National Technical University of Athens
School of Electrical and Computer Engineering
Computer Science Division

Computation and Reasoning Laboratory

Money Burning Mechanism Design for Facility Location

Diploma Thesis

of

Faidra Georgia Monachou

Supervisor: Dimitrios Fotakis
Assistant Professor N.T.U.A.

Athens, November 2015

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Approved by the committee on the 3rd of November 2015.

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Περίληψη

Στη παρούσα διπλωματική εργασία, μελετάμε προβλήματα χωροθέτησης από τη σκοπιά του σχεδιασμού μηχανισμών. Επικεντρωνόμαστε σε παίγνια, όπου ένα ορισμένο πλήθος από εγκαταστάσεις οι οποίες προσφέρουν ένα συγκεκριμένο τύπο υπηρεσίας, πρόκειται να τοποθετηθούν σε ένα μετρικό χώρο, με βάση τη ζήτηση που δηλώνουν ένα σύνολο από στρατηγικούς παίκτες. Οι παίκτες έχουν ως στόχο να ελαχιστοποιήσουν το κόστος σύνδεσής τους, δηλαδή το γινόμενο της ζήτησης τους επί την απόσταση τους από την κοντινότερη εγκατάσταση και για αυτό ενδέχεται να δηλώσουν ψευδή ζήτηση. Ενδιαφερόμαστε ιδίως για μηχανισμούς που είναι φιλαλήθεις, δηλαδή εξασφαλίζουν ότι κανένας παίκτης δε θα ωφεληθεί δηλώνοντας ψευδή ζήτηση, δε χρησιμοποιούν χρηματικά ανταλλάγματα και προσεγγίζουν το βέλτιστο συνολικό κόστος. Πραγματοποιούμε μια ανασκόπηση της βιβλιογραφίας σχετικά με το σχεδιασμό μηχανισμών χωρίς χρήματα για προβλήματα χωροθέτησης. Επίσης, παρουσιάζουμε πρόσφατα αποτελέσματα που αφορούν το σχεδιασμό μηχανισμών με χρήση money burning, όπου οι πληρωμές που οφείλουν να καταβάλουν οι παίκτες είναι στη μορφή μειωμένης ποιότητας της υπηρεσίας που λαμβάνουν. Τέλος, αναπτύσσουμε μια νέα προσέγγιση στο σχεδιασμό φιλαληθών μηχανισμών για το συγκεκριμένο πρόβλημα χωροθέτησης, χρησιμοποιώντας την τεχνική money burning. Παρουσιάζουμε τα αποτελέσματά μας για την ευθεία των πραγματικών αριθμών, όσο και για γενικούς μετρικούς χώρους.

Λέξεις Κλειδιά – Σχεδιασμός Φιλαληθών Μηχανισμών, Χωροθέτηση, Money Burning

Abstract

In this thesis, we study Facility Location problems from a mechanism design perspective. We focus on games, where a number of facilities that provide a certain type of service, are placed in a metric space based on the demands reported by strategic agents. The agents seek to minimize their connection cost, namely the product of their demand times the distance of their location to the nearest facility, and may misreport their demand. We are interested in mechanisms that are truthful, i.e. ensure that no agent can benefit from misreporting his demand, do not resort to monetary transfers, and approximate the optimal social cost. We survey recent results in Mechanism Design without Money for Facility Location problems. We also survey Money Burning Mechanism Design, where the payments charged to the agents are not in the form of monetary transfers, but service degradation. We then develop an approach to designing both deterministic and randomized truthful mechanisms for our Facility Location game, using a money burning technique. We present our results for the real line setting, as well as general metric spaces.

Keywords— Mechanism Design without Money, Facility Location, Money Burning

Ευχαριστίες

Ολοκληρώνοντας τις σπουδές μου στο ΕΜΠ, θα ήθελα να ευχαριστήσω τον επιβλέποντα καθηγητή μου κ.Φωτάκη για την εμπιστοσύνη που έδειξε στις δυνατότητές μου, τη συνεργασία μας, καθώς και την ενθάρρυνσή του σε όλη την ακαδημαϊκή μου πορεία μέχρι τώρα. Την αγάπη μου για τη Θεωρητική Πληροφορική την οφείλω τόσο στον ίδιο, όσο και στον ενθουσιασμό και στη διδασκαλία του κ.Ζάχου από τα πρώτα κιόλας εξάμηνα της σχολής.

Θα ήθελα επίσης να ευχαριστήσω τους φίλους και την οικογένειά μου, και πάνω από όλους τη μητέρα μου για όσα έχει κάνει και συνεχίζει να κάνει για μένα. Τέλος, ευχαριστώ τον Κωστή που είναι πάντα δίπλα μου σε όσα και αν περάσαμε μαζί.

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Chapter 1

Introduction

The advent of the Internet has radically changed the way people interact with each other. This massive network of networks can be viewed as an interconnected system of strategic entities, in which the multiple participants, each one optimizing their own objective, affect each other's outcome. However, the competing incentives of different participants may lead to complex environments that unavoidably create many possible courses of action; agents may even intentionally misreport data in order to optimize their actions and achieve their goals.

From a *Computer Science* perspective, modelling and analyzing systems that are the outcome of the strategic interactions among agents, required a different approach; to optimize system-wide objectives in stategic environments, classical algorithm design is no longer enough. Meanwhile, *Game Theory* studies the general principles of interactions between rational agents and provides solution concepts that prescribe how the rational agents should act when they try to maximize their own payoffs.

The fruitful combination of Game Theory and algorithms gave rise to the development of *Algorithmic Mechanism Design*. Given a set of players with private utilities and a desired outcome, Mechanism Design aims to design a game such that at the equilibrium of the designed game no agent has an incentive to deviate. Therefore, Mechanism Design is sometimes called reverse Game Theory; Game Theory studies the outcome of systems of selfish agents taking the rules of the game as given, while Mechanism Design looks to find processes that can give the agents the incentives to behave in a desired and predictable way, i.e. transforms the selfish behaviour of agents into a favorable outcome for themselves, and the designer of the mechanism as well. In other words, the *mechanism* is a function which based on the information received from the agents, chooses an outcome and probably an appropriate payment scheme.

Computational efficiency issues and algorithmic methods have later been incorporated into Mechanism Design introducing the subfield of Algorithmic Mechanism Design. As a result, the problem of designing a mechanism became actually a *computational problem.*

Algorithmic mechanism design deals with game-theoretic versions of a plethora of optimization problems such as routing in computer networks, auctions and scheduling. In this thesis, we focus on *Facility Location* games. The problem of Facility Location is classical and has a long history within Operations Research, Social Choice Theory and Algorithms. During the past few years, it has received considerable attention in the field of Algorithmic Mechanism Design.

Many variants of the Facility Location problem have been studied under the umbrella of mechanism design. In general, in a Facility Location game one or more facilities are placed in a metric space to serve a set of agents. The goal of the mechanism designer is to guarantee truthfulness, i.e. that agents report to the mechanism their true private information, as well as to optimize a related objective function. In the classical Facility Location setting, each agent's private information is his location. An agent may have the incentive to lie in order to be as close to a facility as possible, thus minimizing his individual cost; the cost of an agent is defined as the distance between his true location and the closest facility. The objective of the mechanism designer may be the minimization of the *social cost* (namely the sum of agents' individual costs) or the maximum cost. For instance, this model fits naturally in real-life problems such as building a public building such as a library based on the preferences of the residents of a city. Nevertheless, this setting is just one of the many other variants of Facility Location that have also been extensively studied.

Although the classic Mechanism Design relies on payments to obtain optimal results, in some settings, including Facility Location games, money transfers are impossible or undesirable. Procaccia and Tennecholtz [30] first proposed that approximation can ensure strategy-proofness without resorting to payments, thus initiating the research on *Approximate Mechanism Design without Money.* Staying within the confines of this *money-free* framework, in this thesis we are interested in the study of *approximate*, *truthful* mechanisms which do not use money.

In addition to Approximate Mechanism Design without Money, we are also interested in *Money Burning* Mechanism Design, introduced by Hartline and Roughgarden in [20]. Inspired by the ability of computer systems such as networks to degrade service quality, Hartline and Roughgarden consider an other way of overriding the use of monetary payments: agents are are required to burn money and the payments take the form of wasted resources instead of actual money. In the last part of this thesis, we apply money burning to obtain truthful approximate mechanisms for a Facility

Location setting.

Scope of this Thesis. In this thesis, we survey classic Mechanism Design literature as well as recent results in Mechanism Design without Money for Facility Location problems. We propose a new Facility Location setting, where a number of facilities that provide a certain type of service, are placed in a metric space based on the demands reported by strategic agents. The agents seek to minimize their connection cost, namely the product of their demand times the distance of their location to the nearest facility. The locations of agents are public knowledge, but their splittable demands are private. Our goal is to design mechanisms that are truthful and approximate the optimal social cost in this Facility Location setting.

1.1 Organization of the Thesis

The purpose of *Chapter 1* is to introduce the basic concepts of Algorithmic Mechanism Design. We will present several classic results and related work on the field. In the last two sections of this Chapter, we focus on two well-known mechanisms: the VCG mechanism and the Exponential Mechanism. We present their properties, applying the theory of *Algorithmic Mechanism Design*. In Chapter 4, we will revisit the VCG mechanism, but from a different perspective.

In *Chapter 2*, we examine problems related to *Facility Location*. We survey the most important results in the rich literature on Facility Location games. We also make a short introduction to *Social Choice Theory* and to results relevant to Facility Location. We are particularly interested in Mechanism Design without Money, thus we focus our attention to recent research in Approximate Mechanism Design without Money, where approximation is used to obtain truthfulness, while sacrificing a fraction of the optimal social cost.

In *Chapter 3*, we survey *Money Burning*. We summarize the essential parts of the paper [20] by Hartline and Roughgarden, who introduced the framework of Money Burning. We then briefly refer to several recent papers, where a money burning technique is applied.

Finally, in *Chapter 4*, we propose a novel approach to designing truthful mechanisms for Facility Location. We consider a Facility Location setting that has not been studied from a mechanism design perspective before, and we combine the main concept of Money Burning with results in Facility Location literature, in order to design mechanims with payments in the form of additional individual (and social) cost. We present a randomized as well a deterministic mechanism and study their properties in the real line setting as well as in general metric spaces.

1.2 Mechanism Design

In this Chapter, we mainly survey some of the classic results in mechanism design and define most of the notation and terminology used in the classic mechanism design literature. We also present some examples of mechanisms, such as the seminal VCG Mechanism and the Exponential Mechanism.

1.2.1 Notation and Basic Concepts in Mechanism Design

Informally, a mechanism could be described in one sentence as a pair of an algorithm and a payment rule that prescribe an allocation and a payment method for the agents. Payments are usually required to achieve computational efficiency. However, monetary transfers are sometimes difficult to be implemented or even illegal to be used. Assuming the existence of payments, we formally define the following model for a general mechanism design problem:

The basic setting. There is a set of n *agents* (players) and a set O of feasible *outcomes* of the mechanism, where outcome is an allocation combined with a price charged to each agent i. Each agent i has a private *valuation* $v_i : O \to [0,1]$ for the outcomes. If an agent does not want to be truthful, he may not reveal its truthful valuation to the mechanism, but strategically report a false valuation $b_i : O \to [0,1]$, that we will call the agent's *bid*. The agents submit their bids to the mechanism. The central entity or the designer of the mechanism collects the reported bids and chooses an allocation rule $x(\mathbf{b}) \in O$ and a payment rule $p(\mathbf{b}) \in \mathbb{R}^n$. Each agent i also has a *quasi-linear utility* $u_i(\mathbf{b}) = v_i(x(\mathbf{b})) - p_i(\mathbf{b})$.

Notation. Let $\mathbf{v}$ denote the vector of agents' valuations and $\mathbf{b}$ the vector of reported bids. We will denote by v_{-i} the vector of all agents' valuations except i and by b_{-i} the vector of all agents' bids except i.

As previously mentioned, an agent may strategically misreport its valuation to satisfy its goals. Mechanism Design theory focuses on the simple idea of *incentive compatibility* or *truthfulness*, i.e. the allocation and payment rules are such that each agent's best strategy is to thruthfully reveal its private information to the mechanism. The agent maximizes its utility by bidding its real valuation regardless of other agents' bids. Formally,

Definition 1.2.1. *An mechanism is* incentive compatible *or* truthful *iff for each agent $i \in [n]$ with valuation v_i, $\forall b_{-i}$:*

$$v_i \in \arg\max_{b_i}\{v_i(x(b_i, b_{-i})) - p_i(b_i, b_{-i})\}$$

An other naturally desirable property of a mechanism would be *individual rationality.* Agents have to pay money to the mechanism, but this payment should not be greater than the valuation they have for the item they get.

Definition 1.2.2. *An mechanism is* individually rational *iff the (expected) utility of each agent $i \in [n]$ is always non-negative assuming truthful bidding, i.e.*

$$v_i(x(v_i, b_{-i})) - p_i(v_i, b_{-i}) \geq 0$$

The Revelation Principle for dominant strategies. It is natural to assume that each agent will follow the best possible strategy that leads to a payoff larger than any other, regardless of the strategies played by the other agents. Such a strategy is a *dominant* strategy.

Definition 1.2.3. *A bid profile b_i is a* dominant strategy *for agent $i \in [n]$ if $\forall b_{-i}, b'_i$:*

$$u(b_i, b_{-i}) \geq u(b'_i, b_{-i})$$

In a *direct revelation mechanism* each agent is asked to report its individual information to the mechanism. Thanks to the powerful Revelation Principle, we can focus our attention on the set of direct revelation mechanisms, where the bidders reveal their true valuation for an outcome $o \in O$. The Revelation Principle states that for every mechanism in which every agent has a dominant strategy, there is an equivalent truthful direct revelation mechanism.

Theorem 1.2.4 (Revelation Principle, [28])**.** *If there exists an mechanism that implements an allocation rule $x(\cdot)$ in dominant strategies, then there exists a direct truthful mechanism that implements $x(\cdot)$.*

Myerson's Lemma. Myerson's Lemma belongs to the foundation theory of Mechanism Design and it is extremely powerful. The Lemma states that a mechanism is dominant strategy truthful if and only if the allocation rule is monotone. On the top of that, there is no ambiguity in how to assign payments to achieve truthfulness. The resulting payment formula has a simple and intuitive format. Indeed, we can interpet the payment formula as follows: as agent i increases his reported value per unit of resource, he pays for each additional part of resource at a rate equal to the minimum report needed to win that part.

Lemma 1.2.5 (Myerson's Lemma, [27])**.** *A mechanism for a single-parameter problem is dominant strategy truthful if and only if for every agents i and fixed reported bids b_{-i} of other agents,*

- *$x_i(b_i)$ is a monotone non-decreasing function of agent i's bid b_i*
- *the payment rule is given by the explicit formula*

$$p_i(b_i, b_{-i}) = \int_0^{b_i} z \frac{\mathrm{d}}{\mathrm{d}z} x_i(z, b_{-i}) dz \tag{1.1}$$

assuming $p(0) = 0$.

Gibbard-Satterthwaite Impossibility Theorem. An allocation rule is *onto* if for any outcome $o \in O$ there is a valuation $v : O \to [0, 1]$ such that $x(v) = o$. An allocation rule $x(\cdot)$ is *dictatorial* if there is no agent i such that for any bid vector b_{-i}, $x(v_i, b_{-i}) = \arg\max_o v_i(o)$. The main impossibility result states that:

Theorem 1.2.6 (Gibbard-Satterthwaite, [18, 32])**.** *There is no thruthful allocation rule $x(\cdot)$ that satisfies all of the following conditions:*

- *there are at least three different outcomes, $|O| \geq 3$*
- *the allocation rule $x(\cdot)$ is onto*
- *the allocation rule $x(\cdot)$ is not dictatorial*

The above result implies that the general mechanism design is an impossible problem. However, we can avoid this negative result if we modify the setting by introducing money. The existence of money is a reasonable and convenient solution in many settings and payments of the agents will extend the range of possible solutions. A second direction is to restrict the domain of agents' utility functions; a very common assumption is that agents have quasilinear utilities.

Objective functions. In a mechanism design problem, the designer has to choose an outcome $o \in O$ that depends on the agents' private valuations. The quality of a possible outcome can be measured using various objectives; considering different objectives there may exist different desired outcomes.

The most common objective fuction is the *social welfare*. In mechanism design, the social welfare is a measure of how much the agents value an outcome $o \in O$. For example, in auction design the goal is to award the item to the agent with the highest valuation, maximizing the social welfare.

Definition 1.2.7. *For an outcome $o \in O$, the* social welfare *of a mechanism is defined as follows:*

$$SW(o) = \sum_{i=1}^{n} v_i(o)$$

1.3 Examples of mechanisms

1.3.1 The VCG Mechanism

For the simple setting of a single item for sale and n bidders, the famous *Vickrey mechanism* or sealed-bid second-price auction is a simple, yet efficient solution. In a Vickrey auction, the item is awarded to the highest bidder, but the amount the mechanism charges the winner is the second highest bid. Every losing bidder pays nothing. Under these allocation and payment rules, a bidder has an incentive to report its true valuation for the item, and thus social welfare optimality is achieved. If the valuation v_i of bidder i is the highest among all bids, then the bidder i maximizes its utility only by bidding thruthfully and winning. If there exists a bid $b_j, j \neq i$ such that $b_j > v_i$, then bidder i has no incentive to receive negative utility by overbidding.

The main idea behind the Vickrey mechanism's payment rule is charging every bidder i his *"externality"* - or the welfare loss the presence of the bidder i causes to the other bidders. In a sequence of works by Vickrey [37], Clarke [10] and Groves [19], this idea becomes the key to solving the more general setting of multiple items for sale, where a bidder may have different valuations for different items or outcomes. A formal statement of the *Vickrey-Clarke-Groves (VCG) mechanism* is the following:

The VCG mechanism requires computing the optimal solution $\arg\max_{o \in O} \sum_{i=1}^{n} b_i(o)$ which usually can not be implemented in polynomial time. Despite its computational inefficiency, the VCG mechanism is a truthful mechanism that has many strong properties and can be applied to a large range of settings. For example, it satisfies the property of *no positive transfers*, where no player is paid anything by the mechanism, The VCG mechanism is a mechanism (x, p) where

- *Allocation rule*: $x(\mathbf{b}) = \arg\max_{o \in O} \sum_{i=1}^{n} b_i(o)$
- *Payment rule*: $p_i(\mathbf{b}) = \max_{o \in O} \sum_{j \neq i} b_j(o) - \sum_{j \neq i}^{n} b_j(x(\mathbf{b}))$

Mechanism 1.1: The VCG Mechanism

i.e. it holds that for every agent i and every bid profile $\mathbf{b} : p_i(\mathbf{b}) \geq 0$. This property follows immediately from the definition of p_i.

Under the assumption that $b_i \geq 0$, the VCG mechanism is also individually rational. To prove this, fix an agent i and let $\omega^* = x(\mathbf{b}) = \arg\max_{o \in O} \sum_{i=1}^{n} b_i(o)$ and $\omega = \arg\max_{o \in O} \sum_{i \neq j} b_i(o)$. We have that

$$u_i(\mathbf{b}) = v_i(x(\mathbf{b})) - p_i(\mathbf{b}) = [v_i(\omega^*) + \textstyle\sum_{j \neq i} b_j(\omega^*)] - \sum_{j \neq i} b_j(\omega) \geq \\ \textstyle\sum_{j=1}^{n} b_j(\omega^*) - \sum_{j=1}^{n} b_j(\omega) \geq 0,$$

since $b_i(\omega) \geq 0$ and ω^* was chosen as to maximize $\arg\max_{o \in O} \sum_{i=1}^{n} b_i(o)$.

Theorem 1.3.1. *The VCG mechanism is truthful.*

Proof. Let $\omega^* = x(\mathbf{b}) = \arg\max_{o \in O} \sum_{j=1}^{n} b_j(o)$. We have to prove that for each agent i, $\forall v_i, b_i, b_{-i}$:

$$u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$$

Indeed,

$$u_i(b_i, b_{-i}) = v_i(\omega^*) - p_i(b_i, b_{-i}) = [v_i(\omega^*) + \textstyle\sum_{j \neq i} b_j(\omega^*)] - [\max_{o \in O} \sum_{j \neq i} b_j(o)]$$

The term $max_{o \in O} \sum_{j \neq i} b_j(o)$ does not depend on b_i, so agent i can only influence - through the choice of ω^* - the term $[v_i(\omega^*) + \sum_{j \neq i} b_j(\omega^*)]$ in order to maximize its utility $u_i(b_i, b_{-i})$. If we set $b_i = v_i$ then the term $[v_i(\omega^*) + \sum_{j \neq i} b_j(\omega^*)]$ becomes equal to $\sum_{j=1}^{n} b_j(\omega^*)$, where $\omega^* = argmax_{o \in O} \sum_{j=1}^{n} b_j(o)$ is the outcome chosen by the mechanism. Therefore, the best strategy for agent i is bidding its true valuation v_i. $\square$

1.3.2 The Exponential Mechanism

Algorithmic Mechanism Design has mainly focused on the design of incentive compatible mechanisms. However, thruthful bidding may not always be a dominant strategy for an agent as he may achieve its optimal payoff by bidding a false valuation. Moreover, an agent might not bid truthfully for privacy reasons; bidding its true valuation implies that the agent reveals personal - probably sensitive or important - information. As a result, we need to design mechanisms that are both incentive compatible and near optimal subject to a social welfare objective and at the same time protect the privacy of agent's personal information.

McSherry and Talwar [24] introduced the *exponential mechanism* and Huang and Kannan [21] used this result to design an incentive compatible and individually rational mechanism that still ensures differential privacy. The ϵ- *differential privacy* of a mechanism means that the probability of an outcome can increase by a factor of $\exp(\epsilon)$ or less when a single agent misrepresents its information.

Definition 1.3.2. *A mechanism is* ϵ-differentially private *if for any set of outcomes* $S \subseteq O$*, and for any two valuation vectors* $\mathbf{v} = (v_1, ..., v_i, ..., v_n)$ *and* $\mathbf{v}' = (v_1, ..., v_i', ..., v_n)$ *that differ only in the value of a single agent* i*, we have*

$$\mathbf{Pr}[x(\mathbf{v}) \in S] \leq \exp(\epsilon) \cdot \mathbf{Pr}[x(\mathbf{v}') \in S]$$

Given an arbitrary range O of outcomes, a set of n inputs each from a domain D and a *quality function* $q : D^n \to O$ that maps a pair of an input data set D and an outcome $o \in O$ to a real number, the exponential mechanism of McSherry and Talwar [24] is a general differential privacy mechanism. Intuitively, the purpose of using a quality function is to assign a score to a pair of an input $x \in D^n$ and an output $o \in O$, giving higher preference to outcomes with higher scores.

If D is a reported bid profile and the quality function q is the social welfare $\sum_{i=1}^n v_i(o)$, then the Exponential Mechanism $EXP_\epsilon^O(\mathbf{v})$ with Social Welfare Quality Score is the following:

$$\textit{Choose outcome } o^* \in O \textit{ with probability } \Pr[o^*] = \frac{exp(\frac{\epsilon}{2}\sum_{i=1}^n v_i(o^*))}{\sum_{o \in O} exp(\frac{\epsilon}{2}\sum_{i=1}^n v_i(o))}$$

The exponential mechanism can be a useful tool because of two basic properties: it is ϵ-differential private and chooses with high probability a high welfare outcome $o \in O$. The next theorem illustrates this properties.

Theorem 1.3.3. ([24], [12]) *Let o be the output of the Exponential Mechanism* $EXP_\epsilon^O(\mathbf{v})$ *and* $OPT(\mathbf{v}) = max_{o^* \in O} \sum_{i=1}^n v_i(o^*)$. *The exponential mechanism is* ϵ*-differentially private and ensures that*

$$\Pr[\sum_{i=1}^n v_i(o) \leq OPT(\mathbf{v}) - \frac{2 \cdot \log |O|}{\epsilon} - \frac{2t}{\epsilon}] \leq \exp(-t)$$

Huang and Kannan [21] used the connection between the exponential mechanism and the Gibbs measure to introduce a payment scheme such that the exponential mechanism becomes truthful. The payment rule is defined below in Mechanism 1.2, where $H(D) = \sum_{r \in R} \Pr_D[r] \cdot \log \frac{1}{\Pr_D[r]}$ is the *Shannon entropy* of a distribution D.

1. Choose outcome $o^* \in O$ with probability $\Pr[o^*] = \dfrac{\exp(\frac{\epsilon}{2} \sum_{i=1}^n b_i(o^*))}{\sum_{o \in O} \exp(\frac{\epsilon}{2} \sum_{i=1}^n b_i(o))}$
2. For $1 \leq i \leq n$, charge agent i price

   $p_i = -\mathbf{E}_{o \sim EXP_\epsilon^O(\mathbf{b})}[\sum_{k \neq i} b_k(o)] - \frac{2}{\epsilon} H(EXP_\epsilon^O(\mathbf{b})) + \frac{2}{\epsilon} \ln(\sum_{o \in O} \exp(\frac{\epsilon}{2} \sum_{k \neq i} b_k(o)))$

Mechanism 1.2: The incentive compatible Exponential Mechanism with payments [21]

We note that the payment formula can be written in a *VCG-like form* as

$$p_i = (\mathbf{E}_{o \sim EXP_\epsilon^O(b_{-i})}[\sum_{k \neq i} b_k(o)] + \frac{2}{\epsilon} H(EXP_\epsilon^O(b_{-i}))) - (\mathbf{E}_{o \sim EXP_\epsilon^O(\mathbf{b})}[\sum_{k \neq i} b_k(o)] + \frac{2}{\epsilon} H(EXP_\epsilon^O(\mathbf{b})))$$

Indeed,

$$\cdot \mathbf{E}_{o \sim EXP_\epsilon^O(b_{-i})}[\sum_{k \neq i} b_k(o)] + \frac{2}{\epsilon} H(EXP_\epsilon^O(b_{-i})) = \frac{2}{\epsilon} \sum_{o \in O} \Pr[o] \cdot (\frac{\epsilon}{2} \sum_{k \neq i} b_k(o) + \ln \frac{1}{\Pr[o]})$$

$$= \frac{2}{\epsilon} \sum_{o \in O} \Pr[o] \cdot \ln(\frac{\exp(\frac{\epsilon}{2} \sum_{k \neq i} b_k(o))}{\Pr[o]}) = \frac{2}{\epsilon} (\sum_{o \in O} \Pr[o]) \cdot \ln\, (\exp(\frac{\epsilon}{2} \sum_{k \neq i} b_k(o)))$$

$$= \frac{2}{\epsilon} \ln(\sum_{o \in O} \exp(\frac{\epsilon}{2} \sum_{k \neq i} b_k(o)))$$

Theorem 1.3.4. ([21]) *The Exponential Mechanism with payments is truthful and individually rational.*

Chapter 2

Mechanism Design for Facility Location Games

In this Chapter, we briefly review the most relevant results in the rich literature on Facility Location games. The basic underlying setting of the papers that we present, consists of n agents located in metric space and k facilities that need to be placed. However, there exist some differences among the various settings, which we analyze throughout this Chapter. At the end of the Chapter, we summarize the main models and variants of Facility Location games.

2.1 Social Choice Theory: Facility Location and Single Peakedness

The first results that are relevant to Facility Location games come from the field of Social Theory. For this reason, this section starts with a brief summary of the basic definitions about social choice functions.

Social Choice. *Social Choice Theory* is the mathematical study of collective decision processes and procedures, i.e. making decisions based on the preferences of multiple agents. In the general setting, there is a set N of n agents and a set A of alternatives. Each agent has a private linear order $\succ_i \in L$ over the alternatives in A. A *social choice function* $f : L^n \to A$ is a mapping from agents' preferences to an alternative.

Some properties of the social functions are the following:

- *Unanimous*: A social choice function f is unanimous if, when all player prefer a certain outcome more than anything else, then that outcome must be the alternative chosen by the mechanism. Formally, if $\exists a \in A$ such that $\forall b \in A$ and $\forall i \in N$ $a \succ_i b$, then $f(\succ_1, ..., \succ_n) = a$.

- *Strategyproof* or *Truthful*: A social choice function f is strategyproof if for all $\succ_1, ..., \succ_n$ and for each agent i: $f(\succ_1, ..., \succ_i, ..., \succ_n) \succ_i f(\succ_1, ..., \succ_i', ..., \succ_n)$, for every $\succ_i'$.

- *Dictatorship*: An agent i is a dictator in a social choice function f if for all $\succ_1, ..., \succ_n$: $f(\succ_1, ..., \succ_n) = a$ where $a \succ_i b$ $\forall b$, $b \neq a$.

- *Onto*: A social choice function f is onto if $\forall a \in A$, $\exists x \in L^n$ such that $f(x) = a$.

It becomes clear that there is a strong relation between Social Choice and Mechanism Design for Facility Location.

The Social Choice setting (related to Facility Location). [26] We consider a set of n agents (n is odd) and a set A of *alternatives*. The set U_i of agent i's possible preferences is a subset of A. We say that alternative *a defeats by majority vote* alternative b if the set of agents that strictly prefer a to b are more than the half of the number of agents. We assume that the sets $U_1, ..., U_n$ are such that for every profile $(u_1, ..., u_n) \in U_1 \times ... \times U_n$ there exists a *unique Condorcet winner*, i.e. an alternative $C(u_1, ..., u_n)$ that defeats any other alternative by majority rule.

A setting where a Condorcet winner always exists is the *real line* ($A = \mathbb{R}$) and we restrict the setting to the real line. An other assumption we make is that agents' preferences have a single most preferred point on the real line. Formally, each agent has a *single peak* $x_i \in A$ such that for all $b < a \leq x_i$: $a \succ_i b$ and for all $x_i \leq a < b$: $a \succ_i b$. We assume that each agent submits only his peak.

In [26], Moulin states that

Theorem 2.1.1. [26] *Assuming single peakedness, a rule f is strategy-proof, onto and anonymous if and only if there exist $a_1, a_2, ..., a_{n-1} \in [0, 1]$ such that for all peaks $(x_1, ..., x_n) \in R^n$,*

$$f(x_1, ..., x_n) = median(x_1, x_2, ..., x_n, a_1, ..., a_{n-1})$$

To prove the strategyproofness, let o be the outcome of the rule. If $o = x_i$ for an agent i, then this outcome implies a zero cost for this agent. Otherwise, without loss of generality we can assume that $o > x_i$. In this case, if agent i reports his peak at

$x'_i \leq o$, then he can not manipulate the mechanism. However, if i submits a peak at $x'_i > o$, then according to the median rule the facility would be placed at a location that is more distant to him than o is.

2.2 Approximate Mechanism Design without Money

As discussed in Chapter 1, the Gibbard- Satterthwaite Theorem shows that for non-restricted settings any non-trivial truthful mechanism is dictatorial. One way to overcome the problem is by introducing monetary payments. Mechanism Design with Money deals with mechanisms which employ payments. When monetary transfers are allowed, the mechanism designer can many times achieve optimality, truthfulness and efficiency. For instance, the VCG mechanism for our Facility Location games is not only strategyproof, but also achieves an optimal solution; of course, the only restriction is that payments are allowed. However, in many social choice settings, including Facility Location problems, monetary payments may be unavailable due to legal or ethical considerations. In addition, the VCG mechanism presupposes the computation of the optimal solution for a given objective function, but for many problems finding the optimal solution in a computationally efficient way is impossible. Therefore, many researchers have turned their interest to mechanisms without monetary payments.

There are types of problems where there exists no optimal truthful mechanism without money. Nevertheless, we can sacrifice to some extent the optimality of the solution in order to achieve truthfulness without money. This was first proposed as *Approximate Mechanism Design without Money* and was initiated by Procaccia and Tennenholtz in their seminal paper [30, 31].

For the next sections, we formally define the setting as follows.

The basic setting (Facility Location). Agents and facilities are located in a *metric space* (X, d), where $d : X \times X \to \mathbb{R}$ is the distance function. The function d is a metric, i.e. it is non-negative, symmetric and satisfies the triangle inequality. There exists a set N of n agents and each agent i has a location $x_i \in X$, which is his private information. We refer to the tuple $(x_1, ..., x_n)$ as the *location profile*. For a location profile $\mathbf{x}$, let x_{-i} denote the locations of all agents except for agent i.

Agent i reports his location to the mechanism M. A number of k facilities should be placed based on agents' reported locations. A *deterministic mechanism* is a mapping from a location profile to to a tuple consisting of the locations of the k facilities. A *randomized mechanism* is a probability distribution over deterministic mechanisms.

In the classical setting, each agent has a cost function $cost[x_i, M(\mathbf{x})]$ that he wants to minimize. A mechanism M is *strategyproof* if for any location profile $\mathbf{x}$, any agent i and any location y, $cost[x_i, F(\mathbf{x})] \leq cost[x_i, F(y, x_{-i})]$. A mechanism M is *group-strategyproof* if for any location profile $\mathbf{x}$, any non-empty set of agents $S \subseteq N$, and any location profile $\mathbf{y}_S$ for them, there exists some agent $i \in S$ such that $cost[x_i, M(\mathbf{x})] \leq cost[x_i, M(\mathbf{y}_S, \mathbf{x}_{-S})]$.

2.2.1 1-Facility and 2-Facility Location

2.2.1.1 Facility Location on the Real Line

In [31], the authors use approximation to obtain truthfulness without payments and apply their framework to Facility Location games. In the general setting, agents are located on the real line and submit their locations to the authority. Given their location profiles, the mechanism selects the location of the facility (or the facilities). In the basic setting, only one facility must be located, but the results are extended to the 2-Facility Location game on the real line and then to domains where each agents controls multiple facilities. We will focus only on the first two facility location settings.

The setting. Procaccia and Tennenholtz study a limited version of the general setting. The underlying metric space is the real line and the *cost function* of agent i is $cost(\mathbf{y}, x_i) = \min\{|y_1 - x_i|, |y_2 - x_i|\}$, where y_1 and y_2 are the locations of facilities. In games where only one facility y is located, $cost(y, x_i) = |y - x_i|$. Two objective functions are considered: minimizing *social cost* and minimizing *maximum cost* in a strategyproof way. The social cost with respect to a location profile $\mathbf{x} \in \mathbb{R}^n$ is defined as $sc(\mathbf{y}, \mathbf{x}) = \sum_{i \in N} cost(\mathbf{y}, x_i)$. The maximum cost is $mc(\mathbf{y}, \mathbf{x}) = \max_{i \in N} cost(\mathbf{y}, x_i)$.

1-Facility Location. The obvious approach to solve the 1-Facility Location game on the real line is to choose the *median* location $med(\mathbf{x})$ in $\mathbf{x}$. Fortunately, this mechansism is also truthful; the proof is as follows. If n is odd, then any agent that is to the left of $med(\mathbf{x})$ has higher cost than that of $med(\mathbf{x})$. The same holds for any agent to the right of $med(\mathbf{x})$. If n is even, then any point in the interval $[x_{n/2}, x_{n/2} + 1]$ is an optimal Facility Location; in this case we consider $med(\mathbf{x}) = x_{n/2}$. Consequently, this mechanism is strategyproof and achieves an approximation ratio of 1 for the social cost. This argument is easily generalized for coalitions of agents, thus the mechanism is also group-strategyproof.

The Two-Extremes Mechanism for 2-Facility Location. For a location profile $\mathbf{x} \in \mathbb{R}^n$, let $lt(\mathbf{x})$ denote the leftmost location in $\mathbf{x}$, namely $lt(\mathbf{x}) = \min_{i \in N} x_i$. The rightmost location is $rt(\mathbf{x}) = \max_{i \in N} x_i$. For the 2-Facility Location game, the mechanism that outputs the optimal solution is not truthful. A group strategyproof $(n-1)$-approximation mechanism is given by choosing $rt(\mathbf{x})$ and $lt(\mathbf{x})$.

Lu et al. in [22] provided a *lower bound* for the 2-Facility Location Game: any *deterministic* strategyproof mechanism for the 2-facility game in the line metric space has an approximation ratio of at least $\frac{n-1}{2}$. The same holds for any metric space which can be locally viewed as a line, such as the circle. The Two-Extremes Mechanism is surprisingly the only deterministic anonymous strategyproof mechanism with a bounded approximation ratio for 2-Facility Location on the line. [17]

2.2.1.2 1-Facility Location on Trees, Circles and Graphs

Trees. In [33], Schummer and Vohra study rules that choose a location on a network, based on agents' single-peaked preferences. Later in [1], Alon et al. provide a mechanism for the 1-Facility Location problem on trees. The mechanism that selects the median of the reported locations is group-strategyproof and achieves the optimal social cost. Finding the median in a tree is simple. We first fix an arbitrary node as the root of the tree. Then, as long as the current node has a subtree that contains more than half of the agents, we move down this subtree, until it not possible to find such a subtree. In this case, we return the current node. Their result follows from similar arguments as the ones given for a median on a line. An agent can only change the location chosen by the mechanism only by pushing the returned Facility Location away from its true location.

Circles and General Graphs. For non-tree networks, the network contains at least a cycle. Any strategy proof mechanism that is onto must is a dictatorship when all agents are located in the cycle [33]. This gives a tight lower bound of $n-1$ on the approximation ratio of any strategyproof mechanism. For randomized mechanisms, Alon et al. design a mechanism that is random dictator, i.e. it chooses one of the agents' locations with probability equal to $\frac{1}{n}$. It follows immediately from the triangle inequality that the random dictator is $(2-2/n)$-approximation mechanism for the social cost. The random dictator is group-strategyproof for the circle, but for general connected graphs if only if the maximum degree is equal to 2.

2.2.1.3 The Proportional Mechanism for the 2-Facility Location game

Lu et al. in [22] also introduced a truthful 4-approximation randomized mechanism for the 2-Facility Location problem in metric spaces. The so called *Proportional Mechanism* proceeds as follows: the first facility is selcted uniformly at ramdom over all reported locations of agents and the second facility is assigned to another agent i with probability proportional to the distance of agent i to the first facility. The approximation ratio of 4 for the proportional mechanism is tight even on the line.

The Proportional Mechanism is strategyproof for the 2-Facility Location, but it is not for 3 or more facilities. An illustrating counterexample for 3 facilities is the following. We consider a location profile where there exist n_0 agents at location 0, n_1 agents at location 1, n_2 agents at location $1+x$ and 1 agent at location $1+x+y$. Let $y = 100$, $n_1 = 50$, $n_2 = 4$ and $x = 10^5$. If n_0 is large enough so that the first facility is always placed at 0. If any of the n_1 agents at location 1 reports $1+x$ (instead of 1) as his true location, then he can manipulate the mechanism.

2.2.1.4 The maximum cost objective function

An other objective that has been considered is minimizing the *maximum cost*. The maximum cost of a Facility Location y with respect to a location profile $\mathbf{x}$ is $mc(y, \mathbf{x}) = max_{i \in N} cost(y, x_i)$, whereas the maximum cost of a distribution P with respect to $\mathbf{x}$ is $mc(P, \mathbf{x}) = \mathbb{E}_{y \sim P}[mc(y, \mathbf{x})]$.

1-Facility Location. [1, 31] For 1-Facility Location games, the approximation ratio achieved by any deterministic mechanism is at least 2. The dictatorship mechanism that chooses the agent at x_1 provides a 2-approximation for the maximum cost (because for all agents $i \in N$: $d(x_1, x_i) \le d(x_1, y) + d(y, x_i) \le 2 \cdot \max\{d(y, x_1), d(y, x_i)\} \le 2 \cdot mc(y, \mathbf{x})$), thus it is the best possible mechanism. Given a location profile $\mathbf{x}$ on the real line setting, a randomized mechanism for the same problem return the leftmost agent $lt(\mathbf{x})$ and the rightmost agent $rt(\mathbf{x})$ with probability $\frac{1}{4}$ respectively and the center $cen(\mathbf{x})$ of the interval $[lt(\mathbf{x}), rt(\mathbf{x})]$ with probability $\frac{1}{2}$. This mechanism a group strategyproof 3/2-approximation mechanism for the maximum cost. When the underlying metric space is a tree, Alon et al. provided a randomized strategyproof $(2 - \frac{2}{n+2})$-approximation mechanism.

Remark 2.2.1. *(VCG-like payments)* We observe that if payments are allowed, we can obtain a truthful optimal solution for the maximum cost, by using VCG-like payments: each agent $i \in N$ pays the distance between the optimal Facility Location when $\mathbf{x}$ is the input location profile and the optimal Facility Location when x_{-i} is the input location profile.

2-Facility Location. [31] For the 2-Facility Location game on the real line, the mechanism that chooses the leftmost $lt(\mathbf{x})$ and the rightmost $rt(\mathbf{x})$ location is a deterministic group-strategyproof 2-approximation mechanism for the maximum cost.

Given a location profile $\mathbf{x}$, let the left boundary location be $lb(\mathbf{x}) = \max\{x_i : i \in N : x_i \leq cen(\mathbf{x})\}$ and the right boundary location be $rb(\mathbf{x}) = \min\{x_i : i \in N : x_i \geq cen(\mathbf{x})\}$. They also define $dist(\mathbf{x}) = \max\{lb(\mathbf{x}) - lt(\mathbf{x}), rt(\mathbf{x}) - rb(\mathbf{x})\}$. A randomized strategyproof 5/3-approximation mechanism for the maximum cost in the two Facility Location on the real line is the following: Compute $dist(\mathbf{x})$ and return $(lt(\mathbf{x}) + dist(\mathbf{x}), rt(\mathbf{x}) - dist(\mathbf{x}))$ with probability 1/6, $(lt(\mathbf{x}), rt(\mathbf{x}))$ with probability 1/2 and $(lt(\mathbf{x}) + dist(\mathbf{x})/2, rt(\mathbf{x}) - dist(\mathbf{x})/2)$ with probability 1/3.

2.2.2 k-Facility Location

2.2.2.1 The Winner-Imposing Proportional Mechanism for k-Facility Location

In [29], Nissim, Smorodinsky and Tennenholtz consider *imposing mechanisms*, namely mechanisms able to penalize lying agents by restricting the set of allowable post-actions for the agents. As a result, liars can not fully exploit their outcome.

This extension to the standard mechanism design model fits naturally in Facility Location games. In [15], Fotakis and Tzamos considered the imposing variant of the k-Facility Location game, where an authority can impose on some agents the facilities where they will be served. In particular, an imposing mechanism requires that an agent must connect only to the facility nearest to his reported location, thus increasing his connection cost if he misreports his location.

Definitions and Notation. The definition of metric distance function d is extended as follows: for $x \in M$ and a non-empty $M' \subseteq M$, $d(x, M') = \inf\{d(x, y) : y \in M'\}$.

A deterministic mechanism F maps a location profile $\mathbf{x}$ to a tuple of non-empty sets $(C, C^1, ..., C^n)$, where $C \subseteq M$ is the facility set of F and each $C^i \subseteq C$ contains the facilities where agent i must connect to. Let $F(x)$ denote the facility set of F and $F^i(\mathbf{x})$ to denote the facility subset of each agent i.

Definition 2.2.2. *A mechanism is* imposing *if each agent i can only connect to the facility in $F(x)$ closest to his reported location, namely where $\{z \in F(\mathbf{x}) : d(x_i, z) = d(x_i, F(\mathbf{x}))\} \subseteq F^i(\mathbf{x})$.*
A mechanism F that allocates facilities to the agents, i.e. $F(\mathbf{x}) \subseteq \{x_1, x_2, ..., x_n\}$, is winner-imposing *if for every agent i, $F^i(\mathbf{x}) = \{x_i\}$ if $x_i \in F(\mathbf{x})$, and $F^i(\mathbf{x}) = F(\mathbf{x})$ otherwise.*

The mechanism. In [15], the authors introduce the winner-imposing version of the Proportional Mechanism [22], to obtain the truthful $4 \log k$-approximation mechanism

WIPROP for the k-Facility Location in metric spaces. The mechanism works in k rounds. For each round $l = 1, ..., k$, let C_l be the set of the l facilities that WIPROP has placed in the first l rounds. Initially, $C_0 = \emptyset$.

INPUT: location profile $(x_1, x_2, ..., x_n)$

Round 1: Choose agent i_1 uniformly at random from N. Place the first facility at x_{i_1}. Connect i_1 to F_1. Set $C_1 = \{x_{i_1}\}$.

Round $l = 2, 3, ..., k$: Select agent i_l with probability equal to $\frac{d(x_{i_l}, C_l)}{\sum_{i \in N} d(x_i, C_l)}$ Place the l-th facility at x_{i_l}. Connect i_l to F_l. Set $C_{l+1} = C_l \cup \{x_{i_l}\}$.

Finally, the set of the facilities that the mechanism outputs, is C_k. Each agent not allocated a facility is served by the facility in C_k that is closest to his true location.

Mechanism 2.1: The Winner-Imposing Proportional Mechanism

Theorem 2.2.3. *For any* $k \geq 1$, WIPROP *is a strategyproof mechanism for the* k-*Facility Location game.*

Proof. For each $l = 0, 1, ..., k$, we let $cost[x_i, f(y, x_{-i})|C_l]$ be the expected connection cost of an agent i at the end of WIPROP, given that i reports the location y and that the facility set of WIPROP at the end of round l is C_l. We have that

$$cost[x_i, f(y, x_{-i})|C_l] = \frac{d(y, C_l)d(x_i, C_l) + \sum_{j \neq i} d(x_j, C_l) cost[x_i, f(y, x_{-i})|C_l \cup \{x_j\}]}{d(y, C_l) + \sum_{j \neq i} d(x_j, C_l)}$$

By induction on l, it can be proved that for any y, any $l = 0, 1..., k$ and any C_l

$$cost[x_i, f(y, x_{-i})|C_l] \geq cost[x_i, f(x)|C_l]$$

implying that the mechanism is truthful.

For the basis case $l = k$, it is easy to see that if agent i's location is not in C_k, his connection cost is $d(x_i, C_k)$ and does not depend on her reported location y.

We inductively assume that

$$cost[x_i, f(y, x_{-i})|C_l] \geq cost[x_i, f(x)|C_l]$$

holds for $l+1$ and any facility set C_{l+1}, and show that it also holds for and any facility set C_l.

If $l \geq 1$,

$$cost[x_i, f(y, x_{-i})|C_l] \geq \frac{d(x_i, y)d(y, C_l) + \sum_{j \neq i} d(x_j, C_l) cost[x_i, f(x)|C_l \cup \{x_j\}]}{d(y, C_l) + \sum_{j \neq i} d(x_j, C_l)} \geq$$

$$\geq \frac{d(x_i, y)d(y, C_l) + (d(x_i, C_l) + \sum_{j \neq i} d(x_j, C_l)) cost[x_i, f(x)|C_l]}{d(y, C_l) + \sum_{j \neq i} d(x_j, C_l)}$$

If $d(x_i, C_l) \geq d(y, C_l)$, the previous equation implies that $cost[x_i, f(y, x_{-i})|C_l] \geq cost[x_i, f(x)|C_l]$. Otherwise, we continue and get that

$$cost[x_i, f(y, x_{-i})|C_l] > \frac{d(x_i, y) + d(x_i, C_l) + \sum_{j \neq i} d(x_j, C_l) cost[x_i, f(x)|C_l]}{d(y, C) + \sum_{j \neq i} d(x_j, C_l)} \geq cost[x_i, f(x)|C_l]$$

For $l = 0$,

$$cost[x_i, f(y, x_{-i})] \geq \frac{1}{n} cost[x_i, f(x)|\{x_j\}] = cost[x_i, f(x)]$$

The proof is complete. □

Approximation ratio. In [36], it is proved that the Winner-Imposing Proportional Mechanism has an approximation ratio of at most $4k$ for the k-Facility Location game. Here, we extend the ideas in [36] by adopting the technique proposed by Arthur and Vassilvitskii [2]. In [2], the authors design *k-means++*, an algorithm that solves the k-means clustering problem. The k-means clustering problem is NP-hard and is defined as follows: given an integer k and a set $\mathcal{X}$ of n points in $\mathbb{R}^d$, choose k centers so as to minimize the total squared distance D^2 between each point and its closest center. The k-means++ algorithm that is $O(\log k)$-competitive with the optimal clustering, is similar to the Proportional Mechanism. The algorithm selects the first center uniformly at random and in $k-1$ rounds it choosed each of the next $k-1$ centers $x \in \mathcal{X}$ with probability equal to $\frac{D(x)^2}{\sum_{x \in \mathcal{X}} D(x)^2}$ (this is the so called D^2 weighting process).

By applying the technique used for the proof of the approximation ratio of the k-means++ algorithm, we have the following theorem.

Theorem 2.2.4. *For any $k \geq 1$,* WIPROP *is a $O(\log k)$-approximation mechanism for the k-Facility Location game.*

Proof. We fix a location profile $(x_1, x_2, ..., x_n)$, and compare the cost of WIPROP with input location profile $(x_1, x_2, ..., x_n)$ to the cost of a set $C^* = \{c_1^*, ..., c_k^*\}$ of optimal Facility Locations for this location profile. The p-th optimal cluster N_p consists of the agents whose nearest facility in C^* is c_p^* and $\text{OPT}_p = \sum_{i \in N_p} d_i^*$ denotes the optimal cost of the agents in cluster N_p, where $d_i^* = d(x_i, C^*)$ is agent i's distance to the nearest facility in C^*.

Given a set C of facilities chosen by WIPROP, we use additional notation for the covered and uncovered clusters. Let $H(C) = \{p \in [k] : C \cap Np = \varnothing\}$ be the set of indices of the optimal clusters covered by C, and let $U(C) = [k] \setminus H(C)$ be the set of indices of the optimal clusters not covered by C. If C_l is the facility set of WIPROP at the end of round l, then let $H_l = H(C_l)$ and $U_l = U(C_l)$ be the sets of indices of the optimal clusters covered and not covered, respectively, by WIPROP at the end of round l. Let also $D(N', C) = \sum_{i \in N'} d(x_i, C)$ for a subset of agents $N' \subseteq N$. For a set of indices $I \subseteq [k]$, we let $N(I) = \cup_{p \in I} N_p$ be the set of agents in the optimal clusters indexed by I.

The expected cost $\mathbb{E}[D(N, C_k)]$ of the agents in WIPROP is equal to the expected cost of the optimal clusters covered and not covered by the mechanism:

$$\mathbb{E}[D(N, C_k)] = \mathbb{E}[D(N(U_k), C_k)] + \sum_{p \in H_k} \mathbb{E}[D(N_p, C_k)|p \in H_k]$$

For the covered clusters, we use the following lemma.

Lemma 2.2.5. [22] *For every optimal cluster N_p, $\mathbb{E}[D(N_p, C_k)|p \in H_k] \leq 4\text{OPT}_p$.*

The proof is by induction on the number of rounds l. For the basis case $l = 1$, we have that $\mathbb{E}[D(N_p, C_1)|c_1 \in N_p] \leq 2\text{OPT}_p$. If N_p is covered at round $l > 1$, then using the triangle inequality we get

$$\mathbb{E}[D(N_p, C_l)|c_l \in N_p] = \sum_{i \in N_p} \frac{d_i}{D(N_p, C_{l-1})} \sum_{j \in N_p} \min\{d_j, d(x_i, x_j)\} \leq \sum_{i \in N_p} \frac{d_i}{D(N_p, C_{l-1})} \sum_{j \in N_p} \min\{d_j, d_i^* + d_j^*\}$$

If $D(N_p, C_{l-1}) \leq \text{OPT}_p$, it follows immediately that $\mathbb{E}[D(N_p, C_l)|c_l \in N_p] \leq \text{OPT}_p$. Otherwise, we have that $\mathbb{E}[D(N_p, C_l)|c_l \in N_p] \leq 4\text{OPT}_p$.

Next, for the optimal clusters not covered by WIProp, we observe that Lemma 3.4 [2] can be applied with the only difference that $\mathbb{E}[D(N_p, C_k)|p \in H_k] \leq 4\text{OPT}_p$, due to Lemma 2.2.5.

Consequently, we conclude that the mechanism $O(\log k)$-approximates the optimal cost. □

Note. In Chapter 4, we revisit the idea of the Proportional Mechanism to design a truthful mechanism without money.

Remark 2.2.6. *(Deterministic Mechanisms for k-Facility Location)*[17] For every $k \geq 3$, there do not exist any deterministic anonymous strategyproof mechanisms with a bounded approximation ratio for k-Facility Location on the line, even there are only $k+1$ agents. We recall that a mechanism is *anonymous* if for all agent profiles $\mathbf{x}$ and permutations on agent locations, the output of the mechanism does depend only on the locations of the agents and not their identities.

2.2.2.2 k-Facility Location with $k+1$ agents

Escoffier et al. [13] consider strategyproof mechanisms for k-Facility Location games with $k+1$ agent and present their results for the general metric spaces as well as for tree metrics. For general metric spaces, Escoffier et al. design the *Inversely Proportional mechanism*, that works as follows. Given a location profile $\mathbf{x}$, if there are at most $n-1$ distinct locations in $\mathbf{x}$ then the mechanism opens facilities at all locations in $\mathbf{x}$. Otherwise, it chooses placement $P_i(\mathbf{x})$ with probability

$$p_i(\mathbf{x}) = \frac{\frac{1}{d(x_i, P_i(\mathbf{x}))}}{\sum_{j=1}^{n} \frac{1}{d(x_j, P_j(\mathbf{x}))}}$$

where $P_i(\mathbf{x})$ denotes the placement of $(n-1)$ facilities at the reported locations of all but agent i, i.e. $P_i(\mathbf{x}) = \{x_1, ..., x_{i-1}, x_{i+1}, ..., x_n\}$. Two kinds of objectives are studied: social cost (*egalitarian social cost*) and maximum cost (*utilitarian social cost*). The Inversely Proportional Mechanism is strategyproof, an $n/2$- approximation with respect to the social cost and an n-approximation with respect to the maximum cost.

Finally, they prove that no randomized strategyproof mechanism on a line metric space has an approximation ratio smaller than $10 - 4\sqrt{5}$.

2.2.2.3 k-Facility Location on the Real Line for Concave Cost Functions

In this section, we summarize the results presented in [16]. The connection cost of each agent is given by a non-negative increasing *concave* function $c(d)$ of the distance d. The main contribution of [16] consists of two randomized mechanisms for the *real line* setting: EqualCost and PickTheLoser.

The mechanism EqualCost. We have a three-phase mechanism. At the first step of EqualCost, the mechanism finds[1] an optimal covering of all agent locations with k disjoint intervals $[a_i, a_i + l]$ that minimizes the interval length. At the second step, the mechanism constructs a random variable $X(l) \in [0, l]$ such that all locations $x \in [0, l]$ have the same expected connection cost $C(l) = \mathbb{E}[c(|x - X(l)|)]$ (see [16] for the technical details). It can be proved that the expected cost $C(l)$ is an increasing function of the interval length l. At the final step and for every interval $[a_i, a_i + l]$, the mechanism places a facility at $a_i + X$, if i is odd, or at $a_i + l - X$, if i is even. EqualCost is group-strategyproof and has an approximation ratio of at most 2 for the objective of maximum cost, for any concave cost function c.

EqualCost has a good performance for the objective of maximum cost, but not for the objective of social cost. For instance, when we have k facilities and only $k + 1$ agents, as in the previous section, then one could easily satisfy all but one agents. Nevertheless, EqualCost has a high cost. For that reason, a second mechanism is considered on the line.

The mechanism PickTheLoser **for k facilities and $k+1$ agents.** The mechanism sorts the locations of agents in an increasing order such that $x_i < x_{i+1}$ and constructs two sets E and O of agents consisting of the even and odd numbered agents, respectively. For every odd numbered agent $i \in O$, the mechanism places a facility at x_i. For each agent $i \in E$, PickTheLoser samples a number s_i uniformly in $(0, 1)$, and computes i's scaled cost $\frac{\min_{j \neq i} c(|x_j - x_i|)}{s_i}$. Finally, PickTheLoser selects the agent with the smallest scaled cost as the loser. All other even numbered agents get a facility.

2.2.2.4 Summary of the approximability results on k-Facility Location games

In the following table, we present the known *precise approximation ratios* for each problem or the best known *lower and upper bounds.*

[1] The minimum feasible interval length l can be computed by checking all $n^2/2$ possible candidate values (the value of l is equal to the distane between two agent locations). Using binary search over the space of candidate values, we can compute the optimal l given the number k of intervals.

SOCIAL COST	k=1	k=2	$2 < k < n-1$	$k = n-1$
Deterministic	1 [26]	$n-2$ [31]	∞ [17]	∞ [17]
Randomized	1 [26]	[1.045, 4] [23, 22]	[1.045, n] [16]	[1.045, 2] [16]

MAXIMUM COST	k=1	k=2	$2 < k < n-1$	$k = n-1$
Deterministic	2 [31]	2 [31]	∞ [17]	∞ [17]
Randomized	1.5 [31]	[1.5, 5/3] [31]	[1.5, 2] [16]	1.5 [13]

Figure 2.1: Summary of the approximability results on k-Facility Location games

Note. The lower bound on the approximation ratio of deterministic mechanisms for $k \geq 3$ is only shown for anonymous mechanisms. The randomized upper bounds for the case of $k \geq 3$ (maximum cost, social cost) and for the case of $k = n-1$ (social cost) hold for any concave cost function.

2.2.3 Variants of Facility Location games

2.2.3.1 Facility Location with Uniform Opening Cost

In [15], a Facility Location game with a *uniform facility opening cost* is also considered. The objective is to place facilities so as to minimize the social cost and the total facility opening cost, which is a constant price multiplied by the number of facilities opened. The number of facilities is not fixed. They present a winner-imposing randomized truthful 8-approximation mechanism (OFL). Given a random permutation on the location profile $\mathbf{x}$, OFL places the first facility at x_1 and connects agent 1 to it. We set $C_1 = \{x_1\}$. For each of the following $i = 2, ..., n$ agents, it opens a facility at x_i and connects agent i to it, with probability $d(x_i, C_{i-1})$. We set $C_i = C_{i-1} \cup \{x_i\}$. Each agent not allocated a facility connects to the facility in C_n closest to his true location. For the real line setting, a deterministic non-imposing group strategyproof $O(\log n)$-approximate mechanism is presented.

2.2.3.2 Obnoxious Facility Location

An other Facility Location game is *obnoxious facility location* where the facility is undesirable and all agents try to be as far away from the facility as possible. Two underlying metric spaces have been considered: the real line and networks.

Real line (path).[7] In the basic setting, the n agents are located on a interval with left and right endpoints a and b, and the mechanism must select the location of a single obnoxious facility. As in [31], the distance function between two points x and y is $d(x,y) = |x-y|$; this function also represents the cost of each agent, but, in contrast to previous work, each agent wants to maximize his distance from the obnoxious facility. For simplicity, we assume that the agents are located in the interval $[0,2]$. The first mechanism designed by Cheng et al. is the following: If the number n_1 of agents located on the interval $[0,1]$ is less or equal than the number $n_2 = n - n_1$ of agents located on $(1,2]$, then select the left endpoint 0; otherwise select the right endpoint 2. This mechanism is group-strategyproof and guarantees an approximation ratio of 3. In fact, they show that any deterministic mechanism which only selects one of endpoints as the facility location cannot do better. For the real line setting, a lower bound of 2 for any strategy-proof deterministic mechanism is proved. They also study randomized mechanisms for the same problem. The best approximation ratio (equal to $\frac{3}{2}$ is achieved by the mechanism that returns the left endpoint with probability $a = \frac{2n_1n_2+n_2^2}{n_1^2+n_2^2+4n_1n_2}$ and the right endpoint with probability $1-a$.

Networks.[8] For trees and circles, Cheng et al. study both deterministic and randomized mechanisms. We can parameterize the *circle* G such that any point $x \in G$ can be viewed as a real number $x \in [0,1]$. Point $x = 1$ coincides with point $x = 0$. The mechanism that chooses the point $\frac{3}{4}$ if the number n_1 of agents located on the arc $[0,1/2]$ is less or equal than the number $n_2 = n - n_1$ of agents located on the arc $(1/2,1)$ and otherwise returns the point $\frac{1}{4}$ is group-strategyproof with tight approximation ratio of 3. For a *tree* $T(V,E)$, a 3-approximate group-strategyproof mechanism is considered. The mechanism constructs two subtrees T_a and T_b using the following method. First, we construct a new graph tree T by adding only the nodes where agents are located as well as their incident edges. Tree T' is then partioned into two subtrees. Let m_{ab} be the midpoint in the diameter $P[a,b]$ of T' with endpoints $[a,b]$. We assume that m_{ab} is on edge $[r,s]$ where r is closer to a than to b, so by deleting $[r,s]$ we get two subtrees T'_a and T'_b. The mechanism given a location profile $\mathbf{x}$ and a diameter $P[a,b]$, places the facility at $y = b$ if the number of agents on T_a is greater or equal to the number of agents on T'_b, otherwise it chooses the location $y = a$.

2.2.3.3 Facility Location games with dual preferences

In a recent result [39], games with agents with dual preferences are considered, meaning that both preferences of agents exist in the Facility Location game, as some agents prefer staying as close as possible to a certain facility (or facilities) but prefer staying as far away as possible from an other facility (or facilities). The underlying metric space is the real line segment $(0,l)$.

In the *dual character Facility Location* game, a single facility has to be placed and each agent i has a preference value p_i which indicates whether he want to stay close to the facility (1 - desirable) or not (0 - obnoxious). Based on their preference, the *utility* $u(x_i, p_i, y)$ of each agent is defined as the distance between the agent and the facility at y, if $p_i = 0$; if $p_i = 1$, his utility is defined $l - d(x_i$,y). Each agents wants to maximize his utility. The *social utility* is defined as $su(\mathbf{x}, \mathbf{p}, y) = \sum_{i=1}^{n} u(x_i, p_i, y)$. An agent may misreport his preference, his location or both. If an agent can misreport both his preference and location, the locations are transformed to $x_i^* = l - x_i$ if $p_i = 1$ and $x_i^* = x_i$ if $p_i = 0$. Given the transformed location profile, the mechanism that builds the facility at 0 if the number of agents in $[0, 1/2)$ is less than or equal to the number of those in $[1/2, l]$ and otherwise it builds the facility at 1, is a group-strategyproof mechanism with an approximation ratio of 1/3 for the objective of social utility.

In the *two-opposite Facility Location game with limited distance*, two facilities need to be built on the real line segment. An additional constraint for the construction of two facilities is that the distance between them cannot exceed a certain value C, $0 < C < l$. The social utility of this game is defined as $su(y_0, y_1, \mathbf{x}) = \sum_{i=1}^{n} u(x_i, y_0, y_1) = \sum_{i=1}^{n}(d(x_i, y_0) - d(x_i, y_1))$, where y_0 and y_1 are the locations of the two facilities. Zou and Li consider two cases. When the number of agents is even ($n = 2k$), they design a deterministic group-strategyproof mechanism with approximation ratio $\frac{1}{k}$. When the number of agents is odd ($n = 2k - 1$), another deterministic group strategy-proof mechanism is given with approximation ratio $\frac{1}{2k-1}$. Furthermore, they prove that the approximation ratios for these mechanisms are the best for any deterministic strategy-proof mechanism in their settings.

2.2.3.4 The least squares objective

In [14], Feldman and Wilf study the sum of squared distances (SOS) function that is $sd(y, \mathbf{x}) = \sum_{i=1}^{n} d(y, x_i)^2$.

On the *real line*, with respect to the *miniSOS objective* (minimizing the sum of squared distances), the mechanism that chooses the median location [31] in $\mathbf{x}$ is a strategyproof 2-approximation mechanism and this ratio is tight with respect to deterministic strategyproof mechanisms. The random dictator mechanism [1] is a 2-approximate randomized mechanism. An other 1.5-approximate randomized mechanism is the one that choose the average point with probability 1/2 and applies the random dictator with probability 1/2.

For the game on a *tree*, the median of a tree is an strategyproof 2-approximation mechanism for the miniSOS objective. Feldman and Wilf also design a randomized mechanism that obtains 1.83-approximation for trees, based on their results for the family of randomized strategyproof mechanisms for locating a facility on a tree.

2.2.3.5 Facility Location games with weighted agents on a line

In addition to the basic setting, in [38] each agent has a *weight* w_i. Let $W_{max} = \max w_i$ denote the maximum weight in the input agent profile, and $W_{min} = \min w_i$ as the minimum weight. Agents prefer to stay close to the facility. The cost of agent i is measured by the weight times the distance away from the facility. The social cost and the maximum cost objectives are defined accordingly.

In the following table, a summary of the results in [38] is given.

1-Facility Location	Upper bound	Lower bound
Social Cost	$\frac{W_{max}}{W_{min}}$	$\frac{W_{max}}{W_{min}}$
Maximum Cost	$\frac{W_{max}}{W_{min}}+1$	$\frac{W_{max}}{W_{min}}+1$

2-Facility Location	Upper bound	Lower bound
Social Cost	$(n-2)\frac{W_{max}}{W_{min}}$	$\frac{n-1}{2}\cdot\frac{W_{max}}{W_{min}}$
Maximum Cost	$\frac{W_{max}}{W_{min}}+1$	$\frac{W_{max}}{W_{min}}+1$

Obnoxious 1-Facility Location	Upper bound	Lower bound
Social Utility	$3\cdot\frac{W_{max}}{W_{min}}$	$\frac{W_{min}}{2W_{max}}+\frac{3}{2}$

Figure 2.2: Summary of the results on Facility Location games with weighted agents

2.3 Summary

In a Facility Location n agents are located in a metric space (X, d) and k facilities need to be placed. The metric function d represents the cost of an agent which is usually equal to the distance between the facility and this agent. Various metric spaces have been considered, such as graphs ([33, 1, 35, 14], Euclidean or general metric spaces (eg. [15]), the real line (eg. [31, 16]), the cycle (eg. [22, 1]). Each agent aims at minimizing or maximizing his individual cost. In the former case, we have the classical Facility Location game; in the latter case, we have an *obnoxious* game (eg. [7, 8]), i.e. a game where an undesirable facility has to be located. In a recent result [39], settings with *dual preferences* are also considered.

The first and classic Facility Location game has its roots in *Social Choice Theory* (eg. [26, 3, 25]), where this type of preferences are known as *single-peaked*. In [30], Facility Location games are first studied under the *Approximate Mechanism Design without Money* perspective. This paper initiated a series of results related to mechanism design without money for Facility Location. Their common goal is to design *strategyproof* (truthful) or even *group-strategyproof mechanisms* that are approximately optimal with respect to a given objective function. Several *objective functions* have been studied including minimizing the social cost and the maximum cost (eg. [31, 22, 13, 16]) or maximizing the obnoxious social welfare (eg. [7, 8]) and the social utility [39]. In [14], the objective of *least squares* has also been considered for trees.

Furthermore, several other extensions of the standard model have been studied. For 1-facility games, in [31, 22] *each agent controls multiple locations.* In [15], a Facility Location game with a *uniform facility opening cost* is considered. The objective is to place facilities so as to minimize the social cost and the total facility opening cost. For the real line setting, facility location games with *weighted agents* and facility location games with *threshold agents* have also been studied [38]. Although at the time that this thesis is written, only an extected abstract of [34] is avalaible, for completeness we also refer to a recent result about richer model of the classical k-Facility Location problem where the facilities are *heterogeneous*, meaning that they serve different purposes, and agents have interest only in some of them.

Chapter 3

Money Burning

As discussed in Chapter 1, monetary transfers are usually essential for Mechanism Design, in order to escape the Gibbard–Satterthwaite impossibility result. However, there are settings where payments are undesirable or infeasible. In Chapter 2, we survey Approximate Mechanism Design without Money where approximation is used to obtain truthful mechanisms that approximate the optimal result and do not resort to payments. In [20], Hartline and Roughgarden consider an other way of overriding the use of payments and introduce the idea of *Money Burning*.

Most computer systems such as networks have the ability to reduce service quality, introduce computational challenges or add delay. Consequently, users are required to burn money, as all these factors are payments that take the form of wasted resources instead of actual money. In such settings, the objective is no more the maximization of social welfare, but becomes the maximization of the *residual surplus* (or *social utility*), measured by the social welfare minus the payments charged.

3.1 Optimal Mechanism Design and Money Burning

In [20], Hartline and Roughgarden consider single-unit and multi-unit auctions and present a template for designing *truthful prior-free mechanisms* that achieve at least a constant fraction of the optimal social utility of Bayesian mechanisms. In addition, they compare the social utility of a truthful mechanism to the maximum social welfare and obtain a logarithmic fraction of the total social welfare. In this section, we review the most important results of the paper [20].

3.1.1 Bayesian Mechanism Design

We extend the model presented in Chapter 1 to the Bayesian setting. For completeness, we recall the basic definitions, where necessary. As considered in Chapter 1, agents aim to maximize their *quasilinear utility*, i.e. $u_i = v_i x_i - p_i$. The mechanism's objective is to maximize the *residual surplus*, defined as $\sum_i (v_i x_i - p_i)$. The main difference between this setting and the setting in Chapter 1 is that the payment p_i of agent i is burnt, meaning that p_i denotes the amount of money that is equivalent to the service degradation.

Definitions and Notation. We consider mechanisms that allocate an item or service to a subset of n agents. Agent valuations $\mathbf{v} = (v_1, ..., v_n)$ are independently drawn from a known *prior distribution* with cumulative distribution function $F(z)$ and probability density function $f(z)$. Let $\mathbf{F}$ denote the product distribution of agent values. Agent i's *strategy* is a mapping from their true valuation v_i to a course of *actions*, thus inducing a distribution on agent actions. Agents' actions are in *Bayes-Nash equilibrium* if no agent, given his own valuation and the distribution on other agents' actions, can increase his expected payoff via alternative actions. Due to the *Revelation Principle* (see Theorem 1.2.4), we can focus only on direct truthful mechanisms, meaning mechanisms in which truthtelling is a Bayes-Nash equilibrium.

Given the distribution of agent valuations, the mechanism provides an outcome, i.e. an *allocation vector* $\mathbf{x}(\mathbf{v}) = (x_1(v_1), ..., x_n(v_n))$ and a *payment vector* $\mathbf{p}(\mathbf{v}) = (p_1(v_1), ..., p_n(v_n))$. In the non-Bayesian setting, if agent i wins the good, then x_i is 1, otherwise x_i is 0. In the Bayesian setting, $x_i(v_i)$ denotes the probability (over other agents' valuations $\mathbf{v}_{-i}$) that agent i gets the item when his valuation is v_i, namely $x_i(v_i) = \mathbb{E}_{\mathbf{v}_{-i}}[x_i(v_i, \mathbf{v}_{-i})]$.

We restate Myerson's Lemma (see Chapter 1) for the Bayesian setting. *Bayesian Incentive Compatibility* is a property of a direct mechanisms requiring that truth be a Bayesian equilibrium.

Theorem 3.1.1 (Myerson's Lemma, [27])**.** *Every Bayesian incentive compatible mechanism satisfies the following two properties:*

1. Allocation monotonicity*: For all i and $v_i > v_i'$, $x_i(v_i) \geq x_i(v_i')$*
2. Payment identity*: For all i and v_i, $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(v) dv$*

Definition 3.1.2. [27] *If agent i's valuation is distributed according to F, where we assume that F has support $[a, b]$ and positive density function f throughout $[a, b]$, then his* vitrual valuation for payment *is*

$$\phi(v_i) = v_i - \frac{1 - F(v_i)}{f(v_i)} \tag{3.1}$$

The virtual surplus *of allocation* $\mathbf{x}$ *is* $\sum_i \phi(v_i)x_i$.

Lemma 3.1.3. [27] *In a Bayesian incentive compatible mechanism with allocation rule* $\mathbf{x}(\cdot)$, *the expected payment of agent i satisfies* $\mathbb{E}_{\mathbf{v}}[p_i(\mathbf{v}] = \mathbf{E}_{\mathbf{v}}[\phi(v_i)x_i(\mathbf{v})]$

3.1.2 Bayesian Optimal Money Burning

Generalization of Myerson's Ironing procedure. Myerson [27] applies an "ironing" procedure to transform a possible non-monotone virtual function into a monotone ironed virtual valuation function. In [20], Hartline and Roughgarden generalize the *ironing technique* of Myerson. The authors define the *virtual valuation for utility* as $\theta(v_i) = \frac{1-F(v_i)}{f(v_i)}$ and extend Lemma 3.1.3 to prove that in a Bayesian incentive-compatible mechanism with allocation rule $\mathbf{x}$, the expected utility of agent i satisfies $\mathbb{E}_{\mathbf{v}}[u_i(\mathbf{v}] = \mathbf{E}_{\mathbf{v}}[\theta(v_i)x_i(\mathbf{v})]$. Consequently, the maximization of the expected virtual utility implies a Bayesian optimal mechanism for residual surplus.

However, $\theta(\cdot)$ is not always a non-decreasing function. In such cases, the ironing procedure is applied and the authors provide a theorem that shows that the maximization of the ironed virtual surplus for utility is equivalent to the maximization of the expected residual surplus subject to incentive compatibility. Let now $\frac{f(v)}{1-F(v)}$ denote the *hazard rate* of distribution F at v. The *monotone hazard rate* (MHR) *assumption* is that the hazard rate is monotone non-decreasing. There exist also settings where the hazard rate is monotone in the opposite direction (anti-MHR) and settings where it is neither monotone increasing nor monotone decreasing. For every case, they obtain the following results:

- For agents with i.i.d valuation distribution functions that satisfy the *MHR condition*, an optimal money-burning mechanism for allocating k units is a k-unit lottery.
- For agents with i.i.d valuation distribution functions that satisfy the *anti-MHR condition*, an optimal money-burning mechanism for allocating k units is a k-unit Vickrey auction.

- For agents with i.i.d valuation distribution functions that satisfy the *non-MHR condition*, an optimal money-burning mechanism for allocating k units is an indirect k-unit Vickrey auction.

3.1.3 Prior-Free Money Burning Mechanism Design

Performance benchmark. As their next step, Hartline and Roughgarden develop a general template for prior-free, i.e. worst-case, optimal mechanism design. They introduce a family of mechanisms $\text{Opt}_{\mathbf{F}}$, where $\text{Opt}_{\mathbf{F}}$ corresponds to a mechanism that maximizes the expected residual surplus for valuations drawn from an i.i.d. distribution $\mathbf{F}$, and then define a prior-free performance benchmark as

$$\mathcal{G}(\mathbf{v}) = \sup_{\mathbf{F}} \text{Opt}_{\mathbf{F}}(\mathbf{v})$$

In other words, the benchmark competes simultaneously with all Bayesian optimal for some i.i.d. distribution on a fixed (worst-case) valuation profile.

A mechanism M *β-approximates the benchmark* $\mathcal{G}$ if for every valuation profile $\mathbf{v}$, its expected residual surplus is at least $\mathcal{G}(\mathbf{v})/\beta$. This implies that on any i.i.d. distribution, the mechanism M achieves at least a β fraction of the expected residual surplus of every mechanism. An example of such a mechanism M is the *k-unit p-lottery*; they show that for every valuation profile $\mathbf{v}$, there is a k-unit p-lottery with expected residual surplus at least $\mathcal{G}(\mathbf{v})/2$. The definition of *$k$-unit p-lotteries* is as follows:

Definition 3.1.4. [20] *The* k-unit p-lottery *allocates to agents with value at least p at price p. If there are more than k such agents, the winning agents are selected uniformly at random.*

As a last step, they design a mechanism that $O(1)$-approximates the benchmark. The mechanism is *Random Sampling Optimal Lottery* (RSOL) and is defined as follows:

Definition 3.1.5. [20] *With a set $S = \{1, ..., n\}$ of n agents and a k identical units of an item, the* Random Sampling Optimal Lottery (RSOL) *is the following mechanism:*

1. *Choose a subset $S_1 \subset S$ of the agents uniformly at random, and let S_2 denote the rest of the agents. Let p_2 denote the price charged by the optimal k-unit p-lottery for S_2.*

2. *With probability $\frac{1}{2}$, run a k-unit p_2-lottery on S_1.*

3. *Otherwise, run a k-unit Vickrey auction on S_1.*

The theorem below states one of the most important results of the paper.

Theorem 3.1.6. [20] RSOL *O(1)-approximates the benchmark $\mathcal{G}$.*

Lower Bounds. The authors also establish a lower bound of 4/3 on the approximation ratio of every prior-free money-burning mechanism. Determining the best-possible approximation ratio is still an open question.

Theorem 3.1.7. [20] *No prior-free mechanism money-burning mechanism can achieve approximation ratio better than $4/3$ with respect to the benchmark $\mathcal{G}$, even for the case of two agents and one unit of an item. For two bidders and one unit of an item, there is a prior-free mechanism that $3/2$-approximates the benchmark $\mathcal{G}$.*

The power of Money Burning. If we compare the social utility of a truthful mechanism to the maximum social welfare, then the best possible approximation guarantee for k-unit auctions is $\Theta(1 + log(\frac{n}{k}))$, where n is the number of agents.

3.2 Applications of Money Burning

In this section, we refer to some recent results where money burning techniques are applied. All of the following papers share in common the following constraint : the settings the study eschew monetary transfers completely. This establishes several limits to what the mechanism designer can achieve since the use of money usually ensures truthfulness.

Mechanism Design for Fair Division. In [11], Cole et al. study the problem of fair division (or cake-cutting) from a mechanism design perspective. In their setting, there exist n agents and m items; each item is divisible, meaning that it can be divided into pieces and then allocated to different agents. Each bidder is assigned a weight $b_i \geq 1$ which allows for comparison of valuations. These weights are defined by the mechanism. They find a truthful way to implement good approximation outcomes in settings where monetary payments are not an option.

Their main result is the *Partial Allocation* mechanism, that discards a fraction of the allocated resources in order to ensure truthfulness by the agents. The Partial Allocation mechanism has 3 steps and uses the proportionally fair allocation, i.e. a feasible allocation x^* such that for any other feasible allocation, it holds that

$\sum_{i\in N} \frac{b_i[v_i(x')-v_i(x^*)]}{v_i(x^*)} \leq 0$. A notable property of the proportionally fair solution is that it gives a good tradeoff between fairness and efficiency. However, it cannot be implemented using truthful mechanisms without the use of payments. The mechanism succeeds to use the proportionally fair solution to obtain a truthful allocation rule without money.

First, the Partial Allocation mechanism computes the proportionally fair allocation x^* based on the reported bids. Second, for each player i, it removes this player and computes the proportionally fair allocation x^*_{-i} that would arise in his absence. Finally, it allocates to each player i a fraction $f_i = \frac{\prod_{i'\neq i}[v_{i'}(x^*)]^{b_{i'}}}{\prod_{i'\neq i}[v_{i'}(x^*_{-i})]^{b_{i'}}}$ of everything that he receives according to x^*. The mechanism is truthful and guarantees that every player will receive at least a $1/e$ fraction of his proportionally fair valuation.

A more careful look at the PA mechanism reveals a connection with the well known VCG mechanism, if we simply consider the surrogate valuation $u_i(\cdot) = b_i \log v_i(\cdot)$ for each player i.

Optimal Provision-After-Wait in Healthcare. In [5], Braverman et al. utility optimization in health care service allocation. Specifically they study the problem of allocating medical treatments to a population of patients. Each patient has a demand for exactly one unit of treatment and chooses to be treated in one of k hospitals. Each patient has a value for each hospital. Patients do not pay anything for a treatment and a third party (the payer) with a budget B covers the different costs of each hospital. Due to the budget constraints, the payer can only cover a limited number of treatments in the more expensive hospitals, thus the access to over-demanded hospitals is regulated through *waiting times*. The objective is to compute the appropriate waiting times and to assign in a feasible way a number of treatments to each hospital, so that in equilibrium social welfare is maximized subject to the budget constraint.

In this setting, money is allowed, but money and waiting times are not interchangeable quantities. Waiting times have a *welfare-burning effect*, because neither the patients or the payer benefit from the waiting times. On the contrary, the waiting times represent a loss in the social welfare. This is the reason why the efficiency of the mechanism is measured again as the sum of the total utility of each agent, where the utility of an agent is defined as the total value he receives minus his total waiting time.

Although Braverman et al. study the welfare-burning phenomenon, they focus on the complexity of computing efficient equilibrium allocations, instead of approximate truthful mechanisms, which we are interested in.

Incentive Compatible Two-Tiered Resource Allocation without Money. In [6], Cavallo considers the resource allocation problem with two types of good. There exists only a single isntance of a high-value good (*type A*) and an unlimited number of identical lower-value goods (*type B*). Each agent has use only for a single good, so it is natural to assume that each agent prefers the good of type A, but he also prefers to receive a good of type B, instead of nothing. Type B goods are in fact a useful substitute for money, serving as a means of incentivizing truthful reporting of agents' private values.

The expected utility of agent i is defined as $u_i(q_i, p_i) = q_i v_i + p_i$, where v_i is agent i's private utility for the type-A good, q_i is the probability of obtaining the type-A good and $p_i \in [0, 1 - q_i]$ the probability of winning a type-B good. The mechanism designer seeks to maximize the social welfare, i.e. the sum of agent utilities. There is a trivial non-monetary mechanism wherein misreporting types can never manipulate the mechanism. The mechanism randomly chooses an agent i, allocating to him the type-A good, and giving to all others a type-B good.

Adapting the theorems proved in [20], Cavallo shows that for any i.i.d. value distribution with monotonically increasing hazard rate, no dominant strategy incentive compatible mechanism can do better than random allocation. Nevertheless, in other cases such mechanisms can have more positive results. For example, we may consider settings where a specific agent has a far higher value than all others for the type-A good.

Chapter 4

Money Burning Mechanism Design for Facility Location

In this Chapter, we study a Facility Location setting from a mechanism design perspective. A set of n agents are located in a metric space and k facilities must be placed, but in addition to the Mechanism Design results for Facility Location that we presented in Chapter 2, facilities offer a certain type of service and agents have private splittable demands for this service. In contrast to the settings previously studied, in our model agents' locations are public knowledge and agents can misreport only their demands.

To solve the problem, we introduce a novel approach that has not been used before in Facility Location games. We combine the Money Burning main ideas with Facility Location, in order to obtain truthful mechanisms that approximate the optimal social cost. Adapting the VCG mechanism, we first design a VCG-like mechanism where payments are in the form of additional individual cost. In the second section of this chapter, we revisit Proportional Mechanism from a Money Burning perspective and design a truthful mechanism. We present our results for the real line setting, as well as general metric spaces.

4.1 Model and Definitions

Let $N = \{1, .., n\}$ be a set of n strategic agents located in a *metric space* (X, d), where $d : X \times X \to \mathbb{R}_+$ is a metric on X, namely a function that is non-negative, symmetric, and satisfies the triangle inequality. In our model the metric $d(x, x')$ represents the distance between the locations $x \in X$ and $x' \in X$.

Each agent $i \in N$ has a *location* $x_i \in X$, which is public information. There is one type of service offered and each agent i has a *demand* $w_i \in [1, +\infty)$ for it. Let $\mathbf{w} = (w_1, ..., w_n)$ denote the input vector of agents' demands. The demand w_i is private, which means that an agent may report a demand that is different from his true value in order to manipulate the mechanism. However, he can only report a demand that is at least equal to his real demand $\overline{w}_j$. This assumption is reasonable, since $\overline{w}_i$ denotes the precise quantity that each agent needs. If agent i reports $w_i \geq \overline{w}_i$, then the entire demand w_i must be served by the mechanism, but agent i does not use the excessive quantity $w_i - \overline{w}_i$ that he gets.

The demand of agents has to be served by a set $F = \{F_1, ..., F_k\}$ of k *facilities* and each facility can serve unlimited amount of demand. Let also $d_i(x)$ denote the distance between the location of agent i and location x. For a set of facilities $C \subseteq F$, we define $d_i(C)$ as $d_i(C) = \inf\{d_i(c) : c \in C\}$.

The demand w_i that an agent $i \in N$ has, can *be split to different facilities.* Let $w_i^{(j)} \geq 0$ denote the part of agent i's total reported demand that is served by the facility F_j; of course, the sum of them satisfies the condition $\sum_{j=1}^{k} w_i^{(j)} = w_i$. The locations of the k facilities which serve each part $w_i^{(j)}$ are decided by the mechanism.

Mechanism design. In the mechanisms that we study, the mechanism may determine the order in which the facilities serve the demand of an agent i. For example, in a setting with $k \geq 2$ facilities, where half of agent's i demand is served by facility F_1 and the other half by facility F_2, such that $d_i(F_1) < d_i(F_2)$, agent i is not allowed to consume the service from F_1 before consuming the complete service available to him from F_2.

In the mechanisms described in this chapter, the demand w_i of an agent i is always split to at most 2 facilities F_1 and F_2, where $d_i(F_1) \leq d_i(F_2)$. The most distant facility F_2 serves agent i first, until all the demand $w_i^{(2)}$ is served. To sum up, the mechanism maps an input demand vector to a tuple $\mathbf{S} = (S_1, ..., S_n)$, where $S_i = ((w_i^{(i_1)}, F_{i_1}), (w_i^{(i_2)}, F_{i_2}))$ such that F_{i_1}, $F_{i_2} \subseteq F$, $d_i(F_{i_1}) \leq d_i(F_{i_2})$ and $w^{(i_1)} + w_i^{(i_2)} = w_i$. We write $M_i(\mathbf{w})$ to denote S_i.

Social Cost. Given a set of facilities F and an assignment $\mathbf{S}$ of demands $\mathbf{w}$ to the facilities in F, the *cost* of agent i is defined as

$$cost_i[M(\mathbf{w})] = cost_i[M_i(\mathbf{w})] = \min\{w_i^{(i_1)} d_i(F_{i_1}) + w_i^{(i_2)} d_i(F_{i_2}),\ w_i^{(i_2)} d_i(F_{i_2}) + \max\{\overline{w}_i - w_i^{(i_2)}, 0\} d_i(F_{i_1}), \min\{w_i^{(i_2)}, \overline{w}_i\} d_i(F_{i_2})\}$$

The *social cost* of a mechanism $M(\mathbf{w})$ is $SC[M(\mathbf{w})] = \sum_{i=1}^{n} cost_i[M(\mathbf{w})]$.

Truthfulness. A mechanism M is *truthful* if for every agent i $cost_i[M(w_i, w_{-i})] \geq cost_i[M(\overline{w}_i, w_{-i})]$. Similarly, a randomized mechanism M is *truthful* (in expectation) if $\mathbb{E}[cost_i[M(w_i, w_{-i})]] \geq \mathbb{E}[cost_i[M(\overline{w}_i, w_{-i})]]$ for every agent i.

Our goal is to design a *truthful* mechanism that picks the locations of the facilities and determines the assignment of agents' demands $w_i^{(j)}$, $i \in N$, $j \in F$ to facilities, while *minimizing the social cost.*

Approximation ratio. A mechanism M achieves an *approximation ratio* a if for all input demand vectors $\mathbf{w}$ the social cost of $M(\mathbf{w})$ is at most a times the optimal social cost for $\mathbf{w}$. For the randomized mechanism that we design, we will use the expected social cost as a measure for the approximation ratio.

4.2 The Money Burning VCG Mechanism

Notation. For each agent $i \in N$ we define an ordering $F_{i,1}, F_{i,2}, ..., F_{i,k}$ of the k facilities according to its increasing distance from each facility. Thus, the closest facility to agent i is $F_{i,1}$ and the most distant facility is $F_{i,k}$. Let also $\textsc{opt}(w_j, w_{-j})$ (or $\textsc{opt}$) denote the total cost of agents in the optimal solution, namely $\min\{\sum_{i=1}^{n} w_i d_i(F_{i,1})\}$.

By excluding agent j, we have a set $N \setminus \{j\}$ of $n-1$ agents other than j. Similarly, for each agent i we define an ordering $F_{i,1}^{-j}, F_{i,2}^{-j}, ..., F_{i,k}^{-j}$ of the k facilities according to its increasing distance from each facility. Let $\textsc{opt}(w_{-j})$ (or $\textsc{opt}^{-j}$) denote the total cost in the optimal solution for agents $N \setminus \{j\}$, namely $\min\{\sum_{i \neq j} w_i d_i(F_{i,1}^{-j})\}$.

Remark 4.2.1. *The locations $F^1, F^2, ..., F^m$ in the solution $\textsc{opt}^{-j}$ may not be unique; for simplicity's sake, in the proof of Lemma 4.2.3 and Theorem 4.2.5 $\textsc{opt}^{-j}$ denotes the optimal solution $\min\{\sum_{i \neq j} w_i d_i(F_{i,1}^{-j})\}$, such that for agent j the distance $d_j(F_{j,1}^{-j})$ is the smallest possible.*

The mechanism. In a general multi-parameter setting with a social welfare maximization objective, the main idea behind the VCG mechanism is that each agent i pays his *"externality"*, i.e. the welfare loss the presence of agent i causes to the other agents. In settings where monetary transfers are permitted, it is easy to interpret this welfare loss as the price each agents pays for the good(s) he receives. Though in our cost minimization setting money is not allowed, we can establish an analogy between the classical VCG mechanism and our *social cost burning VCG mechanism.*

Remark 4.2.2. *Regarding equation 4.2, we note that this is the total cost of an agent that truthfully reports his demand, i.e. $w_j = \overline{w}_j$. However, the real cost of a probably*

Input: agents' reported demands $w_1, w_2, ..., w_n$

Placement of Facilities: The facilities are placed at the locations that the optimal solution indicates:

$$(F^1, F^2, ..., F^m) = \arg \min_{x_1,...,x_k \in X} \sum_{i=1}^{n} w_i \min\{d_i(x_1), d_i(x_2), ..., d_i(x_n)\} \quad (4.1)$$

Cost Computation: For each agent j, facility $F_{j,1}$ serves only $w_j^{(1)}$ of his reported demand w_j. The facility $F_{j,k}$ serves the rest $w_j^{(2)} = w_j - w_j^{(1)}$. From the perspective of the mechanism designer, the total cost of agent j is equal to

$$cost_j(w_j, w_{-j}) = w_j^{(1)} d_j(F_{j,1}) + w_j^{(2)} d_j(F_{j,k}) = w_j d_j(F_{j,1}) + w_j^{(2)}[d_j(F_{j,k}) - d_j(F_{j,1})] \quad (4.2)$$

where $w_j^{(2)}$ satisfies the following formula:

$$w_j^{(2)}[d_j(F_{j,k}) - d_j(F_{j,1})] = \sum_{i \neq j} w_i d_i(F_{i,1}) - \sum_{i \neq j} w_i d_i(F_{i,1}^{-j}) \quad (4.3)$$

Mechanism 4.1: The Money Burning VCG Mechanism

lying agent is equal to

$$\begin{cases} \overline{w}_j d_j(F_{j,1}) + w_j^{(2)}[d_j(F_{j,k}) - d_j(F_{j,1})] & \textit{if } w_j^{(2)} \leq \overline{w}_j \\ \overline{w}_j d_j(F_{j,k}) & \textit{if } w_j^{(2)} > \overline{w}_j \end{cases} \quad (4.4)$$

Equation 4.2 implies that each agent j receives his cost in the OPT solution plus an additional cost introduced by the mechanism as a "payment". The payment does not involve any money; however, the quantity $w_j^{(2)}[d_j(F_{j,m}) - d_j(F_{j,1})]$ indeed corresponds to a payment: a *burnt payment* in the form of service cost. Setting the "payment" $w_j^{(2)}[d_j(F_{j,m}) - d_j(F_{j,1})]$ to be equal to $\sum_{i \neq j} w_i d_i(F_{i,1}) - \sum_{i \neq j} w_i d_i(F_{i,1}^{-j})$, the core idea of the classical VCG mechanism is actually embodied in our social cost burning mechanism.

We next prove some simple properties of the Money Burning VCG (or MBurningVCG) Mechanism.

Lemma 4.2.3. *Let w_j denote the demand that agent j submits to the mechanism and $\overline{w}_j \leq w_j$ his true, private demand. The following properties hold true:*

(i) *The total cost of agent j is at most equal to $w_j d_j(F^{-j}_{j,1})$, i.e.*

$$w_j d_j(F_{j,1}) + \sum_{i \neq j} w_i d_i(F_{i,1}) - \sum_{i \neq j} w_i d_i(F^{-j}_{i,1}) \leq w_j d_j(F^{-j}_{j,1})$$

The above inequality implies that $d_j(F_{j,1}) \leq d_j(F^{-j}_{j,1})$ (see also Remark 4.2.1).

(ii) *If agent j's distance to the closest facility $F^{-j}_{j,1}$ in OPT^{-j} is at most equal to his distance to the most distant facility $F_{j,k}$ in every optimal solution OPT, namely $d_j(F_{j,k}) \geq d_j(F^{-j}_{j,1})$ for each optimal solution OPT, then $w^{(2)}_j \leq w_j$. In other words, the mechanism is* feasible.

(iii) *If $d_j(F_{j,k}) \geq d_j(F^{-j}_{j,1})$ for each optimal solution $\text{OPT}(w_i, w_{-i})$, then the Money Burning VCG Mechanism is* truthful.

Proof. We prove each part separately:

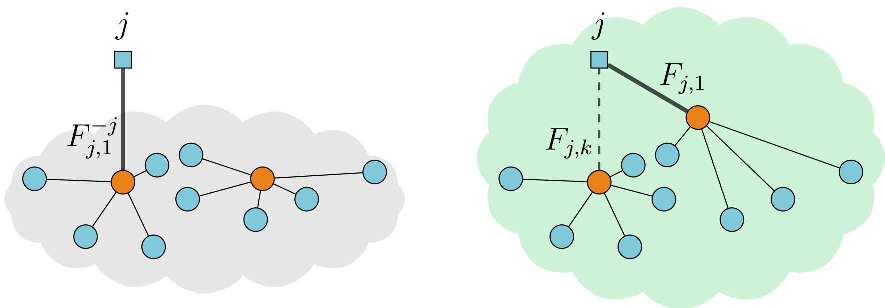


Figure 4.1: Optimal solution for $N \setminus \{j\}$ (left) - Optimal solution for N (right)

(i) According to Remark 4.2.1, we have asumned that $d_j(F^{-j}_{j,1}) := \min_{\text{OPT}^{-j}}\{d_j(F^{-j}_{j,1})\}$. Suppose now for the sake of contradiction that there exists an optimal solution OPT' such that

$$w_j d_j(F^{\text{OPT}'}_{j,1}) + \sum_{i \neq j} w_i d_i(F^{\text{OPT}'}_{i,1}) - \sum_{i \neq j} w_i d_i(F^{-j}_{i,1}) > w_j d_j(F^{-j}_{j,1})$$

Consequently, $w_j d_j(F^{\text{OPT}'}_{j,1}) + \sum_{i \neq j} w_i d_i(F^{\text{OPT}'}_{i,1}) > \sum_{i \neq j} w_i d_i(F^{-j}_{i,1}) + w_j d_j(F^{-j}_{j,1})$, thus the solution OPT^{-j} has a total social cost that is lower than the optimal cost OPT'. This contradicts the optimality of OPT', therefore the proof is complete.

(ii) For the nontrivial case where $d_j(F_{j,k}) > d_j(F_{j,1}^{-j})$, Lemma 4.2.3 (i) imply that

$$\sum_{i \neq j} w_i d_i(F_{i,1}) - \sum_{i \neq j} w_i d_i(F_{i,1}^{-j}) \leq w_j d_j(F_{j,1}^{-j}) - w_j d_j(F_{j,1})$$

We apply equations 4.2 and 4.3 and obtain that

$$w_j^{(2)} = \frac{\sum_{i \neq j} w_i d_i(F_{i,1}) - \sum_{i \neq j} w_i d_i(F_{i,1}^{-j})}{d_j(F_{j,k}^{-j}) - d_j(F_{j,1})} \leq \frac{w_j d_j(F_{j,1}^{-j}) - w_j d_j(F_{j,1})}{d_j(F_{j,k}^{-j}) - d_j(F_{j,1})} = w_j$$

(iii) Let j be an agent that reports a demand w_j other than his true demand $\overline{w}_j$, $\overline{w}_j < w_j$. The real cost of agent j is equal to

$$cost_j(w_j, w_{-j}) = \begin{cases} \overline{w}_j d_j(F_{j,1}) + w_j^{(2)}[d_j(F_{j,k}) - d_j(F_{j,1})] & \text{if } w_j^{(2)} \leq \overline{w}_j \\ \overline{w}_j d_j(F_{j,k}) & \text{if } w_j^{(2)} > \overline{w}_j \end{cases} \tag{4.5}$$

where $w_j^{(2)}[d_j(F_{j,k}) - d_j(F_{j,1})] = \sum_{i \neq j} w_i d_i(F_{i,1}) - \sum_{i \neq j} w_i d_i(F_{i,1}^{-j})$.

We consider two cases.

- If $w_j^{(2)} > \overline{w}_j$, then it holds that

$$cost_j(w_j, w_{-j}) = w_j d_j(F_{j,k}) \geq w_j d_j(F_{j,1}^{-j}) \geq \overline{w}_j d_j(F_{j,1}^{-j}) \geq cost_j(\overline{w}_j, w_{-j})$$

where the last inequality follows from Lemma 4.2.3 (i).

- If $w_j^{(2)} \leq \overline{w}_j$, then

$$cost_j(w_j, w_{-j}) = w_j d_j(F_{j,1}) + \sum_{i \neq j} w_i d_i(F_{i,1}) - \sum_{i \neq j} w_i d_i(F_{i,1}^{-j})$$

The term $\sum_{i \neq j} w_i d_i(F_{i,1}^{-j})$ is independent of w_j, therefore the problem of minimizing $cost_j(w_j, w_{-j})$ reduces to the problem of minimizing the term $w_j d_j(F_{j,1}) + \sum_{i \neq j} w_i d_i(F_{i,1})$. Submitting the true demand $\overline{w}_j$ results in the mechanism choosing an outcome that minimizes agent j's cost; $\overline{w}_j$ is the demand that achieves the optimal sum $\min\{w_j d_j(F_{j,1}) + \sum_{i \neq j} w_i d_i(F_{i,1})\}$, therefore agent j has no incentive to lie. This completes the proof.

□

Theorem 4.2.4. *The social cost of Money Burning VCG Mechanism is at most n times the optimal solution.*

Proof. The proof follows immediately from the formula of agent j's total cost

$$w_j d_j(F_{j,1}) + \sum_{i \neq j} w_i d_i(F_{i,1}) - \sum_{i \neq j} w_i d_i(F_{i,1}^{-j})$$

Summing over all the agents in N, we get that the social cost of the mechanism M is $SC(M(\mathbf{w})) = n\textsc{opt}(\mathbf{w}) - \sum_i \textsc{opt}^{-i}(w_{-i})$.

If we consider $k = n-1$, then $\textsc{opt}^{-i}(w_{-i}) = 0$ for each i. Consequently, the social cost is n times the optimal. □

From Lemma 4.2.3, we have that a *sufficient condition for the truthfulness* of MBurningVCG is $d_j(F_{j,k}) \geq d_j(F_{j,1}^{-j})$ for each optimal solution $\textsc{opt}(w_i, w_{-i})$. In the next section, we apply MBurningVCG to a metric line setting and prove that the mechanism is truthful.

4.2.1 The Money Burning VCG Mechanism on the line

In this section, we will restrict our attention to a simpler setting. When the metric space is the line, we prove that MBurningVCG is truthful. We note that the metric function $d(x, y)$, $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is a non-negative and non-decreasing function of the distance $|x - y|$ of two points x and y on the line.

Theorem 4.2.5. *The Money Burning VCG Mechanism is truthful on the line.*

Proof. Due to Lemma 4.2.3, it suffices to show that $d_j(F_{j,m}) \geq d_j(F_{j,1}^{-j})$ for every optimal solution $\textsc{opt}$.

The proof is by case analysis.

1. The optimal solution $\textsc{opt}^{-j}$ has placed *all the facilities* $F_{j,1}^{-j}, F_{j,2}^{-j}, ..., F_{j,k}^{-j}$ *on the right side* - or without loss of generality on the left side - of the location x_j of agent j.

   For each facility F we use two additional definitions. Let $A_{right}(F) \subseteq N$ denote the set of agents that their nearest facility is F and their location is on the right side of F. Similarly, let $A_{left}(F) \subseteq N$ denote the set of agents that their nearest facility is F and their location is on the left side of F. The quantity c_{min} is defined as $c_{min} = \min_{\textsc{opt}^{-j}}\{d_j^{\textsc{opt}^{-j}}(F_{j,1}^{-j})\}$ (see also Remark 4.2.1).

Suppose that it is possible that the facilities $F_{j,1}, F_{j,2}, ..., F_{j,k}$ of the optimal solution OPT are located within a distance $d_j(F_{j,i}) < c_{min}$, $\forall i \in [k]$.

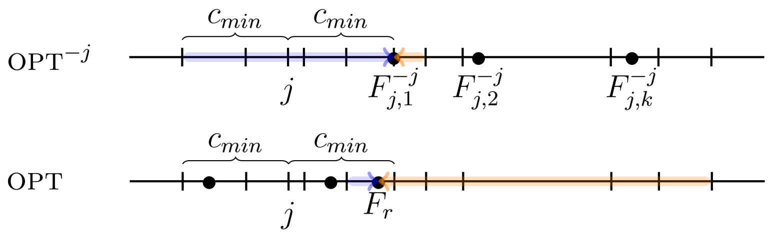


Figure 4.2: Case 1: All facilities $F_{j,1}^{-j}, F_{j,2}^{-j}, ..., F_{j,k}^{-j}$ are on the right side of agent j

We focus on the rightmost facility F_r. (We note that F_r is not necessarily the most distant facility $F_{j,k}$.)

If j does not connect to F_r then we have the following relations

$$A_{right}^{\text{OPT}^{-j}}(F_{j,1}^{-j}) \subseteq A_{right}^{\text{OPT}}(F_r)$$

$$A_{left}^{\text{OPT}^{-j}}(F_{j,1}^{-j}) \supseteq A_{left}^{\text{OPT}}(F_r)$$

From the preceding relations, it can be easily seen that the optimal location of F_r can not be closer to agent j than $F_{j,1}^{-j}$ is. Suppose not. Let $A_{right}^{\text{OPT}^{-j}}(F_{j,1}^{-j}) \equiv A_{right}^{\text{OPT}}(F_r)$ and $A_{left}^{\text{OPT}^{-j}}(F_{j,1}^{-j}) \equiv A_{left}^{\text{OPT}}(F_r)$. Then placing F_r on the left side of $F_{j,1}^{-j}$ is not optimal, due to Remark 4.2.1. A careful examination shows that the same holds for the other cases.

If j connects to F_r, then either $j \in A_{left}^{\text{OPT}}(F_r)$ or $j \in A_{right}^{\text{OPT}}(F_r)$. In the latter case, it still holds that $A_{right}^{\text{OPT}^{-j}}(F_{j,1}^{-j}) \subseteq A_{right}^{\text{OPT}}(F_r)$, so the proof is similar to the previous one.

In the former case ($j \in A_{left}^{\text{OPT}}(F_r)$), we prove by contradiction that F_r can not be the rightmost facility. Indeed, due to the fact that there are $k \geq 2$ facilities and F_r is the rightmost facility, the rest $k-1$ facilities must be on the left side of agent j. These $k-1$ facilities form a set $\mathcal{F}'$.

We define the sets of agents $S_r = A_{right}^{\text{OPT}}(F_r) \setminus (A_{right}^{\text{OPT}}(F_r) \cap A_{right}^{\text{OPT}}(F_r))$ and $S_l = \cup_{F \in \mathcal{F}'}(A_{right}^{\text{OPT}}(F) \cup A_{right}^{\text{OPT}}(F))$. We have that

$$\textstyle\sum_{i \in S_r} d_i(F_r) > \sum_{i \in S_r} d_i(F_{j,1}^{-j}) \text{ and } \sum_{i \in S_l} d_i(\mathcal{F}') < \sum_{i \in S_l} d_i(F_{j-1}^{-j})$$

Due to the optimality of the solution OPT^{-j},

$$\sum_{i\in S_r} d_i(F_r) - \sum_{i\in S_r} d_i(F_{j,1}^{-j}) > \sum_{i\in S_l} d_i(F_{j-1}^{-j}) - \sum_{i\in S_l} d_i(\mathcal{F}')$$

or equivalently,

$$\sum_{i\in S_r} d_i(F_r) + \sum_{i\in S_l} d_i(\mathcal{F}') > \sum_{i\in S_r} d_i(F_{j,1}^{-j}) + \sum_{i\in S_l} d_i(F_{j-1}^{-j})$$

Consequently, placing at least one of the facilities in $\mathcal{F}'$ to the location where facility $F_{j,1}^{-j}$ was, leads to a better solution. Thus, F_r can not be the rightmost facility in an optimal solution OPT.

Therefore, $d_j(F_{j,k}) \geq d_j(F_{j,1}^{-j}) \geq c_{min}$.

2. The optimal solution OPT^{-j} has placed the *facilities* $F_{j,1}^{-j}, F_{j,2}^{-j}, ..., F_{j,k}^{-j}$ *both on the right and left side* of the location x_j of agent j.

   In the optimal solution OPT, let F_r and F_l be the rightmost and leftmost facilities, respectively. We assume that $d_j(F_l) < c_{min}$ and $d_j(F_r) < c_{min}$.

   We consider two subcases.

   (2a) Let $F_r \neq F_{j,1}$ and $F_l \neq F_{j,1}$ (in this case there must be $k \geq 3$ facilities). This means that agent j does not connect to either F_r or F_l. In the optimal solution OPT^{-j}, let also $F_{j,1,left}^{-j}$ denote the closest facility to agent j on his left side. In other words, the definition does not necessarily imply that $F_{j,1,left}^{-j} \equiv F_{j,1}^{-j}$. However, there is no other facility that is located on the left side of agent j and it is closer to j than the facility $F_{j,1,left}^{-j}$ is. In the optimal solution OPT^{-j}, let also $F_{j,1,right}^{-j}$ denote his closest facility on his right side. It is either $F_{j,1,left}^{-j} \equiv F_{j,1}^{-j}$ or $F_{j,1,right}^{-j} \equiv F_{j,1}^{-j}$, but we assume without loss of generality that $F_{j,1,right}^{-j} \equiv F_{j,1}^{-j}$.
   We have again that
   $$A_{right}^{\text{OPT}^{-j}}(F_{j,1}^{-j}) \subseteq A_{right}^{\text{OPT}}(F_r)$$

   - If $A_{left}^{\text{OPT}^{-j}}(F_{j,1}^{-j}) \supseteq A_{left}^{\text{OPT}}(F_r)$, then the proof is similar to the first part of Case 1.
   - If $A_{left}^{\text{OPT}^{-j}}(F_{j,1}^{-j}) \not\supseteq A_{left}^{\text{OPT}}(F_r)$, then a set S of agents, which in OPT^{-j} do not connect to $F_{j,1}^{-j}$, connect to F_r in the optimal solution OPT. Consequently, the agents in S must connect to $F_{j,1,left}^{-j}$ in the OPT^{-j} solution. Formally, for each agent $i \in S$ we have that $d_i(F_r) \leq d_j(F_l) \leq d_i(F_{j,1,left}^{-j}) \leq d_i(F_{j,1}^{-j})$.

The assumptions $F_r \neq F_{j,1}$ and $F_l \neq F_{j,1}$ imply that $A^{\text{OPT}^{-j}}_{right}(F^{-j}_{j,1,left}) \supseteq A^{\text{OPT}}_{right}(F_l)$. Moreover, $A^{\text{OPT}^{-j}}_{left}(F^{-j}_{j,1,left}) \subseteq A^{\text{OPT}}_{left}(F_l)$.
This is symmetrical to case 1, therefore using a similar approach we get that $d_j(F_{j,k}) \geq d_j(F_l) \geq c_{min}$.

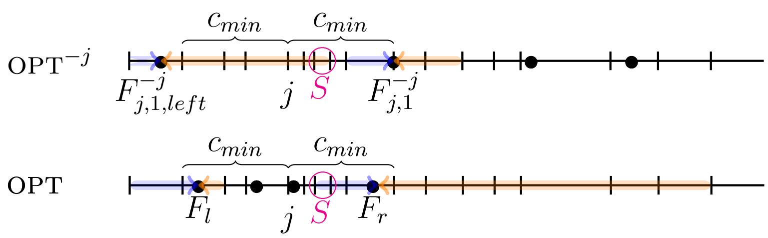


Figure 4.3: Case 2a: Agent j does not connect to either F_r or F_l

(2b) Suppose without loss of generality that $F_l \equiv F_{j,1}$.
We have that $A^{\text{OPT}^{-j}}_{right}(F^{-j}_{j,1,right}) \subseteq A^{\text{OPT}}_{right}(F_r)$.

- If $A^{\text{OPT}^{-j}}_{left}(F^{-j}_{j,1,right}) \supseteq A^{\text{OPT}}_{left}(F_r)$, then the proof is similar to the first part of Case 1.
- If $A^{\text{OPT}^{-j}}_{left}(F^{-j}_{j,1,right}) \not\supseteq A^{\text{OPT}}_{left}(F_r)$, then there exists a set S of agents such that

$$d_i(F_r) \leq d_i(F_{j,1}) < d_i(F^{-j}_{j,1,left}) \leq d_i(F^{-j}_{j,1,right}) \tag{4.6}$$

for each agent $i \in S$.

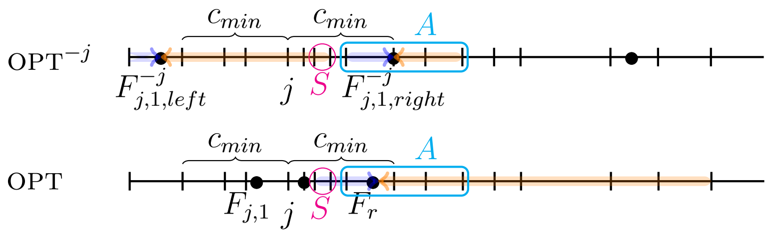


Figure 4.4: Case 2b: Agent j connects to $F_l \equiv F_{j,1}$

Let A be the set of agents that connect to $F^{-j}_{j,1,right}$, i.e.

$$A = A^{\text{OPT}^{-j}}_{right}(F^{-j}_{j,1,right}) \cup A^{\text{OPT}^{-j}}_{left}(F^{-j}_{j,1,right})$$

For the total cost of agents in $A \cup S$ (where $j \notin (A \cup S)$) we have that

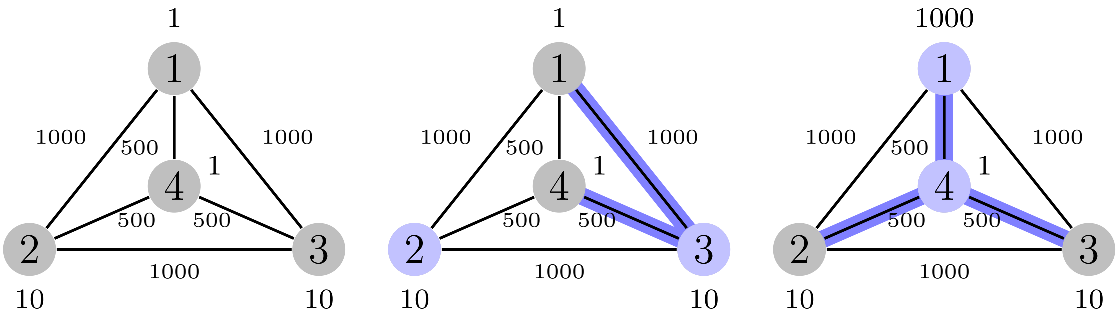
$$\sum_{i \in A} d_i(F^{-j}_{j,1,right}) + \sum_{i \in S} d_i(F^{-j}_{j,1,right}) \geq \sum_{i \in A} d_i(F_r) + \sum_{i \in S} d_i(F_r) \tag{4.7}$$

because of the optimality of OPT.
In the optimal solution OPT^{-j}, if we replace the facility $F^{-j}_{j,1,right}$ with F_r, we create a solution that it is equal or better than OPT^{-j}, but $d_j(F_r) < c_{min}$. Equation 4.7 contradicts our assumption in Remark 4.2.1 and this completes the proof.

□

4.2.2 Applying the Money Burning VCG Mechanism to general metric spaces

Although the Money Burning VCG Mechanism works on the line, it does not work for all metric spaces. The following example illustrates a case, in which the mechanism is not truthful.



(a) The metric graph G (b) Agent 1 reports $w_1 = 1$ (c) Agent 1 reports $w_1 = 1000$

Figure 4.5: Applying the Money Burning VCG Mechanism to metric graph G

Example 4.2.6. Let G be a metric graph. The locations of the 4 agents are x_1, x_2, x_3 and x_4. We define the metric d as follows: $d(x_1, x_2) = d(x_1, x_3) = d(x_2, x_3) = 1000$ and $d(x_1, x_4) = d(x_2, x_4) = d(x_3, x_4) = 500$. We want to locate 2 facilities.

The true demands of agents are $w_1 = 1$, $w_2 = 10$, $w_3 = 10$ and $w_4 = 1$. If all agents report their true demands, then the optimal solution OPT places the two facilities at x_2 and x_3 and we have $\text{OPT} = 1 \cdot 1000 + 10 \cdot 0 + 10 \cdot 0 + 1 \cdot 500 = 1500$. The optimal solution OPT^{-1} places again the two facilities at x_2 and x_3. The optimal cost (for agents 2, 3 and 4) is $\text{OPT}^{-1} = 10 \cdot 0 + 10 \cdot 0 + 1 \cdot 500 = 500$. In this case, the social cost of agent 1 is $cost_1(1, 10, 10, 1) = 1000$.

Next, we assume that agent 1 reports a demand $w_1 = 1000$ instead of $\overline{w}_1 = 1$. The optimal solution OPT places the two facilities at x_1 and x_4 and we have $\text{OPT} =$

$1000 \cdot 0 + 10 \cdot 500 + 10 \cdot 500 + 1 \cdot 0 = 10000$. The optimal cost (for agents 2, 3 and 4) is again $\text{OPT}^{-1} = 500$. The "payment" of agent 1 is equal to $10000 - 500 = 9500$, but his total social cost is $cost_1(1000, 10, 10, 1) = \overline{w}_1 \cdot d(x_1, x_4) = 150$, because $\overline{w}_1 < w_1^{(2)}$. Consequently, we have $cost_1(1000, 10, 10, 1) < cost_1(1, 10, 10, 1)$, thus agent 1 has the incentive to lie. This implies that the mechanism is not truthful in this setting.

4.3 Money Burning Proportional Mechanism

In this section, we follow a different approach to design a truthful mechanism for general metric spaces. We return to the basic idea of the randomized mechanism called Proportional Mechanism which was presented in Chapter 2. In brief, the Money Burning Proportional Mechanism or MBurningPropMech allocates the k facilities to k different agents and as in the Proportional Mechanism, the j-th facility is placed at agent i's location with *probability proportional* to the current cost of agent i. However, we also introduce a *money burning payment scheme*; except for the first round of the mechanism, every agent that gets a facility in rounds 2 to k pays an additional individual cost. The details of the mechanism can be found in Mechanism 4.2.

4.3.1 Payments and truthfulness

The question naturally arises why the already known Proportional Mechanism is truthful or not. The following examples provide intuition as to why payments in the form of additional individual cost are necessary in order to design a truthful proportional-like mechanism.

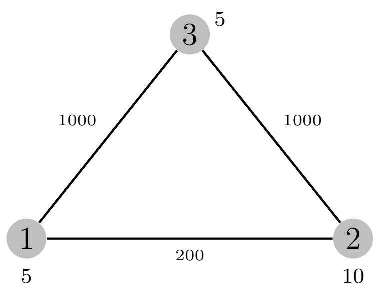


Figure 4.6: *Remark 4.3.1*: The Proportional Mechanism is not truthful

Remark 4.3.1. *(Proportional Mechanism is not truthful without payments.)* While the (winner-imposing) Proportional Mechanism achieves truthfulness in the classic setting of k-Facility Location without resorting to payments, this is not true for our setting.

A *counterexample* for 3 agents and 2 facilities is the following (see also Figure 4.6). All agents that get a facility pay nothing, thus if agent 1 reports his demand truthfully, then agent 1's expected total cost is $\frac{1}{4}\cdot 0+\frac{1}{4}\cdot[\frac{1}{3}\cdot 0+\frac{2}{3}\cdot(5\cdot 200)]+\frac{2}{4}\cdot[\frac{1}{6}\cdot 0+\frac{5}{6}\cdot(5\cdot 200)] = \frac{1750}{3}$. However, if agent 1 reports $w_1 = 10 > \overline{w}_1$, then agent 1's expected total cost is $\frac{2}{5}\cdot 0+\frac{1}{5}[\frac{1}{2}\cdot 0+\frac{1}{2}\cdot(5\cdot 200)]+\frac{2}{5}\cdot[\frac{2}{7}\cdot 0+\frac{5}{7}\cdot(5\cdot 200)] = \frac{2700}{7} < \frac{1750}{3}$. This means that agent 1 has the incentive to lie.

Remark 4.3.2. *(Proportional Mechanism with payments only at Round 1 is not truthful.)* An other observation is that even a relatively high payment in the first round of the mechanism does not guarantee truthfulness. Here, we consider a version of the Proportional Mechanism, where there is a payment only for the agent that gets the (second) facility at Round 2. The payment is equal to the best current cost of this agent.

We provide a counterexample with $k = 3$ facilities and $n = 4$ agents (see Figure 4.7) to show that such a mechanism is not truthful. Indeed, if agent 4 reports a demand $w_4 = 100$ instead of $\overline{w}_4 = 1$, then we can easily see that his expected cost decreases. Though the possible payment at Round 2 was really high, *in expectation* agent 4 will benefit from lying. In fact, due to the fact that all distances are equal to 1, the agent can manipulate the mechanism by misreporting a higher demand. Thus, the probability that he gets a facility is directly proportional to his reported demand.

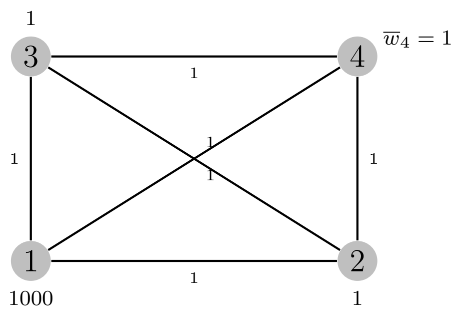


Figure 4.7: *Remark 4.3.2*: The Proportional Mechanism with payments only at Round 2 is not truthful

It becomes clear that Proportional Mechanism without payments is not truthful and that (non-monetary) payments are needed for more than one rounds. In the next section, we present our randomized money burning proportional-like mechanism

MBurningPropMech with payments at all rounds 2 to k; an interesting question is whether payments are necessary for every round $r > 1$.

4.3.2 The Mechanism

Let $\overline{w_j}$ and w_j denote the true and the reported demand of an agent j, respectively. The mechanism takes as input only the demands w_1, ..., w_n reported by the n agents, given that their locations x_1, ..., x_n are public knowledge. The mechanism works in $k+1$ rounds and at each round $r = 1, ..., k$ selects the location of the next facility among the locations x_1, ..., x_n of the agents. The mechanism also decides the single facility or the two facilities that each agent has to connect to. At round $k+1$ the remaining $n-k$ agents, i.e. agents that have not been selected by the mechanism during rounds 1 to k, connect to the facility that it is closest to their location.

For each round $r = 1, ..., k$, we define the set C_r of locations of the r facilities placed in round 1 to r, as well as the set A_r of agents that have not been connected to any facility yet. We set $C_0 = \varnothing$ and $A_0 \equiv A$. The mechanism is given on the next page.

Theorem 4.3.3. MBurningPropMech *is* truthful.

Proof. For each round $r = 1, .., k$, let $cost_j[w_j|C_{r-1}]$ be the expected total cost of agent $j \in A_{r-1}$ with reported demand w_j, given that the mechanism has already placed the first $r-1$ facilities. Thus, the ex-ante total cost of an agent who participates in the mechanism is $cost_j[w_j] = cost_j[w_j|C_0]$. We also assume that $\overline{w}_j \leq w_j$, where $\overline{w}_j$ is agent j's true demand.

At round r, agent $j \in A_{r-1}$ is selected with probability $\frac{w_j d_j(C_{r-1})}{\sum_{i \in A_{r-1}} w_i d_i(C_{r-1})}$ and its final cost is equal to $\min(aw_j, \overline{w}_j) d_j(C_{r-1})$. If, instead of agent j, a different agent $i \in A_{r-1}$, $i \neq j$ is chosen, then the expected cost of agent j is $cost_j[w_j|C_{r-1} \cup \{x_i\}]$. To sum up, we get the following formula:

$$cost_j[w_j|C_{r-1}] = \frac{\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) cost_j[w_j|C_{r-1} \cup \{x_i\}] + \min\{\overline{w}_j, aw_j\} d_j(C_{r-1}) w_j d_j(C_{r-1})}{\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1})} \quad (4.9)$$

We will show that $cost_j[w_j|C_0] \geq cost_j[\overline{w}_j|C_0]$, i.e. the mechanism is truthful. First, we prove, by induction on r, that $cost_j[w_j|C_{r-1}] \geq cost_j[\overline{w}_j|C_{r-1}]$ for every $r = 2, ..., k$, w_j, $\overline{w}_j$ s.t. $w_j \geq \overline{w}_j$. For the next part of the proof, we rest on the *induction hypothesis* that $cost_j[w_j|C_r] \geq cost_j[\overline{w}_j|C_r]$.

Round 1: One location x_{j_1} out of x_1, ..., x_n is chosen uniformly at random. The first facility is placed at x_{j_1} and agent j_1 connects to this facility. The cost of agent j_1 is zero. We set $C_1 = \{x_{j_1}\}$ and $A_1 = A_0 \setminus \{j_1\}$.

Round $r = 2, 3, ..., k$: The location x_{j_r} of the r-th facility is selected with probability equal to $\frac{w_{j_r} d_{j_r}(C_{r-1})}{\sum_{j \in A_r} w_j d_j(C_{r-1})}$. The facility at x_{j_r} serves only $\max(\overline{w}_{j_r} - a w_{j_r}, 0)$ of agent j_r's actual demand $\overline{w}_{j_r}$, where

$$a = \frac{\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) \min\{d_j(C_{r-1}), d_j(x_i))\}}{d_j(C_{r-1})(2 \sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + d_j(C_{r-1}))} \tag{4.8}$$

The facility in C_{r-1} that it is closest to x_{j_r}, serves the rest $\min(a w_{j_r}, \overline{w}_{j_r})$ of agent j_r's demand $\overline{w}_{j_r}$. The total cost of agent j_r is $\min(a w_{j_r}, \overline{w}_{j_r}) d_{j_r}(C_{r-1})$. We also set $C_r = C_{r-1} \cup \{x_{j_r}\}$ and $A_r = A_{r-1} \setminus \{j_r\}$.

Round $k+1$: Finally, the set of the facilities is C_k. The demand of every agent $i \in A_k$ is served by the facility in C_k that is closest to this agent i.

Mechanism 4.2: The Money Burning Proportional Mechanism

Let $\epsilon > 0$ be a small constant, s.t. $w'_j = w_j + \epsilon$.

$$\frac{\mathrm{d}cost_j[w_j|C_{r-1}]}{\mathrm{d}w_j} = \lim_{\epsilon \to 0} \frac{cost_j[w_j + \epsilon|C_{r-1}] - cost_j[w_j|C_{r-1}]}{\epsilon} \tag{4.10}$$

where $\frac{cost_j[w_j+\epsilon|C_{r-1}] - cost_j[w_j|C_{r-1}]}{\epsilon}$ is equal to

$$A(\epsilon) = \frac{1}{\epsilon(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + (w_j + \epsilon) d_j(C_{r-1}))(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1}))}.$$

$$\Big[\Big(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1})\Big) \sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) w_j cost_j[w_j + \epsilon | C_{r-1} \cup \{x_i\}]$$

$$+\Big(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1})\Big) a (w_j + \epsilon) d_j(C_{r-1})(w_j + \epsilon) d_j(C_{r-1})$$

$$\Big(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + (w_j + \epsilon) d_j(C_{r-1})\Big) \sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) w_j cost_j[w_j | C_{r-1} \cup \{x_i\}]$$

$$-(\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + (w_j+\epsilon)d_j(C_{r-1}))aw_j d_j(C_{r-1})w_j d_j(C_{r-1})\Big]$$

As ϵ approaches 0, the limit of the function is

$$\lim_{\epsilon\to 0} A(\epsilon) = \lim_{\epsilon\to 0} \frac{\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1})[\frac{cost_j[w_j+\epsilon|C_{r-1}\cup\{x_i\}]-cost_j[w_j|C_{r-1}\cup\{x_i\}]}{\epsilon}]}{\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + (w_j+\epsilon)d_j(C_{r-1})}$$
$$+\lim_{\epsilon\to 0}\frac{ad_j^2(C_{r-1})w_j^2\epsilon + 2ad_j^2(C_{r-1})w_j}{\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + (w_j+\epsilon)d_j(C_{r-1})}$$

$$-\lim_{\epsilon\to 0}\frac{d_j(C_{r-1})(aw_j^2 d_j^2(C_{r-1}) + \sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1})cost_j[w_j|C_{r-1}\cup\{x_i\}])}{\epsilon(\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + (w_j+\epsilon)d_j(C_{r-1}))(\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1}))} =$$

$$= \frac{\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1})\ \frac{\mathrm{d}cost_j[w|C_{r-1}\cup\{x_i\}]}{\mathrm{d}w}\Big|_{w=w_j}}{\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1})} + \frac{2ad_j^2(C_{r-1})w_j}{\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1})}$$
$$-\frac{d_j(C_{r-1})(aw_j^2 d_j^2(C_{r-1}) + \sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1})cost_j[w_j|C_{r-1}\cup\{x_i\}])}{(\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1}))^2}$$

We require the function $\frac{\mathrm{d}cost_j[w_j|C_{r-1}]}{\mathrm{d}w_j} = \lim_{\epsilon\to 0} A(\epsilon)$ to be positive (or at least zero) for every w_j. By the induction hypothesis it holds that $\frac{\mathrm{d}cost_j[w|C_{r-1}\cup\{x_i\}]}{\mathrm{d}w_j}\Big|_{w=w_j}$ for every $i\in A_{r-1}$, hence it suffices to solve the following inequality for a:

$$\frac{2ad_j^2(C_{r-1})w_j}{\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1})} - \frac{d_j(C_{r-1})(aw_j^2 d_j^2(C_{r-1}) + \sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1})cost_j[w_j|C_{r-1}\cup\{x_i\}])}{(\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1}))^2} \geq 0$$

We obtain that

$$a \geq \frac{\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1})cost_j[w_j|C_{r-1}\cup\{x_i\}]}{w_j d_j(C_{r-1})(2\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1}))} \tag{4.11}$$

Using the inequality $w_j \min\{d_j(C_{r-1}), d_j(x_i))\} \geq cost_j[w_j|C_{r-1}\cup\{x_i\}]$, a should suffice to be equal to:

$$a = \frac{\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1})\min\{d_j(C_{r-1}), d_j(x_i))\}}{d_j(C_{r-1})(2\sum_{i\in A_{r-1},i\neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1}))} \tag{4.12}$$

The parameter a should not depend on demand w_j, therefore we set a to be equal to

$$a = \frac{\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) \min\{d_j(C_{r-1}), d_j(x_i))\}}{d_j(C_{r-1})(2\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + d_j(C_{r-1}))} \tag{4.13}$$

Up to this point, we have proved that $cost_j[w_j|C_{r-1}] \geq cost_j[\overline{w}_j|C_{r-1}]$ for every round $r = 2, .., k$, when $\min\{aw_j, \overline{w}_j\} = aw_j$. We now consider the case $\min\{aw_j, \overline{w}_j\} = \overline{w}_j$. We need to study the monotonicity of the function

$$f(w_j) = \frac{\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) cost_j[w_j|C_{r-1} \cup \{x_i\}] + \overline{w}_j d_j(C_{r-1}) w_j d_j(C_{r-1})}{\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1})} \tag{4.14}$$

The derivative of f is

$$\frac{\mathrm{d}f(w_j)}{\mathrm{d}w_j} = \frac{(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) \frac{\mathrm{d}cost_j[w_j|C_{r-1} \cup \{x_i\}]}{\mathrm{d}w_j} + \overline{w}_j d_j^2(C_{r-1}))(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1}))}{(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1}))^2}$$

$$-\frac{d_j(C_{r-1})(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) cost_j[w_j|C_{r-1} \cup \{x_i\}] + \overline{w}_j w_j d_j^2(C_{r-1}))}{(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1}))^2} =$$

$$= \frac{\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) \frac{\mathrm{d}cost_j[w_j|C_{r-1} \cup \{x_i\}]}{\mathrm{d}w_j}}{\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1})} +$$

$$\frac{\overline{w}_j d_j^2(C_{r-1}) \sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) - d_j(C_{r-1}) \sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) cost_j[w_j|C_{r-1} \cup \{x_i\}]}{(\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1}))^2}$$

and by applying the induction hypothesis, as well as the inequality

$$\overline{w}_j d_j(C_{r-1}) \geq cost_j[w_j|C_{r-1} \cup \{x_i\}],$$

we obtain that $\frac{\mathrm{d}f(w_j)}{\mathrm{d}w_j} \geq 0$.

Consequently, if $w_j \geq \overline{w}_j$, then

$$cost_j[w_j|C_{r-1}] = f(w_j) \geq f(\overline{w}_j) =$$

$$= \frac{\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) cost_j[w_j|C_{r-1} \cup \{x_i\}] + a\overline{w}_j d_j(C_{r-1}) w_j d_j(C_{r-1})}{\sum_{i \in A_{r-1}, i \neq j} w_i d_i(C_{r-1}) + w_j d_j(C_{r-1})}$$

$$\geq cost_j[\overline{w}_j|C_{r-1}]$$

In conclusion, so far we have showed that $cost_j[w_j|C_{r-1}] \geq cost_j[\overline{w}_j|C_{r-1}]$ for rounds $r = 2, ..., k$. To prove the truthfulness of the mechanism, we need to show that $cost_j[w_j] \geq cost_j[\overline{w}_j]$. It is easy to see that for $r = 1$

$$cost_j[w_j] = cost_j[w_j|C_0] = \frac{1}{n}\sum_{i\neq j} cost_j[w_j|C_0 \cup \{x_i\}]$$

$$\geq \frac{1}{n}\sum_{i\neq j} cost_j[\overline{w}_j|C_0 \cup \{x_i\}] = cost_j[\overline{w}_j|C_0] = cost_j[\overline{w}_j] \tag{4.15}$$

The *basis* case is for $r = k$, thus we need to prove that $cost_j[w_j|C_{k-1}] \geq cost_j[\overline{w}_j|C_{k-1}]$. We have the following equations:

$$cost_j[w_j|C_{k-1}] = \frac{\sum_{i\in A_{k-1}, i\neq j} w_i d_i(C_{k-1})\overline{w}_j \min\{d_j(x_i), d_j(C_{k-1})\} + \min\{\overline{w}_j, aw_j\}d_j(C_{k-1})w_j d_j(C_{k-1})}{\sum_{i\in A_{k-1}, i\neq j} w_i d_i(C_{k-1}) + w_j d_j(C_{k-1})}$$

$$cost_j[\overline{w}_j|C_{k-1}] = \frac{\sum_{i\in A_{k-1}, i\neq j} w_i d_i(C_{k-1})\overline{w}_j \min\{d_j(x_i), d_j(C_{k-1})\} + a\overline{w}_j d_j(C_{k-1})\overline{w}_j d_j(C_{k-1})}{\sum_{i\in A_{k-1}, i\neq j} w_i d_i(C_{k-1}) + \overline{w}_j d_j(C_{k-1})}$$

First, we consider the case where $\min\{\overline{w}_j, aw_j\} = \overline{w}_j$ and focus on the monotonicity of the function $f(w_j) = cost_j[w_j|C_{k-1}]$. The derivative of f is

$$\frac{\mathrm{d}f(w_j)}{\mathrm{d}w_j} = d_j(C_{k-1})\frac{\overline{w}_j d_j(C_{k-1})(\sum_{i\in A_{k-1}, i\neq j} w_i d_i(C_{k-1})) - \sum_{i\in A_{k-1}, i\neq j} w_i d_i(C_{k-1})\overline{w}_j \min\{d_j(x_i), d_j(C_{k-1})\}}{(\sum_{i\in A_{k-1}, i\neq j} w_i d_i(C_{k-1}) + \overline{w}_j d_j(C_{k-1}))^2}$$

and $\frac{\mathrm{d}f(w_j)}{\mathrm{d}w_j} \geq 0$, because $d_j(C_{k-1}) \geq \min\{d_j(x_i), d_j(C_{k-1})\}$.

Consequently, we obtain that

$$cost_j[w_j|C_{k-1}] \geq \frac{\sum_{i\in A_{k-1}, i\neq j} w_i d_i(C_{k-1})\overline{w}_j \min\{d_j(x_i), d_j(C_{k-1})\} + \overline{w}_j d_j(C_{k-1})\overline{w}_j d_j(C_{k-1})}{\sum_{i\in A_{k-1}, i\neq j} w_i d_i(C_{k-1}) + \overline{w}_j d_j(C_{k-1})} \geq cost_j[\overline{w}_j|C_{k-1}]$$

Next, we assume that $\min\{\overline{w}_j, aw_j\} = aw_j$. Using a technique similar to the general case, we get that a given by 4.8 will satisfy the inequality $cost_j[w_j|C_{k-1}] \geq cost_j[\overline{w}_j|C_{k-1}]$. □

The theorem below shows that the mechanism is feasible, as well as that an agent does not need to pay more than 1/2 of his current cost at the round that he gets a facility.

Theorem 4.3.4. *The mechanism is* feasible*, i.e. it holds that* $a \leq 1/2 < 1$.

Proof. The property follows immediately from the inequality

$$d_j(C_{r-1}) \geq \min\{d_j(C_{r-1}), d_j(x_i)\}$$

for every $i \in A_{r-1}$. □

4.4 Conclusion and future work

Summary of results. In this Chapter, we studied a Facility Location problem where n agents with private splittable demands are located in a metric space and k facilities that serve a specific type of service must be placed. We combined a money burning technique with well-known results in the field of Mechanism Design in order to design truthful approximate mechanisms for this Facility Location problem.

On our first attempt, we used the VCG mechanism as a basis for designing a VCG-like mechanism (MBurningVCG) that enforces non-monetary payments. We provided a sufficient condition for the truthfulness of MBurningVCG and proved that MBurningVCG is a truthful n-approximate mechanism on the line.

In addition, based on the Proportional Mechanism without money, we designed a truthful Money Burning Proportional Mechanism (MBurningPropMech) which works in k rounds and uses a money burning technique. The main idea is that the r-th facility is placed at agent i's location with probability proportional to the current best cost of agent i. If agent i gets a facility in rounds $r = 2$ to k, then MBurningPropMech requires agent i to serve part of his reported demand at a non-zero total cost, while the rest of agent i's demand is served at zero cost.

Directions for future research. Current work includes modifying MBurningPropMech in order to get a good, bounded approximation ratio; payments should remain as small as possible. One other direction could be studying settings where agents' locations are private, so each agent can lie about his true demand, his true location or both. Furthermore, an extension to our Facility Location could include different types of services. In such a heterogeneous model, each facility will serve a different subset of services and each agent will have a demand for some of the service types offered.

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