	0.077
150	0.010	0.007	0.007	0.007	0.052	0.038	0.040	0.038	0.105	0.081	0.082	0.080
200	0.011	0.007	0.009	0.008	0.054	0.039	0.041	0.040	0.104	0.083	0.085	0.084
250	0.011	0.007	0.008	0.007	0.052	0.040	0.043	0.041	0.107	0.086	0.090	0.088

Right-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
-5	0.010	0.016	0.021	0.019	0.044	0.076	0.084	0.077	0.092	0.153	0.161	0.154
-2.5	0.010	0.017	0.020	0.018	0.043	0.106	0.112	0.106	0.095	0.249	0.254	0.251
0	0.012	0.022	0.027	0.024	0.055	0.111	0.117	0.114	0.117	0.238	0.246	0.241
2.5	0.014	0.024	0.029	0.024	0.063	0.113	0.120	0.118	0.127	0.232	0.237	0.232
5	0.014	0.025	0.028	0.025	0.063	0.110	0.115	0.112	0.127	0.211	0.219	0.213
10	0.014	0.023	0.026	0.024	0.062	0.099	0.105	0.101	0.121	0.184	0.189	0.187
25	0.011	0.018	0.022	0.020	0.057	0.081	0.086	0.083	0.110	0.154	0.157	0.153
50	0.011	0.015	0.017	0.015	0.054	0.073	0.078	0.076	0.106	0.136	0.141	0.137
75	0.011	0.014	0.017	0.015	0.055	0.067	0.072	0.070	0.106	0.131	0.135	0.133
100	0.010	0.014	0.017	0.015	0.054	0.067	0.070	0.067	0.105	0.125	0.129	0.127
125	0.011	0.014	0.017	0.015	0.054	0.065	0.070	0.065	0.105	0.124	0.127	0.125
150	0.010	0.014	0.017	0.014	0.053	0.063	0.065	0.063	0.103	0.121	0.126	0.123
200	0.011	0.014	0.015	0.013	0.053	0.060	0.063	0.062	0.102	0.119	0.123	0.120
250	0.010	0.012	0.014	0.012	0.054	0.058	0.062	0.059	0.104	0.116	0.118	0.113

Two-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
-5	0.010	0.010	0.013	0.011	0.045	0.039	0.046	0.041	0.094	0.080	0.089	0.081
-2.5	0.009	0.010	0.012	0.011	0.040	0.045	0.051	0.048	0.088	0.104	0.112	0.106
0	0.010	0.011	0.014	0.012	0.048	0.053	0.060	0.055	0.097	0.110	0.118	0.115
2.5	0.012	0.013	0.016	0.014	0.054	0.058	0.065	0.061	0.108	0.118	0.125	0.123
5	0.012	0.012	0.016	0.014	0.055	0.060	0.067	0.063	0.108	0.120	0.126	0.122
10	0.012	0.013	0.015	0.014	0.054	0.061	0.066	0.061	0.107	0.117	0.125	0.120
25	0.010	0.011	0.014	0.011	0.056	0.058	0.063	0.061	0.104	0.107	0.115	0.111
50	0.010	0.010	0.012	0.011	0.054	0.054	0.059	0.056	0.104	0.105	0.113	0.108
75	0.010	0.010	0.011	0.010	0.052	0.052	0.057	0.053	0.104	0.102	0.110	0.105
100	0.010	0.009	0.012	0.010	0.048	0.050	0.055	0.051	0.106	0.105	0.110	0.105
125	0.010	0.009	0.012	0.011	0.049	0.049	0.055	0.050	0.105	0.104	0.110	0.104
150	0.011	0.010	0.012	0.012	0.050	0.049	0.054	0.050	0.100	0.100	0.105	0.101
200	0.010	0.009	0.012	0.011	0.050	0.048	0.052	0.050	0.101	0.099	0.104	0.101
250	0.010	0.009	0.011	0.010	0.052	0.048	0.054	0.052	0.102	0.097	0.105	0.101

Left-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
-5	0.008	0.000	0.000	0.000	0.047	0.003	0.003	0.003	0.096	0.013	0.014	0.014
-2.5	0.006	0.000	0.000	0.000	0.047	0.000	0.000	0.000	0.108	0.001	0.001	0.001
0	0.013	0.000	0.000	0.000	0.040	0.002	0.002	0.001	0.066	0.004	0.004	0.004
2.5	0.021	0.001	0.001	0.001	0.058	0.007	0.007	0.007	0.093	0.017	0.017	0.017
5	0.022	0.003	0.002	0.002	0.064	0.014	0.014	0.014	0.105	0.030	0.031	0.030
10	0.019	0.004	0.004	0.004	0.063	0.022	0.022	0.021	0.109	0.046	0.045	0.046
25	0.016	0.007	0.007	0.007	0.060	0.031	0.032	0.031	0.108	0.066	0.064	0.064
50	0.014	0.007	0.008	0.008	0.059	0.037	0.037	0.036	0.108	0.073	0.075	0.073
75	0.013	0.008	0.008	0.007	0.057	0.038	0.039	0.038	0.107	0.080	0.080	0.080
100	0.012	0.008	0.009	0.009	0.056	0.040	0.040	0.039	0.106	0.082	0.082	0.081
125	0.013	0.008	0.008	0.009	0.055	0.041	0.041	0.040	0.106	0.083	0.084	0.083
150	0.012	0.009	0.009	0.009	0.053	0.040	0.041	0.040	0.105	0.085	0.085	0.085
200	0.012	0.009	0.009	0.009	0.053	0.041	0.041	0.041	0.104	0.087	0.086	0.086
250	0.012	0.009	0.009	0.009	0.052	0.043	0.043	0.042	0.105	0.088	0.088	0.087

Right-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
-5	0.007	0.013	0.015	0.014	0.041	0.069	0.069	0.066	0.084	0.145	0.144	0.143
-2.5	0.008	0.016	0.017	0.016	0.041	0.098	0.098	0.097	0.091	0.236	0.235	0.236
0	0.010	0.021	0.021	0.020	0.051	0.104	0.105	0.102	0.107	0.228	0.225	0.225
2.5	0.011	0.022	0.022	0.022	0.059	0.111	0.111	0.109	0.120	0.216	0.218	0.216
5	0.011	0.022	0.021	0.020	0.062	0.105	0.108	0.105	0.119	0.203	0.202	0.201
10	0.010	0.018	0.018	0.017	0.061	0.099	0.099	0.098	0.116	0.179	0.178	0.177
25	0.010	0.015	0.015	0.015	0.055	0.083	0.083	0.082	0.109	0.150	0.151	0.150
50	0.010	0.014	0.014	0.014	0.051	0.069	0.071	0.071	0.104	0.136	0.138	0.136
75	0.010	0.013	0.014	0.014	0.051	0.065	0.066	0.066	0.099	0.128	0.128	0.129
100	0.011	0.013	0.014	0.013	0.051	0.061	0.061	0.061	0.099	0.122	0.125	0.123
125	0.011	0.013	0.013	0.013	0.051	0.060	0.061	0.061	0.099	0.119	0.120	0.120
150	0.010	0.012	0.013	0.012	0.049	0.059	0.061	0.061	0.098	0.116	0.117	0.117
200	0.010	0.012	0.012	0.012	0.050	0.060	0.060	0.060	0.100	0.116	0.116	0.117
250	0.009	0.012	0.012	0.011	0.050	0.060	0.059	0.059	0.100	0.114	0.116	0.117

Two-sided tests- $T = 1000$

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
-5	0.007	0.006	0.007	0.006	0.041	0.034	0.035	0.034	0.087	0.071	0.072	0.070
-2.5	0.007	0.009	0.009	0.008	0.038	0.044	0.045	0.043	0.084	0.096	0.099	0.098
0	0.009	0.010	0.011	0.011	0.044	0.051	0.052	0.051	0.091	0.105	0.106	0.104
2.5	0.009	0.011	0.010	0.010	0.051	0.059	0.060	0.059	0.102	0.117	0.117	0.116
5	0.010	0.010	0.011	0.010	0.052	0.058	0.061	0.061	0.106	0.119	0.122	0.119
10	0.010	0.011	0.011	0.010	0.056	0.060	0.062	0.063	0.109	0.118	0.121	0.120
25	0.010	0.011	0.011	0.010	0.053	0.057	0.057	0.057	0.105	0.113	0.115	0.113
50	0.010	0.011	0.011	0.011	0.051	0.053	0.055	0.054	0.103	0.105	0.107	0.106
75	0.010	0.011	0.010	0.010	0.051	0.053	0.054	0.053	0.101	0.104	0.105	0.104
100	0.010	0.011	0.010	0.010	0.052	0.051	0.054	0.053	0.099	0.101	0.102	0.100
125	0.010	0.010	0.010	0.010	0.052	0.052	0.054	0.053	0.100	0.101	0.102	0.102
150	0.009	0.010	0.011	0.010	0.052	0.052	0.052	0.051	0.100	0.100	0.102	0.101
200	0.010	0.010	0.010	0.010	0.051	0.051	0.052	0.051	0.100	0.100	0.101	0.101
250	0.010	0.010	0.010	0.009	0.050	0.052	0.052	0.051	0.101	0.102	0.102	0.101

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.3: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP3 (Positive Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.50; \quad -0.50 \quad 1]$.

	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
c	1%				5%				10%			
Left-sided tests - $T = 250$												
-5	0.010	0.003	0.005	0.002	0.053	0.020	0.024	0.020	0.104	0.047	0.053	0.048
-2.5	0.010	0.000	0.001	0.000	0.050	0.006	0.007	0.005	0.101	0.017	0.017	0.017
0	0.005	0.001	0.001	0.001	0.030	0.006	0.006	0.005	0.061	0.017	0.017	0.017
2.5	0.010	0.002	0.002	0.002	0.043	0.015	0.016	0.015	0.086	0.035	0.036	0.036
5	0.012	0.004	0.004	0.004	0.048	0.022	0.023	0.023	0.097	0.049	0.050	0.049
10	0.012	0.006	0.006	0.005	0.052	0.029	0.030	0.030	0.098	0.062	0.063	0.061
25	0.011	0.007	0.008	0.007	0.051	0.037	0.038	0.037	0.100	0.074	0.076	0.074
50	0.011	0.007	0.008	0.007	0.050	0.038	0.039	0.037	0.099	0.081	0.082	0.082
75	0.009	0.008	0.008	0.007	0.048	0.036	0.038	0.038	0.097	0.082	0.085	0.082
100	0.009	0.007	0.008	0.007	0.050	0.039	0.041	0.040	0.099	0.083	0.086	0.085
125	0.008	0.006	0.007	0.006	0.049	0.040	0.042	0.040	0.097	0.084	0.086	0.085
150	0.008	0.006	0.007	0.006	0.048	0.040	0.043	0.042	0.096	0.084	0.087	0.085
200	0.009	0.008	0.008	0.007	0.048	0.043	0.044	0.043	0.098	0.088	0.089	0.087
250	0.011	0.009	0.009	0.008	0.051	0.045	0.046	0.045	0.097	0.085	0.088	0.089
Right-sided tests - $T = 250$												
-5	0.010	0.016	0.021	0.016	0.046	0.072	0.081	0.072	0.097	0.144	0.154	0.146
-2.5	0.012	0.022	0.028	0.022	0.053	0.104	0.110	0.106	0.107	0.201	0.208	0.201
0	0.014	0.023	0.025	0.022	0.061	0.100	0.104	0.100	0.123	0.195	0.203	0.195
2.5	0.013	0.021	0.024	0.022	0.060	0.094	0.100	0.094	0.118	0.174	0.178	0.175
5	0.014	0.019	0.021	0.019	0.061	0.084	0.088	0.086	0.115	0.159	0.163	0.158
10	0.014	0.018	0.019	0.018	0.056	0.077	0.079	0.077	0.109	0.147	0.149	0.146
25	0.011	0.015	0.016	0.015	0.054	0.069	0.071	0.068	0.102	0.129	0.134	0.130
50	0.011	0.013	0.015	0.014	0.052	0.061	0.066	0.064	0.106	0.123	0.127	0.124
75	0.010	0.013	0.014	0.014	0.052	0.061	0.064	0.062	0.107	0.120	0.124	0.123
100	0.011	0.013	0.014	0.013	0.054	0.063	0.066	0.064	0.106	0.119	0.123	0.119
125	0.010	0.013	0.014	0.012	0.055	0.061	0.067	0.064	0.107	0.118	0.122	0.120
150	0.010	0.012	0.013	0.012	0.055	0.061	0.065	0.061	0.108	0.118	0.121	0.118
200	0.010	0.011	0.012	0.011	0.055	0.059	0.062	0.060	0.110	0.117	0.120	0.117
250	0.009	0.011	0.012	0.011	0.056	0.058	0.061	0.059	0.107	0.114	0.118	0.114
Two-sided tests - $T = 250$												
-5	0.010	0.008	0.016	0.010	0.048	0.044	0.056	0.045	0.098	0.093	0.105	0.092
-2.5	0.011	0.012	0.016	0.012	0.048	0.054	0.062	0.053	0.100	0.109	0.118	0.111
0	0.012	0.012	0.015	0.013	0.049	0.054	0.058	0.053	0.097	0.107	0.110	0.106
2.5	0.012	0.012	0.014	0.013	0.051	0.053	0.058	0.056	0.104	0.108	0.115	0.109
5	0.012	0.013	0.014	0.014	0.053	0.056	0.061	0.059	0.103	0.107	0.112	0.108
10	0.013	0.012	0.014	0.013	0.051	0.053	0.058	0.055	0.102	0.105	0.110	0.106
25	0.011	0.011	0.013	0.012	0.052	0.052	0.056	0.055	0.102	0.104	0.109	0.105
50	0.010	0.009	0.011	0.010	0.050	0.050	0.055	0.052	0.097	0.099	0.105	0.100
75	0.009	0.010	0.010	0.010	0.049	0.051	0.056	0.051	0.097	0.096	0.102	0.100
100	0.008	0.009	0.011	0.009	0.048	0.050	0.054	0.050	0.103	0.101	0.107	0.104
125	0.009	0.008	0.011	0.010	0.049	0.047	0.053	0.050	0.101	0.101	0.109	0.104
150	0.008	0.008	0.009	0.009	0.051	0.049	0.054	0.051	0.102	0.101	0.108	0.103
200	0.008	0.009	0.011	0.009	0.050	0.048	0.053	0.052	0.103	0.102	0.106	0.103
250	0.008	0.009	0.011	0.009	0.050	0.049	0.055	0.050	0.105	0.102	0.107	0.104

	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
c	1%				5%				10%			
Left-sided tests - $T = 1000$												
-5	0.009	0.002	0.003	0.002	0.050	0.018	0.018	0.019	0.098	0.045	0.045	0.044
-2.5	0.009	0.000	0.000	0.000	0.047	0.004	0.004	0.004	0.098	0.015	0.015	0.016
0	0.007	0.001	0.001	0.001	0.032	0.008	0.008	0.009	0.065	0.021	0.021	0.021
2.5	0.013	0.002	0.002	0.002	0.048	0.018	0.018	0.018	0.091	0.043	0.043	0.043
5	0.013	0.004	0.003	0.003	0.054	0.024	0.025	0.024	0.101	0.055	0.055	0.054
10	0.013	0.006	0.005	0.005	0.054	0.030	0.030	0.030	0.105	0.066	0.067	0.067
25	0.011	0.008	0.008	0.007	0.056	0.040	0.040	0.040	0.104	0.077	0.078	0.078
50	0.012	0.009	0.009	0.009	0.053	0.042	0.041	0.042	0.104	0.084	0.084	0.084
75	0.013	0.010	0.009	0.009	0.053	0.044	0.045	0.044	0.105	0.088	0.088	0.087
100	0.013	0.010	0.010	0.010	0.052	0.045	0.045	0.045	0.104	0.086	0.087	0.088
125	0.013	0.012	0.011	0.011	0.052	0.044	0.044	0.044	0.103	0.089	0.090	0.091
150	0.014	0.011	0.012	0.011	0.051	0.046	0.045	0.045	0.102	0.091	0.092	0.091
200	0.012	0.011	0.011	0.011	0.051	0.046	0.046	0.046	0.101	0.093	0.093	0.092
250	0.012	0.011	0.010	0.010	0.051	0.045	0.046	0.045	0.105	0.094	0.094	0.095
Right-sided tests - $T = 1000$												
-5	0.009	0.014	0.015	0.013	0.046	0.072	0.073	0.072	0.097	0.141	0.143	0.141
-2.5	0.009	0.018	0.017	0.016	0.049	0.098	0.099	0.097	0.102	0.195	0.195	0.193
0	0.012	0.019	0.020	0.019	0.058	0.095	0.094	0.093	0.115	0.193	0.192	0.191
2.5	0.013	0.021	0.020	0.020	0.059	0.088	0.089	0.088	0.115	0.168	0.169	0.169
5	0.012	0.020	0.019	0.018	0.056	0.083	0.083	0.082	0.109	0.156	0.156	0.156
10	0.010	0.016	0.016	0.016	0.054	0.076	0.074	0.073	0.107	0.139	0.140	0.140
25	0.009	0.014	0.014	0.013	0.051	0.064	0.064	0.063	0.101	0.125	0.124	0.124
50	0.011	0.014	0.014	0.013	0.048	0.058	0.058	0.059	0.097	0.117	0.117	0.116
75	0.011	0.013	0.013	0.013	0.049	0.055	0.057	0.057	0.098	0.113	0.113	0.113
100	0.010	0.011	0.012	0.011	0.049	0.056	0.057	0.057	0.098	0.109	0.111	0.110
125	0.010	0.013	0.013	0.012	0.050	0.055	0.055	0.056	0.096	0.109	0.110	0.109
150	0.011	0.013	0.013	0.013	0.050	0.056	0.057	0.056	0.097	0.109	0.107	0.108
200	0.011	0.012	0.012	0.012	0.048	0.054	0.055	0.055	0.099	0.107	0.108	0.108
250	0.012	0.013	0.012	0.012	0.050	0.055	0.054	0.054	0.101	0.109	0.109	0.109
Two-sided tests- $T = 1000$												
-5	0.009	0.008	0.008	0.007	0.045	0.042	0.044	0.041	0.096	0.089	0.091	0.091
-2.5	0.008	0.008	0.008	0.008	0.045	0.048	0.048	0.047	0.095	0.101	0.103	0.101
0	0.010	0.010	0.010	0.010	0.045	0.050	0.051	0.049	0.096	0.102	0.103	0.102
2.5	0.011	0.011	0.011	0.011	0.051	0.052	0.054	0.053	0.101	0.107	0.107	0.106
5	0.012	0.012	0.011	0.011	0.051	0.054	0.056	0.055	0.103	0.106	0.108	0.106
10	0.010	0.011	0.010	0.010	0.051	0.055	0.055	0.053	0.101	0.105	0.104	0.103
25	0.009	0.011	0.010	0.010	0.048	0.050	0.051	0.050	0.101	0.102	0.104	0.102
50	0.011	0.011	0.011	0.011	0.051	0.051	0.052	0.052	0.099	0.099	0.099	0.100
75	0.011	0.012	0.011	0.011	0.053	0.052	0.052	0.053	0.100	0.099	0.101	0.101
100	0.012	0.012	0.011	0.012	0.053	0.053	0.055	0.054	0.102	0.101	0.101	0.102
125	0.011	0.012	0.012	0.012	0.052	0.052	0.053	0.053	0.101	0.098	0.099	0.100
150	0.012	0.011	0.012	0.012	0.052	0.052	0.052	0.052	0.100	0.101	0.102	0.101
200	0.012	0.012	0.013	0.012	0.052	0.053	0.054	0.053	0.099	0.099	0.101	0.100
250	0.011	0.012	0.012	0.012	0.054	0.052	0.055	0.054	0.099	0.099	0.100	0.099

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.4: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP3 (Positive Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad 0; \quad 0 \quad 1]$.

	1%				5%				10%			
c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.010	0.011	0.017	0.010	0.053	0.051	0.060	0.052	0.102	0.101	0.110	0.102
-2.5	0.011	0.011	0.015	0.011	0.052	0.052	0.059	0.050	0.104	0.103	0.111	0.103
0	0.010	0.010	0.013	0.011	0.049	0.049	0.054	0.049	0.100	0.100	0.104	0.099
2.5	0.009	0.009	0.010	0.010	0.051	0.049	0.053	0.051	0.100	0.099	0.102	0.100
5	0.010	0.010	0.011	0.010	0.051	0.050	0.054	0.050	0.098	0.095	0.099	0.096
10	0.011	0.012	0.012	0.011	0.051	0.051	0.054	0.051	0.099	0.099	0.102	0.101
25	0.011	0.010	0.012	0.010	0.050	0.052	0.052	0.051	0.100	0.099	0.102	0.101
50	0.010	0.009	0.011	0.010	0.051	0.052	0.053	0.052	0.097	0.097	0.099	0.097
75	0.009	0.009	0.011	0.009	0.049	0.048	0.051	0.049	0.098	0.098	0.102	0.099
100	0.009	0.009	0.010	0.010	0.049	0.047	0.049	0.048	0.098	0.098	0.101	0.098
125	0.009	0.009	0.010	0.009	0.049	0.046	0.050	0.048	0.097	0.095	0.097	0.098
150	0.009	0.008	0.009	0.009	0.048	0.047	0.050	0.048	0.097	0.095	0.097	0.097
200	0.009	0.010	0.010	0.010	0.048	0.046	0.049	0.046	0.098	0.096	0.098	0.097
250	0.010	0.010	0.011	0.011	0.048	0.047	0.049	0.048	0.095	0.095	0.097	0.096
Right-sided tests - $T = 250$												
-5	0.012	0.012	0.017	0.012	0.051	0.051	0.060	0.050	0.101	0.101	0.110	0.099
-2.5	0.010	0.011	0.015	0.010	0.053	0.052	0.058	0.051	0.099	0.099	0.104	0.097
0	0.011	0.012	0.014	0.011	0.050	0.050	0.053	0.050	0.098	0.100	0.105	0.099
2.5	0.011	0.011	0.012	0.011	0.052	0.051	0.054	0.052	0.102	0.101	0.104	0.102
5	0.011	0.012	0.012	0.011	0.052	0.050	0.053	0.051	0.102	0.101	0.105	0.102
10	0.011	0.011	0.011	0.011	0.052	0.050	0.052	0.051	0.104	0.103	0.104	0.103
25	0.011	0.011	0.012	0.011	0.052	0.052	0.055	0.051	0.102	0.100	0.104	0.102
50	0.011	0.011	0.013	0.012	0.052	0.052	0.055	0.053	0.107	0.104	0.108	0.107
75	0.010	0.012	0.012	0.011	0.055	0.052	0.056	0.054	0.106	0.104	0.109	0.105
100	0.011	0.011	0.011	0.011	0.055	0.054	0.058	0.054	0.105	0.106	0.107	0.105
125	0.011	0.010	0.011	0.010	0.055	0.055	0.058	0.055	0.104	0.103	0.108	0.104
150	0.009	0.010	0.011	0.010	0.054	0.055	0.057	0.055	0.104	0.104	0.108	0.104
200	0.010	0.010	0.011	0.011	0.055	0.054	0.058	0.055	0.107	0.105	0.108	0.105
250	0.011	0.010	0.012	0.011	0.053	0.053	0.056	0.053	0.103	0.101	0.106	0.103
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.020	0.012	0.051	0.051	0.065	0.052	0.102	0.102	0.120	0.103
-2.5	0.011	0.012	0.018	0.011	0.051	0.051	0.067	0.052	0.101	0.103	0.116	0.101
0	0.011	0.011	0.015	0.010	0.051	0.053	0.059	0.052	0.096	0.098	0.107	0.098
2.5	0.010	0.010	0.012	0.010	0.050	0.053	0.057	0.051	0.100	0.101	0.107	0.102
5	0.011	0.010	0.013	0.011	0.050	0.051	0.055	0.052	0.100	0.101	0.107	0.101
10	0.011	0.011	0.013	0.011	0.050	0.049	0.053	0.052	0.101	0.100	0.106	0.102
25	0.012	0.011	0.013	0.012	0.052	0.052	0.056	0.053	0.102	0.102	0.107	0.102
50	0.011	0.011	0.013	0.011	0.051	0.050	0.056	0.054	0.104	0.104	0.108	0.105
75	0.010	0.009	0.012	0.010	0.052	0.049	0.055	0.053	0.103	0.100	0.106	0.104
100	0.010	0.010	0.011	0.010	0.050	0.049	0.053	0.052	0.103	0.100	0.107	0.102
125	0.009	0.008	0.011	0.010	0.050	0.049	0.053	0.051	0.104	0.101	0.108	0.103
150	0.008	0.008	0.010	0.009	0.050	0.051	0.055	0.052	0.102	0.102	0.106	0.102
200	0.009	0.009	0.010	0.009	0.051	0.048	0.054	0.052	0.103	0.099	0.106	0.101
250	0.009	0.010	0.012	0.011	0.049	0.049	0.054	0.050	0.100	0.099	0.105	0.101

	1%				5%				10%			
c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.009	0.009	0.009	0.009	0.049	0.050	0.051	0.051	0.097	0.097	0.098	0.096
-2.5	0.010	0.009	0.011	0.009	0.049	0.050	0.048	0.048	0.096	0.097	0.097	0.095
0	0.012	0.011	0.012	0.011	0.052	0.052	0.052	0.053	0.104	0.104	0.103	0.103
2.5	0.011	0.011	0.012	0.012	0.054	0.054	0.053	0.052	0.102	0.103	0.102	0.103
5	0.012	0.012	0.011	0.011	0.052	0.051	0.051	0.051	0.105	0.105	0.105	0.105
10	0.011	0.010	0.011	0.010	0.053	0.052	0.052	0.052	0.104	0.103	0.104	0.105
25	0.012	0.013	0.012	0.011	0.054	0.052	0.053	0.053	0.103	0.104	0.104	0.104
50	0.011	0.012	0.011	0.011	0.051	0.052	0.052	0.051	0.104	0.106	0.106	0.105
75	0.011	0.011	0.011	0.011	0.052	0.054	0.053	0.052	0.105	0.105	0.106	0.104
100	0.011	0.012	0.011	0.011	0.052	0.052	0.052	0.052	0.105	0.105	0.106	0.106
125	0.012	0.010	0.011	0.011	0.053	0.053	0.052	0.052	0.104	0.103	0.102	0.102
150	0.011	0.011	0.012	0.012	0.053	0.051	0.053	0.053	0.104	0.101	0.102	0.102
200	0.012	0.011	0.011	0.011	0.053	0.051	0.052	0.051	0.103	0.101	0.103	0.101
250	0.011	0.011	0.011	0.011	0.052	0.051	0.051	0.051	0.101	0.102	0.102	0.101
Right-sided tests - $T = 1000$												
-5	0.011	0.011	0.012	0.010	0.049	0.049	0.049	0.048	0.100	0.102	0.103	0.099
-2.5	0.010	0.010	0.011	0.009	0.048	0.048	0.049	0.047	0.096	0.095	0.097	0.094
0	0.008	0.009	0.009	0.009	0.049	0.049	0.048	0.047	0.099	0.100	0.101	0.099
2.5	0.009	0.010	0.010	0.009	0.051	0.051	0.051	0.049	0.102	0.101	0.101	0.100
5	0.010	0.010	0.010	0.009	0.053	0.052	0.051	0.052	0.103	0.103	0.103	0.102
10	0.011	0.010	0.010	0.010	0.052	0.050	0.052	0.051	0.099	0.099	0.101	0.099
25	0.011	0.011	0.011	0.011	0.049	0.049	0.048	0.048	0.097	0.095	0.095	0.096
50	0.011	0.011	0.011	0.011	0.050	0.050	0.050	0.049	0.094	0.093	0.094	0.093
75	0.010	0.010	0.011	0.011	0.049	0.049	0.049	0.049	0.097	0.095	0.097	0.096
100	0.011	0.011	0.011	0.011	0.049	0.051	0.050	0.049	0.099	0.099	0.101	0.100
125	0.011	0.011	0.011	0.011	0.051	0.051	0.052	0.051	0.102	0.101	0.102	0.103
150	0.011	0.011	0.011	0.011	0.051	0.051	0.052	0.051	0.102	0.103	0.102	0.102
200	0.011	0.011	0.012	0.012	0.053	0.052	0.052	0.052	0.101	0.100	0.101	0.100
250	0.010	0.011	0.011	0.010	0.053	0.052	0.054	0.053	0.100	0.100	0.100	0.100
Two-sided tests- $T = 1000$												
-5	0.010	0.009	0.011	0.009	0.048	0.049	0.052	0.049	0.098	0.097	0.100	0.099
-2.5	0.010	0.010	0.011	0.010	0.050	0.050	0.051	0.049	0.095	0.098	0.098	0.096
0	0.010	0.011	0.011	0.010	0.048	0.050	0.053	0.049	0.098	0.099	0.100	0.100
2.5	0.012	0.011	0.011	0.011	0.052	0.051	0.051	0.049	0.102	0.105	0.104	0.102
5	0.011	0.011	0.011	0.010	0.050	0.050	0.049	0.051	0.105	0.103	0.103	0.103
10	0.011	0.012	0.011	0.011	0.051	0.050	0.050	0.049	0.105	0.102	0.103	0.102
25	0.011	0.011	0.011	0.011	0.052	0.052	0.053	0.052	0.100	0.100	0.101	0.101
50	0.012	0.012	0.011	0.011	0.052	0.051	0.053	0.052	0.101	0.101	0.101	0.100
75	0.012	0.013	0.012	0.012	0.052	0.051	0.052	0.051	0.101	0.103	0.103	0.101
100	0.012	0.012	0.012	0.012	0.052	0.050	0.051	0.051	0.100	0.103	0.102	0.101
125	0.012	0.012	0.012	0.012	0.055	0.053	0.054	0.053	0.102	0.102	0.103	0.103
150	0.012	0.012	0.012	0.011	0.053	0.054	0.053	0.054	0.105	0.102	0.104	0.104
200	0.012	0.012	0.011	0.012	0.054	0.052	0.054	0.055	0.104	0.102	0.104	0.103
250	0.012	0.012	0.012	0.012	0.052	0.053	0.053	0.053	0.103	0.104	0.105	0.104

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.5: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP4 (Negative Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = -0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.95; \quad -0.95 \quad 1]$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 250$												
-5	0.008	0.000	0.001	0.000	0.047	0.004	0.005	0.003	0.096	0.013	0.014	0.013
-2.5	0.006	0.000	0.000	0.000	0.042	0.000	0.001	0.001	0.107	0.002	0.002	0.002
0	0.013	0.000	0.000	0.000	0.039	0.001	0.001	0.001	0.063	0.002	0.003	0.003
2.5	0.021	0.001	0.001	0.001	0.060	0.004	0.005	0.005	0.095	0.013	0.013	0.012
5	0.023	0.001	0.001	0.001	0.066	0.011	0.011	0.010	0.112	0.027	0.028	0.026
10	0.018	0.003	0.003	0.003	0.063	0.020	0.021	0.019	0.115	0.044	0.045	0.045
25	0.014	0.005	0.005	0.005	0.058	0.029	0.030	0.031	0.108	0.064	0.065	0.063
50	0.010	0.006	0.006	0.006	0.055	0.037	0.038	0.035	0.106	0.076	0.078	0.079
75	0.010	0.006	0.008	0.006	0.053	0.038	0.040	0.040	0.109	0.085	0.089	0.085
100	0.010	0.007	0.008	0.007	0.054	0.043	0.045	0.043	0.110	0.092	0.095	0.093
125	0.011	0.009	0.010	0.009	0.055	0.046	0.049	0.047	0.109	0.095	0.098	0.096
150	0.011	0.009	0.011	0.010	0.055	0.047	0.051	0.049	0.108	0.099	0.100	0.099
200	0.011	0.011	0.012	0.011	0.055	0.052	0.054	0.051	0.106	0.100	0.103	0.102
250	0.011	0.011	0.013	0.012	0.054	0.054	0.056	0.054	0.108	0.104	0.108	0.107
Right-sided tests - $T = 250$												
-5	0.010	0.016	0.021	0.019	0.045	0.077	0.084	0.077	0.090	0.157	0.162	0.156
-2.5	0.010	0.018	0.021	0.019	0.044	0.105	0.110	0.106	0.094	0.263	0.265	0.262
0	0.014	0.024	0.027	0.025	0.063	0.118	0.125	0.120	0.126	0.251	0.252	0.249
2.5	0.016	0.025	0.027	0.025	0.070	0.117	0.125	0.122	0.136	0.232	0.238	0.236
5	0.015	0.024	0.027	0.024	0.068	0.111	0.116	0.113	0.134	0.209	0.216	0.210
10	0.015	0.021	0.026	0.023	0.064	0.095	0.100	0.096	0.123	0.177	0.183	0.180
25	0.013	0.016	0.018	0.017	0.057	0.073	0.078	0.074	0.109	0.141	0.144	0.141
50	0.011	0.012	0.014	0.012	0.053	0.063	0.065	0.061	0.110	0.125	0.130	0.128
75	0.009	0.011	0.013	0.011	0.054	0.058	0.061	0.058	0.107	0.115	0.121	0.117
100	0.010	0.011	0.011	0.011	0.052	0.053	0.056	0.054	0.107	0.112	0.116	0.113
125	0.010	0.011	0.011	0.011	0.050	0.051	0.054	0.051	0.105	0.109	0.113	0.110
150	0.010	0.010	0.012	0.011	0.050	0.049	0.054	0.049	0.105	0.104	0.108	0.105
200	0.010	0.010	0.010	0.010	0.053	0.049	0.054	0.049	0.105	0.098	0.102	0.099
250	0.011	0.009	0.010	0.009	0.052	0.049	0.051	0.049	0.103	0.096	0.099	0.096
Two-sided tests - $T = 250$												
-5	0.010	0.009	0.012	0.011	0.046	0.041	0.047	0.043	0.092	0.081	0.088	0.081
-2.5	0.009	0.009	0.010	0.010	0.040	0.048	0.053	0.049	0.087	0.105	0.111	0.107
0	0.012	0.012	0.015	0.012	0.054	0.056	0.063	0.059	0.105	0.119	0.125	0.121
2.5	0.014	0.012	0.015	0.014	0.057	0.062	0.066	0.062	0.114	0.121	0.130	0.126
5	0.013	0.013	0.017	0.013	0.056	0.060	0.066	0.060	0.113	0.119	0.127	0.123
10	0.012	0.012	0.014	0.012	0.056	0.057	0.063	0.059	0.108	0.113	0.121	0.115
25	0.010	0.010	0.012	0.011	0.052	0.051	0.056	0.054	0.105	0.100	0.108	0.104
50	0.008	0.008	0.010	0.008	0.049	0.047	0.052	0.049	0.101	0.098	0.103	0.097
75	0.009	0.009	0.010	0.008	0.050	0.045	0.052	0.050	0.102	0.096	0.102	0.099
100	0.009	0.009	0.011	0.009	0.050	0.047	0.051	0.049	0.101	0.096	0.101	0.097
125	0.010	0.009	0.012	0.010	0.049	0.048	0.053	0.048	0.101	0.096	0.103	0.098
150	0.008	0.010	0.012	0.009	0.050	0.048	0.052	0.049	0.100	0.098	0.104	0.098
200	0.010	0.010	0.012	0.010	0.051	0.049	0.054	0.051	0.105	0.101	0.109	0.100
250	0.010	0.010	0.011	0.010	0.052	0.048	0.055	0.050	0.104	0.101	0.107	0.103

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.045	0.003	0.003	0.003	0.094	0.011	0.011	0.011
-2.5	0.005	0.000	0.000	0.000	0.046	0.000	0.000	0.000	0.108	0.001	0.001	0.001
0	0.016	0.000	0.000	0.000	0.042	0.001	0.001	0.001	0.068	0.004	0.003	0.004
2.5	0.024	0.001	0.001	0.001	0.059	0.006	0.006	0.006	0.096	0.016	0.016	0.016
5	0.023	0.002	0.002	0.002	0.066	0.014	0.014	0.013	0.109	0.029	0.029	0.028
10	0.020	0.004	0.004	0.004	0.065	0.021	0.021	0.020	0.112	0.046	0.046	0.045
25	0.017	0.007	0.008	0.008	0.061	0.031	0.031	0.030	0.108	0.064	0.065	0.064
50	0.014	0.008	0.009	0.009	0.057	0.036	0.036	0.036	0.109	0.076	0.076	0.075
75	0.014	0.009	0.008	0.008	0.056	0.039	0.040	0.040	0.106	0.081	0.084	0.082
100	0.013	0.008	0.008	0.008	0.053	0.041	0.042	0.042	0.109	0.085	0.085	0.084
125	0.014	0.009	0.008	0.009	0.053	0.044	0.043	0.043	0.108	0.089	0.088	0.088
150	0.012	0.009	0.009	0.009	0.055	0.044	0.045	0.044	0.107	0.088	0.089	0.089
200	0.012	0.009	0.009	0.009	0.054	0.046	0.047	0.045	0.107	0.090	0.090	0.091
250	0.012	0.010	0.010	0.010	0.053	0.046	0.046	0.046	0.106	0.093	0.095	0.094
Right-sided tests - $T = 1000$												
-5	0.007	0.013	0.014	0.014	0.039	0.066	0.068	0.066	0.084	0.143	0.144	0.140
-2.5	0.008	0.019	0.019	0.018	0.042	0.100	0.098	0.098	0.088	0.244	0.241	0.241
0	0.011	0.021	0.022	0.020	0.052	0.111	0.112	0.110	0.110	0.236	0.237	0.235
2.5	0.011	0.023	0.023	0.022	0.060	0.110	0.114	0.111	0.120	0.224	0.226	0.226
5	0.011	0.021	0.022	0.020	0.062	0.108	0.110	0.108	0.124	0.208	0.209	0.209
10	0.012	0.021	0.019	0.019	0.061	0.098	0.099	0.098	0.117	0.184	0.184	0.183
25	0.011	0.017	0.017	0.017	0.056	0.080	0.081	0.080	0.111	0.151	0.152	0.152
50	0.011	0.014	0.014	0.015	0.052	0.070	0.068	0.069	0.105	0.134	0.134	0.134
75	0.009	0.013	0.012	0.012	0.053	0.066	0.066	0.064	0.105	0.127	0.126	0.126
100	0.010	0.013	0.012	0.012	0.051	0.063	0.064	0.064	0.103	0.121	0.124	0.122
125	0.010	0.012	0.012	0.012	0.052	0.061	0.061	0.061	0.102	0.119	0.120	0.120
150	0.010	0.013	0.011	0.011	0.052	0.057	0.058	0.058	0.104	0.117	0.117	0.116
200	0.010	0.011	0.011	0.011	0.051	0.057	0.058	0.057	0.103	0.113	0.114	0.113
250	0.010	0.011	0.010	0.011	0.051	0.056	0.056	0.054	0.103	0.113	0.113	0.112
Two-sided tests- $T = 1000$												
-5	0.007	0.007	0.007	0.007	0.040	0.032	0.034	0.032	0.085	0.069	0.070	0.069
-2.5	0.007	0.009	0.009	0.009	0.039	0.046	0.046	0.045	0.083	0.098	0.098	0.098
0	0.008	0.010	0.011	0.011	0.045	0.053	0.053	0.053	0.093	0.111	0.113	0.111
2.5	0.009	0.012	0.011	0.011	0.052	0.060	0.061	0.060	0.102	0.116	0.120	0.117
5	0.010	0.011	0.012	0.010	0.055	0.060	0.061	0.061	0.109	0.122	0.124	0.121
10	0.010	0.012	0.011	0.011	0.056	0.062	0.063	0.063	0.107	0.118	0.120	0.118
25	0.011	0.013	0.012	0.012	0.053	0.054	0.056	0.056	0.104	0.111	0.112	0.110
50	0.011	0.012	0.012	0.012	0.052	0.054	0.056	0.054	0.102	0.104	0.104	0.105
75	0.010	0.011	0.011	0.011	0.053	0.053	0.054	0.052	0.104	0.105	0.106	0.104
100	0.010	0.010	0.011	0.010	0.052	0.053	0.054	0.052	0.104	0.105	0.105	0.106
125	0.010	0.011	0.011	0.010	0.050	0.052	0.053	0.051	0.105	0.103	0.104	0.104
150	0.011	0.011	0.011	0.010	0.052	0.052	0.053	0.050	0.103	0.101	0.103	0.102
200	0.011	0.011	0.011	0.010	0.049	0.050	0.052	0.050	0.103	0.103	0.104	0.102
250	0.012	0.012	0.011	0.011	0.050	0.050	0.050	0.050	0.102	0.101	0.102	0.100

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.6: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP4 (Negative Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = -0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.90; \quad -0.90 \quad 1]$.

c	1% $t^{*,RWB}_{zx}$	1% $t^{*,FRWB}_{zx}$	1% t^{EW}_{zx}	1% t_{zx}	5% $t^{*,RWB}_{zx}$	5% $t^{*,FRWB}_{zx}$	5% t^{EW}_{zx}	5% t_{zx}	10% $t^{*,RWB}_{zx}$	10% $t^{*,FRWB}_{zx}$	10% t^{EW}_{zx}	10% t_{zx}
Left-sided tests - $T = 250$												
-5	0.009	0.001	0.001	0.000	0.048	0.005	0.005	0.005	0.098	0.015	0.016	0.014
-2.5	0.007	0.000	0.000	0.000	0.045	0.000	0.001	0.001	0.106	0.002	0.002	0.002
0	0.012	0.000	0.000	0.000	0.037	0.001	0.001	0.001	0.060	0.003	0.004	0.004
2.5	0.019	0.001	0.001	0.001	0.056	0.005	0.005	0.005	0.096	0.014	0.014	0.013
5	0.020	0.002	0.002	0.002	0.066	0.011	0.011	0.011	0.112	0.028	0.029	0.029
10	0.017	0.003	0.003	0.003	0.062	0.021	0.021	0.019	0.111	0.046	0.048	0.047
25	0.012	0.006	0.006	0.005	0.056	0.030	0.032	0.031	0.106	0.066	0.067	0.066
50	0.011	0.006	0.007	0.006	0.054	0.036	0.038	0.036	0.106	0.078	0.080	0.078
75	0.010	0.006	0.008	0.006	0.054	0.039	0.041	0.040	0.108	0.086	0.091	0.086
100	0.010	0.008	0.008	0.007	0.054	0.044	0.047	0.044	0.109	0.091	0.096	0.093
125	0.011	0.009	0.010	0.009	0.055	0.047	0.050	0.047	0.110	0.098	0.100	0.098
150	0.012	0.010	0.012	0.010	0.055	0.049	0.052	0.047	0.109	0.098	0.102	0.101
200	0.013	0.011	0.013	0.012	0.054	0.050	0.053	0.052	0.108	0.101	0.105	0.103
250	0.012	0.012	0.014	0.012	0.053	0.053	0.056	0.055	0.110	0.106	0.111	0.108
Right-sided tests - $T = 250$												
-5	0.009	0.017	0.021	0.018	0.044	0.076	0.084	0.078	0.090	0.155	0.161	0.154
-2.5	0.011	0.019	0.022	0.018	0.045	0.108	0.113	0.107	0.095	0.255	0.258	0.255
0	0.013	0.024	0.028	0.025	0.061	0.116	0.122	0.118	0.128	0.244	0.247	0.245
2.5	0.016	0.025	0.028	0.025	0.071	0.115	0.121	0.117	0.135	0.227	0.231	0.228
5	0.016	0.024	0.027	0.024	0.068	0.107	0.114	0.110	0.133	0.201	0.209	0.203
10	0.014	0.021	0.026	0.022	0.063	0.093	0.098	0.092	0.122	0.172	0.178	0.175
25	0.011	0.016	0.018	0.017	0.056	0.072	0.077	0.073	0.111	0.140	0.144	0.143
50	0.010	0.012	0.013	0.011	0.054	0.064	0.067	0.064	0.107	0.125	0.127	0.125
75	0.010	0.010	0.013	0.011	0.054	0.059	0.063	0.060	0.107	0.116	0.118	0.116
100	0.010	0.011	0.012	0.011	0.051	0.055	0.057	0.055	0.107	0.112	0.116	0.113
125	0.011	0.011	0.012	0.010	0.050	0.050	0.053	0.051	0.107	0.109	0.112	0.108
150	0.011	0.011	0.012	0.010	0.049	0.049	0.053	0.049	0.106	0.105	0.108	0.104
200	0.010	0.009	0.011	0.010	0.051	0.049	0.052	0.048	0.105	0.101	0.103	0.101
250	0.011	0.010	0.011	0.009	0.053	0.050	0.051	0.049	0.105	0.097	0.101	0.099
Two-sided tests - $T = 250$												
-5	0.008	0.009	0.012	0.009	0.045	0.041	0.047	0.041	0.093	0.080	0.090	0.083
-2.5	0.010	0.010	0.011	0.010	0.040	0.048	0.052	0.049	0.089	0.107	0.114	0.108
0	0.011	0.012	0.015	0.012	0.051	0.056	0.062	0.059	0.107	0.117	0.123	0.119
2.5	0.014	0.012	0.016	0.014	0.060	0.061	0.067	0.064	0.113	0.120	0.126	0.122
5	0.012	0.013	0.016	0.014	0.056	0.058	0.065	0.062	0.113	0.117	0.125	0.121
10	0.011	0.012	0.015	0.012	0.055	0.057	0.062	0.059	0.106	0.112	0.119	0.112
25	0.010	0.010	0.012	0.011	0.051	0.051	0.056	0.054	0.103	0.102	0.109	0.104
50	0.009	0.007	0.010	0.009	0.049	0.049	0.053	0.049	0.101	0.099	0.105	0.100
75	0.008	0.008	0.009	0.008	0.050	0.046	0.052	0.048	0.101	0.100	0.104	0.099
100	0.009	0.009	0.011	0.009	0.049	0.047	0.052	0.050	0.101	0.097	0.103	0.099
125	0.009	0.009	0.011	0.010	0.049	0.049	0.052	0.049	0.101	0.096	0.103	0.098
150	0.009	0.009	0.011	0.009	0.051	0.048	0.052	0.050	0.102	0.098	0.104	0.096
200	0.010	0.010	0.012	0.011	0.050	0.049	0.054	0.050	0.103	0.099	0.105	0.100
250	0.010	0.010	0.013	0.011	0.050	0.049	0.054	0.051	0.104	0.101	0.107	0.104

c	1% $t^{*,RWB}_{zx}$	1% $t^{*,FRWB}_{zx}$	1% t^{EW}_{zx}	1% t_{zx}	5% $t^{*,RWB}_{zx}$	5% $t^{*,FRWB}_{zx}$	5% t^{EW}_{zx}	5% t_{zx}	10% $t^{*,RWB}_{zx}$	10% $t^{*,FRWB}_{zx}$	10% t^{EW}_{zx}	10% t_{zx}
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.046	0.003	0.003	0.003	0.095	0.013	0.014	0.014
-2.5	0.004	0.000	0.000	0.000	0.048	0.000	0.000	0.000	0.110	0.001	0.001	0.001
0	0.014	0.000	0.000	0.000	0.040	0.002	0.002	0.002	0.066	0.004	0.004	0.004
2.5	0.022	0.001	0.001	0.001	0.058	0.007	0.007	0.007	0.096	0.017	0.018	0.018
5	0.022	0.003	0.003	0.003	0.064	0.014	0.015	0.014	0.107	0.031	0.031	0.031
10	0.019	0.005	0.005	0.005	0.063	0.021	0.022	0.021	0.109	0.047	0.047	0.046
25	0.016	0.007	0.008	0.008	0.058	0.031	0.032	0.031	0.109	0.064	0.065	0.065
50	0.014	0.008	0.008	0.008	0.056	0.039	0.039	0.038	0.109	0.078	0.079	0.077
75	0.014	0.009	0.008	0.007	0.057	0.042	0.042	0.041	0.105	0.082	0.084	0.083
100	0.013	0.009	0.008	0.008	0.055	0.043	0.044	0.044	0.107	0.084	0.085	0.085
125	0.012	0.009	0.009	0.008	0.054	0.045	0.045	0.045	0.107	0.089	0.089	0.089
150	0.013	0.009	0.009	0.009	0.056	0.045	0.046	0.045	0.107	0.091	0.092	0.091
200	0.012	0.009	0.010	0.010	0.053	0.046	0.046	0.046	0.106	0.093	0.094	0.092
250	0.012	0.010	0.010	0.010	0.053	0.046	0.046	0.046	0.108	0.096	0.097	0.095
Right-sided tests - $T = 1000$												
-5	0.008	0.014	0.014	0.013	0.040	0.069	0.069	0.066	0.084	0.143	0.144	0.143
-2.5	0.009	0.018	0.018	0.017	0.041	0.100	0.100	0.099	0.091	0.239	0.237	0.238
0	0.011	0.022	0.022	0.021	0.052	0.107	0.107	0.104	0.109	0.233	0.229	0.230
2.5	0.011	0.023	0.023	0.021	0.061	0.107	0.109	0.109	0.120	0.219	0.221	0.221
5	0.012	0.022	0.022	0.020	0.061	0.105	0.106	0.105	0.122	0.202	0.202	0.201
10	0.011	0.019	0.019	0.018	0.061	0.095	0.096	0.095	0.116	0.179	0.178	0.176
25	0.010	0.016	0.016	0.016	0.056	0.078	0.080	0.079	0.109	0.149	0.150	0.148
50	0.010	0.014	0.014	0.014	0.052	0.068	0.067	0.067	0.104	0.131	0.133	0.133
75	0.010	0.012	0.012	0.013	0.051	0.062	0.065	0.064	0.104	0.125	0.128	0.125
100	0.011	0.012	0.012	0.012	0.050	0.062	0.064	0.063	0.103	0.120	0.122	0.121
125	0.009	0.012	0.012	0.012	0.052	0.060	0.061	0.060	0.101	0.118	0.118	0.117
150	0.010	0.011	0.012	0.011	0.052	0.060	0.060	0.060	0.101	0.114	0.116	0.115
200	0.009	0.011	0.012	0.010	0.052	0.058	0.059	0.058	0.103	0.111	0.112	0.112
250	0.010	0.011	0.011	0.011	0.051	0.055	0.056	0.056	0.107	0.113	0.114	0.114
Two-sided tests- $T = 1000$												
-5	0.007	0.007	0.007	0.007	0.041	0.034	0.035	0.034	0.086	0.071	0.072	0.070
-2.5	0.007	0.009	0.009	0.009	0.038	0.045	0.046	0.045	0.085	0.098	0.100	0.099
0	0.009	0.010	0.011	0.011	0.045	0.052	0.054	0.053	0.093	0.108	0.109	0.106
2.5	0.009	0.010	0.012	0.011	0.052	0.060	0.059	0.059	0.102	0.115	0.116	0.116
5	0.010	0.012	0.012	0.011	0.054	0.059	0.060	0.060	0.105	0.118	0.121	0.119
10	0.010	0.011	0.011	0.011	0.054	0.061	0.062	0.061	0.106	0.117	0.118	0.116
25	0.010	0.012	0.012	0.011	0.051	0.055	0.055	0.055	0.104	0.109	0.111	0.109
50	0.012	0.011	0.011	0.011	0.052	0.054	0.056	0.054	0.103	0.105	0.106	0.105
75	0.010	0.011	0.011	0.011	0.052	0.053	0.054	0.053	0.105	0.105	0.106	0.105
100	0.010	0.010	0.010	0.009	0.052	0.053	0.053	0.052	0.105	0.105	0.108	0.107
125	0.010	0.011	0.011	0.010	0.053	0.052	0.054	0.052	0.106	0.104	0.107	0.105
150	0.011	0.010	0.010	0.010	0.052	0.054	0.054	0.052	0.106	0.105	0.106	0.105
200	0.010	0.010	0.010	0.009	0.050	0.050	0.051	0.049	0.104	0.102	0.105	0.104
250	0.010	0.011	0.011	0.011	0.051	0.050	0.052	0.051	0.103	0.101	0.102	0.101

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.7: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP4 (Negative Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = -0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.50; \quad -0.50 \quad 1]$.

Left-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
		1%				**5%**				**10%**		
-5	0.010	0.003	0.005	0.002	0.052	0.020	0.024	0.019	0.105	0.048	0.053	0.048
-2.5	0.010	0.000	0.001	0.000	0.050	0.006	0.007	0.005	0.099	0.017	0.018	0.016
0	0.005	0.001	0.001	0.000	0.030	0.006	0.007	0.006	0.060	0.019	0.020	0.019
2.5	0.009	0.002	0.002	0.002	0.043	0.016	0.018	0.017	0.086	0.040	0.040	0.039
5	0.011	0.004	0.005	0.004	0.050	0.024	0.024	0.024	0.095	0.052	0.055	0.053
10	0.012	0.006	0.007	0.007	0.052	0.031	0.032	0.031	0.100	0.063	0.065	0.065
25	0.010	0.007	0.007	0.007	0.052	0.038	0.040	0.039	0.102	0.080	0.081	0.080
50	0.010	0.007	0.009	0.007	0.051	0.041	0.044	0.043	0.102	0.087	0.089	0.087
75	0.010	0.008	0.009	0.008	0.053	0.047	0.048	0.047	0.101	0.089	0.092	0.091
100	0.010	0.008	0.010	0.009	0.053	0.047	0.050	0.049	0.102	0.093	0.095	0.094
125	0.010	0.009	0.011	0.010	0.055	0.050	0.053	0.051	0.104	0.097	0.101	0.099
150	0.010	0.010	0.012	0.010	0.053	0.051	0.053	0.052	0.105	0.099	0.103	0.103
200	0.011	0.011	0.013	0.011	0.055	0.053	0.056	0.054	0.106	0.102	0.106	0.105
250	0.012	0.012	0.013	0.012	0.056	0.056	0.058	0.057	0.106	0.104	0.108	0.107

Right-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.016	0.021	0.016	0.045	0.072	0.081	0.073	0.097	0.144	0.154	0.146
-2.5	0.012	0.023	0.029	0.022	0.054	0.106	0.111	0.105	0.107	0.202	0.211	0.203
0	0.013	0.021	0.024	0.021	0.067	0.101	0.107	0.100	0.126	0.197	0.204	0.197
2.5	0.014	0.020	0.023	0.022	0.065	0.092	0.098	0.093	0.125	0.170	0.177	0.173
5	0.013	0.019	0.021	0.020	0.062	0.082	0.086	0.084	0.119	0.159	0.161	0.158
10	0.012	0.016	0.018	0.017	0.057	0.075	0.078	0.075	0.112	0.139	0.144	0.141
25	0.011	0.014	0.016	0.014	0.056	0.065	0.069	0.067	0.107	0.123	0.129	0.124
50	0.010	0.011	0.013	0.011	0.054	0.058	0.062	0.061	0.108	0.117	0.121	0.117
75	0.009	0.011	0.012	0.011	0.053	0.056	0.062	0.057	0.108	0.113	0.118	0.112
100	0.010	0.011	0.012	0.011	0.051	0.052	0.056	0.053	0.105	0.107	0.112	0.107
125	0.010	0.010	0.011	0.011	0.051	0.054	0.056	0.052	0.105	0.103	0.108	0.105
150	0.010	0.010	0.011	0.010	0.051	0.051	0.055	0.052	0.106	0.103	0.107	0.103
200	0.010	0.010	0.010	0.009	0.050	0.050	0.052	0.049	0.106	0.103	0.106	0.105
250	0.010	0.009	0.011	0.009	0.049	0.049	0.051	0.048	0.102	0.099	0.102	0.099

Two-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.009	0.016	0.009	0.048	0.044	0.057	0.044	0.097	0.090	0.105	0.092
-2.5	0.011	0.011	0.015	0.011	0.049	0.054	0.062	0.054	0.100	0.112	0.118	0.110
0	0.012	0.012	0.013	0.012	0.051	0.054	0.061	0.055	0.101	0.107	0.114	0.106
2.5	0.012	0.012	0.014	0.013	0.051	0.055	0.060	0.056	0.103	0.107	0.115	0.110
5	0.012	0.012	0.013	0.012	0.054	0.054	0.059	0.056	0.104	0.106	0.111	0.107
10	0.012	0.010	0.013	0.013	0.053	0.051	0.057	0.054	0.102	0.104	0.110	0.106
25	0.010	0.011	0.013	0.010	0.050	0.049	0.054	0.051	0.105	0.103	0.109	0.106
50	0.008	0.009	0.009	0.009	0.050	0.050	0.054	0.050	0.103	0.099	0.106	0.104
75	0.009	0.010	0.012	0.009	0.049	0.048	0.052	0.051	0.105	0.103	0.110	0.104
100	0.008	0.009	0.011	0.010	0.050	0.047	0.053	0.051	0.103	0.099	0.106	0.102
125	0.010	0.010	0.011	0.010	0.051	0.048	0.052	0.049	0.105	0.101	0.109	0.102
150	0.009	0.010	0.012	0.010	0.049	0.049	0.053	0.050	0.104	0.102	0.108	0.104
200	0.010	0.010	0.012	0.009	0.052	0.052	0.057	0.053	0.105	0.101	0.108	0.103
250	0.010	0.010	0.012	0.010	0.053	0.053	0.058	0.053	0.104	0.103	0.109	0.105

Left-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
		1%				**5%**				**10%**		
-5	0.009	0.002	0.002	0.002	0.049	0.018	0.018	0.019	0.097	0.044	0.045	0.044
-2.5	0.009	0.001	0.001	0.000	0.050	0.004	0.004	0.004	0.098	0.015	0.015	0.015
0	0.008	0.001	0.000	0.000	0.033	0.008	0.008	0.008	0.065	0.021	0.021	0.020
2.5	0.012	0.002	0.003	0.002	0.048	0.018	0.018	0.018	0.092	0.043	0.044	0.043
5	0.013	0.004	0.003	0.003	0.054	0.025	0.024	0.025	0.100	0.056	0.056	0.057
10	0.014	0.006	0.005	0.005	0.054	0.031	0.030	0.030	0.103	0.067	0.067	0.067
25	0.013	0.008	0.008	0.008	0.054	0.039	0.039	0.038	0.105	0.080	0.080	0.080
50	0.014	0.010	0.010	0.010	0.054	0.043	0.045	0.044	0.104	0.085	0.086	0.085
75	0.014	0.011	0.011	0.011	0.053	0.044	0.046	0.046	0.102	0.090	0.090	0.089
100	0.014	0.011	0.012	0.012	0.053	0.047	0.047	0.047	0.104	0.093	0.092	0.092
125	0.013	0.012	0.011	0.011	0.054	0.048	0.047	0.047	0.105	0.094	0.095	0.094
150	0.013	0.011	0.011	0.011	0.054	0.048	0.049	0.048	0.104	0.096	0.096	0.094
200	0.012	0.011	0.011	0.010	0.055	0.051	0.051	0.050	0.103	0.096	0.097	0.096
250	0.012	0.011	0.010	0.011	0.054	0.050	0.051	0.050	0.104	0.098	0.098	0.098

Right-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.014	0.015	0.013	0.045	0.072	0.073	0.073	0.097	0.141	0.143	0.141
-2.5	0.008	0.019	0.018	0.016	0.050	0.099	0.100	0.098	0.104	0.198	0.197	0.195
0	0.012	0.021	0.020	0.019	0.059	0.095	0.096	0.095	0.118	0.190	0.191	0.189
2.5	0.013	0.020	0.020	0.020	0.059	0.089	0.089	0.089	0.117	0.169	0.170	0.169
5	0.013	0.019	0.019	0.018	0.057	0.081	0.081	0.080	0.110	0.154	0.155	0.154
10	0.011	0.016	0.015	0.015	0.054	0.074	0.075	0.074	0.106	0.140	0.141	0.139
25	0.010	0.013	0.013	0.012	0.053	0.065	0.066	0.065	0.102	0.120	0.122	0.121
50	0.010	0.012	0.012	0.012	0.050	0.059	0.058	0.058	0.099	0.114	0.115	0.116
75	0.010	0.011	0.012	0.012	0.050	0.056	0.057	0.057	0.102	0.114	0.115	0.114
100	0.010	0.012	0.012	0.012	0.050	0.056	0.055	0.056	0.101	0.112	0.113	0.113
125	0.010	0.011	0.011	0.011	0.051	0.056	0.055	0.056	0.101	0.111	0.112	0.112
150	0.010	0.011	0.011	0.011	0.051	0.055	0.055	0.055	0.104	0.111	0.110	0.110
200	0.010	0.011	0.011	0.011	0.052	0.055	0.055	0.055	0.104	0.111	0.110	0.110
250	0.010	0.010	0.011	0.010	0.052	0.056	0.056	0.055	0.106	0.111	0.111	0.110

Two-sided tests- $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.007	0.009	0.007	0.047	0.042	0.043	0.042	0.098	0.089	0.091	0.091
-2.5	0.007	0.008	0.009	0.008	0.046	0.049	0.049	0.047	0.096	0.102	0.104	0.102
0	0.009	0.011	0.010	0.011	0.047	0.051	0.052	0.051	0.097	0.102	0.104	0.103
2.5	0.011	0.010	0.011	0.010	0.050	0.054	0.056	0.053	0.100	0.106	0.107	0.106
5	0.010	0.012	0.011	0.011	0.052	0.055	0.055	0.054	0.098	0.104	0.105	0.104
10	0.010	0.011	0.011	0.011	0.052	0.054	0.054	0.054	0.100	0.104	0.105	0.104
25	0.010	0.011	0.011	0.011	0.051	0.051	0.053	0.051	0.101	0.103	0.105	0.104
50	0.011	0.011	0.012	0.011	0.053	0.052	0.052	0.052	0.101	0.102	0.102	0.102
75	0.011	0.012	0.013	0.012	0.053	0.051	0.053	0.052	0.103	0.100	0.102	0.103
100	0.012	0.013	0.012	0.013	0.053	0.051	0.052	0.052	0.102	0.101	0.102	0.103
125	0.013	0.012	0.012	0.012	0.053	0.052	0.052	0.053	0.103	0.102	0.102	0.103
150	0.012	0.012	0.012	0.012	0.053	0.052	0.052	0.052	0.103	0.102	0.104	0.103
200	0.010	0.010	0.011	0.010	0.052	0.050	0.052	0.052	0.103	0.105	0.106	0.105
250	0.010	0.011	0.010	0.010	0.049	0.047	0.049	0.049	0.104	0.105	0.107	0.105

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.8: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP4 (Negative Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = -0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad 0; \quad 0 \quad 1]$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 250$												
-5	0.011	0.011	0.017	0.010	0.051	0.051	0.059	0.052	0.101	0.103	0.111	0.102
-2.5	0.011	0.012	0.015	0.010	0.051	0.052	0.059	0.049	0.103	0.104	0.110	0.103
0	0.011	0.011	0.014	0.010	0.050	0.051	0.055	0.050	0.099	0.100	0.102	0.100
2.5	0.010	0.010	0.012	0.010	0.049	0.048	0.053	0.051	0.099	0.098	0.101	0.099
5	0.010	0.010	0.012	0.011	0.050	0.051	0.053	0.050	0.100	0.098	0.101	0.099
10	0.011	0.011	0.013	0.011	0.052	0.050	0.053	0.051	0.103	0.102	0.105	0.105
25	0.010	0.010	0.011	0.010	0.049	0.049	0.052	0.051	0.100	0.099	0.102	0.101
50	0.011	0.009	0.010	0.010	0.048	0.046	0.049	0.049	0.098	0.098	0.100	0.099
75	0.010	0.009	0.011	0.011	0.049	0.047	0.050	0.048	0.097	0.094	0.097	0.095
100	0.009	0.009	0.010	0.010	0.048	0.048	0.050	0.048	0.097	0.093	0.097	0.096
125	0.009	0.009	0.011	0.010	0.050	0.049	0.052	0.049	0.095	0.093	0.097	0.094
150	0.009	0.010	0.011	0.011	0.048	0.049	0.052	0.049	0.097	0.096	0.098	0.095
200	0.009	0.009	0.011	0.010	0.048	0.048	0.050	0.048	0.099	0.097	0.099	0.098
250	0.009	0.010	0.011	0.010	0.050	0.049	0.053	0.051	0.101	0.097	0.103	0.098
Right-sided tests - $T = 250$												
-5	0.011	0.012	0.016	0.012	0.052	0.050	0.060	0.051	0.102	0.103	0.110	0.100
-2.5	0.011	0.010	0.016	0.010	0.052	0.051	0.057	0.051	0.102	0.101	0.107	0.100
0	0.012	0.011	0.015	0.011	0.050	0.051	0.055	0.049	0.097	0.096	0.101	0.097
2.5	0.010	0.010	0.012	0.010	0.052	0.052	0.053	0.051	0.103	0.101	0.103	0.102
5	0.011	0.010	0.012	0.010	0.050	0.050	0.052	0.050	0.103	0.102	0.105	0.103
10	0.010	0.011	0.011	0.011	0.052	0.052	0.054	0.052	0.100	0.099	0.102	0.101
25	0.011	0.013	0.014	0.012	0.051	0.049	0.053	0.053	0.103	0.101	0.105	0.103
50	0.010	0.011	0.012	0.011	0.051	0.050	0.054	0.053	0.103	0.101	0.106	0.101
75	0.011	0.010	0.011	0.011	0.051	0.051	0.053	0.051	0.101	0.100	0.104	0.101
100	0.010	0.010	0.011	0.010	0.049	0.050	0.053	0.051	0.101	0.099	0.103	0.100
125	0.010	0.010	0.012	0.011	0.051	0.050	0.052	0.051	0.101	0.099	0.101	0.099
150	0.010	0.010	0.012	0.011	0.050	0.050	0.052	0.050	0.099	0.098	0.101	0.098
200	0.011	0.011	0.011	0.011	0.048	0.048	0.051	0.050	0.099	0.097	0.100	0.098
250	0.010	0.010	0.011	0.010	0.048	0.048	0.051	0.048	0.100	0.096	0.100	0.097
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.020	0.012	0.051	0.050	0.066	0.051	0.102	0.101	0.119	0.103
-2.5	0.011	0.011	0.018	0.011	0.051	0.053	0.066	0.051	0.101	0.101	0.116	0.100
0	0.011	0.011	0.015	0.011	0.050	0.051	0.058	0.050	0.097	0.099	0.109	0.099
2.5	0.009	0.010	0.012	0.010	0.051	0.051	0.058	0.053	0.099	0.099	0.106	0.102
5	0.011	0.010	0.012	0.011	0.050	0.052	0.057	0.052	0.099	0.100	0.105	0.100
10	0.011	0.011	0.014	0.011	0.050	0.048	0.053	0.051	0.103	0.100	0.107	0.104
25	0.012	0.011	0.012	0.013	0.051	0.049	0.055	0.052	0.101	0.100	0.105	0.104
50	0.010	0.009	0.012	0.011	0.049	0.050	0.055	0.051	0.099	0.096	0.103	0.101
75	0.009	0.011	0.012	0.011	0.049	0.049	0.055	0.052	0.098	0.097	0.103	0.099
100	0.009	0.010	0.011	0.011	0.048	0.048	0.054	0.053	0.098	0.096	0.102	0.099
125	0.009	0.010	0.011	0.010	0.049	0.049	0.054	0.051	0.099	0.098	0.104	0.100
150	0.010	0.010	0.012	0.010	0.048	0.047	0.051	0.051	0.097	0.097	0.104	0.099
200	0.009	0.010	0.012	0.010	0.047	0.049	0.052	0.050	0.096	0.094	0.101	0.098
250	0.010	0.011	0.012	0.010	0.049	0.050	0.053	0.050	0.098	0.097	0.103	0.099

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 1000$												
-5	0.009	0.009	0.009	0.009	0.050	0.050	0.051	0.051	0.096	0.096	0.098	0.097
-2.5	0.009	0.009	0.010	0.009	0.050	0.050	0.051	0.050	0.096	0.096	0.096	0.096
0	0.010	0.010	0.011	0.010	0.052	0.053	0.053	0.053	0.102	0.103	0.103	0.102
2.5	0.012	0.011	0.011	0.011	0.053	0.053	0.054	0.053	0.105	0.104	0.105	0.105
5	0.012	0.011	0.010	0.011	0.051	0.053	0.052	0.051	0.106	0.105	0.105	0.105
10	0.011	0.011	0.010	0.010	0.051	0.050	0.052	0.052	0.103	0.104	0.104	0.104
25	0.011	0.012	0.011	0.010	0.054	0.052	0.052	0.053	0.103	0.104	0.104	0.103
50	0.012	0.011	0.011	0.011	0.054	0.052	0.053	0.053	0.102	0.102	0.102	0.102
75	0.011	0.012	0.012	0.012	0.053	0.053	0.053	0.052	0.102	0.102	0.102	0.103
100	0.012	0.013	0.012	0.012	0.053	0.052	0.052	0.051	0.103	0.102	0.103	0.102
125	0.012	0.012	0.012	0.013	0.051	0.051	0.052	0.052	0.102	0.103	0.103	0.103
150	0.012	0.012	0.011	0.012	0.052	0.052	0.051	0.052	0.101	0.102	0.102	0.101
200	0.012	0.011	0.011	0.011	0.051	0.051	0.051	0.051	0.102	0.102	0.101	0.100
250	0.012	0.012	0.011	0.012	0.050	0.051	0.051	0.050	0.103	0.104	0.104	0.102
Right-sided tests - $T = 1000$												
-5	0.011	0.011	0.012	0.010	0.047	0.049	0.049	0.048	0.101	0.100	0.103	0.100
-2.5	0.009	0.009	0.011	0.009	0.048	0.048	0.049	0.047	0.096	0.095	0.098	0.095
0	0.009	0.010	0.010	0.009	0.048	0.049	0.048	0.048	0.098	0.099	0.100	0.098
2.5	0.009	0.010	0.010	0.009	0.051	0.050	0.051	0.049	0.102	0.099	0.102	0.102
5	0.009	0.009	0.009	0.009	0.053	0.054	0.054	0.053	0.101	0.101	0.103	0.102
10	0.010	0.009	0.009	0.009	0.052	0.051	0.053	0.052	0.101	0.101	0.102	0.101
25	0.010	0.009	0.009	0.009	0.050	0.049	0.050	0.050	0.098	0.097	0.098	0.098
50	0.010	0.009	0.009	0.009	0.050	0.049	0.051	0.051	0.097	0.096	0.097	0.097
75	0.010	0.011	0.010	0.010	0.048	0.048	0.049	0.048	0.099	0.099	0.099	0.099
100	0.010	0.011	0.011	0.010	0.048	0.048	0.048	0.048	0.101	0.099	0.100	0.101
125	0.011	0.010	0.010	0.010	0.049	0.050	0.049	0.048	0.101	0.100	0.102	0.101
150	0.009	0.010	0.010	0.009	0.050	0.049	0.050	0.049	0.100	0.100	0.101	0.101
200	0.010	0.010	0.010	0.010	0.050	0.049	0.050	0.049	0.102	0.101	0.102	0.101
250	0.010	0.010	0.010	0.010	0.049	0.048	0.049	0.049	0.101	0.100	0.101	0.100
Two-sided tests- $T = 1000$												
-5	0.010	0.010	0.011	0.009	0.047	0.048	0.052	0.049	0.098	0.098	0.100	0.099
-2.5	0.010	0.010	0.010	0.009	0.050	0.049	0.051	0.049	0.096	0.099	0.100	0.097
0	0.010	0.011	0.010	0.010	0.048	0.048	0.050	0.048	0.101	0.102	0.101	0.101
2.5	0.010	0.011	0.011	0.011	0.051	0.052	0.052	0.051	0.103	0.103	0.105	0.102
5	0.010	0.010	0.011	0.010	0.050	0.051	0.051	0.051	0.105	0.105	0.105	0.104
10	0.010	0.010	0.010	0.010	0.051	0.050	0.051	0.050	0.103	0.102	0.104	0.104
25	0.012	0.010	0.010	0.010	0.052	0.052	0.052	0.051	0.103	0.101	0.102	0.103
50	0.012	0.012	0.011	0.012	0.052	0.051	0.052	0.052	0.103	0.102	0.103	0.104
75	0.012	0.012	0.012	0.011	0.053	0.052	0.052	0.052	0.101	0.101	0.102	0.100
100	0.011	0.011	0.012	0.012	0.051	0.050	0.052	0.051	0.100	0.098	0.099	0.099
125	0.011	0.012	0.011	0.011	0.051	0.050	0.051	0.050	0.100	0.100	0.101	0.100
150	0.012	0.012	0.012	0.011	0.050	0.051	0.052	0.050	0.100	0.100	0.101	0.100
200	0.012	0.011	0.012	0.012	0.049	0.048	0.050	0.049	0.100	0.099	0.101	0.100
250	0.011	0.012	0.012	0.012	0.049	0.047	0.048	0.048	0.099	0.099	0.100	0.099

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.9: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP5 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T$ and $\psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = \left[\sigma^2_{ut} \quad -0.95\sigma_{ut}\sigma_{vt}; \quad -0.95\sigma_{ut}\sigma_{vt} \quad \sigma^2_{vt}\right]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.006	0.001	0.000	0.001	0.040	0.006	0.004	0.011	0.089	0.017	0.013	0.030
-2.5	0.006	0.000	0.000	0.000	0.033	0.002	0.001	0.003	0.082	0.005	0.004	0.008
0	0.014	0.000	0.000	0.001	0.051	0.003	0.003	0.005	0.090	0.008	0.007	0.011
2.5	0.017	0.001	0.001	0.002	0.059	0.005	0.005	0.009	0.102	0.014	0.012	0.018
5	0.019	0.001	0.001	0.002	0.067	0.008	0.008	0.014	0.112	0.022	0.018	0.029
10	0.018	0.002	0.002	0.005	0.067	0.013	0.012	0.024	0.114	0.034	0.033	0.050
25	0.017	0.004	0.004	0.011	0.058	0.025	0.024	0.041	0.110	0.052	0.054	0.075
50	0.014	0.006	0.007	0.015	0.055	0.033	0.034	0.051	0.105	0.066	0.067	0.092
75	0.013	0.006	0.007	0.016	0.055	0.034	0.037	0.057	0.106	0.073	0.076	0.100
100	0.012	0.006	0.008	0.017	0.056	0.039	0.041	0.062	0.106	0.078	0.083	0.109
125	0.011	0.007	0.008	0.018	0.057	0.040	0.043	0.066	0.107	0.084	0.087	0.113
150	0.010	0.007	0.008	0.018	0.057	0.043	0.045	0.068	0.109	0.088	0.091	0.118
200	0.011	0.008	0.009	0.020	0.055	0.046	0.049	0.072	0.109	0.091	0.094	0.122
250	0.011	0.009	0.010	0.021	0.057	0.049	0.052	0.075	0.109	0.093	0.098	0.128
Right-sided tests - $T = 250$												
-5	0.009	0.015	0.011	0.028	0.043	0.072	0.054	0.103	0.088	0.143	0.113	0.182
-2.5	0.008	0.017	0.022	0.031	0.046	0.102	0.086	0.133	0.092	0.244	0.186	0.282
0	0.010	0.020	0.027	0.037	0.054	0.107	0.117	0.150	0.113	0.223	0.223	0.273
2.5	0.011	0.020	0.025	0.037	0.058	0.106	0.119	0.151	0.123	0.218	0.227	0.274
5	0.010	0.021	0.025	0.037	0.056	0.101	0.113	0.149	0.121	0.203	0.211	0.253
10	0.011	0.021	0.025	0.038	0.058	0.0923	0.100	0.132	0.113	0.183	0.189	0.229
25	0.012	0.018	0.022	0.034	0.059	0.082	0.088	0.116	0.114	0.152	0.158	0.195
50	0.011	0.015	0.018	0.032	0.056	0.071	0.076	0.107	0.112	0.135	0.142	0.177
75	0.012	0.015	0.019	0.030	0.056	0.066	0.073	0.101	0.107	0.128	0.133	0.167
100	0.011	0.014	0.016	0.029	0.055	0.063	0.070	0.097	0.107	0.120	0.126	0.158
125	0.012	0.014	0.017	0.028	0.055	0.060	0.066	0.092	0.106	0.116	0.122	0.153
150	0.012	0.014	0.017	0.028	0.053	0.058	0.063	0.089	0.107	0.113	0.120	0.150
200	0.012	0.012	0.015	0.027	0.054	0.056	0.060	0.087	0.107	0.108	0.114	0.146
250	0.012	0.011	0.014	0.025	0.054	0.054	0.060	0.085	0.108	0.105	0.109	0.142
Two-sided tests - $T = 250$												
-5	0.009	0.008	0.007	0.016	0.045	0.037	0.029	0.064	0.092	0.076	0.058	0.114
-2.5	0.007	0.008	0.012	0.017	0.042	0.044	0.048	0.069	0.085	0.102	0.087	0.136
0	0.008	0.009	0.014	0.020	0.048	0.052	0.064	0.083	0.099	0.109	0.120	0.155
2.5	0.009	0.010	0.014	0.021	0.047	0.052	0.062	0.087	0.103	0.110	0.124	0.160
5	0.009	0.010	0.014	0.022	0.048	0.053	0.059	0.087	0.103	0.109	0.121	0.163
10	0.009	0.011	0.012	0.024	0.054	0.056	0.061	0.087	0.102	0.105	0.112	0.156
25	0.012	0.010	0.012	0.025	0.053	0.055	0.060	0.093	0.109	0.106	0.112	0.157
50	0.011	0.011	0.013	0.027	0.053	0.053	0.059	0.091	0.108	0.103	0.110	0.158
75	0.012	0.011	0.013	0.027	0.053	0.052	0.058	0.092	0.106	0.101	0.109	0.158
100	0.011	0.011	0.013	0.028	0.053	0.051	0.057	0.091	0.107	0.100	0.111	0.159
125	0.010	0.010	0.013	0.028	0.054	0.050	0.056	0.091	0.107	0.100	0.109	0.158
150	0.010	0.010	0.014	0.029	0.054	0.050	0.055	0.091	0.106	0.099	0.108	0.157
200	0.010	0.010	0.014	0.027	0.053	0.049	0.057	0.092	0.107	0.100	0.109	0.159
250	0.011	0.010	0.014	0.027	0.053	0.050	0.057	0.095	0.110	0.102	0.112	0.160

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.002	0.045	0.007	0.005	0.014	0.091	0.022	0.016	0.035
-2.5	0.005	0.000	0.000	0.001	0.039	0.002	0.002	0.003	0.088	0.005	0.003	0.009
0	0.014	0.000	0.000	0.001	0.059	0.004	0.003	0.006	0.100	0.007	0.007	0.011
2.5	0.020	0.001	0.000	0.002	0.071	0.005	0.004	0.010	0.118	0.015	0.014	0.022
5	0.024	0.001	0.001	0.003	0.075	0.009	0.008	0.015	0.123	0.023	0.020	0.034
10	0.021	0.002	0.002	0.006	0.070	0.016	0.015	0.026	0.121	0.037	0.033	0.055
25	0.016	0.005	0.005	0.011	0.063	0.024	0.023	0.044	0.115	0.056	0.054	0.080
50	0.013	0.006	0.006	0.014	0.060	0.030	0.031	0.056	0.114	0.070	0.071	0.100
75	0.011	0.006	0.006	0.014	0.058	0.035	0.034	0.062	0.113	0.078	0.078	0.109
100	0.011	0.006	0.006	0.016	0.059	0.037	0.037	0.065	0.110	0.081	0.080	0.112
125	0.012	0.007	0.006	0.018	0.057	0.038	0.038	0.065	0.109	0.083	0.082	0.113
150	0.012	0.008	0.007	0.018	0.056	0.040	0.039	0.067	0.107	0.085	0.084	0.116
200	0.011	0.008	0.008	0.020	0.055	0.042	0.043	0.069	0.104	0.085	0.086	0.117
250	0.012	0.007	0.009	0.021	0.055	0.044	0.044	0.070	0.105	0.088	0.088	0.120
Right-sided tests - $T = 1000$												
-5	0.010	0.015	0.010	0.028	0.046	0.073	0.052	0.106	0.093	0.145	0.113	0.188
-2.5	0.006	0.015	0.015	0.025	0.037	0.093	0.076	0.119	0.080	0.238	0.171	0.275
0	0.006	0.017	0.020	0.033	0.046	0.100	0.103	0.138	0.098	0.216	0.209	0.267
2.5	0.008	0.019	0.023	0.036	0.056	0.107	0.111	0.145	0.114	0.208	0.208	0.256
5	0.010	0.020	0.022	0.035	0.054	0.106	0.109	0.143	0.117	0.197	0.195	0.240
10	0.011	0.022	0.022	0.036	0.056	0.095	0.096	0.131	0.113	0.177	0.175	0.220
25	0.012	0.018	0.019	0.034	0.056	0.080	0.081	0.112	0.104	0.153	0.151	0.193
50	0.011	0.015	0.016	0.032	0.054	0.073	0.073	0.103	0.101	0.136	0.137	0.176
75	0.010	0.015	0.015	0.030	0.053	0.069	0.070	0.101	0.104	0.131	0.132	0.168
100	0.009	0.014	0.014	0.029	0.052	0.067	0.067	0.098	0.105	0.126	0.128	0.162
125	0.010	0.013	0.013	0.029	0.052	0.064	0.065	0.095	0.102	0.123	0.124	0.159
150	0.010	0.013	0.012	0.029	0.053	0.063	0.063	0.094	0.103	0.121	0.122	0.157
200	0.010	0.012	0.012	0.028	0.053	0.061	0.062	0.091	0.100	0.116	0.118	0.153
250	0.011	0.012	0.012	0.027	0.053	0.060	0.061	0.091	0.100	0.113	0.115	0.150
Two-sided tests- $T = 1000$												
-5	0.009	0.007	0.006	0.016	0.049	0.039	0.026	0.066	0.099	0.079	0.056	0.120
-2.5	0.005	0.007	0.008	0.012	0.033	0.040	0.039	0.061	0.073	0.095	0.077	0.122
0	0.005	0.007	0.008	0.015	0.041	0.049	0.054	0.075	0.086	0.105	0.106	0.144
2.5	0.006	0.010	0.011	0.017	0.046	0.055	0.059	0.085	0.098	0.112	0.115	0.155
5	0.008	0.011	0.013	0.021	0.046	0.054	0.057	0.092	0.104	0.116	0.117	0.158
10	0.010	0.013	0.012	0.024	0.051	0.056	0.058	0.091	0.104	0.111	0.111	0.156
25	0.011	0.012	0.012	0.027	0.053	0.056	0.056	0.091	0.101	0.103	0.104	0.156
50	0.010	0.010	0.010	0.027	0.052	0.054	0.054	0.092	0.101	0.102	0.103	0.160
75	0.011	0.011	0.011	0.026	0.051	0.051	0.053	0.093	0.103	0.103	0.104	0.163
100	0.011	0.011	0.010	0.025	0.053	0.050	0.053	0.093	0.103	0.101	0.104	0.163
125	0.011	0.011	0.011	0.026	0.052	0.052	0.053	0.095	0.104	0.103	0.103	0.160
150	0.011	0.010	0.011	0.026	0.054	0.051	0.053	0.094	0.103	0.101	0.103	0.161
200	0.012	0.011	0.010	0.026	0.054	0.051	0.053	0.097	0.105	0.103	0.104	0.160
250	0.011	0.010	0.011	0.027	0.054	0.053	0.054	0.095	0.105	0.104	0.105	0.162

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.10: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP5 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T$ and $\psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = [\sigma^2_{ut} \quad -0.9\sigma_{ut}\sigma_{vt}; \quad -0.9\sigma_{ut}\sigma_{vt} \quad \sigma^2_{vt}]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
c		1%				5%				10%		
Left-sided tests - $T = 250$												
-5	0.007	0.000	0.000	0.002	0.040	0.007	0.005	0.013	0.090	0.019	0.013	0.032
-2.5	0.006	0.000	0.000	0.001	0.035	0.002	0.002	0.004	0.082	0.005	0.004	0.009
0	0.013	0.000	0.000	0.001	0.047	0.003	0.003	0.005	0.085	0.009	0.007	0.012
2.5	0.016	0.001	0.001	0.002	0.057	0.006	0.005	0.010	0.099	0.015	0.013	0.021
5	0.019	0.001	0.001	0.002	0.064	0.010	0.008	0.015	0.110	0.023	0.020	0.032
10	0.018	0.002	0.002	0.006	0.064	0.015	0.014	0.026	0.114	0.038	0.035	0.054
25	0.016	0.004	0.004	0.012	0.057	0.026	0.026	0.042	0.109	0.056	0.056	0.078
50	0.013	0.006	0.006	0.015	0.055	0.033	0.034	0.053	0.103	0.066	0.068	0.094
75	0.013	0.006	0.008	0.016	0.054	0.036	0.038	0.057	0.106	0.074	0.077	0.104
100	0.012	0.007	0.008	0.017	0.056	0.038	0.041	0.063	0.106	0.083	0.084	0.109
125	0.012	0.007	0.008	0.018	0.055	0.041	0.045	0.067	0.107	0.085	0.088	0.113
150	0.011	0.008	0.009	0.018	0.054	0.042	0.046	0.068	0.107	0.087	0.089	0.116
200	0.011	0.008	0.010	0.020	0.056	0.045	0.049	0.072	0.106	0.090	0.094	0.124
250	0.010	0.009	0.010	0.022	0.056	0.048	0.053	0.075	0.108	0.094	0.099	0.128
Right-sided tests - $T = 250$												
-5	0.007	0.015	0.010	0.027	0.044	0.072	0.051	0.103	0.089	0.143	0.113	0.181
-2.5	0.008	0.017	0.021	0.031	0.045	0.101	0.082	0.135	0.096	0.242	0.180	0.276
0	0.010	0.019	0.026	0.037	0.056	0.106	0.115	0.149	0.113	0.219	0.216	0.269
2.5	0.010	0.019	0.025	0.037	0.058	0.103	0.116	0.149	0.122	0.212	0.219	0.266
5	0.011	0.021	0.024	0.037	0.056	0.100	0.108	0.144	0.122	0.197	0.204	0.245
10	0.012	0.019	0.024	0.035	0.058	0.092	0.098	0.132	0.117	0.178	0.185	0.227
25	0.012	0.018	0.021	0.035	0.058	0.080	0.085	0.114	0.112	0.151	0.157	0.193
50	0.012	0.016	0.019	0.032	0.055	0.068	0.073	0.105	0.1108	0.134	0.138	0.173
75	0.011	0.015	0.018	0.029	0.053	0.064	0.068	0.098	0.108	0.125	0.131	0.163
100	0.012	0.014	0.018	0.029	0.054	0.060	0.066	0.094	0.106	0.120	0.125	0.157
125	0.012	0.013	0.017	0.028	0.053	0.059	0.062	0.091	0.107	0.114	0.120	0.153
150	0.012	0.013	0.017	0.028	0.054	0.057	0.061	0.087	0.105	0.113	0.117	0.150
200	0.011	0.012	0.015	0.027	0.054	0.055	0.060	0.087	0.106	0.109	0.114	0.145
250	0.011	0.012	0.014	0.025	0.055	0.052	0.058	0.082	0.107	0.105	0.110	0.141
Two-sided tests - $T = 250$												
-5	0.007	0.008	0.007	0.017	0.045	0.037	0.027	0.063	0.092	0.077	0.056	0.116
-2.5	0.007	0.009	0.012	0.017	0.041	0.044	0.046	0.070	0.089	0.103	0.084	0.139
0	0.008	0.010	0.014	0.021	0.047	0.051	0.062	0.083	0.098	0.109	0.118	0.154
2.5	0.008	0.010	0.013	0.020	0.049	0.054	0.062	0.087	0.102	0.108	0.121	0.158
5	0.009	0.010	0.014	0.021	0.048	0.053	0.061	0.088	0.105	0.109	0.116	0.159
10	0.010	0.010	0.012	0.024	0.051	0.055	0.061	0.088	0.105	0.108	0.112	0.157
25	0.011	0.010	0.013	0.025	0.055	0.055	0.059	0.092	0.106	0.104	0.110	0.156
50	0.011	0.010	0.013	0.028	0.055	0.053	0.059	0.090	0.105	0.100	0.108	0.158
75	0.010	0.010	0.013	0.029	0.053	0.053	0.059	0.091	0.104	0.099	0.106	0.155
100	0.011	0.010	0.013	0.029	0.053	0.050	0.058	0.091	0.105	0.099	0.107	0.156
125	0.011	0.010	0.014	0.030	0.053	0.050	0.057	0.092	0.106	0.099	0.107	0.158
150	0.010	0.011	0.014	0.029	0.053	0.051	0.058	0.091	0.105	0.098	0.107	0.155
200	0.010	0.012	0.014	0.028	0.053	0.050	0.057	0.093	0.107	0.100	0.109	0.159
250	0.010	0.011	0.014	0.028	0.053	0.049	0.058	0.096	0.108	0.101	0.111	0.157

	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
c		1%				5%				10%		
Left-sided tests - $T = 1000$												
-5	0.010	0.001	0.000	0.002	0.045	0.008	0.005	0.016	0.094	0.025	0.017	0.038
-2.5	0.006	0.000	0.000	0.001	0.041	0.002	0.002	0.004	0.091	0.006	0.004	0.010
0	0.013	0.001	0.000	0.002	0.055	0.004	0.003	0.006	0.094	0.008	0.007	0.012
2.5	0.019	0.001	0.000	0.002	0.068	0.006	0.005	0.010	0.116	0.016	0.014	0.024
5	0.020	0.001	0.001	0.003	0.069	0.010	0.008	0.017	0.120	0.024	0.021	0.039
10	0.019	0.002	0.002	0.007	0.067	0.016	0.015	0.028	0.118	0.039	0.035	0.056
25	0.015	0.005	0.005	0.012	0.059	0.025	0.024	0.044	0.113	0.056	0.055	0.080
50	0.012	0.006	0.006	0.014	0.057	0.031	0.031	0.055	0.113	0.070	0.070	0.101
75	0.011	0.006	0.006	0.014	0.059	0.035	0.035	0.063	0.112	0.077	0.078	0.110
100	0.010	0.007	0.006	0.016	0.057	0.039	0.039	0.063	0.110	0.082	0.081	0.113
125	0.011	0.006	0.007	0.017	0.054	0.041	0.040	0.066	0.107	0.083	0.084	0.114
150	0.011	0.007	0.007	0.017	0.056	0.040	0.041	0.068	0.107	0.085	0.086	0.117
200	0.012	0.007	0.008	0.019	0.056	0.043	0.043	0.070	0.104	0.088	0.087	0.119
250	0.012	0.009	0.009	0.021	0.055	0.044	0.044	0.072	0.105	0.090	0.091	0.120
Right-sided tests - $T = 1000$												
-5	0.009	0.015	0.009	0.026	0.046	0.073	0.050	0.105	0.092	0.146	0.111	0.189
-2.5	0.005	0.013	0.015	0.025	0.038	0.095	0.074	0.123	0.082	0.230	0.165	0.266
0	0.007	0.017	0.018	0.030	0.046	0.098	0.101	0.136	0.101	0.212	0.204	0.260
2.5	0.008	0.020	0.021	0.035	0.054	0.103	0.106	0.144	0.113	0.204	0.201	0.252
5	0.010	0.020	0.021	0.035	0.055	0.103	0.103	0.138	0.115	0.189	0.191	0.236
10	0.012	0.020	0.021	0.035	0.055	0.092	0.093	0.127	0.110	0.174	0.173	0.216
25	0.011	0.017	0.019	0.033	0.054	0.078	0.077	0.110	0.103	0.147	0.148	0.190
50	0.011	0.016	0.017	0.032	0.053	0.070	0.071	0.101	0.103	0.134	0.136	0.171
75	0.011	0.015	0.015	0.029	0.053	0.067	0.068	0.101	0.106	0.130	0.131	0.167
100	0.010	0.014	0.014	0.029	0.054	0.066	0.067	0.099	0.104	0.125	0.126	0.160
125	0.010	0.012	0.012	0.028	0.054	0.065	0.067	0.096	0.105	0.123	0.124	0.157
150	0.011	0.013	0.012	0.028	0.054	0.062	0.065	0.093	0.102	0.120	0.122	0.154
200	0.009	0.013	0.012	0.028	0.055	0.063	0.063	0.093	0.103	0.116	0.118	0.150
250	0.010	0.013	0.012	0.027	0.054	0.060	0.061	0.092	0.102	0.113	0.114	0.147
Two-sided tests- $T = 1000$												
-5	0.009	0.008	0.006	0.016	0.049	0.038	0.025	0.068	0.098	0.080	0.055	0.121
-2.5	0.005	0.006	0.008	0.013	0.034	0.040	0.036	0.061	0.077	0.096	0.075	0.126
0	0.005	0.007	0.008	0.015	0.039	0.049	0.052	0.075	0.087	0.102	0.104	0.142
2.5	0.007	0.009	0.011	0.019	0.045	0.055	0.057	0.082	0.095	0.109	0.111	0.154
5	0.007	0.011	0.013	0.020	0.046	0.055	0.055	0.089	0.100	0.111	0.111	0.155
10	0.010	0.012	0.012	0.023	0.049	0.056	0.056	0.091	0.098	0.109	0.108	0.156
25	0.012	0.012	0.012	0.027	0.052	0.053	0.054	0.089	0.100	0.101	0.101	0.154
50	0.011	0.010	0.011	0.028	0.052	0.054	0.053	0.090	0.100	0.101	0.102	0.156
75	0.011	0.011	0.011	0.026	0.050	0.049	0.050	0.092	0.104	0.102	0.103	0.164
100	0.010	0.010	0.010	0.027	0.051	0.049	0.051	0.095	0.106	0.104	0.105	0.162
125	0.011	0.010	0.010	0.027	0.052	0.051	0.051	0.098	0.106	0.104	0.107	0.161
150	0.010	0.011	0.011	0.026	0.052	0.052	0.054	0.096	0.105	0.103	0.106	0.161
200	0.012	0.010	0.011	0.027	0.051	0.051	0.053	0.098	0.107	0.103	0.105	0.162
250	0.011	0.011	0.011	0.029	0.054	0.051	0.054	0.097	0.106	0.103	0.105	0.164

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.11: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP5 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T$ and $\psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = \left[\sigma^2_{ut} \quad -0.5\sigma_{ut}\sigma_{vt}; \quad -0.5\sigma_{ut}\sigma_{vt} \quad \sigma^2_{vt}\right]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.008	0.003	0.002	0.007	0.046	0.021	0.012	0.036	0.098	0.050	0.034	0.068
-2.5	0.008	0.001	0.001	0.002	0.040	0.010	0.004	0.013	0.086	0.022	0.014	0.029
0	0.008	0.002	0.001	0.003	0.038	0.011	0.010	0.017	0.075	0.026	0.022	0.036
2.5	0.010	0.002	0.002	0.005	0.046	0.018	0.014	0.026	0.090	0.042	0.035	0.054
5	0.012	0.003	0.003	0.008	0.051	0.022	0.020	0.035	0.098	0.050	0.045	0.069
10	0.012	0.005	0.004	0.010	0.054	0.027	0.026	0.045	0.103	0.062	0.060	0.085
25	0.011	0.007	0.008	0.014	0.053	0.034	0.035	0.058	0.105	0.075	0.075	0.102
50	0.010	0.007	0.008	0.016	0.050	0.039	0.041	0.062	0.105	0.082	0.084	0.115
75	0.010	0.007	0.008	0.018	0.054	0.042	0.045	0.068	0.103	0.084	0.088	0.117
100	0.010	0.008	0.010	0.019	0.055	0.044	0.048	0.070	0.101	0.086	0.090	0.120
125	0.011	0.009	0.010	0.022	0.052	0.044	0.047	0.072	0.101	0.088	0.092	0.120
150	0.011	0.009	0.011	0.022	0.052	0.045	0.048	0.070	0.101	0.091	0.094	0.122
200	0.011	0.010	0.013	0.022	0.052	0.046	0.050	0.072	0.100	0.092	0.095	0.125
250	0.011	0.011	0.012	0.022	0.052	0.046	0.051	0.076	0.105	0.097	0.102	0.132
Right-sided tests - $T = 250$												
-5	0.010	0.014	0.006	0.028	0.049	0.069	0.042	0.097	0.094	0.131	0.091	0.169
-2.5	0.011	0.020	0.015	0.031	0.052	0.100	0.061	0.126	0.110	0.193	0.132	0.221
0	0.011	0.019	0.025	0.033	0.057	0.094	0.086	0.126	0.117	0.183	0.167	0.218
2.5	0.011	0.017	0.020	0.033	0.057	0.086	0.088	0.124	0.116	0.165	0.163	0.208
5	0.012	0.019	0.021	0.033	0.056	0.080	0.082	0.116	0.113	0.151	0.153	0.195
10	0.012	0.018	0.020	0.033	0.055	0.070	0.074	0.106	0.109	0.143	0.146	0.184
25	0.011	0.014	0.018	0.030	0.055	0.067	0.070	0.100	0.108	0.129	0.132	0.168
50	0.011	0.014	0.015	0.027	0.054	0.063	0.067	0.095	0.106	0.117	0.122	0.156
75	0.011	0.013	0.016	0.027	0.052	0.055	0.061	0.089	0.105	0.113	0.116	0.152
100	0.011	0.012	0.015	0.027	0.052	0.055	0.059	0.086	0.103	0.111	0.115	0.148
125	0.010	0.011	0.013	0.026	0.051	0.055	0.058	0.086	0.105	0.108	0.114	0.145
150	0.010	0.011	0.013	0.026	0.053	0.055	0.059	0.085	0.105	0.107	0.112	0.144
200	0.010	0.011	0.013	0.025	0.053	0.054	0.059	0.085	0.108	0.108	0.112	0.142
250	0.009	0.011	0.013	0.023	0.053	0.053	0.058	0.084	0.106	0.103	0.108	0.142
Two-sided tests - $T = 250$												
-5	0.009	0.008	0.004	0.020	0.050	0.043	0.023	0.075	0.098	0.090	0.053	0.133
-2.5	0.010	0.010	0.008	0.020	0.046	0.053	0.035	0.076	0.100	0.109	0.065	0.139
0	0.010	0.010	0.013	0.021	0.047	0.051	0.051	0.081	0.097	0.104	0.096	0.143
2.5	0.009	0.010	0.012	0.022	0.048	0.051	0.054	0.085	0.101	0.102	0.102	0.150
5	0.009	0.009	0.012	0.023	0.049	0.052	0.052	0.084	0.100	0.102	0.102	0.151
10	0.011	0.010	0.013	0.027	0.049	0.052	0.052	0.086	0.100	0.099	0.100	0.151
25	0.010	0.011	0.013	0.026	0.050	0.049	0.053	0.090	0.102	0.100	0.105	0.157
50	0.011	0.010	0.014	0.027	0.049	0.048	0.053	0.093	0.103	0.101	0.108	0.157
75	0.009	0.009	0.012	0.026	0.050	0.050	0.056	0.092	0.104	0.097	0.106	0.157
100	0.009	0.010	0.012	0.027	0.052	0.050	0.057	0.092	0.104	0.099	0.107	0.156
125	0.010	0.010	0.013	0.027	0.053	0.050	0.060	0.092	0.102	0.098	0.105	0.158
150	0.010	0.010	0.013	0.027	0.051	0.050	0.058	0.092	0.104	0.098	0.107	0.155
200	0.010	0.011	0.014	0.028	0.050	0.050	0.057	0.093	0.104	0.098	0.109	0.157
250	0.010	0.012	0.015	0.027	0.050	0.048	0.055	0.092	0.107	0.099	0.109	0.160

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.010	0.003	0.001	0.008	0.050	0.024	0.012	0.038	0.101	0.053	0.033	0.076
-2.5	0.010	0.001	0.001	0.003	0.047	0.011	0.006	0.017	0.093	0.026	0.015	0.035
0	0.010	0.002	0.001	0.004	0.045	0.013	0.010	0.020	0.086	0.032	0.025	0.042
2.5	0.012	0.003	0.002	0.006	0.052	0.019	0.016	0.029	0.097	0.044	0.037	0.058
5	0.012	0.004	0.004	0.008	0.053	0.023	0.018	0.036	0.101	0.052	0.047	0.070
10	0.012	0.006	0.005	0.011	0.052	0.028	0.025	0.045	0.103	0.061	0.056	0.086
25	0.012	0.008	0.008	0.016	0.053	0.033	0.033	0.056	0.102	0.072	0.071	0.102
50	0.012	0.008	0.008	0.018	0.051	0.038	0.038	0.064	0.104	0.081	0.081	0.112
75	0.011	0.008	0.008	0.018	0.053	0.041	0.040	0.069	0.104	0.086	0.087	0.117
100	0.011	0.008	0.008	0.019	0.053	0.042	0.043	0.070	0.105	0.088	0.089	0.123
125	0.011	0.008	0.009	0.019	0.054	0.045	0.045	0.073	0.105	0.091	0.091	0.127
150	0.011	0.009	0.009	0.020	0.053	0.045	0.045	0.075	0.108	0.096	0.095	0.129
200	0.010	0.009	0.009	0.023	0.052	0.046	0.046	0.077	0.108	0.099	0.098	0.129
250	0.012	0.010	0.010	0.024	0.054	0.048	0.048	0.079	0.106	0.097	0.098	0.131
Right-sided tests - $T = 1000$												
-5	0.010	0.015	0.006	0.029	0.052	0.069	0.040	0.098	0.097	0.133	0.092	0.174
-2.5	0.007	0.018	0.011	0.027	0.046	0.096	0.052	0.118	0.101	0.187	0.120	0.216
0	0.008	0.017	0.015	0.030	0.049	0.087	0.074	0.117	0.106	0.172	0.148	0.206
2.5	0.009	0.017	0.017	0.029	0.052	0.083	0.077	0.112	0.106	0.159	0.151	0.197
5	0.009	0.017	0.016	0.029	0.050	0.078	0.073	0.110	0.105	0.150	0.145	0.189
10	0.010	0.015	0.014	0.029	0.051	0.073	0.073	0.102	0.103	0.137	0.133	0.177
25	0.012	0.016	0.016	0.028	0.049	0.063	0.063	0.093	0.098	0.122	0.122	0.160
50	0.011	0.016	0.015	0.029	0.050	0.061	0.061	0.089	0.097	0.117	0.117	0.153
75	0.011	0.013	0.014	0.027	0.053	0.060	0.060	0.088	0.098	0.115	0.115	0.151
100	0.011	0.013	0.013	0.026	0.052	0.060	0.060	0.091	0.102	0.114	0.115	0.149
125	0.010	0.012	0.012	0.025	0.052	0.059	0.060	0.089	0.102	0.113	0.115	0.149
150	0.010	0.012	0.011	0.025	0.053	0.058	0.061	0.088	0.104	0.113	0.115	0.146
200	0.010	0.011	0.011	0.025	0.054	0.059	0.060	0.088	0.103	0.109	0.111	0.146
250	0.010	0.011	0.011	0.027	0.053	0.059	0.059	0.088	0.104	0.111	0.110	0.145
Two-sided tests- $T = 1000$												
-5	0.010	0.008	0.003	0.021	0.053	0.046	0.021	0.079	0.104	0.093	0.052	0.137
-2.5	0.006	0.009	0.006	0.015	0.044	0.051	0.028	0.070	0.094	0.106	0.058	0.134
0	0.007	0.009	0.009	0.018	0.043	0.049	0.044	0.073	0.089	0.100	0.084	0.137
2.5	0.010	0.010	0.009	0.019	0.045	0.050	0.045	0.078	0.093	0.101	0.093	0.141
5	0.010	0.010	0.010	0.021	0.046	0.049	0.046	0.080	0.095	0.101	0.091	0.146
10	0.011	0.011	0.011	0.023	0.047	0.051	0.048	0.084	0.097	0.101	0.098	0.147
25	0.013	0.012	0.012	0.028	0.049	0.050	0.049	0.084	0.096	0.096	0.095	0.148
50	0.012	0.011	0.011	0.029	0.052	0.052	0.051	0.089	0.098	0.099	0.099	0.153
75	0.011	0.011	0.011	0.027	0.052	0.050	0.051	0.092	0.102	0.102	0.101	0.157
100	0.011	0.011	0.011	0.027	0.050	0.050	0.050	0.094	0.103	0.101	0.103	0.161
125	0.010	0.010	0.011	0.027	0.050	0.049	0.050	0.095	0.104	0.103	0.105	0.162
150	0.009	0.011	0.010	0.027	0.052	0.050	0.052	0.096	0.104	0.103	0.106	0.163
200	0.010	0.010	0.010	0.027	0.053	0.054	0.056	0.098	0.105	0.104	0.106	0.165
250	0.010	0.010	0.009	0.028	0.056	0.056	0.057	0.098	0.106	0.107	0.107	0.168

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.12: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP5 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = \begin{bmatrix}\sigma^2_{ut} & 0; & 0 & \sigma^2_{vt}\end{bmatrix}$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.009	0.009	0.004	0.019	0.050	0.048	0.026	0.076	0.102	0.101	0.064	0.131
-2.5	0.009	0.010	0.004	0.015	0.048	0.050	0.023	0.065	0.100	0.100	0.056	0.121
0	0.010	0.010	0.011	0.017	0.046	0.048	0.039	0.064	0.094	0.098	0.077	0.120
2.5	0.010	0.011	0.011	0.019	0.046	0.049	0.043	0.067	0.095	0.095	0.087	0.123
5	0.010	0.010	0.010	0.020	0.051	0.051	0.047	0.073	0.097	0.096	0.091	0.123
10	0.009	0.011	0.011	0.022	0.052	0.052	0.050	0.076	0.101	0.100	0.097	0.126
25	0.010	0.011	0.013	0.023	0.051	0.049	0.052	0.076	0.098	0.097	0.096	0.129
50	0.010	0.010	0.011	0.022	0.052	0.050	0.053	0.077	0.100	0.099	0.101	0.128
75	0.010	0.009	0.011	0.021	0.052	0.051	0.055	0.078	0.101	0.098	0.102	0.133
100	0.008	0.010	0.011	0.021	0.053	0.051	0.055	0.078	0.100	0.098	0.102	0.132
125	0.010	0.009	0.013	0.023	0.052	0.049	0.053	0.079	0.101	0.098	0.102	0.131
150	0.010	0.010	0.013	0.024	0.052	0.050	0.053	0.079	0.099	0.098	0.101	0.133
200	0.011	0.011	0.013	0.024	0.052	0.052	0.055	0.077	0.099	0.096	0.099	0.132
250	0.010	0.010	0.013	0.023	0.052	0.051	0.055	0.078	0.100	0.097	0.100	0.133
Right-sided tests - $T = 250$												
-5	0.013	0.012	0.003	0.023	0.055	0.053	0.030	0.077	0.106	0.104	0.066	0.133
-2.5	0.011	0.012	0.005	0.019	0.051	0.052	0.028	0.067	0.100	0.106	0.060	0.124
0	0.009	0.010	0.009	0.020	0.051	0.053	0.043	0.070	0.097	0.101	0.086	0.125
2.5	0.011	0.011	0.011	0.020	0.052	0.052	0.046	0.073	0.099	0.100	0.095	0.126
5	0.011	0.012	0.011	0.022	0.052	0.052	0.049	0.073	0.099	0.098	0.094	0.127
10	0.013	0.012	0.012	0.022	0.050	0.050	0.050	0.075	0.101	0.096	0.095	0.131
25	0.012	0.011	0.013	0.024	0.053	0.050	0.052	0.079	0.104	0.100	0.102	0.133
50	0.011	0.011	0.013	0.024	0.051	0.050	0.053	0.079	0.100	0.098	0.102	0.136
75	0.010	0.010	0.011	0.023	0.051	0.050	0.053	0.078	0.101	0.100	0.103	0.136
100	0.010	0.011	0.012	0.023	0.051	0.049	0.053	0.080	0.102	0.099	0.105	0.136
125	0.010	0.010	0.012	0.023	0.052	0.049	0.053	0.079	0.101	0.098	0.103	0.136
150	0.010	0.010	0.012	0.024	0.052	0.048	0.053	0.078	0.103	0.099	0.105	0.137
200	0.010	0.011	0.013	0.026	0.054	0.051	0.055	0.081	0.103	0.100	0.104	0.137
250	0.010	0.012	0.013	0.026	0.054	0.051	0.056	0.080	0.105	0.100	0.105	0.138
Two-sided tests - $T = 250$												
-5	0.011	0.010	0.004	0.025	0.054	0.051	0.025	0.086	0.103	0.100	0.056	0.153
-2.5	0.010	0.011	0.005	0.020	0.050	0.053	0.024	0.073	0.096	0.103	0.051	0.132
0	0.009	0.011	0.012	0.019	0.048	0.052	0.045	0.075	0.094	0.100	0.081	0.134
2.5	0.010	0.011	0.012	0.023	0.049	0.050	0.047	0.083	0.098	0.100	0.090	0.141
5	0.010	0.011	0.011	0.024	0.051	0.053	0.050	0.088	0.103	0.102	0.097	0.146
10	0.010	0.011	0.012	0.026	0.053	0.053	0.052	0.090	0.103	0.102	0.100	0.150
25	0.010	0.011	0.013	0.028	0.052	0.049	0.054	0.090	0.102	0.100	0.104	0.155
50	0.010	0.010	0.014	0.026	0.051	0.049	0.056	0.094	0.103	0.098	0.106	0.155
75	0.010	0.010	0.012	0.026	0.049	0.048	0.055	0.093	0.104	0.100	0.108	0.156
100	0.010	0.009	0.012	0.027	0.049	0.048	0.056	0.096	0.104	0.100	0.107	0.158
125	0.009	0.009	0.012	0.027	0.051	0.049	0.056	0.096	0.103	0.097	0.106	0.158
150	0.009	0.011	0.013	0.029	0.052	0.051	0.057	0.095	0.104	0.096	0.106	0.156
200	0.010	0.011	0.016	0.029	0.054	0.052	0.061	0.095	0.105	0.101	0.110	0.158
250	0.010	0.011	0.015	0.029	0.054	0.053	0.061	0.096	0.105	0.102	0.110	0.158

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.012	0.010	0.003	0.022	0.052	0.050	0.026	0.075	0.100	0.099	0.062	0.130
-2.5	0.010	0.011	0.003	0.019	0.050	0.053	0.026	0.067	0.101	0.103	0.058	0.121
0	0.010	0.012	0.010	0.019	0.048	0.052	0.041	0.069	0.099	0.102	0.084	0.123
2.5	0.010	0.011	0.009	0.018	0.047	0.048	0.044	0.070	0.099	0.101	0.089	0.127
5	0.011	0.010	0.009	0.019	0.049	0.049	0.045	0.071	0.099	0.101	0.091	0.129
10	0.011	0.011	0.010	0.020	0.048	0.050	0.046	0.071	0.098	0.099	0.092	0.128
25	0.011	0.012	0.012	0.023	0.049	0.050	0.048	0.072	0.096	0.096	0.094	0.130
50	0.011	0.011	0.010	0.023	0.052	0.052	0.052	0.078	0.101	0.101	0.099	0.132
75	0.011	0.010	0.010	0.023	0.053	0.051	0.052	0.080	0.103	0.100	0.101	0.132
100	0.012	0.011	0.011	0.023	0.052	0.051	0.052	0.079	0.102	0.102	0.101	0.136
125	0.011	0.012	0.011	0.023	0.052	0.052	0.052	0.080	0.101	0.101	0.101	0.138
150	0.011	0.011	0.011	0.023	0.052	0.052	0.051	0.079	0.103	0.102	0.103	0.138
200	0.010	0.011	0.011	0.024	0.054	0.051	0.053	0.081	0.104	0.101	0.102	0.138
250	0.010	0.010	0.010	0.025	0.053	0.053	0.053	0.080	0.101	0.101	0.101	0.137
Right-sided tests - $T = 1000$												
-5	0.010	0.010	0.003	0.021	0.053	0.052	0.025	0.079	0.105	0.104	0.063	0.133
-2.5	0.008	0.009	0.004	0.015	0.048	0.049	0.021	0.064	0.097	0.101	0.055	0.118
0	0.008	0.010	0.007	0.016	0.043	0.044	0.034	0.063	0.092	0.096	0.073	0.118
2.5	0.009	0.009	0.008	0.018	0.046	0.045	0.038	0.064	0.094	0.097	0.085	0.125
5	0.008	0.009	0.007	0.017	0.046	0.046	0.039	0.069	0.098	0.100	0.089	0.128
10	0.009	0.009	0.008	0.019	0.048	0.045	0.044	0.073	0.096	0.098	0.093	0.128
25	0.010	0.010	0.009	0.021	0.047	0.048	0.046	0.074	0.100	0.099	0.097	0.132
50	0.010	0.010	0.010	0.022	0.050	0.049	0.049	0.080	0.100	0.100	0.099	0.133
75	0.011	0.011	0.011	0.022	0.050	0.051	0.051	0.077	0.099	0.100	0.100	0.136
100	0.011	0.011	0.011	0.022	0.051	0.051	0.051	0.078	0.100	0.099	0.099	0.137
125	0.010	0.010	0.011	0.024	0.050	0.050	0.051	0.080	0.101	0.101	0.103	0.136
150	0.010	0.011	0.011	0.024	0.051	0.051	0.052	0.081	0.102	0.101	0.102	0.138
200	0.010	0.010	0.010	0.024	0.052	0.051	0.052	0.080	0.102	0.102	0.103	0.138
250	0.010	0.010	0.011	0.023	0.052	0.051	0.053	0.080	0.101	0.102	0.102	0.138
Two-sided tests- $T = 1000$												
-5	0.012	0.011	0.002	0.024	0.053	0.050	0.019	0.091	0.104	0.101	0.051	0.154
-2.5	0.010	0.011	0.003	0.018	0.048	0.053	0.020	0.074	0.098	0.101	0.047	0.131
0	0.008	0.011	0.010	0.019	0.045	0.049	0.038	0.073	0.091	0.096	0.075	0.132
2.5	0.009	0.012	0.010	0.021	0.046	0.047	0.040	0.076	0.094	0.093	0.081	0.134
5	0.012	0.011	0.011	0.022	0.044	0.045	0.039	0.077	0.094	0.094	0.084	0.140
10	0.011	0.010	0.010	0.023	0.046	0.047	0.042	0.081	0.094	0.096	0.090	0.144
25	0.011	0.010	0.010	0.026	0.050	0.049	0.048	0.086	0.096	0.097	0.094	0.146
50	0.011	0.011	0.011	0.027	0.050	0.049	0.048	0.092	0.101	0.102	0.100	0.158
75	0.012	0.012	0.011	0.028	0.051	0.052	0.050	0.093	0.102	0.102	0.103	0.157
100	0.012	0.012	0.012	0.028	0.051	0.051	0.052	0.094	0.103	0.102	0.103	0.157
125	0.011	0.012	0.012	0.029	0.052	0.052	0.053	0.094	0.102	0.102	0.103	0.160
150	0.011	0.011	0.012	0.028	0.053	0.051	0.053	0.096	0.103	0.103	0.102	0.160
200	0.011	0.010	0.011	0.027	0.053	0.053	0.054	0.096	0.104	0.103	0.105	0.161
250	0.009	0.010	0.010	0.028	0.052	0.052	0.053	0.098	0.104	0.104	0.105	0.160

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.13: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP6 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = \begin{bmatrix}\sigma_{ut}^2 & -0.95\sigma_{ut}\sigma_{vt}; & -0.95\sigma_{ut}\sigma_{vt} & \sigma_{vt}^2\end{bmatrix}$ and $\sigma_{ut}^2 = \sigma_{vt}^2 = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 1/4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.007	0.000	0.000	0.000	0.043	0.000	0.002	0.000	0.095	0.001	0.009	0.001
-2.5	0.008	0.000	0.000	0.000	0.071	0.000	0.000	0.000	0.152	0.000	0.000	0.000
0	0.005	0.000	0.000	0.000	0.014	0.000	0.000	0.000	0.024	0.000	0.000	0.000
2.5	0.018	0.000	0.000	0.000	0.048	0.001	0.001	0.001	0.080	0.006	0.005	0.006
5	0.028	0.001	0.001	0.001	0.066	0.010	0.008	0.012	0.107	0.024	0.021	0.027
10	0.025	0.003	0.003	0.006	0.069	0.020	0.019	0.027	0.111	0.042	0.041	0.054
25	0.017	0.006	0.007	0.012	0.062	0.029	0.031	0.046	0.110	0.062	0.062	0.082
50	0.013	0.007	0.008	0.014	0.057	0.036	0.038	0.056	0.111	0.072	0.075	0.103
75	0.013	0.006	0.008	0.016	0.055	0.037	0.039	0.061	0.113	0.080	0.083	0.113
100	0.011	0.007	0.008	0.018	0.057	0.040	0.043	0.068	0.109	0.084	0.088	0.114
125	0.012	0.008	0.008	0.019	0.055	0.043	0.046	0.068	0.110	0.087	0.090	0.117
150	0.012	0.008	0.009	0.020	0.057	0.045	0.047	0.070	0.109	0.092	0.094	0.119
200	0.013	0.009	0.011	0.021	0.059	0.049	0.052	0.074	0.107	0.092	0.097	0.124
250	0.011	0.009	0.011	0.022	0.060	0.051	0.056	0.077	0.107	0.096	0.101	0.126
Right-sided tests - $T = 250$												
-5	0.004	0.010	0.051	0.009	0.034	0.062	0.172	0.036	0.078	0.148	0.289	0.088
-2.5	0.008	0.016	0.038	0.023	0.035	0.078	0.214	0.089	0.071	0.213	0.441	0.205
0	0.009	0.019	0.029	0.032	0.055	0.104	0.138	0.142	0.1167	0.220	0.273	0.266
2.5	0.011	0.021	0.028	0.035	0.060	0.111	0.121	0.146	0.126	0.219	0.234	0.254
5	0.012	0.023	0.027	0.034	0.062	0.101	0.109	0.135	0.123	0.198	0.207	0.234
10	0.013	0.021	0.025	0.035	0.060	0.090	0.096	0.121	0.115	0.171	0.177	0.208
25	0.013	0.017	0.021	0.032	0.057	0.075	0.079	0.106	0.108	0.141	0.146	0.177
50	0.012	0.014	0.017	0.029	0.054	0.066	0.072	0.099	0.108	0.128	0.134	0.167
75	0.011	0.015	0.017	0.029	0.054	0.062	0.067	0.094	0.109	0.122	0.128	0.158
100	0.011	0.014	0.017	0.028	0.054	0.060	0.064	0.092	0.107	0.117	0.124	0.154
125	0.011	0.014	0.016	0.027	0.055	0.059	0.063	0.089	0.107	0.115	0.120	0.152
150	0.012	0.014	0.016	0.026	0.055	0.059	0.064	0.089	0.107	0.113	0.117	0.149
200	0.011	0.012	0.015	0.025	0.055	0.054	0.059	0.085	0.110	0.110	0.114	0.147
250	0.011	0.012	0.014	0.025	0.057	0.053	0.058	0.081	0.108	0.104	0.110	0.142
Two-sided tests - $T = 250$												
-5	0.004	0.005	0.033	0.006	0.033	0.028	0.103	0.017	0.077	0.062	0.174	0.036
-2.5	0.007	0.008	0.020	0.013	0.031	0.036	0.101	0.046	0.065	0.078	0.214	0.089
0	0.008	0.009	0.014	0.016	0.043	0.051	0.069	0.074	0.096	0.106	0.138	0.142
2.5	0.009	0.011	0.015	0.018	0.048	0.055	0.066	0.081	0.101	0.110	0.122	0.148
5	0.009	0.012	0.016	0.021	0.048	0.056	0.064	0.082	0.102	0.109	0.117	0.147
10	0.011	0.013	0.015	0.024	0.054	0.059	0.063	0.084	0.103	0.111	0.115	0.148
25	0.011	0.011	0.015	0.025	0.054	0.053	0.059	0.088	0.106	0.104	0.110	0.152
50	0.012	0.011	0.014	0.024	0.051	0.050	0.054	0.089	0.107	0.102	0.110	0.155
75	0.012	0.010	0.014	0.026	0.054	0.049	0.057	0.091	0.106	0.098	0.106	0.155
100	0.012	0.011	0.013	0.028	0.054	0.051	0.058	0.090	0.108	0.100	0.107	0.160
125	0.011	0.010	0.013	0.029	0.054	0.051	0.058	0.094	0.109	0.101	0.108	0.157
150	0.012	0.011	0.014	0.027	0.055	0.052	0.059	0.094	0.1081	0.103	0.111	0.159
200	0.010	0.011	0.014	0.029	0.055	0.053	0.060	0.095	0.110	0.103	0.111	0.158
250	0.011	0.011	0.015	0.027	0.056	0.054	0.061	0.098	0.113	0.103	0.114	0.157

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.009	0.000	0.000	0.000	0.043	0.001	0.005	0.000	0.093	0.004	0.010	0.001
-2.5	0.012	0.000	0.000	0.000	0.072	0.000	0.000	0.000	0.155	0.000	0.000	0.000
0	0.004	0.000	0.000	0.000	0.015	0.000	0.000	0.000	0.024	0.000	0.000	0.000
2.5	0.017	0.000	0.000	0.000	0.048	0.003	0.002	0.003	0.082	0.007	0.007	0.008
5	0.025	0.002	0.001	0.001	0.068	0.010	0.008	0.011	0.112	0.023	0.021	0.029
10	0.023	0.003	0.003	0.006	0.067	0.019	0.017	0.026	0.118	0.043	0.041	0.054
25	0.018	0.006	0.005	0.013	0.063	0.029	0.029	0.047	0.111	0.064	0.064	0.087
50	0.014	0.007	0.007	0.017	0.060	0.037	0.036	0.060	0.111	0.075	0.074	0.102
75	0.012	0.008	0.007	0.017	0.059	0.039	0.039	0.065	0.110	0.081	0.082	0.110
100	0.012	0.007	0.007	0.017	0.056	0.040	0.041	0.066	0.110	0.085	0.085	0.116
125	0.010	0.007	0.007	0.018	0.054	0.039	0.040	0.068	0.109	0.085	0.088	0.120
150	0.011	0.008	0.007	0.019	0.054	0.040	0.042	0.067	0.108	0.087	0.087	0.122
200	0.011	0.009	0.008	0.020	0.053	0.043	0.043	0.069	0.105	0.090	0.090	0.123
250	0.011	0.008	0.008	0.020	0.053	0.044	0.044	0.073	0.107	0.092	0.092	0.123
Right-sided tests - $T = 1000$												
-5	0.005	0.012	0.051	0.009	0.039	0.068	0.174	0.038	0.089	0.156	0.290	0.092
-2.5	0.005	0.012	0.031	0.017	0.027	0.072	0.208	0.080	0.060	0.197	0.434	0.187
0	0.008	0.016	0.021	0.025	0.043	0.095	0.118	0.126	0.100	0.208	0.257	0.254
2.5	0.009	0.019	0.022	0.028	0.052	0.100	0.109	0.131	0.110	0.209	0.217	0.248
5	0.009	0.020	0.022	0.032	0.054	0.095	0.097	0.124	0.108	0.190	0.189	0.225
10	0.011	0.020	0.020	0.031	0.052	0.086	0.086	0.114	0.107	0.165	0.164	0.198
25	0.011	0.016	0.016	0.027	0.050	0.074	0.075	0.102	0.102	0.137	0.139	0.174
50	0.010	0.014	0.015	0.027	0.051	0.068	0.067	0.093	0.097	0.123	0.124	0.158
75	0.009	0.012	0.013	0.026	0.049	0.064	0.062	0.093	0.099	0.119	0.120	0.152
100	0.009	0.012	0.013	0.025	0.049	0.059	0.060	0.088	0.098	0.117	0.120	0.151
125	0.010	0.013	0.013	0.025	0.049	0.057	0.058	0.087	0.098	0.115	0.117	0.151
150	0.010	0.013	0.013	0.025	0.047	0.056	0.056	0.085	0.096	0.114	0.115	0.148
200	0.011	0.012	0.013	0.025	0.047	0.054	0.055	0.083	0.099	0.110	0.113	0.144
250	0.010	0.012	0.013	0.024	0.049	0.055	0.056	0.085	0.098	0.107	0.109	0.143
Two-sided tests- $T = 1000$												
-5	0.005	0.006	0.031	0.006	0.039	0.033	0.102	0.018	0.090	0.068	0.179	0.038
-2.5	0.005	0.005	0.013	0.010	0.024	0.030	0.088	0.038	0.055	0.071	0.208	0.080
0	0.006	0.007	0.012	0.013	0.034	0.042	0.056	0.063	0.080	0.096	0.118	0.126
2.5	0.006	0.009	0.010	0.016	0.038	0.048	0.054	0.071	0.085	0.103	0.111	0.134
5	0.007	0.011	0.012	0.018	0.044	0.053	0.055	0.076	0.090	0.106	0.105	0.136
10	0.009	0.013	0.013	0.020	0.049	0.054	0.054	0.078	0.093	0.105	0.104	0.140
25	0.011	0.011	0.012	0.024	0.049	0.052	0.052	0.086	0.100	0.104	0.104	0.148
50	0.010	0.010	0.010	0.024	0.051	0.051	0.051	0.091	0.102	0.103	0.104	0.153
75	0.010	0.010	0.010	0.024	0.049	0.050	0.051	0.089	0.102	0.100	0.101	0.158
100	0.010	0.011	0.010	0.024	0.049	0.048	0.048	0.088	0.100	0.099	0.101	0.154
125	0.009	0.010	0.010	0.025	0.049	0.048	0.049	0.087	0.098	0.096	0.099	0.154
150	0.010	0.010	0.010	0.025	0.050	0.049	0.049	0.087	0.097	0.094	0.097	0.152
200	0.011	0.010	0.010	0.026	0.050	0.048	0.049	0.089	0.098	0.095	0.098	0.152
250	0.010	0.011	0.011	0.027	0.049	0.047	0.048	0.089	0.100	0.097	0.100	0.157

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.14: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP6 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = \left[\sigma^2_{ut} \quad -0.9\sigma_{ut}\sigma_{vt}; \quad -0.9\sigma_{ut}\sigma_{vt} \quad \sigma^2_{vt}\right]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 1/4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t^{*,RWB}_{zx}$ 1%	$t^{*,FRWB}_{zx}$ 1%	t^{EW}_{zx} 1%	t_{zx} 1%	$t^{*,RWB}_{zx}$ 5%	$t^{*,FRWB}_{zx}$ 5%	t^{EW}_{zx} 5%	t_{zx} 5%	$t^{*,RWB}_{zx}$ 10%	$t^{*,FRWB}_{zx}$ 10%	t^{EW}_{zx} 10%	t_{zx} 10%
Left-sided tests - $T = 250$												
-5	0.007	0.000	0.001	0.000	0.044	0.001	0.005	0.000	0.095	0.003	0.014	0.001
-2.5	0.010	0.000	0.000	0.000	0.067	0.000	0.000	0.000	0.142	0.000	0.000	0.000
0	0.005	0.000	0.000	0.000	0.014	0.000	0.000	0.000	0.024	0.000	0.000	0.000
2.5	0.016	0.000	0.000	0.000	0.045	0.002	0.002	0.002	0.079	0.008	0.007	0.009
5	0.024	0.002	0.001	0.001	0.064	0.011	0.009	0.012	0.108	0.025	0.023	0.029
10	0.023	0.004	0.003	0.006	0.067	0.021	0.020	0.028	0.112	0.045	0.043	0.058
25	0.016	0.007	0.007	0.012	0.060	0.031	0.031	0.047	0.109	0.063	0.063	0.084
50	0.013	0.007	0.007	0.015	0.056	0.037	0.039	0.058	0.108	0.073	0.077	0.103
75	0.012	0.007	0.008	0.016	0.057	0.038	0.040	0.062	0.110	0.082	0.084	0.112
100	0.012	0.008	0.008	0.016	0.055	0.041	0.044	0.067	0.110	0.086	0.088	0.115
125	0.012	0.007	0.009	0.019	0.055	0.044	0.047	0.068	0.109	0.088	0.092	0.119
150	0.012	0.008	0.010	0.019	0.055	0.045	0.048	0.071	0.108	0.090	0.094	0.122
200	0.011	0.010	0.012	0.022	0.057	0.048	0.053	0.075	0.107	0.095	0.099	0.125
250	0.011	0.009	0.012	0.023	0.058	0.052	0.056	0.076	0.105	0.096	0.101	0.128
Right-sided tests - $T = 250$												
-5	0.004	0.010	0.064	0.008	0.033	0.063	0.188	0.034	0.076	0.154	0.302	0.092
-2.5	0.008	0.015	0.053	0.022	0.034	0.081	0.242	0.088	0.072	0.222	0.454	0.213
0	0.010	0.020	0.030	0.032	0.057	0.105	0.138	0.137	0.117	0.216	0.279	0.265
2.5	0.012	0.021	0.028	0.035	0.063	0.108	0.122	0.142	0.127	0.212	0.230	0.251
5	0.013	0.023	0.026	0.035	0.061	0.098	0.105	0.132	0.120	0.192	0.202	0.231
10	0.013	0.021	0.024	0.036	0.058	0.087	0.094	0.119	0.113	0.169	0.173	0.204
25	0.012	0.018	0.020	0.031	0.054	0.073	0.079	0.105	0.108	0.140	0.144	0.175
50	0.010	0.014	0.017	0.028	0.055	0.066	0.071	0.097	0.106	0.124	0.131	0.167
75	0.011	0.014	0.016	0.028	0.056	0.063	0.069	0.094	0.107	0.118	0.125	0.155
100	0.011	0.014	0.016	0.028	0.057	0.060	0.066	0.092	0.106	0.117	0.123	0.154
125	0.012	0.014	0.016	0.027	0.055	0.057	0.064	0.091	0.109	0.115	0.119	0.154
150	0.011	0.014	0.016	0.027	0.056	0.059	0.064	0.089	0.108	0.114	0.119	0.151
200	0.012	0.012	0.016	0.026	0.057	0.057	0.062	0.086	0.107	0.108	0.114	0.148
250	0.010	0.011	0.013	0.025	0.057	0.054	0.059	0.080	0.109	0.104	0.109	0.143
Two-sided tests - $T = 250$												
-5	0.004	0.005	0.039	0.005	0.033	0.028	0.120	0.017	0.075	0.064	0.193	0.034
-2.5	0.007	0.008	0.027	0.013	0.031	0.037	0.129	0.046	0.067	0.080	0.242	0.088
0	0.008	0.010	0.016	0.016	0.046	0.050	0.072	0.077	0.097	0.104	0.138	0.137
2.5	0.009	0.011	0.016	0.020	0.049	0.055	0.067	0.080	0.098	0.109	0.124	0.144
5	0.010	0.011	0.016	0.021	0.048	0.057	0.062	0.083	0.100	0.107	0.114	0.144
10	0.011	0.012	0.015	0.024	0.052	0.056	0.061	0.084	0.102	0.108	0.114	0.147
25	0.012	0.012	0.015	0.025	0.053	0.054	0.058	0.088	0.106	0.105	0.110	0.152
50	0.011	0.011	0.015	0.025	0.053	0.050	0.055	0.090	0.106	0.102	0.110	0.155
75	0.011	0.012	0.014	0.025	0.053	0.048	0.056	0.090	0.104	0.099	0.108	0.156
100	0.010	0.011	0.013	0.027	0.052	0.050	0.057	0.092	0.108	0.101	0.110	0.159
125	0.011	0.010	0.013	0.028	0.054	0.051	0.057	0.093	0.109	0.102	0.111	0.159
150	0.010	0.010	0.013	0.027	0.054	0.051	0.058	0.094	0.108	0.102	0.113	0.160
200	0.010	0.010	0.014	0.029	0.057	0.052	0.061	0.095	0.112	0.104	0.114	0.161
250	0.010	0.011	0.014	0.028	0.056	0.052	0.060	0.097	0.113	0.106	0.115	0.156

c	$t^{*,RWB}_{zx}$ 1%	$t^{*,FRWB}_{zx}$ 1%	t^{EW}_{zx} 1%	t_{zx} 1%	$t^{*,RWB}_{zx}$ 5%	$t^{*,FRWB}_{zx}$ 5%	t^{EW}_{zx} 5%	t_{zx} 5%	$t^{*,RWB}_{zx}$ 10%	$t^{*,FRWB}_{zx}$ 10%	t^{EW}_{zx} 10%	t_{zx} 10%
Left-sided tests - $T = 1000$												
-5	0.010	0.000	0.001	0.000	0.043	0.001	0.006	0.000	0.094	0.004	0.016	0.001
-2.5	0.012	0.000	0.000	0.000	0.069	0.000	0.000	0.000	0.142	0.000	0.000	0.000
0	0.004	0.000	0.000	0.000	0.014	0.000	0.000	0.000	0.025	0.000	0.000	0.000
2.5	0.015	0.000	0.000	0.000	0.046	0.003	0.002	0.004	0.082	0.009	0.008	0.011
5	0.022	0.001	0.001	0.002	0.063	0.011	0.009	0.013	0.111	0.025	0.024	0.031
10	0.020	0.004	0.003	0.007	0.064	0.019	0.018	0.028	0.117	0.044	0.041	0.056
25	0.017	0.006	0.005	0.013	0.060	0.031	0.029	0.049	0.112	0.065	0.065	0.089
50	0.014	0.007	0.006	0.017	0.058	0.037	0.036	0.058	0.111	0.077	0.076	0.105
75	0.012	0.007	0.007	0.016	0.058	0.039	0.039	0.064	0.110	0.083	0.081	0.113
100	0.012	0.008	0.007	0.016	0.056	0.041	0.041	0.067	0.110	0.085	0.086	0.116
125	0.012	0.008	0.007	0.018	0.056	0.042	0.042	0.067	0.108	0.087	0.086	0.119
150	0.010	0.008	0.007	0.018	0.054	0.042	0.042	0.070	0.107	0.088	0.088	0.121
200	0.011	0.008	0.008	0.019	0.053	0.042	0.043	0.070	0.106	0.089	0.091	0.125
250	0.011	0.008	0.009	0.019	0.055	0.044	0.044	0.073	0.106	0.093	0.093	0.125
Right-sided tests - $T = 1000$												
-5	0.005	0.013	0.061	0.009	0.040	0.071	0.190	0.040	0.088	0.162	0.305	0.096
-2.5	0.005	0.012	0.044	0.016	0.027	0.073	0.236	0.080	0.064	0.208	0.447	0.193
0	0.008	0.016	0.024	0.026	0.045	0.094	0.122	0.127	0.101	0.206	0.259	0.256
2.5	0.009	0.019	0.023	0.030	0.052	0.097	0.105	0.128	0.110	0.204	0.207	0.241
5	0.010	0.020	0.021	0.032	0.055	0.093	0.095	0.121	0.108	0.184	0.182	0.218
10	0.011	0.020	0.020	0.032	0.051	0.085	0.083	0.110	0.104	0.159	0.158	0.195
25	0.011	0.016	0.015	0.028	0.049	0.073	0.073	0.101	0.100	0.136	0.137	0.170
50	0.008	0.014	0.013	0.026	0.051	0.065	0.066	0.096	0.100	0.123	0.125	0.156
75	0.010	0.012	0.012	0.024	0.049	0.062	0.064	0.093	0.099	0.119	0.121	0.155
100	0.009	0.013	0.012	0.024	0.047	0.060	0.060	0.091	0.100	0.117	0.118	0.153
125	0.009	0.013	0.012	0.024	0.047	0.057	0.058	0.088	0.098	0.114	0.113	0.150
150	0.010	0.013	0.013	0.024	0.048	0.056	0.057	0.086	0.099	0.113	0.114	0.147
200	0.010	0.013	0.013	0.024	0.048	0.055	0.056	0.084	0.099	0.110	0.113	0.145
250	0.010	0.011	0.013	0.024	0.048	0.056	0.055	0.085	0.097	0.107	0.108	0.144
Two-sided tests- $T = 1000$												
-5	0.005	0.006	0.038	0.006	0.040	0.033	0.116	0.019	0.089	0.072	0.196	0.040
-2.5	0.004	0.006	0.020	0.010	0.024	0.030	0.117	0.037	0.058	0.072	0.236	0.080
0	0.005	0.008	0.012	0.014	0.037	0.044	0.059	0.063	0.081	0.093	0.122	0.127
2.5	0.006	0.008	0.010	0.016	0.040	0.050	0.055	0.070	0.088	0.101	0.107	0.132
5	0.008	0.011	0.012	0.018	0.045	0.052	0.053	0.075	0.090	0.103	0.104	0.134
10	0.011	0.013	0.013	0.021	0.047	0.053	0.053	0.079	0.094	0.103	0.101	0.138
25	0.012	0.012	0.012	0.024	0.049	0.053	0.052	0.083	0.098	0.102	0.103	0.149
50	0.010	0.010	0.011	0.025	0.049	0.053	0.053	0.090	0.101	0.102	0.103	0.154
75	0.010	0.011	0.011	0.024	0.048	0.049	0.050	0.088	0.102	0.101	0.103	0.157
100	0.010	0.010	0.011	0.025	0.048	0.048	0.048	0.089	0.100	0.100	0.101	0.158
125	0.009	0.010	0.010	0.025	0.049	0.047	0.048	0.088	0.097	0.098	0.100	0.155
150	0.010	0.010	0.010	0.024	0.048	0.046	0.048	0.086	0.098	0.099	0.099	0.156
200	0.010	0.010	0.011	0.025	0.048	0.048	0.049	0.088	0.098	0.096	0.099	0.155
250	0.011	0.010	0.011	0.025	0.048	0.047	0.049	0.089	0.098	0.097	0.099	0.158

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.15: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP6 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = \left[\sigma^2_{ut} \quad -0.5\sigma_{ut}\sigma_{vt}; \quad -0.5\sigma_{ut}\sigma_{vt} \quad \sigma^2_{vt}\right]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 1/4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

Left-sided tests - $T = 250$

c	1%				5%				10%			
	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.009	0.001	0.054	0.000	0.045	0.011	0.096	0.004	0.097	0.030	0.127	0.017
-2.5	0.009	0.000	0.020	0.000	0.045	0.002	0.033	0.002	0.101	0.009	0.044	0.008
0	0.004	0.000	0.003	0.000	0.018	0.004	0.013	0.006	0.040	0.012	0.022	0.016
2.5	0.007	0.002	0.002	0.003	0.039	0.014	0.015	0.017	0.076	0.034	0.034	0.041
5	0.010	0.003	0.003	0.005	0.050	0.022	0.022	0.029	0.093	0.051	0.050	0.064
10	0.013	0.006	0.006	0.010	0.053	0.031	0.030	0.044	0.100	0.065	0.065	0.082
25	0.013	0.008	0.009	0.015	0.053	0.037	0.039	0.056	0.102	0.077	0.078	0.103
50	0.011	0.008	0.008	0.018	0.052	0.040	0.042	0.063	0.104	0.085	0.087	0.115
75	0.011	0.009	0.010	0.019	0.051	0.040	0.043	0.069	0.105	0.090	0.093	0.120
100	0.011	0.008	0.010	0.020	0.052	0.043	0.047	0.071	0.104	0.090	0.096	0.124
125	0.011	0.009	0.011	0.022	0.051	0.044	0.047	0.073	0.107	0.094	0.097	0.127
150	0.010	0.008	0.0107	0.022	0.052	0.0468	0.049	0.074	0.105	0.095	0.100	0.130
200	0.011	0.011	0.0129	0.023	0.053	0.049	0.051	0.076	0.106	0.097	0.102	0.133
250	0.011	0.010	0.0135	0.023	0.054	0.047	0.053	0.078	0.105	0.097	0.102	0.133

Right-sided tests - $T = 250$

c	1%				5%				10%			
-5	0.005	0.012	0.186	0.006	0.040	0.073	0.287	0.038	0.087	0.146	0.356	0.097
-2.5	0.006	0.017	0.188	0.019	0.037	0.101	0.288	0.097	0.088	0.207	0.357	0.193
0	0.012	0.021	0.057	0.032	0.062	0.097	0.162	0.127	0.127	0.197	0.261	0.230
2.5	0.011	0.020	0.024	0.031	0.059	0.089	0.094	0.114	0.118	0.171	0.177	0.199
5	0.012	0.018	0.021	0.028	0.057	0.079	0.083	0.104	0.112	0.153	0.156	0.181
10	0.011	0.016	0.017	0.027	0.054	0.073	0.076	0.096	0.105	0.134	0.137	0.166
25	0.011	0.013	0.016	0.027	0.051	0.059	0.063	0.086	0.099	0.118	0.123	0.153
50	0.011	0.013	0.015	0.025	0.051	0.055	0.061	0.084	0.100	0.110	0.113	0.146
75	0.010	0.012	0.014	0.024	0.052	0.057	0.061	0.086	0.102	0.110	0.115	0.146
100	0.010	0.012	0.014	0.025	0.052	0.054	0.060	0.088	0.105	0.111	0.117	0.149
125	0.010	0.012	0.014	0.025	0.051	0.054	0.059	0.086	0.108	0.108	0.114	0.149
150	0.011	0.011	0.014	0.026	0.052	0.054	0.058	0.085	0.105	0.106	0.111	0.147
200	0.011	0.012	0.015	0.026	0.054	0.054	0.058	0.084	0.108	0.105	0.110	0.143
250	0.011	0.012	0.015	0.024	0.055	0.054	0.059	0.083	0.106	0.104	0.110	0.139

Two-sided tests - $T = 250$

c	1%				5%				10%			
-5	0.004	0.006	0.203	0.003	0.039	0.039	0.306	0.016	0.087	0.085	0.382	0.042
-2.5	0.005	0.008	0.177	0.010	0.033	0.048	0.265	0.046	0.077	0.103	0.321	0.099
0	0.009	0.010	0.044	0.019	0.047	0.052	0.108	0.074	0.096	0.100	0.175	0.133
2.5	0.009	0.010	0.015	0.019	0.045	0.051	0.060	0.073	0.096	0.102	0.109	0.131
5	0.009	0.010	0.012	0.020	0.046	0.051	0.053	0.073	0.096	0.101	0.105	0.134
10	0.009	0.011	0.012	0.023	0.050	0.053	0.057	0.081	0.100	0.103	0.106	0.139
25	0.012	0.011	0.012	0.026	0.050	0.051	0.056	0.084	0.100	0.095	0.101	0.142
50	0.011	0.010	0.012	0.026	0.052	0.050	0.055	0.089	0.101	0.096	0.103	0.146
75	0.010	0.010	0.012	0.025	0.051	0.048	0.054	0.091	0.101	0.096	0.103	0.155
100	0.010	0.009	0.012	0.027	0.049	0.048	0.054	0.089	0.103	0.098	0.107	0.159
125	0.010	0.010	0.012	0.027	0.051	0.050	0.056	0.087	0.100	0.097	0.106	0.160
150	0.010	0.009	0.013	0.028	0.053	0.052	0.059	0.091	0.102	0.098	0.107	0.159
200	0.010	0.011	0.015	0.030	0.054	0.052	0.060	0.095	0.105	0.101	0.109	0.160
250	0.011	0.012	0.017	0.030	0.053	0.052	0.058	0.095	0.107	0.101	0.112	0.161

Left-sided tests - $T = 1000$

c	1%				5%				10%			
	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.010	0.001	0.053	0.000	0.048	0.013	0.100	0.004	0.098	0.030	0.136	0.017
-2.5	0.008	0.000	0.026	0.000	0.048	0.003	0.042	0.002	0.100	0.007	0.055	0.006
0	0.004	0.000	0.004	0.001	0.020	0.004	0.014	0.006	0.043	0.013	0.024	0.016
2.5	0.010	0.002	0.002	0.003	0.041	0.015	0.016	0.022	0.081	0.040	0.037	0.048
5	0.012	0.004	0.003	0.006	0.049	0.025	0.022	0.033	0.095	0.053	0.050	0.064
10	0.013	0.005	0.005	0.011	0.052	0.030	0.028	0.042	0.100	0.064	0.060	0.082
25	0.012	0.008	0.007	0.016	0.055	0.039	0.038	0.059	0.107	0.080	0.078	0.108
50	0.011	0.008	0.008	0.019	0.054	0.044	0.044	0.067	0.105	0.088	0.087	0.116
75	0.011	0.009	0.009	0.020	0.054	0.044	0.045	0.070	0.105	0.089	0.090	0.122
100	0.011	0.009	0.008	0.020	0.053	0.046	0.046	0.073	0.105	0.092	0.092	0.124
125	0.010	0.009	0.009	0.020	0.053	0.046	0.046	0.074	0.105	0.095	0.094	0.125
150	0.011	0.009	0.008	0.021	0.054	0.047	0.048	0.075	0.104	0.094	0.095	0.128
200	0.011	0.009	0.009	0.021	0.055	0.048	0.049	0.078	0.104	0.096	0.098	0.127
250	0.011	0.010	0.009	0.022	0.053	0.049	0.050	0.076	0.105	0.096	0.096	0.130

Right-sided tests - $T = 1000$

c	1%				5%				10%			
-5	0.008	0.016	0.195	0.006	0.048	0.078	0.307	0.044	0.093	0.152	0.380	0.099
-2.5	0.006	0.019	0.227	0.018	0.038	0.098	0.323	0.092	0.082	0.204	0.389	0.189
0	0.009	0.018	0.054	0.029	0.057	0.094	0.149	0.122	0.122	0.192	0.252	0.229
2.5	0.010	0.018	0.021	0.029	0.054	0.085	0.087	0.107	0.111	0.165	0.165	0.190
5	0.011	0.017	0.017	0.027	0.052	0.079	0.076	0.098	0.103	0.147	0.143	0.173
10	0.010	0.015	0.014	0.027	0.052	0.069	0.066	0.094	0.100	0.134	0.130	0.162
25	0.010	0.014	0.013	0.026	0.050	0.062	0.060	0.086	0.098	0.120	0.120	0.152
50	0.009	0.012	0.012	0.026	0.049	0.058	0.058	0.086	0.099	0.112	0.112	0.147
75	0.009	0.011	0.012	0.024	0.049	0.055	0.057	0.085	0.101	0.110	0.112	0.145
100	0.010	0.012	0.012	0.023	0.049	0.056	0.056	0.084	0.099	0.109	0.109	0.143
125	0.010	0.012	0.012	0.024	0.048	0.055	0.055	0.085	0.099	0.108	0.108	0.142
150	0.009	0.012	0.012	0.024	0.048	0.055	0.055	0.085	0.101	0.109	0.109	0.143
200	0.010	0.011	0.011	0.022	0.050	0.054	0.054	0.084	0.100	0.107	0.107	0.144
250	0.010	0.011	0.011	0.022	0.049	0.052	0.054	0.084	0.101	0.106	0.109	0.144

Two-sided tests- $T = 1000$

c	1%				5%				10%			
-5	0.007	0.008	0.205	0.003	0.047	0.044	0.327	0.018	0.094	0.091	0.407	0.048
-2.5	0.005	0.008	0.225	0.009	0.034	0.047	0.309	0.045	0.074	0.102	0.365	0.095
0	0.007	0.009	0.038	0.015	0.044	0.047	0.103	0.068	0.092	0.099	0.162	0.129
2.5	0.009	0.010	0.013	0.018	0.043	0.052	0.054	0.071	0.091	0.100	0.103	0.128
5	0.009	0.011	0.011	0.018	0.048	0.052	0.051	0.073	0.095	0.102	0.098	0.131
10	0.010	0.011	0.010	0.020	0.048	0.053	0.050	0.077	0.094	0.098	0.094	0.136
25	0.010	0.011	0.009	0.024	0.050	0.052	0.051	0.084	0.100	0.099	0.098	0.145
50	0.010	0.010	0.010	0.025	0.052	0.050	0.051	0.089	0.102	0.102	0.101	0.153
75	0.010	0.010	0.009	0.026	0.051	0.050	0.051	0.089	0.102	0.100	0.102	0.155
100	0.009	0.010	0.010	0.026	0.050	0.049	0.050	0.091	0.102	0.100	0.102	0.157
125	0.010	0.009	0.010	0.025	0.050	0.050	0.050	0.090	0.101	0.100	0.101	0.159
150	0.011	0.009	0.011	0.026	0.051	0.050	0.051	0.091	0.102	0.100	0.103	0.160
200	0.010	0.010	0.011	0.026	0.049	0.049	0.051	0.091	0.102	0.102	0.103	0.162
250	0.011	0.011	0.011	0.026	0.048	0.050	0.051	0.092	0.100	0.103	0.104	0.160

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.16: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP6 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$, and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = \left[\sigma^2_{ut} \quad 0; \quad 0 \quad \sigma^2_{vt}\right]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 1/4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	1% $t^{*,RWB}_{zx}$	1% $t^{*,FRWB}_{zx}$	1% t^{EW}_{zx}	1% t_{zx}	5% $t^{*,RWB}_{zx}$	5% $t^{*,FRWB}_{zx}$	5% t^{EW}_{zx}	5% t_{zx}	10% $t^{*,RWB}_{zx}$	10% $t^{*,FRWB}_{zx}$	10% t^{EW}_{zx}	10% t_{zx}
Left-sided tests - $T = 250$												
-5	0.010	0.012	0.108	0.004	0.046	0.049	0.150	0.025	0.095	0.099	0.175	0.063
-2.5	0.008	0.012	0.027	0.010	0.040	0.051	0.045	0.046	0.086	0.099	0.059	0.093
0	0.010	0.011	0.032	0.017	0.048	0.047	0.070	0.061	0.092	0.094	0.114	0.110
2.5	0.009	0.009	0.013	0.014	0.045	0.047	0.050	0.061	0.093	0.096	0.097	0.113
5	0.010	0.010	0.011	0.016	0.045	0.046	0.048	0.063	0.095	0.096	0.094	0.115
10	0.011	0.010	0.012	0.019	0.048	0.048	0.049	0.065	0.093	0.094	0.094	0.117
25	0.012	0.011	0.012	0.022	0.050	0.050	0.052	0.073	0.097	0.096	0.098	0.124
50	0.011	0.010	0.011	0.021	0.049	0.047	0.051	0.074	0.098	0.098	0.101	0.131
75	0.010	0.010	0.011	0.023	0.050	0.047	0.052	0.074	0.098	0.097	0.101	0.132
100	0.011	0.011	0.012	0.023	0.050	0.048	0.052	0.075	0.100	0.096	0.102	0.133
125	0.010	0.010	0.013	0.024	0.052	0.050	0.053	0.076	0.102	0.100	0.104	0.139
150	0.010	0.012	0.013	0.024	0.051	0.050	0.054	0.078	0.105	0.100	0.106	0.137
200	0.011	0.011	0.013	0.026	0.053	0.052	0.056	0.080	0.106	0.101	0.106	0.139
250	0.011	0.012	0.014	0.027	0.057	0.055	0.058	0.081	0.107	0.101	0.106	0.136
Right-sided tests - $T = 250$												
-5	0.008	0.009	0.111	0.003	0.048	0.049	0.150	0.026	0.098	0.101	0.175	0.064
-2.5	0.007	0.012	0.031	0.009	0.039	0.049	0.048	0.043	0.086	0.101	0.063	0.092
0	0.011	0.011	0.034	0.017	0.047	0.049	0.075	0.062	0.099	0.098	0.119	0.118
2.5	0.009	0.010	0.014	0.016	0.046	0.049	0.050	0.063	0.093	0.099	0.098	0.115
5	0.009	0.010	0.010	0.016	0.047	0.049	0.049	0.064	0.096	0.097	0.096	0.118
10	0.011	0.011	0.011	0.018	0.047	0.048	0.049	0.066	0.096	0.097	0.096	0.123
25	0.012	0.010	0.013	0.021	0.052	0.051	0.053	0.070	0.096	0.095	0.097	0.122
50	0.011	0.011	0.013	0.024	0.051	0.049	0.052	0.074	0.098	0.095	0.099	0.130
75	0.011	0.010	0.012	0.023	0.051	0.051	0.053	0.077	0.101	0.099	0.102	0.132
100	0.010	0.010	0.013	0.023	0.051	0.050	0.055	0.078	0.103	0.102	0.105	0.132
125	0.010	0.011	0.013	0.024	0.052	0.051	0.056	0.079	0.102	0.100	0.104	0.135
150	0.011	0.011	0.014	0.025	0.053	0.052	0.055	0.078	0.104	0.099	0.104	0.138
200	0.011	0.012	0.015	0.026	0.054	0.053	0.056	0.079	0.104	0.101	0.106	0.137
250	0.013	0.013	0.015	0.026	0.053	0.050	0.055	0.081	0.104	0.100	0.104	0.134
Two-sided tests - $T = 250$												
-5	0.008	0.010	0.196	0.002	0.046	0.049	0.258	0.020	0.093	0.098	0.300	0.051
-2.5	0.007	0.013	0.050	0.010	0.036	0.052	0.075	0.044	0.077	0.099	0.092	0.089
0	0.009	0.011	0.052	0.018	0.049	0.049	0.101	0.068	0.094	0.097	0.145	0.123
2.5	0.010	0.009	0.015	0.017	0.043	0.048	0.053	0.067	0.091	0.096	0.100	0.123
5	0.010	0.010	0.012	0.019	0.046	0.047	0.048	0.069	0.091	0.095	0.096	0.126
10	0.013	0.012	0.013	0.022	0.049	0.049	0.051	0.077	0.094	0.096	0.098	0.131
25	0.012	0.012	0.014	0.026	0.053	0.052	0.055	0.087	0.100	0.100	0.105	0.143
50	0.012	0.011	0.014	0.027	0.051	0.048	0.055	0.087	0.100	0.097	0.103	0.148
75	0.010	0.011	0.013	0.028	0.050	0.049	0.055	0.089	0.100	0.097	0.105	0.151
100	0.010	0.010	0.014	0.028	0.053	0.050	0.056	0.093	0.101	0.099	0.106	0.153
125	0.011	0.010	0.014	0.028	0.054	0.053	0.059	0.095	0.103	0.100	0.109	0.155
150	0.010	0.010	0.014	0.030	0.054	0.054	0.060	0.096	0.103	0.101	0.108	0.156
200	0.009	0.011	0.016	0.032	0.056	0.055	0.063	0.099	0.107	0.104	0.111	0.159
250	0.011	0.012	0.016	0.032	0.055	0.054	0.062	0.101	0.110	0.105	0.113	0.162

c	1% $t^{*,RWB}_{zx}$	1% $t^{*,FRWB}_{zx}$	1% t^{EW}_{zx}	1% t_{zx}	5% $t^{*,RWB}_{zx}$	5% $t^{*,FRWB}_{zx}$	5% t^{EW}_{zx}	5% t_{zx}	10% $t^{*,RWB}_{zx}$	10% $t^{*,FRWB}_{zx}$	10% t^{EW}_{zx}	10% t_{zx}
Left-sided tests - $T = 1000$												
-5	0.008	0.009	0.156	0.002	0.046	0.049	0.197	0.023	0.097	0.098	0.221	0.063
-2.5	0.006	0.010	0.026	0.008	0.039	0.050	0.041	0.045	0.085	0.100	0.054	0.093
0	0.012	0.012	0.036	0.019	0.053	0.051	0.080	0.067	0.102	0.101	0.121	0.124
2.5	0.011	0.011	0.014	0.018	0.050	0.053	0.051	0.066	0.100	0.102	0.100	0.120
5	0.010	0.010	0.010	0.018	0.048	0.050	0.047	0.066	0.098	0.101	0.098	0.121
10	0.010	0.009	0.009	0.018	0.049	0.049	0.047	0.068	0.098	0.100	0.097	0.124
25	0.010	0.010	0.009	0.019	0.051	0.051	0.049	0.073	0.100	0.102	0.100	0.131
50	0.010	0.011	0.010	0.023	0.051	0.051	0.051	0.075	0.101	0.102	0.100	0.134
75	0.011	0.010	0.010	0.024	0.051	0.050	0.050	0.079	0.102	0.102	0.103	0.136
100	0.011	0.011	0.010	0.023	0.051	0.051	0.050	0.079	0.102	0.101	0.102	0.136
125	0.011	0.010	0.010	0.023	0.053	0.052	0.052	0.079	0.103	0.102	0.103	0.137
150	0.010	0.010	0.010	0.023	0.052	0.052	0.052	0.081	0.101	0.102	0.102	0.135
200	0.010	0.011	0.011	0.023	0.055	0.053	0.053	0.081	0.101	0.100	0.101	0.136
250	0.010	0.011	0.011	0.023	0.051	0.052	0.052	0.081	0.102	0.101	0.100	0.137
Right-sided tests - $T = 1000$												
-5	0.009	0.010	0.156	0.002	0.051	0.053	0.195	0.028	0.098	0.101	0.220	0.065
-2.5	0.007	0.012	0.026	0.009	0.039	0.051	0.041	0.045	0.087	0.101	0.055	0.093
0	0.011	0.011	0.036	0.016	0.050	0.049	0.076	0.064	0.098	0.101	0.121	0.120
2.5	0.009	0.010	0.011	0.015	0.047	0.051	0.049	0.062	0.097	0.101	0.097	0.119
5	0.010	0.011	0.009	0.017	0.048	0.048	0.046	0.063	0.094	0.098	0.094	0.117
10	0.012	0.011	0.010	0.020	0.049	0.049	0.047	0.066	0.097	0.095	0.094	0.122
25	0.011	0.010	0.010	0.021	0.052	0.049	0.048	0.073	0.098	0.099	0.097	0.127
50	0.010	0.010	0.010	0.021	0.049	0.048	0.048	0.074	0.099	0.096	0.097	0.133
75	0.011	0.011	0.011	0.020	0.047	0.046	0.047	0.075	0.097	0.098	0.097	0.134
100	0.011	0.011	0.011	0.020	0.047	0.046	0.046	0.077	0.098	0.098	0.098	0.133
125	0.010	0.011	0.011	0.021	0.046	0.045	0.046	0.077	0.101	0.099	0.100	0.133
150	0.011	0.012	0.011	0.021	0.047	0.046	0.047	0.075	0.101	0.099	0.099	0.133
200	0.010	0.011	0.011	0.021	0.046	0.046	0.048	0.078	0.101	0.099	0.099	0.132
250	0.010	0.010	0.011	0.022	0.048	0.049	0.049	0.078	0.099	0.100	0.099	0.134
Two-sided tests- $T = 1000$												
-5	0.007	0.010	0.286	0.001	0.047	0.049	0.353	0.019	0.097	0.101	0.392	0.051
-2.5	0.005	0.011	0.044	0.008	0.035	0.051	0.066	0.043	0.076	0.101	0.081	0.090
0	0.011	0.012	0.057	0.020	0.051	0.050	0.110	0.075	0.103	0.101	0.156	0.131
2.5	0.009	0.009	0.013	0.018	0.049	0.050	0.056	0.070	0.096	0.104	0.100	0.128
5	0.010	0.010	0.009	0.019	0.048	0.051	0.049	0.072	0.095	0.099	0.093	0.129
10	0.011	0.011	0.010	0.022	0.047	0.049	0.048	0.077	0.098	0.098	0.094	0.134
25	0.010	0.011	0.010	0.024	0.050	0.048	0.048	0.085	0.099	0.099	0.097	0.146
50	0.010	0.010	0.010	0.024	0.051	0.050	0.050	0.087	0.099	0.098	0.099	0.149
75	0.011	0.011	0.011	0.026	0.050	0.048	0.050	0.086	0.098	0.096	0.097	0.154
100	0.011	0.010	0.011	0.027	0.049	0.049	0.049	0.087	0.098	0.096	0.097	0.156
125	0.010	0.011	0.011	0.027	0.050	0.049	0.051	0.087	0.098	0.096	0.098	0.156
150	0.010	0.010	0.011	0.027	0.050	0.050	0.051	0.088	0.100	0.097	0.098	0.156
200	0.011	0.011	0.010	0.026	0.049	0.050	0.051	0.090	0.100	0.100	0.101	0.159
250	0.010	0.011	0.012	0.026	0.048	0.050	0.051	0.090	0.100	0.099	0.102	0.158

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.17: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP7 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\,\boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim NIID(\mathbf{0}, \mathbf{\Omega})$ with $\mathbf{\Omega} = [1 \quad -0.95; -0.95 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1e^2_{i,t-1} + 0.85\sigma^2_{i,t-1},\ i = 1, 2.$

	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
c	1%				5%				10%			
Left-sided tests - $T = 250$												
-5	0.009	0.000	0.001	0.001	0.048	0.003	0.004	0.006	0.095	0.013	0.013	0.016
-2.5	0.005	0.000	0.000	0.000	0.043	0.001	0.001	0.001	0.108	0.001	0.001	0.002
0	0.012	0.000	0.000	0.000	0.037	0.001	0.001	0.001	0.063	0.003	0.003	0.003
2.5	0.023	0.001	0.001	0.001	0.057	0.005	0.005	0.006	0.095	0.015	0.015	0.016
5	0.025	0.002	0.002	0.002	0.066	0.012	0.012	0.014	0.114	0.027	0.026	0.030
10	0.022	0.003	0.003	0.006	0.066	0.020	0.020	0.025	0.116	0.041	0.041	0.050
25	0.017	0.006	0.007	0.012	0.062	0.028	0.029	0.040	0.107	0.059	0.060	0.076
50	0.016	0.007	0.008	0.016	0.059	0.034	0.036	0.051	0.108	0.070	0.071	0.089
75	0.015	0.008	0.009	0.016	0.057	0.038	0.040	0.055	0.107	0.077	0.079	0.098
100	0.015	0.008	0.010	0.017	0.057	0.038	0.042	0.058	0.106	0.079	0.083	0.101
125	0.015	0.009	0.010	0.020	0.057	0.039	0.042	0.059	0.106	0.082	0.086	0.107
150	0.015	0.009	0.011	0.020	0.055	0.042	0.045	0.062	0.108	0.084	0.089	0.112
200	0.013	0.010	0.011	0.020	0.056	0.045	0.047	0.067	0.109	0.089	0.092	0.114
250	0.012	0.009	0.011	0.021	0.055	0.046	0.049	0.068	0.106	0.092	0.097	0.121
Right-sided tests - $T = 250$												
-5	0.008	0.016	0.023	0.019	0.044	0.072	0.083	0.074	0.086	0.150	0.164	0.146
-2.5	0.010	0.019	0.024	0.023	0.043	0.100	0.113	0.107	0.091	0.239	0.255	0.246
0	0.011	0.024	0.029	0.031	0.054	0.104	0.117	0.123	0.109	0.228	0.240	0.245
2.5	0.012	0.023	0.027	0.032	0.061	0.115	0.123	0.133	0.125	0.219	0.230	0.242
5	0.012	0.023	0.027	0.032	0.060	0.107	0.118	0.130	0.125	0.207	0.217	0.232
10	0.013	0.022	0.026	0.032	0.059	0.097	0.105	0.119	0.120	0.183	0.192	0.211
25	0.011	0.016	0.020	0.027	0.060	0.081	0.089	0.106	0.114	0.156	0.163	0.185
50	0.010	0.015	0.018	0.024	0.056	0.071	0.079	0.099	0.114	0.138	0.145	0.169
75	0.011	0.013	0.016	0.023	0.056	0.066	0.071	0.093	0.111	0.132	0.138	0.163
100	0.011	0.014	0.016	0.025	0.056	0.063	0.067	0.087	0.112	0.125	0.131	0.156
125	0.011	0.012	0.016	0.025	0.056	0.062	0.067	0.087	0.113	0.119	0.126	0.153
150	0.010	0.013	0.014	0.025	0.058	0.059	0.065	0.087	0.111	0.116	0.120	0.146
200	0.010	0.012	0.014	0.024	0.058	0.058	0.062	0.083	0.109	0.110	0.114	0.139
250	0.012	0.011	0.014	0.024	0.058	0.054	0.059	0.081	0.109	0.107	0.112	0.136
Two-sided tests - $T = 250$												
-5	0.007	0.008	0.013	0.011	0.044	0.038	0.047	0.042	0.088	0.074	0.087	0.080
-2.5	0.009	0.010	0.013	0.014	0.039	0.046	0.056	0.055	0.084	0.102	0.113	0.108
0	0.008	0.011	0.014	0.015	0.046	0.053	0.062	0.066	0.094	0.105	0.118	0.125
2.5	0.011	0.012	0.015	0.018	0.050	0.058	0.067	0.076	0.105	0.119	0.129	0.138
5	0.009	0.012	0.016	0.020	0.051	0.060	0.065	0.077	0.107	0.118	0.130	0.144
10	0.011	0.013	0.016	0.020	0.055	0.060	0.066	0.080	0.108	0.116	0.125	0.144
25	0.011	0.011	0.014	0.022	0.055	0.056	0.063	0.084	0.109	0.109	0.117	0.145
50	0.012	0.011	0.013	0.023	0.055	0.052	0.060	0.083	0.110	0.106	0.114	0.150
75	0.012	0.010	0.014	0.025	0.055	0.051	0.058	0.084	0.109	0.103	0.111	0.148
100	0.012	0.010	0.014	0.025	0.055	0.051	0.058	0.086	0.108	0.101	0.109	0.146
125	0.011	0.010	0.013	0.026	0.058	0.054	0.059	0.085	0.106	0.100	0.108	0.146
150	0.011	0.010	0.013	0.026	0.059	0.053	0.060	0.086	0.110	0.101	0.109	0.149
200	0.011	0.010	0.013	0.027	0.058	0.053	0.058	0.087	0.111	0.102	0.108	0.149
250	0.011	0.011	0.013	0.027	0.057	0.052	0.060	0.086	0.111	0.100	0.108	0.149

	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
c	1%				5%				10%			
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.043	0.003	0.003	0.003	0.093	0.011	0.011	0.010
-2.5	0.007	0.000	0.000	0.000	0.044	0.000	0.000	0.000	0.109	0.001	0.001	0.001
0	0.014	0.000	0.000	0.000	0.041	0.001	0.001	0.002	0.067	0.003	0.003	0.004
2.5	0.023	0.001	0.001	0.001	0.058	0.006	0.005	0.006	0.099	0.015	0.014	0.015
5	0.025	0.002	0.002	0.003	0.067	0.012	0.011	0.013	0.114	0.029	0.029	0.031
10	0.021	0.004	0.004	0.005	0.067	0.019	0.019	0.024	0.113	0.045	0.046	0.053
25	0.018	0.006	0.006	0.010	0.063	0.029	0.029	0.040	0.111	0.061	0.062	0.076
50	0.015	0.007	0.007	0.014	0.059	0.032	0.033	0.049	0.111	0.071	0.072	0.093
75	0.013	0.007	0.007	0.016	0.058	0.034	0.034	0.055	0.111	0.077	0.078	0.102
100	0.013	0.008	0.007	0.017	0.056	0.037	0.038	0.060	0.108	0.081	0.080	0.105
125	0.014	0.008	0.008	0.019	0.055	0.038	0.039	0.061	0.107	0.081	0.081	0.108
150	0.013	0.008	0.008	0.020	0.056	0.040	0.039	0.063	0.105	0.082	0.083	0.112
200	0.013	0.010	0.010	0.021	0.055	0.042	0.042	0.068	0.104	0.086	0.086	0.116
250	0.013	0.010	0.010	0.022	0.054	0.043	0.044	0.069	0.104	0.087	0.086	0.118
Right-sided tests - $T = 1000$												
-5	0.006	0.013	0.013	0.014	0.041	0.066	0.069	0.065	0.086	0.147	0.152	0.144
-2.5	0.007	0.015	0.018	0.018	0.036	0.093	0.096	0.094	0.081	0.234	0.240	0.232
0	0.009	0.021	0.021	0.025	0.047	0.107	0.109	0.113	0.103	0.223	0.229	0.231
2.5	0.010	0.023	0.024	0.026	0.053	0.112	0.117	0.123	0.117	0.217	0.223	0.233
5	0.010	0.021	0.024	0.027	0.056	0.105	0.109	0.117	0.115	0.205	0.207	0.218
10	0.010	0.021	0.022	0.026	0.058	0.096	0.098	0.110	0.111	0.180	0.182	0.198
25	0.010	0.017	0.018	0.025	0.053	0.082	0.084	0.099	0.108	0.155	0.157	0.178
50	0.009	0.015	0.015	0.026	0.055	0.076	0.077	0.099	0.108	0.140	0.141	0.164
75	0.011	0.015	0.016	0.027	0.054	0.072	0.074	0.095	0.106	0.132	0.134	0.162
100	0.011	0.015	0.016	0.028	0.055	0.070	0.070	0.094	0.107	0.128	0.129	0.157
125	0.011	0.015	0.015	0.028	0.055	0.067	0.068	0.092	0.109	0.128	0.130	0.156
150	0.011	0.014	0.014	0.027	0.054	0.065	0.065	0.093	0.107	0.126	0.128	0.156
200	0.011	0.013	0.014	0.026	0.054	0.064	0.065	0.092	0.107	0.121	0.122	0.157
250	0.010	0.012	0.013	0.026	0.054	0.062	0.062	0.091	0.108	0.119	0.121	0.156
Two-sided tests- $T = 1000$												
-5	0.006	0.007	0.007	0.007	0.041	0.033	0.037	0.033	0.087	0.068	0.072	0.068
-2.5	0.006	0.008	0.008	0.009	0.031	0.041	0.044	0.043	0.074	0.093	0.096	0.095
0	0.007	0.010	0.011	0.013	0.039	0.053	0.057	0.057	0.087	0.106	0.110	0.115
2.5	0.009	0.011	0.013	0.014	0.044	0.054	0.058	0.063	0.095	0.117	0.122	0.129
5	0.009	0.012	0.013	0.015	0.048	0.057	0.062	0.069	0.098	0.116	0.120	0.130
10	0.009	0.012	0.013	0.017	0.050	0.057	0.060	0.073	0.103	0.115	0.118	0.134
25	0.010	0.011	0.011	0.020	0.051	0.058	0.059	0.081	0.104	0.111	0.113	0.139
50	0.010	0.010	0.010	0.023	0.053	0.056	0.057	0.085	0.105	0.108	0.110	0.148
75	0.012	0.011	0.011	0.025	0.053	0.054	0.055	0.088	0.105	0.105	0.108	0.150
100	0.011	0.011	0.011	0.026	0.054	0.053	0.054	0.089	0.107	0.106	0.107	0.154
125	0.011	0.011	0.012	0.027	0.055	0.054	0.056	0.091	0.108	0.105	0.107	0.153
150	0.011	0.012	0.012	0.027	0.055	0.053	0.055	0.090	0.105	0.104	0.104	0.156
200	0.012	0.012	0.012	0.030	0.053	0.052	0.053	0.092	0.108	0.104	0.107	0.160
250	0.012	0.011	0.011	0.030	0.054	0.053	0.053	0.093	0.107	0.105	0.106	0.160

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.18: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP7 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\, \boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim NIID(\mathbf{0}, \mathbf{\Omega})$ with $\mathbf{\Omega} = [1 \quad -0.9; -0.9 \quad 1]$ and $\sigma_{it}^2 = 0.05 + 0.1 e_{i,t-1}^2 + 0.85 \sigma_{i,t-1}^2,\ i = 1, 2$.

	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
c		1%				5%				10%		
Left-sided tests - $T = 250$												
-5	0.008	0.000	0.001	0.001	0.048	0.005	0.006	0.007	0.098	0.015	0.017	0.018
-2.5	0.007	0.000	0.000	0.000	0.047	0.001	0.001	0.001	0.108	0.002	0.002	0.003
0	0.012	0.000	0.000	0.000	0.035	0.001	0.001	0.002	0.061	0.004	0.004	0.004
2.5	0.022	0.001	0.001	0.001	0.054	0.006	0.007	0.007	0.094	0.017	0.017	0.018
5	0.024	0.002	0.002	0.003	0.065	0.013	0.014	0.015	0.110	0.028	0.029	0.032
10	0.022	0.004	0.004	0.006	0.065	0.021	0.021	0.026	0.111	0.042	0.042	0.050
25	0.018	0.006	0.008	0.012	0.059	0.028	0.028	0.038	0.107	0.060	0.061	0.074
50	0.014	0.007	0.008	0.014	0.056	0.033	0.035	0.048	0.106	0.071	0.071	0.090
75	0.014	0.006	0.008	0.014	0.054	0.036	0.038	0.052	0.105	0.077	0.079	0.097
100	0.014	0.008	0.010	0.015	0.055	0.038	0.040	0.056	0.104	0.080	0.081	0.101
125	0.014	0.009	0.010	0.018	0.054	0.038	0.042	0.058	0.105	0.083	0.086	0.105
150	0.014	0.009	0.011	0.019	0.053	0.040	0.043	0.059	0.106	0.084	0.088	0.110
200	0.012	0.009	0.011	0.019	0.054	0.043	0.046	0.063	0.105	0.088	0.090	0.113
250	0.011	0.010	0.011	0.020	0.052	0.046	0.049	0.066	0.105	0.093	0.097	0.115
Right-sided tests - $T = 250$												
-5	0.008	0.015	0.022	0.017	0.045	0.074	0.086	0.075	0.087	0.153	0.167	0.147
-2.5	0.009	0.018	0.026	0.022	0.044	0.103	0.117	0.106	0.094	0.239	0.256	0.243
0	0.010	0.022	0.027	0.029	0.053	0.105	0.119	0.121	0.111	0.225	0.240	0.240
2.5	0.013	0.024	0.027	0.031	0.062	0.113	0.122	0.127	0.123	0.215	0.226	0.233
5	0.013	0.023	0.027	0.031	0.062	0.107	0.116	0.124	0.124	0.201	0.210	0.222
10	0.012	0.022	0.025	0.030	0.059	0.094	0.102	0.113	0.118	0.177	0.184	0.202
25	0.011	0.017	0.020	0.026	0.059	0.082	0.086	0.103	0.114	0.154	0.159	0.180
50	0.010	0.014	0.018	0.023	0.058	0.071	0.076	0.093	0.114	0.140	0.145	0.166
75	0.010	0.014	0.016	0.023	0.056	0.065	0.071	0.090	0.113	0.133	0.137	0.158
100	0.010	0.013	0.015	0.022	0.056	0.063	0.068	0.085	0.111	0.125	0.130	0.152
125	0.011	0.012	0.015	0.023	0.056	0.060	0.065	0.084	0.110	0.118	0.123	0.148
150	0.010	0.011	0.014	0.023	0.056	0.059	0.064	0.082	0.108	0.114	0.120	0.143
200	0.011	0.012	0.014	0.022	0.057	0.056	0.061	0.080	0.108	0.109	0.113	0.136
250	0.013	0.012	0.015	0.022	0.057	0.055	0.059	0.077	0.111	0.108	0.113	0.134
Two-sided tests - $T = 250$												
-5	0.008	0.008	0.013	0.010	0.046	0.038	0.049	0.045	0.091	0.078	0.092	0.081
-2.5	0.008	0.009	0.014	0.013	0.040	0.048	0.057	0.052	0.087	0.103	0.118	0.107
0	0.008	0.011	0.015	0.015	0.045	0.053	0.061	0.064	0.094	0.106	0.120	0.123
2.5	0.010	0.012	0.016	0.016	0.052	0.056	0.067	0.073	0.105	0.119	0.128	0.134
5	0.011	0.013	0.016	0.019	0.054	0.060	0.066	0.076	0.109	0.119	0.129	0.140
10	0.011	0.012	0.016	0.019	0.056	0.059	0.066	0.078	0.109	0.115	0.123	0.140
25	0.011	0.012	0.014	0.021	0.055	0.055	0.062	0.080	0.107	0.109	0.114	0.141
50	0.011	0.010	0.013	0.021	0.053	0.053	0.059	0.081	0.108	0.103	0.111	0.141
75	0.011	0.011	0.013	0.023	0.055	0.052	0.057	0.080	0.108	0.101	0.109	0.142
100	0.011	0.009	0.013	0.022	0.055	0.051	0.059	0.082	0.108	0.099	0.108	0.141
125	0.011	0.010	0.013	0.023	0.055	0.052	0.060	0.081	0.107	0.098	0.107	0.141
150	0.011	0.010	0.013	0.024	0.058	0.052	0.059	0.082	0.106	0.099	0.107	0.141
200	0.011	0.011	0.014	0.025	0.058	0.052	0.058	0.083	0.108	0.099	0.108	0.143
250	0.011	0.010	0.014	0.026	0.056	0.051	0.057	0.081	0.108	0.100	0.108	0.143

	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
c		1%				5%				10%		
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.046	0.003	0.004	0.004	0.096	0.013	0.013	0.014
-2.5	0.007	0.000	0.000	0.000	0.048	0.000	0.000	0.001	0.109	0.001	0.001	0.001
0	0.013	0.000	0.000	0.000	0.037	0.001	0.001	0.002	0.064	0.004	0.004	0.004
2.5	0.022	0.001	0.001	0.002	0.056	0.006	0.006	0.007	0.096	0.016	0.017	0.018
5	0.023	0.002	0.002	0.003	0.066	0.012	0.012	0.015	0.109	0.030	0.030	0.033
10	0.020	0.004	0.003	0.005	0.064	0.020	0.020	0.024	0.111	0.048	0.047	0.052
25	0.016	0.006	0.005	0.010	0.060	0.029	0.030	0.040	0.111	0.063	0.062	0.074
50	0.014	0.007	0.007	0.014	0.060	0.036	0.036	0.050	0.109	0.071	0.072	0.091
75	0.013	0.007	0.008	0.015	0.057	0.036	0.036	0.055	0.106	0.077	0.078	0.098
100	0.012	0.008	0.007	0.016	0.056	0.039	0.039	0.059	0.106	0.080	0.081	0.105
125	0.013	0.008	0.008	0.018	0.055	0.038	0.039	0.061	0.106	0.082	0.083	0.107
150	0.014	0.008	0.008	0.019	0.055	0.041	0.040	0.062	0.105	0.083	0.084	0.110
200	0.013	0.009	0.010	0.021	0.056	0.042	0.043	0.065	0.106	0.085	0.086	0.113
250	0.013	0.009	0.010	0.022	0.053	0.043	0.043	0.066	0.105	0.088	0.089	0.117
Right-sided tests - $T = 1000$												
-5	0.007	0.012	0.013	0.013	0.039	0.068	0.071	0.066	0.088	0.147	0.153	0.144
-2.5	0.006	0.015	0.017	0.017	0.037	0.098	0.099	0.097	0.086	0.230	0.239	0.231
0	0.010	0.021	0.022	0.024	0.049	0.108	0.109	0.110	0.104	0.223	0.229	0.229
2.5	0.011	0.022	0.024	0.025	0.054	0.109	0.112	0.119	0.117	0.213	0.215	0.225
5	0.011	0.022	0.023	0.025	0.056	0.104	0.106	0.115	0.116	0.198	0.200	0.210
10	0.011	0.022	0.023	0.025	0.057	0.094	0.094	0.105	0.111	0.177	0.178	0.194
25	0.011	0.016	0.018	0.024	0.055	0.081	0.083	0.097	0.109	0.151	0.152	0.170
50	0.011	0.016	0.015	0.024	0.056	0.075	0.075	0.095	0.109	0.139	0.141	0.161
75	0.010	0.015	0.016	0.027	0.056	0.071	0.071	0.091	0.108	0.134	0.135	0.159
100	0.010	0.016	0.016	0.027	0.053	0.068	0.069	0.092	0.107	0.128	0.129	0.155
125	0.010	0.016	0.015	0.026	0.053	0.067	0.067	0.089	0.109	0.129	0.130	0.155
150	0.011	0.014	0.015	0.025	0.053	0.064	0.065	0.089	0.108	0.126	0.129	0.154
200	0.011	0.014	0.014	0.024	0.054	0.061	0.063	0.087	0.107	0.123	0.123	0.155
250	0.011	0.013	0.013	0.024	0.056	0.063	0.063	0.087	0.107	0.119	0.119	0.151
Two-sided tests- $T = 1000$												
-5	0.006	0.006	0.007	0.007	0.040	0.033	0.036	0.032	0.089	0.070	0.075	0.070
-2.5	0.006	0.008	0.008	0.008	0.032	0.040	0.044	0.042	0.079	0.096	0.100	0.097
0	0.007	0.011	0.011	0.013	0.041	0.053	0.055	0.056	0.088	0.108	0.110	0.112
2.5	0.010	0.013	0.013	0.013	0.044	0.055	0.059	0.062	0.096	0.115	0.118	0.126
5	0.010	0.012	0.013	0.015	0.048	0.057	0.059	0.067	0.101	0.114	0.118	0.130
10	0.010	0.012	0.013	0.017	0.053	0.059	0.061	0.071	0.101	0.113	0.114	0.129
25	0.010	0.011	0.011	0.019	0.052	0.057	0.058	0.077	0.106	0.110	0.113	0.136
50	0.011	0.010	0.010	0.020	0.053	0.054	0.056	0.083	0.107	0.109	0.111	0.145
75	0.011	0.010	0.010	0.022	0.056	0.054	0.055	0.083	0.107	0.106	0.108	0.146
100	0.011	0.012	0.012	0.024	0.056	0.054	0.055	0.086	0.107	0.106	0.108	0.151
125	0.011	0.012	0.012	0.025	0.056	0.055	0.055	0.085	0.106	0.104	0.106	0.150
150	0.012	0.011	0.012	0.025	0.055	0.052	0.055	0.086	0.106	0.104	0.105	0.151
200	0.013	0.013	0.013	0.027	0.053	0.053	0.055	0.088	0.108	0.104	0.105	0.152
250	0.013	0.012	0.013	0.027	0.054	0.053	0.055	0.090	0.107	0.105	0.106	0.152

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.19: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP7 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\, \boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim NIID(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad -0.5; -0.5 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1 e^2_{i,t-1} + 0.85\sigma^2_{i,t-1},\ i = 1, 2$.

	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
c		1%				5%				10%		
Left-sided tests - $T = 250$												
-5	0.010	0.002	0.007	0.004	0.051	0.019	0.027	0.022	0.102	0.047	0.056	0.047
-2.5	0.008	0.001	0.002	0.001	0.049	0.005	0.008	0.008	0.098	0.016	0.021	0.018
0	0.007	0.001	0.002	0.001	0.029	0.007	0.010	0.008	0.058	0.018	0.021	0.021
2.5	0.012	0.003	0.003	0.003	0.044	0.019	0.020	0.020	0.084	0.038	0.039	0.039
5	0.014	0.004	0.005	0.006	0.050	0.024	0.026	0.025	0.095	0.051	0.051	0.053
10	0.014	0.006	0.007	0.008	0.053	0.030	0.032	0.034	0.101	0.064	0.066	0.068
25	0.013	0.007	0.008	0.010	0.053	0.038	0.040	0.044	0.101	0.074	0.077	0.080
50	0.010	0.007	0.008	0.010	0.052	0.039	0.043	0.046	0.099	0.079	0.081	0.086
75	0.010	0.008	0.009	0.010	0.052	0.043	0.046	0.049	0.098	0.083	0.085	0.090
100	0.011	0.009	0.010	0.011	0.052	0.043	0.046	0.049	0.098	0.084	0.087	0.093
125	0.011	0.009	0.010	0.011	0.050	0.043	0.047	0.051	0.098	0.084	0.088	0.094
150	0.010	0.009	0.009	0.011	0.051	0.044	0.047	0.053	0.099	0.089	0.092	0.095
200	0.010	0.009	0.011	0.012	0.050	0.045	0.050	0.054	0.099	0.091	0.094	0.099
250	0.010	0.010	0.011	0.012	0.051	0.047	0.051	0.055	0.100	0.092	0.095	0.102
Right-sided tests - $T = 250$												
-5	0.009	0.015	0.034	0.016	0.047	0.074	0.100	0.073	0.097	0.143	0.167	0.140
-2.5	0.013	0.023	0.043	0.023	0.054	0.103	0.124	0.104	0.111	0.203	0.220	0.200
0	0.013	0.020	0.032	0.021	0.059	0.099	0.113	0.097	0.122	0.194	0.204	0.193
2.5	0.012	0.018	0.025	0.021	0.062	0.092	0.101	0.094	0.120	0.171	0.182	0.173
5	0.012	0.019	0.023	0.020	0.061	0.083	0.091	0.087	0.116	0.159	0.167	0.164
10	0.011	0.015	0.018	0.018	0.057	0.075	0.082	0.079	0.112	0.143	0.150	0.150
25	0.010	0.014	0.017	0.015	0.052	0.065	0.070	0.071	0.110	0.132	0.136	0.137
50	0.010	0.011	0.014	0.014	0.052	0.061	0.067	0.066	0.108	0.121	0.125	0.127
75	0.008	0.010	0.012	0.013	0.054	0.060	0.064	0.067	0.107	0.119	0.122	0.125
100	0.008	0.010	0.011	0.014	0.053	0.057	0.061	0.063	0.109	0.114	0.119	0.121
125	0.009	0.010	0.011	0.012	0.053	0.055	0.059	0.064	0.109	0.113	0.116	0.119
150	0.009	0.010	0.012	0.012	0.054	0.055	0.060	0.063	0.108	0.111	0.114	0.118
200	0.009	0.011	0.012	0.013	0.055	0.056	0.058	0.061	0.107	0.108	0.109	0.115
250	0.012	0.011	0.013	0.015	0.055	0.054	0.058	0.062	0.106	0.104	0.107	0.114
Two-sided tests - $T = 250$												
-5	0.009	0.009	0.029	0.011	0.048	0.045	0.076	0.050	0.100	0.093	0.127	0.095
-2.5	0.011	0.012	0.031	0.012	0.049	0.054	0.082	0.054	0.099	0.106	0.132	0.111
0	0.010	0.011	0.022	0.012	0.048	0.051	0.068	0.054	0.096	0.105	0.122	0.105
2.5	0.010	0.011	0.017	0.012	0.051	0.054	0.066	0.058	0.105	0.109	0.121	0.115
5	0.010	0.011	0.015	0.013	0.053	0.052	0.062	0.059	0.104	0.107	0.117	0.112
10	0.011	0.011	0.014	0.014	0.052	0.053	0.061	0.060	0.103	0.105	0.114	0.112
25	0.011	0.010	0.012	0.014	0.051	0.051	0.058	0.058	0.101	0.102	0.110	0.115
50	0.009	0.009	0.011	0.013	0.050	0.049	0.055	0.058	0.103	0.099	0.109	0.112
75	0.009	0.008	0.011	0.012	0.048	0.048	0.055	0.057	0.104	0.102	0.110	0.116
100	0.008	0.008	0.011	0.011	0.051	0.047	0.052	0.057	0.103	0.100	0.107	0.112
125	0.008	0.009	0.012	0.012	0.050	0.050	0.057	0.060	0.103	0.098	0.105	0.114
150	0.009	0.010	0.012	0.012	0.050	0.050	0.055	0.060	0.104	0.098	0.107	0.116
200	0.009	0.010	0.012	0.013	0.052	0.048	0.058	0.063	0.104	0.101	0.109	0.115
250	0.010	0.011	0.013	0.014	0.051	0.050	0.057	0.062	0.104	0.101	0.109	0.116

	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
c		1%				5%				10%		
Left-sided tests - $T = 1000$												
-5	0.010	0.001	0.002	0.003	0.052	0.019	0.022	0.019	0.101	0.048	0.050	0.048
-2.5	0.010	0.001	0.000	0.001	0.049	0.005	0.005	0.006	0.100	0.017	0.018	0.016
0	0.008	0.001	0.001	0.001	0.034	0.008	0.009	0.009	0.062	0.022	0.022	0.022
2.5	0.011	0.003	0.003	0.004	0.048	0.017	0.017	0.019	0.089	0.042	0.043	0.044
5	0.012	0.004	0.004	0.005	0.052	0.024	0.023	0.026	0.099	0.055	0.056	0.056
10	0.013	0.005	0.005	0.006	0.056	0.031	0.031	0.035	0.103	0.067	0.066	0.069
25	0.011	0.006	0.006	0.009	0.052	0.036	0.037	0.041	0.103	0.078	0.077	0.084
50	0.011	0.007	0.007	0.011	0.053	0.042	0.042	0.047	0.102	0.083	0.083	0.090
75	0.011	0.008	0.007	0.010	0.054	0.042	0.043	0.049	0.103	0.085	0.087	0.094
100	0.010	0.008	0.008	0.011	0.052	0.043	0.043	0.049	0.102	0.087	0.088	0.096
125	0.010	0.008	0.008	0.011	0.053	0.043	0.043	0.051	0.103	0.090	0.090	0.098
150	0.010	0.009	0.008	0.011	0.052	0.044	0.044	0.052	0.103	0.091	0.091	0.100
200	0.011	0.008	0.008	0.012	0.052	0.045	0.046	0.054	0.104	0.093	0.093	0.102
250	0.012	0.009	0.010	0.013	0.051	0.045	0.048	0.055	0.103	0.093	0.094	0.103
Right-sided tests - $T = 1000$												
-5	0.008	0.013	0.017	0.013	0.045	0.071	0.079	0.068	0.095	0.137	0.144	0.134
-2.5	0.008	0.018	0.024	0.017	0.051	0.097	0.108	0.094	0.106	0.200	0.205	0.196
0	0.013	0.020	0.025	0.020	0.060	0.096	0.101	0.095	0.119	0.192	0.196	0.192
2.5	0.012	0.020	0.023	0.021	0.059	0.090	0.092	0.091	0.118	0.169	0.169	0.170
5	0.012	0.019	0.020	0.018	0.059	0.084	0.086	0.086	0.112	0.157	0.157	0.158
10	0.012	0.018	0.019	0.019	0.054	0.074	0.076	0.077	0.108	0.141	0.142	0.146
25	0.012	0.015	0.015	0.015	0.054	0.069	0.069	0.071	0.104	0.128	0.127	0.130
50	0.011	0.014	0.014	0.014	0.054	0.064	0.065	0.069	0.103	0.122	0.121	0.125
75	0.011	0.013	0.014	0.015	0.055	0.063	0.063	0.068	0.102	0.118	0.118	0.124
100	0.011	0.014	0.014	0.015	0.054	0.062	0.062	0.065	0.104	0.117	0.117	0.123
125	0.011	0.013	0.013	0.015	0.053	0.061	0.061	0.067	0.108	0.116	0.118	0.124
150	0.013	0.013	0.013	0.016	0.054	0.061	0.061	0.068	0.106	0.117	0.117	0.126
200	0.011	0.013	0.014	0.016	0.054	0.058	0.059	0.064	0.106	0.114	0.114	0.122
250	0.012	0.014	0.014	0.016	0.054	0.058	0.059	0.064	0.105	0.112	0.112	0.121
Two-sided tests- $T = 1000$												
-5	0.008	0.007	0.010	0.008	0.046	0.041	0.052	0.043	0.099	0.089	0.100	0.087
-2.5	0.007	0.009	0.013	0.007	0.046	0.047	0.055	0.048	0.096	0.103	0.114	0.100
0	0.010	0.011	0.013	0.012	0.046	0.051	0.057	0.050	0.097	0.104	0.110	0.104
2.5	0.010	0.011	0.012	0.012	0.052	0.054	0.057	0.057	0.101	0.108	0.110	0.109
5	0.011	0.010	0.012	0.011	0.053	0.054	0.056	0.059	0.102	0.108	0.109	0.112
10	0.011	0.012	0.012	0.013	0.051	0.054	0.056	0.057	0.102	0.107	0.106	0.111
25	0.011	0.011	0.011	0.013	0.051	0.053	0.054	0.058	0.103	0.105	0.107	0.112
50	0.011	0.010	0.011	0.014	0.053	0.052	0.053	0.060	0.105	0.105	0.108	0.116
75	0.011	0.010	0.011	0.014	0.053	0.053	0.053	0.060	0.105	0.105	0.106	0.117
100	0.011	0.011	0.012	0.014	0.053	0.052	0.051	0.060	0.105	0.103	0.105	0.115
125	0.010	0.011	0.011	0.015	0.052	0.050	0.051	0.060	0.106	0.105	0.104	0.118
150	0.011	0.010	0.011	0.015	0.052	0.050	0.052	0.061	0.105	0.104	0.105	0.120
200	0.011	0.012	0.012	0.015	0.053	0.051	0.052	0.063	0.104	0.104	0.105	0.119
250	0.012	0.012	0.012	0.016	0.054	0.052	0.053	0.064	0.105	0.103	0.106	0.119

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.20: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP7 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\, \boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim NIID(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad 0; 0 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1 e^2_{i,t-1} + 0.85 \sigma^2_{i,t-1},\ i = 1, 2$.

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.011	0.011	0.025	0.012	0.050	0.049	0.070	0.052	0.101	0.101	0.122	0.098
-2.5	0.010	0.010	0.025	0.011	0.048	0.048	0.062	0.046	0.096	0.097	0.104	0.092
0	0.010	0.009	0.018	0.010	0.046	0.046	0.058	0.046	0.096	0.096	0.103	0.094
2.5	0.011	0.010	0.015	0.011	0.048	0.047	0.054	0.047	0.098	0.096	0.103	0.097
5	0.011	0.010	0.013	0.011	0.049	0.047	0.051	0.048	0.099	0.099	0.105	0.098
10	0.010	0.010	0.011	0.011	0.050	0.049	0.054	0.049	0.102	0.103	0.105	0.102
25	0.011	0.011	0.013	0.012	0.053	0.051	0.053	0.053	0.103	0.101	0.103	0.103
50	0.010	0.010	0.012	0.011	0.050	0.048	0.052	0.049	0.101	0.099	0.104	0.100
75	0.010	0.010	0.011	0.010	0.049	0.047	0.050	0.049	0.097	0.096	0.098	0.096
100	0.010	0.010	0.011	0.010	0.048	0.046	0.049	0.047	0.096	0.093	0.096	0.093
125	0.010	0.010	0.011	0.010	0.046	0.045	0.049	0.047	0.096	0.093	0.096	0.094
150	0.010	0.010	0.011	0.010	0.049	0.046	0.050	0.047	0.097	0.094	0.098	0.095
200	0.010	0.009	0.010	0.009	0.048	0.048	0.050	0.049	0.095	0.094	0.097	0.096
250	0.009	0.009	0.010	0.010	0.049	0.048	0.051	0.049	0.099	0.096	0.099	0.098
Right-sided tests - $T = 250$												
-5	0.011	0.011	0.026	0.013	0.049	0.050	0.070	0.050	0.099	0.098	0.116	0.098
-2.5	0.011	0.011	0.025	0.012	0.052	0.052	0.067	0.052	0.103	0.104	0.113	0.102
0	0.011	0.010	0.018	0.012	0.051	0.050	0.057	0.050	0.101	0.100	0.102	0.099
2.5	0.010	0.010	0.014	0.011	0.050	0.049	0.054	0.050	0.103	0.102	0.106	0.100
5	0.009	0.009	0.013	0.010	0.050	0.049	0.055	0.050	0.102	0.100	0.105	0.101
10	0.009	0.009	0.010	0.009	0.051	0.049	0.052	0.051	0.102	0.100	0.105	0.102
25	0.010	0.009	0.011	0.010	0.051	0.050	0.053	0.051	0.103	0.101	0.102	0.103
50	0.009	0.009	0.011	0.009	0.051	0.051	0.052	0.050	0.101	0.099	0.104	0.099
75	0.009	0.008	0.010	0.010	0.053	0.050	0.054	0.053	0.101	0.100	0.103	0.100
100	0.010	0.009	0.010	0.009	0.050	0.050	0.053	0.050	0.102	0.100	0.105	0.101
125	0.008	0.009	0.010	0.009	0.050	0.049	0.052	0.049	0.104	0.100	0.104	0.100
150	0.007	0.008	0.010	0.010	0.051	0.050	0.052	0.050	0.102	0.101	0.104	0.100
200	0.008	0.009	0.010	0.010	0.050	0.049	0.053	0.051	0.104	0.102	0.104	0.102
250	0.009	0.010	0.011	0.011	0.050	0.049	0.051	0.050	0.104	0.102	0.105	0.102
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.038	0.015	0.049	0.049	0.089	0.053	0.099	0.099	0.140	0.102
-2.5	0.011	0.011	0.037	0.012	0.048	0.048	0.082	0.050	0.097	0.101	0.130	0.097
0	0.011	0.010	0.025	0.011	0.047	0.048	0.068	0.048	0.095	0.095	0.115	0.095
2.5	0.011	0.011	0.017	0.012	0.047	0.048	0.060	0.049	0.098	0.096	0.108	0.097
5	0.010	0.010	0.014	0.011	0.050	0.047	0.056	0.050	0.097	0.095	0.106	0.098
10	0.010	0.009	0.012	0.010	0.049	0.047	0.053	0.052	0.100	0.098	0.106	0.100
25	0.011	0.011	0.012	0.012	0.049	0.047	0.053	0.050	0.102	0.100	0.107	0.103
50	0.009	0.009	0.011	0.010	0.051	0.048	0.056	0.052	0.100	0.098	0.104	0.099
75	0.008	0.008	0.010	0.010	0.047	0.046	0.052	0.048	0.100	0.097	0.104	0.101
100	0.009	0.008	0.010	0.009	0.046	0.046	0.051	0.048	0.096	0.095	0.101	0.098
125	0.009	0.008	0.010	0.009	0.047	0.046	0.050	0.047	0.096	0.094	0.100	0.096
150	0.008	0.008	0.011	0.009	0.046	0.046	0.051	0.048	0.099	0.097	0.102	0.097
200	0.008	0.008	0.011	0.009	0.050	0.049	0.054	0.050	0.098	0.097	0.103	0.100
250	0.008	0.009	0.011	0.010	0.049	0.049	0.054	0.051	0.098	0.096	0.103	0.099

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.010	0.001	0.002	0.003	0.052	0.019	0.022	0.019	0.101	0.048	0.050	0.048
-2.5	0.010	0.001	0.000	0.001	0.049	0.005	0.005	0.006	0.100	0.017	0.018	0.016
0	0.008	0.001	0.001	0.001	0.034	0.008	0.009	0.009	0.062	0.022	0.022	0.022
2.5	0.011	0.003	0.003	0.004	0.048	0.017	0.017	0.019	0.089	0.042	0.043	0.044
5	0.012	0.004	0.004	0.005	0.052	0.024	0.023	0.026	0.099	0.055	0.056	0.056
10	0.013	0.005	0.005	0.006	0.056	0.031	0.031	0.035	0.103	0.067	0.066	0.069
25	0.011	0.006	0.006	0.009	0.052	0.036	0.037	0.041	0.103	0.078	0.077	0.084
50	0.011	0.007	0.007	0.011	0.053	0.042	0.042	0.047	0.102	0.083	0.083	0.090
75	0.011	0.008	0.007	0.010	0.054	0.042	0.043	0.049	0.103	0.085	0.087	0.094
100	0.010	0.008	0.008	0.011	0.052	0.043	0.043	0.049	0.102	0.087	0.088	0.096
125	0.010	0.008	0.008	0.011	0.053	0.043	0.043	0.051	0.103	0.090	0.090	0.098
150	0.010	0.009	0.008	0.011	0.052	0.044	0.044	0.052	0.103	0.091	0.091	0.100
200	0.011	0.008	0.008	0.012	0.052	0.045	0.046	0.054	0.104	0.093	0.093	0.102
250	0.012	0.009	0.010	0.013	0.051	0.045	0.048	0.055	0.103	0.093	0.094	0.103
Right-sided tests - $T = 1000$												
-5	0.011	0.011	0.016	0.012	0.049	0.049	0.057	0.048	0.100	0.099	0.107	0.100
-2.5	0.010	0.010	0.018	0.010	0.050	0.050	0.059	0.050	0.095	0.095	0.107	0.095
0	0.011	0.010	0.014	0.011	0.048	0.051	0.054	0.048	0.101	0.103	0.104	0.099
2.5	0.008	0.008	0.011	0.008	0.049	0.049	0.052	0.048	0.103	0.104	0.105	0.101
5	0.009	0.009	0.009	0.009	0.050	0.050	0.050	0.050	0.100	0.101	0.103	0.101
10	0.008	0.010	0.010	0.010	0.052	0.051	0.051	0.051	0.101	0.100	0.101	0.099
25	0.011	0.011	0.011	0.011	0.051	0.050	0.050	0.050	0.100	0.100	0.101	0.100
50	0.012	0.011	0.011	0.012	0.052	0.051	0.052	0.052	0.102	0.101	0.103	0.101
75	0.012	0.011	0.011	0.011	0.053	0.052	0.053	0.052	0.101	0.101	0.102	0.101
100	0.011	0.011	0.012	0.011	0.052	0.052	0.053	0.052	0.100	0.098	0.100	0.099
125	0.012	0.011	0.011	0.011	0.052	0.053	0.052	0.052	0.100	0.101	0.101	0.099
150	0.011	0.012	0.011	0.011	0.053	0.053	0.052	0.053	0.102	0.102	0.102	0.100
200	0.011	0.011	0.011	0.011	0.052	0.051	0.053	0.051	0.104	0.104	0.105	0.102
250	0.011	0.010	0.011	0.010	0.052	0.050	0.051	0.051	0.106	0.105	0.106	0.105
Two-sided tests- $T = 1000$												
-5	0.011	0.011	0.017	0.012	0.049	0.049	0.065	0.049	0.099	0.099	0.116	0.099
-2.5	0.009	0.009	0.022	0.010	0.051	0.049	0.068	0.050	0.097	0.099	0.118	0.098
0	0.011	0.011	0.017	0.011	0.048	0.050	0.062	0.050	0.099	0.101	0.113	0.099
2.5	0.010	0.010	0.011	0.010	0.048	0.049	0.054	0.048	0.099	0.102	0.107	0.100
5	0.009	0.009	0.010	0.008	0.049	0.048	0.051	0.048	0.101	0.100	0.103	0.101
10	0.009	0.009	0.010	0.009	0.050	0.050	0.052	0.050	0.101	0.100	0.101	0.100
25	0.010	0.010	0.010	0.011	0.051	0.051	0.052	0.052	0.100	0.097	0.099	0.099
50	0.010	0.010	0.011	0.011	0.050	0.050	0.050	0.050	0.101	0.099	0.100	0.101
75	0.010	0.010	0.010	0.010	0.051	0.049	0.050	0.051	0.102	0.100	0.102	0.100
100	0.010	0.010	0.010	0.011	0.051	0.051	0.052	0.051	0.102	0.101	0.102	0.101
125	0.010	0.010	0.011	0.011	0.051	0.050	0.051	0.052	0.101	0.101	0.101	0.101
150	0.011	0.011	0.011	0.011	0.053	0.049	0.052	0.051	0.100	0.100	0.100	0.101
200	0.011	0.011	0.011	0.011	0.050	0.050	0.051	0.051	0.102	0.100	0.101	0.100
250	0.011	0.011	0.011	0.011	0.050	0.050	0.051	0.051	0.099	0.097	0.099	0.099

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.21: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP8 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\, \boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim iid t_5(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad -0.95; -0.95 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1 e^2_{i,t-1} + 0.85\sigma^2_{i,t-1},\ i = 1, 2$. $t_5(\mathbf{0}, \boldsymbol{\Omega})$ defines a mean zero Student-t distribution with 5 degrees of freedom and variance matrix $\boldsymbol{\Omega}$).

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.007	0.001	0.001	0.007	0.047	0.005	0.005	0.018	0.096	0.016	0.015	0.032
-2.5	0.007	0.001	0.000	0.003	0.044	0.002	0.002	0.009	0.093	0.005	0.005	0.013
0	0.010	0.001	0.001	0.005	0.038	0.003	0.003	0.009	0.067	0.006	0.007	0.015
2.5	0.019	0.001	0.001	0.006	0.054	0.006	0.006	0.015	0.094	0.016	0.016	0.029
5	0.022	0.002	0.002	0.008	0.064	0.012	0.011	0.025	0.110	0.028	0.027	0.047
10	0.023	0.004	0.003	0.015	0.069	0.018	0.018	0.041	0.116	0.041	0.041	0.071
25	0.019	0.006	0.006	0.025	0.064	0.028	0.027	0.066	0.113	0.057	0.058	0.101
50	0.017	0.006	0.007	0.035	0.064	0.033	0.035	0.083	0.116	0.069	0.072	0.126
75	0.016	0.007	0.009	0.039	0.065	0.036	0.039	0.092	0.117	0.078	0.082	0.140
100	0.016	0.007	0.010	0.041	0.064	0.039	0.045	0.099	0.120	0.083	0.090	0.149
125	0.015	0.007	0.011	0.044	0.063	0.041	0.048	0.101	0.119	0.088	0.094	0.155
150	0.015	0.007	0.011	0.046	0.065	0.045	0.051	0.105	0.119	0.090	0.097	0.160
200	0.013	0.007	0.013	0.049	0.065	0.048	0.055	0.112	0.122	0.096	0.107	0.165
250	0.011	0.008	0.015	0.050	0.065	0.050	0.058	0.114	0.121	0.098	0.108	0.172
Right-sided tests - $T = 250$												
-5	0.008	0.014	0.029	0.038	0.044	0.084	0.106	0.106	0.083	0.173	0.188	0.180
-2.5	0.006	0.017	0.025	0.037	0.035	0.102	0.122	0.141	0.083	0.251	0.257	0.279
0	0.007	0.017	0.024	0.051	0.048	0.101	0.122	0.172	0.105	0.221	0.243	0.300
2.5	0.008	0.018	0.026	0.058	0.056	0.105	0.123	0.183	0.120	0.212	0.227	0.298
5	0.009	0.020	0.027	0.062	0.061	0.100	0.113	0.183	0.123	0.198	0.208	0.286
10	0.011	0.021	0.027	0.066	0.062	0.093	0.104	0.175	0.123	0.177	0.189	0.266
25	0.012	0.018	0.023	0.067	0.062	0.080	0.091	0.165	0.122	0.154	0.166	0.243
50	0.012	0.016	0.020	0.065	0.061	0.075	0.085	0.157	0.123	0.137	0.149	0.229
75	0.011	0.014	0.020	0.064	0.061	0.068	0.080	0.154	0.124	0.133	0.145	0.220
100	0.011	0.012	0.018	0.062	0.061	0.065	0.077	0.147	0.122	0.128	0.139	0.216
125	0.012	0.012	0.017	0.062	0.061	0.062	0.074	0.143	0.122	0.122	0.135	0.207
150	0.011	0.011	0.017	0.062	0.061	0.058	0.072	0.138	0.122	0.119	0.131	0.202
200	0.010	0.011	0.018	0.060	0.062	0.057	0.068	0.131	0.120	0.111	0.123	0.195
250	0.011	0.010	0.016	0.055	0.061	0.053	0.064	0.126	0.120	0.107	0.120	0.187
Two-sided tests - $T = 250$												
-5	0.007	0.007	0.019	0.029	0.043	0.041	0.062	0.081	0.085	0.088	0.112	0.124
-2.5	0.005	0.008	0.014	0.024	0.030	0.043	0.062	0.081	0.075	0.104	0.123	0.149
0	0.006	0.008	0.012	0.035	0.040	0.049	0.064	0.110	0.087	0.104	0.125	0.182
2.5	0.007	0.008	0.013	0.042	0.045	0.052	0.065	0.120	0.098	0.111	0.129	0.197
5	0.008	0.010	0.015	0.047	0.049	0.056	0.067	0.130	0.105	0.111	0.124	0.208
10	0.011	0.012	0.017	0.054	0.055	0.058	0.067	0.141	0.111	0.112	0.122	0.216
25	0.013	0.012	0.017	0.064	0.059	0.054	0.064	0.156	0.118	0.107	0.118	0.230
50	0.014	0.012	0.016	0.070	0.063	0.051	0.063	0.163	0.122	0.107	0.120	0.240
75	0.013	0.011	0.015	0.073	0.062	0.052	0.064	0.168	0.121	0.104	0.119	0.246
100	0.013	0.010	0.016	0.076	0.062	0.050	0.064	0.167	0.122	0.102	0.121	0.245
125	0.012	0.010	0.016	0.076	0.060	0.049	0.063	0.166	0.121	0.103	0.121	0.244
150	0.012	0.009	0.016	0.077	0.061	0.050	0.064	0.165	0.121	0.102	0.123	0.243
200	0.010	0.009	0.017	0.078	0.063	0.049	0.067	0.167	0.124	0.102	0.123	0.243
250	0.009	0.008	0.016	0.076	0.060	0.049	0.068	0.167	0.123	0.102	0.122	0.241

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.001	0.007	0.044	0.004	0.004	0.017	0.094	0.013	0.012	0.028
-2.5	0.007	0.000	0.000	0.003	0.047	0.001	0.001	0.007	0.103	0.003	0.003	0.011
0	0.008	0.000	0.000	0.004	0.033	0.002	0.002	0.009	0.060	0.005	0.005	0.015
2.5	0.016	0.001	0.001	0.006	0.051	0.005	0.004	0.017	0.091	0.013	0.012	0.030
5	0.022	0.001	0.001	0.011	0.066	0.010	0.009	0.030	0.108	0.023	0.020	0.055
10	0.023	0.003	0.002	0.022	0.070	0.015	0.014	0.055	0.117	0.037	0.036	0.088
25	0.023	0.004	0.004	0.048	0.071	0.024	0.023	0.094	0.120	0.054	0.053	0.133
50	0.021	0.006	0.006	0.064	0.068	0.030	0.030	0.118	0.117	0.064	0.064	0.164
75	0.021	0.007	0.007	0.075	0.068	0.034	0.034	0.132	0.116	0.070	0.070	0.178
100	0.020	0.008	0.009	0.082	0.066	0.038	0.037	0.142	0.114	0.075	0.076	0.185
125	0.019	0.009	0.009	0.088	0.064	0.040	0.040	0.149	0.115	0.075	0.078	0.190
150	0.020	0.010	0.010	0.093	0.066	0.040	0.041	0.151	0.115	0.078	0.079	0.197
200	0.019	0.010	0.011	0.096	0.066	0.043	0.044	0.159	0.115	0.082	0.084	0.207
250	0.018	0.011	0.012	0.101	0.064	0.045	0.047	0.166	0.113	0.086	0.087	0.214
Right-sided tests - $T = 1000$												
-5	0.005	0.015	0.054	0.033	0.028	0.083	0.132	0.092	0.070	0.175	0.221	0.156
-2.5	0.003	0.014	0.041	0.044	0.020	0.098	0.146	0.136	0.054	0.241	0.283	0.258
0	0.003	0.019	0.024	0.070	0.027	0.091	0.115	0.191	0.071	0.204	0.234	0.309
2.5	0.004	0.019	0.024	0.086	0.033	0.098	0.110	0.213	0.086	0.197	0.212	0.320
5	0.005	0.020	0.024	0.095	0.040	0.095	0.103	0.218	0.090	0.186	0.195	0.315
10	0.006	0.020	0.023	0.105	0.045	0.091	0.093	0.219	0.095	0.169	0.175	0.308
25	0.007	0.018	0.019	0.115	0.049	0.079	0.082	0.220	0.101	0.153	0.156	0.300
50	0.008	0.017	0.016	0.121	0.051	0.072	0.074	0.222	0.103	0.140	0.145	0.296
75	0.008	0.016	0.017	0.123	0.049	0.070	0.074	0.219	0.108	0.133	0.138	0.290
100	0.009	0.015	0.016	0.125	0.053	0.067	0.072	0.216	0.106	0.129	0.133	0.287
125	0.009	0.015	0.015	0.126	0.055	0.065	0.070	0.214	0.106	0.126	0.131	0.283
150	0.009	0.014	0.016	0.126	0.054	0.065	0.069	0.214	0.107	0.123	0.127	0.280
200	0.010	0.014	0.016	0.125	0.056	0.061	0.066	0.213	0.109	0.120	0.127	0.277
250	0.010	0.013	0.016	0.125	0.054	0.060	0.066	0.213	0.109	0.117	0.123	0.275
Two-sided tests- $T = 1000$												
-5	0.004	0.007	0.041	0.026	0.027	0.038	0.086	0.071	0.070	0.086	0.136	0.109
-2.5	0.002	0.007	0.025	0.032	0.015	0.038	0.081	0.084	0.046	0.097	0.147	0.143
0	0.003	0.010	0.013	0.054	0.021	0.043	0.060	0.126	0.053	0.091	0.117	0.200
2.5	0.003	0.010	0.014	0.066	0.024	0.049	0.058	0.153	0.064	0.102	0.114	0.230
5	0.004	0.011	0.014	0.076	0.030	0.051	0.058	0.171	0.072	0.105	0.111	0.248
10	0.007	0.011	0.013	0.095	0.036	0.053	0.056	0.194	0.084	0.105	0.108	0.274
25	0.011	0.011	0.012	0.125	0.049	0.054	0.056	0.233	0.102	0.104	0.105	0.314
50	0.013	0.011	0.011	0.146	0.055	0.053	0.056	0.260	0.109	0.101	0.104	0.340
75	0.014	0.011	0.012	0.158	0.059	0.053	0.056	0.273	0.112	0.104	0.108	0.352
100	0.014	0.011	0.013	0.168	0.060	0.052	0.058	0.282	0.113	0.102	0.109	0.359
125	0.015	0.012	0.013	0.173	0.061	0.055	0.059	0.286	0.116	0.104	0.110	0.363
150	0.016	0.012	0.014	0.176	0.061	0.055	0.061	0.289	0.116	0.105	0.111	0.365
200	0.015	0.012	0.014	0.181	0.062	0.056	0.062	0.294	0.118	0.103	0.111	0.372
250	0.016	0.012	0.014	0.184	0.065	0.057	0.062	0.300	0.116	0.102	0.113	0.379

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.22: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP8 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\,\boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim iid t_5(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad -0.9; -0.9 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1 e^2_{i,t-1} + 0.85\sigma^2_{i,t-1}$, $i = 1, 2$. $t_5(\mathbf{0}, \boldsymbol{\Omega})$ defines a mean zero Student-t distribution with 5 degrees of freedom and variance matrix $\boldsymbol{\Omega}$).

	1%				5%				10%			
c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.007	0.001	0.001	0.007	0.046	0.006	0.007	0.019	0.097	0.017	0.017	0.035
-2.5	0.007	0.001	0.001	0.003	0.044	0.002	0.002	0.009	0.092	0.006	0.006	0.014
0	0.009	0.001	0.001	0.005	0.035	0.003	0.004	0.010	0.066	0.008	0.008	0.016
2.5	0.017	0.001	0.002	0.006	0.054	0.007	0.008	0.018	0.091	0.019	0.019	0.032
5	0.022	0.002	0.002	0.009	0.065	0.013	0.014	0.028	0.108	0.032	0.031	0.050
10	0.022	0.004	0.004	0.016	0.067	0.021	0.021	0.044	0.113	0.044	0.043	0.074
25	0.018	0.006	0.006	0.026	0.063	0.029	0.029	0.064	0.116	0.058	0.059	0.101
50	0.016	0.006	0.008	0.034	0.064	0.032	0.035	0.081	0.115	0.069	0.073	0.123
75	0.016	0.007	0.009	0.037	0.062	0.036	0.041	0.090	0.117	0.079	0.082	0.136
100	0.016	0.007	0.010	0.040	0.062	0.040	0.045	0.095	0.116	0.083	0.088	0.143
125	0.014	0.007	0.011	0.041	0.062	0.042	0.049	0.098	0.116	0.087	0.093	0.150
150	0.014	0.007	0.011	0.043	0.062	0.043	0.051	0.101	0.119	0.091	0.098	0.153
200	0.013	0.008	0.012	0.046	0.062	0.047	0.056	0.107	0.120	0.096	0.105	0.160
250	0.011	0.008	0.014	0.047	0.063	0.049	0.060	0.109	0.121	0.098	0.109	0.165
Right-sided tests - $T = 250$												
-5	0.009	0.015	0.034	0.037	0.045	0.086	0.110	0.107	0.087	0.173	0.188	0.178
-2.5	0.006	0.016	0.029	0.035	0.038	0.110	0.129	0.139	0.089	0.251	0.257	0.270
0	0.007	0.018	0.024	0.047	0.048	0.101	0.122	0.163	0.106	0.223	0.242	0.291
2.5	0.010	0.019	0.027	0.054	0.056	0.103	0.120	0.173	0.120	0.208	0.221	0.285
5	0.010	0.021	0.027	0.059	0.063	0.099	0.111	0.171	0.122	0.195	0.208	0.267
10	0.011	0.021	0.025	0.061	0.062	0.089	0.101	0.165	0.124	0.175	0.185	0.252
25	0.012	0.018	0.023	0.063	0.060	0.078	0.088	0.154	0.119	0.152	0.161	0.231
50	0.012	0.014	0.019	0.060	0.061	0.071	0.083	0.148	0.120	0.135	0.145	0.217
75	0.011	0.014	0.017	0.058	0.059	0.065	0.078	0.143	0.125	0.131	0.141	0.212
100	0.011	0.012	0.017	0.056	0.061	0.063	0.075	0.137	0.122	0.127	0.137	0.205
125	0.010	0.012	0.016	0.055	0.059	0.060	0.073	0.134	0.122	0.121	0.132	0.200
150	0.009	0.010	0.016	0.058	0.060	0.058	0.071	0.130	0.121	0.119	0.129	0.194
200	0.010	0.009	0.016	0.055	0.061	0.057	0.069	0.126	0.119	0.111	0.125	0.186
250	0.011	0.009	0.015	0.052	0.063	0.056	0.067	0.121	0.119	0.109	0.117	0.179
Two-sided tests - $T = 250$												
-5	0.008	0.007	0.025	0.029	0.044	0.042	0.068	0.080	0.088	0.091	0.117	0.126
-2.5	0.006	0.007	0.017	0.024	0.033	0.046	0.069	0.080	0.079	0.112	0.132	0.148
0	0.006	0.009	0.013	0.034	0.040	0.047	0.065	0.103	0.086	0.104	0.126	0.173
2.5	0.007	0.009	0.014	0.040	0.045	0.052	0.067	0.114	0.100	0.110	0.127	0.190
5	0.008	0.010	0.015	0.043	0.050	0.057	0.068	0.125	0.108	0.111	0.124	0.199
10	0.011	0.012	0.015	0.049	0.056	0.057	0.066	0.135	0.111	0.111	0.122	0.209
25	0.013	0.012	0.016	0.059	0.058	0.056	0.064	0.143	0.115	0.105	0.116	0.218
50	0.013	0.010	0.015	0.066	0.062	0.052	0.063	0.152	0.119	0.104	0.118	0.229
75	0.012	0.010	0.014	0.066	0.060	0.050	0.062	0.157	0.120	0.100	0.119	0.233
100	0.011	0.008	0.014	0.066	0.060	0.047	0.062	0.159	0.120	0.103	0.121	0.232
125	0.010	0.009	0.014	0.069	0.058	0.048	0.061	0.158	0.119	0.101	0.122	0.232
150	0.010	0.009	0.014	0.070	0.060	0.049	0.063	0.158	0.119	0.102	0.121	0.231
200	0.009	0.009	0.015	0.071	0.058	0.050	0.067	0.158	0.122	0.104	0.124	0.233
250	0.009	0.008	0.015	0.070	0.060	0.052	0.066	0.162	0.124	0.105	0.127	0.229

	1%				5%				10%			
c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.002	0.006	0.045	0.005	0.007	0.019	0.097	0.016	0.017	0.032
-2.5	0.009	0.000	0.000	0.003	0.047	0.001	0.001	0.008	0.103	0.003	0.003	0.013
0	0.007	0.001	0.001	0.004	0.030	0.002	0.003	0.010	0.060	0.006	0.006	0.015
2.5	0.014	0.001	0.001	0.006	0.049	0.005	0.005	0.018	0.088	0.016	0.016	0.036
5	0.018	0.002	0.001	0.011	0.062	0.010	0.009	0.034	0.105	0.026	0.024	0.059
10	0.022	0.003	0.003	0.023	0.065	0.016	0.014	0.056	0.112	0.040	0.038	0.089
25	0.019	0.005	0.005	0.044	0.068	0.026	0.023	0.092	0.114	0.054	0.053	0.130
50	0.020	0.007	0.006	0.062	0.064	0.032	0.032	0.115	0.115	0.066	0.065	0.156
75	0.019	0.007	0.007	0.071	0.065	0.036	0.036	0.126	0.114	0.071	0.071	0.169
100	0.019	0.009	0.009	0.079	0.066	0.038	0.039	0.135	0.114	0.076	0.076	0.176
125	0.018	0.009	0.010	0.084	0.064	0.040	0.041	0.139	0.113	0.078	0.080	0.182
150	0.018	0.010	0.011	0.086	0.065	0.042	0.042	0.143	0.115	0.080	0.081	0.189
200	0.018	0.011	0.012	0.089	0.063	0.043	0.045	0.150	0.113	0.083	0.085	0.196
250	0.017	0.010	0.012	0.092	0.064	0.045	0.048	0.155	0.112	0.083	0.086	0.206
Right-sided tests - $T = 1000$												
-5	0.005	0.015	0.054	0.031	0.030	0.085	0.127	0.091	0.071	0.174	0.208	0.156
-2.5	0.004	0.016	0.049	0.042	0.024	0.105	0.150	0.132	0.063	0.241	0.273	0.248
0	0.005	0.019	0.029	0.063	0.031	0.095	0.120	0.179	0.078	0.209	0.236	0.295
2.5	0.005	0.018	0.026	0.077	0.038	0.098	0.111	0.194	0.091	0.198	0.209	0.298
5	0.006	0.020	0.024	0.085	0.042	0.096	0.103	0.198	0.096	0.186	0.193	0.292
10	0.007	0.021	0.023	0.093	0.047	0.089	0.094	0.198	0.099	0.170	0.174	0.285
25	0.007	0.018	0.019	0.102	0.050	0.082	0.084	0.202	0.105	0.150	0.154	0.278
50	0.008	0.016	0.016	0.107	0.051	0.070	0.074	0.203	0.105	0.141	0.145	0.275
75	0.010	0.017	0.017	0.108	0.051	0.068	0.072	0.202	0.107	0.132	0.138	0.270
100	0.009	0.015	0.017	0.109	0.052	0.066	0.070	0.199	0.104	0.125	0.132	0.266
125	0.009	0.015	0.016	0.110	0.054	0.063	0.067	0.196	0.105	0.123	0.128	0.263
150	0.010	0.015	0.015	0.110	0.055	0.062	0.067	0.195	0.105	0.122	0.126	0.260
200	0.011	0.014	0.016	0.109	0.054	0.062	0.067	0.194	0.108	0.116	0.121	0.260
250	0.010	0.014	0.015	0.109	0.054	0.060	0.065	0.195	0.108	0.114	0.120	0.258
Two-sided tests- $T = 1000$												
-5	0.004	0.007	0.042	0.026	0.029	0.042	0.086	0.069	0.073	0.088	0.134	0.110
-2.5	0.002	0.008	0.033	0.029	0.020	0.044	0.089	0.081	0.052	0.105	0.151	0.140
0	0.004	0.010	0.017	0.048	0.023	0.046	0.066	0.114	0.059	0.096	0.123	0.188
2.5	0.004	0.011	0.015	0.060	0.028	0.049	0.060	0.141	0.068	0.104	0.116	0.212
5	0.005	0.011	0.014	0.070	0.031	0.053	0.059	0.158	0.075	0.105	0.112	0.231
10	0.008	0.012	0.013	0.084	0.039	0.054	0.058	0.177	0.085	0.105	0.108	0.254
25	0.011	0.011	0.012	0.110	0.049	0.052	0.054	0.214	0.101	0.107	0.107	0.294
50	0.013	0.012	0.012	0.129	0.055	0.051	0.054	0.239	0.109	0.102	0.106	0.317
75	0.014	0.011	0.012	0.140	0.060	0.052	0.056	0.255	0.114	0.104	0.108	0.327
100	0.014	0.011	0.011	0.149	0.061	0.053	0.057	0.261	0.114	0.103	0.109	0.334
125	0.015	0.011	0.013	0.153	0.061	0.054	0.058	0.266	0.116	0.103	0.108	0.335
150	0.015	0.012	0.013	0.160	0.060	0.056	0.061	0.267	0.117	0.103	0.109	0.338
200	0.016	0.012	0.015	0.163	0.063	0.056	0.061	0.270	0.115	0.104	0.112	0.345
250	0.014	0.013	0.015	0.163	0.064	0.055	0.061	0.276	0.117	0.105	0.114	0.349

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.23: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP8 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\,\boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim iidt_5(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad -0.5; -0.5 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1e^2_{i,t-1} + 0.85\sigma^2_{i,t-1}$, $i = 1, 2$. $t_5(\mathbf{0}, \boldsymbol{\Omega})$ defines a mean zero Student-t distribution with 5 degrees of freedom and variance matrix $\boldsymbol{\Omega}$).

c	$t^{*,RWB}_{zx}$ 1%	$t^{*,FRWB}_{zx}$ 1%	t^{EW}_{zx} 1%	t_{zx} 1%	$t^{*,RWB}_{zx}$ 5%	$t^{*,FRWB}_{zx}$ 5%	t^{EW}_{zx} 5%	t_{zx} 5%	$t^{*,RWB}_{zx}$ 10%	$t^{*,FRWB}_{zx}$ 10%	t^{EW}_{zx} 10%	t_{zx} 10%
Left-sided tests - $T = 250$												
-5	0.008	0.003	0.009	0.013	0.048	0.019	0.027	0.036	0.102	0.046	0.050	0.064
-2.5	0.008	0.002	0.004	0.007	0.043	0.009	0.012	0.019	0.091	0.025	0.024	0.034
0	0.007	0.002	0.005	0.007	0.033	0.012	0.016	0.022	0.070	0.027	0.030	0.039
2.5	0.011	0.005	0.006	0.011	0.048	0.021	0.026	0.035	0.089	0.045	0.049	0.062
5	0.014	0.005	0.007	0.014	0.055	0.027	0.030	0.044	0.097	0.056	0.058	0.073
10	0.017	0.008	0.009	0.020	0.057	0.032	0.034	0.053	0.102	0.064	0.066	0.088
25	0.016	0.010	0.009	0.025	0.058	0.040	0.043	0.063	0.103	0.077	0.078	0.102
50	0.014	0.008	0.009	0.026	0.058	0.042	0.046	0.069	0.106	0.085	0.089	0.114
75	0.012	0.008	0.010	0.024	0.059	0.045	0.048	0.074	0.108	0.087	0.092	0.119
100	0.010	0.008	0.010	0.025	0.057	0.045	0.050	0.076	0.110	0.090	0.095	0.123
125	0.010	0.008	0.011	0.025	0.056	0.045	0.051	0.077	0.109	0.092	0.099	0.124
150	0.010	0.009	0.011	0.026	0.054	0.045	0.052	0.078	0.108	0.092	0.099	0.122
200	0.009	0.009	0.013	0.028	0.055	0.045	0.053	0.077	0.109	0.095	0.103	0.127
250	0.009	0.009	0.013	0.027	0.054	0.048	0.054	0.079	0.109	0.098	0.107	0.131
Right-sided tests - $T = 250$												
-5	0.010	0.018	0.039	0.033	0.048	0.078	0.087	0.096	0.099	0.151	0.146	0.160
-2.5	0.010	0.021	0.037	0.030	0.053	0.102	0.102	0.113	0.110	0.202	0.178	0.203
0	0.011	0.017	0.031	0.029	0.057	0.087	0.104	0.109	0.115	0.185	0.181	0.197
2.5	0.012	0.017	0.027	0.032	0.058	0.084	0.096	0.108	0.116	0.160	0.174	0.187
5	0.012	0.017	0.023	0.032	0.057	0.079	0.088	0.104	0.114	0.149	0.156	0.180
10	0.012	0.014	0.019	0.031	0.055	0.071	0.079	0.101	0.109	0.138	0.147	0.172
25	0.010	0.013	0.016	0.032	0.053	0.062	0.070	0.095	0.109	0.125	0.132	0.157
50	0.010	0.013	0.016	0.032	0.053	0.059	0.066	0.090	0.105	0.116	0.125	0.153
75	0.009	0.011	0.015	0.031	0.054	0.057	0.064	0.090	0.108	0.115	0.121	0.148
100	0.009	0.010	0.014	0.031	0.055	0.056	0.065	0.090	0.110	0.113	0.121	0.144
125	0.009	0.011	0.014	0.031	0.055	0.055	0.062	0.087	0.110	0.110	0.120	0.143
150	0.009	0.009	0.014	0.032	0.054	0.054	0.062	0.087	0.111	0.110	0.119	0.145
200	0.009	0.010	0.015	0.032	0.056	0.054	0.061	0.087	0.113	0.107	0.116	0.145
250	0.010	0.011	0.015	0.032	0.058	0.052	0.061	0.085	0.112	0.103	0.114	0.145
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.035	0.031	0.048	0.049	0.073	0.081	0.096	0.096	0.114	0.132
-2.5	0.008	0.012	0.030	0.020	0.047	0.056	0.071	0.075	0.097	0.111	0.114	0.132
0	0.009	0.010	0.024	0.024	0.044	0.049	0.069	0.076	0.093	0.098	0.120	0.130
2.5	0.011	0.012	0.021	0.028	0.049	0.052	0.068	0.083	0.101	0.104	0.122	0.144
5	0.012	0.012	0.019	0.031	0.055	0.052	0.065	0.089	0.105	0.105	0.119	0.148
10	0.015	0.012	0.015	0.034	0.056	0.054	0.062	0.093	0.108	0.104	0.113	0.154
25	0.014	0.010	0.013	0.037	0.057	0.050	0.060	0.100	0.106	0.103	0.113	0.158
50	0.012	0.010	0.014	0.039	0.056	0.048	0.057	0.102	0.111	0.101	0.112	0.159
75	0.010	0.009	0.014	0.037	0.054	0.048	0.056	0.104	0.111	0.101	0.113	0.163
100	0.009	0.009	0.012	0.037	0.054	0.048	0.059	0.105	0.110	0.101	0.115	0.166
125	0.009	0.008	0.013	0.038	0.052	0.047	0.061	0.104	0.109	0.098	0.113	0.165
150	0.009	0.009	0.014	0.038	0.052	0.049	0.060	0.101	0.109	0.099	0.114	0.165
200	0.008	0.009	0.015	0.040	0.053	0.049	0.062	0.100	0.109	0.098	0.114	0.164
250	0.008	0.010	0.014	0.040	0.052	0.049	0.062	0.102	0.110	0.099	0.115	0.164

c	$t^{*,RWB}_{zx}$ 1%	$t^{*,FRWB}_{zx}$ 1%	t^{EW}_{zx} 1%	t_{zx} 1%	$t^{*,RWB}_{zx}$ 5%	$t^{*,FRWB}_{zx}$ 5%	t^{EW}_{zx} 5%	t_{zx} 5%	$t^{*,RWB}_{zx}$ 10%	$t^{*,FRWB}_{zx}$ 10%	t^{EW}_{zx} 10%	t_{zx} 10%
Left-sided tests - $T = 1000$												
-5	0.010	0.003	0.011	0.013	0.050	0.021	0.030	0.035	0.103	0.051	0.053	0.059
-2.5	0.007	0.002	0.006	0.007	0.042	0.010	0.014	0.018	0.093	0.026	0.026	0.036
0	0.005	0.002	0.006	0.007	0.031	0.011	0.016	0.021	0.066	0.030	0.031	0.042
2.5	0.006	0.003	0.006	0.011	0.042	0.018	0.023	0.037	0.083	0.045	0.047	0.064
5	0.009	0.004	0.006	0.016	0.047	0.022	0.026	0.046	0.092	0.053	0.057	0.079
10	0.010	0.005	0.006	0.021	0.049	0.026	0.028	0.057	0.100	0.061	0.064	0.093
25	0.011	0.007	0.006	0.031	0.050	0.031	0.033	0.071	0.102	0.071	0.070	0.109
50	0.013	0.008	0.008	0.041	0.055	0.038	0.039	0.082	0.103	0.077	0.077	0.126
75	0.015	0.009	0.009	0.044	0.056	0.040	0.041	0.089	0.109	0.081	0.082	0.132
100	0.014	0.009	0.010	0.045	0.057	0.041	0.042	0.094	0.110	0.085	0.085	0.138
125	0.014	0.010	0.011	0.047	0.057	0.042	0.043	0.095	0.110	0.087	0.089	0.141
150	0.014	0.011	0.011	0.046	0.056	0.043	0.044	0.099	0.110	0.089	0.091	0.143
200	0.013	0.010	0.010	0.047	0.054	0.043	0.045	0.100	0.109	0.091	0.092	0.148
250	0.012	0.010	0.011	0.048	0.055	0.045	0.046	0.100	0.111	0.092	0.094	0.150
Right-sided tests - $T = 1000$												
-5	0.008	0.016	0.035	0.028	0.041	0.074	0.081	0.073	0.088	0.144	0.132	0.129
-2.5	0.010	0.020	0.045	0.030	0.049	0.100	0.102	0.098	0.095	0.191	0.162	0.176
0	0.010	0.020	0.040	0.034	0.052	0.098	0.106	0.108	0.106	0.185	0.176	0.190
2.5	0.011	0.020	0.034	0.038	0.053	0.089	0.097	0.112	0.106	0.163	0.169	0.184
5	0.011	0.019	0.029	0.039	0.055	0.081	0.094	0.110	0.103	0.154	0.161	0.177
10	0.011	0.017	0.022	0.041	0.053	0.077	0.085	0.112	0.104	0.145	0.150	0.174
25	0.011	0.014	0.015	0.046	0.054	0.068	0.073	0.111	0.106	0.134	0.136	0.176
50	0.011	0.014	0.014	0.048	0.053	0.065	0.069	0.111	0.105	0.124	0.127	0.171
75	0.010	0.013	0.013	0.049	0.053	0.061	0.064	0.113	0.104	0.118	0.123	0.171
100	0.011	0.012	0.013	0.050	0.052	0.060	0.064	0.112	0.105	0.116	0.119	0.167
125	0.010	0.013	0.013	0.049	0.052	0.057	0.062	0.110	0.106	0.116	0.120	0.168
150	0.010	0.013	0.013	0.049	0.053	0.056	0.060	0.112	0.103	0.113	0.116	0.168
200	0.010	0.013	0.014	0.049	0.052	0.056	0.059	0.110	0.103	0.109	0.113	0.167
250	0.009	0.013	0.015	0.048	0.052	0.055	0.057	0.110	0.105	0.107	0.112	0.165
Two-sided tests- $T = 1000$												
-5	0.008	0.010	0.037	0.029	0.041	0.046	0.074	0.068	0.085	0.095	0.112	0.108
-2.5	0.009	0.011	0.039	0.024	0.042	0.053	0.078	0.069	0.084	0.110	0.116	0.116
0	0.007	0.011	0.033	0.027	0.040	0.055	0.075	0.077	0.082	0.109	0.122	0.129
2.5	0.008	0.011	0.027	0.032	0.044	0.053	0.072	0.092	0.092	0.105	0.119	0.149
5	0.008	0.011	0.023	0.038	0.045	0.054	0.070	0.099	0.094	0.102	0.120	0.156
10	0.009	0.011	0.016	0.045	0.048	0.052	0.061	0.110	0.096	0.102	0.113	0.169
25	0.011	0.012	0.013	0.055	0.050	0.052	0.055	0.125	0.103	0.099	0.105	0.182
50	0.012	0.011	0.012	0.064	0.055	0.051	0.055	0.135	0.105	0.104	0.108	0.193
75	0.012	0.011	0.011	0.068	0.056	0.053	0.056	0.142	0.109	0.102	0.106	0.202
100	0.012	0.011	0.012	0.071	0.054	0.051	0.056	0.144	0.107	0.101	0.106	0.206
125	0.012	0.010	0.011	0.072	0.055	0.052	0.056	0.144	0.107	0.098	0.105	0.205
150	0.012	0.011	0.012	0.072	0.054	0.052	0.056	0.144	0.106	0.097	0.104	0.211
200	0.011	0.011	0.013	0.073	0.054	0.052	0.056	0.145	0.105	0.097	0.104	0.210
250	0.012	0.012	0.012	0.074	0.055	0.052	0.057	0.148	0.109	0.098	0.103	0.210

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.24: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP8 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0,\ \rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\, \boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim iidt_5(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad 0; 0 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1e^2_{i,t-1} + 0.85\sigma^2_{i,t-1},\ i = 1, 2$. $t_5(\mathbf{0}, \boldsymbol{\Omega})$ defines a mean zero Student-t distribution with 5 degrees of freedom and variance matrix $\boldsymbol{\Omega}$).

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 250$												
-5	0.011	0.011	0.022	0.025	0.051	0.049	0.057	0.067	0.101	0.099	0.097	0.107
-2.5	0.012	0.012	0.021	0.019	0.049	0.048	0.047	0.056	0.093	0.095	0.084	0.102
0	0.012	0.012	0.022	0.018	0.046	0.048	0.056	0.057	0.093	0.095	0.093	0.104
2.5	0.012	0.012	0.019	0.021	0.051	0.049	0.057	0.062	0.097	0.097	0.103	0.111
5	0.012	0.012	0.016	0.021	0.052	0.050	0.057	0.066	0.099	0.101	0.105	0.114
10	0.013	0.011	0.013	0.021	0.055	0.051	0.055	0.068	0.103	0.100	0.106	0.115
25	0.014	0.012	0.014	0.023	0.053	0.051	0.057	0.067	0.106	0.102	0.107	0.117
50	0.010	0.009	0.013	0.021	0.053	0.049	0.055	0.069	0.106	0.099	0.107	0.117
75	0.009	0.009	0.012	0.020	0.054	0.050	0.055	0.068	0.108	0.101	0.109	0.120
100	0.010	0.008	0.012	0.020	0.053	0.051	0.055	0.068	0.107	0.101	0.107	0.120
125	0.009	0.009	0.013	0.019	0.053	0.049	0.055	0.067	0.107	0.099	0.106	0.119
150	0.010	0.009	0.013	0.020	0.051	0.049	0.055	0.068	0.105	0.097	0.105	0.117
200	0.010	0.010	0.013	0.021	0.050	0.047	0.053	0.069	0.105	0.098	0.105	0.120
250	0.008	0.010	0.014	0.022	0.051	0.048	0.054	0.070	0.107	0.098	0.108	0.120
Right-sided tests - $T = 250$												
-5	0.011	0.011	0.022	0.026	0.053	0.053	0.057	0.068	0.105	0.101	0.095	0.113
-2.5	0.009	0.010	0.021	0.017	0.053	0.054	0.049	0.061	0.100	0.102	0.086	0.109
0	0.010	0.010	0.016	0.020	0.050	0.051	0.055	0.061	0.096	0.099	0.093	0.109
2.5	0.010	0.010	0.018	0.022	0.051	0.051	0.058	0.065	0.101	0.096	0.103	0.112
5	0.011	0.010	0.016	0.022	0.051	0.050	0.056	0.066	0.103	0.099	0.105	0.115
10	0.011	0.009	0.013	0.021	0.054	0.051	0.057	0.066	0.104	0.100	0.106	0.117
25	0.011	0.010	0.012	0.021	0.053	0.050	0.057	0.066	0.101	0.099	0.105	0.117
50	0.011	0.009	0.012	0.024	0.052	0.052	0.058	0.068	0.103	0.102	0.107	0.119
75	0.011	0.011	0.014	0.024	0.054	0.051	0.056	0.068	0.102	0.098	0.107	0.122
100	0.010	0.011	0.014	0.026	0.055	0.052	0.058	0.071	0.108	0.101	0.107	0.123
125	0.010	0.011	0.014	0.026	0.055	0.054	0.059	0.072	0.108	0.104	0.109	0.124
150	0.010	0.010	0.015	0.025	0.057	0.054	0.062	0.073	0.111	0.106	0.112	0.125
200	0.010	0.010	0.014	0.026	0.058	0.054	0.061	0.074	0.111	0.103	0.112	0.127
250	0.009	0.010	0.013	0.026	0.056	0.053	0.059	0.074	0.115	0.105	0.114	0.127
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.034	0.037	0.051	0.052	0.074	0.084	0.102	0.103	0.114	0.135
-2.5	0.011	0.011	0.031	0.023	0.050	0.051	0.063	0.070	0.098	0.100	0.096	0.117
0	0.011	0.011	0.028	0.024	0.050	0.049	0.068	0.070	0.094	0.099	0.111	0.118
2.5	0.012	0.012	0.025	0.027	0.053	0.053	0.070	0.077	0.101	0.098	0.115	0.127
5	0.013	0.011	0.020	0.026	0.054	0.052	0.065	0.077	0.102	0.098	0.113	0.132
10	0.013	0.011	0.015	0.028	0.052	0.051	0.061	0.079	0.108	0.101	0.113	0.135
25	0.013	0.010	0.014	0.031	0.052	0.049	0.058	0.082	0.106	0.101	0.114	0.133
50	0.010	0.009	0.014	0.029	0.052	0.051	0.059	0.082	0.105	0.101	0.113	0.137
75	0.010	0.008	0.013	0.030	0.053	0.049	0.061	0.082	0.106	0.100	0.111	0.136
100	0.009	0.008	0.014	0.029	0.052	0.049	0.061	0.084	0.108	0.101	0.114	0.139
125	0.008	0.009	0.014	0.029	0.050	0.050	0.059	0.084	0.107	0.103	0.114	0.139
150	0.008	0.010	0.015	0.030	0.050	0.049	0.059	0.085	0.106	0.100	0.116	0.141
200	0.009	0.010	0.015	0.030	0.049	0.049	0.061	0.085	0.107	0.102	0.115	0.143
250	0.008	0.010	0.015	0.032	0.050	0.049	0.061	0.087	0.105	0.099	0.114	0.144

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 1000$												
-5	0.009	0.010	0.023	0.022	0.047	0.048	0.050	0.055	0.099	0.097	0.083	0.096
-2.5	0.009	0.009	0.016	0.017	0.047	0.046	0.040	0.051	0.095	0.099	0.072	0.095
0	0.007	0.007	0.018	0.016	0.047	0.048	0.051	0.056	0.093	0.095	0.088	0.100
2.5	0.009	0.009	0.017	0.019	0.047	0.045	0.055	0.061	0.096	0.097	0.102	0.104
5	0.010	0.009	0.016	0.020	0.049	0.047	0.057	0.060	0.099	0.096	0.103	0.106
10	0.009	0.009	0.014	0.022	0.046	0.046	0.054	0.061	0.097	0.095	0.102	0.107
25	0.010	0.010	0.011	0.025	0.048	0.048	0.051	0.068	0.096	0.093	0.098	0.109
50	0.010	0.011	0.010	0.028	0.048	0.047	0.049	0.071	0.099	0.098	0.101	0.116
75	0.011	0.010	0.011	0.029	0.050	0.049	0.050	0.074	0.101	0.098	0.101	0.118
100	0.011	0.010	0.012	0.031	0.051	0.050	0.053	0.074	0.100	0.096	0.100	0.117
125	0.010	0.010	0.010	0.030	0.051	0.050	0.051	0.075	0.101	0.097	0.099	0.120
150	0.009	0.009	0.010	0.030	0.049	0.050	0.051	0.075	0.101	0.097	0.099	0.122
200	0.009	0.009	0.010	0.030	0.050	0.047	0.051	0.077	0.101	0.098	0.101	0.123
250	0.009	0.008	0.009	0.030	0.050	0.050	0.052	0.078	0.102	0.098	0.101	0.124
Right-sided tests - $T = 1000$												
-5	0.010	0.010	0.024	0.022	0.046	0.048	0.049	0.053	0.092	0.095	0.083	0.094
-2.5	0.010	0.011	0.023	0.017	0.046	0.049	0.049	0.052	0.091	0.098	0.079	0.098
0	0.010	0.011	0.018	0.016	0.045	0.049	0.052	0.056	0.093	0.101	0.091	0.104
2.5	0.010	0.009	0.020	0.020	0.047	0.051	0.059	0.061	0.100	0.102	0.105	0.110
5	0.012	0.012	0.019	0.022	0.050	0.050	0.059	0.063	0.101	0.100	0.110	0.111
10	0.011	0.010	0.015	0.024	0.050	0.051	0.056	0.067	0.100	0.102	0.108	0.115
25	0.011	0.011	0.013	0.027	0.051	0.053	0.055	0.073	0.106	0.102	0.106	0.118
50	0.011	0.011	0.011	0.029	0.055	0.053	0.056	0.076	0.104	0.101	0.106	0.123
75	0.011	0.010	0.011	0.030	0.055	0.054	0.055	0.080	0.106	0.104	0.109	0.125
100	0.010	0.009	0.011	0.031	0.056	0.053	0.056	0.080	0.106	0.104	0.107	0.127
125	0.010	0.010	0.011	0.031	0.055	0.052	0.055	0.079	0.104	0.104	0.107	0.127
150	0.010	0.010	0.011	0.031	0.053	0.051	0.055	0.080	0.104	0.102	0.107	0.128
200	0.010	0.010	0.011	0.031	0.053	0.049	0.054	0.080	0.103	0.101	0.106	0.128
250	0.011	0.010	0.011	0.033	0.054	0.049	0.053	0.079	0.105	0.100	0.104	0.130
Two-sided tests- $T = 1000$												
-5	0.009	0.011	0.037	0.030	0.046	0.047	0.070	0.069	0.094	0.097	0.100	0.108
-2.5	0.008	0.010	0.030	0.023	0.045	0.047	0.061	0.061	0.092	0.095	0.089	0.103
0	0.008	0.010	0.028	0.019	0.043	0.046	0.061	0.063	0.091	0.094	0.103	0.112
2.5	0.010	0.009	0.026	0.027	0.046	0.046	0.066	0.073	0.093	0.096	0.114	0.122
5	0.011	0.010	0.025	0.030	0.047	0.047	0.067	0.076	0.097	0.097	0.116	0.123
10	0.011	0.010	0.016	0.032	0.049	0.047	0.061	0.079	0.098	0.096	0.110	0.127
25	0.012	0.010	0.013	0.037	0.050	0.048	0.054	0.088	0.098	0.100	0.105	0.141
50	0.010	0.010	0.011	0.039	0.052	0.049	0.054	0.097	0.102	0.100	0.105	0.147
75	0.009	0.009	0.010	0.040	0.052	0.051	0.056	0.099	0.104	0.101	0.105	0.154
100	0.009	0.008	0.010	0.042	0.051	0.049	0.054	0.099	0.105	0.102	0.109	0.153
125	0.009	0.009	0.010	0.043	0.051	0.050	0.054	0.099	0.105	0.101	0.106	0.155
150	0.010	0.008	0.010	0.044	0.051	0.049	0.054	0.099	0.104	0.100	0.106	0.155
200	0.009	0.009	0.010	0.044	0.049	0.049	0.054	0.101	0.102	0.098	0.105	0.157
250	0.008	0.009	0.010	0.044	0.048	0.046	0.052	0.103	0.104	0.100	0.105	0.157

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.25: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP9 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.95; -0.95 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma^2_{1t}, \sigma^2_{2t})$, σ^2_{it} are GARCH processes generated as $\sigma^2_{it} = 0.05 + 0.1e^2_{i,t-1} + 0.85\sigma^2_{i,t-1}$, $i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim NIID(\mathbf{0}, \mathbf{I_2})$ where $\mathbf{I_2}$ is a 2×2 identity matrix.

Left-sided tests - $T = 250$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.009	0.000	0.000	0.001	0.047	0.002	0.003	0.004	0.097	0.011	0.013	0.014
-2.5	0.005	0.000	0.000	0.000	0.047	0.000	0.000	0.000	0.109	0.001	0.001	0.001
0	0.012	0.000	0.000	0.000	0.040	0.001	0.001	0.001	0.066	0.003	0.003	0.003
2.5	0.022	0.001	0.001	0.001	0.058	0.007	0.007	0.006	0.095	0.013	0.013	0.014
5	0.024	0.003	0.002	0.003	0.066	0.011	0.011	0.012	0.108	0.026	0.025	0.027
10	0.021	0.003	0.003	0.005	0.067	0.018	0.018	0.021	0.113	0.042	0.042	0.045
25	0.016	0.005	0.006	0.008	0.061	0.029	0.030	0.036	0.109	0.058	0.059	0.067
50	0.014	0.006	0.008	0.010	0.057	0.033	0.035	0.043	0.104	0.071	0.072	0.081
75	0.013	0.008	0.008	0.011	0.055	0.036	0.038	0.049	0.103	0.075	0.077	0.087
100	0.012	0.008	0.009	0.012	0.053	0.039	0.040	0.051	0.102	0.076	0.080	0.091
125	0.011	0.008	0.009	0.012	0.053	0.041	0.043	0.051	0.101	0.079	0.082	0.093
150	0.011	0.009	0.009	0.013	0.053	0.042	0.045	0.054	0.101	0.082	0.083	0.096
200	0.011	0.010	0.011	0.015	0.053	0.044	0.046	0.055	0.100	0.085	0.089	0.101
250	0.011	0.010	0.011	0.014	0.052	0.044	0.048	0.057	0.099	0.087	0.092	0.105

Left-sided tests - $T = 1000$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.009	0.000	0.000	0.000	0.042	0.003	0.003	0.004	0.093	0.012	0.012	0.011
-2.5	0.006	0.000	0.000	0.000	0.043	0.000	0.000	0.000	0.107	0.001	0.001	0.001
0	0.012	0.000	0.000	0.000	0.040	0.001	0.001	0.001	0.066	0.002	0.002	0.003
2.5	0.021	0.001	0.001	0.001	0.060	0.005	0.005	0.005	0.098	0.013	0.012	0.012
5	0.021	0.002	0.002	0.001	0.068	0.010	0.010	0.011	0.111	0.026	0.025	0.028
10	0.019	0.003	0.003	0.003	0.065	0.020	0.019	0.020	0.115	0.040	0.040	0.044
25	0.016	0.006	0.006	0.008	0.060	0.028	0.028	0.033	0.114	0.060	0.060	0.066
50	0.014	0.007	0.007	0.010	0.057	0.035	0.036	0.045	0.106	0.069	0.068	0.081
75	0.014	0.007	0.007	0.012	0.056	0.038	0.038	0.049	0.107	0.074	0.074	0.088
100	0.012	0.008	0.008	0.014	0.055	0.038	0.038	0.050	0.105	0.080	0.080	0.094
125	0.013	0.008	0.007	0.014	0.056	0.039	0.040	0.053	0.106	0.083	0.082	0.098
150	0.012	0.008	0.008	0.014	0.055	0.040	0.040	0.054	0.105	0.083	0.084	0.101
200	0.013	0.008	0.008	0.016	0.054	0.041	0.042	0.056	0.104	0.084	0.086	0.103
250	0.012	0.009	0.009	0.016	0.051	0.041	0.042	0.056	0.104	0.088	0.087	0.107

Right-sided tests - $T = 250$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.009	0.015	0.019	0.017	0.041	0.069	0.080	0.072	0.088	0.146	0.158	0.142
-2.5	0.009	0.016	0.022	0.022	0.044	0.100	0.109	0.104	0.089	0.245	0.254	0.246
0	0.011	0.022	0.026	0.026	0.055	0.107	0.120	0.119	0.110	0.231	0.241	0.242
2.5	0.014	0.024	0.028	0.029	0.061	0.110	0.122	0.123	0.124	0.220	0.232	0.237
5	0.013	0.024	0.030	0.029	0.067	0.110	0.118	0.125	0.126	0.204	0.211	0.219
10	0.014	0.023	0.027	0.030	0.065	0.101	0.109	0.118	0.125	0.184	0.192	0.202
25	0.013	0.020	0.022	0.026	0.062	0.085	0.091	0.099	0.115	0.154	0.158	0.171
50	0.012	0.016	0.019	0.023	0.058	0.072	0.078	0.089	0.114	0.138	0.143	0.155
75	0.011	0.014	0.017	0.021	0.058	0.069	0.073	0.083	0.111	0.129	0.132	0.147
100	0.011	0.014	0.015	0.020	0.055	0.064	0.068	0.080	0.112	0.123	0.129	0.143
125	0.011	0.013	0.014	0.020	0.052	0.059	0.063	0.074	0.109	0.120	0.124	0.141
150	0.010	0.011	0.014	0.018	0.053	0.058	0.061	0.073	0.108	0.115	0.122	0.135
200	0.010	0.011	0.013	0.017	0.053	0.054	0.059	0.071	0.108	0.110	0.115	0.130
250	0.011	0.010	0.012	0.017	0.055	0.052	0.056	0.069	0.107	0.106	0.110	0.125

Right-sided tests - $T = 1000$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.007	0.014	0.015	0.015	0.040	0.068	0.070	0.068	0.089	0.148	0.150	0.146
-2.5	0.010	0.019	0.020	0.021	0.042	0.095	0.095	0.094	0.085	0.242	0.244	0.238
0	0.008	0.021	0.020	0.022	0.054	0.113	0.115	0.114	0.109	0.229	0.233	0.233
2.5	0.010	0.023	0.022	0.024	0.060	0.116	0.119	0.122	0.123	0.226	0.228	0.231
5	0.011	0.021	0.021	0.024	0.059	0.111	0.115	0.118	0.123	0.213	0.213	0.220
10	0.011	0.020	0.021	0.024	0.058	0.099	0.101	0.108	0.115	0.190	0.191	0.198
25	0.011	0.018	0.018	0.022	0.056	0.083	0.084	0.093	0.108	0.154	0.155	0.166
50	0.010	0.016	0.016	0.022	0.055	0.075	0.077	0.088	0.108	0.144	0.145	0.158
75	0.010	0.015	0.015	0.021	0.055	0.071	0.072	0.086	0.109	0.138	0.136	0.151
100	0.011	0.014	0.014	0.020	0.055	0.070	0.070	0.083	0.108	0.131	0.132	0.148
125	0.011	0.015	0.015	0.021	0.054	0.066	0.067	0.081	0.106	0.125	0.128	0.145
150	0.012	0.014	0.015	0.022	0.053	0.064	0.064	0.080	0.105	0.124	0.124	0.140
200	0.013	0.014	0.015	0.021	0.052	0.061	0.063	0.078	0.102	0.118	0.120	0.138
250	0.012	0.014	0.014	0.019	0.053	0.060	0.061	0.075	0.104	0.117	0.117	0.135

Two-sided tests - $T = 250$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.009	0.009	0.012	0.010	0.041	0.036	0.042	0.039	0.090	0.072	0.083	0.076
-2.5	0.008	0.009	0.011	0.011	0.039	0.045	0.054	0.051	0.083	0.100	0.109	0.104
0	0.010	0.011	0.014	0.014	0.046	0.053	0.061	0.060	0.093	0.108	0.121	0.120
2.5	0.011	0.013	0.016	0.016	0.053	0.061	0.070	0.071	0.106	0.116	0.128	0.129
5	0.011	0.013	0.017	0.017	0.058	0.063	0.072	0.076	0.112	0.122	0.129	0.137
10	0.012	0.013	0.017	0.019	0.059	0.063	0.069	0.073	0.112	0.119	0.127	0.139
25	0.012	0.013	0.016	0.019	0.057	0.059	0.065	0.076	0.111	0.113	0.120	0.134
50	0.011	0.011	0.014	0.019	0.055	0.055	0.060	0.072	0.109	0.105	0.113	0.132
75	0.012	0.011	0.014	0.018	0.052	0.050	0.056	0.070	0.109	0.104	0.110	0.132
100	0.011	0.011	0.014	0.018	0.048	0.047	0.053	0.067	0.106	0.102	0.108	0.131
125	0.010	0.011	0.013	0.017	0.050	0.046	0.053	0.069	0.102	0.099	0.106	0.125
150	0.010	0.010	0.012	0.017	0.052	0.047	0.053	0.070	0.101	0.099	0.106	0.126
200	0.010	0.010	0.013	0.017	0.052	0.049	0.053	0.069	0.104	0.097	0.104	0.126
250	0.010	0.010	0.012	0.018	0.052	0.049	0.054	0.069	0.104	0.097	0.104	0.126

Two-sided tests - $T = 1000$

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
-5	0.007	0.008	0.008	0.009	0.040	0.033	0.034	0.034	0.091	0.070	0.072	0.071
-2.5	0.008	0.011	0.012	0.012	0.038	0.046	0.048	0.046	0.079	0.094	0.096	0.094
0	0.007	0.010	0.010	0.010	0.045	0.056	0.057	0.060	0.094	0.114	0.116	0.115
2.5	0.008	0.011	0.011	0.012	0.049	0.058	0.061	0.065	0.103	0.119	0.124	0.127
5	0.009	0.010	0.011	0.012	0.050	0.058	0.059	0.065	0.103	0.119	0.125	0.128
10	0.009	0.012	0.012	0.013	0.051	0.057	0.058	0.065	0.103	0.118	0.120	0.128
25	0.010	0.011	0.011	0.015	0.054	0.056	0.059	0.069	0.104	0.110	0.111	0.126
50	0.010	0.011	0.010	0.017	0.053	0.059	0.058	0.073	0.108	0.111	0.112	0.132
75	0.010	0.010	0.009	0.017	0.054	0.056	0.057	0.074	0.109	0.108	0.110	0.135
100	0.009	0.010	0.009	0.018	0.056	0.054	0.057	0.076	0.108	0.107	0.108	0.133
125	0.010	0.009	0.009	0.019	0.054	0.055	0.056	0.076	0.106	0.104	0.107	0.134
150	0.010	0.009	0.009	0.019	0.053	0.054	0.055	0.075	0.104	0.103	0.104	0.134
200	0.010	0.010	0.010	0.020	0.054	0.051	0.053	0.074	0.104	0.102	0.104	0.134
250	0.012	0.011	0.011	0.021	0.053	0.052	0.053	0.075	0.103	0.103	0.103	0.131

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.26: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP9 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.9; -0.9 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2,\ i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim NIID(\mathbf{0}, \mathbf{I_2})$ where $\mathbf{I_2}$ is a 2×2 identity matrix.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests												
-5	0.008	0.000	0.001	0.001	0.048	0.003	0.005	0.005	0.098	0.013	0.015	0.016
-2.5	0.007	0.000	0.000	0.000	0.046	0.000	0.000	0.001	0.110	0.001	0.001	0.002
0	0.011	0.000	0.000	0.000	0.039	0.001	0.001	0.001	0.064	0.003	0.004	0.003
2.5	0.020	0.001	0.001	0.001	0.055	0.007	0.007	0.007	0.092	0.015	0.014	0.016
5	0.023	0.003	0.003	0.003	0.063	0.011	0.012	0.013	0.106	0.028	0.027	0.030
10	0.019	0.004	0.004	0.005	0.064	0.018	0.019	0.023	0.110	0.044	0.044	0.049
25	0.016	0.006	0.006	0.008	0.060	0.029	0.030	0.035	0.108	0.062	0.061	0.068
50	0.013	0.006	0.007	0.010	0.056	0.035	0.036	0.043	0.103	0.071	0.072	0.081
75	0.012	0.007	0.009	0.010	0.055	0.037	0.038	0.048	0.103	0.076	0.077	0.087
100	0.011	0.008	0.009	0.011	0.053	0.038	0.041	0.049	0.101	0.079	0.081	0.090
125	0.012	0.007	0.009	0.012	0.054	0.039	0.042	0.051	0.100	0.079	0.082	0.093
150	0.011	0.008	0.009	0.012	0.052	0.041	0.044	0.052	0.098	0.081	0.084	0.096
200	0.012	0.009	0.011	0.014	0.052	0.042	0.047	0.054	0.099	0.086	0.088	0.099
250	0.011	0.010	0.010	0.014	0.051	0.045	0.048	0.055	0.098	0.089	0.092	0.104
Right-sided tests												
-5	0.009	0.015	0.020	0.018	0.043	0.074	0.084	0.076	0.091	0.150	0.162	0.146
-2.5	0.010	0.018	0.023	0.020	0.044	0.102	0.111	0.105	0.094	0.244	0.253	0.243
0	0.011	0.023	0.026	0.026	0.053	0.107	0.116	0.115	0.111	0.232	0.240	0.239
2.5	0.013	0.023	0.028	0.029	0.063	0.110	0.122	0.122	0.123	0.217	0.226	0.231
5	0.013	0.024	0.028	0.030	0.067	0.107	0.114	0.119	0.127	0.200	0.207	0.215
10	0.014	0.022	0.026	0.028	0.065	0.099	0.106	0.114	0.121	0.179	0.187	0.195
25	0.011	0.017	0.020	0.025	0.060	0.082	0.087	0.096	0.115	0.151	0.157	0.167
50	0.011	0.015	0.017	0.022	0.058	0.071	0.077	0.086	0.111	0.136	0.141	0.152
75	0.010	0.013	0.016	0.020	0.056	0.068	0.073	0.080	0.109	0.126	0.133	0.145
100	0.010	0.013	0.015	0.019	0.053	0.063	0.068	0.078	0.110	0.123	0.127	0.140
125	0.010	0.012	0.014	0.018	0.053	0.058	0.064	0.074	0.110	0.119	0.123	0.138
150	0.009	0.011	0.013	0.017	0.054	0.057	0.061	0.072	0.108	0.112	0.119	0.133
200	0.009	0.011	0.012	0.017	0.053	0.054	0.058	0.069	0.107	0.110	0.114	0.127
250	0.012	0.011	0.012	0.017	0.054	0.053	0.056	0.066	0.108	0.105	0.110	0.123
Two-sided tests												
-5	0.009	0.008	0.012	0.010	0.044	0.035	0.045	0.040	0.093	0.076	0.089	0.080
-2.5	0.009	0.009	0.012	0.011	0.041	0.044	0.053	0.049	0.085	0.100	0.112	0.106
0	0.010	0.011	0.014	0.015	0.045	0.053	0.061	0.059	0.093	0.106	0.117	0.117
2.5	0.010	0.013	0.017	0.016	0.054	0.060	0.066	0.069	0.104	0.117	0.129	0.129
5	0.011	0.013	0.016	0.016	0.057	0.062	0.070	0.073	0.110	0.118	0.125	0.132
10	0.012	0.013	0.017	0.018	0.058	0.063	0.068	0.073	0.111	0.117	0.125	0.137
25	0.012	0.012	0.014	0.018	0.055	0.058	0.063	0.073	0.110	0.111	0.117	0.131
50	0.011	0.011	0.013	0.017	0.055	0.053	0.059	0.070	0.109	0.105	0.113	0.130
75	0.010	0.009	0.013	0.017	0.051	0.050	0.055	0.067	0.107	0.103	0.111	0.128
100	0.010	0.010	0.013	0.017	0.050	0.046	0.051	0.065	0.105	0.101	0.109	0.127
125	0.010	0.010	0.013	0.016	0.049	0.045	0.051	0.066	0.103	0.098	0.106	0.125
150	0.010	0.011	0.012	0.016	0.052	0.047	0.053	0.068	0.103	0.100	0.106	0.124
200	0.011	0.010	0.012	0.016	0.050	0.048	0.052	0.066	0.102	0.098	0.105	0.123
250	0.010	0.010	0.012	0.017	0.051	0.048	0.054	0.068	0.103	0.096	0.104	0.122

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests												
-5	0.009	0.000	0.000	0.000	0.044	0.004	0.003	0.004	0.094	0.013	0.013	0.014
-2.5	0.006	0.000	0.000	0.000	0.046	0.000	0.000	0.001	0.105	0.001	0.001	0.001
0	0.012	0.000	0.000	0.000	0.036	0.001	0.001	0.001	0.064	0.003	0.003	0.004
2.5	0.020	0.001	0.001	0.001	0.059	0.005	0.005	0.006	0.097	0.014	0.014	0.015
5	0.020	0.002	0.002	0.002	0.068	0.011	0.011	0.012	0.109	0.028	0.028	0.029
10	0.017	0.003	0.003	0.004	0.063	0.020	0.020	0.022	0.115	0.044	0.043	0.047
25	0.015	0.005	0.006	0.008	0.058	0.028	0.028	0.034	0.111	0.062	0.062	0.068
50	0.013	0.007	0.007	0.010	0.058	0.036	0.036	0.045	0.107	0.070	0.070	0.082
75	0.012	0.007	0.007	0.012	0.056	0.038	0.038	0.047	0.105	0.076	0.076	0.089
100	0.012	0.007	0.007	0.012	0.056	0.037	0.038	0.049	0.106	0.079	0.080	0.093
125	0.012	0.007	0.007	0.013	0.054	0.040	0.040	0.052	0.105	0.082	0.083	0.097
150	0.013	0.008	0.007	0.013	0.054	0.040	0.041	0.052	0.104	0.083	0.083	0.099
200	0.011	0.008	0.007	0.015	0.052	0.041	0.041	0.054	0.103	0.086	0.086	0.102
250	0.011	0.008	0.008	0.016	0.048	0.042	0.041	0.054	0.102	0.087	0.088	0.103
Right-sided tests												
-5	0.008	0.014	0.015	0.015	0.043	0.070	0.073	0.070	0.092	0.149	0.154	0.146
-2.5	0.009	0.020	0.021	0.020	0.040	0.098	0.099	0.099	0.090	0.240	0.242	0.237
0	0.009	0.021	0.020	0.021	0.054	0.108	0.111	0.112	0.111	0.228	0.230	0.231
2.5	0.011	0.023	0.022	0.023	0.059	0.113	0.114	0.117	0.120	0.220	0.224	0.226
5	0.011	0.022	0.022	0.023	0.059	0.108	0.111	0.113	0.123	0.207	0.209	0.212
10	0.011	0.022	0.022	0.023	0.059	0.097	0.098	0.104	0.116	0.182	0.183	0.192
25	0.012	0.019	0.019	0.021	0.057	0.080	0.081	0.090	0.107	0.150	0.152	0.162
50	0.010	0.016	0.016	0.021	0.057	0.075	0.075	0.084	0.109	0.139	0.139	0.151
75	0.010	0.014	0.014	0.020	0.056	0.071	0.073	0.085	0.109	0.134	0.134	0.147
100	0.010	0.014	0.014	0.019	0.057	0.069	0.069	0.081	0.107	0.129	0.130	0.144
125	0.011	0.014	0.015	0.020	0.056	0.067	0.067	0.079	0.105	0.124	0.125	0.140
150	0.011	0.014	0.015	0.020	0.054	0.063	0.064	0.077	0.106	0.123	0.123	0.136
200	0.012	0.014	0.015	0.020	0.054	0.061	0.062	0.075	0.104	0.116	0.118	0.135
250	0.013	0.015	0.014	0.020	0.053	0.058	0.059	0.073	0.105	0.114	0.116	0.134
Two-sided tests												
-5	0.008	0.009	0.009	0.009	0.044	0.034	0.036	0.036	0.095	0.074	0.076	0.074
-2.5	0.008	0.011	0.011	0.011	0.036	0.045	0.046	0.044	0.084	0.099	0.100	0.099
0	0.008	0.010	0.010	0.011	0.044	0.053	0.055	0.057	0.092	0.110	0.112	0.113
2.5	0.009	0.011	0.011	0.011	0.048	0.059	0.060	0.063	0.103	0.117	0.120	0.123
5	0.009	0.011	0.011	0.012	0.049	0.058	0.060	0.063	0.103	0.117	0.121	0.125
10	0.009	0.012	0.012	0.013	0.050	0.056	0.057	0.065	0.102	0.116	0.118	0.126
25	0.011	0.011	0.011	0.014	0.054	0.055	0.057	0.069	0.103	0.107	0.109	0.124
50	0.010	0.011	0.010	0.016	0.053	0.055	0.057	0.071	0.108	0.109	0.111	0.129
75	0.011	0.009	0.010	0.017	0.055	0.055	0.057	0.073	0.109	0.108	0.111	0.132
100	0.010	0.009	0.010	0.017	0.055	0.055	0.058	0.074	0.107	0.106	0.108	0.130
125	0.010	0.009	0.009	0.018	0.055	0.055	0.056	0.074	0.106	0.106	0.106	0.131
150	0.011	0.010	0.010	0.018	0.055	0.054	0.055	0.073	0.105	0.102	0.105	0.129
200	0.010	0.010	0.010	0.019	0.054	0.054	0.054	0.072	0.104	0.101	0.102	0.129
250	0.011	0.011	0.011	0.020	0.053	0.053	0.054	0.072	0.100	0.099	0.100	0.126

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.27: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP9 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.5; -0.5 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2,\ i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim NIID(\mathbf{0}, \mathbf{I_2})$ where $\mathbf{I_2}$ is a 2×2 identity matrix.

Left-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.002	0.005	0.003	0.050	0.017	0.022	0.019	0.101	0.042	0.051	0.044
-2.5	0.010	0.001	0.002	0.001	0.048	0.005	0.008	0.005	0.100	0.014	0.018	0.015
0	0.007	0.001	0.002	0.001	0.028	0.006	0.007	0.006	0.061	0.017	0.019	0.017
2.5	0.014	0.002	0.003	0.003	0.042	0.017	0.018	0.018	0.082	0.036	0.036	0.037
5	0.014	0.004	0.005	0.005	0.050	0.024	0.024	0.025	0.092	0.048	0.050	0.049
10	0.014	0.006	0.007	0.007	0.053	0.030	0.031	0.032	0.102	0.061	0.064	0.065
25	0.013	0.007	0.008	0.010	0.051	0.038	0.039	0.040	0.100	0.073	0.076	0.077
50	0.013	0.008	0.009	0.010	0.051	0.039	0.042	0.044	0.096	0.078	0.081	0.083
75	0.011	0.008	0.009	0.009	0.049	0.041	0.044	0.044	0.097	0.081	0.084	0.086
100	0.010	0.008	0.009	0.010	0.049	0.041	0.042	0.044	0.097	0.085	0.089	0.090
125	0.011	0.009	0.010	0.009	0.048	0.042	0.045	0.046	0.097	0.085	0.089	0.091
150	0.009	0.009	0.010	0.010	0.049	0.043	0.046	0.048	0.097	0.086	0.089	0.093
200	0.010	0.009	0.010	0.010	0.051	0.045	0.048	0.050	0.099	0.089	0.093	0.094
250	0.010	0.010	0.011	0.012	0.050	0.046	0.049	0.051	0.098	0.090	0.094	0.095

Right-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.015	0.033	0.018	0.049	0.075	0.098	0.072	0.095	0.144	0.163	0.141
-2.5	0.010	0.020	0.040	0.019	0.054	0.104	0.124	0.100	0.109	0.206	0.220	0.201
0	0.012	0.019	0.029	0.021	0.060	0.098	0.110	0.100	0.123	0.199	0.213	0.200
2.5	0.012	0.018	0.024	0.020	0.063	0.092	0.100	0.097	0.121	0.173	0.181	0.175
5	0.012	0.018	0.022	0.020	0.059	0.085	0.091	0.087	0.117	0.160	0.167	0.163
10	0.011	0.017	0.019	0.018	0.057	0.074	0.081	0.079	0.109	0.145	0.149	0.146
25	0.010	0.013	0.016	0.014	0.054	0.065	0.071	0.071	0.108	0.131	0.136	0.135
50	0.009	0.011	0.013	0.012	0.053	0.060	0.064	0.065	0.107	0.122	0.128	0.127
75	0.008	0.010	0.012	0.011	0.052	0.058	0.063	0.064	0.108	0.116	0.121	0.122
100	0.008	0.009	0.011	0.011	0.050	0.056	0.061	0.062	0.107	0.115	0.119	0.120
125	0.008	0.010	0.012	0.012	0.052	0.056	0.060	0.060	0.105	0.112	0.118	0.118
150	0.008	0.010	0.012	0.011	0.052	0.055	0.059	0.060	0.107	0.109	0.114	0.113
200	0.010	0.010	0.011	0.012	0.054	0.052	0.056	0.059	0.105	0.106	0.109	0.111
250	0.010	0.011	0.013	0.013	0.051	0.053	0.056	0.056	0.106	0.104	0.107	0.109

Two-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.009	0.025	0.012	0.049	0.044	0.068	0.046	0.099	0.090	0.120	0.091
-2.5	0.009	0.010	0.026	0.010	0.049	0.051	0.075	0.052	0.097	0.109	0.132	0.106
0	0.009	0.009	0.019	0.011	0.047	0.052	0.067	0.055	0.098	0.104	0.117	0.106
2.5	0.009	0.010	0.015	0.012	0.052	0.056	0.066	0.057	0.105	0.110	0.118	0.115
5	0.010	0.010	0.014	0.012	0.052	0.056	0.064	0.059	0.103	0.106	0.115	0.112
10	0.011	0.011	0.015	0.013	0.052	0.052	0.058	0.058	0.102	0.104	0.112	0.110
25	0.011	0.010	0.013	0.013	0.051	0.050	0.056	0.056	0.102	0.101	0.109	0.110
50	0.009	0.009	0.010	0.011	0.050	0.048	0.055	0.055	0.102	0.098	0.106	0.109
75	0.009	0.008	0.010	0.010	0.050	0.047	0.053	0.056	0.100	0.098	0.107	0.108
100	0.008	0.008	0.010	0.010	0.047	0.046	0.051	0.054	0.099	0.096	0.104	0.105
125	0.008	0.008	0.010	0.010	0.046	0.044	0.048	0.053	0.102	0.097	0.105	0.107
150	0.009	0.009	0.010	0.011	0.045	0.046	0.050	0.054	0.102	0.098	0.105	0.108
200	0.010	0.010	0.012	0.012	0.050	0.049	0.055	0.055	0.102	0.096	0.104	0.108
250	0.009	0.011	0.013	0.013	0.050	0.048	0.054	0.058	0.101	0.098	0.105	0.106

Left-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.002	0.004	0.004	0.051	0.017	0.021	0.018	0.102	0.043	0.049	0.044
-2.5	0.008	0.000	0.001	0.001	0.047	0.005	0.006	0.005	0.099	0.014	0.016	0.014
0	0.006	0.001	0.001	0.001	0.029	0.006	0.006	0.007	0.059	0.017	0.018	0.017
2.5	0.010	0.002	0.003	0.002	0.044	0.015	0.015	0.015	0.086	0.037	0.038	0.038
5	0.011	0.003	0.004	0.004	0.049	0.021	0.022	0.021	0.096	0.047	0.049	0.050
10	0.012	0.005	0.005	0.006	0.049	0.026	0.027	0.028	0.101	0.060	0.061	0.063
25	0.011	0.006	0.007	0.008	0.050	0.035	0.035	0.036	0.100	0.073	0.073	0.077
50	0.011	0.008	0.008	0.009	0.049	0.039	0.039	0.042	0.099	0.079	0.080	0.084
75	0.013	0.009	0.010	0.011	0.052	0.042	0.043	0.046	0.100	0.082	0.083	0.089
100	0.012	0.009	0.010	0.011	0.051	0.043	0.044	0.047	0.101	0.089	0.089	0.091
125	0.012	0.010	0.010	0.011	0.052	0.043	0.044	0.047	0.101	0.091	0.091	0.094
150	0.011	0.010	0.010	0.012	0.052	0.044	0.046	0.048	0.103	0.091	0.092	0.097
200	0.010	0.008	0.010	0.011	0.053	0.045	0.047	0.051	0.101	0.090	0.093	0.096
250	0.009	0.008	0.009	0.010	0.052	0.048	0.048	0.051	0.102	0.093	0.094	0.099

Right-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.014	0.017	0.015	0.047	0.073	0.079	0.071	0.098	0.141	0.149	0.138
-2.5	0.010	0.020	0.024	0.019	0.054	0.106	0.113	0.105	0.113	0.204	0.210	0.202
0	0.010	0.018	0.019	0.018	0.058	0.101	0.104	0.097	0.123	0.199	0.205	0.196
2.5	0.011	0.020	0.020	0.018	0.061	0.093	0.093	0.093	0.119	0.174	0.178	0.175
5	0.011	0.019	0.019	0.019	0.057	0.084	0.084	0.082	0.114	0.160	0.163	0.164
10	0.013	0.018	0.017	0.018	0.054	0.075	0.075	0.076	0.107	0.145	0.144	0.144
25	0.013	0.016	0.016	0.016	0.050	0.063	0.065	0.067	0.101	0.127	0.128	0.128
50	0.012	0.014	0.014	0.015	0.053	0.062	0.062	0.066	0.104	0.121	0.121	0.124
75	0.010	0.012	0.012	0.014	0.055	0.061	0.064	0.065	0.107	0.119	0.121	0.125
100	0.010	0.012	0.013	0.013	0.056	0.061	0.062	0.065	0.105	0.117	0.118	0.121
125	0.011	0.013	0.013	0.014	0.055	0.061	0.061	0.065	0.106	0.117	0.118	0.119
150	0.012	0.013	0.013	0.014	0.054	0.060	0.060	0.064	0.104	0.114	0.115	0.119
200	0.012	0.012	0.014	0.015	0.054	0.056	0.058	0.062	0.102	0.112	0.112	0.117
250	0.012	0.013	0.013	0.015	0.053	0.057	0.059	0.061	0.103	0.112	0.111	0.116

Two-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.008	0.010	0.008	0.048	0.043	0.051	0.045	0.098	0.090	0.098	0.088
-2.5	0.009	0.009	0.012	0.009	0.050	0.052	0.060	0.050	0.103	0.111	0.119	0.109
0	0.008	0.009	0.011	0.009	0.045	0.050	0.054	0.051	0.097	0.109	0.112	0.105
2.5	0.010	0.012	0.012	0.011	0.052	0.054	0.057	0.055	0.105	0.109	0.112	0.111
5	0.010	0.012	0.013	0.012	0.050	0.054	0.056	0.055	0.100	0.106	0.108	0.106
10	0.010	0.012	0.012	0.011	0.051	0.054	0.054	0.055	0.100	0.104	0.104	0.108
25	0.011	0.012	0.013	0.014	0.051	0.052	0.053	0.054	0.098	0.100	0.101	0.106
50	0.011	0.011	0.011	0.012	0.052	0.053	0.055	0.058	0.102	0.102	0.104	0.111
75	0.011	0.010	0.010	0.012	0.053	0.051	0.053	0.059	0.104	0.103	0.107	0.111
100	0.010	0.010	0.011	0.012	0.054	0.052	0.054	0.058	0.105	0.103	0.104	0.111
125	0.010	0.009	0.010	0.012	0.052	0.053	0.054	0.059	0.105	0.103	0.104	0.114
150	0.011	0.011	0.010	0.012	0.051	0.051	0.053	0.058	0.104	0.101	0.104	0.111
200	0.011	0.010	0.011	0.014	0.053	0.051	0.053	0.059	0.102	0.101	0.102	0.110
250	0.011	0.011	0.011	0.013	0.053	0.050	0.052	0.060	0.104	0.102	0.104	0.110

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.28: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP9 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad 0; 0 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma^2_{1t}, \sigma^2_{2t})$, σ^2_{it} are GARCH processes generated as $\sigma^2_{it} = 0.05 + 0.1 e^2_{i,t-1} + 0.85\sigma^2_{i,t-1}$, $i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim NIID(\mathbf{0}, \mathbf{I_2})$ where $\mathbf{I_2}$ is a 2×2 identity matrix.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.010	0.011	0.025	0.012	0.049	0.049	0.070	0.052	0.101	0.101	0.122	0.098
-2.5	0.010	0.010	0.025	0.011	0.049	0.048	0.062	0.046	0.098	0.096	0.104	0.092
0	0.009	0.010	0.018	0.010	0.045	0.047	0.058	0.046	0.094	0.097	0.103	0.094
2.5	0.010	0.010	0.015	0.011	0.048	0.047	0.054	0.047	0.097	0.097	0.103	0.097
5	0.010	0.010	0.013	0.011	0.048	0.048	0.051	0.048	0.100	0.099	0.105	0.098
10	0.010	0.010	0.011	0.011	0.051	0.050	0.054	0.049	0.103	0.102	0.105	0.102
25	0.011	0.010	0.013	0.012	0.053	0.051	0.053	0.053	0.102	0.100	0.103	0.103
50	0.011	0.010	0.012	0.011	0.049	0.049	0.052	0.049	0.101	0.100	0.104	0.100
75	0.010	0.010	0.011	0.010	0.048	0.046	0.050	0.049	0.097	0.095	0.098	0.096
100	0.010	0.009	0.011	0.010	0.048	0.046	0.049	0.047	0.096	0.093	0.096	0.093
125	0.009	0.010	0.011	0.010	0.047	0.046	0.049	0.047	0.096	0.095	0.096	0.094
150	0.009	0.009	0.011	0.010	0.047	0.047	0.050	0.047	0.096	0.094	0.098	0.095
200	0.009	0.009	0.010	0.009	0.050	0.048	0.050	0.049	0.096	0.094	0.097	0.096
250	0.009	0.009	0.010	0.010	0.049	0.048	0.051	0.049	0.098	0.093	0.099	0.098
Right-sided tests - $T = 250$												
-5	0.011	0.011	0.026	0.013	0.051	0.049	0.070	0.050	0.098	0.100	0.116	0.098
-2.5	0.012	0.011	0.025	0.012	0.053	0.053	0.067	0.052	0.105	0.104	0.113	0.102
0	0.011	0.009	0.018	0.012	0.050	0.050	0.057	0.050	0.100	0.100	0.102	0.099
2.5	0.009	0.009	0.014	0.011	0.051	0.048	0.054	0.050	0.103	0.101	0.106	0.100
5	0.009	0.009	0.013	0.010	0.051	0.049	0.055	0.050	0.102	0.101	0.105	0.101
10	0.008	0.008	0.010	0.009	0.050	0.047	0.052	0.051	0.102	0.100	0.105	0.102
25	0.009	0.010	0.011	0.010	0.052	0.051	0.053	0.051	0.101	0.099	0.102	0.103
50	0.009	0.009	0.011	0.009	0.052	0.050	0.052	0.050	0.102	0.099	0.104	0.099
75	0.009	0.009	0.010	0.010	0.053	0.051	0.054	0.053	0.102	0.099	0.103	0.100
100	0.008	0.009	0.010	0.009	0.050	0.048	0.053	0.050	0.101	0.102	0.105	0.101
125	0.008	0.009	0.010	0.009	0.049	0.049	0.052	0.049	0.104	0.101	0.104	0.100
150	0.008	0.008	0.010	0.010	0.051	0.049	0.052	0.050	0.103	0.101	0.104	0.100
200	0.008	0.009	0.010	0.010	0.050	0.049	0.053	0.051	0.105	0.101	0.104	0.102
250	0.009	0.009	0.011	0.011	0.049	0.050	0.051	0.050	0.105	0.101	0.105	0.102
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.038	0.015	0.049	0.050	0.089	0.053	0.100	0.099	0.140	0.102
-2.5	0.010	0.011	0.037	0.012	0.048	0.048	0.082	0.050	0.098	0.100	0.130	0.097
0	0.010	0.010	0.025	0.011	0.047	0.047	0.068	0.048	0.094	0.096	0.115	0.095
2.5	0.011	0.011	0.017	0.012	0.047	0.047	0.060	0.049	0.099	0.096	0.108	0.097
5	0.010	0.010	0.014	0.011	0.049	0.046	0.056	0.050	0.097	0.096	0.106	0.098
10	0.010	0.010	0.012	0.010	0.049	0.047	0.053	0.052	0.100	0.097	0.106	0.100
25	0.011	0.010	0.012	0.012	0.050	0.048	0.053	0.050	0.104	0.101	0.107	0.103
50	0.009	0.010	0.011	0.010	0.050	0.049	0.056	0.052	0.100	0.098	0.104	0.099
75	0.008	0.009	0.010	0.010	0.048	0.047	0.052	0.048	0.100	0.095	0.104	0.101
100	0.008	0.009	0.010	0.009	0.047	0.046	0.051	0.048	0.098	0.095	0.101	0.098
125	0.008	0.009	0.010	0.009	0.047	0.044	0.050	0.047	0.096	0.093	0.100	0.096
150	0.008	0.009	0.011	0.009	0.046	0.045	0.051	0.048	0.098	0.096	0.102	0.097
200	0.008	0.009	0.011	0.009	0.050	0.051	0.054	0.050	0.100	0.097	0.103	0.100
250	0.009	0.009	0.011	0.010	0.048	0.048	0.054	0.051	0.099	0.097	0.103	0.099

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.011	0.010	0.013	0.010	0.051	0.051	0.059	0.051	0.102	0.101	0.110	0.099
-2.5	0.010	0.011	0.018	0.011	0.048	0.048	0.059	0.048	0.100	0.096	0.105	0.096
0	0.011	0.011	0.015	0.012	0.051	0.052	0.059	0.051	0.099	0.102	0.109	0.098
2.5	0.010	0.011	0.013	0.011	0.053	0.053	0.055	0.052	0.101	0.101	0.101	0.101
5	0.010	0.010	0.010	0.010	0.052	0.051	0.053	0.051	0.105	0.102	0.105	0.103
10	0.009	0.010	0.010	0.010	0.049	0.049	0.051	0.048	0.101	0.101	0.103	0.101
25	0.011	0.010	0.010	0.010	0.048	0.049	0.049	0.049	0.100	0.100	0.099	0.100
50	0.010	0.009	0.009	0.009	0.050	0.049	0.049	0.049	0.102	0.100	0.101	0.098
75	0.010	0.010	0.010	0.010	0.048	0.048	0.049	0.048	0.098	0.098	0.099	0.097
100	0.010	0.011	0.010	0.010	0.049	0.048	0.049	0.048	0.096	0.095	0.098	0.094
125	0.010	0.010	0.010	0.010	0.049	0.047	0.049	0.049	0.099	0.097	0.096	0.096
150	0.010	0.010	0.010	0.010	0.047	0.048	0.049	0.048	0.098	0.099	0.099	0.096
200	0.010	0.009	0.010	0.010	0.050	0.047	0.049	0.049	0.099	0.099	0.100	0.097
250	0.010	0.010	0.010	0.010	0.048	0.049	0.048	0.048	0.102	0.100	0.101	0.101
Right-sided tests - $T = 1000$												
-5	0.011	0.011	0.016	0.012	0.049	0.050	0.057	0.048	0.099	0.099	0.107	0.100
-2.5	0.011	0.010	0.018	0.010	0.050	0.050	0.059	0.050	0.095	0.096	0.107	0.095
0	0.010	0.010	0.014	0.011	0.048	0.051	0.054	0.048	0.101	0.102	0.104	0.099
2.5	0.008	0.008	0.011	0.008	0.049	0.049	0.052	0.048	0.103	0.104	0.105	0.101
5	0.008	0.009	0.009	0.009	0.051	0.050	0.050	0.050	0.101	0.102	0.103	0.101
10	0.008	0.010	0.010	0.010	0.052	0.051	0.051	0.051	0.102	0.100	0.101	0.099
25	0.011	0.010	0.011	0.011	0.051	0.049	0.050	0.050	0.099	0.101	0.101	0.100
50	0.012	0.011	0.011	0.012	0.052	0.051	0.052	0.052	0.102	0.101	0.103	0.101
75	0.012	0.011	0.011	0.011	0.054	0.052	0.053	0.052	0.101	0.101	0.102	0.101
100	0.011	0.011	0.012	0.011	0.053	0.052	0.053	0.052	0.100	0.099	0.100	0.099
125	0.012	0.011	0.011	0.011	0.052	0.052	0.052	0.052	0.101	0.101	0.101	0.099
150	0.012	0.011	0.011	0.011	0.053	0.052	0.052	0.053	0.101	0.103	0.102	0.100
200	0.011	0.011	0.011	0.011	0.053	0.051	0.053	0.051	0.103	0.104	0.105	0.102
250	0.011	0.010	0.011	0.010	0.051	0.050	0.051	0.051	0.105	0.105	0.106	0.105
Two-sided tests - $T = 1000$												
-5	0.012	0.011	0.017	0.012	0.049	0.050	0.065	0.049	0.100	0.100	0.116	0.099
-2.5	0.009	0.010	0.022	0.010	0.050	0.049	0.068	0.050	0.098	0.100	0.118	0.098
0	0.011	0.012	0.017	0.011	0.049	0.049	0.062	0.050	0.099	0.101	0.113	0.099
2.5	0.010	0.009	0.011	0.010	0.048	0.049	0.054	0.048	0.101	0.102	0.107	0.100
5	0.009	0.009	0.010	0.008	0.049	0.049	0.051	0.048	0.102	0.101	0.103	0.101
10	0.009	0.009	0.010	0.009	0.049	0.050	0.052	0.050	0.101	0.100	0.101	0.100
25	0.010	0.010	0.010	0.011	0.051	0.051	0.052	0.052	0.099	0.098	0.099	0.099
50	0.011	0.009	0.011	0.011	0.050	0.049	0.050	0.050	0.101	0.099	0.100	0.101
75	0.010	0.009	0.010	0.010	0.050	0.049	0.050	0.051	0.102	0.101	0.102	0.100
100	0.010	0.010	0.010	0.011	0.051	0.050	0.052	0.051	0.102	0.100	0.102	0.101
125	0.010	0.010	0.011	0.011	0.051	0.050	0.051	0.052	0.101	0.099	0.101	0.101
150	0.011	0.011	0.011	0.011	0.052	0.051	0.052	0.051	0.100	0.100	0.100	0.101
200	0.011	0.011	0.011	0.011	0.050	0.050	0.051	0.051	0.102	0.099	0.101	0.100
250	0.011	0.011	0.011	0.011	0.050	0.049	0.051	0.051	0.099	0.098	0.099	0.099

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.29: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP10 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.95; -0.95 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma^2_{1t}, \sigma^2_{2t})$, σ^2_{it} are GARCH processes generated as $\sigma^2_{it} = 0.05 + 0.1e^2_{i,t-1} + 0.85\sigma^2_{i,t-1}$, $i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim iidt_5(\mathbf{0}, \mathbf{I_2})$ where $t_5(\mathbf{0}, \mathbf{I_2})$ defines a mean zero Student-t distribution with 5 degrees of freedom and an 2×2 identity variance matrix.

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.007	0.001	0.000	0.006	0.045	0.004	0.004	0.018	0.095	0.013	0.012	0.031
-2.5	0.007	0.000	0.000	0.003	0.041	0.002	0.002	0.008	0.098	0.005	0.004	0.014
0	0.010	0.001	0.001	0.005	0.042	0.003	0.004	0.010	0.073	0.007	0.007	0.017
2.5	0.019	0.001	0.001	0.006	0.060	0.007	0.007	0.014	0.100	0.014	0.013	0.026
5	0.024	0.002	0.001	0.008	0.068	0.012	0.011	0.023	0.115	0.023	0.022	0.041
10	0.024	0.002	0.002	0.012	0.069	0.016	0.016	0.037	0.118	0.038	0.036	0.066
25	0.019	0.003	0.004	0.022	0.068	0.025	0.028	0.062	0.116	0.055	0.057	0.100
50	0.018	0.005	0.006	0.030	0.066	0.032	0.035	0.078	0.113	0.068	0.070	0.119
75	0.016	0.006	0.008	0.034	0.063	0.037	0.039	0.086	0.115	0.074	0.079	0.133
100	0.015	0.007	0.009	0.037	0.062	0.039	0.043	0.093	0.113	0.078	0.084	0.141
125	0.013	0.008	0.011	0.040	0.060	0.041	0.045	0.096	0.112	0.081	0.088	0.146
150	0.014	0.008	0.012	0.044	0.061	0.042	0.049	0.099	0.113	0.085	0.091	0.152
200	0.012	0.008	0.012	0.046	0.062	0.046	0.054	0.106	0.116	0.091	0.099	0.159
250	0.011	0.008	0.013	0.048	0.059	0.046	0.055	0.110	0.118	0.097	0.107	0.165
Right-sided tests - $T = 250$												
-5	0.009	0.013	0.022	0.038	0.042	0.078	0.092	0.110	0.085	0.170	0.176	0.184
-2.5	0.007	0.016	0.023	0.039	0.037	0.104	0.112	0.145	0.083	0.258	0.244	0.292
0	0.009	0.020	0.025	0.052	0.047	0.101	0.117	0.173	0.104	0.222	0.236	0.303
2.5	0.009	0.020	0.025	0.060	0.052	0.099	0.115	0.180	0.115	0.210	0.224	0.298
5	0.010	0.021	0.026	0.063	0.058	0.100	0.114	0.177	0.119	0.193	0.209	0.283
10	0.011	0.021	0.027	0.065	0.061	0.094	0.105	0.172	0.122	0.178	0.189	0.261
25	0.012	0.018	0.023	0.067	0.062	0.084	0.093	0.160	0.121	0.154	0.166	0.237
50	0.009	0.013	0.018	0.065	0.059	0.071	0.081	0.150	0.118	0.140	0.151	0.224
75	0.010	0.012	0.017	0.062	0.059	0.065	0.078	0.145	0.120	0.132	0.142	0.213
100	0.009	0.011	0.017	0.060	0.057	0.060	0.074	0.141	0.121	0.128	0.139	0.210
125	0.009	0.011	0.018	0.059	0.058	0.059	0.069	0.137	0.120	0.122	0.135	0.205
150	0.010	0.010	0.017	0.059	0.059	0.058	0.069	0.135	0.120	0.118	0.131	0.202
200	0.010	0.010	0.017	0.058	0.062	0.057	0.068	0.129	0.120	0.112	0.124	0.193
250	0.012	0.011	0.017	0.054	0.063	0.056	0.066	0.126	0.118	0.107	0.119	0.184
Two-sided tests - $T = 250$												
-5	0.009	0.006	0.013	0.030	0.042	0.037	0.049	0.079	0.086	0.081	0.096	0.128
-2.5	0.006	0.009	0.013	0.027	0.033	0.044	0.055	0.085	0.075	0.104	0.114	0.153
0	0.008	0.011	0.015	0.036	0.041	0.051	0.062	0.112	0.088	0.103	0.121	0.183
2.5	0.009	0.011	0.015	0.042	0.045	0.054	0.065	0.119	0.096	0.106	0.122	0.194
5	0.010	0.011	0.015	0.048	0.050	0.056	0.066	0.126	0.102	0.109	0.124	0.199
10	0.011	0.011	0.015	0.052	0.054	0.057	0.065	0.138	0.110	0.108	0.121	0.208
25	0.012	0.011	0.014	0.061	0.057	0.054	0.062	0.148	0.116	0.108	0.120	0.222
50	0.011	0.009	0.012	0.065	0.061	0.050	0.063	0.157	0.117	0.102	0.115	0.228
75	0.011	0.008	0.013	0.067	0.058	0.050	0.062	0.156	0.117	0.101	0.117	0.231
100	0.011	0.009	0.014	0.069	0.056	0.048	0.061	0.156	0.114	0.099	0.117	0.234
125	0.010	0.008	0.014	0.072	0.058	0.048	0.061	0.158	0.114	0.100	0.114	0.232
150	0.010	0.010	0.015	0.074	0.059	0.049	0.066	0.160	0.117	0.099	0.117	0.234
200	0.010	0.009	0.017	0.077	0.060	0.049	0.067	0.164	0.121	0.102	0.122	0.235
250	0.009	0.009	0.017	0.074	0.058	0.051	0.071	0.163	0.119	0.101	0.121	0.237

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.007	0.000	0.000	0.006	0.041	0.004	0.002	0.015	0.092	0.011	0.010	0.025
-2.5	0.005	0.000	0.000	0.003	0.044	0.001	0.001	0.007	0.098	0.004	0.003	0.011
0	0.009	0.000	0.000	0.004	0.036	0.002	0.002	0.008	0.064	0.005	0.005	0.014
2.5	0.019	0.001	0.001	0.006	0.057	0.005	0.005	0.016	0.094	0.014	0.013	0.028
5	0.027	0.001	0.002	0.011	0.068	0.009	0.009	0.030	0.111	0.023	0.022	0.050
10	0.029	0.003	0.002	0.022	0.071	0.016	0.014	0.053	0.116	0.036	0.034	0.081
25	0.024	0.004	0.004	0.045	0.070	0.023	0.022	0.090	0.118	0.049	0.047	0.128
50	0.021	0.006	0.006	0.063	0.064	0.029	0.028	0.115	0.113	0.060	0.060	0.158
75	0.020	0.007	0.007	0.072	0.063	0.033	0.034	0.129	0.112	0.066	0.066	0.173
100	0.019	0.007	0.007	0.080	0.064	0.037	0.037	0.137	0.110	0.069	0.070	0.182
125	0.019	0.008	0.008	0.087	0.065	0.038	0.040	0.144	0.114	0.073	0.075	0.188
150	0.018	0.009	0.009	0.091	0.064	0.040	0.042	0.149	0.116	0.077	0.079	0.193
200	0.018	0.008	0.010	0.098	0.066	0.042	0.044	0.158	0.115	0.081	0.085	0.202
250	0.018	0.009	0.010	0.100	0.066	0.044	0.046	0.164	0.118	0.085	0.087	0.210
Right-sided tests - $T = 1000$												
-5	0.004	0.013	0.042	0.039	0.030	0.081	0.129	0.099	0.067	0.184	0.225	0.166
-2.5	0.003	0.014	0.027	0.049	0.018	0.096	0.134	0.149	0.052	0.248	0.289	0.281
0	0.003	0.016	0.020	0.078	0.027	0.090	0.112	0.203	0.068	0.207	0.233	0.330
2.5	0.004	0.019	0.022	0.094	0.033	0.098	0.109	0.223	0.084	0.198	0.215	0.337
5	0.005	0.019	0.022	0.104	0.039	0.095	0.102	0.228	0.093	0.191	0.198	0.333
10	0.006	0.020	0.022	0.110	0.043	0.089	0.095	0.230	0.094	0.172	0.179	0.317
25	0.007	0.018	0.019	0.121	0.046	0.082	0.083	0.229	0.103	0.154	0.158	0.306
50	0.008	0.016	0.017	0.127	0.049	0.073	0.078	0.227	0.104	0.139	0.144	0.300
75	0.009	0.015	0.016	0.129	0.052	0.069	0.074	0.226	0.108	0.134	0.140	0.293
100	0.009	0.015	0.015	0.130	0.054	0.067	0.072	0.223	0.110	0.133	0.140	0.289
125	0.010	0.014	0.016	0.133	0.056	0.068	0.071	0.223	0.112	0.131	0.137	0.287
150	0.011	0.014	0.016	0.133	0.055	0.067	0.071	0.224	0.113	0.129	0.135	0.289
200	0.011	0.014	0.016	0.131	0.056	0.064	0.070	0.223	0.115	0.126	0.133	0.285
250	0.011	0.014	0.017	0.131	0.057	0.063	0.066	0.221	0.118	0.124	0.129	0.282
Two-sided tests - $T = 1000$												
-5	0.003	0.006	0.032	0.029	0.028	0.038	0.078	0.074	0.069	0.084	0.131	0.115
-2.5	0.001	0.007	0.013	0.036	0.014	0.037	0.067	0.093	0.045	0.096	0.135	0.156
0	0.002	0.008	0.010	0.057	0.021	0.042	0.054	0.135	0.053	0.090	0.114	0.211
2.5	0.003	0.010	0.011	0.074	0.026	0.049	0.058	0.163	0.064	0.100	0.113	0.239
5	0.004	0.010	0.012	0.085	0.029	0.051	0.058	0.179	0.073	0.103	0.111	0.258
10	0.006	0.010	0.012	0.099	0.038	0.054	0.055	0.203	0.086	0.105	0.109	0.283
25	0.011	0.012	0.012	0.129	0.047	0.053	0.055	0.238	0.097	0.103	0.105	0.319
50	0.013	0.010	0.011	0.150	0.054	0.052	0.056	0.261	0.104	0.103	0.106	0.342
75	0.014	0.011	0.011	0.163	0.057	0.051	0.054	0.273	0.109	0.102	0.108	0.356
100	0.015	0.011	0.013	0.171	0.058	0.052	0.056	0.285	0.114	0.103	0.110	0.361
125	0.016	0.011	0.013	0.176	0.058	0.054	0.057	0.291	0.116	0.106	0.111	0.367
150	0.016	0.012	0.014	0.183	0.060	0.054	0.058	0.296	0.116	0.106	0.113	0.373
200	0.015	0.013	0.014	0.185	0.063	0.054	0.061	0.302	0.121	0.105	0.114	0.381
250	0.014	0.012	0.013	0.189	0.064	0.055	0.062	0.305	0.119	0.106	0.113	0.385

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (9) and (13) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.30: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP10 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.9; -0.9 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma^2_{1t}, \sigma^2_{2t})$, σ^2_{it} are GARCH processes generated as $\sigma^2_{it} = 0.05 + 0.1e^2_{i,t-1} + 0.85\sigma^2_{i,t-1},\ i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim iidt_5(\mathbf{0}, \mathbf{I_2})$ where $t_5(\mathbf{0}, \mathbf{I_2})$ defines a mean zero Student-t distribution with 5 degrees of freedom and an 2×2 identity variance matrix.

Left-sided tests

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.007	0.001	0.000	0.007	0.045	0.004	0.005	0.018	0.094	0.016	0.015	0.032
-2.5	0.007	0.001	0.000	0.003	0.043	0.003	0.002	0.008	0.095	0.006	0.005	0.015
0	0.009	0.001	0.001	0.004	0.039	0.003	0.004	0.011	0.070	0.007	0.007	0.017
2.5	0.018	0.001	0.001	0.006	0.059	0.006	0.007	0.015	0.099	0.015	0.015	0.028
5	0.023	0.002	0.002	0.008	0.068	0.012	0.011	0.023	0.112	0.026	0.024	0.043
10	0.022	0.003	0.002	0.013	0.069	0.018	0.017	0.037	0.117	0.038	0.037	0.067
25	0.018	0.004	0.004	0.023	0.067	0.027	0.028	0.061	0.115	0.057	0.056	0.098
50	0.017	0.006	0.007	0.029	0.063	0.033	0.035	0.077	0.114	0.069	0.071	0.118
75	0.015	0.007	0.008	0.034	0.062	0.036	0.041	0.085	0.114	0.076	0.079	0.132
100	0.015	0.006	0.009	0.036	0.061	0.039	0.043	0.092	0.114	0.080	0.084	0.139
125	0.013	0.008	0.010	0.039	0.061	0.041	0.046	0.094	0.111	0.083	0.088	0.143
150	0.012	0.008	0.011	0.042	0.061	0.042	0.048	0.097	0.113	0.086	0.092	0.150
200	0.011	0.008	0.012	0.045	0.061	0.046	0.055	0.106	0.114	0.092	0.100	0.156
250	0.012	0.008	0.014	0.046	0.060	0.048	0.055	0.107	0.117	0.098	0.107	0.163

Right-sided tests

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.015	0.027	0.040	0.043	0.083	0.096	0.113	0.089	0.172	0.181	0.189
-2.5	0.007	0.018	0.025	0.038	0.038	0.103	0.114	0.144	0.085	0.253	0.247	0.284
0	0.009	0.021	0.025	0.051	0.048	0.102	0.119	0.168	0.106	0.221	0.235	0.296
2.5	0.009	0.020	0.026	0.059	0.055	0.102	0.115	0.175	0.114	0.206	0.221	0.290
5	0.010	0.020	0.026	0.062	0.059	0.099	0.112	0.173	0.119	0.194	0.203	0.278
10	0.011	0.020	0.027	0.065	0.062	0.095	0.106	0.166	0.121	0.174	0.186	0.254
25	0.011	0.018	0.022	0.064	0.063	0.082	0.091	0.156	0.121	0.153	0.163	0.231
50	0.010	0.015	0.019	0.061	0.060	0.071	0.080	0.145	0.119	0.138	0.150	0.220
75	0.008	0.012	0.017	0.059	0.058	0.067	0.076	0.138	0.120	0.129	0.142	0.209
100	0.009	0.011	0.017	0.058	0.056	0.060	0.074	0.136	0.120	0.124	0.136	0.205
125	0.009	0.011	0.017	0.058	0.056	0.059	0.071	0.134	0.119	0.120	0.134	0.200
150	0.009	0.010	0.017	0.057	0.058	0.058	0.069	0.131	0.121	0.119	0.131	0.197
200	0.010	0.011	0.017	0.056	0.060	0.058	0.069	0.126	0.119	0.111	0.124	0.190
250	0.012	0.012	0.017	0.055	0.063	0.056	0.066	0.123	0.119	0.107	0.119	0.181

Two-sided tests

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.008	0.009	0.017	0.031	0.043	0.040	0.054	0.082	0.089	0.087	0.100	0.131
-2.5	0.006	0.009	0.014	0.027	0.035	0.046	0.059	0.084	0.077	0.105	0.116	0.153
0	0.008	0.011	0.015	0.035	0.040	0.051	0.062	0.107	0.090	0.104	0.123	0.179
2.5	0.009	0.011	0.014	0.042	0.047	0.055	0.064	0.119	0.096	0.106	0.122	0.190
5	0.009	0.012	0.014	0.045	0.050	0.056	0.064	0.124	0.104	0.109	0.123	0.196
10	0.011	0.011	0.015	0.051	0.055	0.057	0.065	0.133	0.110	0.112	0.123	0.203
25	0.011	0.011	0.014	0.058	0.059	0.054	0.064	0.145	0.115	0.107	0.119	0.217
50	0.012	0.009	0.013	0.062	0.061	0.051	0.063	0.152	0.117	0.103	0.115	0.222
75	0.011	0.008	0.013	0.065	0.059	0.048	0.061	0.151	0.114	0.101	0.117	0.223
100	0.010	0.010	0.013	0.066	0.056	0.047	0.060	0.152	0.118	0.099	0.116	0.228
125	0.010	0.009	0.014	0.069	0.058	0.048	0.062	0.154	0.116	0.098	0.116	0.229
150	0.010	0.009	0.015	0.072	0.058	0.047	0.065	0.158	0.116	0.100	0.117	0.228
200	0.009	0.009	0.016	0.074	0.059	0.049	0.069	0.161	0.119	0.104	0.123	0.231
250	0.009	0.009	0.017	0.072	0.059	0.050	0.069	0.157	0.119	0.103	0.122	0.229

Left-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.007	0.000	0.000	0.006	0.042	0.004	0.003	0.016	0.092	0.012	0.011	0.027
-2.5	0.006	0.000	0.000	0.004	0.044	0.001	0.001	0.007	0.097	0.004	0.003	0.011
0	0.008	0.000	0.000	0.004	0.035	0.002	0.002	0.008	0.064	0.006	0.006	0.014
2.5	0.019	0.001	0.001	0.006	0.055	0.005	0.005	0.016	0.093	0.015	0.014	0.030
5	0.026	0.001	0.002	0.011	0.067	0.010	0.009	0.031	0.110	0.025	0.023	0.051
10	0.029	0.003	0.002	0.022	0.070	0.017	0.014	0.052	0.115	0.038	0.035	0.083
25	0.024	0.005	0.004	0.044	0.068	0.023	0.023	0.088	0.118	0.051	0.049	0.126
50	0.021	0.006	0.006	0.061	0.064	0.029	0.029	0.112	0.113	0.061	0.060	0.155
75	0.019	0.007	0.007	0.071	0.062	0.033	0.035	0.126	0.113	0.065	0.065	0.173
100	0.020	0.008	0.008	0.078	0.065	0.037	0.037	0.135	0.111	0.071	0.072	0.182
125	0.020	0.008	0.008	0.085	0.066	0.038	0.039	0.142	0.113	0.075	0.075	0.185
150	0.019	0.009	0.009	0.090	0.066	0.040	0.041	0.146	0.116	0.077	0.079	0.190
200	0.018	0.009	0.010	0.094	0.065	0.043	0.044	0.155	0.115	0.083	0.085	0.197
250	0.018	0.010	0.010	0.099	0.065	0.044	0.047	0.161	0.116	0.084	0.087	0.206

Right-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.004	0.015	0.049	0.038	0.032	0.084	0.136	0.098	0.072	0.187	0.231	0.166
-2.5	0.003	0.014	0.032	0.048	0.020	0.099	0.141	0.144	0.053	0.247	0.286	0.278
0	0.005	0.016	0.023	0.073	0.030	0.091	0.115	0.197	0.072	0.206	0.234	0.322
2.5	0.004	0.019	0.023	0.091	0.036	0.098	0.109	0.219	0.085	0.196	0.211	0.332
5	0.005	0.019	0.022	0.100	0.039	0.094	0.102	0.219	0.093	0.189	0.196	0.326
10	0.006	0.020	0.021	0.105	0.043	0.089	0.093	0.224	0.097	0.172	0.177	0.310
25	0.007	0.018	0.018	0.115	0.045	0.081	0.083	0.222	0.101	0.151	0.156	0.300
50	0.008	0.015	0.018	0.121	0.048	0.071	0.076	0.220	0.103	0.138	0.142	0.294
75	0.010	0.014	0.016	0.124	0.052	0.070	0.073	0.219	0.106	0.132	0.138	0.285
100	0.010	0.014	0.015	0.127	0.055	0.067	0.071	0.217	0.108	0.133	0.138	0.283
125	0.010	0.015	0.016	0.128	0.056	0.066	0.070	0.216	0.111	0.130	0.135	0.282
150	0.010	0.014	0.016	0.129	0.055	0.065	0.070	0.217	0.114	0.128	0.135	0.280
200	0.012	0.014	0.015	0.126	0.056	0.064	0.069	0.217	0.114	0.127	0.132	0.280
250	0.010	0.013	0.016	0.127	0.056	0.061	0.066	0.216	0.115	0.123	0.129	0.277

Two-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.003	0.007	0.036	0.029	0.030	0.040	0.086	0.076	0.073	0.089	0.139	0.114
-2.5	0.002	0.006	0.018	0.035	0.016	0.039	0.073	0.091	0.046	0.097	0.142	0.151
0	0.003	0.008	0.011	0.055	0.021	0.042	0.054	0.131	0.057	0.093	0.117	0.206
2.5	0.003	0.010	0.012	0.070	0.026	0.049	0.057	0.157	0.068	0.101	0.114	0.235
5	0.004	0.011	0.012	0.080	0.030	0.052	0.058	0.173	0.075	0.104	0.111	0.250
10	0.006	0.011	0.013	0.096	0.038	0.052	0.054	0.196	0.086	0.106	0.107	0.276
25	0.011	0.012	0.012	0.125	0.047	0.051	0.054	0.231	0.096	0.103	0.106	0.310
50	0.013	0.010	0.010	0.146	0.054	0.051	0.055	0.252	0.103	0.099	0.105	0.331
75	0.015	0.011	0.011	0.156	0.057	0.052	0.055	0.265	0.108	0.103	0.108	0.345
100	0.015	0.011	0.012	0.162	0.056	0.052	0.055	0.279	0.113	0.102	0.108	0.352
125	0.015	0.012	0.012	0.168	0.056	0.053	0.056	0.282	0.117	0.104	0.110	0.357
150	0.016	0.012	0.014	0.174	0.058	0.051	0.058	0.288	0.116	0.105	0.111	0.363
200	0.016	0.013	0.014	0.180	0.061	0.053	0.059	0.292	0.119	0.105	0.112	0.372
250	0.014	0.012	0.013	0.182	0.064	0.054	0.060	0.294	0.119	0.106	0.113	0.376

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.31: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP10 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.5; -0.5 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2,\ i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim iidt_5(\mathbf{0}, \mathbf{I_2})$ where $t_5(\mathbf{0}, \mathbf{I_2})$ defines a mean zero Student-t distribution with 5 degrees of freedom and an 2×2 identity variance matrix.

Left-sided tests

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.002	0.006	0.012	0.050	0.015	0.020	0.031	0.097	0.039	0.040	0.056
-2.5	0.010	0.001	0.002	0.006	0.044	0.008	0.007	0.016	0.087	0.018	0.016	0.026
0	0.008	0.001	0.003	0.007	0.034	0.009	0.010	0.017	0.069	0.020	0.023	0.030
2.5	0.012	0.003	0.003	0.009	0.050	0.015	0.017	0.026	0.092	0.033	0.034	0.051
5	0.014	0.004	0.004	0.011	0.056	0.020	0.021	0.036	0.101	0.044	0.044	0.064
10	0.016	0.005	0.005	0.016	0.059	0.026	0.026	0.046	0.105	0.058	0.058	0.081
25	0.016	0.006	0.006	0.022	0.060	0.035	0.036	0.063	0.108	0.068	0.072	0.102
50	0.014	0.007	0.008	0.026	0.061	0.037	0.040	0.072	0.111	0.079	0.081	0.117
75	0.014	0.007	0.009	0.027	0.060	0.040	0.044	0.077	0.111	0.082	0.088	0.122
100	0.013	0.007	0.010	0.030	0.059	0.043	0.048	0.078	0.109	0.086	0.090	0.126
125	0.011	0.008	0.011	0.031	0.059	0.044	0.050	0.081	0.110	0.089	0.094	0.130
150	0.011	0.008	0.011	0.032	0.059	0.046	0.052	0.085	0.111	0.091	0.095	0.131
200	0.010	0.008	0.013	0.033	0.058	0.047	0.054	0.088	0.113	0.096	0.102	0.138
250	0.010	0.010	0.013	0.034	0.057	0.046	0.056	0.089	0.112	0.097	0.106	0.142

Right-sided tests

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.017	0.037	0.035	0.047	0.081	0.096	0.101	0.094	0.160	0.168	0.170
-2.5	0.009	0.020	0.042	0.036	0.051	0.109	0.117	0.128	0.105	0.221	0.205	0.228
0	0.010	0.018	0.032	0.039	0.056	0.098	0.111	0.132	0.116	0.198	0.204	0.233
2.5	0.010	0.018	0.028	0.042	0.057	0.090	0.104	0.130	0.116	0.178	0.185	0.220
5	0.011	0.019	0.025	0.041	0.059	0.087	0.097	0.132	0.117	0.166	0.175	0.209
10	0.011	0.017	0.021	0.042	0.061	0.081	0.088	0.124	0.114	0.153	0.162	0.201
25	0.010	0.015	0.018	0.042	0.058	0.072	0.080	0.116	0.113	0.135	0.144	0.185
50	0.011	0.012	0.016	0.039	0.056	0.064	0.072	0.111	0.110	0.123	0.133	0.177
75	0.011	0.011	0.015	0.040	0.055	0.059	0.068	0.106	0.110	0.120	0.129	0.172
100	0.009	0.011	0.017	0.041	0.057	0.058	0.067	0.105	0.113	0.116	0.127	0.166
125	0.009	0.011	0.016	0.041	0.057	0.058	0.068	0.105	0.116	0.115	0.126	0.163
150	0.010	0.010	0.017	0.040	0.057	0.057	0.067	0.104	0.115	0.115	0.126	0.162
200	0.011	0.011	0.016	0.041	0.060	0.056	0.067	0.101	0.117	0.112	0.121	0.161
250	0.011	0.010	0.017	0.040	0.059	0.055	0.065	0.097	0.117	0.109	0.120	0.156

Two-sided tests

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.009	0.032	0.032	0.047	0.047	0.072	0.081	0.096	0.095	0.116	0.132
-2.5	0.008	0.010	0.030	0.024	0.046	0.060	0.077	0.084	0.096	0.115	0.124	0.144
0	0.008	0.009	0.022	0.028	0.048	0.050	0.072	0.088	0.095	0.107	0.122	0.149
2.5	0.009	0.010	0.018	0.031	0.049	0.052	0.066	0.093	0.100	0.105	0.121	0.156
5	0.010	0.011	0.015	0.034	0.051	0.055	0.065	0.099	0.106	0.107	0.117	0.167
10	0.011	0.011	0.014	0.038	0.056	0.053	0.062	0.106	0.109	0.106	0.115	0.170
25	0.011	0.009	0.013	0.041	0.058	0.052	0.059	0.113	0.109	0.107	0.116	0.179
50	0.013	0.009	0.013	0.045	0.055	0.048	0.058	0.115	0.110	0.101	0.112	0.183
75	0.011	0.009	0.013	0.045	0.056	0.049	0.059	0.117	0.111	0.100	0.113	0.183
100	0.010	0.008	0.014	0.046	0.058	0.048	0.061	0.118	0.111	0.100	0.115	0.184
125	0.010	0.008	0.014	0.048	0.055	0.047	0.061	0.120	0.113	0.101	0.117	0.185
150	0.008	0.009	0.014	0.049	0.055	0.047	0.062	0.121	0.115	0.102	0.119	0.189
200	0.008	0.010	0.016	0.051	0.054	0.048	0.062	0.122	0.116	0.103	0.121	0.189
250	0.009	0.010	0.016	0.052	0.055	0.050	0.065	0.123	0.116	0.101	0.121	0.186

Left-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.001	0.005	0.011	0.048	0.012	0.017	0.028	0.098	0.030	0.034	0.045
-2.5	0.007	0.000	0.001	0.005	0.043	0.004	0.005	0.012	0.092	0.011	0.013	0.022
0	0.007	0.001	0.002	0.005	0.029	0.005	0.007	0.013	0.062	0.013	0.014	0.025
2.5	0.011	0.001	0.002	0.008	0.044	0.012	0.012	0.026	0.086	0.027	0.026	0.047
5	0.016	0.002	0.002	0.013	0.052	0.017	0.016	0.038	0.101	0.036	0.036	0.066
10	0.019	0.003	0.003	0.021	0.058	0.022	0.021	0.056	0.106	0.048	0.047	0.089
25	0.017	0.005	0.004	0.036	0.057	0.028	0.026	0.079	0.107	0.061	0.059	0.117
50	0.017	0.006	0.006	0.045	0.057	0.033	0.033	0.093	0.105	0.068	0.068	0.139
75	0.015	0.007	0.007	0.051	0.057	0.036	0.036	0.105	0.106	0.074	0.074	0.149
100	0.015	0.008	0.007	0.055	0.059	0.037	0.038	0.111	0.107	0.079	0.079	0.157
125	0.015	0.007	0.008	0.058	0.059	0.040	0.041	0.117	0.110	0.082	0.082	0.161
150	0.015	0.008	0.009	0.060	0.060	0.041	0.043	0.120	0.110	0.084	0.084	0.166
200	0.014	0.010	0.010	0.063	0.062	0.044	0.046	0.127	0.110	0.085	0.088	0.174
250	0.014	0.009	0.010	0.068	0.062	0.045	0.048	0.129	0.111	0.089	0.091	0.176

Right-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.007	0.017	0.054	0.033	0.038	0.084	0.115	0.090	0.083	0.170	0.186	0.155
-2.5	0.007	0.020	0.051	0.040	0.038	0.110	0.141	0.130	0.083	0.223	0.234	0.231
0	0.007	0.020	0.037	0.053	0.041	0.099	0.123	0.152	0.095	0.205	0.219	0.256
2.5	0.009	0.020	0.031	0.063	0.047	0.095	0.110	0.156	0.099	0.186	0.196	0.257
5	0.010	0.020	0.026	0.063	0.046	0.088	0.097	0.161	0.101	0.173	0.181	0.247
10	0.010	0.018	0.021	0.066	0.047	0.080	0.084	0.162	0.101	0.157	0.164	0.239
25	0.010	0.016	0.017	0.071	0.048	0.071	0.073	0.159	0.099	0.138	0.143	0.238
50	0.008	0.014	0.015	0.076	0.049	0.066	0.069	0.160	0.102	0.127	0.132	0.229
75	0.010	0.014	0.014	0.079	0.050	0.063	0.066	0.158	0.104	0.125	0.130	0.227
100	0.009	0.014	0.015	0.081	0.052	0.063	0.065	0.157	0.105	0.124	0.128	0.226
125	0.010	0.013	0.014	0.081	0.054	0.059	0.065	0.157	0.106	0.123	0.126	0.226
150	0.010	0.014	0.014	0.083	0.054	0.060	0.066	0.159	0.108	0.122	0.127	0.227
200	0.011	0.012	0.014	0.082	0.055	0.060	0.065	0.158	0.110	0.120	0.127	0.224
250	0.011	0.013	0.014	0.084	0.054	0.061	0.066	0.161	0.111	0.119	0.124	0.222

Two-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.006	0.009	0.047	0.029	0.038	0.046	0.090	0.077	0.083	0.095	0.132	0.117
-2.5	0.005	0.008	0.039	0.028	0.031	0.055	0.090	0.082	0.071	0.113	0.146	0.142
0	0.006	0.010	0.026	0.040	0.031	0.051	0.077	0.103	0.070	0.103	0.129	0.165
2.5	0.007	0.012	0.020	0.050	0.037	0.053	0.066	0.117	0.078	0.106	0.121	0.181
5	0.008	0.013	0.016	0.056	0.039	0.053	0.060	0.129	0.083	0.105	0.113	0.198
10	0.011	0.012	0.014	0.063	0.043	0.052	0.055	0.141	0.090	0.102	0.105	0.218
25	0.011	0.010	0.011	0.080	0.047	0.052	0.053	0.165	0.096	0.098	0.099	0.239
50	0.012	0.009	0.010	0.092	0.052	0.049	0.051	0.181	0.101	0.098	0.102	0.253
75	0.013	0.010	0.011	0.099	0.053	0.049	0.052	0.193	0.104	0.098	0.102	0.263
100	0.012	0.010	0.011	0.104	0.054	0.050	0.053	0.200	0.107	0.098	0.103	0.268
125	0.013	0.010	0.013	0.109	0.056	0.050	0.053	0.203	0.110	0.099	0.106	0.273
150	0.013	0.011	0.013	0.111	0.056	0.050	0.053	0.205	0.110	0.101	0.109	0.279
200	0.013	0.011	0.013	0.115	0.056	0.051	0.055	0.210	0.115	0.104	0.110	0.284
250	0.014	0.010	0.013	0.118	0.055	0.051	0.054	0.213	0.114	0.107	0.113	0.290

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.32: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP10 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad 0; 0 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2$, $i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim iidt_5(\mathbf{0}, \mathbf{I_2})$ where $t_5(\mathbf{0}, \mathbf{I_2})$ defines a mean zero Student-t distribution with 5 degrees of freedom and an 2×2 identity variance matrix.

Left-sided tests

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.011	0.011	0.022	0.025	0.051	0.050	0.057	0.067	0.099	0.098	0.097	0.107
-2.5	0.013	0.012	0.021	0.019	0.048	0.048	0.047	0.056	0.094	0.096	0.084	0.102
0	0.012	0.012	0.022	0.018	0.047	0.049	0.056	0.057	0.093	0.095	0.093	0.104
2.5	0.013	0.012	0.019	0.021	0.053	0.049	0.057	0.062	0.098	0.096	0.103	0.111
5	0.013	0.012	0.016	0.021	0.053	0.050	0.057	0.066	0.100	0.099	0.105	0.114
10	0.013	0.012	0.013	0.021	0.055	0.051	0.055	0.068	0.103	0.101	0.106	0.115
25	0.013	0.012	0.014	0.023	0.052	0.052	0.057	0.067	0.107	0.103	0.107	0.117
50	0.011	0.010	0.013	0.021	0.053	0.050	0.055	0.069	0.107	0.101	0.107	0.117
75	0.010	0.008	0.012	0.020	0.054	0.050	0.055	0.068	0.109	0.102	0.109	0.120
100	0.009	0.009	0.012	0.020	0.054	0.050	0.055	0.068	0.108	0.101	0.107	0.120
125	0.008	0.009	0.013	0.019	0.052	0.047	0.055	0.067	0.106	0.100	0.106	0.119
150	0.009	0.009	0.013	0.020	0.050	0.048	0.055	0.068	0.105	0.096	0.105	0.117
200	0.009	0.010	0.013	0.021	0.050	0.047	0.053	0.069	0.105	0.098	0.105	0.120
250	0.009	0.010	0.014	0.022	0.050	0.047	0.054	0.070	0.107	0.100	0.108	0.120

Right-sided tests

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.011	0.022	0.026	0.055	0.054	0.057	0.068	0.104	0.103	0.095	0.113
-2.5	0.010	0.009	0.021	0.017	0.053	0.052	0.049	0.061	0.101	0.102	0.086	0.109
0	0.010	0.010	0.016	0.020	0.051	0.050	0.055	0.061	0.096	0.100	0.093	0.109
2.5	0.010	0.011	0.018	0.022	0.052	0.049	0.058	0.065	0.099	0.096	0.103	0.112
5	0.011	0.011	0.016	0.022	0.051	0.050	0.056	0.066	0.103	0.099	0.105	0.115
10	0.011	0.010	0.013	0.021	0.052	0.049	0.057	0.066	0.103	0.100	0.106	0.117
25	0.010	0.010	0.012	0.021	0.053	0.051	0.057	0.066	0.101	0.097	0.105	0.117
50	0.010	0.009	0.012	0.024	0.051	0.051	0.058	0.068	0.103	0.101	0.107	0.119
75	0.011	0.010	0.014	0.024	0.053	0.052	0.056	0.068	0.105	0.100	0.107	0.122
100	0.011	0.010	0.014	0.026	0.054	0.053	0.058	0.071	0.105	0.101	0.107	0.123
125	0.010	0.010	0.014	0.026	0.054	0.053	0.059	0.072	0.109	0.104	0.109	0.124
150	0.011	0.010	0.015	0.025	0.056	0.054	0.062	0.073	0.112	0.106	0.112	0.125
200	0.010	0.011	0.014	0.026	0.057	0.055	0.061	0.074	0.113	0.104	0.112	0.127
250	0.010	0.011	0.013	0.026	0.057	0.054	0.059	0.074	0.115	0.105	0.114	0.127

Two-sided tests

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.011	0.010	0.034	0.037	0.051	0.053	0.074	0.084	0.102	0.103	0.114	0.135
-2.5	0.011	0.011	0.031	0.023	0.049	0.052	0.063	0.070	0.097	0.100	0.096	0.117
0	0.011	0.011	0.028	0.024	0.049	0.049	0.068	0.070	0.095	0.098	0.111	0.118
2.5	0.012	0.011	0.025	0.027	0.054	0.052	0.070	0.077	0.101	0.099	0.115	0.127
5	0.013	0.012	0.020	0.026	0.053	0.051	0.065	0.077	0.102	0.100	0.113	0.132
10	0.012	0.010	0.015	0.028	0.053	0.049	0.061	0.079	0.106	0.100	0.113	0.135
25	0.013	0.011	0.014	0.031	0.053	0.049	0.058	0.082	0.107	0.102	0.114	0.133
50	0.011	0.010	0.014	0.029	0.052	0.051	0.059	0.082	0.105	0.101	0.113	0.137
75	0.009	0.008	0.013	0.030	0.053	0.049	0.061	0.082	0.106	0.101	0.111	0.136
100	0.009	0.009	0.014	0.029	0.052	0.050	0.061	0.084	0.109	0.102	0.114	0.139
125	0.009	0.009	0.014	0.029	0.050	0.051	0.059	0.084	0.107	0.101	0.114	0.139
150	0.009	0.009	0.015	0.030	0.049	0.050	0.059	0.085	0.106	0.101	0.116	0.141
200	0.007	0.011	0.015	0.030	0.050	0.049	0.061	0.085	0.106	0.100	0.115	0.143
250	0.008	0.011	0.015	0.032	0.050	0.047	0.061	0.087	0.106	0.100	0.114	0.144

Left-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.010	0.023	0.022	0.047	0.050	0.050	0.055	0.097	0.098	0.083	0.096
-2.5	0.009	0.009	0.016	0.017	0.048	0.046	0.040	0.051	0.094	0.101	0.072	0.095
0	0.008	0.008	0.018	0.016	0.046	0.048	0.051	0.056	0.091	0.096	0.088	0.100
2.5	0.009	0.010	0.017	0.019	0.047	0.046	0.055	0.061	0.096	0.095	0.102	0.104
5	0.009	0.010	0.016	0.020	0.049	0.048	0.057	0.060	0.098	0.095	0.103	0.106
10	0.009	0.009	0.014	0.022	0.049	0.047	0.054	0.061	0.097	0.096	0.102	0.107
25	0.010	0.010	0.011	0.025	0.049	0.048	0.051	0.068	0.096	0.094	0.098	0.109
50	0.010	0.010	0.010	0.028	0.047	0.048	0.049	0.071	0.098	0.096	0.101	0.116
75	0.010	0.010	0.011	0.029	0.049	0.047	0.050	0.074	0.101	0.097	0.101	0.118
100	0.010	0.011	0.012	0.031	0.050	0.050	0.053	0.074	0.101	0.099	0.100	0.117
125	0.010	0.010	0.010	0.030	0.049	0.050	0.051	0.075	0.101	0.097	0.099	0.120
150	0.010	0.009	0.010	0.030	0.050	0.049	0.051	0.075	0.101	0.098	0.099	0.122
200	0.009	0.009	0.010	0.030	0.050	0.048	0.051	0.077	0.101	0.098	0.101	0.123
250	0.009	0.008	0.009	0.030	0.049	0.049	0.052	0.078	0.101	0.099	0.101	0.124

Right-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.011	0.024	0.022	0.046	0.046	0.049	0.053	0.092	0.096	0.083	0.094
-2.5	0.010	0.010	0.023	0.017	0.046	0.051	0.049	0.052	0.091	0.098	0.079	0.098
0	0.009	0.010	0.018	0.016	0.046	0.048	0.052	0.056	0.093	0.100	0.091	0.104
2.5	0.010	0.010	0.020	0.020	0.047	0.048	0.059	0.061	0.099	0.100	0.105	0.110
5	0.011	0.011	0.019	0.022	0.049	0.050	0.059	0.063	0.099	0.099	0.110	0.111
10	0.011	0.010	0.015	0.024	0.051	0.050	0.056	0.067	0.100	0.099	0.108	0.115
25	0.012	0.011	0.013	0.027	0.051	0.052	0.055	0.073	0.106	0.101	0.106	0.118
50	0.010	0.010	0.011	0.029	0.054	0.053	0.056	0.076	0.105	0.104	0.106	0.123
75	0.010	0.010	0.011	0.030	0.054	0.053	0.055	0.080	0.105	0.105	0.109	0.125
100	0.010	0.010	0.011	0.031	0.055	0.053	0.056	0.080	0.106	0.103	0.107	0.127
125	0.010	0.011	0.011	0.031	0.055	0.052	0.055	0.079	0.103	0.103	0.107	0.127
150	0.011	0.011	0.011	0.031	0.055	0.052	0.055	0.080	0.104	0.102	0.107	0.128
200	0.011	0.010	0.011	0.031	0.053	0.050	0.054	0.080	0.103	0.100	0.106	0.128
250	0.011	0.010	0.011	0.033	0.052	0.049	0.053	0.079	0.104	0.102	0.104	0.130

Two-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.008	0.010	0.037	0.030	0.047	0.048	0.070	0.069	0.093	0.097	0.100	0.108
-2.5	0.010	0.010	0.030	0.023	0.045	0.046	0.061	0.061	0.092	0.097	0.089	0.103
0	0.007	0.009	0.028	0.019	0.044	0.046	0.061	0.063	0.089	0.096	0.103	0.112
2.5	0.010	0.009	0.026	0.027	0.045	0.047	0.066	0.073	0.094	0.094	0.114	0.122
5	0.011	0.009	0.025	0.030	0.047	0.047	0.067	0.076	0.097	0.097	0.116	0.123
10	0.011	0.010	0.016	0.032	0.049	0.047	0.061	0.079	0.099	0.097	0.110	0.127
25	0.010	0.011	0.013	0.037	0.050	0.048	0.054	0.088	0.099	0.099	0.105	0.141
50	0.011	0.010	0.011	0.039	0.051	0.049	0.054	0.097	0.101	0.100	0.105	0.147
75	0.009	0.009	0.010	0.040	0.052	0.050	0.056	0.099	0.102	0.101	0.105	0.154
100	0.008	0.010	0.010	0.042	0.050	0.049	0.054	0.099	0.105	0.100	0.109	0.153
125	0.009	0.009	0.010	0.043	0.052	0.050	0.054	0.099	0.105	0.102	0.106	0.155
150	0.008	0.010	0.010	0.044	0.051	0.049	0.054	0.099	0.103	0.101	0.106	0.155
200	0.010	0.008	0.010	0.044	0.050	0.049	0.054	0.101	0.103	0.098	0.105	0.157
250	0.009	0.008	0.010	0.044	0.048	0.047	0.052	0.103	0.102	0.098	0.105	0.157

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.33: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP11 (Stochastic Volatility):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)'$ follow from a first-order AR stochastic volatility process as $(u_t = e_{1t}\texttt{exp}(h_{1t}), v_t = e_{2t}\texttt{exp}(h_{2t}))'$ with $h_{it} = \lambda h_{i,t-1} + 0.5\xi_{it},\ i = 1, 2$ and $(\xi_{it}, e_{it})' \sim NIID(0, diag(\sigma^2_{\xi}, 1))$, independent across i = 1,2. Results are reported for $(\lambda, \sigma_{\xi}) = (0.951, 0.314)$ and $(e_{1t}, e_{2t})' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = [1 \quad -0.95; -0.95 \quad 1]$.

Left-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.000	0.002	0.002	0.050	0.005	0.009	0.010	0.099	0.018	0.024	0.022
-2.5	0.009	0.000	0.000	0.001	0.049	0.001	0.001	0.002	0.103	0.004	0.004	0.006
0	0.008	0.000	0.000	0.000	0.035	0.001	0.002	0.002	0.064	0.005	0.007	0.006
2.5	0.016	0.001	0.002	0.001	0.055	0.007	0.009	0.009	0.095	0.019	0.020	0.021
5	0.019	0.002	0.002	0.003	0.061	0.014	0.014	0.016	0.107	0.032	0.032	0.032
10	0.017	0.004	0.003	0.005	0.062	0.021	0.021	0.023	0.109	0.046	0.047	0.048
25	0.014	0.005	0.006	0.008	0.060	0.031	0.034	0.034	0.111	0.062	0.064	0.066
50	0.013	0.007	0.007	0.009	0.061	0.038	0.040	0.038	0.112	0.075	0.078	0.076
75	0.013	0.008	0.009	0.010	0.062	0.039	0.043	0.042	0.112	0.082	0.085	0.082
100	0.013	0.008	0.010	0.011	0.058	0.041	0.044	0.043	0.115	0.086	0.091	0.084
125	0.013	0.009	0.010	0.011	0.056	0.042	0.046	0.045	0.113	0.088	0.093	0.084
150	0.013	0.009	0.011	0.011	0.057	0.045	0.048	0.045	0.113	0.090	0.095	0.088
200	0.011	0.010	0.011	0.011	0.056	0.047	0.051	0.047	0.111	0.094	0.100	0.091
250	0.012	0.010	0.013	0.011	0.057	0.049	0.053	0.047	0.115	0.100	0.105	0.094

Right-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.011	0.015	0.041	0.019	0.045	0.083	0.117	0.078	0.094	0.160	0.199	0.149
-2.5	0.015	0.021	0.046	0.020	0.055	0.115	0.157	0.109	0.113	0.251	0.284	0.235
0	0.012	0.021	0.037	0.017	0.062	0.116	0.143	0.103	0.127	0.243	0.265	0.229
2.5	0.013	0.022	0.034	0.017	0.067	0.113	0.135	0.102	0.130	0.221	0.241	0.211
5	0.014	0.021	0.031	0.018	0.064	0.106	0.121	0.096	0.125	0.200	0.216	0.190
10	0.013	0.021	0.026	0.016	0.059	0.089	0.103	0.083	0.118	0.174	0.188	0.166
25	0.012	0.017	0.022	0.014	0.055	0.076	0.083	0.068	0.106	0.146	0.157	0.135
50	0.011	0.017	0.019	0.012	0.057	0.071	0.076	0.063	0.109	0.131	0.137	0.124
75	0.010	0.014	0.018	0.011	0.057	0.067	0.074	0.060	0.110	0.126	0.133	0.121
100	0.011	0.014	0.016	0.013	0.055	0.062	0.070	0.057	0.112	0.121	0.129	0.115
125	0.011	0.013	0.016	0.014	0.054	0.060	0.066	0.056	0.110	0.119	0.126	0.110
150	0.010	0.012	0.015	0.014	0.054	0.058	0.065	0.053	0.109	0.114	0.120	0.106
200	0.010	0.012	0.014	0.013	0.054	0.056	0.062	0.053	0.108	0.107	0.115	0.103
250	0.011	0.011	0.013	0.013	0.054	0.054	0.058	0.051	0.106	0.105	0.110	0.100

Two-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.009	0.029	0.013	0.045	0.039	0.079	0.044	0.096	0.086	0.126	0.088
-2.5	0.012	0.012	0.031	0.011	0.051	0.049	0.091	0.052	0.103	0.116	0.158	0.111
0	0.009	0.010	0.022	0.008	0.050	0.055	0.081	0.050	0.101	0.117	0.145	0.105
2.5	0.010	0.010	0.020	0.010	0.052	0.059	0.079	0.053	0.111	0.119	0.143	0.112
5	0.012	0.011	0.018	0.010	0.054	0.058	0.073	0.053	0.109	0.118	0.135	0.111
10	0.012	0.012	0.017	0.011	0.054	0.057	0.066	0.053	0.104	0.109	0.124	0.106
25	0.011	0.011	0.015	0.012	0.055	0.055	0.062	0.051	0.103	0.104	0.117	0.101
50	0.010	0.010	0.013	0.012	0.054	0.052	0.062	0.052	0.111	0.108	0.116	0.101
75	0.011	0.011	0.014	0.012	0.055	0.053	0.060	0.052	0.111	0.105	0.117	0.102
100	0.011	0.011	0.014	0.014	0.057	0.054	0.060	0.051	0.108	0.103	0.114	0.100
125	0.011	0.011	0.014	0.014	0.054	0.051	0.059	0.050	0.110	0.101	0.111	0.100
150	0.011	0.011	0.014	0.014	0.054	0.052	0.059	0.052	0.109	0.101	0.114	0.098
200	0.012	0.012	0.015	0.014	0.052	0.052	0.059	0.053	0.109	0.102	0.113	0.099
250	0.011	0.011	0.015	0.013	0.053	0.051	0.058	0.053	0.108	0.101	0.111	0.098

Left-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.001	0.001	0.002	0.046	0.008	0.009	0.009	0.095	0.021	0.023	0.023
-2.5	0.008	0.000	0.000	0.000	0.050	0.001	0.001	0.001	0.105	0.004	0.004	0.004
0	0.010	0.000	0.000	0.000	0.035	0.002	0.002	0.003	0.062	0.006	0.006	0.008
2.5	0.018	0.001	0.001	0.001	0.054	0.010	0.010	0.012	0.094	0.023	0.024	0.025
5	0.020	0.002	0.003	0.004	0.059	0.017	0.017	0.019	0.105	0.036	0.037	0.038
10	0.018	0.004	0.005	0.006	0.063	0.024	0.024	0.027	0.107	0.050	0.050	0.054
25	0.014	0.006	0.007	0.009	0.057	0.031	0.030	0.033	0.111	0.065	0.065	0.068
50	0.012	0.007	0.007	0.010	0.057	0.035	0.036	0.039	0.107	0.073	0.073	0.076
75	0.012	0.008	0.008	0.010	0.055	0.037	0.037	0.039	0.108	0.077	0.077	0.077
100	0.012	0.008	0.008	0.010	0.053	0.038	0.039	0.040	0.106	0.078	0.080	0.082
125	0.011	0.007	0.007	0.010	0.053	0.040	0.041	0.042	0.103	0.079	0.080	0.082
150	0.012	0.007	0.008	0.010	0.054	0.041	0.042	0.044	0.104	0.081	0.081	0.082
200	0.011	0.008	0.008	0.010	0.054	0.043	0.043	0.044	0.104	0.085	0.085	0.086
250	0.011	0.008	0.008	0.010	0.054	0.043	0.045	0.046	0.104	0.086	0.085	0.087

Right-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.014	0.020	0.017	0.044	0.072	0.087	0.072	0.094	0.154	0.167	0.147
-2.5	0.009	0.019	0.023	0.015	0.052	0.104	0.121	0.099	0.101	0.232	0.244	0.224
0	0.014	0.022	0.027	0.019	0.061	0.113	0.121	0.105	0.122	0.231	0.237	0.222
2.5	0.015	0.026	0.028	0.020	0.067	0.115	0.121	0.107	0.129	0.214	0.221	0.206
5	0.017	0.027	0.030	0.021	0.066	0.106	0.112	0.097	0.126	0.196	0.203	0.193
10	0.015	0.024	0.026	0.019	0.062	0.095	0.097	0.084	0.118	0.180	0.185	0.171
25	0.013	0.019	0.021	0.015	0.057	0.081	0.083	0.072	0.108	0.153	0.155	0.141
50	0.011	0.015	0.017	0.013	0.053	0.073	0.076	0.063	0.106	0.139	0.141	0.128
75	0.010	0.014	0.014	0.010	0.054	0.067	0.070	0.061	0.104	0.133	0.135	0.123
100	0.010	0.013	0.014	0.010	0.054	0.066	0.067	0.057	0.104	0.128	0.130	0.118
125	0.010	0.013	0.014	0.010	0.052	0.063	0.064	0.056	0.104	0.124	0.126	0.116
150	0.009	0.014	0.014	0.010	0.053	0.061	0.064	0.056	0.105	0.123	0.124	0.114
200	0.010	0.012	0.013	0.010	0.051	0.060	0.061	0.054	0.106	0.119	0.121	0.113
250	0.009	0.012	0.012	0.011	0.052	0.059	0.060	0.053	0.105	0.117	0.119	0.111

Two-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.008	0.012	0.010	0.045	0.038	0.048	0.041	0.096	0.080	0.095	0.082
-2.5	0.008	0.008	0.013	0.008	0.046	0.049	0.057	0.047	0.093	0.105	0.123	0.100
0	0.010	0.012	0.014	0.011	0.049	0.054	0.064	0.052	0.100	0.116	0.124	0.108
2.5	0.012	0.014	0.017	0.011	0.056	0.061	0.068	0.058	0.110	0.123	0.131	0.119
5	0.013	0.016	0.018	0.011	0.057	0.064	0.069	0.060	0.109	0.123	0.129	0.116
10	0.013	0.016	0.018	0.013	0.058	0.061	0.064	0.059	0.109	0.118	0.121	0.111
25	0.013	0.014	0.015	0.013	0.052	0.058	0.059	0.052	0.105	0.110	0.113	0.106
50	0.012	0.013	0.012	0.012	0.051	0.053	0.053	0.051	0.105	0.107	0.112	0.101
75	0.011	0.011	0.012	0.012	0.051	0.052	0.053	0.052	0.105	0.103	0.107	0.100
100	0.009	0.011	0.012	0.011	0.052	0.052	0.053	0.050	0.105	0.103	0.106	0.097
125	0.009	0.011	0.011	0.010	0.051	0.051	0.052	0.050	0.102	0.102	0.105	0.098
150	0.009	0.010	0.010	0.010	0.052	0.051	0.053	0.051	0.104	0.102	0.106	0.099
200	0.009	0.010	0.010	0.011	0.054	0.051	0.052	0.051	0.103	0.103	0.104	0.099
250	0.010	0.009	0.010	0.011	0.052	0.050	0.054	0.048	0.105	0.101	0.105	0.099

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.34: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP11 (Stochastic Volatility):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T$, $\psi = 0$ and $(u_t, v_t)'$ follow from a first-order AR stochastic volatility process as $(u_t = e_{1t}\mathtt{exp}(h_{1t}), v_t = e_{2t}\mathtt{exp}(h_{2t}))'$ with $h_{it} = \lambda h_{i,t-1} + 0.5\xi_{it},\ i = 1,2.$ and $(\xi_{it}, e_{it})' \sim NIID(0, diag(\sigma_\xi^2, 1))$, independent across i = 1,2. Results are reported for $(\lambda, \sigma_\xi) = (0.951, 0.314)$ and $(e_{1t}, e_{2t})' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = [1 \quad -0.9; -0.9 \quad 1]$.

Left-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.000	0.003	0.002	0.050	0.006	0.012	0.012	0.100	0.021	0.030	0.025
-2.5	0.009	0.000	0.001	0.001	0.050	0.001	0.002	0.002	0.106	0.004	0.006	0.006
0	0.008	0.000	0.000	0.001	0.034	0.002	0.002	0.002	0.064	0.007	0.008	0.007
2.5	0.015	0.001	0.002	0.002	0.051	0.008	0.010	0.010	0.092	0.022	0.023	0.023
5	0.018	0.003	0.003	0.003	0.058	0.014	0.016	0.017	0.104	0.033	0.035	0.034
10	0.016	0.003	0.004	0.006	0.060	0.021	0.023	0.025	0.108	0.047	0.048	0.049
25	0.014	0.005	0.006	0.008	0.059	0.032	0.034	0.033	0.110	0.065	0.066	0.067
50	0.014	0.007	0.007	0.009	0.059	0.038	0.040	0.039	0.112	0.077	0.079	0.077
75	0.012	0.008	0.009	0.010	0.060	0.039	0.042	0.042	0.113	0.082	0.085	0.082
100	0.014	0.008	0.010	0.011	0.059	0.041	0.044	0.043	0.116	0.089	0.093	0.085
125	0.013	0.009	0.010	0.011	0.059	0.042	0.045	0.044	0.113	0.092	0.094	0.088
150	0.013	0.008	0.011	0.011	0.058	0.043	0.047	0.045	0.115	0.096	0.098	0.090
200	0.011	0.010	0.011	0.012	0.057	0.046	0.051	0.047	0.113	0.096	0.100	0.093
250	0.011	0.010	0.013	0.011	0.056	0.048	0.053	0.049	0.114	0.099	0.105	0.097

Right-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.014	0.042	0.019	0.044	0.077	0.114	0.075	0.091	0.157	0.198	0.145
-2.5	0.012	0.018	0.050	0.019	0.056	0.117	0.160	0.110	0.113	0.249	0.282	0.233
0	0.012	0.020	0.040	0.019	0.065	0.115	0.144	0.105	0.129	0.242	0.263	0.224
2.5	0.014	0.021	0.036	0.018	0.067	0.115	0.134	0.104	0.132	0.216	0.239	0.206
5	0.013	0.020	0.031	0.017	0.064	0.104	0.118	0.096	0.128	0.196	0.210	0.184
10	0.012	0.020	0.025	0.017	0.061	0.090	0.101	0.083	0.116	0.170	0.181	0.161
25	0.012	0.017	0.021	0.014	0.054	0.074	0.081	0.066	0.105	0.140	0.151	0.132
50	0.010	0.015	0.018	0.012	0.057	0.068	0.075	0.063	0.107	0.130	0.135	0.120
75	0.010	0.014	0.016	0.011	0.056	0.065	0.071	0.061	0.108	0.125	0.131	0.118
100	0.010	0.013	0.016	0.012	0.054	0.062	0.069	0.058	0.110	0.121	0.127	0.114
125	0.011	0.013	0.016	0.013	0.054	0.058	0.064	0.054	0.108	0.118	0.125	0.108
150	0.010	0.012	0.015	0.012	0.054	0.057	0.062	0.052	0.108	0.113	0.119	0.106
200	0.010	0.011	0.014	0.013	0.054	0.055	0.060	0.052	0.106	0.107	0.113	0.103
250	0.010	0.011	0.013	0.012	0.054	0.053	0.058	0.052	0.107	0.104	0.109	0.100

Two-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.008	0.008	0.033	0.012	0.044	0.038	0.081	0.045	0.093	0.083	0.126	0.087
-2.5	0.010	0.010	0.032	0.010	0.051	0.052	0.093	0.053	0.104	0.117	0.162	0.112
0	0.009	0.010	0.024	0.009	0.053	0.058	0.083	0.054	0.104	0.117	0.146	0.107
2.5	0.010	0.010	0.022	0.010	0.056	0.059	0.080	0.056	0.111	0.121	0.143	0.113
5	0.011	0.011	0.019	0.010	0.055	0.058	0.071	0.055	0.109	0.117	0.135	0.113
10	0.012	0.012	0.017	0.011	0.052	0.055	0.066	0.052	0.105	0.110	0.124	0.107
25	0.011	0.012	0.015	0.012	0.053	0.054	0.061	0.053	0.102	0.106	0.115	0.099
50	0.011	0.011	0.013	0.012	0.054	0.055	0.061	0.050	0.110	0.106	0.115	0.101
75	0.011	0.010	0.013	0.012	0.053	0.053	0.059	0.051	0.110	0.103	0.114	0.103
100	0.010	0.011	0.014	0.014	0.052	0.050	0.057	0.051	0.109	0.102	0.112	0.101
125	0.011	0.012	0.015	0.014	0.053	0.051	0.058	0.051	0.106	0.100	0.109	0.098
150	0.011	0.010	0.015	0.014	0.053	0.051	0.058	0.052	0.107	0.100	0.109	0.097
200	0.011	0.012	0.015	0.013	0.052	0.051	0.056	0.053	0.108	0.101	0.112	0.100
250	0.011	0.012	0.015	0.014	0.053	0.050	0.057	0.053	0.109	0.101	0.111	0.100

Left-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.011	0.001	0.002	0.002	0.049	0.009	0.010	0.011	0.097	0.024	0.027	0.026
-2.5	0.008	0.000	0.000	0.000	0.050	0.001	0.002	0.002	0.105	0.004	0.005	0.005
0	0.010	0.000	0.000	0.000	0.034	0.003	0.003	0.003	0.062	0.007	0.008	0.009
2.5	0.016	0.001	0.001	0.001	0.053	0.011	0.012	0.012	0.093	0.025	0.025	0.027
5	0.018	0.003	0.003	0.004	0.058	0.018	0.017	0.020	0.104	0.037	0.038	0.040
10	0.016	0.005	0.005	0.007	0.062	0.025	0.025	0.027	0.106	0.052	0.052	0.057
25	0.014	0.007	0.007	0.009	0.058	0.032	0.031	0.034	0.110	0.067	0.068	0.070
50	0.013	0.008	0.007	0.010	0.056	0.035	0.036	0.039	0.109	0.075	0.076	0.077
75	0.012	0.007	0.008	0.010	0.055	0.036	0.037	0.041	0.107	0.077	0.077	0.078
100	0.012	0.007	0.008	0.010	0.055	0.039	0.040	0.042	0.106	0.080	0.080	0.081
125	0.011	0.008	0.008	0.010	0.055	0.040	0.042	0.043	0.103	0.080	0.082	0.082
150	0.011	0.008	0.008	0.010	0.055	0.041	0.043	0.043	0.104	0.082	0.083	0.085
200	0.010	0.007	0.008	0.009	0.055	0.044	0.044	0.046	0.103	0.085	0.086	0.085
250	0.011	0.008	0.008	0.010	0.056	0.044	0.044	0.047	0.103	0.088	0.086	0.088

Right-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.013	0.020	0.016	0.047	0.079	0.089	0.077	0.097	0.156	0.168	0.146
-2.5	0.010	0.017	0.024	0.016	0.051	0.107	0.123	0.100	0.104	0.229	0.243	0.220
0	0.013	0.023	0.026	0.020	0.058	0.111	0.121	0.103	0.122	0.226	0.234	0.216
2.5	0.014	0.026	0.028	0.020	0.067	0.111	0.118	0.103	0.125	0.208	0.217	0.198
5	0.016	0.025	0.028	0.020	0.065	0.103	0.108	0.095	0.124	0.194	0.198	0.186
10	0.015	0.023	0.026	0.018	0.060	0.093	0.095	0.085	0.116	0.175	0.181	0.166
25	0.013	0.018	0.019	0.013	0.055	0.079	0.082	0.071	0.110	0.150	0.154	0.140
50	0.011	0.016	0.015	0.012	0.053	0.070	0.073	0.061	0.106	0.137	0.139	0.125
75	0.009	0.014	0.014	0.010	0.052	0.067	0.068	0.059	0.103	0.130	0.130	0.120
100	0.010	0.014	0.014	0.010	0.052	0.064	0.065	0.058	0.103	0.125	0.127	0.117
125	0.011	0.013	0.014	0.011	0.052	0.062	0.064	0.055	0.102	0.122	0.124	0.113
150	0.010	0.013	0.014	0.010	0.051	0.061	0.062	0.054	0.102	0.120	0.122	0.112
200	0.010	0.013	0.013	0.011	0.053	0.059	0.061	0.054	0.102	0.116	0.118	0.111
250	0.010	0.012	0.012	0.011	0.052	0.059	0.060	0.054	0.103	0.116	0.118	0.110

Two-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.007	0.012	0.010	0.048	0.041	0.052	0.043	0.100	0.086	0.099	0.088
-2.5	0.008	0.009	0.014	0.009	0.045	0.050	0.063	0.046	0.093	0.108	0.125	0.102
0	0.009	0.011	0.014	0.010	0.048	0.057	0.063	0.051	0.099	0.112	0.124	0.107
2.5	0.011	0.014	0.017	0.011	0.056	0.060	0.068	0.058	0.107	0.122	0.130	0.115
5	0.013	0.014	0.018	0.012	0.056	0.062	0.069	0.058	0.110	0.120	0.125	0.115
10	0.014	0.015	0.018	0.013	0.056	0.060	0.063	0.057	0.107	0.115	0.120	0.112
25	0.013	0.014	0.014	0.013	0.052	0.055	0.057	0.054	0.104	0.110	0.114	0.105
50	0.011	0.012	0.012	0.012	0.052	0.054	0.056	0.052	0.103	0.105	0.109	0.100
75	0.011	0.011	0.011	0.012	0.053	0.052	0.054	0.050	0.104	0.103	0.105	0.100
100	0.011	0.011	0.011	0.011	0.051	0.051	0.054	0.050	0.103	0.103	0.105	0.099
125	0.010	0.010	0.011	0.010	0.052	0.052	0.054	0.051	0.104	0.103	0.105	0.098
150	0.009	0.009	0.010	0.010	0.053	0.050	0.053	0.051	0.105	0.102	0.105	0.097
200	0.009	0.009	0.010	0.011	0.052	0.050	0.052	0.049	0.106	0.103	0.105	0.100
250	0.010	0.009	0.010	0.012	0.051	0.049	0.052	0.049	0.105	0.102	0.104	0.100

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.35: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP11 (Stochastic Volatility):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T$, $\psi = 0$ and $(u_t, v_t)'$ follow from a first-order AR stochastic volatility process as $(u_t = e_{1t}\texttt{exp}(h_{1t}), v_t = e_{2t}\texttt{exp}(h_{2t}))'$ with $h_{it} = \lambda h_{i,t-1} + 0.5\xi_{it},\ i = 1, 2.$ and $(\xi_{it}, e_{it})' \sim NIID(0, diag(\sigma_\xi^2, 1))$, independent across i = 1,2. Results are reported for $(\lambda, \sigma_\xi) = (0.951, 0.314)$ and $(e_{1t}, e_{2t})' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = [1 \quad -0.5; -0.5 \quad 1]$.

Left-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.225	0.003	0.015	0.005	0.235	0.020	0.037	0.026	0.247	0.051	0.067	0.053
-2.5	0.010	0.001	0.007	0.002	0.047	0.008	0.016	0.011	0.097	0.022	0.030	0.024
0	0.006	0.001	0.004	0.002	0.032	0.010	0.015	0.012	0.063	0.024	0.030	0.026
2.5	0.010	0.003	0.005	0.004	0.042	0.018	0.022	0.020	0.085	0.041	0.048	0.043
5	0.011	0.004	0.006	0.005	0.049	0.024	0.028	0.026	0.094	0.054	0.057	0.057
10	0.010	0.005	0.007	0.008	0.053	0.032	0.035	0.032	0.100	0.067	0.070	0.067
25	0.011	0.008	0.009	0.009	0.053	0.039	0.041	0.041	0.107	0.080	0.084	0.080
50	0.011	0.009	0.010	0.011	0.054	0.043	0.047	0.045	0.109	0.091	0.094	0.090
75	0.010	0.009	0.010	0.011	0.055	0.045	0.050	0.046	0.111	0.093	0.097	0.093
100	0.011	0.008	0.011	0.011	0.055	0.047	0.052	0.048	0.109	0.095	0.101	0.095
125	0.010	0.008	0.011	0.010	0.055	0.047	0.052	0.049	0.112	0.100	0.104	0.096
150	0.011	0.009	0.011	0.011	0.053	0.046	0.052	0.048	0.112	0.100	0.104	0.098
200	0.010	0.010	0.011	0.010	0.056	0.048	0.054	0.050	0.114	0.103	0.108	0.102
250	0.010	0.009	0.012	0.011	0.054	0.051	0.056	0.054	0.114	0.103	0.110	0.102

Right-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.016	0.048	0.019	0.049	0.074	0.109	0.069	0.095	0.141	0.169	0.130
-2.5	0.013	0.022	0.055	0.022	0.059	0.107	0.131	0.100	0.115	0.199	0.214	0.190
0	0.014	0.022	0.045	0.021	0.069	0.100	0.124	0.096	0.131	0.193	0.208	0.185
2.5	0.014	0.020	0.036	0.020	0.066	0.093	0.107	0.089	0.127	0.169	0.184	0.164
5	0.012	0.018	0.028	0.018	0.060	0.083	0.096	0.080	0.120	0.154	0.169	0.151
10	0.011	0.015	0.020	0.016	0.058	0.073	0.083	0.071	0.111	0.141	0.148	0.136
25	0.010	0.013	0.017	0.012	0.050	0.061	0.066	0.058	0.101	0.118	0.128	0.117
50	0.010	0.012	0.015	0.013	0.052	0.058	0.062	0.054	0.098	0.108	0.114	0.106
75	0.011	0.013	0.015	0.013	0.051	0.054	0.060	0.052	0.098	0.106	0.112	0.102
100	0.011	0.012	0.014	0.012	0.051	0.056	0.061	0.052	0.099	0.107	0.112	0.103
125	0.010	0.010	0.013	0.012	0.050	0.053	0.060	0.052	0.102	0.109	0.112	0.103
150	0.009	0.011	0.013	0.011	0.050	0.052	0.059	0.053	0.104	0.107	0.113	0.102
200	0.008	0.009	0.011	0.011	0.053	0.054	0.060	0.052	0.106	0.105	0.111	0.101
250	0.009	0.010	0.012	0.011	0.051	0.051	0.056	0.050	0.107	0.103	0.110	0.100

Two-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.008	0.008	0.047	0.016	0.049	0.049	0.098	0.053	0.095	0.092	0.146	0.095
-2.5	0.011	0.011	0.045	0.013	0.053	0.060	0.100	0.057	0.105	0.114	0.148	0.111
0	0.011	0.012	0.033	0.012	0.054	0.056	0.087	0.056	0.106	0.111	0.139	0.108
2.5	0.010	0.011	0.027	0.013	0.054	0.055	0.076	0.057	0.108	0.111	0.129	0.109
5	0.011	0.011	0.020	0.012	0.053	0.053	0.069	0.057	0.106	0.108	0.124	0.107
10	0.009	0.009	0.014	0.011	0.053	0.050	0.061	0.055	0.106	0.104	0.118	0.103
25	0.010	0.011	0.014	0.012	0.052	0.049	0.055	0.051	0.100	0.097	0.107	0.099
50	0.011	0.011	0.014	0.012	0.049	0.048	0.054	0.051	0.102	0.100	0.109	0.099
75	0.011	0.011	0.013	0.013	0.052	0.049	0.056	0.052	0.104	0.100	0.110	0.098
100	0.010	0.010	0.013	0.013	0.052	0.048	0.056	0.052	0.104	0.101	0.112	0.100
125	0.008	0.009	0.013	0.013	0.051	0.049	0.056	0.052	0.104	0.100	0.112	0.101
150	0.007	0.009	0.011	0.011	0.049	0.049	0.056	0.053	0.104	0.098	0.111	0.101
200	0.007	0.008	0.012	0.012	0.047	0.048	0.056	0.052	0.105	0.101	0.115	0.102
250	0.008	0.009	0.013	0.013	0.051	0.051	0.059	0.051	0.105	0.101	0.112	0.104

Left-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.011	0.004	0.009	0.005	0.055	0.026	0.036	0.028	0.106	0.059	0.070	0.057
-2.5	0.011	0.002	0.005	0.002	0.051	0.011	0.016	0.011	0.101	0.026	0.032	0.027
0	0.009	0.001	0.003	0.003	0.036	0.013	0.015	0.014	0.067	0.031	0.034	0.031
2.5	0.012	0.004	0.006	0.006	0.048	0.024	0.026	0.026	0.091	0.050	0.051	0.052
5	0.013	0.006	0.005	0.007	0.053	0.030	0.032	0.031	0.096	0.061	0.063	0.062
10	0.013	0.006	0.007	0.008	0.053	0.035	0.036	0.037	0.103	0.069	0.070	0.071
25	0.011	0.008	0.008	0.010	0.057	0.039	0.040	0.043	0.102	0.080	0.081	0.079
50	0.011	0.008	0.008	0.010	0.053	0.042	0.044	0.046	0.103	0.084	0.086	0.083
75	0.011	0.008	0.008	0.010	0.053	0.045	0.045	0.046	0.103	0.089	0.089	0.087
100	0.010	0.009	0.008	0.009	0.055	0.046	0.048	0.047	0.102	0.088	0.091	0.089
125	0.010	0.008	0.008	0.009	0.054	0.047	0.047	0.047	0.104	0.090	0.092	0.090
150	0.010	0.008	0.008	0.010	0.053	0.046	0.047	0.046	0.104	0.092	0.094	0.092
200	0.010	0.008	0.008	0.009	0.053	0.047	0.048	0.047	0.105	0.094	0.095	0.094
250	0.009	0.008	0.009	0.010	0.053	0.047	0.048	0.048	0.104	0.096	0.097	0.095

Right-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.008	0.014	0.029	0.016	0.046	0.067	0.088	0.067	0.095	0.136	0.157	0.127
-2.5	0.010	0.018	0.040	0.018	0.052	0.096	0.116	0.092	0.104	0.187	0.201	0.181
0	0.013	0.019	0.029	0.017	0.059	0.095	0.109	0.089	0.124	0.181	0.193	0.178
2.5	0.013	0.019	0.024	0.018	0.062	0.087	0.093	0.082	0.114	0.158	0.165	0.152
5	0.013	0.018	0.021	0.017	0.057	0.080	0.086	0.075	0.111	0.145	0.151	0.142
10	0.012	0.017	0.017	0.015	0.055	0.071	0.074	0.067	0.106	0.137	0.139	0.133
25	0.011	0.014	0.015	0.012	0.052	0.062	0.065	0.060	0.101	0.123	0.126	0.122
50	0.010	0.013	0.013	0.010	0.050	0.060	0.061	0.055	0.103	0.117	0.120	0.113
75	0.008	0.011	0.011	0.009	0.050	0.058	0.061	0.054	0.099	0.114	0.116	0.107
100	0.008	0.011	0.011	0.009	0.051	0.058	0.060	0.054	0.099	0.112	0.114	0.106
125	0.009	0.011	0.011	0.009	0.051	0.058	0.059	0.053	0.100	0.111	0.114	0.107
150	0.010	0.012	0.011	0.010	0.052	0.058	0.059	0.052	0.102	0.113	0.113	0.106
200	0.010	0.012	0.013	0.010	0.049	0.055	0.057	0.050	0.103	0.110	0.113	0.106
250	0.011	0.012	0.012	0.011	0.049	0.053	0.055	0.051	0.103	0.109	0.111	0.105

Two-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.008	0.024	0.011	0.048	0.045	0.074	0.046	0.099	0.093	0.124	0.094
-2.5	0.009	0.010	0.030	0.011	0.047	0.052	0.082	0.050	0.097	0.106	0.132	0.103
0	0.010	0.010	0.021	0.010	0.049	0.052	0.065	0.051	0.100	0.108	0.125	0.103
2.5	0.012	0.011	0.018	0.012	0.053	0.056	0.062	0.055	0.105	0.111	0.118	0.108
5	0.010	0.012	0.015	0.012	0.055	0.054	0.059	0.055	0.105	0.109	0.118	0.106
10	0.012	0.012	0.013	0.013	0.055	0.054	0.056	0.056	0.104	0.107	0.110	0.104
25	0.011	0.011	0.011	0.012	0.052	0.051	0.054	0.053	0.104	0.103	0.105	0.102
50	0.011	0.010	0.010	0.011	0.052	0.050	0.052	0.050	0.101	0.102	0.105	0.100
75	0.010	0.010	0.009	0.011	0.051	0.050	0.052	0.049	0.103	0.103	0.106	0.100
100	0.009	0.009	0.010	0.010	0.049	0.048	0.051	0.050	0.105	0.105	0.107	0.101
125	0.009	0.009	0.010	0.010	0.050	0.049	0.051	0.050	0.105	0.104	0.107	0.100
150	0.010	0.009	0.010	0.010	0.050	0.049	0.049	0.049	0.104	0.102	0.105	0.098
200	0.010	0.010	0.010	0.010	0.049	0.049	0.051	0.050	0.102	0.100	0.105	0.097
250	0.010	0.010	0.011	0.011	0.050	0.048	0.050	0.050	0.102	0.099	0.103	0.099

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.36: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP11 (Stochastic Volatility):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)'$ follow from a first-order AR stochastic volatility process as $(u_t = e_{1t}\texttt{exp}(h_{1t}), v_t = e_{2t}\texttt{exp}(h_{2t}))'$ with $h_{it} = \lambda h_{i,t-1} + 0.5\xi_{it},\ i = 1, 2.$ and $(\xi_{it}, e_{it})' \sim NIID(0, diag(\sigma_\xi^2, 1))$, independent across i = 1,2. Results are reported for $(\lambda, \sigma_\xi) = (0.951, 0.314)$ and $(e_{1t}, e_{2t})' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = [1 \quad 0; 0 \quad 1]$.

Left-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.011	0.010	0.036	0.014	0.052	0.050	0.080	0.051	0.103	0.103	0.124	0.098
-2.5	0.011	0.011	0.028	0.012	0.051	0.050	0.063	0.051	0.100	0.100	0.103	0.097
0	0.011	0.011	0.024	0.011	0.051	0.051	0.061	0.050	0.099	0.097	0.101	0.096
2.5	0.010	0.009	0.017	0.010	0.050	0.047	0.058	0.049	0.099	0.097	0.106	0.095
5	0.009	0.009	0.014	0.010	0.051	0.048	0.057	0.049	0.099	0.096	0.108	0.094
10	0.010	0.010	0.012	0.011	0.051	0.049	0.055	0.049	0.098	0.099	0.106	0.096
25	0.011	0.010	0.012	0.011	0.051	0.051	0.055	0.052	0.102	0.102	0.110	0.099
50	0.011	0.010	0.012	0.011	0.052	0.049	0.055	0.051	0.106	0.104	0.111	0.101
75	0.009	0.009	0.011	0.011	0.053	0.052	0.057	0.052	0.108	0.105	0.110	0.104
100	0.009	0.009	0.011	0.011	0.054	0.052	0.058	0.053	0.108	0.104	0.109	0.103
125	0.009	0.009	0.012	0.011	0.053	0.052	0.058	0.054	0.110	0.105	0.113	0.104
150	0.009	0.010	0.013	0.012	0.054	0.051	0.058	0.052	0.112	0.106	0.111	0.105
200	0.009	0.011	0.013	0.012	0.052	0.051	0.057	0.051	0.112	0.106	0.112	0.107
250	0.009	0.010	0.013	0.012	0.055	0.052	0.057	0.054	0.111	0.105	0.110	0.106

Left-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.011	0.025	0.012	0.053	0.055	0.071	0.052	0.105	0.104	0.122	0.103
-2.5	0.010	0.012	0.028	0.012	0.051	0.052	0.066	0.050	0.102	0.104	0.111	0.101
0	0.012	0.012	0.022	0.013	0.053	0.053	0.063	0.051	0.104	0.105	0.107	0.101
2.5	0.013	0.013	0.016	0.013	0.054	0.052	0.058	0.053	0.100	0.101	0.105	0.099
5	0.012	0.012	0.015	0.012	0.052	0.052	0.055	0.051	0.099	0.097	0.102	0.096
10	0.012	0.012	0.012	0.013	0.053	0.053	0.054	0.051	0.099	0.100	0.102	0.096
25	0.012	0.012	0.012	0.013	0.051	0.053	0.055	0.051	0.101	0.099	0.100	0.097
50	0.012	0.011	0.012	0.013	0.050	0.051	0.051	0.049	0.099	0.099	0.099	0.095
75	0.012	0.010	0.011	0.012	0.051	0.051	0.052	0.050	0.100	0.100	0.102	0.095
100	0.011	0.011	0.011	0.012	0.053	0.053	0.052	0.051	0.102	0.100	0.101	0.097
125	0.012	0.010	0.010	0.012	0.054	0.051	0.053	0.052	0.102	0.101	0.102	0.097
150	0.011	0.011	0.010	0.012	0.052	0.052	0.052	0.051	0.103	0.101	0.103	0.098
200	0.011	0.010	0.011	0.011	0.051	0.051	0.052	0.051	0.101	0.102	0.103	0.097
250	0.010	0.010	0.011	0.010	0.051	0.051	0.052	0.050	0.100	0.100	0.102	0.098

Right-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.010	0.010	0.035	0.014	0.051	0.051	0.074	0.051	0.100	0.101	0.116	0.097
-2.5	0.010	0.010	0.027	0.012	0.052	0.051	0.061	0.050	0.104	0.102	0.098	0.101
0	0.011	0.011	0.023	0.011	0.056	0.054	0.062	0.053	0.107	0.104	0.107	0.102
2.5	0.012	0.012	0.020	0.012	0.055	0.055	0.065	0.055	0.110	0.105	0.114	0.106
5	0.011	0.012	0.016	0.013	0.058	0.056	0.062	0.054	0.107	0.103	0.112	0.106
10	0.011	0.010	0.014	0.012	0.056	0.054	0.060	0.053	0.107	0.105	0.111	0.104
25	0.010	0.009	0.011	0.010	0.053	0.052	0.056	0.053	0.106	0.105	0.111	0.102
50	0.010	0.010	0.011	0.010	0.051	0.051	0.055	0.053	0.103	0.101	0.106	0.099
75	0.010	0.008	0.011	0.010	0.051	0.050	0.056	0.052	0.103	0.100	0.105	0.097
100	0.010	0.009	0.012	0.010	0.051	0.050	0.054	0.054	0.101	0.098	0.104	0.097
125	0.009	0.010	0.012	0.011	0.051	0.048	0.054	0.050	0.103	0.098	0.105	0.096
150	0.009	0.009	0.012	0.011	0.051	0.048	0.054	0.051	0.103	0.099	0.106	0.097
200	0.008	0.009	0.012	0.010	0.051	0.051	0.055	0.053	0.105	0.099	0.104	0.098
250	0.008	0.010	0.013	0.011	0.050	0.050	0.054	0.050	0.103	0.099	0.106	0.099

Right-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.009	0.025	0.012	0.050	0.049	0.070	0.050	0.099	0.099	0.117	0.094
-2.5	0.011	0.011	0.026	0.012	0.050	0.050	0.063	0.050	0.097	0.096	0.109	0.095
0	0.010	0.010	0.019	0.010	0.047	0.047	0.056	0.046	0.097	0.100	0.101	0.095
2.5	0.011	0.011	0.014	0.011	0.050	0.049	0.056	0.048	0.099	0.099	0.104	0.097
5	0.010	0.010	0.013	0.011	0.049	0.050	0.052	0.049	0.100	0.098	0.104	0.097
10	0.010	0.010	0.011	0.011	0.049	0.050	0.050	0.048	0.100	0.101	0.102	0.099
25	0.010	0.009	0.010	0.011	0.049	0.048	0.049	0.047	0.102	0.100	0.100	0.097
50	0.010	0.010	0.009	0.010	0.050	0.049	0.050	0.048	0.098	0.098	0.100	0.096
75	0.010	0.010	0.011	0.010	0.050	0.049	0.049	0.048	0.099	0.098	0.099	0.097
100	0.010	0.010	0.010	0.010	0.049	0.048	0.049	0.046	0.099	0.098	0.100	0.099
125	0.009	0.009	0.011	0.011	0.050	0.047	0.050	0.049	0.099	0.099	0.099	0.096
150	0.010	0.009	0.010	0.011	0.050	0.049	0.050	0.049	0.098	0.098	0.098	0.095
200	0.009	0.010	0.011	0.010	0.049	0.050	0.051	0.049	0.099	0.098	0.100	0.096
250	0.010	0.011	0.011	0.011	0.050	0.050	0.051	0.048	0.100	0.096	0.099	0.098

Two-sided tests - $T = 250$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.009	0.057	0.017	0.051	0.051	0.106	0.060	0.101	0.100	0.154	0.103
-2.5	0.009	0.010	0.043	0.013	0.051	0.053	0.087	0.055	0.102	0.101	0.124	0.101
0	0.011	0.011	0.035	0.013	0.053	0.052	0.077	0.054	0.104	0.105	0.123	0.103
2.5	0.010	0.011	0.023	0.012	0.054	0.053	0.071	0.055	0.105	0.103	0.123	0.104
5	0.010	0.010	0.017	0.013	0.054	0.053	0.066	0.055	0.107	0.104	0.119	0.103
10	0.010	0.009	0.014	0.013	0.053	0.050	0.058	0.056	0.104	0.103	0.115	0.102
25	0.010	0.009	0.012	0.012	0.051	0.048	0.058	0.053	0.103	0.103	0.112	0.104
50	0.010	0.010	0.013	0.012	0.050	0.048	0.055	0.053	0.102	0.101	0.110	0.104
75	0.009	0.010	0.012	0.011	0.048	0.049	0.056	0.053	0.104	0.101	0.113	0.104
100	0.009	0.009	0.012	0.011	0.048	0.049	0.058	0.054	0.105	0.102	0.112	0.107
125	0.008	0.008	0.012	0.011	0.050	0.047	0.056	0.054	0.103	0.100	0.111	0.104
150	0.007	0.009	0.013	0.012	0.048	0.048	0.057	0.053	0.103	0.099	0.112	0.102
200	0.008	0.009	0.013	0.013	0.049	0.049	0.058	0.054	0.104	0.100	0.112	0.104
250	0.007	0.009	0.013	0.014	0.049	0.051	0.060	0.053	0.104	0.100	0.112	0.103

Two-sided tests - $T = 1000$

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
-5	0.009	0.010	0.034	0.013	0.051	0.052	0.089	0.055	0.102	0.105	0.141	0.102
-2.5	0.011	0.012	0.039	0.013	0.051	0.051	0.082	0.051	0.101	0.102	0.129	0.100
0	0.012	0.013	0.029	0.013	0.053	0.053	0.071	0.052	0.099	0.100	0.119	0.097
2.5	0.012	0.012	0.020	0.013	0.052	0.050	0.063	0.052	0.102	0.101	0.113	0.101
5	0.012	0.013	0.016	0.014	0.052	0.051	0.057	0.053	0.100	0.101	0.108	0.099
10	0.012	0.013	0.013	0.014	0.052	0.051	0.053	0.052	0.101	0.100	0.104	0.099
25	0.011	0.010	0.011	0.012	0.049	0.051	0.052	0.051	0.102	0.100	0.104	0.098
50	0.011	0.011	0.011	0.014	0.050	0.052	0.053	0.050	0.099	0.099	0.101	0.097
75	0.010	0.010	0.011	0.012	0.051	0.052	0.051	0.052	0.101	0.100	0.101	0.097
100	0.010	0.009	0.011	0.013	0.050	0.049	0.053	0.051	0.100	0.099	0.101	0.097
125	0.010	0.009	0.011	0.012	0.051	0.050	0.051	0.050	0.102	0.100	0.103	0.100
150	0.010	0.010	0.011	0.012	0.050	0.050	0.052	0.049	0.101	0.100	0.102	0.100
200	0.010	0.010	0.011	0.012	0.050	0.050	0.050	0.050	0.099	0.102	0.103	0.100
250	0.010	0.010	0.011	0.012	0.052	0.051	0.052	0.053	0.102	0.099	0.103	0.098

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (9) and (13) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (9) computed as described in Algorithms 1 and 2 of Section 4 in the main text.