Zero rest-mass fields and the Newman-Penrose constants on flat space

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Abstract

Zero rest-mass fields of spin 1 (the electromagnetic field) and spin 2 propagating on flat space and their corresponding Newman-Penrose (NP) constants are studied near spatial infinity. The aim of this analysis is to clarify the correspondence between data for these fields on a spacelike hypersurface and the value of their corresponding NP constants at future and past null infinity. To do so, Friedrich's framework of the cylinder at spatial infinity is employed to show that, expanding the initial data in terms spherical harmonics and powers of the geodesic spatial distance ρ to spatial infinity, the NP constants correspond to the data for the second highest possible spherical harmonic at fixed order in ρ. In addition, it is shown that for generic initial data within the class considered in this article, there is no natural correspondence between the NP constants at future and past null infinity —for both the Maxwell and spin-2 field. However, if the initial data is time-symmetric then the NP constants at future and past null infinity have the same information.

Keywords: Conformal methods, spinors, Newman-Penrose constants, cylinder at spatial infinity, soft-hair.

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1 Introduction

The concept of asymptotic simplicity is central for the understanding of isolated systems in general relativity. In this regard, Penrose's proposal [20] is an attempt to characterise the fall-off behaviour of the gravitational field in a geometric manner —see also [11]. The essential mathematical idea behind for the Penrose proposal is that of a *conformal transformation*: given a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfying the Einstein field equations (*the physical spacetime*) one considers a 4-dimensional Lorentzian manifold $\mathcal{M}$ equipped with a metric $\boldsymbol{g}$ such that $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$ are conformal to each other, in other words

$$\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}},$$

where Ξ is the so-called *conformal factor*. The pair $(\mathcal{M}, \boldsymbol{g})$ can be called the *unphysical spacetime*. The set of points where $\Xi = 0$ but $\mathbf{d}\Xi \neq 0$ is called the *null infinity* and is denoted by $\mathscr{I}$. If

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$\tilde{\boldsymbol{g}}$ satisfies the vacuum Einstein field equations (with vanishing Cosmological constant) near $\mathscr{I}$, then the conformal boundary defines a smooth null hypersurface of $\mathcal{M}$ —see [11, 25]. One can identify two disjoint pieces of $\mathscr{I}$: $\mathscr{I}^-$ and $\mathscr{I}^+$ correspond to the past and future end points of null geodesics. If every null geodesic acquires two distinct endpoints at $\mathscr{I}$, the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is said to be *asymptotically simple* —see [11, 25, 33] for precise definitions. The Minkowski spacetime, $(\mathbb{R}^4, \tilde{\boldsymbol{\eta}})$ is the prototypical example of an asymptotically simple spacetime. In the standard conformal representation of the Minkowski spacetime, the unphysical spacetime can be identified with the Einstein cylinder $(\mathcal{M}_E, \boldsymbol{g}_E)$ where $\mathcal{M}_{\mathcal{E}} \approx \mathbb{R} \times \mathbb{S}^3$ and

$$\boldsymbol{g}_E = \mathbf{d}T \otimes \mathbf{d}T - \mathbf{d}\psi \otimes \mathbf{d}\psi - \sin^2\psi \boldsymbol{\sigma}, \qquad \Xi = \cos(T) + \cos(\psi),$$

where $-\pi < T < \pi$, $0 < \psi < \pi$ and $\boldsymbol{\sigma}$ is the standard metric on $\mathbb{S}^2$. In this conformal representation $\mathscr{I}^{\pm}$ correspond to the sets of points on the Einstein cylinder, $\mathcal{M}_{\mathcal{E}} \equiv \mathbb{R} \times \mathbb{S}^3$, for which $0 < \psi < \pi$ and $T = \pm(\pi - \psi)$. One can directly verify that $\Xi|_{\mathscr{I}^{\pm}} = 0$ while $\mathbf{d}\Xi|_{\mathscr{I}^{\pm}} \neq 0$ —see [25]. Consequently, a distinguished region in the conformal structure of the Minkowski spacetime is *spatial infinity* i^0 for which both $\Xi|_{i^0}$ and $\mathbf{d}\Xi|_{i^0}$ vanish. In this conformal representation, spatial infinity corresponds to a point in the Einstein cylinder with coordinates $\psi = \pi$ and $T = 0$.

A natural problem to be considered is the existence of spacetimes whose conformal structure resembles that of the Minkowski spacetime. In this setting, the conformal Einstein field equations introduced originally in [6] provide a convenient framework for discussing global existence of *asymptotically simple* solutions to the Einstein field equations. An important application of these equations is the proof of the semi-global non-linear stability of the Minkowski spacetime given in [6]. In the latter work, the evolution of perturbed initial data close to exact Minkowski data is analysed. Nevertheless, the initial data is not prescribed on a Cauchy hypersurface $\tilde{\mathcal{S}}$ but in an hyperboloid $\tilde{\mathcal{H}}$ whose conformal extension in $\mathcal{M}$ intersects $\mathscr{I}$. Therefore, an open problem in the framework of the conformal Einstein field equations is the analysis of the evolution of initial data prescribed on a Cauchy hypersurface $\mathcal{S}$ intersecting i^0 —see [5] for the proof of the global non-linear stability of the Minkowski spacetime employing different methods. One of the main difficulties in establishing a global result for the stability of the Minkowski spacetime using conformal methods lies on the fact that the initial data for the conformal Einstein field equations is not smooth at i^0. This is not unexpected since, as observed by Penrose —see [20, 21], the conformal structure of spacetimes with non-vanishing mass becomes singular at spatial infinity. A milestone in the resolution of this problem is the construction, originally introduced in [8], of a new representation of spatial infinity known as the *cylinder at spatial infinity*. In this representation, spatial infinity is not represented as a point but as set whose topology is that of a cylinder. This representation is well adapted to exploit the properties of curves with special conformal properties: *conformal geodesics*. In addition, it allows to formulate a regular finite initial value problem for the conformal Einstein field equations —other approaches for analysing the gravitational field near spatial infinity using different representations of spatial infinity have been also proposed in literature —see [23, 4, 3, 24].

The framework of the cylinder at spatial infinity and its connection with the conformal Einstein field equations have been exploited in an analysis of the *gravitational Newman-Penrose (NP) constants* in [12]. The NP constants, originally introduced in [19], are defined in terms of integrals over cuts $\mathcal{C} \approx \mathbb{S}^2$ of $\mathscr{I}$. The integrands in the expressions defining the NP constants are, however, written in a particular gauge adapted to $\mathscr{I}$ (the so-called *NP-gauge*) while the natural gauge used in the framework of the cylinder at spatial infinity (the so-called *F-gauge* in [12]), is adapted to a congruence of conformal geodesics and hinged at a Cauchy hypersurface $\mathcal{S}$. This fact, which in first instance looks as an obstacle to analyse the NP constants, turns out to be advantageous since, once the relation between the NP-gauge and the F-gauge is clarified, one can relate the initial data prescribed on $\mathcal{S}$ with the gravitational NP constants at $\mathscr{I}$.

In [17], the authors exploited the notion of these conserved quantities at $\mathscr{I}$ to make inroads into the problem of the information paradox —see [16, 14, 15]. In the latter work, the concept of *soft hair* is motivated by means of an analysis of the conservation laws and symmetries of abelian gauge theories in Minkowski space. These conservation laws correspond essentially to the electromagnetic version of the gravitational NP constants. With this motivation, in the present article zero rest-mass fields propagating on flat space and their corresponding NP constants are studied. Two physically relevant fields are analysed: the spin-1 and spin-2 zero rest-mass fields. The spin-1 field provides a description of the electromagnetic field while the spin-2 field on the Minkowski spacetime describes linearised gravity.

In this article it is shown how the framework of the cylinder at spatial infinity can be exploited to relate the corresponding NP constants with the initial data on a Cauchy hypersurface intersecting i^0 —see Propositions 6 and 8 for the spin-1 case and Proposition 9 and 10 for the spin-2 case. Additionally, it is shown that, for generic initial data within the class considered, the NP constants at $\mathscr{I}^+$ and $\mathscr{I}^-$ do not coincide. However, the correspondence between the NP constants at $\mathscr{I}^+$ and $\mathscr{I}^-$ is fulfilled for time-symmetric initial data —see Theorems 1 and 2.

1.1 Outline of the paper

Section 2 contains a general discussion of the cylinder at spatial infinity and the F-gauge in the Minkowski spacetime. In Section 3, the Maxwell equations are written in the F-gauge and the initial data for the electromagnetic field on a spacelike hypersurface is discussed. In Section 4, the equations governing the massless spin-2 field are expressed in the F-gauge and the corresponding initial data is discussed. In Section 5, Bondi coordinates and a NP-frame for a conformal extension of the Minkowski spacetime is derived. Additionally, the relation between this frame and the one introduced in Section 2 is determined explicitly. In Section 6 the electromagnetic NP constants are introduced and written in the F-gauge. This construction is exploited to identify the electromagnetic NP constants with part of the initial data introduced in Section 3. In Section 7 a similar analysis is carried out for the spin-2 field; the corresponding NP constants are found and written in terms of the initial data. Section 8 provides with some concluding remarks. In addition, a general discussion of the connection on $\mathbb{S}^2$ is given in Appendix A and a discussion of the $\eth$ and $\bar{\eth}$ operators of Newman and Penrose is provided in Appendix B.

1.2 Notations and Conventions

The signature convention for (Lorentzian) spacetime metrics will be $(+,-,-,-)$. In the rest of this article $\{{}_a,{}_b,{}_c,...\}$ denote abstract tensor indices and $\{{}_{\boldsymbol{a}},{}_{\boldsymbol{b}},{}_{\boldsymbol{c}},...\}$ will be used as spacetime frame indices taking the values $0,...,3$. In this way, given a basis $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ a generic tensor is denoted by T_{ab} while its components in the given basis are denoted by $T_{\boldsymbol{ab}} \equiv T_{ab}\boldsymbol{e}_{\boldsymbol{a}}{}^a\boldsymbol{e}_{\boldsymbol{b}}{}^b$. Part of the analysis will require the use of spinors. In this respect, the notation and conventions of Penrose & Rindler [22] will be followed. In particular, capital Latin indices $\{{}_A,{}_B,{}_C,...\}$ will denote abstract spinor indices while boldface capital Latin indices $\{{}_{\boldsymbol{A}},{}_{\boldsymbol{B}},{}_{\boldsymbol{C}},...\}$ will denote frame spinorial indices with respect to a specified spin dyad $\{\delta_{\boldsymbol{A}}{}^A\}$. The conventions for the curvature tensors are fixed by the relation

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)v^c = R^c{}_{dab}v^d.$$

2 The cylinder at spatial infinity and the F-Gauge

In this section a conformal representation of the Minkowski spacetime that is adapted to a congruence of conformal geodesics is discussed. This conformal representation, introduced originally in [8], is particularly suited for analysing the behaviour of fields near spatial infinity. In broad terms, in this representation spatial infinity i^0, which corresponds to a point in the standard compactification of the Minkowski spacetime, is blown up to a two-sphere $\mathbb{S}^2$. In the subsequent discussion this representation will be referred as the *cylinder at spatial infinity.* The discussion of the cylinder at spatial infinity as presented in [8] is given in the language of fibre bundles. In particular, the construction of the so-called *extended bundle space* is required —see [8, 1]. Nevertheless, a discussion which does not make use of this construction is presented in the following.

2.1 The cylinder at spatial infinity

Consider the Minkowski metric $\tilde{\boldsymbol{\eta}}$ in cartesian coordinates $\tilde{x}^\alpha = (\tilde{t}, \tilde{x}^i)$,

$$\tilde{\boldsymbol{\eta}} = \eta_{\mu\nu}\mathbf{d}\tilde{x}^\mu \otimes \mathbf{d}\tilde{x}^\nu,$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Introducing polar coordinates defined by $\tilde{\rho} = \delta_{ij}\tilde{x}^i\tilde{x}^j$ where $\delta_{ij} = \mathrm{diag}(1,1,1)$, and an arbitrary choice of coordinates on $\mathbb{S}^2$, the metric $\tilde{\boldsymbol{\eta}}$ can be written as

$$\tilde{\boldsymbol{\eta}} = \mathbf{d}\tilde{t}\otimes\mathbf{d}\tilde{t} - \mathbf{d}\tilde{\rho}\otimes\mathbf{d}\tilde{\rho} - \tilde{\rho}^2\boldsymbol{\sigma}, \tag{1}$$

with $\tilde{t} \in (-\infty)$, $\tilde{\rho} \in [0,\infty)$ and $\boldsymbol{\sigma}$ denotes the standard metric on $\mathbb{S}^2$. A common procedure to obtain a conformal representation of the Minkowski spacetime close to i^0 is to introduce *inversion coordinates* $x^\alpha = (t, x^i)$ defined by —see [25],

$$x^\mu = -\tilde{x}^\mu/\tilde{X}^2, \qquad \tilde{X}^2 \equiv \tilde{\eta}_{\mu\nu}\tilde{x}^\mu\tilde{x}^\nu.$$

The inverse transformation is given by

$$\tilde{x}^\mu = -x^\mu/X^2, \qquad X^2 = \eta_{\mu\nu}x^\mu x^\nu.$$

Using these coordinates one readily identifies the following conformal representation of the Minkowski spacetime

$$\boldsymbol{g}_I = \Xi^2\,\tilde{\boldsymbol{\eta}}, \tag{2}$$

where $\boldsymbol{g}_I = \eta_{\mu\nu}\mathbf{d}x^\mu\otimes\mathbf{d}x^\nu$ and $\Xi = X^2$. Notice, additionally that, $X^2 = 1/\tilde{X}^2$. Introducing an *unphysical polar coordinate* defined as $\rho = \delta_{ij}x^ix^j$, one observes that the rescaled metric $\boldsymbol{g}_I$ and conformal factor Ξ read

$$\boldsymbol{g}_I = \mathbf{d}t\otimes\mathbf{d}t - \mathbf{d}\rho\otimes\mathbf{d}\rho - \rho^2\boldsymbol{\sigma}, \qquad \Xi = t^2 - \rho^2, \tag{3}$$

with $t \in (-\infty,\infty)$ and $\rho \in (0,\infty)$. In this conformal representation, spatial infinity i^0 corresponds to a point located at the origin. For future reference, observe that $\tilde{t}$ and $\tilde{\rho}$ are related to t and ρ via

$$\tilde{t} = -\frac{t}{t^2-\rho^2}, \qquad \tilde{\rho} = \frac{\rho}{t^2-\rho^2}. \tag{4}$$

Then, one introduces a time coordinate τ defined via $t = \rho\tau$. In the coordinate system determined by τ and ρ the metric $\boldsymbol{g}_I$ is written as

$$\boldsymbol{g}_I = \rho^2\mathbf{d}\tau\otimes\mathbf{d}\tau - (1-\tau^2)\mathbf{d}\rho\otimes\mathbf{d}\rho + \rho\tau\mathbf{d}\rho\otimes\mathbf{d}\tau + \rho\tau\mathbf{d}\tau\otimes\mathbf{d}\rho - \rho^2\boldsymbol{\sigma}.$$

The required conformal representation is obtained by considering the rescaled metric

$$\boldsymbol{g} \equiv \frac{1}{\rho^2}\boldsymbol{g}_I. \tag{5}$$

Explicitly one has that

$$\boldsymbol{g} = \mathbf{d}\tau\otimes\mathbf{d}\tau - \frac{(1-\tau^2)}{\rho^2}\mathbf{d}\rho\otimes\mathbf{d}\rho + \frac{\tau}{\rho}(\mathbf{d}\tau\otimes\mathbf{d}\rho + \mathbf{d}\rho\otimes\mathbf{d}\tau) - \boldsymbol{\sigma}. \tag{6}$$

Observe that spatial infinity i^0, which is at infinity respect to the metric $\boldsymbol{g}$, corresponds to a set which has the topology of $\mathbb{R}\times\mathbb{S}^2$ —see [8, 1]. In what follows we continue using the coordinates

(τ, ρ) and call them the *F-coordinates*. Following the conformal rescalings previously introduced one considers the conformal extension $(\mathcal{M}, \boldsymbol{g})$ where

$$\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{\eta}}, \qquad \Theta = \rho(1 - \tau^2), \tag{7}$$

and

$$\mathcal{M} \equiv \{p \in \mathbb{R}^4 \mid -1 \leq \tau \leq 1, \ \ \rho(p) \geq 0\}.$$

In this representation future and past null infinity are located at

$$\mathscr{I}^+ \equiv \{p \in \mathcal{M} \mid \tau(p) = 1\}, \qquad \mathscr{I}^- \equiv \{p \in \mathcal{M} \mid \tau(p) = -1\},$$

and the physical Minkowski spacetime can be identified with the region

$$\tilde{\mathcal{M}} \equiv \{p \in \mathcal{M} \mid -1 < \tau(p) < 1, \ \ \rho(p) > 0\},$$

In addition, the following sets will be distinguished:

$$I \equiv \{p \in \mathcal{M} \mid \ |\tau(p)| < 1, \ \rho(p) = 0\}, \qquad I^0 \equiv \{p \in \mathcal{M} \mid \tau(p) = 0, \ \rho(p) = 0\},$$

$$I^+ \equiv \{p \in \mathcal{M} \mid \tau(p) = 1, \ \rho(p) = 0\}, \qquad I^- \equiv \{p \in \mathcal{M} \mid \tau(p) = -1, \ \rho(p) = 0\}.$$

Notice that spatial infinity i^0, which originally was a point in the $\boldsymbol{g}_I$−representation, can be identified with the set I in the $\boldsymbol{g}$−representation. In addition, one can intuitively think of the *critical sets* I^+ and I^- as the region where spatial infinity "touches" $\mathscr{I}^+$ and $\mathscr{I}^-$ respectively. Similarly, I^0 represents the intersection of i^0 and the initial hypersurface $\mathcal{S} \equiv \{\tau = 0\}$. See [8, 12] and [1] for further discussion of the framework of the cylinder at spatial infinity implemented for stationary spacetimes.

2.2 The F-gauge

In this section a brief discussion of the so-called F-gauge is provided —see [12, 1] for a discussion of the F-gauge in the language of fibre bundles. Following the philosophy of the previous section, the discussion presented here will not make use of the extended bundle space —see [12, 1] for definitions. One of the motivations for the introduction of this gauge is that it exploits the properties of conformal geodesics. More precisely, in this framework, one introduces a null frame whose timelike leg corresponds to the tangent of a conformal geodesic starting from a fiduciary spacelike hypersurface $\mathcal{S} = \{\tau = 0\}$. The notion of conformal geodesics, however, will not be discussed here —see [7, 10, 26, 33] for definitions and further discussion.

To start the discussion, consider the conformal extension $(\mathcal{M}, \boldsymbol{g})$ of the Minkowski spacetime and the F-coordinate system introduced in Section 2.1. Observe that the induced metric on the surface $\mathcal{Q} \equiv \{\tau = \tau_\star, \rho = \rho_\star, \}$, with $\tau_\star, \rho_\star$ fixed, is the standard metric on $\mathbb{S}^2$. Consequently, one can introduce a complex null frame $\{\boldsymbol{\partial}_+, \boldsymbol{\partial}_-\}$ on $\mathcal{Q}$ as described in Appendix A. To propagate this frame off $\mathcal{Q}$ one requires that

$$[\boldsymbol{\partial}_\tau, \boldsymbol{\partial}_\pm] = 0, \qquad [\boldsymbol{\partial}_\rho, \boldsymbol{\partial}_\pm] = 0.$$

Taking into account the above construction one writes, in spinorial notation, the following spacetime frame

$$\boldsymbol{e}_{\boldsymbol{00'}} = \frac{\sqrt{2}}{2}\big((1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho\big), \qquad \boldsymbol{e}_{\boldsymbol{11'}} = \frac{\sqrt{2}}{2}\big((1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho\big), \tag{8a}$$

$$\boldsymbol{e}_{\boldsymbol{01'}} = \frac{\sqrt{2}}{2}\boldsymbol{\partial}_+, \qquad \boldsymbol{e}_{\boldsymbol{10'}} = \frac{\sqrt{2}}{2}\boldsymbol{\partial}_-. \tag{8b}$$

The corresponding dual coframe is given by

$$\boldsymbol{\omega}^{\mathbf{00}'} = \frac{\sqrt{2}}{2}\Big(\mathbf{d}\tau - \frac{1}{\rho}(1-\tau)\mathbf{d}\rho\Big), \qquad \boldsymbol{\omega}^{\mathbf{11}'} = \frac{\sqrt{2}}{2}\Big(\mathbf{d}\tau + \frac{1}{\rho}(1+\tau)\mathbf{d}\rho\Big),$$
$$\boldsymbol{\omega}^{\mathbf{01}'} = \sqrt{2}\boldsymbol{\omega}^{+}, \qquad \boldsymbol{\omega}^{\mathbf{10}'} = \sqrt{2}\boldsymbol{\omega}^{-}.$$

One can directly verify that

$$\boldsymbol{g} = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A}'\boldsymbol{B}'}\boldsymbol{\omega}^{\boldsymbol{AA}'}\boldsymbol{\omega}^{\boldsymbol{BB}'}.$$

The above construction and frame will be referred in the following discussion as the *F-gauge*. A direct computation using the Cartan structure equations shows that the only non-zero reduced connection coefficients are given by

$$\Gamma_{\mathbf{00}'}{}^{\mathbf{1}}{}_{\mathbf{1}} = \Gamma_{\mathbf{11}'}{}^{\mathbf{1}}{}_{\mathbf{1}} = \frac{\sqrt{2}}{4}, \qquad \Gamma_{\mathbf{00}'}{}^{\mathbf{0}}{}_{\mathbf{0}} = \Gamma_{\mathbf{11}'}{}^{\mathbf{0}}{}_{\mathbf{0}} = -\frac{\sqrt{2}}{4},$$
$$\Gamma_{\mathbf{10}'}{}^{\mathbf{1}}{}_{\mathbf{1}} = -\Gamma_{\mathbf{10}'}{}^{\mathbf{0}}{}_{\mathbf{0}} = \frac{\sqrt{2}}{4}\omega, \qquad \Gamma_{\mathbf{01}'}{}^{\mathbf{0}}{}_{\mathbf{0}} = -\Gamma_{\mathbf{01}'}{}^{\mathbf{1}}{}_{\mathbf{1}} = \frac{\sqrt{2}}{4}\overline{\omega}.$$

3 The electromagnetic field in the F-gauge

In this section the Maxwell equations on $(\mathcal{M}, \boldsymbol{g})$ are discussed. After rewriting the equations in terms of the ð and $\bar{ð}$ operators, a general solution is obtained by expanding the fields in spin-weighted spherical harmonics. The resulting equations for the coefficients of the expansion, satisfy ordinary differential equations which can be explicitly solved in terms of special functions. The analysis given here is similar to the one for the Maxwell field on a Schwarzschild background in [31] and the gravitational field in [8]. Notice that, in contrast with the analysis presented in this section, in the latter references the equations and relevant structures are lifted to the extended bundle space. Additionally, the initial data considered in this analysis is generic and in particular is not assumed to be time-symmetric.

3.1 The spinorial Maxwell equations

The Maxwell equations in the 2-spinor formalism take the form of the spin-1 equation

$$\nabla_{A'}{}^{A}\phi_{AB} = 0. \tag{9}$$

Let $\epsilon_{\boldsymbol{A}}{}^{A}$ with $\epsilon_{\mathbf{0}}{}^{A} = o^{A}$ and $\epsilon_{\mathbf{1}}{}^{A} = \iota^{A}$ denote a spin dyad adapted to the F-gauge so that $e_{\boldsymbol{AA}'}{}^{AA'} = \epsilon_{\boldsymbol{A}}{}^{A}\epsilon_{\boldsymbol{A}'}{}^{A}$, corresponds to the null frame introduced in Section 2.2. A direct computation shows that equation (9) implies a set of equations for the components of ϕ_{AB} respect to $\epsilon_{\boldsymbol{A}}{}^{A}$: $\phi_0 \equiv \phi_{AB}o^{A}o^{B}$, $\phi_1 \equiv \phi_{AB}o^{A}\iota^{B}$ and $\phi_2 \equiv \phi_{AB}\iota^{A}\iota^{B}$, which can be split into a system of evolution equations

$$(1+\tau)\boldsymbol{\partial}_{\tau}\phi_0 - \rho\boldsymbol{\partial}_{\rho}\phi_0 - \boldsymbol{\partial}_{+}\phi_1 = -\phi_0, \tag{10a}$$
$$\boldsymbol{\partial}_{\tau}\phi_1 - \frac{1}{2}\big(\boldsymbol{\partial}_{+}\phi_2 + \boldsymbol{\partial}_{-}\phi_0\big) = \frac{1}{2}\big(\overline{\varpi}\phi_2 + \varpi\phi_0\big), \tag{10b}$$
$$(1-\tau)\boldsymbol{\partial}_{\tau}\phi_2 + \rho\boldsymbol{\partial}_{\rho}\phi_2 - \boldsymbol{\partial}_{-}\phi_1 = \phi_2, \tag{10c}$$

and a constraint equation

$$\tau\boldsymbol{\partial}_{\tau}\phi_1 - \rho\boldsymbol{\partial}_{\rho}\phi_1 + \frac{1}{2}\big(\boldsymbol{\partial}_{-}\phi_0 - \boldsymbol{\partial}_{+}\phi_2\big) = \frac{1}{2}\big(\overline{\varpi}\phi_2 - \varpi\phi_0\big). \tag{10d}$$

One can systematically solve the above equations decomposing the fields ϕ_0, ϕ_1, ϕ_2 in spin-weighted spherical harmonics. To do so, one has to rewrite these equations in terms of the ð and

$\bar{\eth}$ operators of Newman and Penrose. Using (B.4) of Appendix B and the fact that ϕ_0, ϕ_1 and ϕ_2 have spin weights 1, 0 and -1, respectively, one finds that equations (10a)-(10d) can be rewritten as the following evolution equations

$$(1+\tau)\boldsymbol{\partial}_\tau\phi_0 - \rho\boldsymbol{\partial}_\rho\phi_0 + \eth\phi_1 = -\phi_0, \tag{11a}$$

$$\boldsymbol{\partial}_\tau\phi_1 + \frac{1}{2}\big(\eth\phi_2 + \bar{\eth}\phi_0\big) = 0, \tag{11b}$$

$$(1-\tau)\boldsymbol{\partial}_\tau\phi_2 + \rho\boldsymbol{\partial}_\rho\phi_2 + \bar{\eth}\phi_1 = \phi_2, \tag{11c}$$

and the constraint equation

$$\tau\boldsymbol{\partial}_\tau\phi_1 - \rho\boldsymbol{\partial}_\rho\phi_1 + \frac{1}{2}\big(\eth\phi_2 - \bar{\eth}\phi_0\big) = 0. \tag{11d}$$

3.2 The transport equations for the electromagnetic field on the cylinder at spatial infinity

In order to analyse the behaviour of solutions of the Maxwell equations in a neighbourhood of the cylinder at spatial infinity it will be assumed that ϕ_0, ϕ_1 and ϕ_2 are smooth functions of τ and ρ. Moreover, taking into account equation (B.6) of Appendix B the following Ansatz is made:

Assumption 1. The components of the Maxwell field admit a Taylor-like expansion around $\rho = 0$ of the form

$$\phi_n = \sum_{p=|1-n|}^{\infty} \sum_{\ell=|1-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p;\ell,m}(\tau) Y_{1-n;\ell,m}\rho^p, \tag{12}$$

where $a_{n,p;\ell m} : \mathbb{R} \to \mathbb{C}$ and with $n = 0, 1, 2$.

Remark 1. For the purposes pursued in this article the expansion (12) is understood as an Ansatz for the solution. Nevertheless, the structure of the expansion (12) can be motivated from analysing the electromagnetic constraint equations. The formal nature of the above Ansatz can be controlled making use of the theory developed in [9] —see also [31, 32]. This, in turn, allows to find conditions on the freely specifiable initial data for the Maxwell equations ensuring the existence of expansions of the from given by (12). Obtaining such conditions, however, goes beyond the scope of the present article and will be discussed elsewhere.

To simplify the notation of the subsequent analysis let

$$\phi_n^{(p)} \equiv \left.\frac{\partial^p \phi_n}{\partial \rho^p}\right|_{\rho=0}, \tag{13}$$

with $n = 0, 1, 2$. Formally differentiating equations (11a)-(11d) respect to ρ and evaluating at the cylinder I one obtains

$$(1+\tau)\dot{\phi}_0^p - (p-1)\phi_0^p + \eth\phi_1^p = 0, \tag{14a}$$

$$\dot{\phi}_1^{(p)} + \frac{1}{2}\big(\eth\phi_2^{(p)} + \bar{\eth}\phi_0^{(p)}\big) = 0, \tag{14b}$$

$$(1-\tau)\dot{\phi}_2^{(p)} + (p-1)\phi_2^{(p)} + \bar{\eth}\phi_1^{(p)} = 0, \tag{14c}$$

$$\tau\dot{\phi}_1^{(p)} - p\phi_1^{(p)} + \frac{1}{2}\big(\eth\phi_2^{(p)} - \bar{\eth}\phi_0^{(p)}\big) = 0, \tag{14d}$$

where the dot denotes a derivative respect to τ. Using equations (B.7a)-(B.7b) of Appendix B and expansions encoded in equation (12) one obtains the following equations for $a_{n,p;\ell m}$

$$(1+\tau)\dot{a}_{0,p;\ell m} + \sqrt{\ell(\ell+1)}a_{1,p;\ell m} - (p-1)a_{0,p;\ell m} = 0, \tag{15}$$

$$\dot{a}_{1,p;\ell m} + \frac{1}{2}\sqrt{\ell(\ell+1)}(a_{2,p;\ell m} - a_{0,p;\ell m}) = 0, \tag{16}$$

$$(1-\tau)\dot{a}_{2,p;\ell m} - \sqrt{\ell(\ell+1)}a_{1,p;\ell m} + (p-1)a_{2,p;\ell,m} = 0, \tag{17}$$

$$\tau\dot{a}_{1,p;\ell m} - \frac{1}{2}\sqrt{\ell(\ell+1)}(a_{2,p;\ell m} + a_{0,p;\ell m}) - pa_{1,p;\ell m} = 0, \tag{18}$$

for $p \geq 1$, $1 \leq \ell \leq p$, $-\ell \leq m \leq \ell$. Notice that equations (15)-(18) correspond, essentially, to the homogeneous part of the equations reported in [31]. Furthermore, $a_{1,p;\ell,m}$ can be solved from (16) and (18) in terms of $a_{0,p;\ell m}$ and $a_{2,p;\ell,m}$ to obtain

$$a_{1,p;\ell m} = \frac{\sqrt{\ell(\ell+1)}}{2p}\big((1-\tau)a_{2,p;\ell.m} + (1+\tau)a_{0,p;\ell,m}\big). \tag{19}$$

Substituting $a_{1,p;\ell,m}$ as given in (19) into equations (15) and (17) one obtains

$$(1+\tau)\dot{a}_{0,p;\ell,m} + \Big(\frac{1}{2p}\ell(\ell+1)(1+\tau) - (p-1)\Big)a_{0,p;\ell,m} + \frac{1}{2p}\ell(\ell+1)(1-\tau)a_{2,p;\ell m} = 0, \tag{20a}$$

$$(1-\tau)\dot{a}_{2,p;\ell m} - \frac{1}{2p}\ell(\ell+1)(1+\tau)a_{0,p;\ell m} - \Big(\frac{1}{2p}\ell(\ell+1)(1-\tau) - (p-1)\Big)a_{2,p;\ell m} = 0. \tag{20b}$$

At this point one can proceed in analogous way as in [31] to obtain a fundamental matrix for the system (20a)-(20b): a direct computation shows that one can decouple the last system of first order equations and obtain the following second order equations

$$(1-\tau^2)\ddot{a}_{0,p;\ell,m} + 2(1-(1-p)\tau)\dot{a}_{0,p;\ell,m} + (p+\ell)(\ell-p+1)a_{0,p;\ell,m} = 0, \tag{21a}$$

$$(1-\tau^2)\ddot{a}_{2,p;\ell,m} - 2(1+(1-p)\tau)\dot{a}_{2,p;\ell,m} + (p+\ell)(\ell-p+1)a_{2,p;\ell,m} = 0. \tag{21b}$$

Dropping temporarily the subindices p, ℓ, m observe that, if $a_2(\tau)$ solves (21b) then $a_2^s(\tau) \equiv a_2(-\tau)$ solves equation (21a). Equations (21a)-(21b) are particular examples of so-called *Jacobi ordinary differential equations*. Following the discussion of [31] one obtains the following:

Proposition 1. *For* $p \geq 1$, $\ell < p$, $-\ell \leq m \leq \ell$ *the solutions to the Jacobi equations* (21a)-(21b) *are polynomial in* τ. *Moreover,*

$$\begin{pmatrix} a_{0,p;l,m}(\tau) \\ a_{2,p;l,m}(\tau) \end{pmatrix} = X_{p,\ell}(\tau)\left(X^{-1}_{p,\ell}(0)\begin{pmatrix} a_{0,p;l,m}(0) \\ a_{2,p;l,m}(0) \end{pmatrix}\right),$$

where the fundamental matrix is given by

$$X_{p,\ell}(\tau) = \begin{pmatrix} Q^1_{p,\ell}(\tau) & (-1)^{\ell+1}Q^3_{p,\ell}(\tau) \\ (-1)^{\ell+1}Q_3(-\tau) & Q^1_{p,\ell}(-\tau) \end{pmatrix},$$

$$\text{with} \qquad Q^1_{p,\ell}(\tau) = \Big(\frac{1-\tau}{2}\Big)^{p+1}P^{p+1,-p+1}_{\ell-1}(\tau) \qquad Q^3_{p,\ell}(\tau) = \Big(\frac{1+\tau}{2}\Big)^{p-1}P^{-p-1,p-1}_{\ell+1}(\tau).$$

For future identification of the NP constants in terms of initial data it is convenient to introduce some notation at this point. The matrix $X^{-1}_{p,l}(0)$ is a symmetric matrix whose components will be represented as

$$X^{-1}_{p,l}(0) = \begin{pmatrix} X^{-1}_A & X^{-1}_B \\ X^{-1}_B & X^{-1}_A \end{pmatrix}. \tag{22}$$

The explicit values of X^{-1}_A, X^{-1}_B can be determined by inverting the fundamental matrix and direct evaluation the the Jacobi polynomials at $\tau = 0$. The only relevant feature for the subsequent discussion is that $X^{-1}_A \neq X^{-1}_B$. Expanding the matrix expression given in the above proposition one obtains:

Lemma 1. *The solutions of Proposition 1 can be written as:*

$$a_{0,p;l,m}(\tau) = C_{p,\ell,m} Q^1_{p,\ell}(\tau) + (-1)^{1+\ell} D_{p,\ell,m} Q^3_{p,\ell}(\tau),$$
$$a_{2,p;l,m}(\tau) = D_{p,\ell,m} Q^1_{p,\ell}(-\tau) + (-1)^{1+\ell} C_{p,\ell,m} Q^3_{p,\ell}(-\tau),$$

where $C_{p,\ell,m} \equiv X_A^{-1} a_{0,p;l,m}(0) + X_B^{-1} a_{2,p;l,m}(0), \quad D_{p,\ell,m} \equiv X_B^{-1} a_{0,p;l,m}(0) + X_A^{-1} a_{2,p;l,m}(0).$

Proposition 1 encodes the solution for $p > l$. The $p = l$ is a special case for which one has the following:

Proposition 2. *For* $p \geq 1$, $\ell = p$, $-p \leq m \leq p$ *one has:*

$$a_{0,p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{p+1} \left(\frac{1+\tau}{2}\right)^{p-1} \left(E_{p,m} + E^*_{p,m} \int_0^\tau \frac{ds}{(1+s)^p (1-s)^{p+2}}\right), \tag{23a}$$

$$a_{2,p;p,m}(\tau) = \left(\frac{1+\tau}{2}\right)^{p+1} \left(\frac{1-\tau}{2}\right)^{p-1} \left(I_{p,m} + I^*_{p,m} \int_0^\tau \frac{ds}{(1-s)^p (1+s)^{p+2}}\right), \tag{23b}$$

where $E_{p,m}$, $E^*_{p,m}$ *and* $I_{p,m}$, $I^*_{p,m}$ *are integration constants.*

Remark 2. For non-vanishing $E^*_{p,m}$ and $I^*_{p,m}$ the solutions $a_{0,p;p,m}(\tau)$ and $a_{2,p;p,m}(\tau)$ with $p \geq 1$, $-p \leq m \leq p$, contain terms which diverge logarithmically near $\tau = \pm 1$.

3.3 Initial data for the Maxwell equations

Evaluating the constraint equation (11d) at $\tau = 0$ gives the following equation

$$\rho \boldsymbol{\partial}_\rho \phi_1 - \frac{1}{2}\left(\eth \phi_2 - \bar{\eth} \phi_0\right) = 0. \tag{24}$$

Consistent with the expressions encoded in equation (12) one considers on the initial hypersurface $\mathcal{S}$ fields $\phi_n|_{\mathcal{S}}$, with $n = 0, 1, 2$, which can be expanded as

$$\phi_n|_{\mathcal{S}} = \sum_{p=|1-n|}^{\infty} \sum_{\ell=|1-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p;\ell,m}(0) Y_{1-n;\ell,m} \rho^p, \tag{25}$$

Observe that once $a_{0,p;\ell,m}(0)$ and $a_{2,p;\ell,m}(0)$ are given, $a_{1,p;\ell,m}(0)$ is already determined by virtue of equation (19) as

$$a_{1,p;\ell m}(0) = \frac{\sqrt{\ell(\ell+1)}}{2p}\left(a_{2,p;\ell,m}(0) + a_{0,p;\ell,m}(0)\right)$$

In addition, observe that equations (20a)-(20b) are first order while equations (21a)-(21b) are second order. Consequently, the initial data $\dot{a}_{0,p,\ell,m}(0)$ and $\dot{a}_{2,p,\ell,m}(0)$ are determined, by virtue of equations (20a)-(20b) restricted to $\mathcal{S}$, by the initial data $a_{0,p,\ell,m}(0)$ and $a_{2,p,\ell,m}(0)$.

Remark 3. Although identifying $a_{0,p,\ell,m}(0)$ and $a_{2,p,\ell,m}(0)$ as the free specifiable initial data is sufficient for the purposes of this article, a systematic and geometric way to parametrise the initial data is to write the solution to the constraint equation (11d) in terms of Hertz potentials. Exploiting the results of [2] one has that the general solution to the constraint equation (11d) can be written as

$$\phi_{AB} = (\mathcal{G}_2 \varphi)_{AB},$$

where $\varphi_{AB} = \varphi_{(AB)}$ is a symmetric but otherwise arbitrary spinor encoding the free specifiable data. The operator $\mathcal{G}_2$ is defined as $(\mathcal{G}_2\varphi)_{AB} \equiv D_{(A}{}^Q \varphi_{B)Q}$ where D_{AB} denotes the spinorial counterpart of the Levi-Civita connection associated with metric $\boldsymbol{h}$ intrinsic to the hypersurface $\mathcal{S}$. The details of this construction are not necessary for the main discussion of this article and will be presented elsewhere.

Assumption 2. Although general initial data allows for solutions with $E^{\star}_{p,m} \neq 0$ and $I^{\star}_{p,m} \neq 0$ for the calculation of the NP constants it will be assumed that

$$E^{*}_{p,m} = I^{*}_{p,m} = 0.$$

In other words, it will be assumed that the fields ϕ_n do not contain the diverging terms of Proposition 2.

Remark 4. The spin-1 field ϕ_{AB} (Maxwell spinor) can be decomposed into its electric and magnetic parts, denoted as η_{AB} and μ_{AB} respectively, as follows:

$$\phi_{AB} = \eta_{AB} + \mathrm{i}\mu_{AB},$$

with

$$\eta_{AB} = \frac{1}{2}(\phi_{AB} + \phi^{\dagger}_{AB}), \qquad \mu_{AB} = -\frac{1}{2}\mathrm{i}\,(\phi_{AB} - \phi^{\dagger}_{AB}).$$

where $\phi^{\dagger}_{AB} \equiv \tau_A{}^{A'}\tau_B{}^{B'}\bar{\phi}_{A'B'}$ with $\tau_A{}^{A'} \equiv o_A o^{A'} + \iota_A\bar{\iota}^{A'}$. Here $\tau^{AA'}$ corresponds to the spinorial counterpart of the vector $\tau^a = \sqrt{2}\boldsymbol{\partial}_\tau$ —see [33] for further discussion on the space spinor formalism. Initial data for which $\mu_{AB}|_{\mathcal{S}} = 0$ will be called time-symmetric. A calculation then shows that time-symmetric data satisfy,

$$\phi_0 = \bar{\phi}_2, \qquad \phi_1 = -\bar{\phi}_1 \qquad \text{on} \quad \mathcal{S}.$$

The latter conditions imply that $a_{0,p,\ell,m}(0) = a_{2,p,\ell,m}(0)$.

Remark 5. If the data is time-symmetric then $C_{p,\ell,m} = D_{p,\ell,m}$. Nevertheless, for generic initial data one has $C_{p,\ell,m} \neq D_{p,\ell,m}$.

Remark 6. The convergence of the expansions encoded in (12) follows from the results of [32].

4 The massless spin-2 field equations in the F-gauge

In Section 3 the Maxwell equations (in the F-gauge) were discussed, these correspond in spinorial formalism to the spin-1 equations. In this section, a similar analysis is performed but now for a spin-2 field propagating on the Minkowski spacetime. As discussed in [29] the spin-2 equations where the background geometry is that of the Minkowski spacetime can be used to describe the linearised gravitational field. In [29] these equations were written in terms the lifts of the relevant structures to the extended bundle space. In this section, following the spirit of the present article, the equations will be discussed without making use of these structures. In a similar way as in the electromagnetic case studied in Section 3, after rewriting the equations in terms of the $\eth$ and $\bar{\eth}$ operators, a general solution is obtained by expanding the fields in spin-weighted spherical harmonics. The resulting equations for the coefficients of the expansion satisfy ordinary differential equations which can be explicitly solved in terms of special functions.

4.1 The spin-2 equation

As discussed in [29], the linearised gravitational field over the Minkowski spacetime can be described with the so-called massless spin-2 field equation

$$\nabla_{A'}{}^{A}\phi_{ABCD} = 0. \tag{26}$$

Following an approach analogous to the one described in Section 3.1 for the electromagnetic field, it can be shown that equation (26) implies the following evolution equations for the components of the spinor ϕ_{ABCD}

$$(1+\tau)\boldsymbol{\partial}_\tau\phi_0 - \rho\boldsymbol{\partial}_\rho\phi_0 - \boldsymbol{\partial}_+\phi_1 + \bar{\varpi}\phi_1 = -2\phi_0, \tag{27a}$$

$$\boldsymbol{\partial}_\tau\phi_1 - \frac{1}{2}\boldsymbol{\partial}_+\phi_2 - \frac{1}{2}\boldsymbol{\partial}_-\phi_0 - \varpi\phi_0 = -\phi_1, \tag{27b}$$

$$\boldsymbol{\partial}_\tau\phi_2 - \frac{1}{2}\boldsymbol{\partial}_-\phi_1 - \frac{1}{2}\boldsymbol{\partial}_+\phi_3 - \frac{1}{2}\varpi\phi_1 - \frac{1}{2}\bar{\varpi}\phi_3 = 0, \tag{27c}$$

$$\boldsymbol{\partial}_\tau\phi_3 - \frac{1}{2}\boldsymbol{\partial}_+\phi_4 - \frac{1}{2}\boldsymbol{\partial}_-\phi_2 - \bar{\varpi}\phi_4 = \phi_3, \tag{27d}$$

$$(1-\tau)\boldsymbol{\partial}_\tau\phi_4 + \rho\boldsymbol{\partial}_\rho\phi_4 - \boldsymbol{\partial}_-\phi_3 + \varpi\phi_3 = 2\phi_4, \tag{27e}$$

and the constraint equations

$$\tau\boldsymbol{\partial}_\tau\phi_1 - \rho\boldsymbol{\partial}_\rho\phi_1 - \frac{1}{2}\boldsymbol{\partial}_+\phi_2 + \frac{1}{2}\boldsymbol{\partial}_-\phi_0 + \varpi\phi_0 = 0, \tag{28a}$$

$$\tau\boldsymbol{\partial}_\tau\phi_2 - \rho\boldsymbol{\partial}_\rho\phi_2 - \frac{1}{2}\boldsymbol{\partial}_+\phi_3 + \frac{1}{2}\boldsymbol{\partial}_-\phi_1 - \frac{1}{2}\bar{\varpi}\phi_3 + \frac{1}{2}\varpi\phi_1 = 0, \tag{28b}$$

$$\tau\boldsymbol{\partial}_\tau\phi_3 - \rho\boldsymbol{\partial}_\rho\phi_3 - \frac{1}{2}\boldsymbol{\partial}_+\phi_4 + \frac{1}{2}\boldsymbol{\partial}_-\phi_2 - \bar{\varpi}\phi_4 = 0, \tag{28c}$$

where the five components $\phi_0, \phi_1, \phi_2, \phi_3$ and ϕ_4, given by

$$\phi_0 \equiv \phi_{ABCD}o^A o^B o^C o^D, \qquad \phi_1 \equiv \phi_{ABCD}o^A o^B o^C \iota^D,$$
$$\phi_2 \equiv \phi_{ABCD}o^A o^B \iota^C \iota^D, \qquad \phi_3 \equiv \phi_{ABCD}o^A \iota^B \iota^C \iota^D,$$
$$\phi_4 \equiv \phi_{ABCD}\iota^A \iota^B \iota^C \iota^D,$$

have spin weight of $2, 1, 0, -1, -2$ respectively. Taking into account this observation and equations (B.4) and (B.5) given in Appendix B one can rewrite (27a)-(28c) in terms of the $\eth$ and $\bar{\eth}$ as done for the electromagnetic case. A direct computation renders the following evolution equations

$$(1+\tau)\boldsymbol{\partial}_\tau\phi_0 - \rho\boldsymbol{\partial}_\rho\phi_0 + \eth\phi_1 = -2\phi_0, \tag{29a}$$

$$\boldsymbol{\partial}_\tau\phi_1 + \frac{1}{2}\bar{\eth}\phi_0 + \frac{1}{2}\eth\phi_2 = -\phi_1, \tag{29b}$$

$$\boldsymbol{\partial}_\tau\phi_2 + \frac{1}{2}\bar{\eth}\phi_1 + \frac{1}{2}\eth\phi_3 = 0, \tag{29c}$$

$$\boldsymbol{\partial}_\tau\phi_3 + \frac{1}{2}\bar{\eth}\phi_2 + \frac{1}{2}\eth\phi_4 = \phi_3, \tag{29d}$$

$$(1-\tau)\boldsymbol{\partial}_\tau\phi_4 + \rho\boldsymbol{\partial}_\rho\phi_4 + \bar{\eth}\phi_3 = 2\phi_4, \tag{29e}$$

and the constraint equations

$$\tau\boldsymbol{\partial}_\tau\phi_1 - \rho\boldsymbol{\partial}_\rho\phi_1 + \frac{1}{2}\eth\phi_2 - \frac{1}{2}\bar{\eth}\phi_0 = 0, \tag{30a}$$

$$\tau\boldsymbol{\partial}_\tau\phi_2 - \rho\boldsymbol{\partial}_\rho\phi_2 + \frac{1}{2}\eth\phi_3 - \frac{1}{2}\bar{\eth}\phi_1 = 0, \tag{30b}$$

$$\tau\boldsymbol{\partial}_\tau\phi_3 - \rho\boldsymbol{\partial}_\rho\phi_3 + \frac{1}{2}\eth\phi_4 - \frac{1}{2}\bar{\eth}\phi_2 = 0. \tag{30c}$$

With the equations already written in this way, one can follow the discussion of [29] for parametrising the solutions to equations (29a)-(30c).

4.2 The transport equations for the massless spin-2 field on the cylinder at spatial infinity

One proceeds in analogous way as in the electromagnetic case and assumes that the fields ϕ_n with $n = 0, 1, 2, 3, 4$, are smooth functions of τ and ρ. Taking into account equation (B.6) of Appendix B, it is assumed that one can express the components the of the linearised gravitational field in a Taylor-like expansion around $\rho = 0$. More precisely, the following Ansatz is made:

Assumption 3. *In what follows it will be assumed that the components of the spin-2 field have the expansions*

$$\phi_n = \sum_{p=|2-n|}^{\infty} \sum_{\ell=|2-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p;\ell,m}(\tau) Y_{2-n;\ell,m} \rho^p \tag{31}$$

where $a_{n,p;\ell,m} : \mathbb{R} \to \mathbb{C}$ *and* $n = 0, \ldots, 4$.

Remark 7. As in the case of the Maxwell equations (cf. Remark 1), these above Ansatz for the spin-2 field can be expressed in terms of condition on the initial data. This analysis falls beyond the scope of the present article and will be discussed elsewhere.

For the remaining part of this section, the p-th derivative respect to ρ of the fields ϕ_n with $n = 0, 1, 2, 3, 4$ evaluated at the cylinder I, is denoted using the same notation as in equation (13). Then, by formally differentiating equations (29a)- (30c) respect to ρ and evaluating at the cylinder I, one obtains the following equations

$$(1+\tau)\boldsymbol{\partial}_\tau \phi_0^{(p)} + \eth\phi_1^{(p)}(p-2)\phi_0^{(p)} = 0, \tag{32a}$$

$$\boldsymbol{\partial}_\tau \phi_1^{(p)} + \frac{1}{2}\bar{\eth}\phi_0^{(p)} + \frac{1}{2}\eth\phi_2^{(p)} + \phi_1^{(p)} = 0, \tag{32b}$$

$$\boldsymbol{\partial}_\tau \phi_2 + \frac{1}{2}\bar{\eth}\phi_1^{(p)} + \frac{1}{2}\eth\phi_3^{(p)} = 0, \tag{32c}$$

$$\boldsymbol{\partial}_\tau \phi_3 + \frac{1}{2}\bar{\eth}\phi_2^{(p)} + \frac{1}{2}\eth\phi_4^{(p)} - \phi_3^{(p)} = 0, \tag{32d}$$

$$(1-\tau)\boldsymbol{\partial}_\tau \phi_4^{(p)} + \bar{\eth}\phi_3^{(p)} + (p-2)\phi_4^{(p)} = 0, \tag{32e}$$

and

$$\tau\boldsymbol{\partial}_\tau \phi_1 + \frac{1}{2}\eth\phi_2^{(p)} - \frac{1}{2}\bar{\eth}\phi_0^{(p)} - p\phi_1^{(p)} = 0, \tag{33a}$$

$$\tau\boldsymbol{\partial}_\tau \phi_2 + \frac{1}{2}\eth\phi_3^{(p)} - \frac{1}{2}\bar{\eth}\phi_1^{(p)} - p\phi_2^{(p)} = 0, \tag{33b}$$

$$\tau\boldsymbol{\partial}_\tau \phi_3 + \frac{1}{2}\eth\phi_4^{(p)} - \frac{1}{2}\bar{\eth}\phi_2^{(p)} - p\phi_3^{(p)} = 0. \tag{33c}$$

The last set of equations along with the expansion (31), in turn, imply the following equations for $a_{n,p;\ell,m}$ with $p \geq 2$ and $2 \leq \ell \leq p$:

$$(1+\tau)\dot{a}_0 + \lambda_1 a_1 - (p-2)a_0 = 0, \tag{34a}$$

$$\dot{a}_1 - \frac{1}{2}\lambda_1 a_0 + \frac{1}{2}\lambda_0 a_2 + a_1 = 0, \tag{34b}$$

$$\dot{a}_2 - \frac{1}{2}\lambda_0 a_1 + \frac{1}{2}\lambda_0 a_3 = 0, \tag{34c}$$

$$\dot{a}_3 - \frac{1}{2}\lambda_0 a_2 + \frac{1}{2}\lambda_1 a_4 - a_3 = 0, \tag{34d}$$

$$(1-\tau)\dot{a}_4 - \lambda_1 a_3 + (p-2)a_4 = 0, \tag{34e}$$

and

$$\tau\dot{a}_1 + \frac{1}{2}\lambda_0 a_2 + \frac{1}{2}\lambda_1 a_0 - pa_1 = 0, \tag{35a}$$

$$\tau\dot{a}_2 + \frac{1}{2}\lambda_0 a_3 + \frac{1}{2}\lambda_0 a_1 - pa_2 = 0, \tag{35b}$$

$$\tau\dot{a}_3 + \frac{1}{2}\lambda_1 a_4 + \frac{1}{2}\lambda_0 a_2 - pa_3 = 0, \tag{35c}$$

where $\lambda_1 \equiv \sqrt{(\ell-1)(\ell+2)}$ and $\lambda_0 \equiv \sqrt{\ell(\ell+1)}$ and the labels $p;\ell,m$ have been suppressed for conciseness. From equations (34b)-(34d) and (35a)-(35c) one obtains an algebraic system which can be written succinctly as

$$\begin{pmatrix} p+\tau & -\frac{1}{2}(1-\tau)\lambda_0 & 0 \\ -\frac{1}{2}(1+\tau)\lambda_0 & p & -\frac{1}{2}(1-\tau)\lambda_0 \\ 0 & -\frac{1}{2}(1+\tau)\lambda_0 & p-\tau \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{1}{2}\lambda_1 \begin{pmatrix} (1+\tau)a_0 \\ 0 \\ (1-\tau)a_4 \end{pmatrix}. \tag{36}$$

Solving the above system and substituting a_0, a_1 and a_3 written in terms of a_0 and a_4 into equations (34a) and (34e) one obtains

$$(1+\tau)\dot{a}_0 + (-(p-2) + f(\tau,p,\ell))a_0 + g(\tau,p,\ell)a_4 = 0, \tag{37a}$$

$$(1-\tau)\dot{a}_4 + (-(p-2) + f(-\tau,p,\ell))a_4 + g(-\tau,p,\ell)a_0 = 0, \tag{37b}$$

where

$$f(\tau,p,\ell) \equiv \frac{(1+\tau)(\ell-1)(\ell+2)[4p^2 - 4p\tau + \ell(\ell+1)(\tau^2-1)]}{4p(2p^2 - \ell(\ell+1) + (\ell-1)(\ell+2)\tau^2)},$$

$$g(\tau,p,\ell) \equiv \frac{(1-\tau)^3\ell(\ell+1)(\ell-1)(\ell+2)}{4p(2p^2 - \ell(\ell+1) + (\ell-1)(\ell+2)\tau^2)}.$$

Together, the last equations entail the following decoupled equations

$$(1-\tau^2)\ddot{a}_0 + (4 + 2(p-1)\tau)\dot{a}_0 + (p+\ell)(p-\ell+1)a_0 = 0, \tag{38a}$$

$$(1-\tau^2)\ddot{a}_4 + (-4 + 2(p-1)\tau)\dot{a}_4 + (p+\ell)(p-\ell+1)a_4 = 0. \tag{38b}$$

It can be verified that if $a_0(\tau)$ solves (38a) then $a_0^s(\tau) \equiv a_0(-\tau)$ solves equation (38b). As in the electromagnetic case, these equations are Jacobi ordinary differential equations. For the solutions to these equations one has the following:

Proposition 3. *For $p \geq 2$, $\ell < p$, $-\ell \leq m \leq \ell$ the solutions to the Jacobi equations* (38a)-(38b) *are polynomial in τ. Moreover,*

$$\begin{pmatrix} a_{0,p;l,m}(\tau) \\ a_{4,p;l,m}(\tau) \end{pmatrix} = X_{p,\ell}(\tau) \left(X_{p,\ell}^{-1}(0) \begin{pmatrix} a_{0,p;l,m}(0) \\ a_{4,p;l,m}(0) \end{pmatrix} \right),$$

where the fundamental matrix is given by

$$X_{p,\ell}(\tau) = \begin{pmatrix} Q^1_{p,\ell}(\tau) & (-1)^\ell Q^3_{p,\ell}(\tau) \\ (-1)^\ell Q_3(-\tau) & Q^1_{p,\ell}(-\tau) \end{pmatrix},$$

with $$Q^1_{p,\ell}(\tau) = \left(\frac{1-\tau}{2}\right)^{p+2} P^{p+2,-p+2}_{\ell-2}(\tau), \qquad Q^3_{p,\ell}(\tau) = \left(\frac{1+\tau}{2}\right)^{p-2} P^{-p-2,p-2}_{\ell+2}(\tau).$$

Notice that identical notation as in Proposition 1 has been used despite that the fundamental matrices are different. Whether the solutions given in Propositions1 or 3 are being referred to, should be clear from the context. As in the discussion of the electromagnetic case the components of the matrix $X_{p,l}^{-1}(0)$ will be represented as

$$X_{p,l}^{-1}(0) = \begin{pmatrix} X_A^{-1} & X_B^{-1} \\ X_B^{-1} & X_A^{-1} \end{pmatrix},$$

As before, the explicit form of X_A^{-1} and X_B^{-1} can be obtained inverting the fundamental matrix and by direct evaluation of the Jacobi polynomials. The only relevant observation is to be made here is that $X_A^{-1} \neq X_B^{-1}$. Expanding the matrix expression given in Proposition 3 one has the following:

Lemma 2. *The solutions of Proposition 3 can be written as*

$$a_{0,p;l,m}(\tau) = C_{p,\ell,m} Q^1_{p,\ell}(\tau) + (-1)^\ell D_{p,\ell,m} Q^3_{p,\ell}(\tau),$$
$$a_{4,p;l,m}(\tau) = D_{p,\ell,m} Q^1_{p,\ell}(-\tau) + (-1)^\ell C_{p,\ell,m} Q^3_{p,\ell}(-\tau),$$

where $C_{p,\ell,m} \equiv X_A^{-1} a_{0,p;l,m}(0) + X_B^{-1} a_{4,p;l,m}(0), \quad D_{p,\ell,m} \equiv X_B^{-1} a_{0,p;l,m}(0) + X_A^{-1} a_{4,p;l,m}(0).$

The $p = l$ is a special case for which one has the following:

Proposition 4. *For $p \geq 2$, $\ell = p$, $-p \leq m \leq p$ one has:*

$$a_{0,p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{p+2} \left(\frac{1+\tau}{2}\right)^{p-2} \left(E_{p,m} + E^*_{p,m} \int_0^\tau \frac{ds}{(1+s)^{p-1}(1-s)^{p+3}}\right), \tag{39}$$

$$a_{4,p;p,m}(\tau) = \left(\frac{1+\tau}{2}\right)^{p+2} \left(\frac{1-\tau}{2}\right)^{p-2} \left(I_{p,m} + I^*_{p,m} \int_0^\tau \frac{ds}{(1-s)^{p-1}(1+s)^{p+3}}\right). \tag{40}$$

where $E_{p,l,m}$, $E^*_{p,l,m}$ and $I_{p,l,m}$, $I^*_{p,l,m}$ are integration constants.

Remark 8. Notice that for non-vanishing $E^*_{p,m}$ and $I^*_{p,m}$ the solutions $a_{0,p;p,m}(\tau)$ and $a_{4,p;p,m}(\tau)$ contain terms which diverge logarithmically near $\tau = \pm 1$.

4.3 Initial data for the spin-2 equations

Consistent with equations (31) one considers on the initial hypersurface $\mathcal{S}$ fields $\phi_n|_{\mathcal{S}}$, with $n = 0, 1, 2, 3, 4$ which can be expanded as

$$\phi_n|_{\mathcal{S}} = \sum_{p=|2-n|}^{\infty} \sum_{\ell=|2-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p;\ell,m}(0) Y_{2-n;\ell m} \rho^p. \tag{41}$$

Observe that, by virtue of equation (36), the initial data $a_{1,p;\ell,m}(0)$, $a_{2,p;\ell,m}(0)$ and $a_{3,p;\ell,m}(0)$ is determined by $a_{0,p,\ell,m}(0)$ and $a_{4,p,\ell,m}(0)$. In addition, notice that, equations (37a)-(37b) are first order while equations (38a)-(38b) are second order. Therefore, the initial data $\dot{a}_{0,p,\ell,m}(0)$ and $\dot{a}_{4,p,\ell,m}(0)$ is determined, as a consequence of equations (37a)-(37b) restricted to $\mathcal{S}$, by the initial data $a_{0,p,\ell,m}(0)$ and $a_{4,p,\ell,m}(0)$. The latter considerations are succinctly incorporated in the fundamental matrix.

Remark 9. Similar to the electromagnetic case, the identification of the coefficients $a_{0,p,\ell,m}(0)$ and $a_{4,p,\ell,m}(0)$ as the freely specifiable data is enough for the purposes of this article. However, the general parametrisation of the solutions to the constraint equations (30a)-(30c) can be given in terms of Hertz potentials exploiting the results of [2]. The general solution to the constraint equation (30a)-(30c) can be written as

$$\phi_{ABCD} = (\mathcal{G}_4 \varphi)_{ABCD},$$

where $\varphi_{ABCD} = \varphi_{(ABCD)}$ is totally symmetric but otherwise arbitrary spinor encoding the freely specifiable data and $\mathcal{G}_4$ is an operator built using the Levi-Civita connection D associated with metric $\boldsymbol{h}$ intrinsic to the hypersurface $\mathcal{S}$. Since the detailed form of the *fundamental operator* $\mathcal{G}_4$ and the construction of initial data using this approach is not required for the main discussion of this article, this will be presented elsewhere.

Assumption 4. Although general initial data allows for solutions with $E^\star_{p,m} \neq 0$ and $I^\star_{p,m} \neq 0$ for the calculation of the NP constants it will be assumed that

$$E^*_{p,m} = I^*_{p,m} = 0.$$

In other words, it will be assumed that the fields ϕ_n do not contain the diverging terms of Proposition 4.

Remark 10. The spinor ϕ_{ABCD} can be decomposed into its electric and magnetic parts, denoted as η_{ABCD} and μ_{ABCD}, as follows:

$$\phi_{AB} = \eta_{ABCD} + \mathrm{i}\mu_{ABCD},$$

with

$$\eta_{ABCD} = \frac{1}{2}(\phi_{ABCD} + \phi^{\dagger}_{ABCD}), \qquad \mu_{ABCD} = -\frac{1}{2}\mathrm{i}\,(\phi_{ABCD} - \phi^{\dagger}_{ABCD}),$$

where $\phi^{\dagger}_{ABCD} \equiv \tau_A{}^{A'}\tau_B{}^{B'}\tau_C{}^{C'}\tau_D{}^{D'}\bar{\phi}_{A'B'C'D'}$ with $\tau^{AA'}$ as defined in Remark 4 —see [33] for further discussion on the space spinor formalism. Initial data for which $\mu_{ABCD}|_{\mathcal{S}} = 0$ will be called time-symmetric. A calculation then shows that time-symmetric data satisfy,

$$\phi_0 = \bar{\phi}_4, \qquad \phi_1 = -\bar{\phi}_3, \quad \phi_2 = \bar{\phi}_2 \qquad \text{on} \quad \mathcal{S}$$

The latter conditions imply that $a_{0,p,\ell,m}(0) = a_{4,p,\ell,m}(0)$.

Remark 11. If the data is time-symmetric then $C_{p,\ell,m} = D_{p,\ell,m}$. Nevertheless, for generic initial data one has $C_{p,\ell,m} \neq D_{p,\ell,m}$.

Remark 12. The convergence of the expansions (31) follows from the results given in [29].

5 The NP-gauge

In this section, an adapted frame satisfying the NP-gauge conditions and Bondi coordinates are constructed for the conformal extension introduced in Section 2.1. For convenience of the reader, a general discussion of the NP-gauge conditions and the construction of Bondi coordinates is provided in the first part of this section.

5.1 The NP-gauge conditions and Bondi coordinates

After a brief description of the construction the NP-frame in general asymptotically simple spacetimes, the discussion is particularised to the case of the Minkowski spacetime. A more comprehensive discussion of the NP gauge conditions in general asymptotically simple spacetimes can be found in [25, 12, 33].

Let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ denote a conformal extension of an asymptotically simple spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ where $\tilde{\boldsymbol{g}}$ satisfies the vacuum Einstein field equations with vanishing Cosmological constant. It is a general result in the theory of asymptotics that for vacuum spacetimes with vanishing Cosmological constant the conformal boundary $\mathscr{I}$, with locus given by $\Xi = 0$, consists of two disjoint null hypersurfaces $\mathscr{I}^+$ and $\mathscr{I}^-$ each one having the topology of $\mathbb{R} \times \mathbb{S}^2$—see [25, 33]. In this section the discussion will be particularised to $\mathscr{I}^+$. Nevertheless, similar results and constructions can be formulated, *mutatis mutandi*, for $\mathscr{I}^-$. To simplify the notation, the symbol $\simeq$ will be used to denote equality at $\mathscr{I}$, e.g. if w is a scalar field on $\mathcal{M}$ that vanishes at $\mathscr{I}$ one writes $w \simeq 0$. Let $\{\boldsymbol{e}'_{\boldsymbol{AA'}}\}$ denote a frame satisfying $\boldsymbol{g}(\boldsymbol{e}'_{\boldsymbol{AA'}}, \boldsymbol{e}'_{\boldsymbol{BB'}}) = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}$ in a neighbourhood $\mathcal{U} \subset \mathcal{M}$ of $\mathscr{I}^+$. Additionally, let $\Gamma'_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$ denote the reduced connection coefficients of the Levi-Civita connection of $\boldsymbol{g}$ defined respect to $\boldsymbol{e}'_{\boldsymbol{AA'}}$. The frame $\boldsymbol{e}'_{\boldsymbol{AA'}}$ is an adapted frame at $\mathscr{I}^+$ if the following conditions hold:

(i) The vector $\boldsymbol{e}'_{\boldsymbol{11'}}$ is tangent to and parallely propagated along $\mathscr{I}^+$, i.e.,

$$\nabla_{\boldsymbol{11'}}\boldsymbol{e}'_{\boldsymbol{11'}} \simeq 0.$$

(ii) On $\mathcal{U}$ there exists a smooth function u inducing an affine parameter on the null generators of $\mathscr{I}^+$, namely $\boldsymbol{e}'_{\mathbf{11}'}(u) \simeq 1$. The vector $\boldsymbol{e}'_{\mathbf{00}'}$ is then defined as $\boldsymbol{e}'_{\mathbf{00}'} = \boldsymbol{g}(\mathbf{d}u, \cdot)$ so that it is tangent to the null generators of the hypersurfaces transverse to $\mathscr{I}$ defined by

$$\mathcal{N}_{u_\circ} \equiv \{p \in \mathcal{U} \mid u(p) = u_\circ\},$$

with constant $u_\circ$.

(iii) The frame $\{\boldsymbol{e}'_{\boldsymbol{AA}'}\}$ is tangent to the cuts $\mathcal{C}_{u_\circ} \equiv \mathcal{N}_{u_\circ} \cap \mathscr{I}^+ \approx \mathbb{S}^2$ and parallely propagated along $\mathcal{N}_{u_\circ}$, namely

$$\nabla_{\mathbf{00}'} \boldsymbol{e}'_{\boldsymbol{AA}'} = 0 \qquad \text{on} \qquad \mathcal{N}_{u_\circ}.$$

Starting from conditions (i)-(iii) above, in the remaining of this section the relation between the NP and F-frame in the Minkowski spacetime will be established. This discussion follows closely that of [12]. The starting point is to set

$$\boldsymbol{e}'^{a}_{\mathbf{11}'} \simeq f g^{ab} \nabla_b \Theta. \tag{42}$$

A calculation using equation (7) renders

$$g^{ab}\nabla_b\Theta = -\rho\tau(1-\tau^2)\boldsymbol{\partial}^a_\tau - \rho^2(1+\tau^2)\boldsymbol{\partial}^a_\rho.$$

Thus one has $\boldsymbol{e}'^{a}_{\mathbf{11}'} \simeq -f\rho^2\boldsymbol{\partial}^a_\rho$. Equation (42) and condition (i) imply the equation

$$g^{ac}(f\nabla_b\nabla_c\Theta + \nabla_b f \nabla_c\Theta)\boldsymbol{e}'^{b}_{\mathbf{11}'} \simeq 0.$$

One way to solve this equation is to consider a vector z^a such that $\nabla_z\Theta \not\simeq 0$. Transvecting the last equation with z_a and rearranging renders

$$g^{ab}\nabla_a \ln f \nabla_b\Theta + \frac{\nabla_z g^{ab}\nabla_a\Theta\nabla_b\Theta}{2\nabla_z\Theta} \simeq 0. \tag{43}$$

Observe that $\boldsymbol{\partial}_\tau\Theta = -2\rho\tau$ hence $z = \boldsymbol{\partial}_\tau$ is admissible. Taking this z, assuming that $f = f(\rho)$ and replacing $\simeq$ with $=$ in equation (43) a direct calculation gives,

$$\frac{\mathrm{d}\ln f}{\mathrm{d}\rho} = -\frac{(1-\tau^2)}{\rho(1+\tau^2)}.$$

Although the last equation could be readily solved, one is only interested in evaluating this expression at $\mathscr{I}^+$ namely, the condition is $\mathrm{d}\ln f/\mathrm{d}\rho \simeq 0$. To adhere to the conventions of [12] one sets $f = f_\star$, with $f_\star = -1/2\sqrt{2}$, which trivially solves the latter equation at $\mathscr{I}^+$. Thus, overall, one has the following condition

$$\boldsymbol{e}'_{\mathbf{11}'} \simeq -f_\star\rho^2\boldsymbol{\partial}_\rho. \tag{44}$$

It is worth stressing that the above condition only holds at $\mathscr{I}^+$ and that the leg $\boldsymbol{e}'_{\mathbf{11}'}$ of the NP-frame has not been determined on the interior of the spacetime yet. Now, for imposing condition (ii) it is necessary for find a coordinate u such that $\boldsymbol{e}'_{\mathbf{11}'}(u) \simeq 1$, this is $\frac{1}{2\sqrt{2}}\rho^2\boldsymbol{\partial}_\rho u \simeq 1$. Direct integration yields

$$u \simeq -\frac{\sqrt{2}}{\rho} + u_\star. \tag{45}$$

Again notice that this is an equality at $\mathscr{I}^+$ satisfying that $u \to -\infty$ as $\rho \to 0$. To determine u in the interior of the spacetime one needs to solve the *eikonal equation*

$$g^{ab}\nabla_a\nabla_b u = 0.$$

Writing the metric in terms of the F-frame $\boldsymbol{e}_{\boldsymbol{AA'}}$ namely, $g^{ab} = \epsilon^{\boldsymbol{AB}}\epsilon^{\boldsymbol{A'B'}}\boldsymbol{e}^a_{\boldsymbol{AA'}}\boldsymbol{e}^b_{\boldsymbol{BB'}}$, the eikonal equation reads

$$\boldsymbol{e}^a_{\boldsymbol{00'}}\boldsymbol{e}^b_{\boldsymbol{11'}}\nabla_a u \nabla_b u - \boldsymbol{e}^a_{\boldsymbol{01'}}\boldsymbol{e}^b_{\boldsymbol{10'}}\nabla_a u \nabla_b u = 0.$$

Recall that the F-frame $\boldsymbol{e}^a_{\boldsymbol{AA'}}$, unlike the NP-frame $\boldsymbol{e}'^a_{\boldsymbol{AA'}}$, is known on the interior of the spacetime —see equation (8)— and hence the above equation can be explicitly solved. Assuming that $u = u(\tau, \rho)$ then one can instead look for the solution to the simpler equation

$$\boldsymbol{e}^a_{\boldsymbol{11'}}\nabla_a u = 0.$$

The latter equation can be solved using the method of characteristics and the boundary condition (45). A direct calculation renders

$$u = u_\star - \frac{2\sqrt{2}}{\rho(1+\tau)}.$$

With this expression at hand one is now in position to determine the $\mathbf{00'}$ leg of the NP-frame in interior of the spacetime using that

$$\boldsymbol{e}'^a_{\boldsymbol{00'}} = g^{ab}\nabla_b u.$$

A straightforward calculation then gives

$$\boldsymbol{e}'^a_{\boldsymbol{00'}} = \frac{4}{\rho(1+\tau)^2}\boldsymbol{e}^a_{\boldsymbol{11'}}. \tag{46}$$

With the latter result one can readily check that the angular sector of the F-frame is already parallely propagated along $\boldsymbol{e}'^a_{\boldsymbol{00'}}$. In other words

$$\boldsymbol{e}'^b_{\boldsymbol{00'}}\nabla_b \boldsymbol{e}^a_{\boldsymbol{01'}} = \boldsymbol{e}'^b_{\boldsymbol{00'}}\nabla_b \boldsymbol{e}^a_{\boldsymbol{10'}} = 0.$$

Hence, condition (iii) is automatically satisfied. Thus, one can set

$$\boldsymbol{e}'^a_{\boldsymbol{01'}} = e^{-2\mathrm{i}\omega}\boldsymbol{e}^a_{\boldsymbol{10'}}, \qquad \boldsymbol{e}'^a_{\boldsymbol{10'}} = e^{-2\mathrm{i}\omega}\boldsymbol{e}^a_{\boldsymbol{01'}}. \tag{47}$$

where ω is a real number encoding a general spin rotation of the frame. It only remains to determine $\boldsymbol{e}'^a_{\boldsymbol{11'}}$, to do so one could parallely propagate $\boldsymbol{e}'^a_{11'}|_{\mathscr{I}^+}$ along $\boldsymbol{e}'^a_{\boldsymbol{00'}}$ namely, solve

$$\boldsymbol{e}'^b_{\boldsymbol{00'}}\nabla_b \boldsymbol{e}^a_{\boldsymbol{11'}} = 0, \quad \text{with} \quad \boldsymbol{e}^a_{\boldsymbol{11'}} \simeq -f_\star \rho^2 \boldsymbol{\partial}^a_\rho.$$

An alternative computationally simpler approach is to solve for $\boldsymbol{e}'^a_{\boldsymbol{11'}}$ algebraically using that $g^{ab} = \epsilon^{\boldsymbol{AB}}\epsilon^{\boldsymbol{A'B'}}\boldsymbol{e}'^a_{\boldsymbol{AA'}}\boldsymbol{e}'^b_{\boldsymbol{BB'}}$ and exploiting that $\boldsymbol{e}'^a_{\boldsymbol{00'}}$, $\boldsymbol{e}'^a_{\boldsymbol{01'}}$ and $\boldsymbol{e}'^a_{\boldsymbol{10'}}$ had already been determined. A calculation using the latter approach renders

$$\boldsymbol{e}'^a_{\boldsymbol{11'}} = \frac{1}{4}\rho(1+\tau)^2\boldsymbol{e}^a_{\boldsymbol{00'}}. \tag{48}$$

In the discussion of the NP-gauge, is customary to complete the construction introducing *Bondi coordinates* (u, r). Recall that the coordinate u has already been obtained. The radial Bondi coordinate r is determined by the condition

$$\boldsymbol{e}'^a_{\boldsymbol{00'}}\nabla_a r = r_\star,$$

where $r_\star$ is a normalisation constant. A direct calculation using equation (46) and direct integration renders

$$r = \frac{r_\star \rho(\tau^2 - 1)}{2\sqrt{2}}.$$

Rewriting the last expression in terms of the physical coordinates exploiting equation (4) and $t = \rho\tau$, one gets

$$r = \frac{r_\star}{2\sqrt{2}\tilde{\rho}}.$$

To get cleaner expressions with the current normalisation of the frame, in the remaining $r_\star = 2\sqrt{2}$ will be set so that $r = 1/\tilde{\rho}$. Therefore, the $\mathbf{00'}$ leg of the NP-frame can be written in terms of the Bondi coordinate as

$$\boldsymbol{e}'_{\mathbf{00'}} = \boldsymbol{\partial}_r = -\tilde{\rho}^2 \boldsymbol{\partial}_{\tilde{\rho}}. \tag{49}$$

To put the calculations of this section in the more general context of asymptotically simple spacetimes, recall that the F-frame and the NP-frame do not coincide because while the former is based on a Cauchy hypersurface, the latter is adapted to $\mathscr{I}^+$. However, these frames are null frames respect to some metric $\boldsymbol{g}$ and $\boldsymbol{g}'$, respectively, where $\boldsymbol{g}' = \kappa^2 \boldsymbol{g}$, for some conformal factor κ. Therefore, the frames $\boldsymbol{e}_{\boldsymbol{AA'}}$ and $\boldsymbol{e}'_{\boldsymbol{AA'}}$ are, in general, related through a conformal rescaling and a Lorentz transformation

$$\boldsymbol{e}'_{\boldsymbol{AA'}} = \kappa^{-1} \Lambda^{\boldsymbol{B}}{}_{\boldsymbol{A}} \bar{\Lambda}^{\boldsymbol{B}'}{}_{\boldsymbol{A}'} \boldsymbol{e}_{\boldsymbol{BB'}}. \tag{50}$$

For the case of the Minkowski spacetime the conformal factor κ and the Lorentz transformation $\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{B}}$ can be directly read from equations (46), (47) and (48). Furthermore, direct inspection the frames reveals that the metric $\boldsymbol{g}'$ associated to the NP-frame, is related to the physical Minkowski metric $\tilde{\boldsymbol{\eta}}$ via

$$\boldsymbol{g}' = \frac{1}{\tilde{\rho}^2} \tilde{\boldsymbol{\eta}}. \tag{51}$$

The discussion of this subsection can be summarised in the following:

Proposition 5. *The NP and F-frames are null frames respect to the metrics $\boldsymbol{g}'$ and $\boldsymbol{g}$ with $\boldsymbol{g}' = \kappa^2 \boldsymbol{g}$ and related via*

$$\boldsymbol{e}'_{\boldsymbol{AA'}} = \kappa^{-1} \Lambda^{\boldsymbol{B}}{}_{\boldsymbol{A}} \bar{\Lambda}^{\boldsymbol{B}'}{}_{\boldsymbol{A}'} \boldsymbol{e}_{\boldsymbol{BB'}}. \tag{52}$$

In the case of the Minkowski spacetime in the horizontal representation of the cylinder at spatial infinity, the metric $\boldsymbol{g}$ is given by the line element (6) *and $\boldsymbol{g}' = \tilde{\rho}^{-2} \tilde{\boldsymbol{\eta}}$ where $\tilde{\boldsymbol{\eta}}$ is the physical Minkowski metric as written in equation* (1)*. Moreover, the NP-frame hinged at $\mathscr{I}^+$ is related to the F-frame via,*

$$\boldsymbol{e}'_{\mathbf{00'}} = \frac{4}{\rho(1+\tau)^2} \boldsymbol{e}_{\mathbf{11'}}, \qquad \boldsymbol{e}'_{\mathbf{11'}} = \frac{1}{4}\rho(1+\tau)^2 \boldsymbol{e}_{\mathbf{00'}},$$
$$\boldsymbol{e}'_{\mathbf{01'}} = e^{-2i\omega} \boldsymbol{e}_{\mathbf{10'}}, \qquad \boldsymbol{e}'_{\mathbf{10'}} = e^{-2i\omega} \boldsymbol{e}_{\mathbf{01'}}.$$

The Lorentz transformation and conformal factor κ relating the frames is given by

$$\Lambda^{\mathbf{1}}{}_{\mathbf{0}} = \frac{2e^{i\omega}}{\sqrt{\rho}(1+\tau)}, \qquad \Lambda^{\mathbf{0}}{}_{\mathbf{1}} = \frac{e^{-i\omega}\sqrt{\rho}(1+\tau)}{2}, \qquad \Lambda^{\mathbf{0}}{}_{\mathbf{0}} = \Lambda^{\mathbf{1}}{}_{\mathbf{1}} = 0, \qquad \kappa = 1. \tag{53}$$

Observe that the latter expressions coincide to leading order with those reported in [12] —see also Proposition 3 of [13]. The remaining freedom encoded in ω will be fixed by setting $\omega = \pi$ to align with the conventions of [12].

6 The electromagnetic NP constants

Consider the Minkowski spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{\eta}})$ as defined in Section 2.1 and introduce the physical retarded time $\tilde{u} = \tilde{t} - \tilde{\rho}$. In these coordinates one has

$$\tilde{\boldsymbol{\eta}} = \mathbf{d}\tilde{u} \otimes \mathbf{d}\tilde{u} + \mathbf{d}\tilde{u} \otimes \mathbf{d}\tilde{\rho} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{u} - \tilde{\rho}^2 \boldsymbol{\sigma}.$$

and the metric $\boldsymbol{g}'$ as given in equation (51). Let $\epsilon'_{\boldsymbol{A}}{}^{A}$, with $\epsilon'_{\mathbf{0}}{}^{A} = o'^{A}$ and $\epsilon'_{\mathbf{1}}{}^{A} = \iota'^{A}$, denote a spin dyad so that $e'_{\boldsymbol{AA'}}{}^{AA'} = \epsilon'_{\boldsymbol{A}}{}^{A} \epsilon'_{\boldsymbol{A}'}{}^{A'}$ constitutes the NP-frame given in Proposition 5. Let $\{\tilde{o}^A, \tilde{\iota}^A\}$ denote a spin dyad denoted by $\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}$ and defined via

$$o^A = \tilde{\rho}\tilde{o}^A, \qquad \iota^A = \tilde{\iota}^A. \tag{54}$$

Notice that, by virtue of equation (51), the spin dyad $\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}$ is normalised respect to $\tilde{\boldsymbol{\eta}}$. To introduce the electromagnetic NP constants as defined in [19] consider the physical Maxwell spinor $\tilde{\phi}_{AB}$ satisfying

$$\tilde{\nabla}_{A'}{}^{A}\tilde{\phi}_{AB} = 0,$$

where $\tilde{\nabla}_{AA'}$ denotes the Levi-Civita connection respect to $\tilde{\boldsymbol{\eta}}$. The components the physical Maxwell spinor respect to the spin dyad $\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}$ will be denoted, as usual, by $\tilde{\phi}_0 \equiv \tilde{\phi}_{AB}\tilde{o}^A\tilde{o}^B$, $\tilde{\phi}_1 \equiv \tilde{\phi}_{AB}\tilde{o}^A\tilde{\iota}^B$, $\tilde{\phi}_2 \equiv \tilde{\phi}_{AB}\tilde{\iota}^A\tilde{\iota}^B$.

Assumption 5. Following [19], the $\tilde{\phi}_0$ component is assumed to have an expansion

$$\tilde{\phi}_0 = \sum_{n=0}^{N} \frac{\tilde{\phi}_0^n}{\tilde{\rho}^{3+n}} + \mathcal{O}\Big(\frac{1}{\tilde{\rho}^{3+N}}\Big), \tag{55}$$

where the coefficients $\tilde{\phi}_0^n$ do not depend on $\tilde{\rho}$.

The electromagnetic NP constants are defined through the following integrals over cuts $\mathcal{C}$ of null infinity:

$$F_m^{n,k} \equiv \int_{\mathcal{C}} \bar{Y}_{1;n+1,m}\tilde{\phi}_0^{n+1}\mathrm{d}S,$$

where n, $m \in \mathbb{Z}$ with $n \geq 0$, $|m| \leq n+1$ and $\mathrm{d}S$ denotes the area element respect to $\boldsymbol{\sigma}$. In flat space, F_m^n are absolutely conserved in the sense that their value is independent of the cut $\mathcal{C}$ on which they are evaluated —see [19]. From these, only those given by $n = 0$ and $m = -1, 0, 1$ are conserved in the general non-linear Einstein Maxwell theory —see [19].

6.1 Translation to the F-gauge

In view of equation (51), one has that, as a consequence of the standard conformal transformation law for the spin-1 equation —see [25], the spinor ϕ'_{AB}, satisfying

$$\nabla'_{A'}{}^{A}\phi'_{AB} = 0,$$

where $\nabla'_{AA'}$ is the Levi-Civita connection of $\boldsymbol{g}'$, is related to $\tilde{\phi}_{AB}$ via

$$\phi'_{AB} = \tilde{\rho}\tilde{\phi}_{AB}. \tag{56}$$

Therefore, using equations (55), (54) and (56), one obtains

$$\phi'_0 = \sum_{n=0}^{N} \frac{\tilde{\phi}_0^n}{\tilde{\rho}^n} + \mathcal{O}\Big(\frac{1}{\tilde{\rho}^N}\Big),$$

where $\phi'_0 \equiv \phi'_{AB}o'^A o'^B$. Applying $\boldsymbol{e}'_{\boldsymbol{00}'}$ to the last expression and using equation (49) one gets

$$\boldsymbol{e}'_{\boldsymbol{00}'}(\phi'_0) = \tilde{\phi}_0^1 + \mathcal{O}(\tilde{\rho}^{-1}).$$

The repeated application of $\boldsymbol{e}'_{\boldsymbol{00}'}$ to the above relation shows that in general

$$\boldsymbol{e}'^{(q)}_{\boldsymbol{00}'}(\phi'_0) = q!\,\tilde{\phi}_0^q + \sum_{i=q+1}^{N} \frac{(i+1)!}{(i-q+1)!}\,\frac{\tilde{\phi}_0^i}{\tilde{\rho}^{i-q}} + \mathcal{O}\Big(\frac{1}{\tilde{\rho}^{N-q}}\Big),$$

where $\boldsymbol{e}^{(q)}_{\boldsymbol{00}'}(\phi'_0)$ denotes q consecutive applications of $\boldsymbol{e}'_{\boldsymbol{00}'}$ to ϕ_0. Thus, the quantities F_m^n can be written as

$$F_m^n = \frac{1}{(n+1)!}\int_{\mathcal{C}} \bar{Y}_{1;n+1,m}\,\boldsymbol{e}'^{(n+1)}_{\boldsymbol{00}'}(\phi'_0)\,\mathrm{d}S. \tag{57}$$

Observe that the constants F_m^n in the previous equation are expressed in terms of $\boldsymbol{g}'$-associated quantities. In order to obtain a general expression for the electromagnetic NP quantities in the F-gauge one has to rewrite expression (57) in terms of $\boldsymbol{g}$-related quantities. As discussed before, the frames $\boldsymbol{e_{AA'}}$ and $\boldsymbol{e}'_{\boldsymbol{AA'}}$ are related through a conformal rescaling and a Lorentz transformation as given in equation (52). For the sake of generality, the first part of the discussion will be carried out for general κ and $\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{B}}$.

6.1.1 Explicit computation of the first three constants

Let $\epsilon_{\boldsymbol{A}}{}^A$, with $\epsilon_{\boldsymbol{0}}{}^A = o^A$ and $\epsilon_{\boldsymbol{1}}{}^A = \iota^A$, denote a spin dyad normalised respect to $\boldsymbol{g}$ as defined in Section 3. As a consequence of equation (52), the spin dyads $\epsilon_{\boldsymbol{A}}{}^A$ and $\epsilon'{}_{\boldsymbol{A}}{}^A$, giving rise to $\boldsymbol{e_{AA'}}$ and $\boldsymbol{e}'_{\boldsymbol{AA'}}$, are related via

$$\epsilon'{}_{\boldsymbol{A}}{}^A = \kappa^{-1/2}\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{A}}\epsilon_{\boldsymbol{B}}{}^A. \tag{58}$$

Additionally, the spinor field ϕ_{AB}, satisfying

$$\nabla_{A'}{}^A\phi_{AB} = 0,$$

where $\nabla_{AA'}$ is the Levi-Civita connection respect to $\boldsymbol{g}$, is related to ϕ'_{AB} via

$$\phi'_{AB} = \kappa^{-1}\phi_{AB}.$$

Therefore, one has that

$$\phi'_0 = \kappa^{-2}\Lambda^{\boldsymbol{C}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{D}}{}_{\boldsymbol{0}}\phi_{\boldsymbol{CD}}, \tag{59}$$

where $\phi_{\boldsymbol{CD}} \equiv \epsilon_{\boldsymbol{C}}{}^C\epsilon_{\boldsymbol{D}}{}^D\phi_{CD}$. Using the Leibniz rule one obtains

$$\boldsymbol{e}'_{\boldsymbol{00}}(\phi'_0) = \kappa^{-2}\Big(\Lambda^{\boldsymbol{C}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{D}}{}_{\boldsymbol{0}}\boldsymbol{e}'_{\boldsymbol{00'}}(\phi_{\boldsymbol{CD}}) + 2\phi_{\boldsymbol{CD}}\Lambda^{\boldsymbol{C}}{}_{\boldsymbol{0}}\boldsymbol{e}'_{\boldsymbol{00'}}(\Lambda^{\boldsymbol{D}}{}_{\boldsymbol{0}}) - 2\kappa^{-1}\Lambda^{\boldsymbol{C}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{D}}{}_{\boldsymbol{0}}\phi_{\boldsymbol{CD}}\boldsymbol{e}'_{\boldsymbol{00'}}(\kappa)\Big). \tag{60}$$

Notice that, in the above expression, all the quantities except for the frame derivative $\boldsymbol{e}'_{\boldsymbol{00}}$ are $\boldsymbol{g}$ related quantities, namely, given in the F-gauge and the F-coordinates. Using equation (52) one can expand expression (60). This leads to the following expression for the conserved quantities:

$$\begin{aligned} F_m^0 = \int_{\mathcal{C}} \bar{Y}_{1;1,m}\kappa^{-3}\Big(&\Lambda^{\boldsymbol{C}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{D}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{0}}\bar{\Lambda}^{\boldsymbol{B}'}{}_{\boldsymbol{0}'}\boldsymbol{e_{BB'}}(\phi_{\boldsymbol{CD}}) \\ &+ 2\kappa\phi_{\boldsymbol{CD}}\Lambda^{\boldsymbol{C}}{}_{\boldsymbol{0}}\boldsymbol{e}'_{\boldsymbol{00'}}(\Lambda^{\boldsymbol{D}}{}_{\boldsymbol{0}}) - 2\Lambda^{\boldsymbol{C}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{D}}{}_{\boldsymbol{0}}\phi_{\boldsymbol{CD}}\boldsymbol{e}'_{\boldsymbol{00}}(\kappa)\Big)\mathrm{d}S, \end{aligned} \tag{61}$$

for $m = -1, 0, 1$. These correspond to the three electromagnetic NP quantities that remain conserved in the non-linear Einstein Maxwell theory. The last expression represent the electromagnetic counterpart of the gravitational NP quantities in the F-gauge as reported in [12] in equation (III.5). To simplify and prepare the notation for the calculation of higher the order constants F_m^n, it is convenient to introduce the following short-hands:

$$\Lambda \equiv \Lambda^{\boldsymbol{1}}{}_{\boldsymbol{0}} = \frac{-2}{\sqrt{\rho}(1+\tau)}, \qquad \boldsymbol{e} \equiv \sqrt{2}\boldsymbol{e_{11'}} = (1+\tau)\partial_\tau - \rho\partial_\rho. \tag{62}$$

Then, using the results of Proposition 5, equation (59) and the notation introduced in equation (62) one has

$$\phi'_0 = \Lambda^2\phi_2, \qquad \boldsymbol{e}'_{\boldsymbol{00'}} = \frac{\sqrt{2}}{2}\Lambda^2\boldsymbol{e}. \tag{63}$$

With this notation, equation (60) reduces to

$$\boldsymbol{e}'_{\boldsymbol{00'}}(\phi'_0) = \frac{\sqrt{2}}{2}(\Lambda^4\boldsymbol{e}\phi_2 + 2\Lambda^3\phi_2\boldsymbol{e}\Lambda). \tag{64}$$

Additionally, using equation (62), one gets

$$\boldsymbol{e}\Lambda = -\tfrac{1}{2}\Lambda. \tag{65}$$

Thus, together equations (65) and (64) read

$$\boldsymbol{e}'_{\mathbf{00}'}(\phi'_0) = \frac{\sqrt{2}}{2}\Lambda^4(\boldsymbol{e}\phi_2 - \phi_2). \tag{66}$$

Using that

$$\boldsymbol{e}(a_{2,p,\ell,m}(\tau)\rho^p) = ((1+\tau)\dot{a}_{2,p,l,m}(\tau) - pa_{2,p,\ell,m}(\tau))\rho^p, \tag{67}$$

and the Ansatz (12) a calculation renders

$$\boldsymbol{e}'_{\mathbf{00}'}(\phi'_0) = \frac{\sqrt{2}}{2}\sum_{p=1}^{\infty}\sum_{\ell=1}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}\Lambda^4\rho^p((1+\tau)\dot{a}_{2,p,l,m}(\tau) - (1+p)a_{2,p,\ell,m}(\tau))Y_{1;\ell,m}.$$

Separating the dependence in τ and ρ explicitly, the latter expression can be written as

$$\boldsymbol{e}'_{\mathbf{00}'}(\phi'_0) = 2^4\Big(\frac{\sqrt{2}}{2}\Big)\sum_{p=1}^{\infty}\sum_{\ell=1}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}(1+\tau)^{-4}\rho^{(p-2)}A^1_{2,p,l,m}(\tau)Y_{1;\ell,m}, \tag{68}$$

where

$$A^1_{2,p,l,m}(\tau) \equiv ((1+\tau)\dot{a}_{2,p,l,m}(\tau) - (1+p)a_{2,p,\ell,m}(\tau)). \tag{69}$$

The subindices in $A^1_{2,p,l,m}$ are a copy of those of $a_{2,p,l,m}$ while the superindex counts the number of times that the $\boldsymbol{e}'_{\mathbf{00}'}$-derivative was applied. Let $A^1_{2,p,l,m}|_{\mathscr{I}^+}$ denote $A^1_{2,p,l,m}(\tau=1)$. Using equation (68) and (57) computing the first set of NP constants reduces to the evaluating the following integral

$$F^0_m = 2^4\Big(\frac{\sqrt{2}}{2}\Big)\lim_{\substack{\rho\to\rho_\star\\ \tau\to 1}}\left(\int_{\mathbb{S}^2}\sum_{p=1}^{\infty}\sum_{\ell=1}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}(1+\tau)^{-4}\rho^{(p-2)}A^1_{2,p,l,m}(\tau)Y_{1;\ell,m}\bar{Y}_{1;1,m}\mathrm{d}S\right). \tag{70}$$

where $\rho_\star$ is a constant that parametrises the choice of cut, $\mathcal{C}$, of $\mathscr{I}^+$. In particular, $\rho_\star = 0$ represents the choice $\mathcal{C} = I^+$. Using the orthogonality relation,

$$\int_{\mathbb{S}^2} Y_{s;\ell',m'}\bar{Y}_{s;\ell,m} = \delta_{\ell,\ell'}\delta_{m,m'}, \tag{71}$$

one obtains,

$$F^0_m = 2^4\Big(\frac{\sqrt{2}}{2}\Big)\sum_{p=1}^{\infty}\frac{1}{p!}\rho_\star^{(p-2)}A^1_{2,p,1,m}|_{\mathscr{I}^+}. \tag{72}$$

Naively one would conclude that F^0_m is singular if the cut I^+ is chosen and that equation (72) contains an infinite number of terms, however, a direct calculation using the explicit form of $a_{2,p,l,m}(\tau)$ shows that

Remark 13. *$A^1_{2,p,1,m}|_{\mathscr{I}^+} = 0$ for $p=1$ and $p \geq 3$.*

Exploiting Remark 13, one concludes that

$$F^0_m = -\frac{3\sqrt{2}}{2}C_{2,1,m}, \tag{73}$$

where $C_{2,1,m}$ is a constant determined by the initial data as in Lemma 1. Before computing the next set of constants in the hierarchy, F^1_m, a couple of observations are in order. The NP constant F^0_m comes from the term with $p = 2$ in equation (70). All the terms with $p \geq 3$ and $l = 1$ that could potentially contribute to the NP constant vanish when $A^1_{2,p,l,m}$ is evaluated at $\mathscr{I}^+$. This is not surprising as the NP constants are independent of the cut $\mathcal{C}$ on which they are computed (constancy). In doing this calculation Assumption 2 has been used in order to discard the logarithmic terms of Proposition 2. This is a necessary restriction on the initial data to get well defined NP constants —see [27] and [28] for a discussion about the connection between the Peeling theorem and the *classical NP constants* and how an alternative set of *logarithmic NP constants* can be defined in the polyhomogeneous case. Additionally, notice that, if one selects the cut $\mathcal{C} = I^+$ — hence $\rho_\star = 0$— and invoke the *finiteness and constancy* of the NP constants, Remark 13 is not needed. In fact, Remark 13 can be thought as an alternative proof of the finiteness and constancy of the first set of electromagnetic NP constants in Minkowski spacetime.

To compute the next set of constants in the hierarchy F^1_m observe that applying $\boldsymbol{e}'_{\mathbf{00'}}$ to equation (66) gives

$$\boldsymbol{e}'^{(2)}_{\mathbf{00'}}\phi'_0 = \Big(\frac{\sqrt{2}}{2}\Big)^2\Lambda^6(\boldsymbol{e}^{(2)}\phi_2 - 3\boldsymbol{e}\phi_2 + 2\phi_2). \tag{74}$$

Then, using that

$$\boldsymbol{e}^{(2)}(a_{2,p,\ell,m}(\tau)\rho^p) = \Big((1+\tau)^2\ddot{a}_{2,p,l,m}(\tau) + (1+\tau)(1-2p)\dot{a}_{2,p,\ell,m}(\tau) + p^2 a_{2,p,\ell,m}(\tau)\Big)\rho^p, \tag{75}$$

and equation (67), a calculation renders

$$\boldsymbol{e}'^{(2)}_{\mathbf{00'}}(\phi'_0) = 2^6\Big(\frac{\sqrt{2}}{2}\Big)^2\sum_{p=1}^{\infty}\sum_{\ell=1}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}(1+\tau)^{-6}\rho^{(p-3)}A^2_{2,p,l,m}(\tau)Y_{1;\ell,m}, \tag{76}$$

where

$$A^2_{2,p,l,m}(\tau) \equiv (1+\tau)^2\ddot{a}_{2,p,l,m}(\tau) - 2(1+\tau)(1+p)\dot{a}_{2,p,\ell,m}(\tau) + (1+p)(2+p)a_{2,p,\ell,m}(\tau). \tag{77}$$

Thus, using equation (57) gives

$$F^1_m = 2^6\Big(\frac{\sqrt{2}}{2}\Big)^2\Big(\frac{1}{2!}\Big)\lim_{\substack{\rho\to\rho_\star\\ \tau\to 1}}\left(\int_{\mathbb{S}^2}\sum_{p=1}^{\infty}\sum_{\ell=1}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}(1+\tau)^{-6}\rho^{(p-3)}A^2_{2,p,l,m}(\tau)Y_{1;\ell,m}\bar{Y}_{1;2,m}\mathrm{d}S\right). \tag{78}$$

Exploiting the orthogonality condition (71) renders

$$F^1_m = 2^6\Big(\frac{\sqrt{2}}{2}\Big)^2\Big(\frac{1}{2!}\Big)\sum_{p=1}^{\infty}\frac{1}{p!}\rho_\star^{(p-3)}A^2_{2,p,2,m}|_{\mathscr{I}^+}. \tag{79}$$

A direct calculation using equation (77) and the explicit form of the solution $a_{2,p,2,m}(\tau)$ gives the following:

Remark 14. $A^2_{2,p,2,m}|_{\mathscr{I}^+} = 0$ *for* $p \leq 2$ *and* $p \geq 4$.

Using equations (14) and (72) one concludes that

$$F^1_m = \frac{20}{3}C_{3,2,m}. \tag{80}$$

Notice again that the constant comes from the $p = 3$ term in equation (78) and that Remark 14 can be avoided if one invokes the finiteness and constancy of the NP constants F_m^1 in order to evaluate them directly at $\rho_\star = 0$. Before discussing the general case, it is instructive to compute the next constant in the hierarchy: F_m^2. Proceeding as with the previous set of constants, applying $\boldsymbol{e}'_{\mathbf{00}'}$ to equation (74) gives

$$\boldsymbol{e}'^{(3)}_{\mathbf{00}'}\phi'_0 = \Big(\frac{\sqrt{2}}{2}\Big)^3 \Lambda^8(\boldsymbol{e}^{(3)}\phi_2 - 6\boldsymbol{e}^{(2)}\phi_2 + 11\boldsymbol{e}\phi_2 - 6\phi_2). \tag{81}$$

A calculation using that

$$\boldsymbol{e}^{(3)}(a_{2,p,\ell,m}(\tau)\rho^p) = \Big((1+\tau)^3 \dddot{a}_{2,p,l,m}(\tau) - 3(1+\tau)(p-1)\ddot{a}_{2,p,l,m}(\tau) + (1+\tau)(1-3p+3p^2)\dot{a}_{2,p,l,m}(\tau) - p^3 a_{2,p,l,m}(\tau)\Big)\rho^p, \tag{82}$$

renders

$$\boldsymbol{e}'^{(3)}_{\mathbf{00}'}(\phi'_0) = 2^8\Big(\frac{\sqrt{2}}{2}\Big)^3 \sum_{p=1}^{\infty}\sum_{\ell=1}^{p}\sum_{m=-\ell}^{\ell} \frac{1}{p!}(1+\tau)^{-8}\rho^{(p-4)} A^3_{2,p,l,m}(\tau) Y_{1;\ell,m}, \tag{83}$$

where

$$A^3_{2,p,l,m}(\tau) \equiv (1+\tau)^3 \dddot{a}_{2,p,l,m}(\tau) - 3(1+\tau)^2(1+p)\ddot{a}_{2,p,\ell,m}(\tau) + 3(1+\tau)(1+p)(2+p)\dot{a}_{2,p,\ell,m}(\tau) - (3+p)(2+p)(1+p)a_{2,p,\ell,m}(\tau). \tag{84}$$

Then, the orthogonality condition (71) and expression (57) give

$$F_m^2 = 2^8\Big(\frac{\sqrt{2}}{2}\Big)^3\Big(\frac{1}{3!}\Big)\sum_{p=1}^{\infty}\frac{1}{p!}\rho_\star^{(p-4)} A^3_{2,p,3,m}|_{\mathscr{I}^+}. \tag{85}$$

To simplify the latter equation, a direct calculation using equation (84) and the explicit form of the solution $a_{2,p,3,m}$ gives the following:

Remark 15. *$A^2_{2,p,3,m}|_{\mathscr{I}^+} = 0$ for $p \leq 3$ and $p \geq 5$.*

Using Remark 15 one finally obtains

$$F_m^2 = -\frac{35\sqrt{2}}{6}C_{4,3,m}. \tag{86}$$

For the calculation of F_m^n instead of proving the generalisation of Remark 15, the calculation will be simplified invoking the finiteness and constancy of the NP constants to compute them directly at I^+.

6.1.2 The general case

The previous discussion suggests that, in principle, it should be possible to obtain a general formula for F_m^n. Revisiting the calculation of F_m^0, F_m^1 and F_m^2 one can obtain the following results concerning the overall structure of the electromagnetic NP constants in flat space:

Lemma 3. *For any integer $n \geq 1$,*

$$\boldsymbol{e}'^{(n)}_{\mathbf{00}'}(\phi'_0) = 2^{2(n+1)}\Big(\frac{\sqrt{2}}{2}\Big)^n \sum_{p=1}^{\infty}\sum_{\ell=1}^{p}\sum_{m=-\ell}^{\ell} \frac{1}{p!}(1+\tau)^{-2(n+1)}\rho^{p-(n+1)} A^n_{2,p,l,m}(\tau) Y_{1;\ell,m}, \tag{87}$$

with,

$$A^n_{2,p,l,m}(\tau) \equiv \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(k+p)!}{p!}(1+\tau)^{n-k} a^{(n-k)}_{2,p,l,m}, \tag{88}$$

where $a^{(k)}_{2,p,l,m} \equiv (\boldsymbol{\partial}_\tau)^k a_{2,p,l,m}$.

Proof. To prove this result one proceeds by induction. Equations (69), (77) and (84) already show that the result is valid for $n=1$, $n=2$ and $n=3$. This constitutes the basis of induction. Now, assume that expressions (87)-(88) hold (induction hypothesis), then applying $\boldsymbol{e}'_{\mathbf{00}'}$ to equation (87), a direct calculation exploiting equations (62) and (63) renders

$$\boldsymbol{e}'^{(n+1)}_{\mathbf{00}'}(\phi'_0) = 2^{2(n+2)}\left(\frac{\sqrt{2}}{2}\right)^{(n+1)} \sum_{p=1}^{\infty}\sum_{\ell=1}^{p}\sum_{m=-\ell}^{\ell} \frac{1}{p!}(1+\tau)^{-2(n+2)}\rho^{p-(n+2)} R^n_{2,p,l,m}(\tau) Y_{1;\ell,m}, \tag{89}$$

where

$$R^n_{2,p,l,m}(\tau) = (1+\tau)\dot{A}^n_{2,p,l,m}(\tau) - (p+n+1)A^n_{2,p,l,m}(\tau).$$

Substituting (88) into the last expression gives

$$\begin{aligned} R^n(\tau) = (1+\tau)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(k+p)!}{p!}\Big((n-k)(1+\tau)^{n-k-1}a^{(n-k)} + (1+\tau)^{n-k}a^{(n-k-1)}\Big) \\ - (p+n+1)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(k+p)!}{p!}(1+\tau)^{n-k}a^{(n-k)}, \end{aligned} \tag{90}$$

where the subindices in $R^n_{2,p,\ell,m}$ and $a_{2,p,\ell,m}$ have been omitted for conciseness. Reorganising the terms one gets

$$\begin{aligned} R^n(\tau) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(k+p)!}{p!}(1+\tau)^{n-k+1}a^{(n-k+1)} \\ - \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(k+p)!}{p!}(p+k+1)(1+\tau)^{n-k}a^{(n-k)}. \end{aligned} \tag{91}$$

Expanding the first term in the first sum of equation (91) renders

$$\begin{aligned} R^n(\tau) = (1+\tau)^{n+1}a^{(n+1)} + \sum_{\lambda=0}^{n-1}(-1)^{\lambda+1}\binom{n}{\lambda+1}\frac{(p+\lambda+1)!}{p!}(1+\tau)^{n-\lambda}a^{(n-\lambda)} \\ + \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{(p+k+1)!}{p!}(1+\tau)^{n-k}a^{(n-k)}. \end{aligned}$$

Separating the last term in the second sum and rearranging gives

$$\begin{aligned} R^n(\tau) = (1+\tau)^{n+1}a^{(n+1)} + \sum_{k=0}^{n-1}(-1)^{k+1}\frac{(p+k+1)!}{p!}(1+\tau)^{n-k}a^{(n-k)}\left(\binom{n}{k} + \binom{n}{n+1}\right) \\ + (-1)^{n+1}\frac{(p+n+1)!}{p!}a. \end{aligned}$$

Using the recursive identity of the binomial coefficients

$$\binom{i}{j} = \binom{i-1}{j} + \binom{i-1}{j-1}, \tag{92}$$

renders

$$R^n(\tau) = (1+\tau)^{n+1}a^{(n+1)} + \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n+1}{k+1}\frac{(p+k+1)!}{p!}(1+\tau)^{n-k}a^{(n-k)} + (-1)^{n+1}\frac{(p+n+1)!}{p!}a.$$

Equivalently,

$$R^n_{2,p,l,m}(\tau) = (1+\tau)^{n+1}a^{(n+1)} + \sum_{i=0}^{n-1}(-1)^{i}\binom{n+1}{i}\frac{(p+i)!}{p!}(1+\tau)^{n-i-1}a^{(n-i-1)} + (-1)^{n+1}\frac{(p+n+1)!}{p!}a.$$

Relabelling the counter to absorb the first and last term and using the definition given in equation (88) one gets

$$R^n_{2,p,\ell,m}(\tau) = A^{n+1}_{2,p,\ell,m}(\tau).$$

Finally, substituting the last expression into equation (89) one concludes that

$$\boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0) \;=\; 2^{2(n+2)}\left(\frac{\sqrt{2}}{2}\right)^{(n+1)}\sum_{p=1}^{\infty}\sum_{\ell=1}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}(1\;+\;\tau)^{-2(n+2)}\rho^{p-(n+2)}A^{n+1}_{2,p,l,m}(\tau)Y_{1;\ell,m},$$

which completes the induction step. □

With Lemma 3 at hand, it is straightforward to determine the general structure of the NP constants:

Proposition 6. *The NP constants in Minkowski spacetime F^{n-1}_m are given by*

$$F^{n-1}_m = Q^+(n)C_{n+1,n,m},$$

where $Q^+(n)$ is a numerical factor and $C_{n+1,n,m}$ is determined by the initial data $a_{0,n+1,n,m}(0)$ and $a_{2,n+1,n,m}(0)$ as given in Lemma 1.

Proof. Exploiting the orthogonality condition (71), the expression for the electromagnetic NP constants given in equation (57) and equations (87) and (88) render

$$F^{n-1}_m = 2^{n+1}\left(\frac{\sqrt{2}}{2}\right)^{n}\left(\frac{1}{n!}\right)\sum_{p=1}^{\infty}\frac{1}{p!}\rho_\star^{(p-(n+1))}A^n_{2,p,n,m}|_{\mathscr{I}^+}. \tag{93}$$

Invoking the finiteness and constancy of the NP constants in order to evaluate the last expression on the cut I^+ one concludes that

$$F^{n-1}_m = 2^{n+1}\left(\frac{\sqrt{2}}{2}\right)^{n}\left(\frac{1}{n!(n+1)!}\right)A^n_{2,n+1,n,m}|_{\mathscr{I}^+}. \tag{94}$$

Direct inspection of $A^n_{2,n+1,n,m}|_{\mathscr{I}^+}$ renders

$$F^{n-1}_m = Q^+(n)C_{n+1,n,m}, \tag{95}$$

where $Q^+(n)$ is an irrelevant numerical factor. □

6.2 The constants at $\mathscr{I}^-$

The analysis carried out in Sections 5 and 6 for the electromagnetic constants defined at $\mathscr{I}^+$, can be performed in a completely analogous way for $\mathscr{I}^-$. To do so, consider a formal replacement $\tau \to -\tau$. Upon this formal replacement the roles of $\boldsymbol{\ell} = \boldsymbol{e_{00'}}$ and $\boldsymbol{n} = \boldsymbol{e_{11'}}$ as defined in (8a) and ϕ_0 and ϕ_2 are essentially interchanged. Then, following the discussion of Sections 5 and 6.1 one obtains *mutatis mutandis* the time dual of Proposition 5 and 6:

Proposition 7. *The NP-frame hinged at* $\mathscr{I}^-$ *is related to the F-frame via*

$$\boldsymbol{e}'_{\boldsymbol{11'}} = \frac{4}{\rho(1-\tau)^2}\boldsymbol{e_{00'}} \qquad \boldsymbol{e}'_{\boldsymbol{00'}} = \frac{1}{4}\rho(1-\tau)^2\boldsymbol{e_{11'}},$$
$$\boldsymbol{e}'_{\boldsymbol{10'}} = e^{-2i\omega}\boldsymbol{e_{01'}}, \qquad \boldsymbol{e}'_{\boldsymbol{01'}} = e^{-2i\omega}\boldsymbol{e_{10'}}.$$

The Lorentz transformation and conformal factor κ *relating the frames is given by*

$$\Lambda^{\boldsymbol{0}}{}_{\boldsymbol{1}} = \frac{2e^{i\omega}}{\sqrt{\rho}(1-\tau)} \qquad \Lambda^{\boldsymbol{1}}{}_{\boldsymbol{0}} = \frac{e^{-i\omega}\sqrt{\rho}(1-\tau)}{2}, \qquad \Lambda^{\boldsymbol{0}}{}_{\boldsymbol{0}} = \Lambda^{\boldsymbol{1}}{}_{\boldsymbol{1}} = 0, \qquad \kappa = 1. \tag{96}$$

Proposition 8. *The electromagnetic constants* $\bar{F}^n_m$ *at* $\mathscr{I}^-$ *are given by*

$$\bar{F}^{n-1}_m = Q^-(n)D_{n+1,n,m},$$

where $Q^-(n)$ *is a numerical coefficient.*

Finally, recalling the results of Propositions 6, and 8, the expressions for $C_{p,\ell,m}$ and $D_{p,\ell,m}$ given in Lemma 1 and Remark 5 one obtains the following:

Theorem 1. *Initial data for the spin-1 field (Maxwell's equations) on the Minkowski spacetime, satisfying the regularity condition of Assumption 2 give rise to a solution* ϕ_{AB} *whose electromagnetic NP constants* F^n_m *and* $\bar{F}^n_m$ *at* $\mathscr{I}^+$ *and* $\mathscr{I}^-$*, that in general, do not coincide. If the data is time-symmetric, the electromagnetic NP constants at* $\mathscr{I}^+$ *and* $\mathscr{I}^-$*, correspond to the same part of the initial data and, hence, up to a numerical factor* $Q^+(n)/Q^-(n)$*, coincide.*

7 The NP constants for the massless spin-2 field

In this section an analogous analysis to that given in Section 6 is performed for the case of the spin-2 massless field. The same notation as the one introduced in Section 6 will be used. In particular, the spin dyads $\tilde{\epsilon}_{\boldsymbol{A}}{}^A$, $\epsilon'_{\boldsymbol{A}}{}^A$ and $\epsilon_{\boldsymbol{A}}{}^A$ associated to $\tilde{\boldsymbol{\eta}}$, $\boldsymbol{g}'$ and $\boldsymbol{g}$ will be employed. To introduce the gravitational NP constants originally introduced in [19], let $\tilde{\phi}_0$, $\tilde{\phi}_1$, $\tilde{\phi}_2$, $\tilde{\phi}_3$ and $\tilde{\phi}_4$ denote the components of the spin-2 massless field $\tilde{\phi}_{ABCD}$ respect to $\tilde{\epsilon}_{\boldsymbol{A}}{}^A$. The spin-2 equation reads

$$\tilde{\nabla}_{A'}{}^A\tilde{\phi}_{ABCD} = 0. \tag{97}$$

Assumption 6. Following [19], the component ϕ_0 is assumed to have the expansion

$$\tilde{\phi}_0 = \sum_{n=0}^{N}\frac{\tilde{\phi}_0^n}{\tilde{\rho}^{5+n}} + \mathcal{O}\Big(\frac{1}{\tilde{\rho}^{5+N}}\Big), \tag{98}$$

where the coefficients $\tilde{\phi}_0^n$ do not depend on $\tilde{\rho}$.

As already mentioned, the field $\tilde{\phi}_{ABCD}$ provides a description of the linearised gravitational field over the Minkowski spacetime. In the full non-linear theory, the linear field $\tilde{\phi}_{ABCD}$ is replaced by the Weyl spinor Ψ_{ABCD} and the analogue of equation (97) encodes the second Bianchi identity in vacuum —see [19]. The spin-2 NP quantities are defined through the following integrals over cuts $\mathcal{C}$ of null infinity:

$$G^n_m \equiv \int_{\mathcal{C}} \bar{Y}_{2;n+2,m}\tilde{\phi}_0^{n+1}\mathrm{d}S,$$

where $n, m \in \mathbb{Z}$ with $n \geq 0$, $|m| \leq n+2$ and $\mathrm{d}S$ denotes the area element respect to $\boldsymbol{\sigma}$. The NP constants G^n_m are absolutely conserved in the sense that their value is independent on the cut $\mathcal{C}$ on which they are evaluated.

Remark 16. In particular, the constants G^0_m are also conserved in the full non-linear case of the gravitational field where $\tilde{\phi}_0$ is replaced by the component Ψ_0 of the Weyl spinor Ψ_{ABCD} —see [19]. These are the only constants of the hierarchy which are generically inherited in the non-linear case.

7.1 Translation to the F-gauge

An expression for the gravitational NP constants in the F-gauge has been given in Section III of [12]. In order to provide a self-contained discussion and for the ease of comparison with the analysis made in Section 6 the analogue of Formula (III.5) of [12] will be derived in accordance with the notation and conventions used in this article. In view of equation (51), one has that, as a consequence of the standard conformal transformation law for the spin-2 equation —see [25], the spinor ϕ'_{ABCD}, satisfying

$$\nabla'_{A'}{}^{A}\phi'_{ABCD} = 0,$$

where $\nabla'_{AA'}$ is the Levi-Civita connection of $\boldsymbol{g}'$, is related to ϕ_{ABCD} via

$$\phi'_{ABCD} = \tilde{\rho}\tilde{\phi}_{ABCD}. \tag{99}$$

Therefore, using equations (98),(54) and (99), one obtains

$$\phi'_0 = \sum_{n=0}^{N} \frac{\tilde{\phi}_0^n}{\tilde{\rho}^n} + \mathcal{O}\Big(\frac{1}{\tilde{\rho}^N}\Big),$$

where $\phi'_0 \equiv \phi'_{ABCD}o'^{A}o'^{B}o'^{C}o'^{D}$. Using the last expansion and recalling that $\boldsymbol{e}'_{\mathbf{0}\mathbf{0}'} = -\tilde{\rho}^2\boldsymbol{\partial}_{\tilde{\rho}}$ one obtains, after consecutive applications of $\boldsymbol{e}'_{\mathbf{0}\mathbf{0}'}$, the expression

$$G^n_m = \frac{1}{(n+1)!}\int_{\mathcal{C}} \bar{Y}_{2;n+2,m}\boldsymbol{e}'^{(n+1)}_{\mathbf{0}\mathbf{0}'}(\phi'_0)\mathrm{d}S. \tag{100}$$

To derive an expression for the spin-2 NP constants in the F-gauge one recalls the relation between the $\boldsymbol{g}'$ and $\boldsymbol{g}$ representations and their associated spin dyads encoded in equation (58). Once again, as a consequence of the conformal transformation laws for the spin-2 equation one has that the spinor field ϕ_{ABCD} related to ϕ'_{ABCD} through

$$\phi'_{ABCD} = \kappa^{-1}\phi_{ABCD},$$

satisfies

$$\nabla_{A'}{}^{A}\phi_{ABCD} = 0,$$

where $\nabla_{AA'}$ represents the Levi-Civita connection respect to $\boldsymbol{g}$. Additionally, one has that

$$\phi'_0 = \kappa^{-3}\Lambda^{\boldsymbol{A}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{B}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}}\phi_{\boldsymbol{ABCD}},$$

where $\phi_{\boldsymbol{ABCD}} \equiv \epsilon_{\boldsymbol{A}}{}^{A}\epsilon_{\boldsymbol{B}}{}^{B}\epsilon_{\boldsymbol{C}}{}^{C}\epsilon_{\boldsymbol{D}}{}^{D}\phi_{ABCD}$.

7.1.1 Explicit computation of the first constant

Using equation (100) and the Leibniz rule one obtains the analogue of Equation (III.5) of [12] written in accordance with the notation and conventions used in this article

$$G^0_m = \int_{\mathcal{C}} \bar{Y}_{2;2,m}\kappa^{-4}\Big(\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{C}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{D}}{}_{\boldsymbol{0}}\big(\Lambda^{\boldsymbol{E}}{}_{\boldsymbol{0}}\bar{\Lambda}^{\boldsymbol{E}'}{}_{\boldsymbol{0}'}\boldsymbol{e}_{\boldsymbol{EE}'}(\phi_{\boldsymbol{ABCD}})$$

$$- 3\phi_{\boldsymbol{ABCD}}\boldsymbol{e}'_{\boldsymbol{00}'}(\kappa)\big) + 4\kappa\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{0}}\Lambda^{\boldsymbol{C}}{}_{\boldsymbol{0}}\phi_{\boldsymbol{ABCD}}\boldsymbol{e}'_{\boldsymbol{00}'}(\Lambda^{\boldsymbol{D}}{}_{\boldsymbol{0}})\Big)\mathrm{d}S. \quad (101)$$

Particularising the discussion to the case of the Minkowski spacetime, simplifies the expressions considerably. To see this, observe that, using Proposition 5 and the notation introduced in equation (62), a direct calculation reveals that $\phi'_0 = \Lambda^4\phi_4$. Using this relation, and the short-hands introduced in equation (62) one has that

$$\boldsymbol{e}'_{\boldsymbol{00}'}(\phi'_0) = \frac{\sqrt{2}}{2}\Lambda^6(\boldsymbol{e}\phi_4 - 2\phi_4). \quad (102)$$

Using that

$$\boldsymbol{e}(a_{4,p,\ell,m}(\tau)\rho^p) = ((1+\tau)\dot{a}_{4,p,l,m}(\tau) - pa_{4,p,\ell,m}(\tau))\rho^p, \quad (103)$$

and the Ansatz (31) a calculation gives

$$\boldsymbol{e}'_{\boldsymbol{00}'}(\phi'_0) = \frac{\sqrt{2}}{2}\sum_{p=2}^{\infty}\sum_{\ell=2}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}\Lambda^6\rho^p((1+\tau)\dot{a}_{4,p,l,m}(\tau) - (p+2)a_{4,p,\ell,m}(\tau))Y_{2;\ell,m}.$$

Separating the dependence in τ and ρ explicitly, the latter expression can be written as

$$\boldsymbol{e}'_{\boldsymbol{00}'}(\phi'_0) = 2^6\Big(\frac{\sqrt{2}}{2}\Big)\sum_{p=2}^{\infty}\sum_{\ell=2}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}(1+\tau)^{-6}\rho^{(p-3)}A^1_{4,p,l,m}(\tau)Y_{2;\ell,m}, \quad (104)$$

where

$$A^1_{4,p,l,m}(\tau) \equiv ((1+\tau)\dot{a}_{4,p,l,m}(\tau) - (2+p)a_{4,p,\ell,m}(\tau)). \quad (105)$$

Using equations (104) and (100) computing the first set of NP constants reduces to the evaluating the following integral

$$G^0_m = 2^6\Big(\frac{\sqrt{2}}{2}\Big)\lim_{\substack{\rho\to\rho_\star \\ \tau\to 1}}\left(\int_{\mathbb{S}^2}\sum_{p=2}^{\infty}\sum_{\ell=2}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}(1+\tau)^{-6}\rho^{(p-3)}A^1_{4,p,l,m}(\tau)Y_{2;\ell,m}\bar{Y}_{2;2,m}\mathrm{d}S\right). \quad (106)$$

Using the orthogonality relation (71) one obtains

$$G^0_m = 2^6\Big(\frac{\sqrt{2}}{2}\Big)\sum_{p=2}^{\infty}\frac{1}{p!}\rho_\star^{(p-3)}A^1_{4,p,2,m}|_{\mathscr{I}^+}. \quad (107)$$

A direct calculation using the explicit form of $a_{4,p,l,m}(\tau)$ gives the following:

Remark 17. $A^1_{4,p,1,m}|_{\mathscr{I}^+} = 0$ *for* $p = 2$ *and* $p \geq 4$.

Exploiting Remark 17 a calculation one concludes that,

$$G^0_m = -\frac{80\sqrt{2}}{3}C_{3,2,m}. \quad (108)$$

Completely analogous remarks as those made for the calculation of the electromagnetic NP constants can be made in this case: the constant comes from the $p = 3$ term and all the terms with $p \geq 4$ that could potentially contribute to the NP constant identically vanish by virtue of Remark 17. The regularity condition of Assumption 4 has been used to have well defined classical NP constants. As in the electromagnetic case, one can avoid using the results of Remark 17 by simply invoking the finiteness and constancy of the NP constants and evaluating at the cut $\mathcal{C} = I^+$.

7.1.2 The general case

Revisiting the calculation of G_m^0 one can see that the mechanism is completely analogous to the electromagnetic case:

Lemma 4. *For any integer* $n \geq 1$

$$\boldsymbol{e}'^{(n)}_{\mathbf{00}'}(\phi'_0) = 2^{2(n+2)}\left(\frac{\sqrt{2}}{2}\right)^n \sum_{p=2}^{\infty}\sum_{\ell=2}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}(1+\tau)^{-2(n+2)}\rho^{p-(n+2)}A^n_{4,p,l,m}(\tau)Y_{2;\ell,m}, \tag{109}$$

with,

$$A^n_{4,p,l,m}(\tau) \equiv \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(k+p+1)!}{(p+1)!}(1+\tau)^{n-k}a^{(n-k)}_{4,p,l,m}, \tag{110}$$

where $a^{(k)}_{4,p,l,m} \equiv (\boldsymbol{\partial}_\tau)^k a_{4,p,l,m}$.

Proof. To prove this result one proceeds by induction. Equations (104) and (105) already show that the result is valid for $n = 1$. This constitutes the basis of induction. Now, assume that equations (109) and (110) hold (induction hypothesis), then, applying $\boldsymbol{e}'_{\mathbf{00}'}$ to equation (109) renders

$$\boldsymbol{e}'^{(n+3)}_{\mathbf{00}'}(\phi'_0) = 2^{2(n+3)}\left(\frac{\sqrt{2}}{2}\right)^{(n+3)} \sum_{p=2}^{\infty}\sum_{\ell=2}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}(1+\tau)^{-2(n+3)}\rho^{p-(n+3)}R^n_{4,p,l,m}(\tau)Y_{2;\ell,m}, \tag{111}$$

where

$$R^n_{4,p,l,m}(\tau) = (1+\tau)\dot{A}^n_{4,p,l,m}(\tau) - (p+n+2)A^n_{4,p,l,m}(\tau).$$

Substituting (110) into the last expression and omitting the subindices gives

$$R^n(\tau) = (1+\tau)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(k+p+1)!}{(p+1)!}\Big((n-k)(1+\tau)^{n-k-1}a^{(n-k)} + (1+\tau)^{n-k}a^{(n-k-1)}\Big)$$
$$- (p+n+2)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(k+p+1)!}{(p+1)!}(1+\tau)^{n-k}a^{(n-k)}.$$

Reorganising the last expression one gets

$$R^n(\tau) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(k+p+1)!}{(p+1)!}(1+\tau)^{n-k+1}a^{(n-k+1)}$$
$$- \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(k+p+1)!}{(p+1)!}(p+k+2)(1+\tau)^{n-k}a^{(n-k)}. \tag{112}$$

Separating the first term in the first sum of equation (112) gives

$$R^n(\tau) = (1+\tau)^{n+1}a^{(n+1)} + \sum_{\lambda=0}^{n-1}(-1)^{\lambda+1}\binom{n}{\lambda+1}\frac{(p+\lambda+2)!}{(p+1)!}(1+\tau)^{n-\lambda}a^{(n-\lambda)}$$
$$+ \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{(p+k+2)!}{(p+1)!}(1+\tau)^{n-k}a^{(n-k)}.$$

Expanding the last term in the second sum and rearranging renders

$$R^n(\tau) = (1+\tau)^{n+1}a^{(n+1)} + \sum_{k=0}^{n-1}(-1)^{k+1}\frac{(p+k+2)!}{(p+1)!}(1+\tau)^{n-k}a^{(n-k)}\left(\binom{n}{k}+\binom{n}{n+1}\right)$$
$$+ (-1)^{n+1}\frac{(p+n+2)!}{(p+1)!}a.$$

Applying the recursive identity of the binomial coefficients (92) one gets

$$R^n(\tau) = (1+\tau)^{n+1}a^{(n+1)} + \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n+1}{k+1}\frac{(p+k+2)!}{(p+1)!}(1+\tau)^{n-k}a^{(n-k)}$$
$$+ (-1)^{n+1}\frac{(p+n+2)!}{(p+1)!}a.$$

The last expression can be rewritten as

$$R^n(\tau) = (1+\tau)^{n+1}a^{(n+1)} + \sum_{i=0}^{n-1}(-1)^{i}\binom{n+1}{i}\frac{(p+i+1)!}{(p+1)!}(1+\tau)^{n-i-1}a^{(n-i-1)}$$
$$+ (-1)^{n+1}\frac{(p+n+2)!}{(p+1)!}a.$$

Reshuffling the counter to absorb the first and last term and using the definition given in equation (110) renders

$$R^n_{4,p,\ell,m}(\tau) = A^{n+1}_{4,p,\ell,m}(\tau).$$

Finally, substituting the last expression into equation (111) one concludes that

$$\boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0) = 2^{2(n+3)}\left(\frac{\sqrt{2}}{2}\right)^{n+1}\sum_{p=2}^{\infty}\sum_{\ell=2}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}(1+\tau)^{-2(n+3)}\rho^{p-(n+3)}A^{n+1}_{4,p,l,m}(\tau)Y_{2;\ell,m},$$

which completes the induction step. □

Exploiting Lemma 4 to determine the general structure of the NP constants is straightforward:

Proposition 9. *The NP constants in Minkowski spacetime G^{n-1}_m are given by*

$$G^{n-1}_m = Q^+(n)C_{n+2,n+1,m}.$$

where $Q^+(n)$ is a numerical factor and $C_{n+1,n,m}$ is determined by the initial data $a_{0,n+1,n,m}(0)$ and $a_{4,n+1,n,m}(0)$ as given in Lemma 2.

Proof. The orthogonality condition (71), along with the expression for the NP constants given in equation (100) and the equations (109) and (110) render

$$G^{n-1}_m = 2^{n+2}\left(\frac{\sqrt{2}}{2}\right)^{n}\left(\frac{1}{n!}\right)\sum_{p=2}^{\infty}\frac{1}{p!}\rho_\star^{(p-(n+2))}A^n_{4,p,n+1,m}|_{\mathscr{I}^+}. \tag{113}$$

Invoking the finiteness and constancy of the NP constants in order to evaluate the last expression on the cut I^+ one concludes that

$$G^{n-1}_m = 2^{n+2}\left(\frac{\sqrt{2}}{2}\right)^{n}\left(\frac{1}{n!(n+2)!}\right)A^n_{4,n+2,n+1,m}|_{\mathscr{I}^+}. \tag{114}$$

direct inspection of $A^n_{4,p,n+2,m}|_{\mathscr{I}^+}$ renders

$$G^{n-1}_m = Q^+(n)C_{n+2,n+1,m}, \tag{115}$$

where $Q^+(n)$ is an irrelevant numerical factor. □

7.2 The constants at $\mathscr{I}^-$

The time dual result can be obtained in a similar way as it was done for the in the electromagnetic case. Using Propositions 7 and 9, one obtains:

Proposition 10. *The NP constants $\bar{G}^n_m$ at $\mathscr{I}^-$ are given by*

$$\bar{G}^{n-1}_m = Q^-(n) D_{n+2,n+1,m},$$

where $Q^-(n)$ is a numerical coefficient.

Recalling the results of Propositions 9, 10, the expressions for $C_{p,\ell,m}$ and $D_{p,\ell,m}$ given in Lemma 2 and Remark 11 one obtains the following:

Theorem 2. *Initial data for the spin-2 field on the Minkowski spacetime, satisfying the regularity condition of Assumption 4 give rise to a solution ϕ_{ABCD} whose associated NP constants G^n_m and $\bar{G}^n_m$ at $\mathscr{I}^+$ and $\mathscr{I}^-$, that in general, do not coincide. If the data is time-symmetric, the NP constants at $\mathscr{I}^+$ and $\mathscr{I}^-$, correspond to the same part of the initial data and, hence, up to a numerical factor $Q^+(n)/Q^-(n)$, coincide.*

Remark 18. A similar symmetric behaviour has been observed in the gravitational case in [30]. In that reference the Newman-Penrose constants at future and past null infinity of the spacetime arising from Bowen-York initial data have been computed.

8 Conclusions

In this article the correspondence between initial data given on a Cauchy hypersurface $\mathcal{S}$ intersecting i^0 on Minkowski spacetime for the spin-1 (electromagnetic) and spin-2 fields and their associated NP constants is analysed. This analysis has been done for the full hierarchy of NP constants F^n_m and G^n_m in the Minkowski spacetime.

For the electromagnetic case, it was shown that, once the initial data for the Maxwell spinor is written as an expansion of the form (25), the electromagnetic NP constants F^{n-1}_m at $\mathscr{I}^+$ can be identified with a particular combination —denoted as $C_{p,\ell,m}$— of the initial data $a_{0,p;\ell,m}(0)$ and $a_{2,p;\ell,m}(0)$ with $p = n+1$ and $\ell = n$ —see Lemma 1 and Proposition 6. Since $1 \leq \ell \leq p$, one concludes that F^n_m are in correspondence with the second highest harmonic but are insensitive to the initial data for lower modes $\ell \leq p-2$ or the highest mode $p = \ell$. Notice, that it has been assumed that the logarithmic terms appearing at $p = \ell$ are not present in the solution which, of course, restricts the initial data —see Assumption 2. This restriction is necessary, otherwise the NP constants are not well defined —see discussion in [27] and [28]. In an analogous way, one can identify the electromagnetic NP constants $\bar{F}^{n-1}_m$ at $\mathscr{I}^-$ with a different combination —denoted as $D_{p,\ell,m}$— of the initial data $a_{0,p;\ell,m}(0)$ and $a_{2,p;\ell,n}(0)$ and similar remarks apply. The crucial observation to be made is that in general $C_{p,\ell\,m} \neq D_{p,\ell\,m}$, and hence, the NP constants at $\mathscr{I}^+$ and $\mathscr{I}^-$ do not coincide —see Theorem 1. Notice however that if the initial data considered is time-symmetric then, $C_{p,\ell,m} = -D_{p,\ell,m}$, and consequently, there is a correspondence between the NP constants defined at $\mathscr{I}^+$ and $\mathscr{I}^-$.

The same analysis was performed for a spin-2 field on a Minkowski background. The conclusion being that the NP constants G^{n-1}_m at $\mathscr{I}^+$ and $\mathscr{I}^-$ depend on the initial data for $a_{0,p;\ell,m}(0)$ and $a_{4,p;\ell,m}(0)$ with $p = n+2$ and $\ell = n+1$ via the expressions for $C_{p,\ell,m}$ and $D_{p,\ell,m}$ as given in Lemma 2 —see Propositions 9 and 10. The main point to be stressed about the particular form of $C_{p,\ell,m}$ and $D_{p,\ell,m}$ is that $C_{p,\ell,m} \neq D_{p,\ell,n}$ —see Theorem 2. As in the electromagnetic case, in order to have well defined NP constants it is necessary to assume that the initial data satisfy the regularity condition of Assumption 4 —see [29], [8] and [33] for further discussion on this regularity condition. The overall conclusion is then then that, for *truly generic initial data* the *classical NP constants* are not defined and even if one considers data satisfying the regularity condition of Assumption 4 —or Assumption 2 for the electromagnetic case— , the NP constants at $\mathscr{I}^+$ and $\mathscr{I}^-$ do not coincide. However, restricting the initial data to be time-symmetric is a sufficient condition to achieve this correspondence.

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Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

A The connection on $\mathbb{S}^2$

In this section expressions for the connection coefficients —of the Levi-Civita connection— respect to a complex null frame which do not make reference to any particular coordinate system on $\mathbb{S}^2$ are obtained. To set up the notation, it is convenient to start the discussion writing the Cartan structure equations in accordance with the conventions used in this article:

$$\mathbf{d}\boldsymbol{\omega}^{\boldsymbol{a}} = -\boldsymbol{\gamma}^{\boldsymbol{a}}{}_{\boldsymbol{b}} \wedge \boldsymbol{\omega}^{\boldsymbol{b}}, \tag{A.1a}$$

$$\mathbf{d}\boldsymbol{\gamma}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = -\boldsymbol{\gamma}^{\boldsymbol{a}}{}_{\boldsymbol{d}} \wedge \boldsymbol{\gamma}^{\boldsymbol{d}}{}_{\boldsymbol{b}} + \boldsymbol{\Omega}^{\boldsymbol{a}}{}_{\boldsymbol{b}}. \tag{A.1b}$$

In the last expressions the curvature 2-form $\boldsymbol{\Omega}^{\boldsymbol{a}}{}_{\boldsymbol{b}}$ and connection 1-form $\boldsymbol{\gamma}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ are defined via

$$\boldsymbol{\Omega}^{\boldsymbol{a}}{}_{\boldsymbol{b}} \equiv \frac{1}{2} R^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} \boldsymbol{\omega}^{\boldsymbol{c}} \wedge \boldsymbol{\omega}^{\boldsymbol{d}}, \qquad \boldsymbol{\gamma}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \equiv \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \wedge \boldsymbol{\omega}^{\boldsymbol{a}}. \tag{A.2}$$

The connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ of the Levi-Civita connection ∇ respect to a given frame $\boldsymbol{e}_{\boldsymbol{a}}$ are defined as $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \equiv \langle \boldsymbol{\omega}^{\boldsymbol{c}}, \nabla_{\boldsymbol{a}} \boldsymbol{e}_{\boldsymbol{b}} \rangle$.

In the remaining part of this appendix the discussion is particularised to the case of $\mathbb{S}^2$. Let $\{\boldsymbol{\partial}_+, \boldsymbol{\partial}_-\}$ be a complex null frame on $\mathbb{S}^2$ with corresponding dual covectors $\{\boldsymbol{\omega}^+, \boldsymbol{\omega}^-\}$. Namely, one considers

$$\boldsymbol{\sigma} = 2(\boldsymbol{\omega}^+ \otimes \boldsymbol{\omega}^- + \boldsymbol{\omega}^- \otimes \boldsymbol{\omega}^+), \qquad \boldsymbol{\sigma}^\flat = \frac{1}{2}(\boldsymbol{\partial}_+ \otimes \boldsymbol{\partial}_- + \boldsymbol{\partial}_- \otimes \boldsymbol{\partial}_+),$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^\flat$ denote the covariant and contravariant version of the standard metric on $\mathbb{S}^2$. Furthermore, one assumes that

$$\boldsymbol{\partial}_+ = \overline{\boldsymbol{\partial}_-}, \tag{A.3}$$

and consequently $\boldsymbol{\omega}^+ = \overline{\boldsymbol{\omega}^-}$. To start the discussion observe that $[\boldsymbol{\partial}_+, \boldsymbol{\partial}_-]$ and its complex conjugate can be expressed as a linear combination of the basis vectors $\boldsymbol{\partial}_+$ and $\boldsymbol{\partial}_-$. A direct inspection, taking into account the condition encoded in equation (A.3), reveals that

$$[\boldsymbol{\partial}_+, \boldsymbol{\partial}_-] = \varpi \boldsymbol{\partial}_+ - \overline{\varpi} \boldsymbol{\partial}_-, \tag{A.4}$$

where ω is a scalar field over $\mathbb{S}^2$. Using the no-torsion condition of the Levi-Civita connection $\eth\!\!\!/\nabla$ on $\mathbb{S}^2$ we get from equation (A.4) that

$$\nabla\!\!\!\!/_{+}\boldsymbol{\partial}_{-} - \nabla\!\!\!\!/_{-}\boldsymbol{\partial}_{+} = \varpi\boldsymbol{\partial}_{+} - \overline{\varpi}\boldsymbol{\partial}_{-}, \tag{A.5}$$

where $\nabla\!\!\!\!/_{+}$ and $\nabla\!\!\!\!/_{-}$ denote a covariant derivative in the direction of $\boldsymbol{\partial}_{+}$ and $\boldsymbol{\partial}_{-}$ respectively. Using equation (A.5) and the metricity conditions $\nabla\!\!\!\!/_{+}\boldsymbol{\sigma} = 0, \nabla\!\!\!\!/_{-}\boldsymbol{\sigma} = 0$, one finds that the only non-zero connection coefficients are all encoded in the scalar field ω:

$$\Gamma_{-}{}^{-}{}_{-} = \overline{\Gamma_{+}{}^{+}{}_{+}} = -\Gamma_{-}{}^{+}{}_{+} = -\overline{\Gamma_{+}{}^{-}{}_{-}} = \varpi.$$

The connection can be compactly encoded in the curvature 1-form $\boldsymbol{\gamma^{a}{}_{b}}$ as defined in equation (A.2). A direct computation renders

$$\boldsymbol{\gamma}^{+}{}_{+} = \overline{\boldsymbol{\gamma}^{-}{}_{-}} = \overline{\varpi}\boldsymbol{\omega}^{+} - \varpi\boldsymbol{\omega}^{-}, \qquad \boldsymbol{\gamma}^{+}{}_{-} = \boldsymbol{\gamma}^{-}{}_{+} = 0.$$

Using the first Cartan structure equation encoded in (A.1a), one obtains

$$\mathbf{d}\boldsymbol{\omega}^{+} = -\varpi\boldsymbol{\omega}^{+}\wedge\boldsymbol{\omega}^{-}, \qquad \mathbf{d}\boldsymbol{\omega}^{-} = \overline{\varpi}\boldsymbol{\omega}^{+}\wedge\boldsymbol{\omega}^{-}. \tag{A.6}$$

For completeness, using the above expressions and the second Cartan structure equation encoded in (A.1b), one can directly compute the curvature form $\boldsymbol{\Omega^{a}{}_{b}}$:

$$\boldsymbol{\Omega}^{+}{}_{+} = \overline{\boldsymbol{\Omega}^{-}{}_{-}} = -2(|\varpi|^2 + \frac{1}{2}(\boldsymbol{\partial}_{+}\varpi + \boldsymbol{\partial}_{-}\overline{\varpi}))\boldsymbol{\omega}^{+}\wedge\boldsymbol{\omega}^{-}.$$

Notice that, in order to find further information about ω one can exploit the fact that the Riemann curvature for maximally symmetric spaces $(\mathcal{N}, \boldsymbol{h})$ is given by

$$R_{\boldsymbol{abcd}} = \frac{1}{2}R(h_{\boldsymbol{ab}}h_{\boldsymbol{bd}} - h_{\boldsymbol{ad}}h_{\boldsymbol{bc}}),$$

where R is the Ricci scalar of the Levi-Civita connection of the metric $\boldsymbol{h}$ on $\mathcal{N}$. Since the Ricci scalar for $\mathbb{S}^2$ is $R = -2$, using equation (A.2) one finds that

$$\boldsymbol{\Omega}^{+}{}_{+} = \overline{\boldsymbol{\Omega}^{-}{}_{-}} = 2\boldsymbol{\omega}^{+}\wedge\boldsymbol{\omega}^{-}.$$

Consequently, one concludes that the scalar field ω satisfies

$$|\varpi|^2 + \frac{1}{2}(\boldsymbol{\partial}_{+}\varpi + \boldsymbol{\partial}_{-}\overline{\varpi}) = -1. \tag{A.7}$$

B The $\eth$ and $\bar{\eth}$ operators

In this appendix, the operators $\boldsymbol{\partial}_{+}$ and $\boldsymbol{\partial}_{-}$ are written in terms of the $\eth$ and $\bar{\eth}$ operators of Newman and Penrose. To fix the notation and conventions, let $\eth_P$ and $\bar{\eth}_P$ denote the $\eth$ and $\bar{\eth}$ operators [22] as defined in [25]. In the language of the NP-formalism [18, 22, 25], given a null frame represented by $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ their corresponding covariant directional derivatives are denoted by $\{D, \Delta, \delta, \bar{\delta}\}$. The operators $\eth_P$ and $\bar{\eth}_P$ acting on a quantity η with spin weight s can be written in terms of the δ and $\bar{\delta}$ derivatives as —see [25],

$$\eth_P\eta = \delta\eta + s(\bar{\alpha} - \beta)\eta, \qquad \bar{\eth}_P\eta = \bar{\delta}\eta - s(\alpha - \bar{\beta})\eta, \tag{B.1}$$

where α and β denote the spin coefficients as defined in the NP formalism. The action of the directional derivatives δ and $\bar{\delta}$ on the vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$, projected into the tangent space $T(\mathcal{Q}) \subset T(\mathcal{M})$ spanned by $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$, is encoded in

$$\delta m^a = -(\bar{\alpha} - \beta)m^a, \qquad \delta\bar{m}^a = (\bar{\alpha} - \beta)\bar{m}^a \qquad \text{on} \qquad \mathcal{Q}. \tag{B.2}$$

The directional derivatives $\not\nabla_+$ and $\not\nabla_-$ as defined on Appendix A are related to δ and $\bar{\delta}$ via

$$\delta = \frac{1}{\sqrt{2}}\not\nabla_+, \qquad \bar{\delta} = \frac{1}{\sqrt{2}}\not\nabla_-.$$

It follows from the discussion of Appendix A and equation (B.2) that

$$\bar{\alpha} - \beta = -\frac{1}{\sqrt{2}}\overline{\varpi}, \qquad \text{on} \qquad \mathcal{Q} \tag{B.3}$$

Using equations (B.2) and (B.3) one obtains

$$\not\nabla_+\eta = \sqrt{2}\eth_P\eta + s\overline{\varpi}\eta, \qquad \not\nabla_-\eta = \sqrt{2}\bar{\eth}_P\eta - s\varpi\eta, \tag{B.4}$$

To align the discussion with the conventions of [12, 31, 29] is convenient to define $\eth$ and $\bar{\eth}$ by rescaling $\eth_P$ and $\bar{\eth}_P$ as

$$\eth \equiv -\frac{1}{\sqrt{2}}\eth_P, \qquad \bar{\eth} \equiv -\frac{1}{\sqrt{2}}\bar{\eth}_P. \tag{B.5}$$

The corresponding eigenfunctions $Y_{s;\ell m}$ of the operator $\eth\bar{\eth}$, defining the spin-weighted spherical harmonics, will be assumed to be rescaled in accordance with equation (B.5). Exploiting that $\{Y_{s;\ell m}\}$, with $0 \leq |s| \leq \ell$ and $-\ell \leq m \leq \ell$, form a complete basis for functions of spin-weight s over $\mathbb{S}^2$, given a scalar field $\xi : \mathcal{Q} \to \mathbb{R}$, with spin-weight s, one can expand ξ as

$$\xi = \sum_{\ell=s}^{\infty}\sum_{m=-\ell}^{\ell} C_{s\ell m}\, Y_{s;\ell m}. \tag{B.6}$$

In addition, one has that

$$\eth(Y_{s;\ell m}) = \sqrt{(\ell - s)(\ell + s + 1)}\; Y_{s+1;\ell,m}, \tag{B.7a}$$

$$\bar{\eth}(Y_{s;\ell,m}) = -\sqrt{(\ell + s)(\ell - s + 1)}\; Y_{s-1;\ell m}. \tag{B.7b}$$

Notice that equation (B.6) as well as equations (B.7a)-(B.7b) do not depend on the specific choice of coordinates on $\mathcal{Q}$.

References

[1] A. Aceña & J. A. Valiente Kroon, *Conformal extensions for stationary spacetimes*, Class. Quantum Grav. **28**, 225023 (2011).

[2] L. Andersson, T. Bäckdahl, & J. Joudioux, *Hertz potentials and asymptotic properties of massless fields*, Commun. Math. Phys. **331**, 755–803 (2014).

[3] R. Beig, *Integration of Einstein's equations near spatial infinity*, Proc. Roy. Soc. Lond. A **391**, 295 (1984).

[4] R. Beig & B. G. Schmidt, *Einstein's equation near spatial infinity*, Comm. Math. Phys. **87**, 65 (1982).

[5] D. Christodoulou & S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton University Press, 1993.

[6] H. Friedrich, *On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations*, Proc. Roy. Soc. Lond. A **375**, 169 (1981).

[7] H. Friedrich, *Einstein equations and conformal structure: existence of anti-de Sitter-type space-times*, J. Geom. Phys. **17**, 125 (1995).

[8] H. Friedrich, *Gravitational fields near space-like and null infinity*, J. Geom. Phys. **24**, 83 (1998).

[9] H. Friedrich, *Spin-2 fields on Minkowski space near space-like and null infinity*, Class. Quantum Grav. **20**, 201 (2003).

[10] H. Friedrich, *Conformal geodesics on vacuum spacetimes*, Comm. Math. Phys. **235**, 513 (2003).

[11] H. Friedrich, *Smoothness at null infinity and the structure of initial data*, in *50 years of the Cauchy problem in general relativity*, edited by P. T. Chruściel & H. Friedrich, Birkhausser, 2004.

[12] H. Friedrich & J. Kánnár, *Bondi-type systems near space-like infinity and the calculation of the NP-constants*, J. Math. Phys. **41**, 2195 (2000).

[13] E. Gasperin & J. A. V. Kroon, *Polyhomogeneous expansions from time symmetric initial data*, Class. Quant. Grav. **34**(19), 195007 (2017).

[14] S. W. Hawking, *Black hole explosions*, Nature **248**, 30–31 (1974).

[15] S. W. Hawking, *Particle creation by black holes*, Comm. Math. Phys. **43**(3), 199–220 (1975).

[16] S. W. Hawking, *Breakdown of predictability in gravitational collapse*, Phys. Rev. D **14**, 2460–2473 (Nov 1976).

[17] S. W. Hawking, M. J. Perry, & A. Strominger, *Soft Hair on Black Holes*, Phys. Rev. Lett. **116**, 231301 (Jun 2016).

[18] E. T. Newman & R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, J. Math. Phys. **3**, 566 (1962).

[19] E. T. Newman & R. Penrose, *New conservation laws for zero rest-mass fields in asymptotically flat space-time*, Proc. Roy. Soc. Lond. A **305**, 175 (1968).

[20] R. Penrose, *Asymptotic properties of fields and space-times*, Phys. Rev. Lett. **10**, 66 (1963).

[21] R. Penrose, *Zero rest-mass fields including gravitation: asymptotic behaviour*, Proc. Roy. Soc. Lond. A **284**, 159 (1965).

[22] R. Penrose & W. Rindler, *Spinors and space-time. Volume 1. Two-spinor calculus and relativistic fields*, Cambridge University Press, 1984.

[23] B. G. Schmidt, *The decay of the gravitational field*, Comm. Math. Phys. **78**, 447 (1981).

[24] B. G. Schmidt, *Gravitational radiation near spatial and null infinity*, Proc. Roy. Soc. Lond. A **410**, 201 (1987).

[25] J. Stewart, *Advanced general relativity*, Cambridge University Press, 1991.

[26] K. P. Tod, *Isotropic cosmological singularities*, in *The Conformal structure of space-time. Geometry, Analysis, Numerics*, edited by J. Frauendiener & H. Friedrich, Lect. Notes. Phys. **604**, page 123, 2002.

[27] J. A. Valiente Kroon, *Conserved Quantities for polyhomogeneous spacetimes*, Class. Quantum Grav. **15**, 2479 (1998).

[28] J. A. Valiente Kroon, *Logarithmic Newman-Penrose Constants for arbitrary polyhomogeneous spacetimes*, Class. Quantum Grav. **16**, 1653 (1999).

[29] J. A. Valiente Kroon, *Polyhomogeneous expansions close to null and spatial infinity*, in *The Conformal Structure of Spacetimes: Geometry, Numerics, Analysis*, edited by J. Frauendiener & H. Friedrich, Lecture Notes in Physics, page 135, Springer, 2002.

[30] J. A. Valiente Kroon, *Asymptotic properties of the development of conformally flat data near spatial infinity*, Class. Quantum Grav. **24**, 3037 (2007).

[31] J. A. Valiente Kroon, *The Maxwell field on the Schwarzschild spacetime: behaviour near spatial infinity*, Proc. Roy. Soc. Lond. A **463**, 2609 (2007).

[32] J. A. Valiente Kroon, *Estimates for the Maxwell field near the spatial and null infinity of the Schwarzschild spacetime*, J. Hyp. Diff. Eqns. **6**, 229 (2009).

[33] J. A. Valiente Kroon, *Conformal methods in General Relativity*, Cambridge University Press, 2016.