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An Application of Space Filling Curves to Substitution Tilings

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Abstract

We present an order structure for tiling substitution systems of the plane. The order structure gives rise to a space filling curve which is defined over an iterative system akin to the given tiling substitution. We use this space filling curve to define a label set on the original tiles, inducing a new tiling with a factor map to the original. On the other hand, our new tiling also defines an almost one-to-one factor map to a one-dimensional tiling obtained from 'flattening' the space filling curve. We view this as a way of reducing dimension, giving new insights on 2-dimensional substitution tilings using symbolic dynamics.

Contents

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Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

A *tiling* is a covering of an n-dimensional Euclidean space $\mathbb{R}^n$, for some $n \in \mathbb{Z}^+$, with a countable non-overlapping collection of pieces called *tiles*. A *tile* is a subset of $\mathbb{R}^n$ that is homeomorphic to the closed unit ball. In order to distinguish translations of identical tiles we add labels to them. In this project we only consider tilings of $\mathbb{R}^n$ for $n \in \{1, 2\}$, which consist of finitely many different types of tiles up to translation called *prototiles*. A tiling is said to be *aperiodic* if it lacks translational symmetry. One important class of aperiodic tilings are the substitution tilings. A *substitution tiling* is a tiling formed by a *substitution rule*; a map which expands each tile by a fixed scaling number greater than 1 and divides them into smaller pieces, each of which is a congruent copy of a prototile. An example of a substitution rule is given in Figure 1.1. The substitution rule ω is defined over two unit squares with different colour labels; white and gray. Both white and gray unit squares are expanded by a scaling factor $\lambda = 2$. They are substituted into a collection of four unit squares as shown in the figure. This substitution rule is called the *2-dimensional Thue-Morse substitution rule* (2DTM substitution rule in short).

Figure 1.1: 2-dimensional Thue-Morse substitution rule

Let p denote the white tile. Applying the substitution rule ω twice to it, we obtain a patch $\omega^2(p)$. The substituted patch is shown in the middle patch of Figure 1.2. Substituting every tile in the middle patch twice (again) leads to the rightmost patch in Figure 1.2. The middle patch in the figure is contained in the centre of the rightmost patch in the figure. Therefore, we get a nested sequence of patches by continuing substituting the tiles twice ad infinitum. The generated patches expand in every direction, at every step. They converge to a collection which covers the whole plane, defining a tiling $T = \bigcup_{n=1}^{\infty} \omega^{2 \cdot n}(p)$.

We call a tiling constructed in this way *self-similar* because $\omega^2(T) = T$. This tiling T is called *2-dimensional Thue-Morse tiling* (2DTM in short).

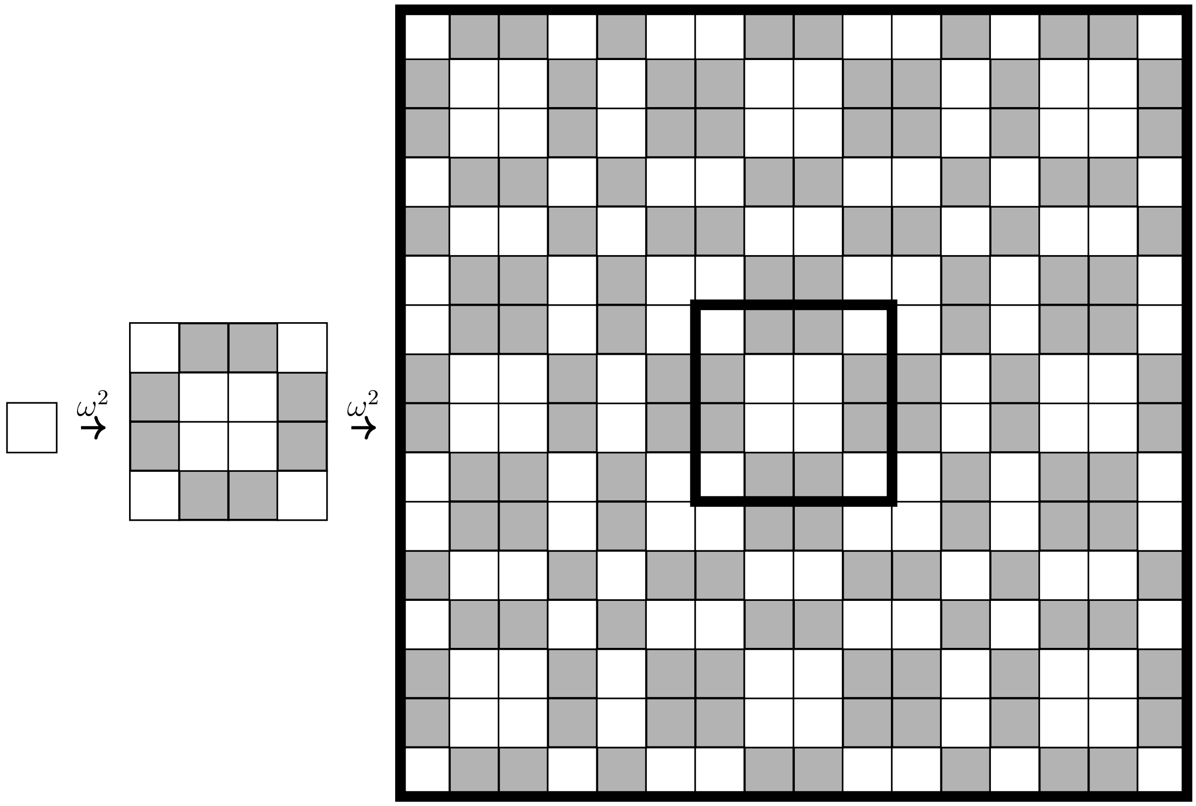


Figure 1.2: A self-similar construction of 2DTM

A *space filling curve* is a continuous surjection from the unit interval to a subset of the plane with a positive Jordan content. Space filling curves are usually generated by iterative systems, analogous to the substitution rules of two dimensional tilings. One of the most famous example of a space filling curve was given by David Hilbert in [12]. Since Hilbert's curve fills the whole unit square, it is illustrated by the curves converging to it through an iterative system. These curves are referred as *approximant curves.* The first three approximant curves are demonstrated in Figure 1.3. The first approximant curve is

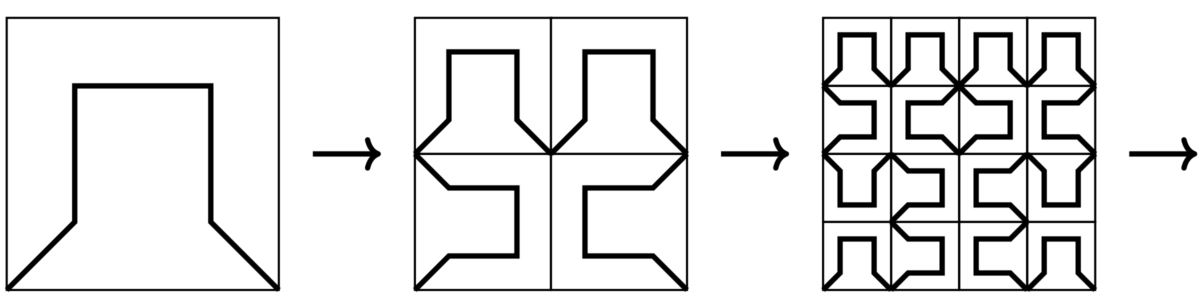


Figure 1.3: Approximant curves of Hilbert's space filling curve.

iterated to a curve which is formed by concatenation of 4 curves, each of which is a scaled isometric copy of the first approximant curve. The second approximant curve is substituted to the third approximant curve in the same fashion. This process continues to infinity. The sequence of approximant curves converges to a curve which fills the unit square. More precisely, let $\gamma_n : [0,1] \mapsto [0,1] \times [0,1]$ for $n \in \mathbb{Z}^+$ denote the n-th approximant curve, the first three of which are demonstrated in Figure 1.3. We have that $\lim_{n\to\infty} \gamma_n(x)$ exists for all $x \in [0,1]$. Moreover, $\gamma : [0,1] \mapsto [0,1] \times [0,1]$ defined by $\gamma(x) = \lim_{n\to\infty} \gamma_n(x)$ is a (well-defined) space filling curve, called *Hilbert's space filling curve.* The iteration process that leads to Hilbert's space filling curve in Figure 1.3 is called *Hilbert's iteration system.*

Hilbert's iteration system induces a substitution rule ω_H whenever an expansion factor $\lambda = 2$ is applied in each of its steps. The substitution rule is depicted in Figure 1.4. There exists a tiling T_H that is formed by this substitution, through the same method as in the construction of T for the 2DTM tiling. It is called *Hilbert's substitution tiling.* A patch of it is demonstrated in Figure 1.5. The tiling T_H consists of square tiles with unit area. The tiles carry curve labels such that they can be concatenated to produce a curve $\mathcal{D}$. Since T_H is constructed as a limit of nested sequence of patches that expand in every direction to cover the whole plane, the curve $\mathcal{D}$ can be written as a limit of nested sequence of curves that expand in every direction. That is, $\mathcal{D} = \bigcup_{k=1}^{\infty} C_k$ where C_k are curves so that $C_j \subseteq C_{j+1}$ for all $j \in \mathbb{Z}^+$. The topology on $\mathcal{D}$ is the subspace topology induced by the tiling topology. The curves C_k are nothing but the expanded versions of approximant curves of the Hilbert's space filling curve.

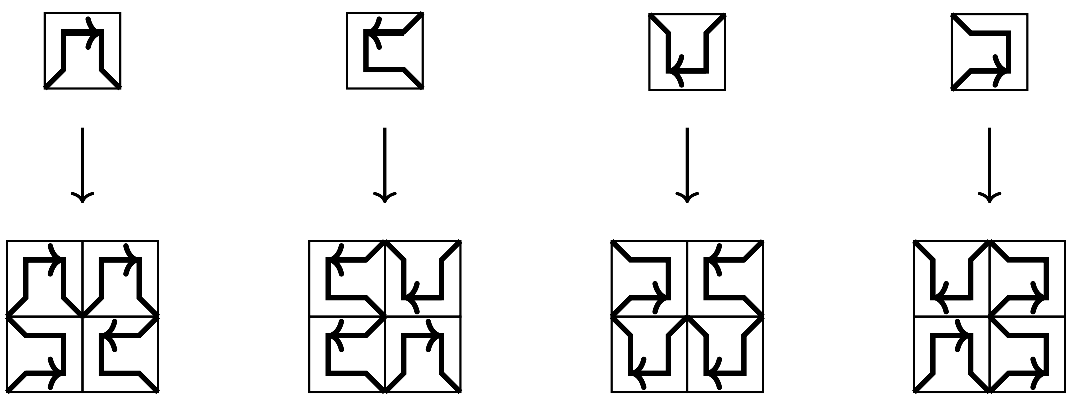


Figure 1.4: The substitution rule ω_H.

The curve $\mathcal{D}$ is relatively dense in the plane; i.e. there exists $R > 0$ such that every 2-dimensional ball with radius R intersects with $\mathcal{D}$. The construction steps of $\mathcal{D}$ and γ are akin in the sense that the curves C_k are expanded versions of approximants of γ. In general, all space filling curves considered in this thesis are associated with relatively dense curves in the same fashion. Therefore, throughout the thesis we refer to space filling curves as continuous images either from the unit interval or from the real line. While the former indicates a compact region of the plane as a proper space filling curve, the latter indicates an unbounded subset of the plane as an associated relatively dense curve.

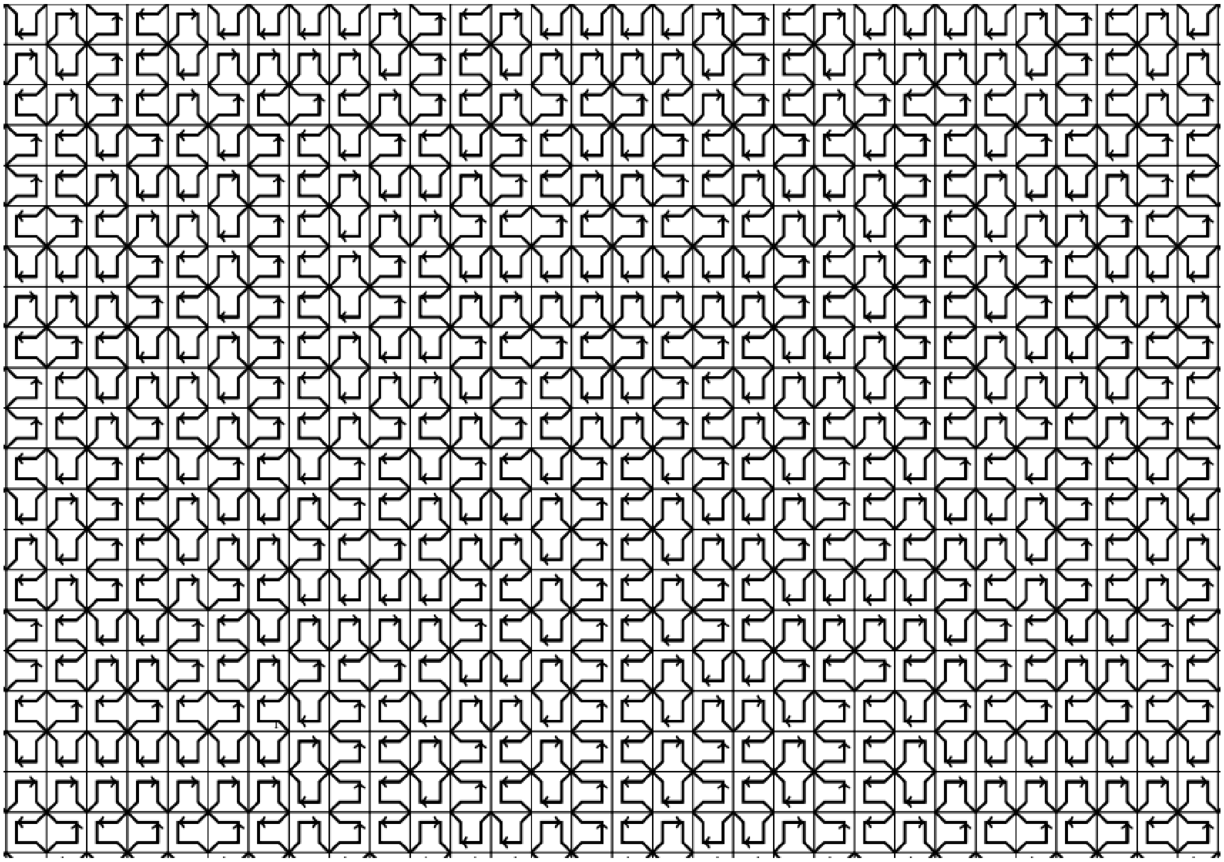


Figure 1.5: A patch of Hilbert's substitution tiling.

Both substitution tilings T and T_H consist of square tiles with unit area. While the labels of T are colours, the labels of T_H are directed curves which can be concatenated to form the relatively dense curve $\mathcal{D}$. One fundamental difference between T and T_H is the order structure between their tiles. There is no obvious order structure defined between the tiles of T. On the other hand, the curve $\mathcal{D}$ induces a total order between the tiles of T_H. For any $t_1, t_2 \in T_H$, define $t_1 \lesssim t_2$ if t_1 is visited by $\mathcal{D}$ before t_2. Since every tile of T_H is visited exactly once by $\mathcal{D}$, this is a well defined total order. Moreover, this order structure induces a 1-dimensional tiling. In particular, flattening the curve $\mathcal{D}$ defines a 1-dimensional substitution tiling V_H. The substitution rule σ_H of V_H can be read from Figure 1.4. If the four square tiles on the top of Figure 1.4 are labelled as a, b, c, d from left to right respectively, then the substitution rule σ_H is the following:

$$\sigma_H(a) = d, a, a, b, \qquad \sigma_H(b) = c, b, b, a, \qquad \sigma_H(c) = b, c, c, d, \qquad \sigma_H(d) = a, d, d, c.$$

One dimensional substitution tilings can be investigated using symbolic dynamics, which has been a focus of research going back almost 100 years. The goal of this project is to define machinery that provides new insights for studying 2-dimensional substitution tilings from the 1-dimensional symbolic dynamics point of view. We propose a method which portrays 2-dimensional substitution tilings as 1-dimensional substitution tilings through the lens of space filling curves. We explain the process by depicting the 2DTM substitution tiling as the doubled 1-dimensional Thue-Morse (doubled-1DTM) substitution tiling through the lens of Hilbert's space filling curve, where the doubled-1DTM substitution tiling is defined with the following substitution rule σ:

$$\sigma(a) = a, b, a, b, \qquad \sigma(b) = b, a, b, a.$$

The main idea is to mimic the total order structure of T_H on the tiling T. More precisely, consider the substitution rule ω' shown in Figure 1.6. This substitution rule defines a tiling T' of the plane via the usual method. This tiling can be viewed as a decorated version of T with the relatively dense curve $\mathcal{D}$ attached as a decoration. In particular, if the curve labels are omitted, then T' becomes T, and if the colour labels are omitted then T' becomes T_H. We illustrate a patch of T, T_H and T' in Figure 1.7, from left to right respectively. If the segments of $\mathcal{D}$ are relabelled according to the correspondence

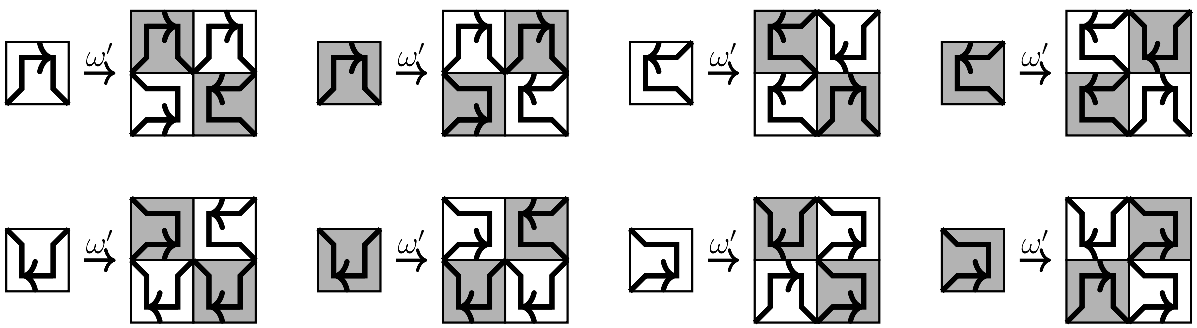


Figure 1.6: The substitution rule ω'.

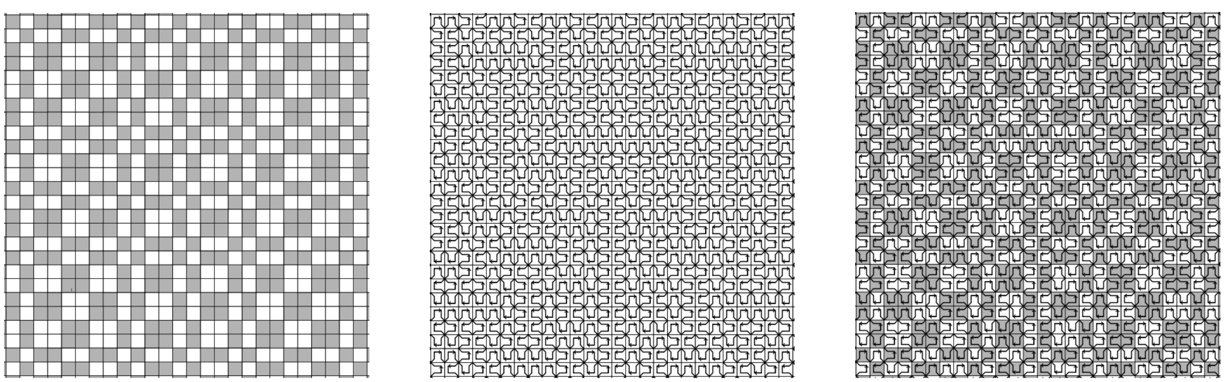


Figure 1.7: A patch of the tilings T, T_H and T', from left to right respectively.

shown in Figure 1.8, then flattening the curve $\mathcal{D}$ defines a 1-dimensional substitution tiling V'. The substitution rule σ' of V' can be read from Figure 1.6 using the labels shown in

Figure 1.8 as follows:

$$\sigma'(a) = d, e, a, f, \quad \sigma'(b) = c, f, b, e, \quad \sigma'(c) = b, g, c, h, \quad \sigma'(d) = a, h, d, g,$$
$$\sigma'(e) = h, a, e, b, \quad \sigma'(f) = g, b, f, a, \quad \sigma'(g) = f, c, g, d, \quad \sigma'(h) = e, d, h, c.$$

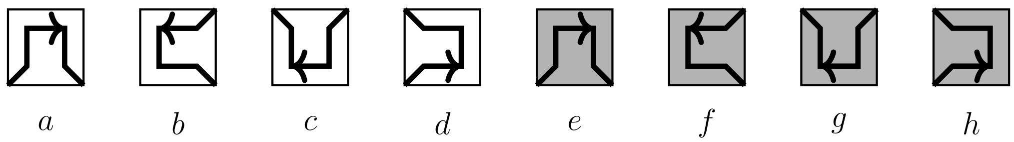


Figure 1.8: The correspondence between tiles

The tiling T' can be described by placing the tiling T_H 'over' the tiling T. By the same token, the tiling V' can be described by placing the tiling V_H 'over' the tiling V, where V is the doubled-1DTM tiling. More precisely, define the maps $\phi_1, \phi_2 : \{a, b, c, d, e, f, g\} \mapsto \{a, b, c, d, e, f, g\}$ by the following relations:

$$\phi_1(a) = \phi_1(e) = a, \quad \phi_1(b) = \phi_1(f) = b, \quad \phi_1(c) = \phi_1(g) = c, \quad \phi_1(d) = \phi_1(h) = d,$$
$$\phi_2(a) = \phi_2(b) = \phi_2(c) = \phi_2(d) = a, \quad \phi_2(e) = \phi_2(f) = \phi_2(g) = \phi_2(h) = b.$$

The maps ϕ_1 and ϕ_2 modify the labels of the tiles given in Figure 1.8 such that ϕ_1 forgets the colour labels of the tiles and ϕ_2 forgets the curve labels of the tiles. We get that $\sigma_H = \phi_1 \circ \sigma'$ and $\sigma = \phi_2 \circ \sigma'$. That is, if we forget the colour labels, then T' becomes T_H and V' becomes V_H. Similarly, if we forget the curve labels, then T' becomes T and V' becomes V.

We view the method above as a way of dimension reduction. The fundamental ingredient for making this technique possible is the existence of such a relatively dense curve, which induces a (decorated) 2-dimensional substitution tiling when it is attached to a given tiling of the plane, and a 1-dimensional substitution tiling when it is flattened. In general, it is not obvious whether there exists such a relatively dense curve for any given substitution tiling of the plane. We refer to the problem as the *existence problem*. We provide an affirmative answer to the problem through an algorithm called the *travelling algorithm*, by regarding tiles as neighbourhoods and curves visiting them as a traveller visiting the neighbourhoods.

The travelling algorithm starts with any given 2-dimensional substitution rule which induces a tiling satisfying generic conditions. We decorate each tile with either a single curve or a pair of non-crossing curves. These curves are completely contained inside the tiles and have end points at the vertices of the tiles. The algorithm defines a substitution for each decorated tile, whenever the substitutions of the tiles form sufficiently large patches. Since the substitutions of tiles can be constructed as large as possible, by applying the substitution rule enough times, our largeness criteria is assured.

The substitution structure induced by the algorithm can be described as follows. Picture the tiles as neighbourhoods and their associated substitutions as cities. The given substitution rule defines a correspondence between the neighbourhoods and cities, such that every point in a neighbourhood corresponds to a point in the associated city. Suppose we want to define a decorated tile by attaching a simple curve decoration e_t to a tile t so that e_t has end points A and B. This decorated tile is thought of as a map with a traveller moving through the neighbourhoods of t, from point A to point B. The travelling algorithm finds a path in the associated city, whenever the city is divided into enough neighbourhoods, for the traveller to move from the point A' to the point B' so that it visits every neighbourhood in the city at least once and at most twice, where A', B' are the corresponding points of A and B in the city. Since the traveller visits every neighbourhood at least once and at most twice, each visited neighbourhood corresponds to a decorated tile, where the decoration is induced according to the movements of the traveller. This process characterises a substitution structure for the decorated version of t, with the decoration e_t. A similar argument is applied when e_t is not a single curve but a pair of non-crossing curves, by interpreting the two non-crossing curves as two travellers whose paths do not cross each other. This construction defines a substitution for every decorated tile. The generated substitution rule is the key to compose a relatively dense curve in the plane. In particular, when the decorated substitution is applied ad infinitum, the cities are fit together in the same way neighbourhoods inside the cities fit together. Consequently, the traveller algorithm induces a relatively dense curve in the plane, which visits every tile of the given tiling at least once and at most twice.

We interpret the constructed relatively dense curves as order systems for tilings of the plane. In particular, if a relatively dense curve visits every tile of a given 2-dimensional tiling exactly once, then we can assign a total order between the tiles of the tiling, according to which tile is visited first. On the other hand, if at least one tile is visited twice, then we define a total order using curve labels as follows. If a tile t is visited twice, then we define two pairs $[t, e_1]$ and $[t, e_2]$ corresponding to the tile t, where e_1 and e_2 are two simple curves associated with the double-visit of the tile t. We identify every other singly visited tile with a pair $[u, e]$ where e corresponds to the single-visit of u. Let $[x, \alpha], [y, \beta]$ be any given pairs where x, y are tiles and α, β are curves associated with the visits of the tiles x, y, respectively. We define $[x, \alpha] \lesssim [y, \beta]$ if α comes before β, according to the given relatively dense curve. This defines a total order between the pairs, which is the main ingredient to construct a 1-dimensional tiling. For that reason we view the travelling algorithm as an order system either over the tiles or over the corresponding pairs.

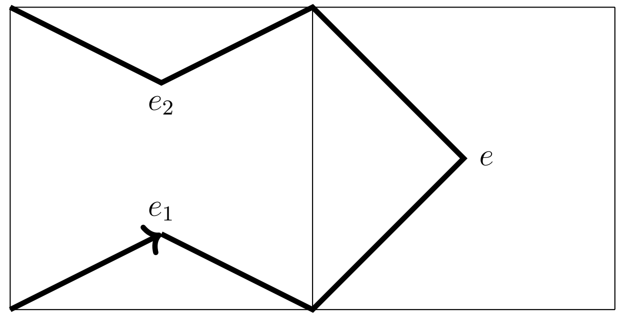


Figure 1.9: The total order between the pairs is demonstrated over the two tile patch $\{t, u\}$. The total order over $\{t, u\}$ is defined as $[t, e_1] \lesssim [u, e] \lesssim [t, e_2]$.

The key argument of this thesis is the details of the travelling algorithm. These details are explained in three sections in Chapter 3. At the end of Chapter 3, we also provide a relation between the travelling algorithm and the Hamiltonian path problem. In Section 3.1, we explain the algorithm for a special class of patches called *circle patches* (See Definition 3.1.1). This is the simplest version of the algorithm. In particular, using the relation at the end of the chapter, the algorithm for circle patches provides an answer to the Hamiltonian path problem for complete and finite graphs. In Section 3.2, we expand the algorithm over a larger class of patches called *cyclic patches* (See Definition 3.2.1). We define composition of tilings and patches, where we expose a special composition method called *circle composition* (Section 3.2.1). A circle composition of a patch replaces a circle subpatch with a single tile. We prove in this section that every cyclic patch can be transformed into a circle patch, after finitely many circle composition steps. These circle composition steps are the main ingredients which makes the methods in Section 3.1 applicable for cyclic patches. In the final section we generalise the algorithm to any patch that is formed by a substitution rule defined over convex shapes of the plane. The final version of the algorithm implies that any given 2-dimensional substitution tiling satisfying mild conditions generates a decorated substitution tiling of the plane, which also satisfies the same assumptions. Lastly, at the end of the chapter, we relate the travelling algorithm and the Hamiltonian path problem. We show that the algorithm provides an answer to the Hamiltonian path problem for special cases with strong assumptions.

The outline of the thesis is as follows. We provide the basics of tilings in Chapter 2. The material in this chapter is classical, except for a new algorithm, which we call *the primitive core algorithm*. The primitive core algorithm returns a primitive substitution from any given non-primitive one. In Chapter 3, we explain the gory details of the travelling algorithm. This algorithm provides a relatively dense curve for any given 2-dimensional substitution tiling satisfying mild conditions. The generated relatively dense curve induces an order structure between the tiles of the given tiling. In Chapter 4, we flatten the constructed relatively dense curve in order to form a 1-dimensional substitution tiling. We also establish a factor map between the associated (discrete) tiling dynamical systems of the two generated tilings; the decorated 2-dimensional substitution tiling and the 1-dimensional substitution tiling. We further show that this map is almost one-to-one; i.e. one-to-one over a dense subset of its domain. In the final chapter we describe how the travelling algorithm generates space filling curves. We also provide several examples of how the technique is applied to the known substitution tilings of the plane.

Chapter 2

Tiling Theory

The material in this chapter splits into two sections. The material in Section 2.1 is standard, which can be found in the survey textbook [28] and/or the references therein. On the other hand, Section 2.2 is a contribution to the tiling literature, up to the author's knowledge. It is dedicated to the primitive core algorithm. This algorithm returns a primitive substitution from any given substitution.

2.1 Preliminary Definitions and Results

A *tile* consists of a compact subset of $\mathbb{R}^d$, denoted by supp t, and a label $l(t)$ that distinguish identical sets. For a given tile t and $x \in \mathbb{R}^d$ we define a new tile $t + x$ by the relations supp $(t + x) =$ supp $(t) + x$ and $l(t + x) = l(t)$. A *partial tiling* is a collection of non-overlapping tiles. We define the support of a partial tiling T as the union of supports of its component tiles. A *patch* P is a partial tiling with a compact and simply connected support in $\mathbb{R}^d$. A *tiling* T is a partial tiling with supp $T = \mathbb{R}^d$. If $T = \{t_n\}$ is a tiling and $x \in \mathbb{R}^d$, then $T + x$ is a tiling with the collection $\{t_n + x\}$.

A *prototile set* $\mathcal{P}$ of a tiling T consists of tiles such that for all $t \in T$ there exists a unique $p \in \mathcal{P}$ so that $t = p + x$ for some $x \in \mathbb{R}^d$. Elements of $\mathcal{P}$ will be called *prototiles* of the tiling T. We denote by $\mathcal{P}^*$ the set of all patches generated by $\mathcal{P}$.

For each prototile $p \in \mathcal{P}$ we fix a point $x(p)$ such that $x(p) \in$ int (supp p). The point $x(p)$ is called the *puncture* of p. We extend the punctures of prototiles to every tile in a tiling. More precisely, if $t \in T$ and $t = p + x$ for some $p \in \mathcal{P}$ and $x \in \mathbb{R}^d$, then we say $x(p) + x$ is the *puncture* of the tile t and is denoted by $x(t)$.

For a given tiling T and a (fixed) prototile set $\mathcal{P}$, we define *punctured tiling* T_p as the same collection of tiles of T where each tile has puncture $x(t)$ induced from $\mathcal{P}$.

Tiling Spaces

Let T be a tiling and $B(x, R)$ be the Euclidean ball of radius R around $x \in \mathbb{R}^d$. We denote the collection of tiles whose support intersect with the ball $B(x, R)$ by

$$T \sqcap B(x, R) := \{t \in T : \text{ supp } t \cap B(x, R) \neq \emptyset\}.$$

Lemma 2.1.1 ([26], Lemma 2.7)**.** *Let U, V be two tilings.*

$$d(U, V) = \inf \left\{ \frac{\sqrt{2}}{2}, \epsilon : \exists x \in \mathbb{R}^d \text{ so that } |x| < \epsilon \text{ and } (U - x) \sqcap B(0, 1/\epsilon) = V \sqcap B(0, 1/\epsilon) \right\}$$

defines a metric on tilings in $\mathbb{R}^d$.

We call d in Lemma 2.1.1 the *tiling metric*. Since every iteration of a tiling T is also a tiling, d defines a metric space on the translation orbit $T + \mathbb{R}^d$ of T. From there, we can define two complete spaces; the continuous hull and the discrete hull of a tiling. The *continuous hull* of a tiling T is the completion of all translations of T in the tiling metric. It is denoted by $\Omega(T)$. We have $\Omega(T) = \overline{T + \mathbb{R}^d}$, where the completion is taken with respect to the tiling metric. The *punctured hull* of a tiling T (with punctures induced by a fixed prototile set $\mathcal{P}$) is the collection of tilings in $\Omega(T)$ that has a puncture at the origin. It is denoted by $\Omega_p(T)$. We have that $\Omega_p(T) = \{A \in \Omega(T) : \exists a \in A \text{ with } x(a) = 0\}$. Completeness of $\Omega_p(T)$ follows by the following lemma.

Lemma 2.1.2. *Let T be a tiling and let $\Omega(T)$ be its continuous hull. Then its punctured hull $\Omega_p(T)$ (with punctures induced by a fixed prototile set $\mathcal{P}$) is a closed subspace of $\Omega(T)$.*

Proof. Suppose $\{A_n\}$ is a sequence of tilings in $\Omega_p(T)$ which converges to a tiling $A \in \Omega(T)$. There exists $N \in \mathbb{Z}^+$ such that every tiling A_n for $n \geq N$ must contain the same tile a which has the puncture at the origin. Because A_n converges to A, we must have $a \in A$ and $x(a) = 0$ (the puncture of tile a in the tiling A). Hence, $A \in \Omega_p(T)$. □

For any given tiling T, elements of $\Omega(T)$ and $\Omega_p(T)$ can be identified by examining the patches of T. In particular, we have the following lemma and theorem.

Lemma 2.1.3. *Let T be any given tiling of $\mathbb{R}^d$ for some $d \in \mathbb{Z}^+$. Then $A \in \Omega(T)$ if and only if A is a tiling of $\mathbb{R}^d$ such that $A = \lim_{n \to \infty} (T + x_n)$ where $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}^d$.*

Proof. It is enough to prove that A is a tiling of $\mathbb{R}^d$ whenever $A = \lim_{n \to \infty} (T + x_n)$ for some sequence $\{x_n\}$ in $\mathbb{R}^d$.

Assume that the sequence $\{T + x_n\}$ converges in $\Omega(T)$. Because $\Omega(T)$ is complete, $\{T + x_n\}$ is a Cauchy sequence. For each $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $d(T +$

$x_n, T+x_m) < \epsilon$ whenever $n, m \geq N$. Then, $T+x_n$ and $T+x_{n+1}$ match around the origin up to some wiggle for each $n \in \mathbb{Z}^+$. Define $P_n = (T+x_n) \sqcap B(0,n)$ for $n \in \mathbb{Z}^+$, and define $\{y_n\}$ to be a sequence of real numbers such that $P_n + y_n \subseteq P_{n+1}$ for $n \in \mathbb{Z}^+$. Such a sequence $\{y_n\}$ exists since $T+x_n$ and $T+x_{n+1}$ match around the origin up to some wiggle for each $n \in \mathbb{Z}^+$. Therefore, $\{P_n + y_n\}$ is a nested sequence of increasing patches such that $A' = \bigcup_n (P_n + y_n)$ defines a tiling. Because the limit is unique, $A' = A$, and A is a tiling of $\mathbb{R}^d$. □

Theorem 2.1.4. *For any given tiling T, we have $A \in \Omega(T)$ if and only if every patch of A appears in T.*

Proof. Suppose $A \in \Omega(T)$ is a tiling in the continuous hull of T and P is a patch appearing in A. There exists $x \in \mathbb{R}^d$ such that P appears around the origin in the tiling $A+x$. Let $R > 0$ be sufficiently large enough so that supp $P \subseteq B(0,R)$. Because $A+x \in \Omega(T)$, for any given $\epsilon > 0$, there exists a tiling $T+y$ such that $d(A+x, T+y) < \epsilon$. Choose $\epsilon_0 < 1/R$. Then $T+y$ contains the patch P around the origin. That is, P appears in $T+y$. Thus, P appears in T.

Conversely, define the patch $P_R = A \sqcap B(0,R)$ for $R > 0$. We have that P_R appears in A. By the assumption it also appears in T. Then there exists $x_R \in \mathbb{R}^d$ such that $(T+x_R) \sqcap B(0,R) = P_R$. Hence, $A = \lim_{R \to \infty} (T+x_R)$ and $A \in \Omega(T)$. □

Substitution Tilings

Definition 2.1.5. Let $\mathcal{P}$ be a finite set of tiles and let $\mathcal{P}^*$ denote all partial tilings generated by the tiles in $\mathcal{P}$. A *substitution rule* ω^* with a scaling factor $\lambda > 1$ is a map from $\mathcal{P}$ to $\mathcal{P}^*$ such that $\omega^*(p)$ is a patch with supp $\omega^*(p) = \lambda \cdot$ supp p for all $p \in \mathcal{P}$.

We can extend the substitution rules defined on prototiles to tiles. More precisely, we can extend ω^* to $\mathcal{P} + \mathbb{R}^d$ by defining $\omega^* : \mathcal{P} + \mathbb{R}^d \mapsto \mathcal{P}^*$ as $\omega^*(p+x) := \omega^*(p) + \lambda x$.

Definition 2.1.6. A substitution rule ω^* is said to be primitive if there exists $n \in \mathbb{N}$ such that for any pair $p, q \in \mathcal{P}$, $(\omega^*)^n(p)$ contains a translate of q.

Construction of a tiling from a primitive substitution

It is a commonly used result that any primitive substitution rule gives rise to a tiling. We present the details of this process.

Let $\omega^* : \mathcal{P} \mapsto \mathcal{P}^*$ be a given primitive substitution rule. For $p \in \mathcal{P}$ there exists $n \in \mathbb{N}$ such that $(\omega^*)^n(p)$ contains a translate of p. Assume without loss of generality n is sufficiently large so that a copy of p completely contained in $\text{int}\Big(\text{supp } (\omega^*)^n(p)\Big)$. Denote the copy of p as $p+x$. we have, $p+x \in (\omega^*)^n(p)$ and supp $(p+x) \subseteq \text{int}\Big(\text{supp } (\omega^*)^n(p)\Big)$. Let

$z = \frac{x}{\lambda^n - 1} \in \mathbb{R}^d$. Then, $x = \lambda^n z - z$ and $p + (\lambda^n z - z) \in (\omega)^n(p)$. Thus, $p - z \in (\omega^*)^n(p - z)$. We arrive at the following "increasing" nested sequence of patches:

$$\{p - z\} \subseteq (\omega^*)^n(p - z) \subseteq (\omega^*)^{2n}(p - z) \subseteq (\omega^*)^{3n}(p - z) \ldots$$

Because $p + x$ is contained in $(\omega^*)^n(p)$, $p - z + x$ is contained inside of $(\omega^*)^n(p - z)$. That is, supports of those patches form a nested (increasing) sequence of sets whose union has support which covers $\mathbb{R}^d$. Hence, $T = \bigcup_{k \in \mathbb{N}} (\omega^*)^{kn}(p - z)$ is a tiling. The tiling T generated by a primitive substitution rule as in above is called a *primitive substitution tiling.*

Lemma 2.1.7. *The map $\omega : \Omega(T) \mapsto \Omega(T)$ defined by $\omega(\{t_n\}_n) := \bigcup_n \omega^*(t_n)$ is a (well-defined) continuous map.*

Proof. The image of a tiling under ω is a tiling. Moreover, if P is a patch appearing in the collection $\bigcup_n \omega^*(t_n)$, then P is contained in $(\omega^*)^N(p)$ for sufficiently large N and for any prototile $p \in \mathcal{P}$ (using primitivity). Therefore, P appears in T. That is, $\bigcup_n \omega^*(t_n) \in \Omega(T)$ and ω is well defined.

Moreover, if U and V match around the origin up to some wiggle then the substituted tilings $\omega(U), \omega(V)$ also match around the origin, up to some wiggle. More precisely, if $d(U, V) < \epsilon$, then $d(\omega(U), \omega(V)) < \lambda \cdot \epsilon$, where λ is the expansion factor of the substitution ω. Hence, ω is a continuous map. □

The map ω will be called the *substitution map* for the primitive substitution tiling T, whereas ω^* will be called the *substitution rule* for the tiling T. From now on, we will denote both functions by ω for simplicity.

Observe that the tiling T constructed previous to Lemma 2.1.7 satisfies $\omega^n(T) = T$. That is, T is a fixed point for the substitution ω^n. In particular, we have the following lemma.

Lemma 2.1.8. *Let ω be a primitive substitution rule and let T be a tiling generated by ω. Then there exits $N \in \mathbb{Z}^+$ such that $\omega^N(T) = T$*

Proof. Because ω is primitive, T can be written as $\bigcup_{k=1}^{\infty} \omega^{k \cdot N}(p - z)$ for some $z \in \mathbb{R}^d$ and $N \in \mathbb{Z}^+$. Since $\omega^{k \cdot N}(p - z) \subseteq \omega^{(k+1) \cdot N}(p - z)$ for each $k \in \mathbb{Z}^+$, we get $\omega^N(T) = T$. □

Definition 2.1.9. Two patches P and P' are said to be *equivalent* if there exists $x \in \mathbb{R}^d$ such that $P = P' + x$. That is, the two patches P and P' are said to be equivalent if for all $p \in P$ there exists unique $p' \in P'$ with $p = p' + x$. We denote equivalent patches as $P \sim P'$.

This is an equivalence relation.

Definition 2.1.10 (FLC)**.** We say a tiling T has *finite local complexity* (FLC in short) if the set $\{(T+x)\sqcap B(0,R): \ x\in\mathbb{R}^d\}$ is finite up to equivalence for any given $R>0$.

FLC is commonly accepted to be a standard assumption being made in the theory of tilings, due to the following lemma.

Lemma 2.1.11. *A tiling T has FLC iff $\Omega(T)$ is compact.*

Proof. Suppose T has FLC and $\{A_k\}_k \subseteq \Omega(T)$ is an infinite sequence of tilings in $\Omega(T)$. For any fixed $R>0$, The set $\{A_k\sqcap B(0,R)\}_k$ has a finite size (up to equivalence), and the patch $A_k\sqcap B(0,R)$ appears in T by Theorem 2.1.4.

Let $\{A_k\sqcap B(0,R)\}/\sim \,=\{P_1,\dots,P_s\}$ for some $s\in\mathbb{Z}^+$. For any given fixed $R>0$, consider the collections of tilings $C_i^0=\{A_k: \ A_k\sqcap B(0,R)=P_i\}$ for $i=1,\dots,s$. We have $\{A_k\}_k=\bigcup_{i=1}^{s} C_i^0$. There exists $i_0\in\{1,\dots,s\}$ such that $C_{i_0}^0$ has infinite cardinality. Choose $B_0\in C_{i_0}^0$. The set $\{A_k\sqcap B(0,2R): \ A_k\in C_{i_0}^0\}$ has a finite size (up to equivalence). Let $\{A_k\sqcap B(0,2R): \ A_k\in C_{i_0}^0\}/\sim\,=\{P_1^1,\dots,P_{s_1}^1\}$ and $C_i^1=\{A_k\in C_{i_0}^0: \ A_k\sqcap B(0,2R)=P_i^1\}$ for $i\in\{1,\dots,s_1\}$. There exists $i_1\in\{1,\dots,s_1\}$ so that $C_{i_1}^1$ has infinitely many elements. Choose $B_1\in C_{i_1}^1$. Continuing this process, we obtain a subsequence B_n, which is a Cauchy sequence. We get that B_n is a convergent subsequence for the given infinite sequence of tilings $\{A_k\}_k$. Hence, $\Omega(T)$ is (sequentially) compact.

The converse implication follows by a similar argument. If T does not have FLC, then there exists $R>0$ such that there are infinitely many different (up to equivalence) patches with radius R. For each patch choose a tiling that contains the chosen patch around its origin. This is an infinite collection of tilings which does not converge to any tiling in $\Omega(T)$. □

Lemma 2.1.11 implies that $(\Omega(T),\mathbb{R}^d)$ is a dynamical system for any given tiling T.

Definition 2.1.12. A tiling T is called *aperiodic* if

$$T+x=T \Longleftrightarrow x=0.$$

A tiling T is called *strongly aperiodic* if for all $T'\in\Omega(T)$

$$T'+x=T' \Longleftrightarrow x=0.$$

Definition 2.1.13 (Repetitive)**.** A tiling T of $\mathbb{R}^d$ is said to be *repetitive* if for any patch $\mathcal{P}$ there exists $R>0$ such that a copy of $\mathcal{P}$ is contained in the patch $T\sqcap B(x,R)$ for all $x\in\mathbb{R}^d$.

Lemma 2.1.14. *Let T be a repetitive tiling. Then for any $T'\in\Omega(T)$ we have $\Omega(T)=\Omega(T')$.*

Proof. Suppose T' is a given tiling in $\Omega(T)$ such that $T' = \lim_k A_k$ for some sequence $\{A_k\}_k \subseteq \Omega(T)$. For any $x \in \mathbb{R}^d$, $T' + x = \lim_k (A_k + x)$. Thus, $T' + \mathbb{R}^d \subseteq \Omega(T)$ and $\Omega(T') \subseteq \Omega(T)$. Conversely, suppose $A \in \Omega(T)$ and T is repetitive. Let P be a patch in A which also appears in T. Because T is repetitive, there exists $R > 0$ so that $B(x, R)$ contains a copy of P for any given $x \in \mathbb{R}^d$. Define $P' = T' \sqcap B(y, 3R)$ for some $y \in \mathbb{R}^d$. We have that P' is a patch in T' and $T' \in \Omega(T)$. Therefore, P' appears in T. Because P' contains a ball of radius R, P' contains a copy of P. Hence, P appears in T' and $\Omega(T') = \Omega(T)$. □

Definition 2.1.15. A substitution tiling T with a substitution map ω is called *recognisable* if ω is invertible. A substitution rule ω is called *recognisable* if it only admits recognisable substitution tilings.

Theorem 2.1.16. *Suppose T is a recognisable primitive substitution tiling with FLC. Then*

(1) *T is strongly aperiodic.*
(2) *$\omega : \Omega(T) \mapsto \Omega(T)$ is a homeomorphism.*
(3) *T is repetitive.*
(4) *$\Omega_p(T)$ is homeomorphic to the Cantor set.*

Proof. Suppose ω is a given primitive substitution rule with an expansion factor λ over a finite collection of prototiles $\mathcal{P}$. Suppose further T is a recognisable primitive substitution tiling with FLC and is generated by the substitution rule ω.

(1) Let $A \in \Omega(T)$ and $x \in \mathbb{R}^d$ be given. There exists a sufficiently large integer $n \in \mathbb{Z}^+$ such that some ball of diameter $\lambda^{-n}||x||$ is contained in every prototile in $\mathcal{P}$. Let $t \in \omega^{-n}(A)$ be any tile in the tiling $\omega^{-n}(A)$. The interiors of t and $t + \lambda^{-n}x$ has to overlap because of the choice of n. Therefore, $\omega^{-n}(A) \neq \omega^{-n}(A + x)$. Since the substitution map ω is injective, T is strongly aperiodic.

(2) Continuity of ω follows by the fact that $d(\omega(T), \omega(T')) < \lambda\epsilon$ whenever $d(T, T') < \epsilon$. Because T is a primitive substitution tiling, there exists $N \in \mathbb{N}$ such that $\omega^N(T) = T$, by Lemma 2.1.8. Therefore, $T \in \text{Ran } \omega$. Similarly, $\omega^N(T + \lambda^{-N}x) = T + x$ implies that $T + \mathbb{R}^d \subseteq \text{Ran } \omega$. Since ω is continuous and $\Omega(T)$ is compact, $\text{Ran } \omega$ is closed. That is, $\Omega(T) = \overline{T + \mathbb{R}^d} \subseteq \text{Ran } \omega$. We get that ω is a continuous bijection. Hence, ω is a homeomorphism since $\Omega(T)$ is compact.

(3) Let P be a given patch in T which appears in $\omega^k(p_0)$ for some $p_0 \in \mathcal{P}(T)$ and sufficiently large $k \in \mathbb{Z}^+$. Because T is a primitive substitution tiling, there exists $N \in \mathbb{N}$ such that for any $p, q \in \mathcal{P}(T)$, $\omega^N(q)$ contains a translate of p. So, a copy of p is contained in $\omega^{k+N}(p)$ for all $p \in \mathcal{P}(T)$. We proved in (ii) that ω is a

homeomorphism. Therefore, there exists a unique $T' \in \Omega(T)$ such that $\omega^{k+N}(T') = T$. Choose $R' = \max_{p \in \mathcal{P}(T)} \left(\text{diam supp } \omega^{k+N}(p)\right)$. For $R = 3 \cdot R'$, $B(x, R)$ contains a copy of P for every $x \in \mathbb{R}^d$. Hence, T is repetitive.

(4) Recall that a metric space X is homeomorphic to the Cantor set if and only if X is compact, totally disconnected and has no isolated points. We have that $\Omega(T)$ is compact by Lemma 2.1.11 and $\Omega_p(T)$ is closed by Lemma 2.1.2. Thus, $\Omega_p(T)$ is compact.

Let $A_p \in \Omega_p(T)$ be a given punctured tiling. We have that T is repetitive. So, for any given $\epsilon > 0$, there exists $x \in \mathbb{R}^d$ such that $d(A+x, A) < \epsilon$ and $A_p + x \in \Omega_p(T)$. We get that T is strongly aperiodic, by (1). Therefore, $A + x \neq A$ and $0 < d(A+x, A) < \epsilon$. The choice of ϵ was arbitrary. Thus, $\Omega_p(T)$ contains no isolated points.

Let $T_1, T_2 \in \Omega_p(T)$ be any given distinct pair of tilings. There exists $R > 0$ such that $T_1 \sqcap B(0, R) \neq T_2 \sqcap B(0, R)$. We get $T_1 \notin U = \{A \in \Omega_p(T) : \ A \sqcap B(0, R) = T_2 \sqcap B(0, R)\}$. Notice that U is closed. For a given tiling $B \in U$, choose $\epsilon_0 < \frac{1}{2R}$. Then $C \sqcap B(0, R) = B \sqcap B(0, R) = T_2 \sqcap B(0, R)$ for any tiling C with $d(C, B) < \epsilon$. Such a tiling C exists since $\Omega_p(T)$ do not contain any isolated points. Therefore, $C \in U$ whenever $d(C, B) < \epsilon$. That is, U is open. We get $T_2 \in U$, $T_1 \notin U$ and U is a clopen set. Thus, $\Omega_p(T)$ is totally disconnected.

Hence, $\Omega_p(T)$ is homeomorphic to the Cantor set. □

Corollary 2.1.17. *Let T be a given recognisable primitive substitution tiling with FLC. Then there exists $k \in \mathbb{Z}^+$ and a prototile set $\mathcal{P}$ so that $\omega^k : \Omega_p(T) \mapsto \Omega_p(T)$ is a homeomorphism, where the punctured tilings in $\Omega_p(T)$ have punctures induced by $\mathcal{P}$.*

Proof. By primitivitiy of the substitution, there exists $k \in \mathbb{Z}^+$ such that $\omega^k(p)$ contains a copy of p inside its patch, for each $p \in \mathcal{P}$. That is, for each $p \in \mathcal{P}$, there exists $x_d \in \mathbb{R}^d$ such that $p + x_d \in \omega^k(p)$ and supp $(p + x) \subseteq$ int (supp $\omega^k(p)$). For each $p \in \mathcal{P}$ there exists a fixed point $a_p \in$ supp $(p + x_d)$ such that a_p stay invariant under the substitution ω^k. Puncture the point a_p for the tile $p + x_d$, for each $p \in \mathcal{P}$. This defines a (punctured) prototile set and a punctured hull $\Omega_p(T)$ of T. We have $\omega^k(A) \in \Omega_p(T)$ for each punctured tiling $A \in \Omega_p(T)$. Hence the corollary follows by (2) in Theorem 2.1.16. □

Finally we provide the notion of equivalence in tilings and tiling spaces, from the textbooks [3] and [28], respectively.

Definition 2.1.18. A tiling T' is *locally derivable* from a tiling T if there is a unique set of local rules to obtain T' from T. More precisely, if there exists $R > 0$ such that, whenever $x, y \in \mathbb{R}^d$ and satisfy that $T \sqcap B(x, R) = T \sqcap B(y, R) + (y - x)$, then $T \sqcap B(x, 1) = T \sqcap B(y, 1) + (y - x)$. If both T and T' are locally derivable from each other then we say T and T' are *mutually locally derivable* (MLD in short).

Definition 2.1.19. Two tiling spaces $\Omega(T)$ and $\Omega(T')$ are called *mutullay locally derivable* (MLD in short) if there exists a topological conjugacy $f : \Omega(T) \mapsto \Omega(T')$ and $R > 0$ such that, whenever $x \in \mathbb{R}^d$ and satisfies that $T_1, T_2 \in \Omega(T)$ agree on $B(x, R)$, then $f(T_1), f(T_2)$ agree on $B(x, 1)$.

The MLD notion of tilings and tiling spaces are equivalent in the sense that two tilings T, T' are MLD if and only if the tiling spaces $\Omega(T), \Omega(T')$ are MLD (through a topological conjugation f). In fact, if T and T' are MLD, then, by defining $f(T) = T'$ over the orbit $T + \mathbb{R}^d$, we get a uniformly continuous map f defined on $T + \mathbb{R}^d$ (See [28, P: 9] for details). Then f can be continuously extended to $\Omega(T)$. Therefore MLD relation of $\Omega(T)$ and $\Omega(T')$ can be acquired. Conversely, if $\Omega(T)$ and $\Omega(T')$ are MLD (through a topological conjugacy f), then the tilings $A \in \Omega(T)$ and $f(A) \in \Omega(T')$ are MLD.

It should be noted that the notion MLD in tilings (or tiling spaces) implies the existence of a topological conjugacy between tiling spaces. However, the converse is not always true (for example [7], [21], [22]).

2.2 Primitive Core Algorithm

In this section we construct an algorithm which generates a primitive substitution from any given substitution. The generated substitution is a power of the given one. Moreover, it is defined over a (possibly) smaller collection.

The function f in the following lemma can be interpreted as a choice function over a finite set of prototiles. On the other hand, the functions g and h in the following lemmas/corollaries can be interpreted as functions recording the distinct prototiles appearing in the substitution of prototiles and substitution of patches, respectively.

Lemma 2.2.1. *Suppose $f : S \mapsto S$ is a map defined on a finite set $S = \{a_1, a_2, \ldots, a_n\}$ for $n \in \mathbb{Z}^+$. Then there exist $a_j \in S$ and $m \in \mathbb{Z}^+$ such that $f^m(a_j) = (a_j)$.*

Proof. The pigeonhole principle implies $\{f^k(a_1)\}_{k=1}^{n+1}$ must have at least one multiple entry; say $f^r(a_1) = f^s(a_1)$ for some $r, s \in \{1, \ldots, n+1\}$ with $r < s$. Hence, the pair $a_j = f^r(a_1)$ and $m = s - r$ satisfy the conclusion. □

Lemma 2.2.2. *Let $h : S^* \mapsto S^*$ be a map defined on the set of all subsets of the finite set S such that*

(i) There exists $a \in S$ with $a \in h(\{a\})$,

(ii) If $A \subseteq B \subseteq S$ then $h(A) \subseteq h(B)$.

Then we have the following:

(1) $h^n(\{a\}) \subseteq h^{n+1}(\{a\})$ for all $n = 0, 1, \ldots,$

(2) There exists $N \in \mathbb{Z}^+$ such that $h^n(\{a\}) = h^N(\{a\})$ for all $n \geq N$. Moreover, $(h^N(\{a\}))^*$ is invariant under h^N.

Proof. (1) directly follows by the assumptions (i) and (ii). As for (2), $\{h^k(\{a\})\}_{k=1}^{\infty}$ is an increasing sequence of subsets of S, by (1). It converges since the power set of S is a finite set. Thus, there exists $N \in \mathbb{Z}^+$ such that $h^n(\{a\}) = h^N(\{a\})$ for all $n \geq N$. Moreover, for all $U \subseteq h^N(\{a\})$ we have $h^{N \cdot s}(U) \subseteq h^{N\cdot(s+1)}(\{a\})$ for each $s \in \mathbb{Z}^+$, by the assumption (ii). Hence, $(h^N(\{a\}))^*$ is invariant under h^N. □

Corollary 2.2.3. *Let $g : S \mapsto S^*$ be a map defined on a finite set S such that there exists $a \in S$ with $a \in g(a)$. Suppose further that $h : S^* \mapsto S^*$ is a map defined by $h(A) = \bigcup_{x \in A} g(x)$ for $A \subseteq S$. Then there exist $N \in \mathbb{Z}^+$ such that $h^n(\{a\}) = h^N(\{a\})$ for all $n \geq N$. Moreover, $(h^N(\{a\}))^*$ is invariant under h^N.*

Proof. We have that h satisfies the conditions (i) and (ii) of Lemma 2.2.2, by construction. Hence, the result follows by Lemma 2.2.2. □

Lemma 2.2.4. *Let $g : S \mapsto S^*$ be a map defined on a finite set S such that there exists $a \in S$ with $g(a) = S$. Assume further, $h : S^* \mapsto S^*$ is defined by $h(A) = \bigcup_{x \in A} g(x)$ for $A \subseteq S$. Then (exactly) one of the following holds:*

(1) There exists $n \in \mathbb{Z}^+$ such that $h^n(\{x\}) = S$ for all $x \in S$.

(2) There exists $b \in S \backslash \{a\}$ such that $a \notin h^s(\{b\})$ for all $s \in \mathbb{Z}^+$.

Proof. Suppose without loss of generality that S contains at least two elements. Assume to the contrary that either both (1) and (2) hold, or both (1) and (2) do not hold. Observe that if (2) holds, then (1) cannot hold. Since (1) and (2) cannot be true at the same time, both (1) and (2) must not hold. Then, for all $c \in S \backslash \{a\}$, there exists $s_c \in \mathbb{Z}^+$ such that $a \in h^{s_c}(\{c\})$ where h satisfies the condition (ii) of Lemma 2.2.2 by construction. Therefore, for all $c \in S \backslash \{a\}$, there exists $s_c \in \mathbb{Z}^+$ such that $S = h(\{a\}) \subseteq h^{s_c+1}(\{c\})$. That is, for each $c \in S \backslash \{a\}$, there exists $s_c \in \mathbb{Z}^+$ such that $h^{s_c+1}(\{c\}) = S$. Choose n to be the maximum of all such $s_c + 1$; i.e. $n = \max_{c \in S \backslash \{a\}} (s_c + 1)$. Then $h^n(\{x\}) = S$ for all $x \in S$, a contradiction. □

Lemma 2.2.5. *Let $g : S \mapsto S^*$ be a map defined on a finite set S such that there exists $a \in S$ with $g(a) = S$, and $h : S^* \mapsto S^*$ be a map defined by $h(A) = \bigcup_{x \in A} g(x)$ for $A \subseteq S$. Assume further, there exists $b \in S \backslash \{a\}$ such that $a \notin h^s(\{b\})$ for all $s \in \mathbb{Z}^+$. Then $W = \{x \in S : \ a \notin h^s(x) \ for \ all \ s \in \mathbb{Z}^+\} = S \backslash \{x \in S : \ a \in h^s(x) \ for \ some s \in \mathbb{Z}^+\}$ is a non-empty subset of S such that W^* is invariant under h.*

Proof. We have $W \neq \emptyset$ since $b \in W$. It is enough to show that $h(W) \subseteq W$. For any $x \in W$, we have $a \notin h^s(\{x\})$ for every $s \in \mathbb{Z}^+$. Therefore, $a \notin \bigcup_{x \in W} h^s(\{x\}) = h^s(W)$ for all $s \in \mathbb{Z}^+$. Let $y \in h(W)$ be given. Then $h^s(\{y\}) \subseteq h^{s+1}(W)$ for all $s \in \mathbb{Z}^+$, by construction of h. Because $a \notin h^{s+1}(W)$ for all $s \in \mathbb{Z}^+$, $a \notin h^s(\{y\})$ for all $s \in \mathbb{Z}^+$. That is, $y \in W$ and $h(W) \subseteq W$. □

Finally, we are ready to define an algorithm that finds a primitive substitution from a non-primitive one. We first sketch the algorithm in steps, before justifying its aspects in Proposition 2.2.6.

The Primitive Core Algorithm Suppose we are given a collection of prototiles $\mathcal{P}_0$ and a substitution ω_0 defined on $\mathcal{P}_0$. We apply the following steps 1 to 3 for each $j = 0, 1, 2, \ldots$ consecutively and claim the algorithm terminates at a primitive substitution in finite time.

Step 1 : Find $p \in \mathcal{P}_j$ such that a translate of p appears in $\omega_j^{n_j}(p)$ for some $n_j \in \mathbb{Z}^+$.

Step 2 : Find $m_j \in \mathbb{Z}^+$ such that the collection $\mathbf{I}(p)$ of all prototiles in $\mathcal{P}_0$ that appear in $\omega_j^{n_j \cdot m_j}(p)$ is invariant under the substitution $\omega_j^{n_j \cdot m_j}$ (in the sense that $\omega_j^{n_j \cdot m_j}(x)$ only consists of translates of prototiles in $\mathbf{I}(p)$, for each $x \in \mathbf{I}(p)$), and every prototile in $\mathbf{I}(p)$ appears in the patch $\omega_j^{n_j \cdot m_j}(p)$.

Step 3 : Check if $\omega_j^{n_j \cdot m_j}|_{\mathbf{I}(p)}$ is primitive on $\mathbf{I}(p)$. If not, go to Step 1 with the collection $\mathcal{P}_{j+1} = \mathbf{I}(p) \backslash \{x \in \mathbf{I}(p) : \ p \in \omega_j^{n_j \cdot m_k \cdot s}(x) \ \mathit{for\ some}\ s \in \mathbb{Z}^+\}$ with substitution $\omega_{j+1} := \omega_j^{n_j \cdot m_j}|_{\mathcal{P}_{j+1}}$.

To see the algorithm terminates in finite time note that $\mathcal{P}_{j+1} \neq \emptyset$ and $\mathcal{P}_{j+1} \subsetneq \mathcal{P}_j$.

Proposition 2.2.6. *Every substitution $\omega : \mathcal{P} \mapsto \mathcal{P}^\star$ over a finite prototile set $\mathcal{P}$ admits a primitive substitution $\omega' : \mathcal{A} \mapsto \mathcal{A}^*$, which is a restriction of the substitution ω^n to $\mathcal{A}$ for some $n \in \mathbb{Z}^+$ and $\mathcal{A} \subseteq \mathcal{P}$. Moreover, $\Omega_{\omega'} \subseteq \Omega_\omega$ where $\Omega_{\omega'}, \Omega_\omega$ are collection of tilings that are generated by the substitutions ω', ω, respectively. In particular, if the substitution ω is recognisable then so is ω'.*

Proof. Suppose ω is a non-primitive substitution defined on a finite prototile set $\mathcal{P}$ with an expansion factor λ. For each prototile $p \in \mathcal{P}$ choose a prototile q that appears in $\omega(p)$. This defines a map $f : \mathcal{P} \mapsto \mathcal{P}$ such that $f(p) = q$ for $p \in \mathcal{P}$.

To see that there is always a solution to Step 1, by Lemma 2.2.1 , there exist $n \in \mathbb{Z}^+$ and a prototile $p_0 \in \mathcal{P}$ such that $f^n(p_0) = p_0$. That is, p_0 appears in the n supertile $\omega^n(p_0)$.

We now claim there is always a solution to Step 2. Indeed, because $p_0 + x_0 \in \omega^n(p_0)$ for some $x_0 \in \mathbb{R}^2$, we have $(\omega^{n \cdot s}(p_0) + \lambda^{n \cdot s} \cdot x_0) \subseteq \omega^{n \cdot (s+1)}(p_0)$ for all $s \in \mathbb{Z}^+$. Define $\mathbf{I}(p_0) = \{q \in \mathcal{P} : \ q + x_q \in \omega^{n \cdot s}(p_0) \ \mathit{for\ some}\ x_q \in \mathbb{R}^2 \ \mathit{and}\ s \in \mathbb{Z}^+\}$ and g to be a map from $\mathcal{P}$ to the subsets of $\mathcal{P}$ (subsets of $\mathcal{P}$ is different from our notation of $\mathcal{P}^*$, which may contain duplicate prototiles, whereas the subsets of $\mathcal{P}$ cannot) such that $g(p)$ is the collection of prototiles in $\mathcal{P}$ whose copies appear in the patch $\omega^n(p)$ for $p \in \mathcal{P}$. $p_0 \in g(p_0)$ since $p_0 + x_0 \in \omega^n(p_0)$. Therefore, the map h defined on the subsets of $\mathcal{P}$ by $h(A) := \bigcup_{x \in A} g(x)$ for $A \subseteq \mathcal{P}$ satisfy the following, by Corollary 2.2.3:

$$\exists m \in \mathbb{Z}^+ \textit{ so that } (h^m(\{p_0\}))^* \textit{ is invariant under } h^m \textit{ and } h^s(\{p_0\}) = h^m(\{p_0\}) \ \forall s \geq m.$$

Notice that $h^s(\{p\})$ records the prototiles appearing in the patch $\omega^{n \cdot s}(p)$ for $p \in \mathcal{P}$ and $s \in \mathbb{Z}^+$. Consequently, $\mathbf{I}(p_0)$ is nothing but $h^m(\{p_0\})$, by the correlation of h and ω^n. Thus, $\mathbf{I}(p_0)$ is invariant under the map $\omega^{n \cdot m}$ (in the sense that $\omega^{n \cdot m}(x)$ only consists of translates of prototiles in $\mathbf{I}(p_0)$, for each $x \in \mathbf{I}(p_0)$) and $\omega^{n \cdot m}(p_0)$ contains a copy of every prototile in $\mathbf{I}(p_0)$, proving our claim that Step 2 has a solution.

Lastly, we show Step 3 either terminates or produces a new prototile set $\mathcal{P}_{j+1}$. Since $\mathbf{I}(p_0)$ is invariant under the map $\omega^{n \cdot m}$, $\omega' = \omega^{n \cdot m}|_{\mathbf{I}(p_0)}$ is a well-defined substitution defined on $\mathbf{I}(p_0)$. If ω' is not a primitive substitution on $\mathbf{I}(p_0)$, then there must exists a prototile $p_1 \in \mathbf{I}(p_0)$ such that $\omega^{n \cdot m \cdot s}(p_1)$ does not contain a copy of p_0 for each $s \in \mathbb{Z}^+$, by Lemma 2.2.4. Hence, by Lemma 2.2.5, the collection $\mathcal{P}' = \mathbf{I}(p_0) \backslash \{x \in \mathbf{I}(p_0) : \ p_0 + y_0 \in \omega^{n \cdot m \cdot s}(x) \ \textit{for some } y_0 \in \mathbb{R}^2 \textit{ and } s \in \mathbb{Z}^+\}$ is invariant under ω'. Therefore, $\omega'|_{\mathcal{P}'}$ defines a substitution rule on the (sub)collection $\mathcal{P}'$. Note that $\emptyset \neq \mathcal{P}' \subsetneq \mathcal{P}$, by Lemma 2.2.5. We construct a substitution ω' on a proper subcollection $\mathcal{P}'$ of $\mathcal{P}$.

Hence, applying the same argument above whenever ω' is not a primitive substitution on $\mathcal{P}'$, we construct smaller and smaller subcollections, and thereby form a primitive substitution eventually, which is a power of the given substitution over a restricted (sub)collection.

For the last part, note that every k-supertile with respect to the substitution ω' for $k \in \mathbb{Z}^+$ is a $k \cdot n$-supertile with respect to the substitution ω. Thus, $\Omega_{\omega'} \subseteq \Omega_\omega$ where $\Omega_{\omega'}, \Omega_\omega$ are set of tilings that are generated by the substitutions ω, ω', respectively. □

Remark 2.2.7. The inclusion $\Omega_{\omega'} \subseteq \Omega_\omega$ in Proposition 2.2.6 can be strict. In particular, let ω be the substitution defined over $\{0, 1, A, B\}$ as follows:

$$\omega(0) = 0, 1, \qquad \omega(1) = 1, 0, \qquad \omega(A) = A, B, \qquad \omega(B) = B, A.$$

We can induce two primitive substitutions μ, ν from ω using the primitive core algorithm such that μ is defined on $\{0, 1\}$, ν is defined on $\{A, B\}$ and

$$\mu(0) = 0, 1, \qquad \mu(1) = 1, 0, \qquad \nu(A) = A, B, \qquad \nu(B) = B, A.$$

Observe that $\Omega_\mu \subsetneq \Omega_\omega$ and $\Omega_\nu \subsetneq \Omega_\omega$.

Chapter 3

The Travelling Algorithm

Our goal in this chapter is to define over travelling algorithm in detail to produce a decorated substitution rule from any given 2-dimensional substitution rule in the plane. Our method of construction will guarantee that the new decorated substitution rule forms relatively dense curves, which are akin to space filling curves. The generated relatively dense curve is the primary ingredient for producing 1-dimensional tilings, which will be discussed in Chapter 4.

The algorithm will be explained in three sections. The first two sections are devoted to constructing the algorithm in special cases. In the first section, we study the algorithm for a special class of patches called *circle patches* (See Definition 3.1.1). These patches are of special interest to us, as they lie at the centre of the travelling algorithm. In the second section, we investigate the algorithm for a larger class, called *cyclic patches* (See Definition 3.2.1). In this section, we explain composition of tilings and patches, where we expose a special composition method called *circle composition* (Section 3.2.1). Circle compositions of a patch return a reformed version of the given patch. It turns out that these circle composition systems form iterative structures. These iterative structures transform any given cyclic patch into a single tile patch. The transformation procedure is what is needed to generalise the results in the first section. In the third section, we finalise the travelling algorithm for any given patch satisfying mild conditions. Lastly, we examine the relations between the travelling algorithm and the Hamiltonian path problem.

The travelling algorithm induces an order structure for any given patch Q. In particular, depending on which tile (or which pair $[u, e]$ where e is a curve that corresponds to a visit of u) is visited by the traveller first, we can define an order system between the tiles of Q, as explained with an example in the introduction. For that reason, throughout the chapter, we refer to the travelling algorithm applied for a patch Q as the order structure of Q. In words, we think of the travelling algorithm as an order system.

Preliminary Definitions

In this section we provide preliminary definitions. We explain decorations, decorated tiles and decorated patches. Order structures for patches are defined through decorations.

Definition 3.0.1. Let T' be a collection of tiles such that supp T' is bounded and int (supp T') has n simply connected components for some $n \in \mathbb{Z}^+$. Then there is a unique sequence of mutually disjoint patches $P_1, \ldots, P_n$ so that $P_1 \cup \cdots \cup P_n = T'$. We call P_i's the *components* of T'.

Definition 3.0.2. The *vertex set* of a patch is the collection of all vertices of tiles inside the patch. A vertex of a patch is called an *exterior vertex* if it intersects the boundary of the patch. It is called an *interior vertex* otherwise. Similarly, the *edge set* of a patch is the collection of all edges of tiles contained in the patch. An edge of a patch is called an *exterior edge* if it is completely contained in the boundary of the patch. It is called an *interior edge* if it does not intersect with the boundary of the patch. An edge is called a *slice edge* if it is not an exterior edge and both its end points are exterior vertices.

Definition 3.0.3. A tile t of a patch Q is called a *slice tile* if int (supp $(Q\backslash\{t\})$) is disconnected. It is called a *non-slice tile* otherwise. Similarly, a subpatch S of a patch Q is called a *slice subpatch* of Q if int (supp $(Q\backslash S)$) is disconnected. It is called a *non-slice subpatch* of Q otherwise.

Decorations

We define decorations for tiles/patches through curves that move in between common vertices of tiles.

Definition 3.0.4. An n-curve is a union of n mutually disjoint simple curves. The corresponding disjoint simple curves are called *components* of the given n-curve. A *decoration* for a prototile p is an n-curve e_p such that $n = 1$ or 2 and $e_p \cap \partial \text{supp } p \subseteq V_p$. We call e_p a *simple decoration* for p if e_p is a simple curve ($n = 1$).

Definition 3.0.5. Suppose $\mathcal{P}$ is a given finite set of prototiles. Let $p \in \mathcal{P}$ be a prototile and e_p a decoration for p. Then the prototile p_d with supp $p_d =$ supp p and $l(p_d) = (l(p), e_p)$ where $l(p)$ is the label set for p and $l(p_d)$ is the label set for p_d, is called a *decorated copy of* p. The decoration e_p is called *the decoration of* p_d.

Similarly, if $t = p + x$ is a translation of a prototile p, then the tile $t_d = p_d + x$ where p_d is a decorated copy of p, is called a *decorated copy of* t. The decoration $e_t = e_p + x$, where e_p is the decoration of p_d, is called *the decoration of* t_d or is called *a decoration of* t.

Figure 3.1: Examples of decorated tiles

Remark 3.0.6. Notice that if p_d is a decorated copy of a prototile p such that it has a label set $l(p_d) = (l(p), e_p)$, then we call e_p *a decoration for* p or *the decoration of* p_d. Similarly, $t_d = p_d + x$ is a decorated copy of a tile $t = p + x$ with label $l(t_d) = l(p_d) = (l(p), e_p)$. We call $e_t = e_p + x$ *a decoration for* t or *the decoration of* t_d.

By the same token, we define decorated supertiles and decorations for supertiles as well. In order to be able to do that we first define corners of supertiles, which are the images of the vertices of the prototiles under the corresponding substitution rule.

Definition 3.0.7. Suppose $\mathcal{P}$ is a finite collection of prototiles and ω is a substitution rule defined on $\mathcal{P}$ with an expansion factor λ. If $p \in \mathcal{P}$ and V_p is the set of vertices of p, then the collection $\{\lambda^k \cdot x : \ x \in V_p\}$ is called the set of *corners* of the k-supertile $\omega^k(p)$ for $k \in \mathbb{Z}^+$.

Figure 3.2: Corners of 1-supertiles of 2DTM are shown. $2 \cdot A,\ 2 \cdot B,\ 2 \cdot C$ and $2 \cdot D$ are the corners of the 1-supertile $\omega(p)$ whereas $2 \cdot E,\ 2 \cdot F,\ 2 \cdot G$ and $2 \cdot H$ are the corners of the 1-supertile $\omega(q)$.

Definition 3.0.8. Assume that $\mathcal{P}$ is a finite collection of prototiles and ω is a substitution rule defined on $\mathcal{P}$ with an expansion factor λ. Suppose further, $p \in \mathcal{P}$, $k \in \mathbb{Z}^+$, e_p is a decoration for p and $\mathcal{C}$ is a subset of supp $\omega^k(p)$ satisfying the following:

(1) If e_p is a simple decoration with end points A, B so that $s(e_p) = A$ and $r(e_p) = B$, then $\mathcal{C}$ is a simple curve with end points $\lambda^k \cdot A, \lambda^k \cdot B$ such that $s(\mathcal{C}) = \lambda^k \cdot A$ and $r(\mathcal{C}) = \lambda^k \cdot B$. If e_p is a 2-curve decoration with components e_p^1, e_p^2 so that $s(e_p^1) = A$, $r(e_p^1) = B$, $s(e_p^2) = C$ and $r(e_p^2) = D$, then $\mathcal{C} = \mathcal{C}^1 \cup \mathcal{C}^2$ such that $\mathcal{C}^1, \mathcal{C}^2$ are curves with end points $\lambda^k \cdot A, \lambda^k \cdot B$ and $\lambda^k \cdot C, \lambda^k \cdot D$, respectively, so that $s(\mathcal{C}^1) = \lambda^k \cdot A$, $r(\mathcal{C}^1) = \lambda^k \cdot B$, $s(\mathcal{C}^2) = \lambda^k \cdot C$ and $r(\mathcal{C}^2) = \lambda^k \cdot D$.

(2) If $t \in \omega^k(p)$ with $t = q + x$ for some $q \in \mathcal{P}$ and $x \in \mathbb{R}^2$, then $\overline{\mathcal{C} \cap \text{int (supp } t)} = e_q + x$ for some decoration e_q for q.

(3) If $\mathcal{C}$ is a curve, then $\mathcal{C}$ does not have any crossing points. If $\mathcal{C} = \mathcal{C}^1 \cup \mathcal{C}^2$ where $\mathcal{C}^1, \mathcal{C}^2$ are curves as in (1), then $\mathcal{C}^1$ and $\mathcal{C}^2$ are non-crossing curves and they do not contain any crossing points (in themselves).

Then $\mathcal{C}$ is called *a decoration for* the supertile $\omega^k(p)$. It is called *a simple decoration for* the supertile $\omega^k(p)$ if, in addition, $\mathcal{C}$ is a curve (i.e. e_p is a simple decoration), and called *a non-simple decoration for* $\omega^k(p)$ otherwise.

If $Q = \omega^k(p)$ is a k-supertile and $\mathcal{C}$ is a decoration for Q, then we can generate a decorated copy Q_d for Q, just like for prototiles, using the curve $\mathcal{C}$. In particular, $\mathcal{C}$ induces decorations for each $t \in \omega^k(p)$. For $t \in \omega^k(p)$, define t_d to be a decorated copy of t with the decoration $e_t = \overline{\mathcal{C} \cap \text{int (supp } t)}$. Then the collection Q_d of all such decorated tiles is called a *decorated copy* of $Q = \omega^k(p)$ and $\mathcal{C}$ is called the *decoration of* Q_d. As in the case of prototiles, we call $\mathcal{C}$ *a decoration for* $Q = \omega^k(p)$ or *the decoration of* Q_d.

Figure 3.3: A 1-supertile of 2DTM and a decorated copy of it are shown. The decoration on the decorated supertile is the first iteration of Hilbert's Space-Filling Curve.

An Order Structure by Decorations

Order structures in patches are defined through the decoration curves of tiles in patches. For a patch to admit an order structure through a curve, the patch must contain distinct exterior vertices satisfying some conditions.

Definition 3.0.9. Suppose Q is a collection of tiles constructed by translations of prototiles of a finite prototile set $\mathcal{P}$ and A, B are given distinct exterior vertices of Q. We say (A, B) is a *valid pair* for Q if there is a curve $\mathcal{C}^{A,B}$ such that

(1) $\mathcal{C}^{A,B}$ is a concatenation of simple curves with $s(\mathcal{C}^{A,B}) = A$ and $r(\mathcal{C}^{A,B}) = B$,

(2) If $t \in Q$ with $t = p + x$ for some $p \in \mathcal{P}$ and $x \in \mathbb{R}^2$, then $\overline{\mathcal{C}^{A,B} \cap \text{int (supp } t)} = e_p + x$ for some decoration e_p of p,

(3) $\mathcal{C}^{A,B}$ does not contain any crossing points.

If, in addition, the decorations $e_t = \overline{\mathcal{C}^{A,B} \cap \text{int (supp } t)}$ for $t \in Q$ are all simple curves, then (A, B) is called a *simple valid pair*. It is called a *non-simple valid pair* otherwise.

Figure 3.4 shows an example of a non-simple valid pair and a simple valid pair. The patch Q shown on the left of the figure has exterior vertices A, B, C, D, E, F. We have that (A, E) is a simple valid pair for Q by the curve demonstrated in the rightmost patch in the figure. On the other hand, (A, B) is a non-simple valid pair for Q since the only curve $\mathcal{C}^{A,B}$ with end points A, B and satisfies the conditions in the Definition 3.0.9 is the one illustrated in the middle patch in the figure. Notice also that there is no curve $\mathcal{C}^{C,F}$ with end points C, F which makes (C, F) a valid pair for Q. That is, (C, F) is not a valid pair for Q.

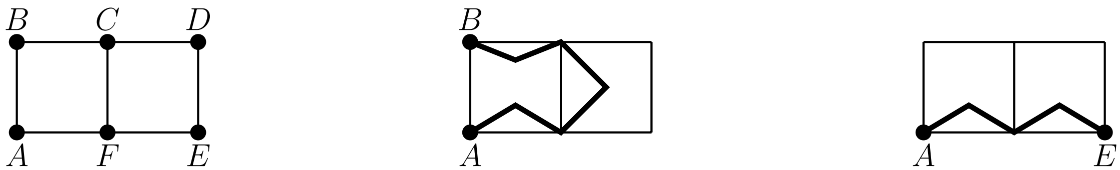


Figure 3.4: Examples of a non-simple valid pair and a simple valid pair

3.1 Circle Patches

In this section we explain the order system for a special category of patches called circle patches. Circle patches are analogous to complete graphs in the graph theory (See the discussion at the end of Section 3.3.1).

Definition 3.1.1. A patch Q is called a *circle patch* if there exists an interior vertex X of Q that belongs to every tile in Q and every tile in Q has an edge completely contained in ∂supp Q. We call the vertex X the *centre* of Q and denote Q as $\mathcal{O}_X$ for simplicity.

The leftmost patch $Q_1 = \{t_1, t_2, t_3, t_4\}$ in Figure 3.5 is an example of a circle patch whereas the patches $Q_3 = \{v_1, v_2, v_3, v_4\}$ and $Q_4 = \{w_1, w_2, w_3\}$ in the Figure are not circle patches since v_3 and w_3 do not have edges over the boundaries ∂supp Q_3 and ∂supp Q_4, respectively. On the other hand, the collection $Q_2 = \{u_1, u_2, u_3\}$ in the Figure does not satisfy our definition of a patch since int (supp Q_2) is not simply connected.

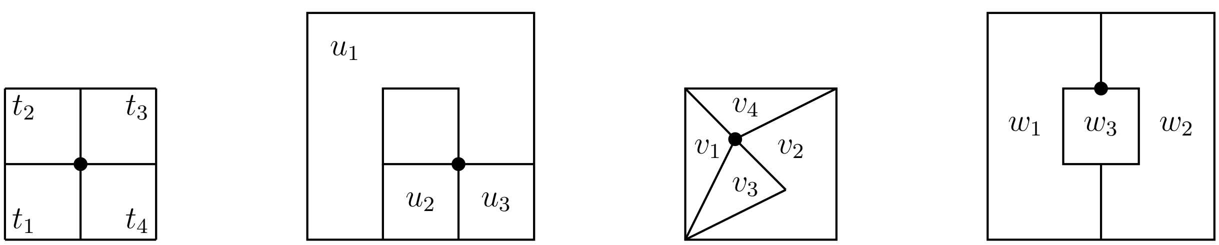


Figure 3.5

Definition 3.1.2. Let Q be a circle patch and let $\gamma \subseteq \partial$supp Q be an arc over the boundary ∂supp Q with end points A, B where $A, B \in V_Q$. Then there exists a unique subpatch S of Q such that every tile t in S has an edge completely contained in γ. We call S the *subpatch generated by* γ.

For example, the patch Q on the left of Figure 3.6 is a circle path with centre X. The bold arc represents the arc γ with end points A, B where A, B are exterior points of Q. The subpatch generated by γ is demonstrated on the right of the figure.

Figure 3.6

3.1.1 Properties of Circle Patches

As circle patches contain a common interior vertex, every tile is connected with every other. This connectedness of circle patches reveals many significant properties that can be used to construct an order system. Before explaining the order systems for circle patches, we provide an analysis of their geometric framework.

Lemma 3.1.3. *Let Q be a circle patch with centre X and $S \subsetneq Q$ be a subpatch of Q. Then X is an exterior vertex of S such that every tile of S contains X as a vertex.*

Proof. Every tile of S contains the centre of Q since Q is a circle patch. As $S \subsetneq Q$, let t be a tile in $Q \backslash S$. The centre X is a vertex of t. So, there are edges e_1, e_2 of t such that $X \in e_i$ for $i = 1, 2$. Because $t \notin S$, e_1 and e_2 have to be exterior edges of S. Therefore, X is an exterior vertex of S. □

Lemma 3.1.4. *Circle patches do not contain slice tiles or slice edges.*

Proof. Let Q be a circle patch with centre X and let t be a tile in Q. By Lemma 3.1.3, X is an exterior vertex of the subpatch $Q \backslash \{t\}$. Because $Q \backslash \{t\}$ is a subpatch, int(supp $Q \backslash \{t\}$) is connected. That is, t is not a slice tile. Similarly, if e is an edge in Q belonging to a tile $t_e \in Q$, then e is not a slice edge since int(supp $Q \backslash \{t_e\}$) is connected. □

Lemma 3.1.5. *Let Q be a circle patch with centre X and S be a subpatch of Q. Then every tile of S contains an edge which is completely contained over the boundary of* supp S.

Proof. Since Q is a circle patch, every tile in Q has at least one edge that is contained in ∂supp Q. Since $S \subseteq Q$, any tile in S also has an edge over the boundary supp Q. This is also on the boundary of supp S. □

The following lemma is an essential argument for several proofs throughout this chapter. It is the only result we provide in this section that is not only valid for circle patches.

Lemma 3.1.6. *Let Q be a patch (not necessarily a circle patch), let A, B be exterior vertices of Q and let $\gamma \subseteq \partial\text{supp } Q$ be a simple curve with end points at A and B. Suppose further for each tile $t \in Q$, $\gamma_t = \gamma \cap \partial\text{supp } t$ is a non-trivial arc with (distinct) end points X_t, Y_t. Then (A, B) is a valid pair for Q. Moreover, (A, B) is a simple valid pair for Q if and only if, in addition, there are distinct tiles $t_A, t_B \in Q$ such that $A \in V_{t_A}$ and $B \in V_{t_B}$.*

Proof. Suppose Q is a given patch, A, B are exterior vertices of Q and γ is an arc with end points at A and B satisfying the assumptions in the lemma. Let $\Psi = \{x_1, \ldots, x_m\}$ denote the set of shared exterior vertices of Q that are lying over the open arc $\gamma \backslash \{A, B\}$. Assume further without loss of generality that $x_1, \ldots, x_m$ are located over γ such that there are arcs $\gamma_1, \ldots, \gamma_{m+1} \subseteq \gamma$ so that γ_1 has end points A and x_1, γ_i has end points x_i and x_{i+1} for each $i \in \{1, \ldots, m-1\}$, and γ_{m+1} has end points x_m and B, as demonstrated in Figure 3.7.

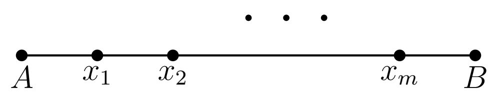


Figure 3.7: Locations of $x_1, \ldots, x_m$ over γ

Since $x_1, \ldots, x_m$ are the shared exterior vertices of Q lying in $\gamma \backslash \{A, B\}$, we have two cases; either $|Q| = m$ or $|Q| = m + 1$, as illustrated in Figure 3.8 with examples. Observe that the former can only happen if there is a unique tile of Q that contains both vertices A and B, as shown on the right of Figure 3.8, and the latter can only happen if there are two distinct tiles of Q containing the vertices A and B, respectively, as shown on the left of Figure 3.8.

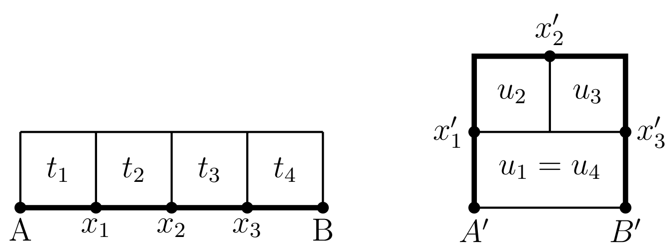


Figure 3.8: $Q = \{t_1, t_2, t_3, t_4\}$ and $Q' = \{u_1, u_2, u_3\}$ are illustrated with arcs γ, γ' over their boundaries, respectively. γ and γ' are the bold arcs as demonstrated, respectively.

Suppose first $|Q| = m + 1$ and $Q = \{t_1, \ldots, t_{m+1}\}$ so that $\{A, x_1\} \subseteq V_{t_1}$, $\{B, x_m\} \subseteq V_{t_{m+1}}$ and $\{x_{i-1}, x_i\} \subseteq V_{t_i}$ for $i \in \{2, \ldots, m\}$. Set $x_0 = A$ and $x_{m+1} = B$. Define $e_i \subseteq \text{supp } t_i$ so that $s(e_i) = x_{i-1}$ and $r(e_i) = x_i$ for $i \in \{1, \ldots, m+1\}$.

Then we can define a curve $C^{A,B}$ which is the concatenation of $m + 1$ simple curves $\{e_i\}_{i=1}^{m+1}$ defined above. $C^{A,B}$ has end points at A and B. Thus, (A, B) is a valid pair for Q. Moreover, it is a simple valid pair, by construction, since A and B belong to different tiles.

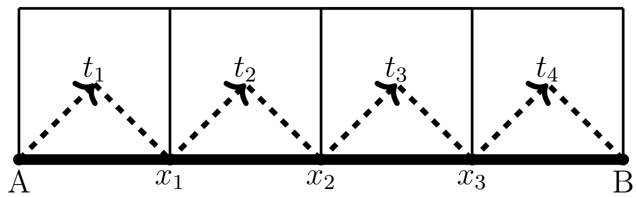


Figure 3.9: Illustration of the process for a patch $Q = \{t_1, t_2, t_3, t_4\}$. $\Psi = \{x_1, x_2, x_3\}$ and γ is the bold arc over the boundary $\partial \text{supp } Q$ with end points A and B.

Assume now $|Q| = m$ and $Q = \{t_1, \ldots, t_m\}$ so that $\{A, B, x_1, x_m\} \subseteq V_{t_1}$ and $\{x_{i-1}, x_i\} \subseteq V_{t_i}$ for $i \in \{2, \ldots, m-1\}$. Define the following:

(1) $e_1 \subseteq \text{supp } t_1$ is a 2-curve with components e_1^1, e_1^2 so that $s(e_1^1) = A$, $r(e_1^1) = x_1$ and $s(e_1^2) = x_m$, $r(e_1^2) = B$.

(2) $e_i \subseteq \text{supp } t_i$ is a simple curve so that $s(e_i) = x_{i-1}$ and $r(e_i) = x_i$ for $i \in \{2, \ldots, m-1\}$.

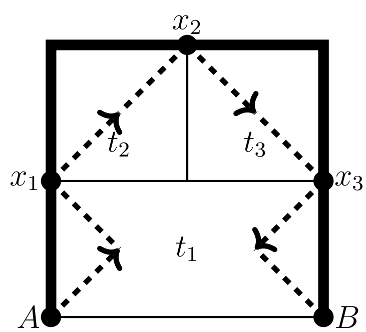


Figure 3.10: Illustration of the process for a patch $Q = \{t_1, t_2, t_3\}$. $\Psi = \{x_1, x_2, x_3\}$ and γ is the bold arc over the boundary $\partial \text{supp } Q$ with end points A and B.

Then we can define a curve $C^{A,B}$ which is the concatenation of $m+1$ simple curves $e_1, e_2, \ldots, e_{m-1}, e_1^1, e_1^2$ defined above. $C^{A,B}$ has end points at A and B. Thus, (A, B) is a valid pair for Q. Moreover, it is not a simple valid pair, by construction, since A and B belong to a same tile. □

Lemma 3.1.7. *Let Q be a circle patch and $S \subseteq Q$ be a (non-empty) subpatch of Q. If A, B are distinct isolated vertices of S that belong to the same tile $u_{AB} \in S$, then (A, B) is a valid pair for S.*

Proof. Because A, B are isolated vertices of Q that belong to the same tile in S (or in Q), there exists an arc γ with end points A, B such that $\gamma \subseteq \partial \text{supp } u_{AB}$ and γ do not contain any shared exterior vertices of Q. Define γ' to be an arc with end points A, B such that $\gamma' \cap \gamma = \{A, B\}$ and $\gamma' \cup \gamma = \partial \text{supp } S$. Then every tile in S has an edge completely contained in γ' by construction. Hence, (A, B) is a valid pair for S by Lemma 3.1.6. □

Lemma 3.1.8. *Let Q be a circle patch with centre X and let A be a shared exterior vertex of Q. Then every tile t with $A \in V_t$ contains an edge e_t such that $A \in e_t$ and $e_t \subseteq \partial\text{supp } Q$. In particular, for every shared exterior vertex of Q, there are exactly two tiles containing the vertex.*

Proof. Assume to the contrary. Let $t \in Q$ be a tile such that $A \in V_t$ and every edge e_t of t with $A \in V_t$ satisfies $e_t \subsetneqq \partial\text{supp } Q$.

Because A is a shared exterior vertex of Q, there are two (distinct) tiles $t_1, t_2 \in Q$ such that there exist edges $e_{t_i} \in E_{t_i}$ for $i = 1, 2$, so that $A \in e_{t_i}$ and $e_{t_i} \subseteq \partial\text{supp } Q$ for $i = 1, 2$. Let $t \in Q$ be a tile such that $A \in V_t$ and every edge e_t of t with $A \in V_t$ satisfies $e_t \subsetneqq \partial\text{supp } Q$. Since Q is a circle patch, t must contain an edge e_0 such that $e_0 \subseteq \partial\text{supp } Q$. Then supp $Q\backslash$supp t must separate tiles t_1 and t_2 as supp t contains both the vertex A and the edge e_0 that does not intersect with A. This is a contradiction. Hence, if a tile contains a shared exterior vertex of Q, then it must also have an edge over the boundary which contains the vertex as an end point. $\square$

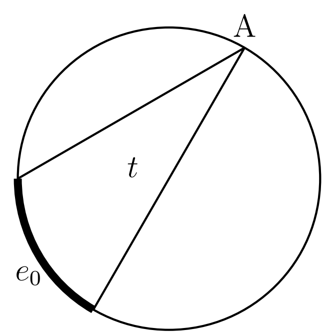


Figure 3.11: t is a slice tile since it has an exterior edge e_0 and an exterior vertex A so that $A \notin e_0$.

Since every tile in a circle patch has an edge completely contained in the boundary of the circle patch, every tile contains at least two shared exterior vertices of the circle patch. Moreover, every tile in the circle patch must contain at most two shared exterior vertices of the circle patch, since otherwise it would contradict the non-existence of slice tiles. Therefore, every tile in a circle patch must contain exactly two shared exterior vertices of a circle patch. In particular, using this fact, tiles in circle patches can be identified with topological triangles. For example, if a tile t of a circle patch Q with centre X, contains the shared exterior vertices M, N of Q, then there exists a topological triangle with vertices M, N, X that covers the area supp t, as shown in Figure 3.12.

Using the shared exterior vertices of a circle patch, we define the concept of neighbour tiles for circle patches. Two tiles in a circle patch are said to be *neighbour tiles* if they have a shared exterior vertex of the circle patch. Similarly, a tile t is said to be a *neighbour* of a tile u if they have a common shared exterior vertex of the circle patch. Every tile in a

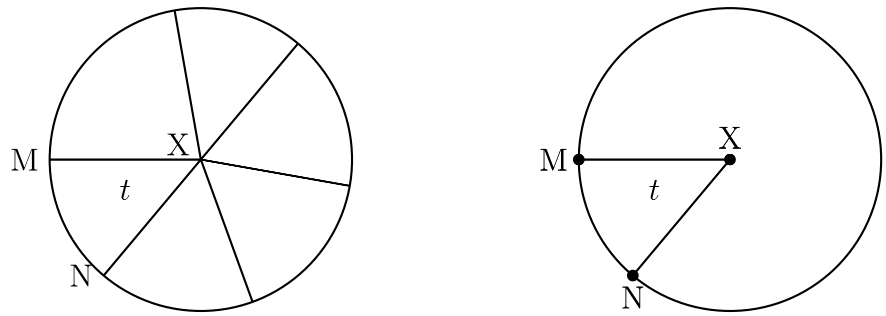


Figure 3.12: Tile t can be identified with the topological triangle with vertices M, N, X.

circle patch, therefore, has exactly two 'edge-neighbours'; i.e, there are exactly two tiles that share a common edge with the given tile.

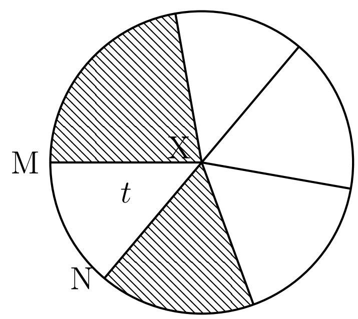


Figure 3.13: Neighbours of t are illustrated with dashed area.

3.1.2 An Order Structure on Circle Patches

We are now ready to define an order structure for circle patches.

Lemma 3.1.9. *For any given circle patch Q with centre X and distinct exterior vertices A, B, there are two connected arcs c_1, c_2 over the boundary of* supp Q *such that $c_1 \cap c_2 \subseteq \{A, B\}$, and there are two subpatches Q^1, Q^2 generated by c_1, c_2, respectively, so that $Q^1 \cap Q^2 = \emptyset$ and $Q^1 \cup Q^2 = Q$.*

Proof. Let Q be a circle patch with a centre X and let A, B be distinct exterior vertices of Q. We have four cases:

(1) Both A and B are shared exterior vertices of Q,

(2) A is an isolated vertex of Q whereas B is a shared exterior vertex of Q,

(3) Both A and B are isolated vertices of Q and belong to the same tile in Q,

(4) Both A and B are isolated vertices of Q and they belong to different tiles in Q.

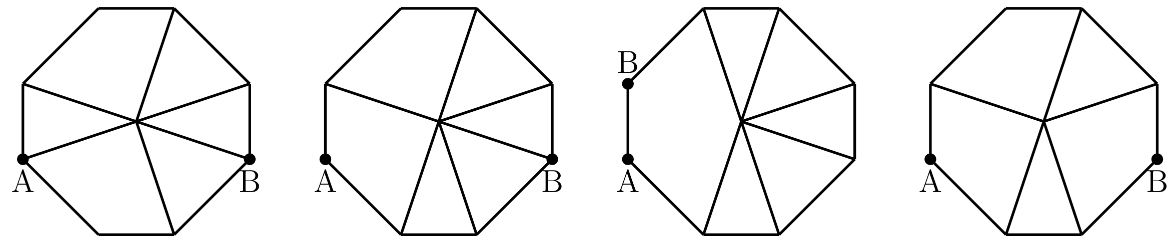


Figure 3.14: Examples of cases (1) – (4) in the proof of Lemma 3.1.9

Case (1): We can define two arcs c_1, c_2 with end points A, B by connecting the vertices A and B over the boundary of supp Q in two different orientations as is shown in the middle patch of Figure 3.15. Let $t_A, t_B \in Q$ be (not necessarily distinct) tiles containing A and B respectively so that t_A and t_B both contain an edge over c_1. Such tiles exist as $A \neq B$ and A, B are shared exterior vertices of Q. Recalling that X is a centre vertex, notice that since A and B are shared exterior vertices of Q, we can define two arcs e_A and e_B with end points A, X and B, X, respectively so that $e_A \cap \partial \text{supp } Q = \{A\}$, $e_B \cap \partial \text{supp } Q = \{B\}$, $e_A \subseteq \partial \text{supp } t_A$ and $e_B \subseteq \partial \text{supp } t_B$ as is shown in the rightmost patch of Figure 3.15. Then $e_A \cup e_B$ is an arc with end points A, B and separates supp Q as shown in the rightmost patch of Figure 3.15. So, if a tile of Q has an edge completely contained in c_1, then it cannot have another edge completely contained in c_2, and vice versa. Define Q^i to be the subpatch generated by c_i for $i = 1, 2$. Then Q^1 and Q^2 are disjoint, since tiles of Q can have edge over only one of c_1 or c_2. Moreover, $Q^1 \cup Q^2 = Q$ as we have $c_1 \cup c_2 = \partial \text{supp } Q$. Therefore, c_1, c_2 and Q^1, Q^2 satisfy the conclusion.

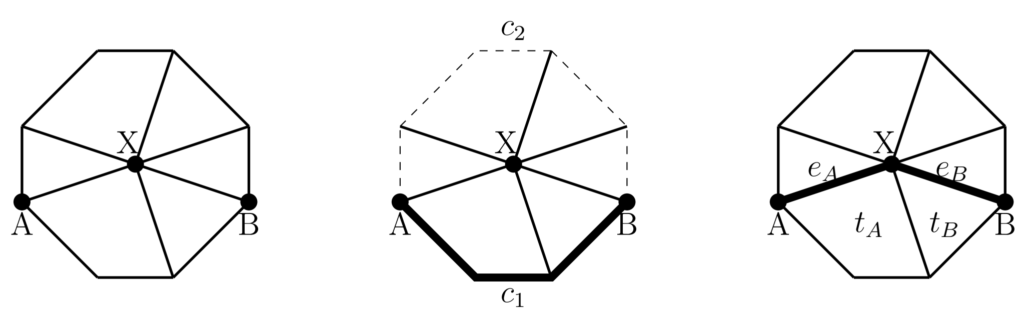


Figure 3.15: c_1 is the bold arc whereas c_2 is the dashed arc in the middle patch. c_1 and c_2 are arcs over the boundary with end points A and B. e_A and e_B are bold arcs with end point pairs A, X and B, X, respectively, as shown in the rightmost patch. $e_A \cup e_B$ separates support of the patch in the rightmost patch.

Case (2): Let A be an isolated exterior vertex of Q and B a shared exterior vertex of Q. Pick an exterior vertex C such that $C \neq B$, and A and C belong to a same tile in Q. Applying Case (1) for the shared exterior vertices B and C, we get arcs d_1, d_2 and subpatches Q^1, Q^2 such that $d_1 \cap d_2 = \{B, C\}$, $d_1 \cup d_2 = \partial \text{supp } Q$ and Q^i is the set of tiles in Q that has an edge completely contained in d_i for $i = 1, 2$ so that $Q^1 \cap Q^2 = \emptyset$ and $Q^1 \cup Q^2 = Q$. Assume without loss of generality $A \in d_1$. Define $c_1 \subseteq d_1$ to be the subarc of d_1 with end points A, B and $c_2 = d_2$. Then we have

$$Q^1 = \{t \in Q : \ t \text{ contains an edge that completely sits in } c_1\}.$$

Moreover, $c_1 \cap c_2 = \{B\}$. Thus, c_1, c_2 and Q^1, Q^2 satisfy the conclusion, as demonstrated in Figure 3.16.

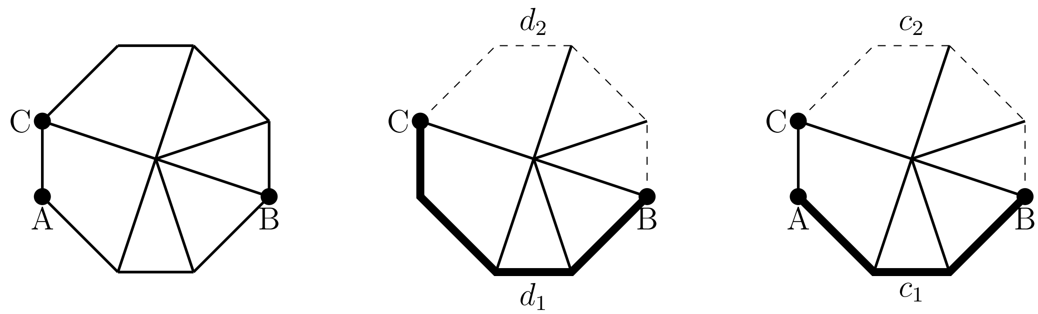


Figure 3.16: d_1, d_2 are arcs over the boundary with end points B, C whereas c_1 is an arc over the boundary with end points A, B and $c_2 = d_2$.

Case (3): Suppose A and B are isolated vertices of Q and belong to $t_{A,B} \in Q$. Let $X_{A,B}, Y_{A,B}$ denote the (distinct) shared exterior vertices of Q belonging to tile $t_{A,B}$. Applying Case (1) to the shared exterior vertices $X_{A,B}, Y_{A,B}$, we get arcs d_1, d_2 and subpatches Q^1, Q^2 such that $d_1 \cap d_2 = \{X_{A,B}, Y_{A,B}\}$, $d_1 \cup d_2 = \partial \text{supp } Q$ and Q^i is the set of tiles in Q that has an edge completely contained in d_i for $i = 1, 2$ so that $Q^1 \cap Q^2 = \emptyset$ and $Q^1 \cup Q^2 = Q$. Assume without loss of generality $A, B \in d_1$. Define c_1 to be the subarc of d_1 with end points A and B and $c_2 = d_2$. Because A and B are isolated vertices belonging to the same tile $t_{A,B}$, we have

$$\begin{aligned} Q^1 &= \{t \in Q : \ t \text{ contains an edge that completely sits in } d_1\} = \{t_{A,B}\} \\ &= \{t \in Q : \ t \text{ contains an edge that completely sits in } c_1\}. \end{aligned}$$

Moreover, $c_1 \cap c_2 = \emptyset$. Thus, c_1, c_2 and Q^1, Q^2 satisfy the conclusion.

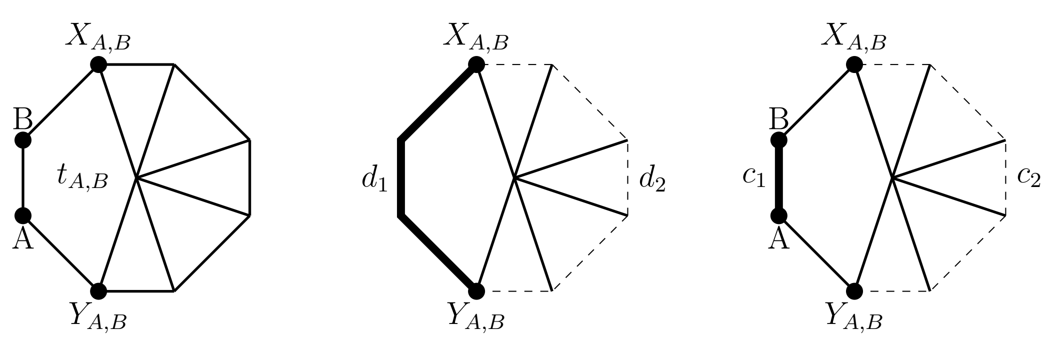


Figure 3.17: d_1, d_2 are arcs over the boundary with end points $X_{A,B}, Y_{A,B}$ whereas c_1 is an arc over the boundary with end points A, B and $c_2 = d_2$.

Case (4): Suppose A, B are isolated vertices of Q, t_A, t_B are the distinct tiles containing A, B respectively and C, D are distinct shared exterior vertices belonging to t_A. Applying Case (2) to the vertices B, C, we get arcs $d_1, d_2 \subseteq \partial\text{supp } Q$ and disjoint subpatches Q^1, Q^2 so that $d_1 \cap d_2 = \{C\}$ and $Q^1 \cup Q^2 = Q$. In particular, by Case (2), we can assume without loss of generality d_2 is an arc with end points B, C and d_1 is an arc with end points C and the shared exterior vertex of t_B that does not belong to d_2. The process is illustrated in Figure 3.18. For the given circle patch on the left of Figure 3.18, d_1 and d_2 are the arcs over the boundary of the middle patch in Figure 3.18 with end points C, B' and C, B respectively.

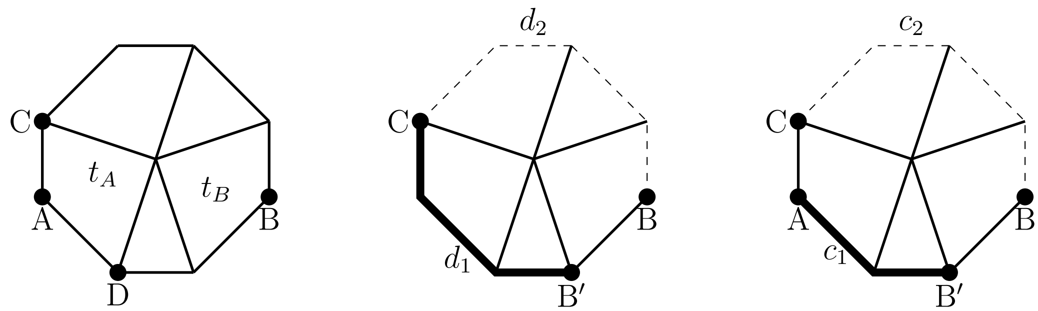


Figure 3.18: c_1 is an arc over the boundary with end points B, C whereas c_2 is an arc over the boundary with end points A, B'.

Define $c_2 = d_2$ and c_1 to be the subarc of d_1 with end points A and the shared exterior vertex of t_B that does not belong to d_2. Because A is an isolated vertex of Q and A and C belong to a same tile in Q, once again, we have $Q^1 = \{t \in Q : t \textit{ contains an edge completely sits in } c_1\}$. Moreover, $c_1 \cap c_2 = \emptyset$. Thus, c_1, c_2 and Q^1, Q^2 satisfy the conclusion. □

Proposition 3.1.10. *Suppose Q is a circle patch with centre X and A, B are distinct exterior vertices of Q. Then (A, B) is a valid pair for Q. In particular, (A, B) is a simple valid pair of Q if and only if there exist distinct tiles t_A, t_B of Q such that $A \in V_{t_A}$ and $B \in V_{t_B}$.*

Proof. Let Q be a circle patch with a centre X and distinct exterior vertices A and B. We have two cases; either there exist two distinct tiles t_A, t_B containing A and B as a vertex, respectively, or there exists a unique tile containing both A and B as a vertex.

Assume first that A and B both belong to a same tile and are isolated vertices of Q. Then there exists an arc $\gamma \subseteq \partial\text{supp } Q$ with end points A, B so that every tile in Q has an edge completely contained in γ. That is, (A, B) is a non-simple valid pair for Q, by Lemma 3.1.6, as demonstrated in Figure 3.19.

Suppose now, there exist two distinct tiles t_A, t_B containing the vertices A and B respectively. By Lemma 3.1.9, we can define arcs c_1, c_2 over the boundary of supp Q with

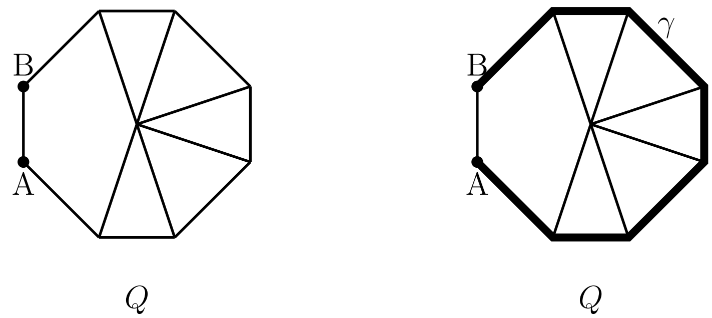


Figure 3.19: An example showing (A, B) is a valid pair for Q via the bold arc γ over the boundary of the right patch, whenever A, B are isolated vertices belonging to same tile.

$c_1 \cap c_2 \subseteq \{A, B\}$ and $c_1 \cup c_2 = \partial \text{supp } Q$ such that the subpatches

$$Q^i = \{t \in Q : \ t \text{ contains an edge that completely sits in } c_i\} \text{ for } i = 1, 2$$

satisfying $Q^1 \cap Q^2 = \emptyset$ and $Q^1 \cup Q^2 = Q$. Assume without loss of generality that $A \in c_1$ and $B \in c_2$. Define arcs $d_i \supseteq c_i$ for $i = 1, 2$ such that $d_i \subseteq \partial \text{supp } Q^i$ for $i = 1, 2$ and d_1 has end points at A, X whereas d_2 has end points at X, B. The process is illustrated with an example in Figure 3.20. Applying Lemma 3.1.9 to the circle patch $\{u_1, \ldots, u_6\}$ on the left of Figure 3.20, we get arcs c_1, c_2 as are shown with bold and dashed arcs over the boundary of the leftmost patch, respectively. We also get subpatches Q^1 and Q^2 such that $Q^1 = \{u_1, u_5, u_6\}$ and $Q^2 = \{u_2, u_3, u_4\}$. We can then define arcs d_1 and d_2 with end points A, X and X, B, respectively as are demonstrated over the boundaries of the middle and right patches in Figure 3.20. It can be readily seen in Figure 3.20 that $c_i \subseteq d_i \subseteq \partial \text{supp } Q^i$ for $i = 1, 2$.

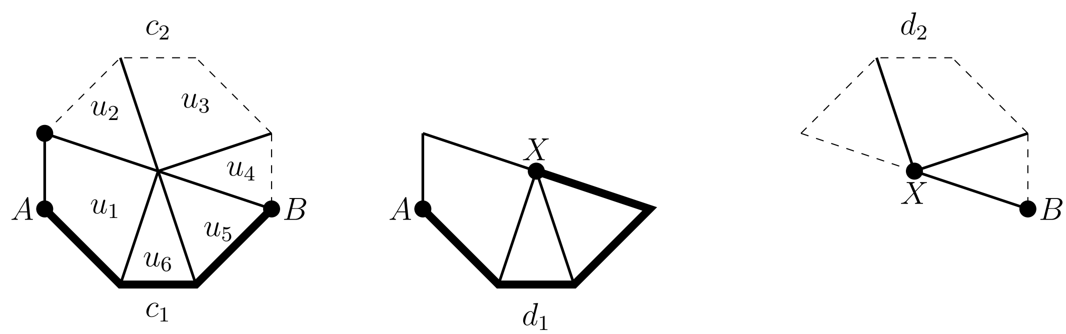


Figure 3.20: An example of the process for $k = 1$. c_1 and d_1 are illustrated with bold arcs whereas c_2 and d_2 are illustrated with dashed arcs.

Every tile in Q^i contains an edge completely contained in d_i since $c_i \subseteq d_i$ for $i = 1, 2$. Then (A, X) is a valid pair for Q^1 and (X, B) is a valid pair for Q^2, by Lemma 3.1.6. In particular, because every tile contains X as a vertex, X is a shared exterior vertex for Q^1 and Q^2 whenever they contain more than one tile. That is, (A, X) is a simple valid pair for Q^1 and (X, B) is a simple valid pair for Q^2, by Lemma 3.1.6. Thus, (A, B) is a simple valid pair for $Q = Q^1 \cup Q^2$. Hence, the proposition follows. □

Corollary 3.1.11. *Let Q be a circle patch, A, B distinct exterior vertices of Q and $\mathcal{C}^{A,B}$ a curve that makes (A, B) a valid pair for Q. Then exactly one of the following holds:*

(1) *$\mathcal{C}^{A,B}$ makes (A, B) a simple valid pair for Q,*

(2) *There exists a unique tile $t \in Q$ containing the vertices A, B such that its decoration induced by $\mathcal{C}^{A,B}$ is a 2-curve. That is, $\overline{\mathcal{C}^{A,B} \cap \text{int supp } t}$ is a 2-curve.*

Proof. By Proposition 3.1.10, a valid pair (A, B) for Q is a non-simple valid pair for Q if and only if A and B are isolated vertices of Q that belong to the same tile. Moreover, whenever it happens, the tile containing both vertices A and B is visited twice. That is, its decoration is a 2-curve. □

Corollary 3.1.12. *For any circle patch Q with a shared exterior vertex A, (A, B) is a simple valid pair for Q for any $B \neq A$ that is an exterior vertex of Q.*

Proof. Since A is a shared exterior vertex, we can find distinct $t_A, t_B \in Q$ such that $A \in V_{t_A}$ and $B \in V_{t_B}$. Thus (A, B) is a simple valid pair for Q by Proposition 3.1.10. □

Using Proposition 3.1.10, we are able to generate order structures in 2-dimensional tilings satisfying fairly generic conditions. For example, consider the 2DTM tiling. Let p, q denote the two square prototiles with different labels and let ω denote the substitution rule (with expansion factor $\lambda = 2$) demonstrated in Figure 3.21.

Figure 3.21: 2DTM Substitution Rule

We choose a prototile and a simple curve decoration in a random fashion. For example, suppose we choose the simple curve decoration e_p and the prototile p as shown over the decorated prototile in the left of Figure 3.22. Then $(2 \cdot A, 2 \cdot B)$ is a simple valid pair for the 1-supertile $\omega(p)$ by Proposition 3.1.10 since $2 \cdot A$ and $2 \cdot B$ belong to different tiles in the 1-supertile. In fact, the method illustrated in the proof of Proposition 3.1.10 generates the decoration over the 1-supertile on the right of the Figure 3.22. The decorated 1-supertile on the right of Figure 3.22 can be regarded as a substitution rule for the decorated prototile on the left of Figure 3.22. The decoration over the supertile on the right of the Figure induces decorated tiles which are illustrated in Figure 3.23.

Next, in the same way, we will define substitution rules for the decorated prototiles in Figure 3.23. Appealing to the same argument above, together with the method in

Figure 3.22: A decorated prototile and supertile

Figure 3.23: Decorated tiles generated from the decorated supertile in the right of Figure3.22

the proof of Proposition 3.1.10, we get the decorated 1-supertiles in Figure 3.24. These decorated 1-supertiles, once again, can be regarded as substitution rules for the decorated prototiles in Figure 3.23, respectively. Continuing this process we get a collection of decorated prototiles which are listed in Figure 3.25 and a (primitive) substitution rule defined on those prototiles which are illustrated in Figure 3.26 (figure is scaled for illustration purposes), respectively.

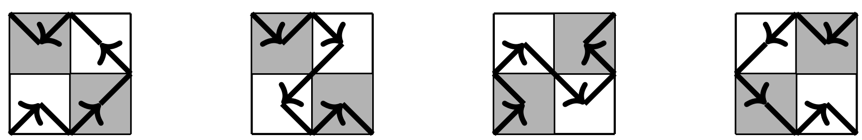


Figure 3.24: Decorated 1-supertiles which can be regarded as substitution rules of the decorated prototiles in Figure 3.23, respectively

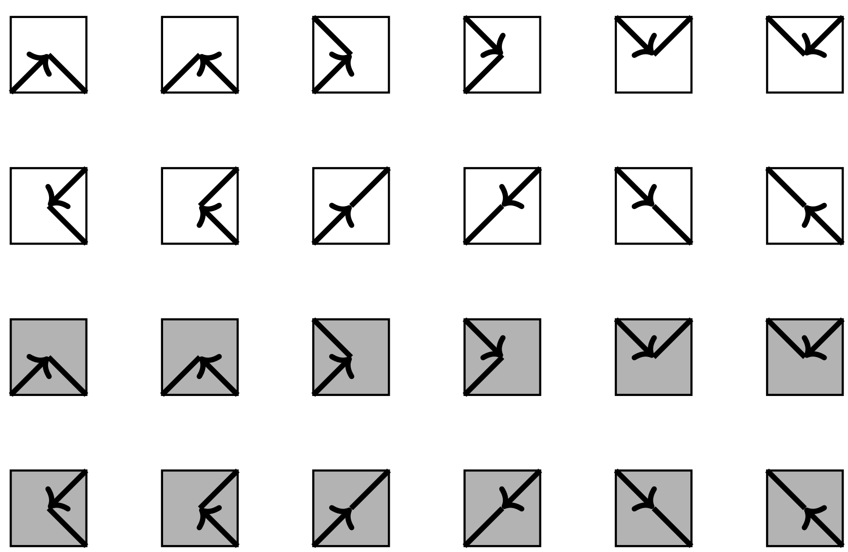


Figure 3.25: Collection of decorated prototiles

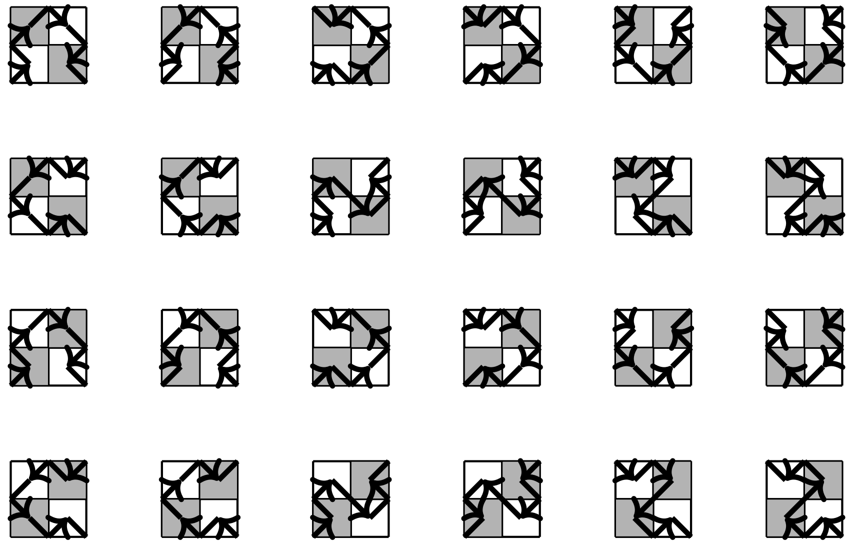


Figure 3.26: Collection of decorated 1-supertiles which can be regarded as substitution rules of the decorated prototiles in Figure 3.25, respectively. Decorated 1-supertiles are scaled for illustration purposes.

In fact, under the light of Proposition 3.1.10, we have the following theorem.

Theorem 3.1.13. *Suppose $\mathcal{P}$ is a given finite collection of tiles (not necessarily convex), ω is a primitive substitution rule defined on $\mathcal{P}$ and T is a recognisable substitution tiling generated by the substitution rule ω. Assume further, every 1-supertile of T is a circle patch such that for any given distinct corners A, B of a 1-supertile, there are two different tiles t_A, t_B in the 1-supertile patch, containing A and B as a vertex, respectively. Then there exist a finite collection of decorated prototiles $\mathcal{P}'$ and a primitive substitution rule ω' defined on $\mathcal{P}'$ such that the following holds:*

(1) For any $p' \in \mathcal{P}'$ there exists a unique $p \in \mathcal{P}$ such that p' is a decorated copy of p with label $l(p') = (l(p), e_p)$ for some simple decoration e_p for p. That is, every decoration is a simple curve decoration.

(2) There exists $k \in \mathbb{Z}^+$ such that if p' is a decorated copy of p with decoration e_p, then

(i) $\text{supp}\ \omega'(p') = \text{supp}\ \omega^k(p)$,

(ii) $\omega'(p')$ is a decorated copy of $\omega^k(p)$ with decoration $\mathcal{C}^{\lambda^k \cdot A, \lambda^k \cdot B}$ which has end points $\lambda^k \cdot A, \lambda^k \cdot B$ where A, B are end points of e_p and λ is the expansion factor of ω.

In particular, there exists a recognisable, primitive, self-similar substitution tiling T' with a prototile set $\mathcal{P}'$ and a substitution rule ω'.

Proof. Let $\mathcal{P}$ be a given finite collection of prototiles and let ω be a primitive substitution defined on $\mathcal{P}$. Choose $q \in \mathcal{P}$ and a simple decoration e_q for q, with end points A, B.

This generates a decorated prototile q' with a label set $l(q') = (l(q), e_q)$ and support $\text{supp } q' = \text{supp } q$. Then $Q = \omega(q)$ is a circle patch such that $\lambda \cdot A$ and $\lambda \cdot B$ are exterior points belonging to two different tiles in $\omega(q)$. So, $(\lambda \cdot A, \lambda \cdot B)$ is a simple valid pair for the circle patch Q by Proposition 3.1.10. Let $\mathcal{C}$ denote a curve that makes $(\lambda \cdot A, \lambda \cdot B)$ a simple valid pair according to the instructions in the proof of Proposition 3.1.10 (recall that such a curve is not necessarily unique). There exists a decorated copy Q' of Q with the decoration $\mathcal{C}$. Tiles in Q' have decorations induced from $\mathcal{C}$. Moreover, all these induced decorations are simple decorations since $(\lambda \cdot A, \lambda \cdot B)$ is a simple valid pair for Q. Record Q' to be the substitute of p'.

For each prototile $p \in \mathcal{P}$ and every simple decoration e_p for p, construct the decorated prototile p' and the decorated 1-supertile Q'_p by the same argument applied for q' above. Record each of the generated decorated 1-supertiles Q'_p as the substitutions of p'. For every distinct pair of corners in each 1-supertile of T, there are distinct tiles containing these vertices respectively. Thus, each of these generated decorated 1-supertiles Q'_p (for $p \in \mathcal{P}$) only consist of decorated tiles with simple decorations. That is, for every prototile p in $\mathcal{P}$ and every simple decoration e_p on it, we have constructed a decorated 1-supertile Q'_p such that we can regard Q'_p as the substitution of p' and whose tiles contain only simple decorations. This yields a collection of decorated prototiles $\mathcal{S}$ and a substitution rule σ defined on $\mathcal{S}$ such that

$$\mathcal{S} = \{p' : \ l(p') = (l(p), e_p), \ \text{supp } p' = \text{supp } p \ \textit{for } p \in \mathcal{P} \textit{ and simple decoration } e_p\}$$

Notice that $\mathcal{S}$ is the set of all prototiles which are decorated copies of prototiles in $\mathcal{P}$ with simple decoration. Therefore, σ and $\mathcal{S}$ satisfy (1) and (2) in the theorem, but do not necessarily define a primitive substitution. However, Proposition 2.2.6 ensures that there exists $\mathcal{P}' \subseteq \mathcal{S}$ and $n \in \mathbb{Z}^+$ such that $\omega' = \sigma^n|_{\mathcal{S}}$ is a primitive substitution over $\mathcal{P}'$ satisfying the conditions (1) and (2). We can now generate a tiling T' from the primitive substitution ω' by applying the standard argument explained in Chapter 2. Hence, T' is recognisable, primitive and self-similar substitution tiling by construction. $\square$

3.2 Cyclic Patches

Circle patches provide an order structure no matter how we choose end point pairs. In this section, we will generalise this concept to a new class of patches, namely cyclic patches. Cyclic patches are patches that lack both slice tiles and slice edges. They are generalisations of circle patches. We show in this section that, after applying finitely many reformation steps, called circle compositions, every cyclic patch can be transformed into a circle patch, which can then be transformed into a single tile patch following the same algorithm. This finite circle composition process generates iterative systems and thereby form space filling curves. We will discuss space filling curves that arise in Chapter 5.

Definition 3.2.1. A patch Q with $|Q| > 1$ is called *cyclic* if it does not contain any slice tiles or slice edges. That is, Q is a cyclic patch if the following holds:

(1) $\forall t \in Q$, the subset $\text{int}(\text{supp } Q \backslash \{t\})$ of $\mathbb{R}^2$ is connected.

(2) For all edges e of Q, $\text{supp } Q \backslash e$ is connected.

The leftmost patch in Figure 3.27 is an example of a cyclic patch. The other two patches in the Figure are not cyclic patches since they both contain slice tiles. The rightmost patch also contains slice edges.

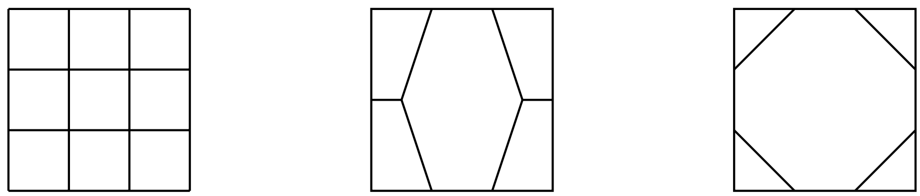


Figure 3.27: Examples of cyclic and non-cyclic patches

Lemma 3.2.2. *Every circle patch is a cyclic patch.*

Proof. Circle patches must contain at least two tiles because of the existence of an interior vertex, namely the centre of the circle patch. Moreover, circle patches do not contain slice tiles or edges by Lemma 3.1.4. Hence every circle patch is a cyclic patch. □

Remark 3.2.3. Every circle patch is a cyclic patch by Lemma 3.2.2. However, the converse is not true in general. For example, the patch $Q = \{v_1, v_2, v_3, v_4\}$ in Figure 3.28 is a cyclic patch which is not a circle patch since v_3 does not contain any edge over the boundary $\partial\text{supp } Q$. In fact, there is no circle subpatch of Q with centre Y.

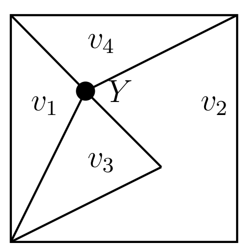


Figure 3.28: A cyclic patch with centre Y, which is not a circle patch

3.2.1 Compositions of Patches

A tiling T_2 is said to be a *composition* of a given tiling T_1 if the tiles of T_2 are union of the tiles of T_1. Similarly, we define a patch P_2 to be a *composition* of a given patch P_1 if the tiles of P_2 are a union of tiles from P_1. We do not insist that the new tiling T_2 is singly edge-to-edge, so tiles of the new patch P_2 do not necessarily meet along one edge. Finally, we further assume new tiles are not necessarily convex and vertices of new tiles are induced by the exterior vertices of the patch (or subpatch) generated by the corresponding union of tiles. An example of patch composition is presented in Figure 3.29. The left patch in the Figure consists of 4 tiles whereas the patch on the right contains a single tile ω. Vertices of ω and exterior vertices of the patch on the left coincide as illustrated.

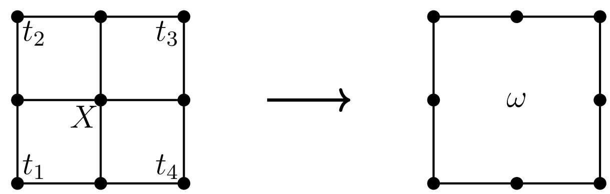


Figure 3.29: An example of a composition of a patch

We call the single tile patch on the right of Figure 3.29 a circle composition of the patch $P = \{t_1, t_2, t_3, t_4\}$ because the patch composed is a circle patch and the composition process generated a single tile patch. More precisely, a patch Q' is called a *circle composition* of a patch Q by the vertex X if X is an interior vertex of Q, $\mathcal{O}_X$ is a (circle) subpatch of Q and $Q' = (Q \backslash \mathcal{O}_X) \cup R_{\mathcal{O}_X}$ where $R_{\mathcal{O}_X}$ is a single tile patch that is a composition of $\mathcal{O}_X$. For simplicity, we denote the patch Q' as $\Theta_X(Q)$.

Figure 3.30 provides an example of a circle composition and a (general) composition process. The patch $Q = \{v_0, v_1, v_2, v_3, v_4\}$ on the left can be composed by the circle subpatch $\mathcal{O}_X = \{v_2, v_3\}$ of Q as illustrated in the top right patch, or can be composed by the subpatch $S = \{v_1, v_2, v_3, v_4\}$ of Q as shown in the bottom right patch.

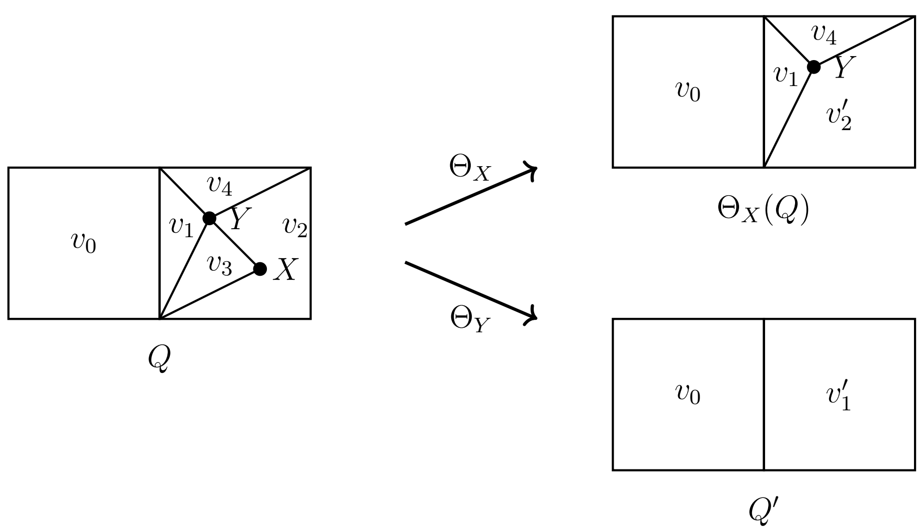


Figure 3.30: Examples for circle composition and (general) composition

Note that circle composition can be applied twice to the patch $Q = \{v_0, v_1, v_2, v_3, v_4\}$

in Figure 3.30, by the vertices X and Y consecutively. More precisely, because $\mathcal{O}_Y = \{v_1, v_2', v_4\}$ is a circle subpatch of $\Theta_X(Q) = \{v_0, v_1, v_2', v_4\}$, we can apply circle composition to $\Theta_X(Q)$ by the vertex Y. Then we get that $\Theta_Y \circ \Theta_X(Q)$ is nothing but Q'. In fact, composing the cyclic patch $S = \{v_1, v_2, v_3, v_4\}$ into a single tile patch $\{v_1'\}$ and composing circle patches $\mathcal{O}_X$ and $\mathcal{O}_Y$ ($\mathcal{O}_X$ as a subcollection in $\{v_1, v_2, v_3, v_4\}$ and $\mathcal{O}_Y$ as a subcollection in $\{v_1, v_2', v_4\}$) consecutively are the same process. The latter is illustrated in Figure 3.31.

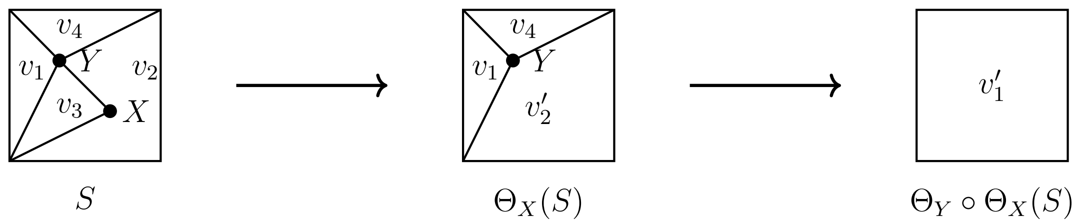


Figure 3.31: Circle compositions applied to $S = \{v_1, v_2, v_3, v_4\}$ by the vertices X and Y consecutively.

Composing cyclic patches into circle patches or single tile patches is essentially the goal of this section. In fact, We will define order structures for cyclic patches through space filling curves that are generated by iterative circle composition steps.

Remark 3.2.4. For a patch Q with an interior vertex Y, the subcollection of all tiles in Q that contains the vertex Y is not necessarily a subpatch of Q. For example the patch $Q = \{t_1, t_2, t_3, t_4\}$ in Figure 3.32 is a patch where Y is an interior vertex of it. The subcollection of tiles in Q whose support intersects with Y is $S_Y = \{t_1, t_2, t_3\}$. Because the support of S_Y is not simply connected, it does not define a subpatch of Q. Therefore, circle composition by the interior vertex Y of Q is not well defined. In fact, the circle composition $\Theta_X(Q)$ for a given patch Q means two things; $\mathcal{O}_X$ is a well defined circle subpatch of Q and $\Theta_X(Q)$ is a circle composition of Q.

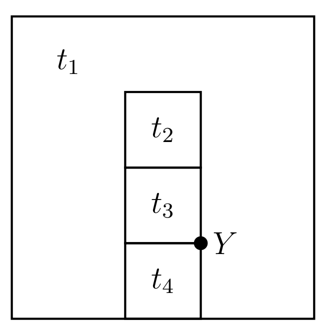


Figure 3.32: The subcollection $S_Y = \{t_1, t_3, t_4\}$ is not a subpatch of the patch $Q = \{t_1, t_2, t_3, t_4\}$.

Lemma 3.2.5. *Let Q be a patch with an interior vertex Y so that $\Theta_Y(Q)$ is a (well-defined) circle composition of Q. Then*

(1) *the edge set of* $\Theta_Y(Q)$ *is a (non-empty) proper subset of the edge set of* Q,

(2) *the vertex set of* $\Theta_Y(Q)$ *is a (non-empty) proper subset of the vertex set of* Q,

(3) *the set of all slice edges of* Q *and the set of all slice edges of* $\Theta_Y(Q)$ *coincide.*

In particular, $\Theta_Y(Q)$ *does not contain the interior vertex* Y *of* Q *and also does not contain edges* $e \in E_Q$ *with* $Y \in e$.

Proof. (1) and (2) follow by the definition of the circle composition process. (3) follows by the fact that slice edges cannot contain any interior vertices and so cannot contain Y. □

Corollary 3.2.6. *If* Q *is a cyclic patch,* X *is an interior vertex of* Q *and* $\Theta_X(Q)$ *is a (well-defined) circle composition of* Q *with no slice tiles, then* $\Theta_X(Q)$ *is a cyclic patch as well.*

Proof. The result follows from conclusion (3) of Lemma 3.2.5 □

Corollary 3.2.7. *Let* Q *be a patch with no slice edges. If* X *is an interior vertex of* Q *such that* $\mathcal{O}_X$ *is a circle subpatch of* Q*, then* $\Theta_X(Q)$ *is a (well-defined) composed patch with no slice edges.*

Proof. The result follows from Lemma 3.2.5. □

Corollary 3.2.8. *Suppose* Q *is a patch,* S *is a subpatch of* Q *and* Q' *is a composition of* Q *that is generated by composing* S *into a single tile. Then* S *is a non-slice subpatch of* Q *if and only if* Q' *is a cyclic patch.*

Proof. By Lemma 3.2.5, Q' does not contain any slice edges. Hence Q' is a cyclic patch if and only if Q' does not contain a slice tile. □

Circle compositions absorb interior vertices. Indeed, the centre of a circle patch disappears in the generated composed patch, as explained in Lemma 3.2.5. In particular, circle compositions absorb all the edges that contain the interior vertex of the composed circle patch. Therefore, considering cyclic patches do not have any slice edges or tiles, it is not a dazzling result that after finitely many circle compositions, cyclic patches form a single tile patch. Though, it is not so straightforward to form such a well-defined circle composition process so that all the composed patches in the middle steps are cyclic, thereby lacking of slice tiles and edges. This is an essential requirement for defining an order system for cyclic patches and it will be explained later in Example 3.2.22.

We will prove that for any given cyclic patch there exists a (finite) sequence of well defined circle compositions that compose the given cyclic patch into a single tile patch. In fact, we will define an algorithm that composes circle patches in a specific order so that by backtracking circle compositions at every step, we will be able to define an order system for cyclic patches. Before exemplifying the algorithm, we define the inverse of the circle composition process and 2-curve decorations of patches that will be needed for the inverse process of circle composition steps.

Definition 3.2.9. Let $Q = \Theta_X(Q')$ be a (well-defined) circle composition of Q' by the vertex X. Then the patch Q' is called the *circle decomposition* of Q and is denoted by $\Theta_X^{-1}(Q)$.

Figure 3.33 shows an example of the process. The circle patch Q with centre X on the left of the Figure is a circle composition of the patch Q' on the right of the Figure. The dashed area on patch Q is decomposed into a circle subpatch $\mathcal{O}_{x'}$ of Q'. In particular, Q' is the circle decomposition of Q.

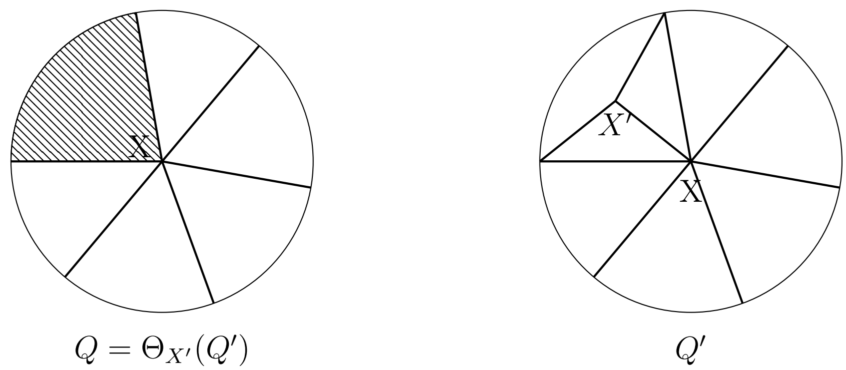


Figure 3.33: Q is a circle patch with centre X and $Q' = \Theta_{X'}^{-1}(Q)$ is the circle decomposition of Q by the vertex X'. Dashed area is decomposed into a circle subpatch $\mathcal{O}_{X'}$ of Q'.

More Decorations

In Section 3.1, we defined an order system for circle patches through valid pairs. If Q is a circle patch and A, B are chosen distinct exterior vertices of Q such that $A \in V_{t_A}$ and $B \in V_{t_B}$ for some distinct tiles $t_A, t_B \in Q$, then (A, B) is a simple valid pair of Q (Proposition 3.1.10). Because (A, B) is a simple valid pair of Q, there exists a curve $\mathcal{C}$ that makes (A, B) a simple valid pair for Q and induces only simple decorations for the tiles of Q. That is, if the chosen exterior vertices A, B are sufficiently 'far away' from each other, then the order system for circle patches only generates decorated tiles with simple decorations.

To avoid using 2-curve decorations for circle patches depends on the positions of the chosen exterior vertices A, B. This essentially follows from the fact that circle patches have a common interior vertex. Unlike in circle patches, cyclic patches do not necessarily contain a common interior vertex. Their geometric framework is, in general, more diverse. For that reason we cannot ensure the same result for cyclic patches. In fact, we cannot avoid using 2-curve decorations for some cyclic patches even if the chosen exterior vertices A, B are 'far away' from each other. We explain this with two examples. The first patch $Q_1 = \{t_A, t_B, u_1, u_2, u_3, u_4\}$ on the left of Figure 3.34 contains distinct exterior vertices A, B. There are unique tiles t_A, t_B that contain the vertices A, B, respectively. These tiles share a common edge. Moreover, $S = \{t_A, t_B\}$ is a slice subpatch of Q_1. Suppose we want to define a curve $\mathcal{C}$ such that $s(\mathcal{C}) = A$, $r(\mathcal{C}) = B$ and $\mathcal{C}$ makes (A, B) a simple valid pair for Q_1. Because A and B are isolated vertices of Q_1, $\mathcal{C}$ must visit t_A first and t_B last. After visiting t_A, $\mathcal{C}$ must visit either u_1 and u_2 or u_3 and u_4. In either choice, in order to visit the other two tiles u_3 and u_4 or u_1 and u_2, $\mathcal{C}$ must pass through either t_A or t_B. For both choices $\mathcal{C}$ will induce a 2-curve decoration either for t_A or t_B. Thus, such a curve $\mathcal{C}$ cannot exits and (A, B) is a non-simple valid pair for Q_1. For the second example, consider the patch Q_2 on the right of Figure 3.34. The patch Q_2 has distinct exterior vertices C, D. There is a unique tile t_C containing the vertex C whereas there are two tiles t_D^1, t_D^2 containing the vertex D. Suppose we want to define a curve $\mathcal{C}'$ such that $s(\mathcal{C}') = C$, $r(\mathcal{C}') = D$ and $\mathcal{C}'$ makes (C, D) a simple valid pair for Q_2. Because C is an isolated vertex, $\mathcal{C}'$ must start at the tile t_C. After visiting the tile t_C, $\mathcal{C}'$ must pass to either v_1 or v_2. For either choice, in order to pass to the other tile and end at t_D^i for some $i = 1, 2$, $\mathcal{C}'$ must double hit (at least) a tile twice. Thus, such a curve $\mathcal{C}'$ cannot exits and (C, D) is a non-simple valid pair for Q_2.

Figure 3.34: Examples of cyclic patches where (A, B) is a non-simple valid pair

In the first example we have that the tiles containing A and B are 'too close' to each other. In particular, $S = \{t_A, t_B\}$ is a slice subpatch of Q_1. That is the reason why (A, B) is a not a simple valid pair for Q_1. In the second example we have that the tiles containing C and D are 'far away' from each other. However, the position of the tile t_C means that (C, D) cannot be a simple valid pair for Q_2. Thus, unlike in circle patches, the decorations of cyclic patches are not solely dependent on the positions of the chosen exterior vertices, but also depends on the geometric locations of its tiles.

Valid pairs in cyclic patches might induce 2-curve decorations for the tiles as explained above. In order to construct a substitution rule for a decorated prototile collection (as in Theorem 3.1.13), we must also be able to define a substitution rule for the tiles that have 2-curve decorations. For that, we define 2-curve decorations of patches, that are called split pairs (Definition 3.2.11).

Definition 3.2.10. Suppose Q is a patch and u is a tile such that the single tile patch $\{u\}$ is a composition of Q. Then we call the pair (Q, u) as a *composition pair*. It is called a *circle composition pair* if Q is a circle patch.

Definition 3.2.11. Let (Q, u) be a composition pair and A, B, C, D be a distinct exterior vertices of u such that e_u is a 2-curve decoration of u with component curves e_u^1, e_u^2 so that $s(e_u^1) = A, r(e_u^1) = B$ and $s(e_u^2) = C, r(e_u^2) = D$. If there exist (not necessarily disjoint) subpatches Q_1, Q_2 of Q such that

(1) $Q_1 \cup Q_2 = Q$,

(2) (A, B) is a valid pair for Q_1 and (C, D) is a valid pair for Q_2 by some curves $\mathcal{C}^{A,B}$ and $\mathcal{C}^{C,D}$, respectively, so that $\mathcal{C}^{A,B}$ and $\mathcal{C}^{C,D}$ do not cross each other,

(3) $\overline{(\mathcal{C}^{A,B} \cup \mathcal{C}^{C,D})} \cap \operatorname{int} \operatorname{supp} t$ is a decoration for t, for each $t \in Q$.

Then we say $\{(A, B), (C, D)\}$ is a *split pair* for Q.

If (A, B) and (C, D) are simple valid pairs for Q_1 and Q_2, respectively, and $Q_1 \cap Q_2 = \emptyset$ then we say $\{(A, B), (C, D)\}$ is a *simple split pair* for Q. We say $\{(A, B), (C, D)\}$ is a *non-simple split pair* otherwise.

Note that condition (3) ensures that each tile in the patch has to be visited at most twice. Note also that, $\{(A, B), (C, D)\}$ is a simple split pair for Q if and only if every tile in Q is visited once and only once.

Observe that $\{(A, B), (C, D)\}$ on the left of Figure 3.35 is an example of a simple split pair whereas $\{(A', B'), (C', D')\}$ on the right of Figure 3.35 is an example of a non-simple split pair.

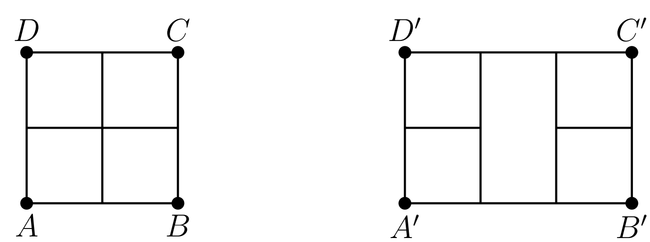
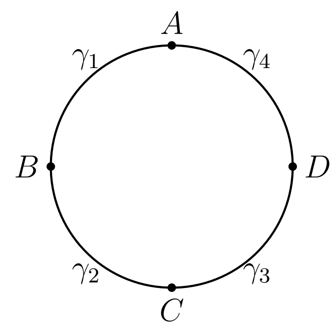


Figure 3.35: Examples of simple split pair and non-simple split pair

We study the split order framework for circle patches before delving into the order systems of cyclic patches.

Proposition 3.2.12. *Let (Q, u) be a circle composition pair and let e_u be a decoration of u with component curves e_u^1, e_u^2 so that $s(e_u^1) = A, r(e_u^1) = B$ and $s(e_u^2) = C, r(e_u^2) = D$. Then $\{(A, B), (C, D)\}$ is a split pair for Q if and only if there are (at least) two distinct tiles t_1, t_2 of Q such that $V_{t_i} \cap \{A, B, C, D\} \neq \emptyset$ for $i = 1, 2$.*

Proof. Assume that (Q, u) is a circle composition pair, X is the centre of Q and e_u is a decoration of u with components e_u^1, e_u^2 so that $s(e_u^1) = A, r(e_u^1) = B$ and $s(e_u^2) = C, r(e_u^2) = D$. Assume further without loss of generality that A, B, C, D are positioned so that we can define arcs $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \subseteq \partial \text{supp } Q$ such that $\gamma_1 \cap \gamma_2 = \{B\}$, $\gamma_2 \cap \gamma_3 = \{C\}$, $\gamma_3 \cap \gamma_4 = \{D\}$, $\gamma_4 \cap \gamma_1 = \{A\}$ and $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 = \partial \text{supp } Q$.



($\Longrightarrow$) Suppose first there are no distinct tiles t_1, t_2 such that $V_{t_i} \cap \{A, B, C, D\} \neq \emptyset$ for $i = 1, 2$. Then A, B, C, D are all isolated vertices of Q that belong to the same tile t_0 of Q. Because all vertices are isolated vertices, in order to visit every tile in Q, curves must visit the tile t_0 at least three times. Thus, $\{(A, B), (C, D)\}$ is not a split pair for Q.

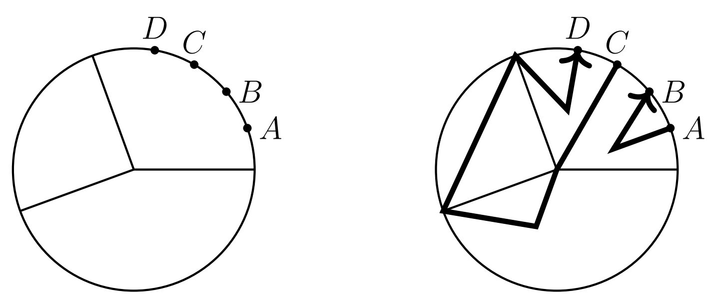
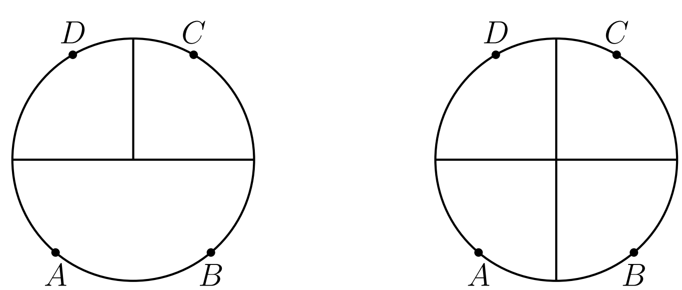


Figure 3.36: $\{(A, B), (C, D)\}$ is not a split pair

($\Longleftarrow$) Suppose now that there are two distinct tiles t_1, t_2 such that $V_{t_i} \cap \{A, B, C, D\} \neq \emptyset$ for $i = 1, 2$. We have two cases:

(i) A and B (or C and D) are isolated vertices of Q that belong to the same tile,

(ii) There exist tiles $t_A, t_B, t_C, t_D \in Q$ such that $t_A \neq t_B$, $t_C \neq t_D$ and $i \in V_{t_i}$ for $i \in \{A, B, C, D\}$.



Case (i) Case (ii)

Case (i): Suppose A and B are isolated vertices of Q that belong to a same tile t_{AB}. Either C or D (or both) must belong to a tile other than t_{AB}. We have two cases in the light of Proposition 3.1.10; either (C, D) is a simple valid pair for Q, or (C, D) is a non-simple valid pair for Q. For both cases, the curves that make (C, D) a valid pair for Q visit the tile t_{AB} once. Define $Q_1 = \{t_{AB}\}$ and $Q_2 = Q$. Then (A, B) is a valid pair for Q_1 and (C, D) is a valid pair for Q_2 by Proposition 3.1.10. Moreover, the curves that makes (A, B) and (C, D) a valid pair for Q_1 and Q_2, respectively, visit every tile at most twice and can be arranged to not cross each other. Thus, $\{(A, B), (C, D)\}$ is a split pair for Q.

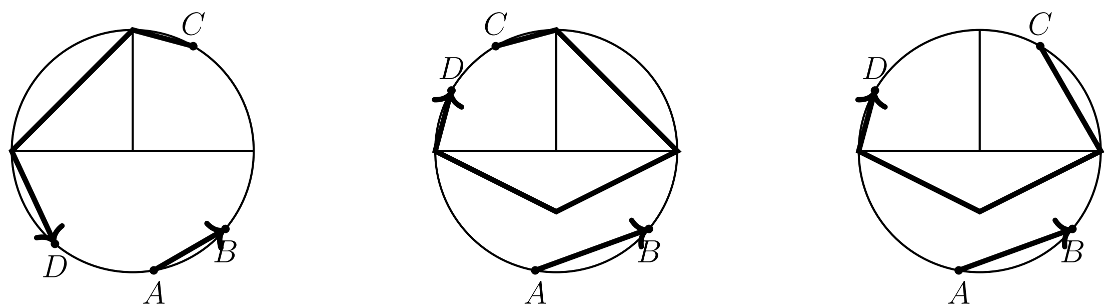


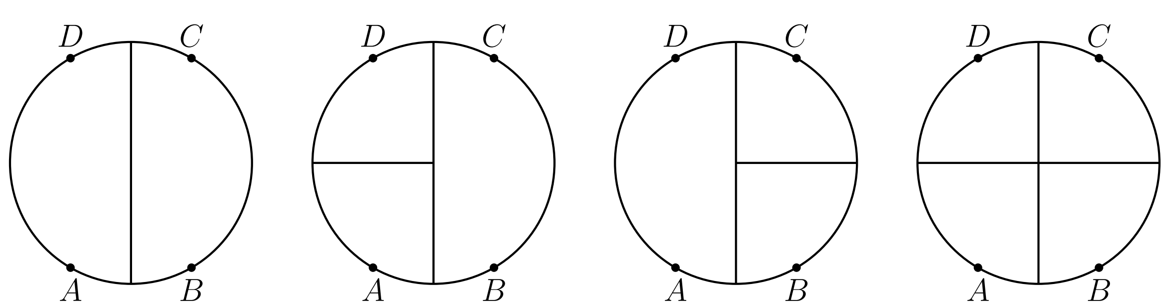
Figure 3.37: Illustration of the proof of Case (i) with some examples

Case (ii): This case has three subcases.

(ii-a) A, B, C, D are all isolated vertices of Q such that there exists a unique tile $t_{BC} \in Q$ containing the vertices B, C and there exists a unique tile $t_{AD} \in Q$ that contains the vertices A, D (i.e, B and C are isolated vertices belonging to a same tile in Q, and A and D are isolated vertices belonging to a same tile in Q).

(ii-b) B, C are isolated vertices of Q that belong to a same tile $t_{BC} \in Q$, and there are distinct tiles $t_A, t_D \in Q$ such that $A \in V_{t_A}$ and $D \in V_{t_D}$.

(ii-c) There exist distinct tiles $t_B, t_C \in Q$ such that $B \in V_{t_B}$ and $C \in V_{t_C}$, and there exist tiles $t_A, t_D \in Q$ such that $t_A \neq t_D$, $A \in V_{t_A}$ and $D \in V_{t_D}$



Case-$(ii - a)$ Case-$(ii - b)$ Case-$(ii - b)$(Symmetric) Case-$(ii - c)$

Case (ii-a): Define Q_1 to be the subpatch generated by γ_1 and Q_2 to be the subpatch generated by γ_3. We have that A, D and B, C are isolated vertices belonging to the same tiles, respectively. Because A and B belong to different tiles, (A, B) is a simple valid pair for Q_1 by Lemma 3.1.6. Similarly, (C, D) is a simple valid pair for Q_2, by Lemma 3.1.6 as well. Since A and D, and B and C belong to the same tiles as isolated vertices, $Q_1 \cup Q_2 = Q$. Moreover, the curves that make (A, B) and (C, D) simple valid pairs for Q_1 and Q_2, respectively, can be arranged to be non-crossing each other. Therefore, $\{(A, B), (C, D)\}$ is a split pair for Q.

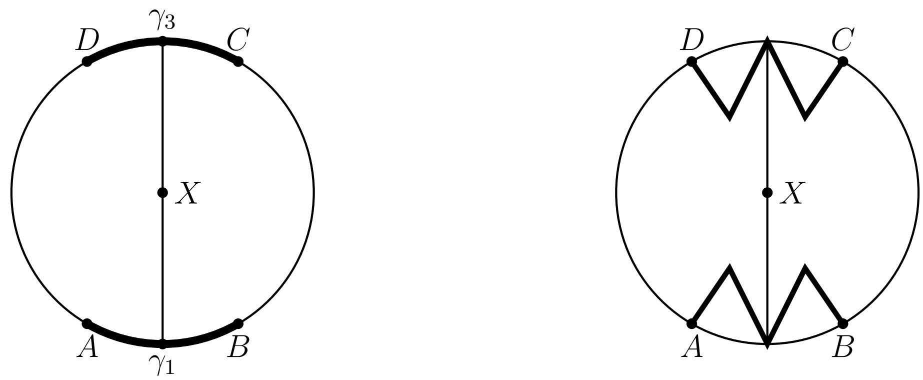


Figure 3.38: An example of Case (ii-a)

Case (ii-b): Define Q_2 to be a subpatch generated by the arc γ_3, as in the Case (ii-a). By the same reasoning in Case (ii-a), (C, D) is a simple valid pair for Q_2. Define further P_1 to be the subpatch generated by γ_1 and Q_1^1 to be a subpatch as follows:

$$Q_1^1 = \begin{cases} P_1 & \textit{if } A \textit{ is a shared exterior vertex of } Q \\ P_1 \backslash \{t_A\} & \textit{if } t_A \textit{ is the only tile in } Q \textit{ containing } A. \end{cases}$$

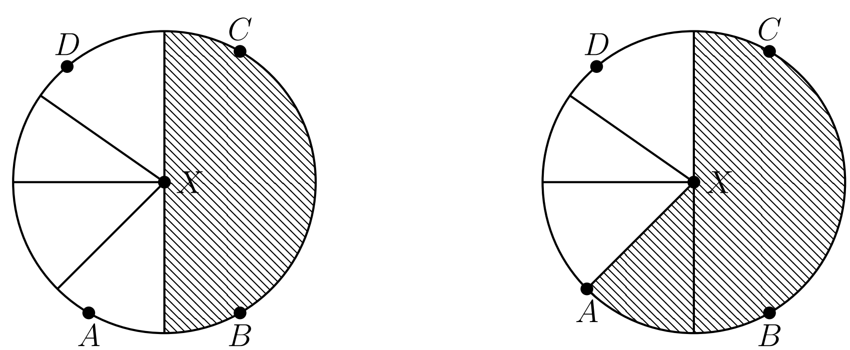


Figure 3.39: Q_1^1 is highlighted for both cases

Note that $Q_1^1 \neq \emptyset$, since the unique tile t_{BC} containing the vertices B, C belongs to Q_1^1. We have either $Q_1^1 \cup Q_2 = Q$ or $Q_1^1 \cup Q_2 \subsetneq Q$.

Suppose first $Q_1^1 \cup Q_2 = Q$. Then A must be a shared exterior of Q in order that $Q_1^1 \cup Q_2 = Q$ holds. Therefore, (A, B) is a simple valid pair for Q_1^1 by Lemma 3.1.6, because γ_1 has end points A, B and every tile in Q_1^1 contains an edge which is completely contained in the curve γ_1. We get that (A, B) is a simple valid pair for Q_1^1, (C, D) is a simple valid pair for Q_2 and $Q_1^1 \cup Q_2 = Q$. Thus, $\{(A, B), (C, D)\}$ is a split pair for Q.

Suppose now $Q_1^1 \cup Q_2 \subsetneq Q$. By construction of Q_1^1, there is an arc over the boundary ∂supp Q_1^1 with end points B and X such that every tile in Q_1^1 has an edge completely contained in it. Existence of such arcs (for both cases) is demonstrated in Figure 3.40 (for both cases in in Figure 3.39).

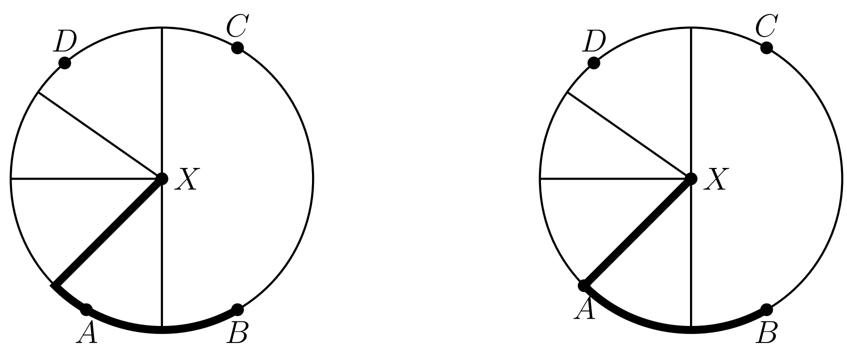


Figure 3.40: (X, B) is a simple valid pair for Q_1^1 (for both cases)

Therefore, (X, B) is a simple valid pair for Q_1^1 by Lemma 3.1.6. Define $Q_1^2 = Q \backslash (Q_1^1 \cup Q_2)$. Q_1^2 is non-empty since $Q_1^1 \cup Q_2 \neq Q$. In addition, by the construction, there exists an arc with end points A, X over the boundary ∂supp Q_1^2 such that every tile in Q_1^2 has an edge contained in it, as illustrated in Figure 3.42 (for both cases shown in Figure 3.41). In particular, (A, X) is a simple valid pair for Q_1^2 by Lemma 3.1.6, because X is a shared exterior vertex of Q_1^2 whenever Q_1^2 contains at least two tiles. So, (A, B) is a simple valid pair for $Q_1 = Q_1^1 \cup Q_1^2$. Furthermore, the curves that makes (A, B) and (C, D) simple valid pairs for Q_1 and Q_2, respectively, can be arranged to be non-crossing. Thus, $\{(A, B), (C, D)\}$ is a split pair for Q.

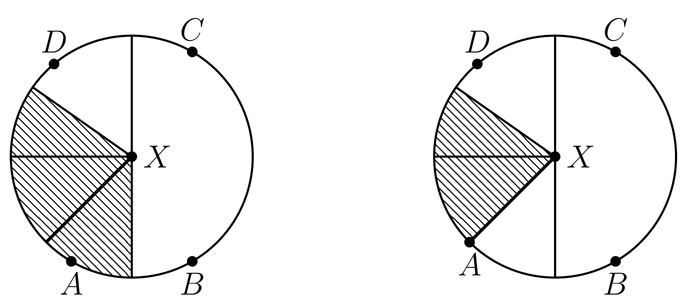


Figure 3.41: Q_1^2 is highlighted for both cases whether A is an isolated vertex or not

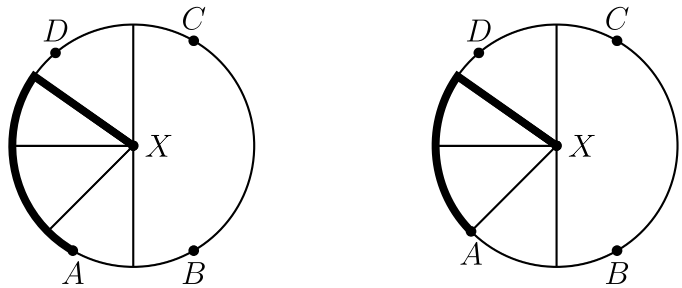


Figure 3.42: (A, X) is a simple valid pair for Q_1^2 for both cases

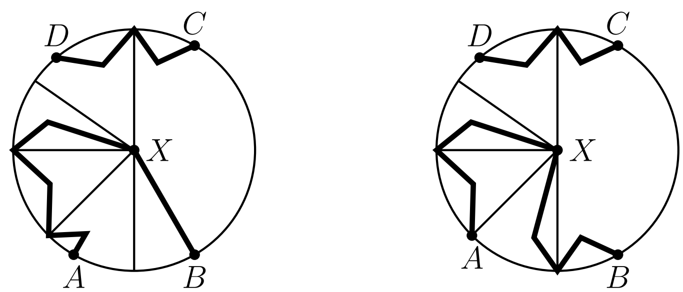


Figure 3.43: An example of Case (ii-b)

Case (ii-c): Define P_1 to be the subpatch generated by $\gamma_1 \cup \gamma_4$ and define Q_1 to be a patch as follows:

$$Q_1 = \begin{cases} P_1 & if\ D\ is\ a\ shared\ exterior\ vertex\ of\ Q \\ P_1 \backslash \{t_D\} & if\ t_D\ is\ the\ only\ tile\ in\ Q\ containing\ D. \end{cases}$$

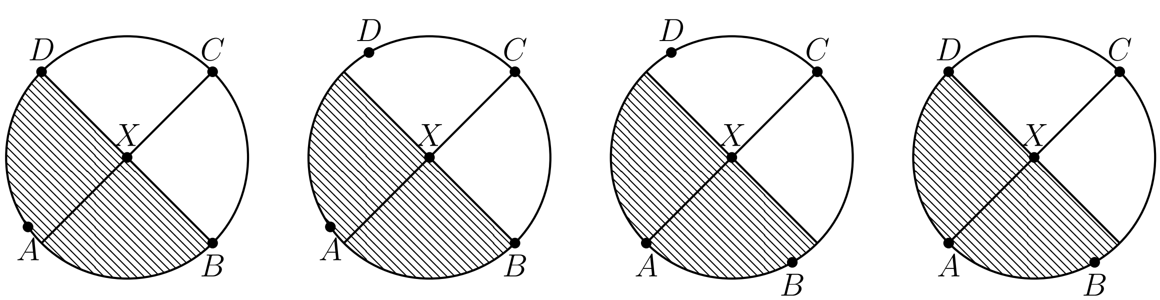


Figure 3.44: Q_1 is highlighted for different cases

Similarly, define P_2 to be the subpatch generated by $\gamma_2 \cup \gamma_3$ and define Q_2 to be a patch as follows:

$$Q_2 = \begin{cases} P_2 & if\ B\ is\ a\ shared\ exterior\ vertex\ of\ Q \\ P_2 \backslash \{t_B\} & if\ t_B\ is\ the\ only\ tile\ in\ Q\ containing\ B. \end{cases}$$

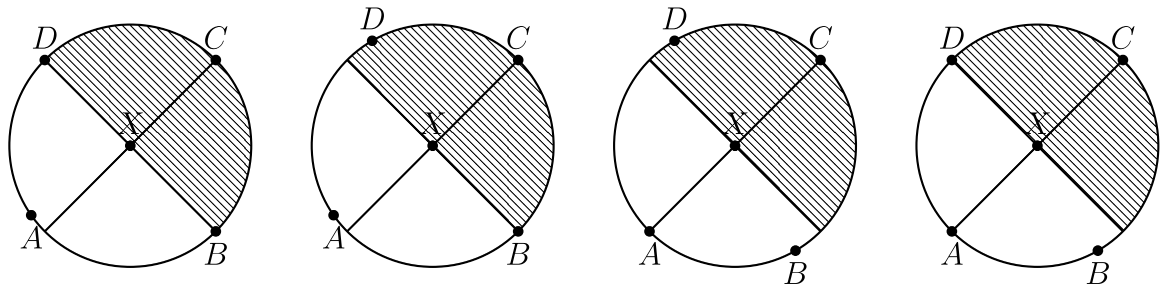


Figure 3.45: Q_2 is highlighted for different cases

By construction, $Q_1 \cap Q_2 = \emptyset$ and $Q_1 \cup Q_2 = Q$. The possible cases for Q_1 and Q_2 are shown in Figure 3.46 and illustrated in Figure 3.44 and Figure 3.45.

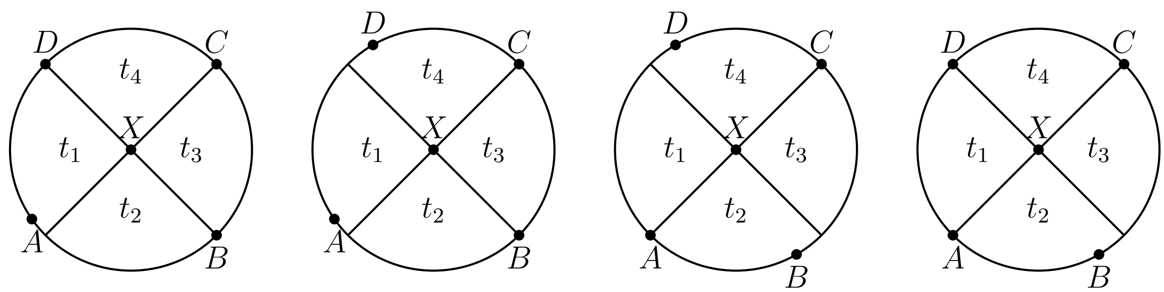


Figure 3.46

We first show that (A, B) is a simple valid pair for Q_1. Let $u_A, u_B \in Q_1$ be distinct tiles in Q_1 such that $A \in V_{u_A}$ and $B \in V_{u_B}$ (such tiles exists since we are proving Case (ii)). Let R_1 denotes the subpatch generated by γ_1. Define a patch S_1 as follows:

$$S_1 = \begin{cases} R_1 & \textit{if } A \textit{ is a shared exterior vertex of } Q \\ R_1 \backslash \{u_A\} & \textit{if } u_A \textit{ is the only tile in } Q \textit{ containing } A. \end{cases}$$

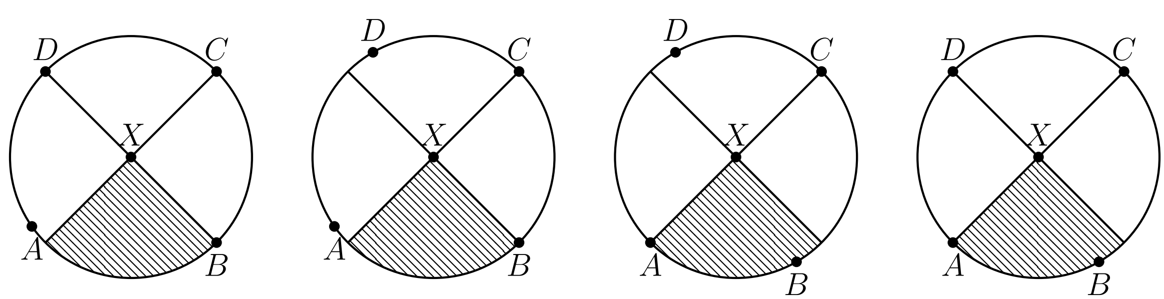


Figure 3.47: S_1 is highlighted for the different cases in Figure 3.46

We have that B is an isolated vertex for Q_1. Therefore, S_1 is a non-empty subset of Q_1 since it contains a tile which has a vertex B and different than u_A. Moreover, by construction of S_1, there is an arc over the boundary $\partial\text{supp}\ S_1$ with end points B and X such that every tile in S_1 has an edge completely contained in it. That is, (X, B) is simple a valid pair for S_1 by Lemma 3.1.6. By the same token, (A, X) is a simple valid pair for $S_2 = Q_1 \backslash S_1$ as illustrated in Figure 3.49.

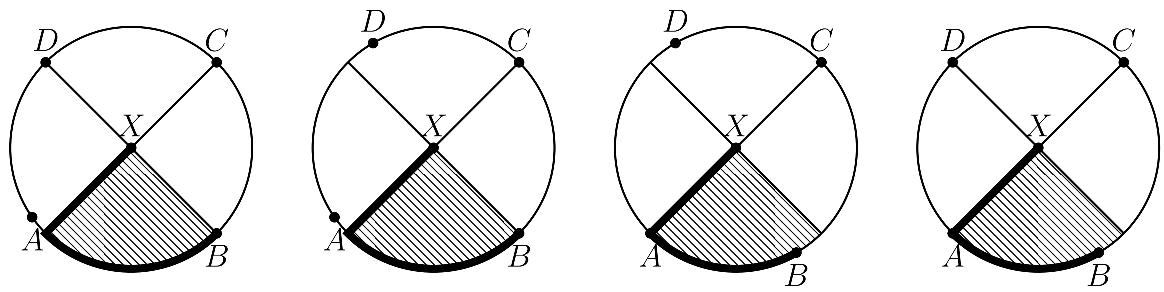


Figure 3.48: (X, B) is a simple valid pair for S_1

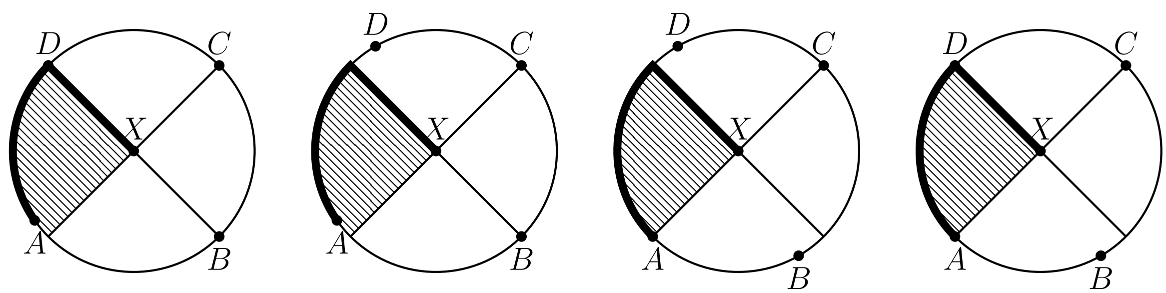


Figure 3.49: (A, X) is a simple valid pair for $S_2 = Q_1 \backslash S_1$

Since S_1 and S_2 are disjoint, (A, B) is a simple valid pair for Q_1. Similarly, (C, D) is a simple valid pair for Q_2. Thus, $\{(A, B), (C, D)\}$ is a (simple) split pair for Q. □

Corollary 3.2.13. *Let (Q, u) be a circle composition pair and e_u be a decoration of u with component curves e_u^1, e_u^2 so that $s(e_u^1) = A, r(e_u^1) = B$ and $s(e_u^2) = C, r(e_u^2) = D$. Then $\{(A, B), (C, D)\}$ is a split pair for Q whenever at least one of A, B, C, D is a shared exterior vertex of Q.*

Convexity Assumption

We have not made any assumption about the tiles in circle patches to define an order system for circle patches. The order system for circle patches only depends on the positioning of its exterior vertices (Proposition 3.1.10). However, this is not the case for cyclic patches. The order system of cyclic patches also depends on the geometric permutation of tiles. We showed two examples in Figure 3.34 such that the geometric locations of tiles in the patches require 2-curve decorations. In particular, convexity of tiles is an essential assumption for defining an order system for cyclic patches. Even though the convexity assumption of tiles may become non-valid after circle compositions applied to a given cyclic patch, it affects the combinatorics of the given cyclic patch by avoiding patches as in Figure 3.50. The patch in the figure has no order structure for any given distinct pair of exterior vertices. For example, (A, B) is not a valid pair for the patch, because there is no curve with end points A, B and visiting the tiles at most twice. Furthermore, we show in Lemma 3.2.14 that convexity also ensures that for any interior vertex X of a given cyclic patch Q, the collection $S_X = \{t \in Q : X \in V_t\}$ is a circle subpatch of Q. This is not true in general without assuming convexity.

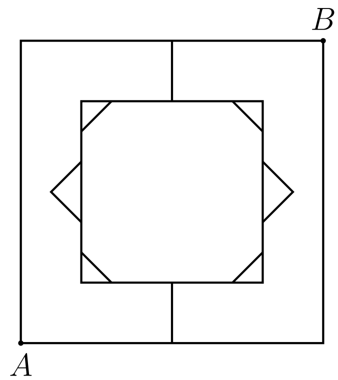


Figure 3.50: (A, B) is not a valid pair.

Lemma 3.2.14. *Let Q be a patch whose tiles are convex. Then the subcollection of all tiles containing the vertex X, for any given interior vertex X of Q, is a circle subpatch of Q.*

Proof. Let S_X denotes the subcollection of tiles in Q that contains the vertex X. Assume without loss of generality, the following holds:

(1) $S_X = \{t_1, \ldots, t_n : n \in \{3, 4, \ldots\}\}$,

(2) $e_1, \ldots, e_n$ are the only edges in Q such that $X \in e_i$ for $i \in \{1, \ldots, n\}$,

(3) $e_i \in V_{t_i} \cap V_{t_{i+1}}$ for $i \in \{1, \ldots, n-1\}$ and $e_n \in V_{t_1} \cap V_{t_n}$.

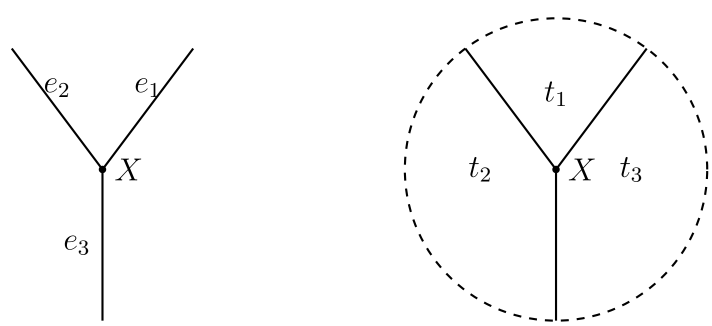


Figure 3.51: Assumptions demonstrated for the case $n = 3$.

Since supp t_i for $i = 1, \ldots, n$ are convex, supp S_X must be simply connected and every tile t_i for $i = 1, \ldots, n$ must contain an edge over the boundary ∂supp S_X. Hence, S_X is a circle subpatch of Q. □

Lemma 3.2.15. *Let Q be a patch of convex tiles. If tiles of Q do not intersect along a single edge, then there exists a patch Q' of convex tiles that is generated by deleting some of the interior vertices of Q, and its tiles intersect along a single edge.*

Proof. Let $t_1, t_2 \in Q$ be tiles that meet along more than one edge. Because both supp t_1 and supp t_2 are convex sets, ∂supp $t_1 \cap \partial$supp t_2 must be a straight line, which is a union of edges of Q. Delete all the interior vertices of Q over that line (except the end points of the line) in order to make the line a single edge that is shared by t_1 and t_2.

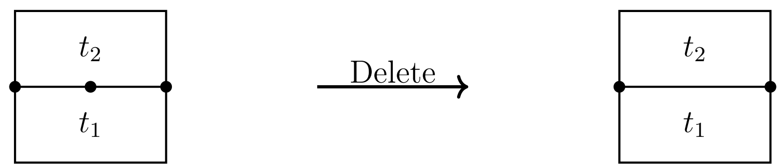


Figure 3.52: Deleting the vertex in the middle

Apply the same process for each non-singly edge-to-edge neighbour tiles. Then the generated patch Q' is the desired patch of convex tiles. □

Corollary 3.2.16. *With the same notation as in Lemma 3.2.15, suppose u is a tile so that (Q, u) is a composition pair.*

(1) *If e_u is a simple decoration of u with end points A, B, then (A, B) is a valid pair for Q whenever it is a valid pair for Q'. Moreover, (A, B) is a simple valid pair for Q whenever it is a simple valid pair for Q'.*

(2) *If e_u is a 2-curve decoration of u with end point pairs A, B and C, D, then $\{(A, B), (C, D)\}$ is a split pair for Q whenever it is a split pair for Q'. Moreover, $\{(A, B), (C, D)\}$ is a simple split pair for Q whenever it is a simple split pair for Q'.*

Proof. Notice that Q' consists of same tiles as Q with a smaller vertex set. Hence, any (valid) decoration on a tile of Q' is also a decoration on the same tile of Q. □

An Example of a Circle Composition Process

Consider the cyclic patch Q on the top left corner of Figure 3.53. We can apply circle compositions by the vertices X_1, X_2 and X_3 consecutively as illustrated in Figure 3.53. After three successive circle compositions, we arrive at the single tile patch $Q' = \Theta_{X_3} \circ \Theta_{X_2} \circ \Theta_{X_1}(Q)$. Next, we will reverse the circle composition steps. We will implement the order structure of circle patches in each (reverse) step. Obviously, (A, B) is a valid pair for the single tile patch Q' as illustrated on the lower right corner of Figure 3.54. Applying a circle decomposition by the vertex X_3, we arrive the circle patch $\Theta_{X_3}^{-1}(Q') = \Theta_{X_2} \circ \Theta_{X_1}(Q)$. We have that (A, B) is a valid pair for the circle patch $\Theta_{X_3}^{-1}(Q')$, by Proposition 3.1.10. A curve that makes (A, B) a valid pair for $\Theta_{X_3}^{-1}(Q')$ is demonstrated in the lower left corner of Figure 3.54. Next we circle decompose the shaded tile on the lower left corner of Figure 3.54. The tile is decomposed into a circle patch with centre X_2, in the upper right corner of the figure. This circle patch has a split pair $\{(A, A'), (B, B')\}$, by Proposition 3.2.12. Therefore, we can reform the decoration of $\Theta_{X_3}^{-1}(Q)$ and define a decoration for $\Theta_{X_2}^{-1} \circ \Theta_{X_3}^{-1}(Q)$. This decoration is presented in the upper right corner of the figure. Similarly, the dashed tile on the top right patch of the figure is decomposed into a circle patch. We apply the order structure of circle patches one more time. We arrive the decoration shown at the top left corner of Figure 3.54. This is a typical example of how circle composition/decomposition gives rise to an order system in cyclic patches by means of defining an order framework by circle compositions.

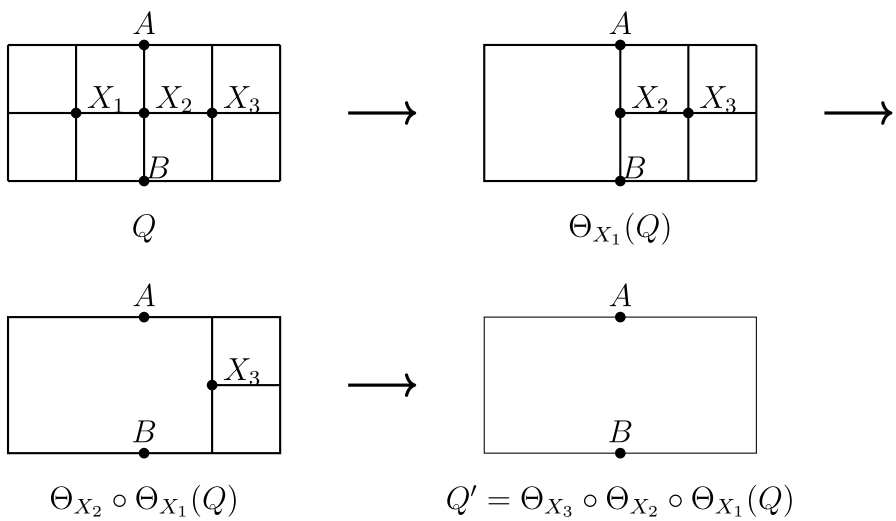


Figure 3.53: An example of a circle composition of a cyclic patch

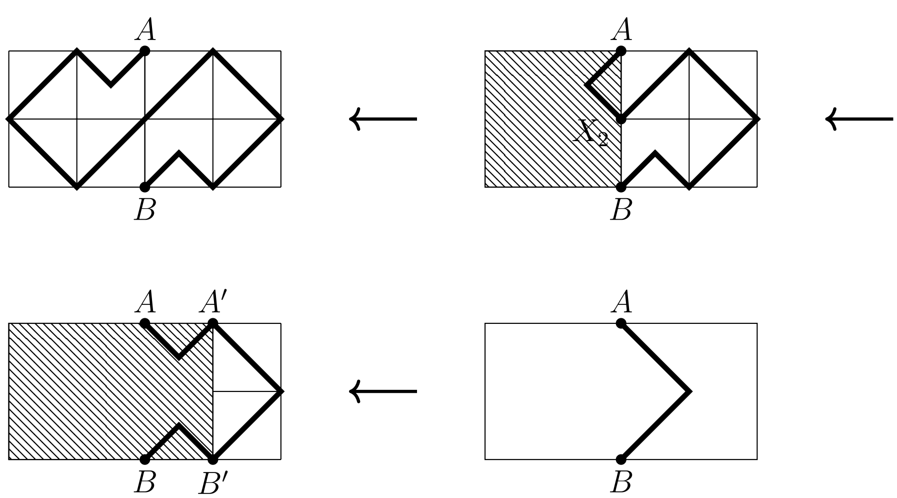


Figure 3.54: An example of a circle decomposition process

3.2.2 Geometry of Cyclic Patches

Both circle patches and cyclic patches lack slice tiles and slice edges. Therefore, they have a similar property of connectedness of their tiles. However, while tiles of circle patches are connected by a common interior vertex, tiles of cyclic patches do not necessarily have a common interior vertex. For that we first examine the combinatorics of cyclic patches.

Lemma 3.2.17. *Let Q be a singly edge-to-edge cyclic patch. Then it contains an interior edge if and only if it is not a circle patch.*

Proof. If Q is a circle patch and its tiles meet singly edge-to-edge, then it contains a unique interior vertex. Thus, it does not contain an interior edge.

If Q is not a circle patch and its tiles meet singly edge-to-edge, then there must exists at least two interior vertices. Because Q does not contain any slice edges, there must exists an interior edge of Q. □

Lemma 3.2.17 ensures that the set of interior edges of a cyclic patch that is not a circle patch and whose tiles meet singly edge-to-edge, is non-empty. We will characterise the geometric structure of cyclic patches through their interior edges.

Definition 3.2.18. For a given patch Q with distinct interior vertices X, Y, we say $(X : Y)$ is a *connected pair* of Q, if there exists a finite sequence of interior edges $\{[Z_iZ_{i+1}]\}_{i=1}^{k}$ such that $\gamma = \bigcup_{i=1}^{k}[Z_iZ_{i+1}]$ is a simple curve with end points X and Y. It is called a *disconnected pair* otherwise.

Lemma 3.2.19. *A patch Q is a cyclic patch if and only if Q does not contain any slice edges and $(X : Y)$ is a connected pair for any distinct interior vertices X, Y of Q.*

Proof. Cyclic patches do not contain any slice tiles or edges. Thus, $(X : Y)$ is a connected pair for any distinct interior vertices X, Y of a given cyclic patch.

On the other hand, suppose Q is a patch without a slice edge and $(X : Y)$ is a connected pair for any distinct interior vertices X, Y of Q. In order for Q to be a non-cyclic patch, Q must contain a slice tile t that does not contain any slice edges, as in Figure 3.55.

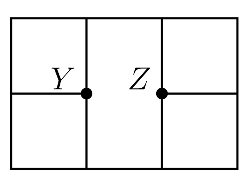


Figure 3.55

Since t does not contain any slice edges and is a slice tile of Q, there must exist two distinct interior vertices Y, Z of Q such that $Y, Z \in V_t$ and $(Y : Z)$ is a disconnected pair of Q, as illustrated in Figure 3.55. This is a contradiction. Hence, Q must be a cyclic patch. □

Lemma 3.2.20. *Suppose Q is a cyclic patch, S is a subpatch of Q and Q' is a composition of Q that is constructed by composing S into a single tile. Then S is a slice subpatch of Q if and only if there exist distinct interior vertices X, Y of Q such that $(X : Y)$ is a disconnected pair of Q'.*

Proof. We have by Corollary 3.2.8 that S is a slice subpatch of Q if and only if Q' is a non-cyclic patch. By Lemma 3.2.5, Q' cannot contain any slice edges. Therefore, by Lemma 3.2.19, Q' is a non-cyclic patch if and only if there exist distinct interior vertices X, Y of Q such that $(X : Y)$ is a disconnected pair of Q. Hence, S is a slice subpatch of Q if and only if there exist distinct interior vertices X, Y of Q such that $(X : Y)$ is a disconnected pair of Q'. □

Lemma 3.2.21. *Let Q be a cyclic patch with an interior vertex X and let $\Theta_X(Q)$ be a circle composition of Q with distinct interior vertices Y, Z such that $Y \neq X$ and $Z \neq X$. Then $(Y : Z)$ is a disconnected pair of $\Theta_X(Q)$ if and only if for all arcs γ which are the union of interior edges of Q, and connect Y and Z in Q, we have $X \in \gamma$.*

Proof. In order for $(Y : Z)$ to be a disconnected pair of $\Theta_X(Q)$, $\mathcal{O}_X$ must separate the vertices Y and Z in the patch Q, by Lemma 3.2.20. Therefore, any arc that is a finite union of interior edges of Q and connect Y and Z in Q must pass through the vertex X.

For the converse, suppose every arc that is a finite union of interior edges of Q and connects Y and Z in Q pass through the vertex X. Then $(Y : Z)$ is a disconnected pair of $\Theta_X(Q)$ since $X \notin V_{\Theta_X(Q)}$. □

Lemma 3.2.20 and Lemma 3.2.21 identify slice circle patches within cyclic patches. We will show with an example why detecting slice subpatches that appear in circle composition steps is crucial.

Example 3.2.22. Consider the cyclic patch on the top left corner of Figure 3.53 (and Figure 3.56). Apply circle compositions by the vertices X_2, X_1, X_3, respectively, as demonstrated in Figure 3.56. We apply circle decomposition steps as in Figure 3.54. However, we fail to create the order structure in one of the middle steps since we generated a slice tile in the middle of the patch. Again, the step illustrated on the top right corner of Figure 3.57 does not impose an order structure through decorated curves. This is due to the fact that $\{(A, A'), (B, B')\}$ is a non-split pair for the dashed circle subpatch shown in the top right corner of Figure 3.57.

Note that applying the first circle composition step by the vertex X_2 in Figure 3.56 made $(X_1 : X_3)$ a disconnected pair in the composed patch, since $\mathcal{O}_{X_2}$ is a slice subpatch. Therefore, it is essential that we avoid slice tiles while circle composing cyclic patches.

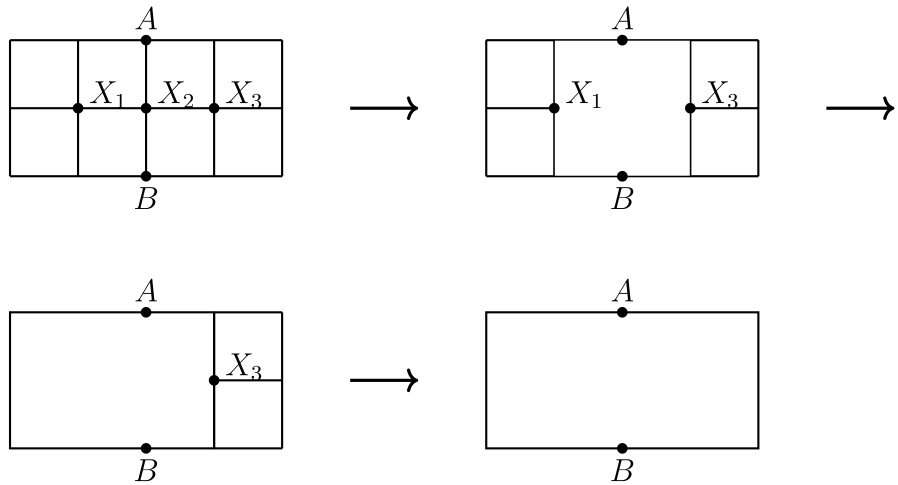


Figure 3.56: A false circle composition process

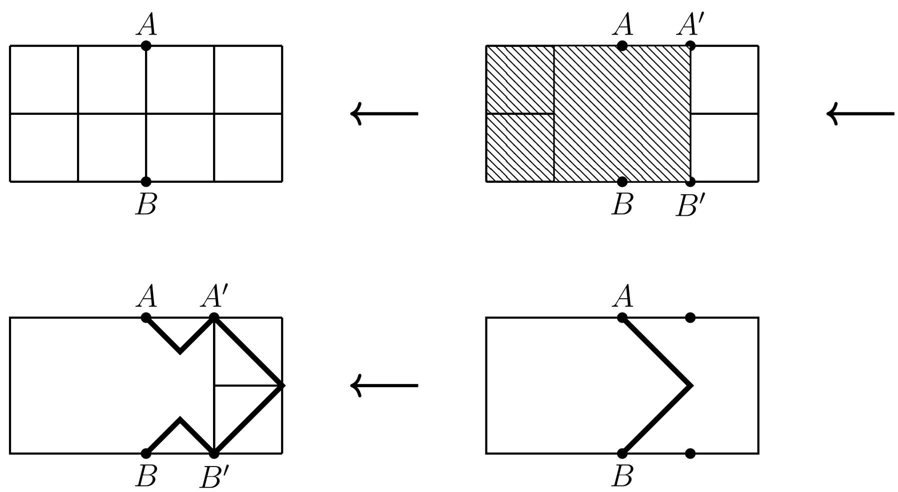


Figure 3.57: Problem with the middle step on the top right corner

3.2.3 A Circle Composition Algorithm for Cyclic Patches

We define an algorithm to circle compose cyclic patches such that no non-cyclic patch can be generated in the middle steps. First, we give some basic definitions from graph theory.

A (undirected) *graph* is a pair consisting of a vertex set and an edge set. All the graphs we consider are finite (undirected) graphs; i.e. the vertex and edge sets are finite. A *subgraph* of a given graph is a pair consisting of a vertex set and an edge set that are subsets of the vertex and edge sets of the given graph, respectively. A graph is called *connected* if any two vertices can be connected through finitely many edges from the edge set of the graph. A connected graph is called a *tree* if it does not contain any closed loop. A *subtree* of a given tree is a connected subgraph of the given tree. A *spanning tree* of a given connected graph is a tree that has the same vertex set of the given connected graph.

For any graph $G = (V, E)$ with a vertex $v \in V$, the *degree* of v is the number of edges in E that intersects with v.

Lemma 3.2.23 ([17], Proposition 4.2.1). *A graph is a tree if and only if every pair of vertices are connected through a unique sequence of edges of the graph.*

Lemma 3.2.24 ([17], Proposition 4.2.3). *Any tree that has at least one edge contains at least two vertices of degree* 1*.*

Lemma 3.2.25 ([6], Theorem 4.6). *For any given graph G, the following are equivalent:*

(1) *G is connected,*

(2) *G contains a spanning tree H.*

Let Q be a given cyclic patch which is not a circle patch and let $G = (V_0, E_0)$ denote an undirected graph with vertex set V_0 and edge set E_0 such that V_0 is the collection of interior vertices of Q and E_0 is the collection of interior edges of Q. By Lemma 3.2.17, E_0 and V_0 are non-empty sets. Moreover, G is a connected graph since any interior vertex pair of Q is a connected pair of Q, by Lemma 3.2.19. Therefore, there exists a spanning tree H of G, by Lemma 3.2.25. In fact, we have the following lemma.

Lemma 3.2.26 (Combinatorics of Cyclic Patches). *A patch Q is a cyclic patch if and only if there exists a tree $H = (V, E)$ such that V is the collection of all interior vertices of Q and E is a collection of interior edges of Q.*

Proof. If Q is a cyclic patch, then there exists such a tree by the argument given immediately before the lemma. Conversely, if there is such a tree H whose vertex set is the collection of all interior vertices of the given patch Q, then Q is a cyclic patch by Lemma 3.2.19. □

Corollary 3.2.27. *Let Q be any given patch such that* $\text{supp}\, t \cap \partial \text{supp}\, Q \neq \emptyset$ *for each $t \in Q$. Then Q is a cyclic patch if and only if there exists a unique tree $H = (V, E)$ such that V is the collection of all interior vertices of Q and E is a collection of interior edges of Q.*

Proof. Since every tile intersects the boundary of the patch, the graph $G = (V, E)$ where V is the set of interior vertices of Q and E is the set of interior edges of Q cannot contain any cycles. So, G is a tree. □

The circle composition process of cyclic patches consists of two algorithms. We explain these two algorithms before delving into details. Suppose Q is a given cyclic patch. We first apply a sequence of circle compositions to Q until it is eventually composed to a single tile patch. This circle composition steps are followed through an algorithm called the *tree generator algorithm*. The circle composition steps in the tree generator algorithm may generate non-cyclic patches in the middle steps of the circle composition process. Therefore, the circle composition of Q by the tree generator algorithm does not form the circle composition steps we need. We apply this algorithm to generate a tree from any given cyclic patch. The tree generator algorithm induces a tree $H = (V, E)$ from the given cyclic patch Q. Next we reform the circle composition steps in order to prevent non-cyclic patches occurring in the circle composition steps. This can be done with the help of H. In fact, H can be regarded as the map of Q. By analysing H, we can detect when a slice subpatch occurs in a circle composition as well as distinguishing the non-cyclic patches that may appear in the circle compositions. This second (reformed) circle composition process is defined by an algorithm called *circle composition algorithm*.

The Tree Generator Algorithm

We define an algorithm that constructs a tree from a given cyclic patch. This constructed tree will give the desired order to circle compose the cyclic patch. Note that every cyclic patch generates a tree by Lemma 3.2.26. However, we are not going to use these trees. Instead we will construct an algorithm to create a new tree carefully so that a (well-defined) sequence of circle compositions can be applied to the given cyclic patches. The result of this (well-defined) sequence of circle compositions will be a single tile patch. Before explaining the tree generator algorithm, we first prove the following results.

Lemma 3.2.28. *Let Q be a patch and X be an interior vertex of Q. Suppose S_X denotes the collection of tiles that contain the vertex X. Then exactly one of the following holds:*

(1) supp S_X *contains a hole.*

(2) S_X is a subpatch of Q which is not a circle patch.

(3) S_X is a circle subpatch of Q.

Proof. We have that X is an interior vertex of Q. Therefore, the proof follows from the fact that the collection S_X is a subpatch if and only if supp S_X does not contain any holes. □

The following two lemmas analyse the conclusions (1) and (2) of Lemma 3.2.28, respectively. The reader may find it useful to check Figure 3.58 while reading over Lemma 3.2.29 and check Figure 3.61 while reading over Lemma 3.2.30. We also note that the assumptions in Lemma 3.2.29 and Lemma 3.2.30 are the same, except (3). The assumptions (3) in these two lemmas refer to the conditions (1) and (2) in Lemma 3.2.28, respectively.

Lemma 3.2.29. *Suppose Q is a cyclic patch consisting of convex tiles and t is a tile in Q that contains an edge e which is completely contained in the boundary ∂*supp *Q. Suppose further Q' is a composition of Q that is generated by finitely many circle composition steps such that*

(1) Q' is not a single tile patch,

(2) There exists a subpatch S of Q and a tile t' of Q' such that $t \in S$ and $Q'\backslash\{t'\} = Q\backslash S$,

(3) There exists $X \in V_{t'}\backslash\partial\text{supp } Q'$ so that $\text{supp } S_X$ contains a hole, where $S_X = \{u \in Q' :\ X \in V_u\}$.

Then, there exists a vertex $Y \in V_{t'}\backslash\partial\text{supp } Q'$ so that the following holds:

(1) $\text{supp } S_Y$ is a subpatch of Q', where $S_Y = \{u \in Q' :\ Y \in V_u\}$.

(2) $[YZ] \in E_Q$ for some $Z \in V_S\backslash V_{Q'}$ (i.e. Z is an interior vertex of S).

Proof. Let Q be a cyclic patch which consists of convex tiles and let t be a tile in Q that contains an edge e so that $e \subseteq \partial\text{supp } Q$. Assume further Q' is a composition of Q that is generated by finitely many circle composition steps and satisfies the conditions (1) to (3) in the lemma. Since $Q'\backslash\{t'\} = Q\backslash S$, the only possible non-convex tile in Q' is t'. Moreover, because S_X is not a circle patch, t' has to be a non-convex tile by Lemma 3.2.14, as illustrated with an example in Figure 3.58.

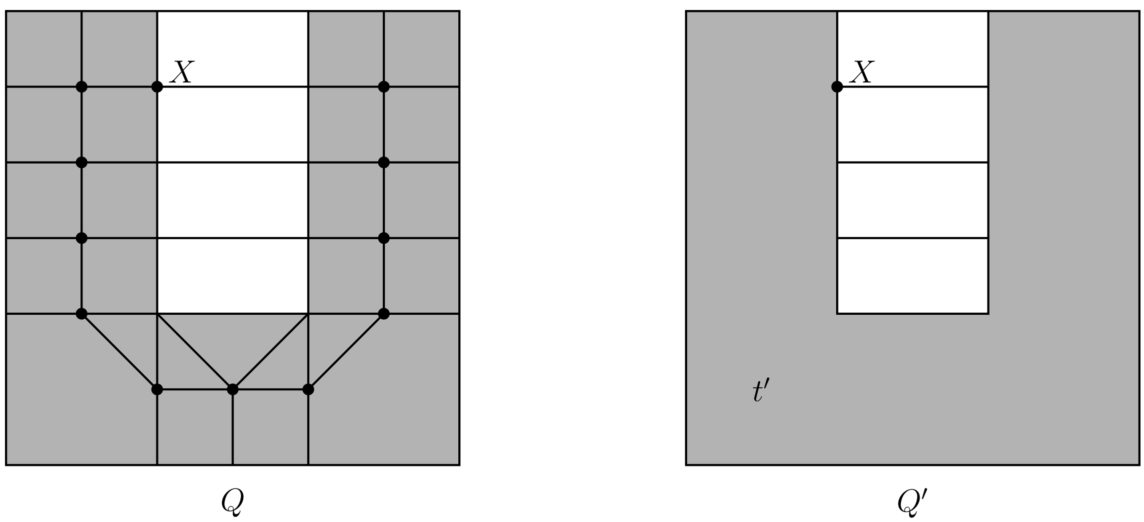


Figure 3.58: An example of patch Q and Q' is illustrated, respectively. The patch Q' is generated from the circle composition steps by the interior vertices of the highlighted patch on the left. The highlighted patch on the left is the subpatch S and the highlighted tile on the right is the tile $t' \in Q'$.

Assume without loss of generality $\text{supp } S_X$ has a single hole $R \subseteq \mathbb{R}^2$. Define arcs γ_1, γ_2 such that $\gamma_1 = \partial R \cap \partial\text{supp } t'$ and $\gamma_2 = \partial R \cap \partial\text{supp } (S_X\backslash\{t'\})$, as illustrated in Figure 3.59. Then we have that $\gamma_1 \cap \gamma_2 = \{a_1, a_2\}$ for some $a_1, a_2 \in V_Q$ and $\gamma_1 \cup \gamma_2 = \partial R$. Note that γ_1, γ_2 are non-trivial (i.e. neither a singleton nor an empty set). In particular, if $\gamma_1 = \partial R$, then $\text{supp } t'$ has the hole R, a contradiction. On the other hand, because $S_X\backslash\{t'\}$ consists of convex tiles that contain the vertex X, we cannot have $\gamma_2 = \partial R$ as well. Thus, γ_1, γ_2 are non-trivial.

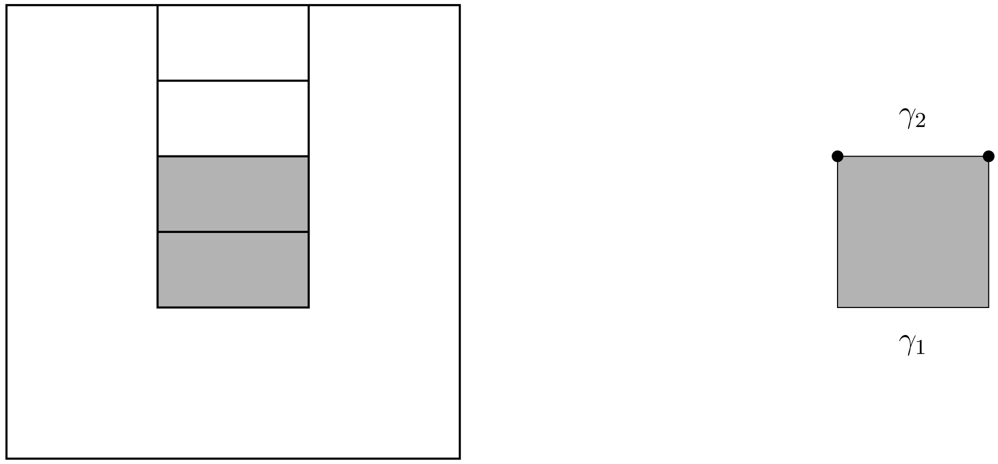


Figure 3.59: For the patch Q' in Figure 3.58, the hole R of supp S_X is the support of the highlighted patch on the left. The arcs γ_1, γ_2 over the boundary ∂R are demonstrated on the right.

Next we show that there exists a vertex $X' \in \gamma_1 \backslash \{a_1, a_2\}$ such that X' is a shared exterior vertex of S. Consider the tiles in S that have an edge e' so that $e' \subseteq \gamma_1$. We must have that $|\{u \in S : \exists e_u \in E_u \text{ with } e_u \subseteq \gamma_1\}| > 1$. That is, γ_1 cannot belong to a single tile in S. This is because of the fact that tiles in S have convex supports. If γ_1 belongs to a single tile u_0 in S, then there exists a line L with end points b_1, b_2 so that $L \subsetneq$ supp u_0 and $b_1, b_2 \in$ supp u_0, as shown in Figure 3.60. Contradicting the fact that $u_0 \in S$ has convex support.

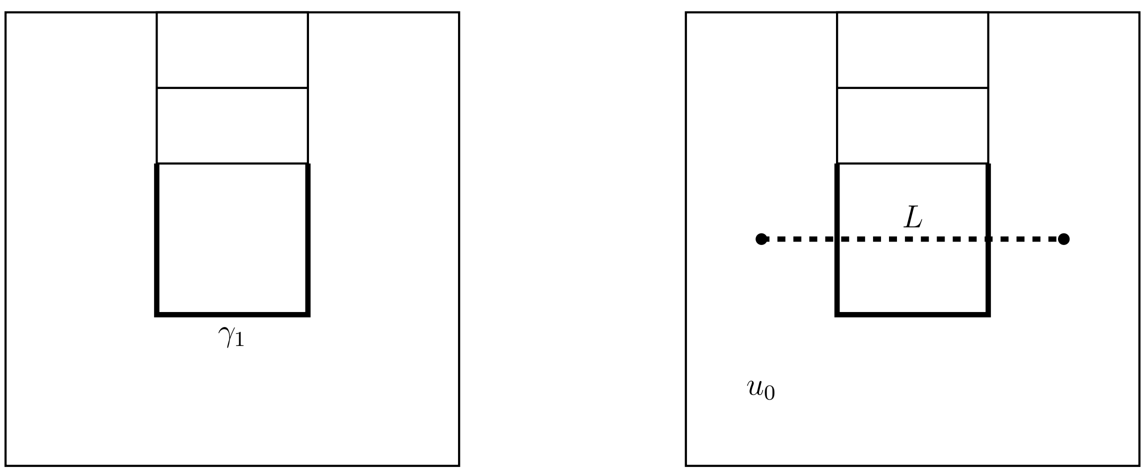


Figure 3.60: The arc γ_1 highlighted on the left cannot belong to a single convex tile $u_0 \in S$, as demonstrated on the right of the figure.

Since we get $|\{u \in S : \exists e_u \in E_u \text{ with } e_u \subseteq \gamma_1\}| > 1$, there exists $X' \in \gamma_1 \backslash \{a_1, a_2\}$ such that X' is a shared exterior vertex of S. Thus, X' is a vertex over the boundary ∂R so that $[X'Y'] \in E_Q$ for some $Y' \in V_S \backslash V_{t'}$. Since $X' \in \partial R$, we have two cases:

(i) supp $S_{X'}$ is a subpatch of Q', where $S_{X'} = \{u \in Q' : X' \in V_u\}$,

(ii) supp $S_{X'}$ contains the holes $R'_1, \ldots, R'_k$ for $k \in \mathbb{Z}^+$ such that $\bigcup_{i=1}^{k} R'_i \subsetneq R$.

For (i) the proof is complete. For (ii), we apply the same argument again, for the collection $S_{X'}$ whose holes are strictly smaller than R. Since $|Q| < +\infty$, the process eventually leads to the case (i). Hence, the proof is complete. □

Lemma 3.2.30. *Assume that Q is a cyclic patch consisting of convex tiles and t is a tile in Q that contains an edge e so that $e \subseteq \partial\text{supp } Q$. Suppose further Q' is a composition of Q that is generated by finitely many circle composition steps such that*

(1) Q' is not a single tile patch,

(2) There exists a subpatch S of Q and a tile t' of Q' such that $t \in S$ and $Q' \backslash \{t'\} = Q \backslash S$,

(3) There exists $X \in V_{t'} \backslash \partial\text{supp } Q'$ so that supp S_X is a subpatch of Q' which is not a circle patch, where $S_X = \{u \in Q' : X \in V_u\}$.

Then, there exists a vertex $Y \in V_{t'} \backslash \partial\text{supp } Q'$ so that the following holds:

(1) supp S_Y is a circle subpatch of Q', where $S_Y = \{u \in Q' : Y \in V_u\}$.

(2) $[YZ] \in E_Q$ for some $Z \in V_S \backslash V_{Q'}$.

Proof. Suppose Q is a cyclic patch that consists of convex tiles and t is a tile in Q such that there exists an edge $e \in E_t$ with $e \subseteq \partial\text{supp } Q$. Assume further Q' is a composition of Q that is generated by finitely many circle composition steps and satisfies conditions (1) to (3) in the lemma. Since $t \in S$, we have that $e \in E_{t'}$. Since S_X is not a circle patch, t' is a non-convex tile by Lemma 3.2.14. Define arcs $\gamma_1, \gamma_2 \subseteq \partial\text{supp } t'$ such that γ_i has end points X, a_i for $i = 1, 2$ so that $a_i \in \partial\text{supp } S_X$ and $a_1 \neq a_2$. Such distinct arcs exist because X is a common interior vertex in S_X and t' contains the edge e.

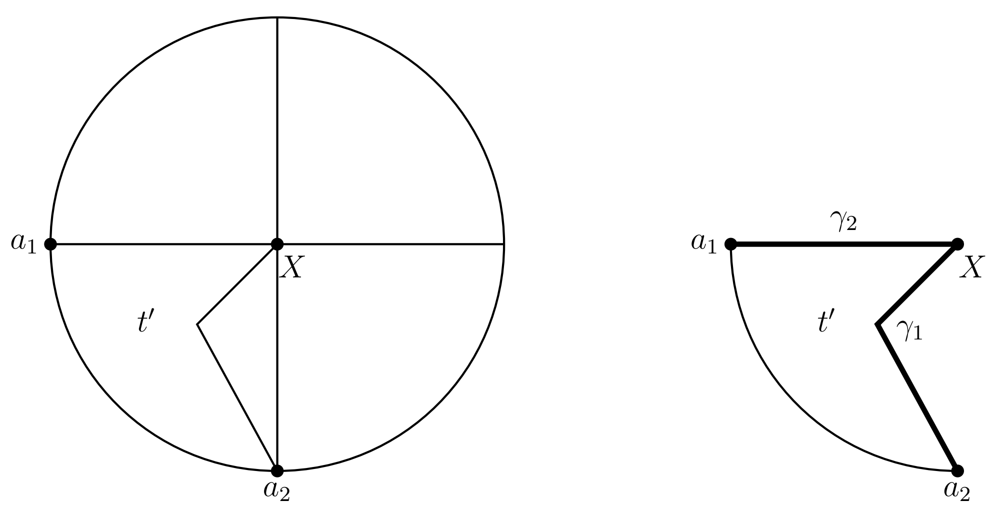


Figure 3.61: The tile t' is the only non-convex tile in the collection S_X which is a subpatch of Q'. The patch S_X is not a circle patch.

By the same reasoning as in the proof of Lemma 3.2.29, we have that one (or both) of the following holds:

(1) $|\{u \in S : \exists e_u \in E_u \text{ with } e_u \subseteq \gamma_1\}| > 1$,

(2) $|\{u \in S : \exists e_u \in E_u \text{ with } e_u \subseteq \gamma_2\}| > 1$.

Assume without loss of generality that (1) holds. The rest of the proof now follows from the exact same argument as the one of the proof of Lemma 3.2.29. □

Corollary 3.2.31. *Let Q be a cyclic patch consisting of convex tiles and let t be a tile in Q which contains an edge e so that $e \subseteq \partial\text{supp } Q$. Suppose that Q' is a composition of Q that is generated by finitely many circle composition steps such that*

(1) Q' is not a single tile patch,

(2) there exists a subpatch S of Q and a tile t' of Q' such that $t \in S$ and $Q' \backslash \{t'\} = Q \backslash S$.

Then, there exists a vertex $X \in V_{t'} \backslash \partial\text{supp } Q'$ so that the following holds:

(1) $\text{supp } S_X$ is a circle subpatch of Q', where $S_X = \{u \in Q' : X \in V_u\}$.

(2) $[XY] \in E_Q$ for some $Y \in V_S \backslash V_{Q'}$.

Proof. The proof follows by Lemma 3.2.28, Lemma 3.2.29 and Lemma 3.2.30. □

Finally we are ready to explain the tree generator algorithm.

The Tree Generator Algorithm :

Assume that Q is a cyclic patch consisting of convex tiles and t is a tile in Q such that there exists $e \in E_t$ with $e \subseteq \partial\text{supp } Q$. Assume further that $G = (V, E)$ is the empty graph (i.e. $V = E = \emptyset$). Apply the following steps until Q composed into a single tile patch.

Step − 0 : Define $W = \{X_1, \ldots, X_n\}$ for $n \in \mathbb{Z}^+$ to be the collection vertices of t which are interior vertices of Q. Suppose without loss of generality $[X_i X_{i+1}]$ for $i \in \{1, \ldots, n-1\}$ is an edge of Q. Apply circle compositions by the vertices $X_1, \ldots, X_n$, consecutively. Simultaneously, add the vertices $X_1, \ldots, X_n$ to the vertex set V of G, and the edges $[X_1 X_2], \ldots, [X_{n-1} X_n]$ to the edge set E of G. Denote the composed patch as Q' and the composed tile as t'. Check whether Q' is a single tile patch. If it is a single tile patch, then terminate the algorithm. The desired tree is the graph $G = (V, E)$. If Q' is not a single tile patch, then move to the next step.

Step − 1 : Find a vertex X such that circle composition by X is well defined and $[XX']$ is an edge of Q where X' is a vertex so that $X' \in V_S \backslash V_{t'}$. Such a vertex exists by Corollary 3.2.31. Simultaneously, add the vertex X to the vertex set V and the edge $[XX']$ to the edge set E.

Step − 2 : Check whether the generated patch is a single tile patch. If it is not a single tile patch, then go back to Step - 1. If it is a single tile patch, then terminate the algorithm. The desired tree is the graph $G = (V, E)$.

The algorithm terminates in finite time.

Corollary 3.2.32. *Assume that Q is a cyclic patch consisting of convex tiles and $G = (V, E)$ is a graph that is generated by the tree generator algorithm from the patch Q. Then G is a tree.*

Proof. We have that $G = (V, E)$ is connected by construction. Suppose now $X_1, \ldots, X_n$ for $n \in \mathbb{Z}^+$ is a collection of interior vertices of Q such that the edges $\{[X_i X_{i+1}]\}_{i=1}^n$ forms a cycle (of edges), with the convention that $X_{n+1} = X_1$. Assume further that $\{X_1, \ldots, X_n\} \subseteq V$. We will show that $\{[X_i X_{i+1}]\}_{i=1}^n \subsetneq E$. Assume without loss of generality $X_1, \ldots, X_n$ are composed by the order $X_1, X_2, \ldots, X_n$. We first circle compose by the vertices X_1 and X_2. We add X_1 and X_2 into the vertex set V whereas we add the edge $[X_1 X_2]$ to the edge set E. However, because $X_1 \in [X_n X_1]$, the edge $[X_n X_1]$ disappears in the composed patch. Since we didn't add the edge $[X_n X_1]$ into E and it does not exist in the composed patch, we will never add $[X_n X_1]$ into E. Thus, $\{[X_i X_{i+1}]\}_{i=1}^n \subsetneq E$. Hence, G cannot contain any cycle, and is a tree. □

Next we illustrate the tree generator algorithm with an example.

Example 3.2.33. Consider the cyclic patch Q and the tile $t \in Q$ given in Figure 3.62. By Step - 0 of the algorithm, we first apply circle composition by the vertex X_1. Then move to Step - 1. We apply Step - 1 to the suitable vertices until we end up a single tile patch. For that, we apply circle compositions by the vertices X_2 and X_3 consecutively, since $[X_1 X_2]$ is the only interior edge of Q with $X_2 \in [X_1 X_2]$ and $[X_2 X_3]$ is the only interior edge of Q (after $[X_1 X_2]$ is composed) with $X_2 \in [X_2 X_3]$. The tree we form from these initial steps consists of the vertices X_1, X_2, X_3 and the edges $[X_1 X_2], [X_2 X_3]$. After that we arrive two options; either circle compose by the vertex X_4 or circle compose by the vertex X_6. Subsequently, regardless of which option we pick, we get other choices to make as well. Continuing circle compositions according to the instructions in Step - 1 in the tree generator algorithm, we arrive a tree. This tree is not unique due to the choices we can make during the process. There are several possible trees that can be generated. We provide all the trees that is generated by the tree algorithm which start with X_1 in Figure 3.63 for illustration.

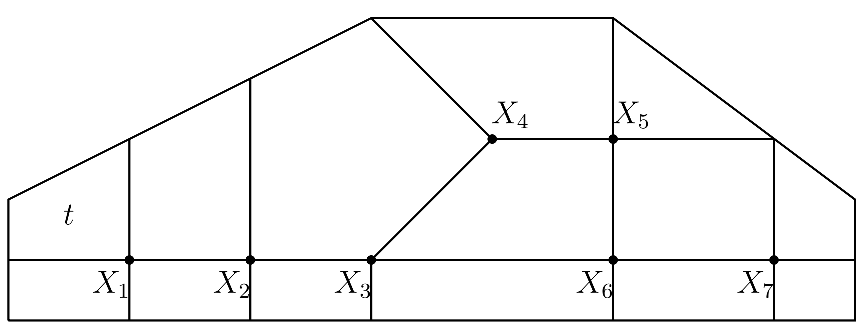


Figure 3.62: A cyclic patch Q and a tile $t \in Q$.

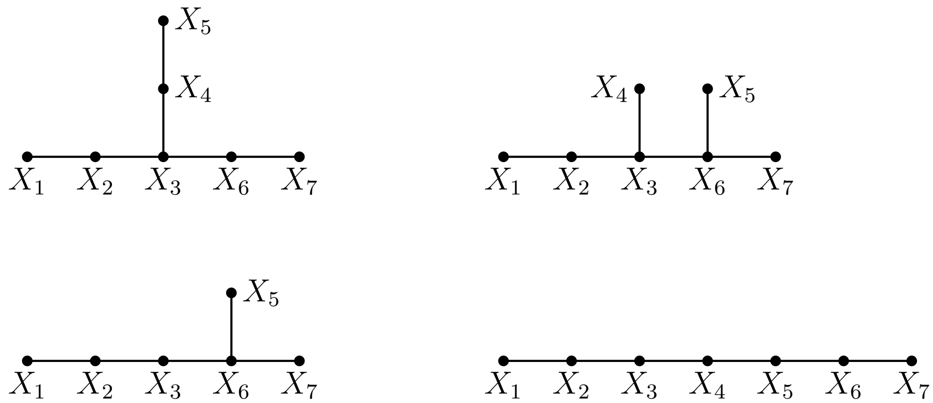


Figure 3.63

Suppose we choose the tree on the top left of Figure 3.63. Denote the chosen tree of the cyclic patch Q by $H = (V, E)$. We have that X_1 is a vertex of degree 1 (in H). If we apply a circle composition by the vertex X_1, then we get a composed patch $Q' = \Theta_{X_1}(Q)$ and a subtree $H' = (V', E') \subseteq H$ which corresponds to the cyclic patch Q', as shown on the right side of Figure 3.64. Notice that we can deduce Q' is a cyclic patch by looking at the tree H', with the help of Lemma 3.2.19. Therefore, we know how the circle composition steps proceed by looking at a tree of a cyclic patch. Finally, we initialised the tree generator algorithm for $t \in Q$ in Figure 3.62. This tile t was deliberately chosen to contain one of the corners of Q. The importance of the choice of t will be explained in Remark 3.2.36 (and will become completely apparent after Corollary 3.2.41).

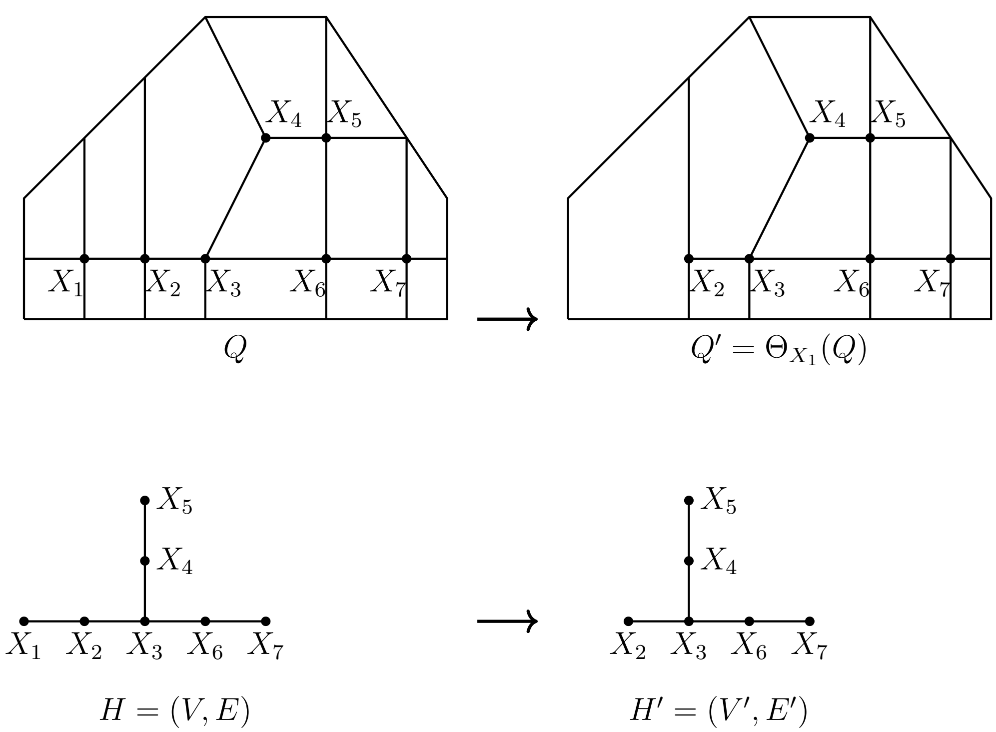


Figure 3.64

The Circle Composition Algorithm

Theorem 3.2.34 (Circle Compositions of Cyclic Patches). *For any given cyclic patch Q that consists of convex tiles, there is a finite sequence of interior vertices $X_1, \ldots, X_n$ (for $n \in \mathbb{Z}^+$) of Q such that $\Theta_{X_n} \circ \cdots \circ \Theta_{X_1}(Q)$ is a single tile patch and $\Theta_{X_i} \circ \cdots \circ \Theta_{X_1}(Q)$ is a cyclic patch for any $i = 1, \ldots, n-1$.*

The proof of this theorem is given by the following circle composition algorithm. The algorithm ensures that every cyclic patch that consists of convex tiles can be composed into a circle patch, after a finite sequence of circle compositions. Moreover, all the composed patches appearing in the (middle) composition process are cyclic patches.

The Circle Composition Algorithm :

Suppose Q is a given cyclic patch which is not a circle patch and consists of convex tiles. Suppose further $G = (V, E)$ is the tree formed by the tree generator algorithm from Q. Apply the following steps to the pair (Q, G) until Q eventually composes into a circle patch.

Step − 1 : Identify the vertices in G that are of degree 1. There are at least two such vertices by Lemma 3.2.24. Let X be a such vertex.

Step − 2 : Apply circle composition by the vertex X. Let $G' = (V', E')$ be a subtree of $G = (V, E)$ such that $V' = V \backslash \{X\}$ and E' is the subcollection of edges in E which do not intersect with the vertex X. Note that $\Theta_X(Q)$ is a cyclic patch by Lemma 3.2.19.

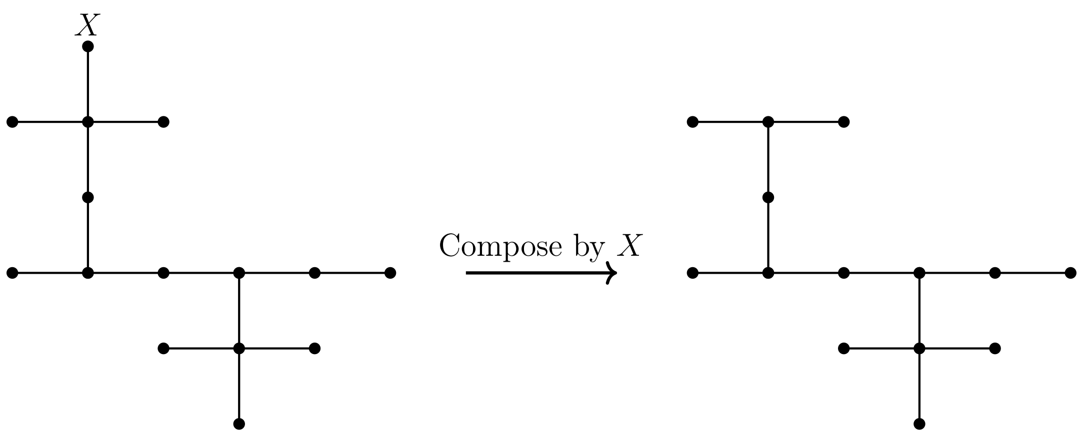


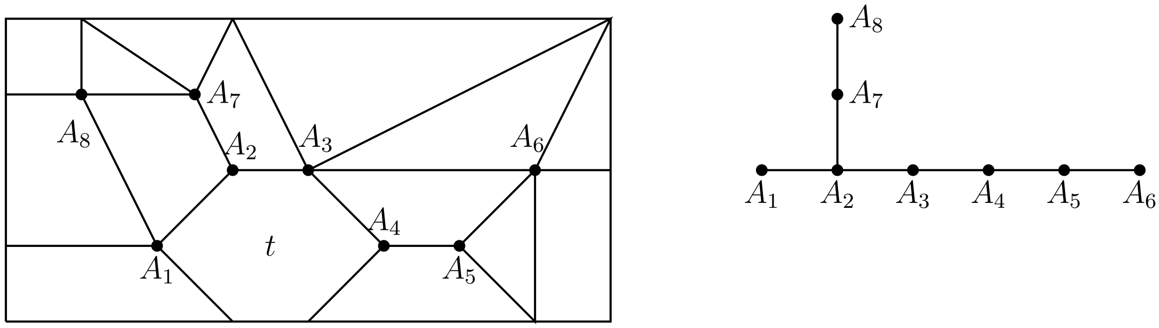
Figure 3.65: An example of a circle composition explained in Step - 2

Step − 3 : Check whether $\Theta_X(Q)$ is a circle patch. If it is a circle patch, then terminate the algorithm. Otherwise, go back to Step - 1 with the pair $(\Theta_X(Q), G')$.

The algorithm terminates in a finite time. It terminates when we arrive at a circle patch. Therefore, after one more circle composition, we arrive a single tile patch. Hence, the proof of the Theorem 3.2.34 follows.

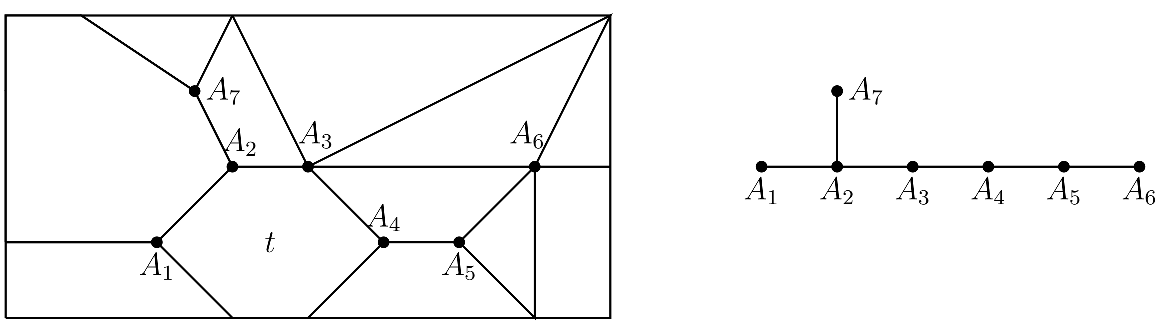
We will explain the circle composition algorithm with an example. In order to provide more variety in terms of examples, we use a different cyclic patch than given in Figure

3.62. Consider the patch Q and a tree generated by Q in Figure 3.66. We first check the vertices of degree 1 in the tree in Figure 3.66. There are three such vertices; A_1, A_6 and A_8. Suppose we choose the vertex A_8. This is Step - 1 of the algorithm. We apply the circle composition by the vertex A_8. This process generates a (composed) cyclic patch and a subtree that corresponds to the cyclic patch generated. The generated cyclic patch and the subtree are illustrated in Figure 3.67.



Q

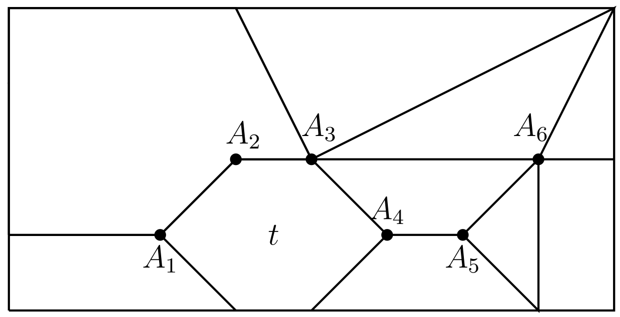
Figure 3.66: A cyclic patch Q and a tree generated by Q

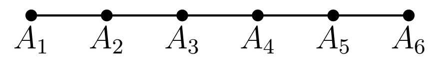


$\Theta_{A_8}(Q)$

Figure 3.67: Patch $\Theta_{Y_3}(Q)$ and the tree corresponds to $\Theta_{Y_3}(Q)$

Next we move to Step - 3. We check whether $\Theta_{A_8}(Q)$ is a circle patch. Because it is not a circle patch we go back to Step - 1. Then we identify the vertices of degree 1 in the tree in Figure 3.67. There are three such vertices; A_7, A_1 and A_6. Suppose we choose the vertex A_7. This is Step - 1 of the algorithm. Next we apply circle composition by the vertex A_7. We arrive the cyclic patch and the tree in Figure 3.68. We continue to apply circle compositions by the vertices A_6, A_5, A_4, A_3, respectively. The generated composed cyclic patch after these steps is a circle patch with centre A_1. The algorithm terminates. Moreover, after applying a circle composition by the vertex A_1 we arrive a single tile patch.





$\Theta_{A_7} \circ \Theta_{A_8}(Q)$

Figure 3.68: Patch $\Theta_{Y_2} \circ \Theta_{Y_3}(Q)$ and the tree corresponds to $\Theta_{Y_3}(Q)$

Definition 3.2.35. A graph G is called a *path graph* if every vertex of G is of degree at most two.

Remark 3.2.36. The tree generator algorithm starts with a tile t such that t has an edge which is completely contained over the boundary of the cyclic patch Q. In particular, Step - 0 of the tree algorithm are the circle compositions by the vertices $X_1, \ldots, X_n$ for $n \in \mathbb{Z}^+$ where $\{X_i : \ i = 1, \ldots, n\}$ is the collection of interior vertices of Q that belong to t. Simultaneously, we add the vertices $X_1, \ldots, X_n$ to the vertex set and the edges $[X_1X_2], \ldots, [X_{n-1}X_n]$ to the edge set. This initial step generates a path graph $H' = (V', E')$ such that $V' = \{X_i : \ i = 1, \ldots, n\}$ and $E' = \{[X_iX_{i+1}] : \ i = 1, \ldots, n-1\}$. Therefore, every tree H, which is formed from the tree generated algorithm starting with the tile t, must satisfy that $H' \subseteq H$. This construction assures that we can modify Step - 2 of the circle composition algorithm (i.e. circle composing by a vertex of degree 1) such that the tree H first reduces to the subtree H' during the circle composition process. For example, for the cyclic patch Q and tile t given in Figure 3.66, we have that $H' = (V', E')$ where $V' = \{A_1, A_2, A_3, A_4\}$. Notice that A_1 is a vertex of degree 1 for the tree in Figure 3.66. Even though it is a vertex of degree 1, we do not choose A_1 to be circle composed. Instead we choose A_8 to start with. After applying circle compositions by the vertices A_8, A_7, A_6, A_5, the tree we started in Figure 3.66 reduces to the path graph H'. Finally, we can keep circle composing by the vertices A_4, A_3, A_2, A_1, consecutively, in order to arrive a single tile patch. By doing so, we forced the circle composition process to end with the step that circle compose by the vertex X_1. This final circle composition step is important because it will be the first circle decomposition step when we start circle decomposing the generated single tile patch. The details of why the first circle decomposition step is significant will be explained after Corollary 3.2.41.

3.2.4 Order Structure by Iterative Circle Decomposition Systems

Circle compositions of cyclic patches form an iterative structure. This iterative structure composes a cyclic patch into a single tile patch in finite steps. Taking the inverse of the circle compositions, we get an iterative system of circle decompositions. We show that these circle decomposition steps will define the desired order system for cyclic patches.

Lemma 3.2.37 and Proposition 3.2.38 are the main ingredient of how to implement the order structure of circle patches within the circle decomposition process.

Lemma 3.2.37 (Reforming the Curves in Decomposition Steps). *Let Q be a patch and S be a subpatch of Q. Define a composition Q' of Q by composing S into a single tile $u \in Q'$. Then the following holds:*

(1) If (A, B) is a valid pair for Q' via some curve $\mathcal{C}$ such that the decoration e_u of u induced by $\mathcal{C}$ has end points M, N so that (M, N) is a valid pair for S, then (A, B) is a valid pair for Q as well.

(2) If (A, B) is a valid pair for Q' via some curve $\mathcal{C}$ such that the decoration e_u of u induced by $\mathcal{C}$ has end point pairs M, N and M', N' so that $\{(M, N), (M', N')\}$ is a split pair for S, then (A, B) is a valid pair for Q as well.

(3) If $\{(A, B), (C, D)\}$ is a split pair for Q' via some curves $\mathcal{C}_1, \mathcal{C}_2$ such that the decoration e_u of u induced by $\mathcal{C}_1 \cup \mathcal{C}_2$ has end points M, N so that (M, N) is a valid pair for S, then $\{(A, B), (C, D)\}$ is a split pair for Q as well.

(4) If $\{(A, B), (C, D)\}$ is a split pair for Q' via some curves $\mathcal{C}_1, \mathcal{C}_2$ such that the decoration e_u of u induced by $\mathcal{C}_1 \cup \mathcal{C}_2$ has end point pairs M, N and M', N' so that $\{(M, N), (M', N')\}$ is a split pair for S, then $\{(A, B), (C, D)\}$ is a split pair for Q as well.

Proof. We will only prove (1). The rest of the cases can be proven in a similar fashion. Let $\mathcal{C}^{M,N}$ denote the curve that makes (M, N) a valid pair for S. Define $\mathcal{C}' = (\mathcal{C} \backslash e_u) \cup \mathcal{C}^{M,N}$. Because $\mathcal{C}^{M,N}$ makes (M, N) a valid pair for S, $\mathcal{C}'$ makes (A, B) a valid pair for Q. □

We illustrate the idea in Lemma 3.2.37 with an example. Consider the circle patch Q, the leftmost patch of Figure 3.69. We have that (A, B) is a valid pair for Q, by a curve $\mathcal{C}$ illustrated in the third patch from the left of the figure. Then $\mathcal{C}$ induces a decoration e_u for the tile u, which has end points M, N, as shown in the figure. The patch Q on the rightmost of the figure has (A, B) as a valid pair. This is because of the fact that the tile u in Q' decomposes into a circle subpatch S of Q such that (M, N) is a valid pair for S. Hence, we can replace the decoration e_u of u, which has end points M, N as demonstrated on the third patch from the left of the figure, with a decoration over the circle subpatch S of Q, as shown on the rightmost patch of the figure. This is the main idea of circle decomposition process.

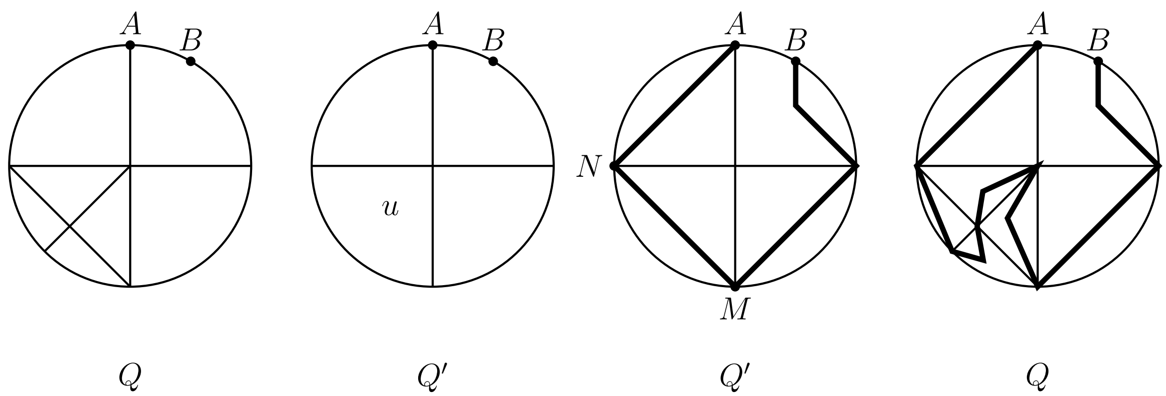


Figure 3.69

The following proposition ensures that if we apply the circle composition process in a specific order, then we can define an order system for cyclic patches with the help of Lemma 3.2.37.

Proposition 3.2.38. *Suppose Q is a patch with interior vertices X, Y such that $\Theta_X(Q)$ is a circle patch with centre Y, and $(X : Y)$ is a connected pair for Q (demonstrated in Figure 3.70). Then*

(1) *Q is a cyclic patch,*

(2) *If (A, B) is a valid pair for $\Theta_X(Q)$, then it is a valid pair for Q.*

(3) *If $\{(A, B), (C, D)\}$ is a split pair for $\Theta_X(Q)$, then it is a split pair for Q.*

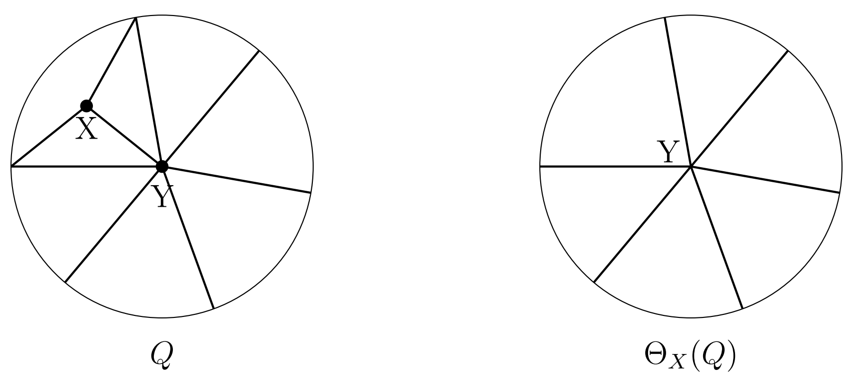


Figure 3.70: $\mathcal{O}_X$ is a circle subpatch of a patch Q such that $\Theta_X(Q)$ is a circle patch with centre Y

Before giving the proof of the proposition, we will make some simplifications about the possible 2-curve decorations that we use to decorate the tiles. The order system of circle patches are defined by Proposition 3.1.10 and Proposition 3.2.12. Recall that we form valid/split pairs for circle patches through curves, which are not uniquely defined for most of the cases. For example, given the exterior vertices A, B given in the leftmost patch of Figure 3.71, we can construct two curves that make (A, B) a valid pair for the circle patch Q, which are illustrated in the middle and rightmost patch of the figure. Both of these curves are defined according to the instructions in the proof of Proposition 3.2.12. So, there is not a unique curve that makes (A, B) a valid pair. Due to these (non-unique) choices, we can pick whichever curve we prefer to implement, or even reform the constructed curves with simple modifications in the decorations. This is the main idea of why the simplification process is applicable.

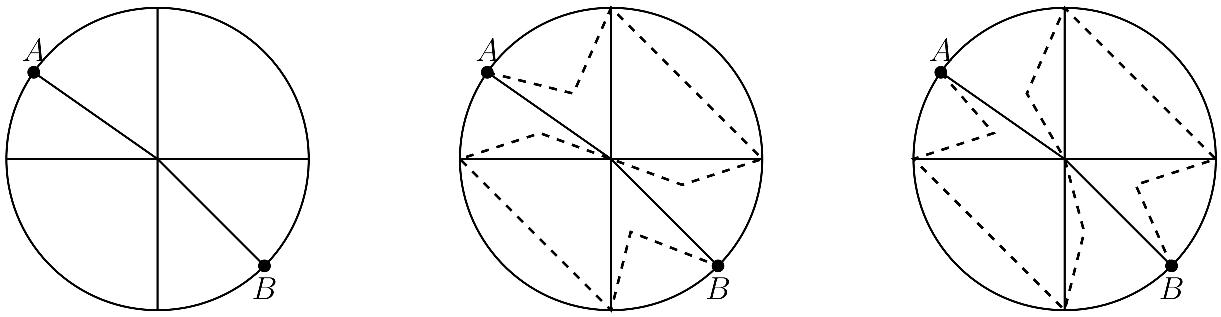


Figure 3.71: Different choices to make (A, B) a valid pair

Proposition 3.2.39. *[Simplification of 2-Curve Decorations] Let Q be a circle patch with centre X, (Q, u) be a circle composition pair and e_u be a decoration of u.*

(1) If e_u is a simple decoration of u with end points A, B, then there exists a curve $\mathcal{C}$ that makes (A, B) a valid pair for Q such that for each $t \in Q$, $\overline{\mathcal{C} \cap \text{int (supp } t)}$ is either a simple curve or $X \in \overline{\mathcal{C} \cap \text{int (supp } t)}$. That is, the decoration of t is either a simple curve or it hits the centre X.

(2) If e_u is a 2-curve decoration of u with end point pairs A, B and C, D, then there exists curves $\mathcal{C}_1, \mathcal{C}_2$ that make $\{(A, B), (C, D)\}$ a split pair for Q such that for each $t \in Q$, $\overline{(\mathcal{C}_1 \cup \mathcal{C}_2) \cap \text{int (supp } t)}$ is either a simple curve or $X \in \overline{(\mathcal{C}_1 \cup \mathcal{C}_2) \cap \text{int (supp } t)}$. That is, the decoration of t is either a simple curve or it hits the centre X.

Proof. Let Q be a circle patch with centre X and u be a tile so that (Q, u) is a circle composition pair.

(1) Assume that e_u is a decoration of u with end points A, B. There exists a curve $\mathcal{C}^{A,B}$ that makes (A, B) a valid pair, by Proposition 3.1.10. If each tile has a simple decoration induced by $\mathcal{C}^{A,B}$, then define $\mathcal{C} = \mathcal{C}^{A,B}$. Otherwise, there exists a unique tile $t \in Q$ with a 2-curve decoration, by Corollary 3.1.11. Suppose t can be identified with a topological triangle with end points M, N, X. The 2-curve decoration e_t of t must have end point pairs M, A and N, B, by Proposition 3.1.10, where A, B are isolated vertices of Q belonging to the tile t. Assume further without loss of generality t_1, t_2 are neighbour tiles of t such that $N \in V_{t_1}$ and $M \in V_{t_2}$, as illustrated in the leftmost patch of Figure 3.72. Then decoration e_{t_1} of t_1 has to be a simple curve and have end points N and N' so that $N' \neq X$, as shown in the middle patch of Figure 3.72. Replace the component simple curve of e_t that has end points B, N, with a curve that has end points B, X. Replace also the decoration e_{t_1} of

t_1 with a curve that has end points X, N', as illustrated in the rightmost patch of Figure 3.72. This process generates a (reformed) curve which still makes (A, B) a valid pair for Q. Moreover, the decoration of t is now set to hit the centre X. Thus, (1) holds.

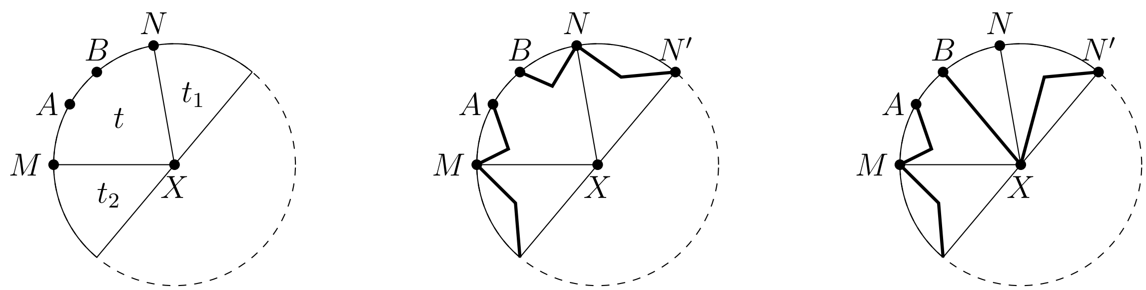


Figure 3.72

(2) Let e_u be a decoration of u with end point pairs A, B and C, D. By Proposition 3.2.12, there exist curves $\mathcal{C}_1, \mathcal{C}_2$ and subpatches Q_1, Q_2 such that $\mathcal{C}_1$ makes (A, B) a valid pair for Q_1, $\mathcal{C}_2$ makes (C, D) a valid pair for Q_2, $Q_1 \cup Q_2 = Q$ and $\mathcal{C}_1$ and $\mathcal{C}_2$ are non-crossing. For each $x \in Q$, denote the decoration induced by $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ as e_x. Suppose $t \in Q$ is a tile that can be characterised by a topological triangle with vertices M, N, X and e_t is a 2-curve decoration such that $X \notin e_t$. The possible 2-curve decorations e_t of t are demonstrated in Figure 3.73.

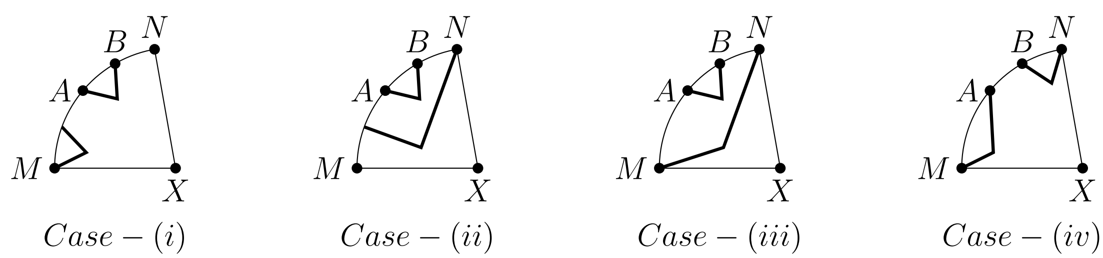


Figure 3.73

The cases (i), (ii), (iii) are directly related with the proof in part (1). In particular, $\mathcal{C}_1$ is a curve in the single tile patch $\{t\}$ that connects A and B, and makes (A, B) a valid pair for $Q_1 = \{t\}$. So, for the cases (i), (ii), (iii), $\mathcal{C}_2$ is a curve that makes (C, D) a valid pair for $Q_2 = Q$. By reforming the decoration of t and also decoration of one of its neighbour tiles, just like in part (1) (See Figure 3.72), we can reform $\mathcal{C}_2$ into a curve $\mathcal{C}_2'$ which still makes (C, D) a valid pair for Q, whereas the decoration induced on t by $\mathcal{C}_2'$ hits the centre X. Furthermore, the proof of the case (iv) is also similar to part (1). Hence, every 2-curve decoration of tiles in a circle patch can be reformed to hit the centre of the given circle patch. □

Corollary 3.2.40. *One can always assume that every 2-curve decoration appearing in a circle patch hits the centre vertex.*

Proof. Proof follows by (1) and (2) of Proposition 3.2.39. □

Proof of Proposition 3.2.38. Suppose $\mathcal{O}_X$ is a circle subpatch of a patch Q, $(\mathcal{O}_X, u)$ is a circle composition pair, $\Theta_X(Q)$ is a circle patch with centre Y, $u \in \Theta_X(Q)$ and $[XY]$ is an edge (or a finite union of edges) in Q. Then u can be regarded as a topological triangle with vertices M, N, Y where M, N are shared exterior vertices of Q with $M, N \in V_u$. Assume further that u_1, u_2 are neighbour tiles of u such that $N \in V_{u_1}$ and $M \in V_{u_2}$, as demonstrated in Figure 3.74.

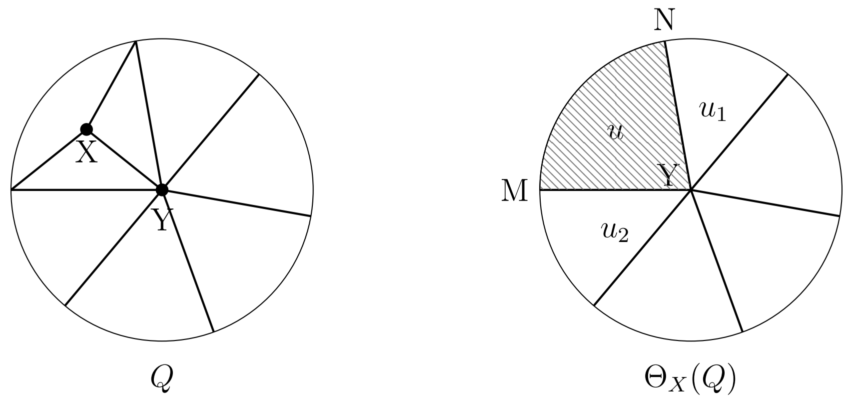


Figure 3.74

(1) We have that $(X : Y)$ is a connected pair for Q. Thus, Q is a cyclic patch, by Lemma 3.2.19.

(2) Suppose (A, B) is a valid pair for $\Theta_X(Q)$ by some curve $\mathcal{C}^{A,B}$. Then $\mathcal{C}^{A,B}$ induces a decoration for u, say e_u. If e_u is a simple curve decoration with end points A', B', then (A', B') is valid pair for $\mathcal{O}_X$ via some curve $\mathcal{C}^{A',B'}$ by Proposition 3.1.10. Thus, by replacing the decoration e_u in $\mathcal{C}^{A,B}$, with the curve $\mathcal{C}^{A',B'}$, we define a curve $\mathcal{C}$ which makes (A, B) a valid pair for Q. Similarly, if e_u is a 2-curve decoration for u, then we can assume without loss of generality that $Y \in e_u$, by Proposition 3.2.39. Suppose e_u has end point pairs M', N' and Y, Y'. Because Y is a shared exterior vertex of $\mathcal{O}_X$, $\{(M', N'), (Y, Y')\}$ is a split pair for $\mathcal{O}_X$, by Corollary 3.2.13. Thus, there exists curves $\mathcal{C}_1, \mathcal{C}_2$ that make $\{(M', N'), (Y, Y')\}$ a split pair for $\mathcal{O}_X$. Replace the decoration e_u in $\mathcal{C}^{A,B}$ with the curves $\mathcal{C}_1, \mathcal{C}_2$. Then the generated curve makes (A, B) a valid pair for Q.

(3) The proof of this case is similar with (2). In particular, because Y is a shared exterior vertex of $\mathcal{O}_X$, and the 2-curve decoration that can be defined for e_u intersects with the vertex Y, the same argument in (2) can be applied. □

Notice that $\Theta_X(Q)$ in Proposition 3.2.38 is a circle patch with centre Y, whereas Q is the circle decomposition of $\Theta_X(Q)$. Therefore, Proposition 3.2.38 states that if the vertices we applied circle decompositions are connected pairs, such as X and Y in the proposition, then the set of valid and split pairs is invariant under the circle decomposition process in the sense that valid and split pairs of $\Theta_X(Q)$ are also valid and split pairs of Q. Similarly, we can apply the argument in the proposition twice. For example, suppose Q is a given cyclic patch such that $\Theta_{X_2} \circ \Theta_{X_1}(Q)$ is a circle patch with centre Y, and $(X_2 : X_1)$ and $(X_1 : Y)$ are both connected pairs of Q. Then valid and split pairs of $\Theta_{X_2} \circ \Theta_{X_1}(Q)$ are also valid and split pairs of Q. Therefore, by investigating the valid and split pairs of the circle patch $\Theta_{X_2} \circ \Theta_{X_1}(Q)$, we can find valid and split pairs for the patch Q. Recall that the valid and split pairs of circle patches are explicitly defined by Proposition 3.1.10 and Proposition 3.2.12, respectively. Therefore, using these propositions, we generalise Proposition 3.2.38, with the following corollary.

Corollary 3.2.41. *Assume that Q is a cyclic patch which is not a circle patch, and has interior vertices $X_1, X_2, \ldots, X_n$ for $n \in \{2, 3, \ldots\}$ such that $Q' = \Theta_{X_n} \circ \cdots \circ \Theta_{X_1}(Q)$ is a single tile patch and $[X_i X_{i+1}]$ is an edge of Q for each $i = 1, \ldots, n-1$. Then*

(1) $\Theta_{X_j} \circ \cdots \circ \Theta_{X_1}(Q)$ is a cyclic patch for each $j \in \{1, \ldots, n-1\}$.

(2) (A, B) is a valid pair for Q for each distinct exterior vertex pair A, B of Q.

(3) If (Q, u) is a composition pair and e_u is a 2-curve decoration with end point pairs A, B and C, D such that there is no single tile $t \in \Theta_{X_n}^{-1}(Q')$ so that $\{A, B, C, D\} \cap V_t \neq \emptyset$ (i.e. there are at least two tiles that intersect with the vertices A, B, C, D), then $\{(A, B), (C, D)\}$ is a split pair for Q.

Proof. Let $Q' = \Theta_{X_n} \circ \cdots \circ \Theta_{X_1}(Q)$ be a single tile patch and let $\Theta_{X_n}^{-1}(Q')$ be the first circle decomposition of Q', that is a circle patch.

(1) Since $Q' = \Theta_{X_n} \circ \cdots \circ \Theta_{X_1}(Q)$ is a single tile patch, every tile in Q must intersects with at least one of the vertices $X_1, \ldots, X_n$. Since $(X_i : X_{i+1})$ is a connected pair of Q for each $i = 1, \ldots, n-1$, we have that $\Theta_{X_j} \circ \cdots \circ \Theta_{X_1}(Q)$ is a cyclic patch for each $j \in \{1, \ldots, n-1\}$, by Lemma 3.2.19.

(2) (A, B) is valid pair for the circle patch $\Theta_{X_n}^{-1}(Q')$, by Proposition 3.1.10. Therefore, (A, B) is a valid pair for $\Theta_{X_{n-1}}^{-1} \circ \Theta_{X_n}^{-1}(Q')$ as well, by Proposition 3.2.38. Apply Proposition 3.2.38 consecutively with the fact that $(X_i : X_{i+1})$ is a connected pair of Q for each $i = 1, \ldots, n-1$. We get (A, B) is a valid pair for $Q = \Theta_{X_1}^{-1} \circ \cdots \circ \Theta_{X_{n-1}}^{-1} \circ \Theta_{X_n}^{-1}(Q')$ as well.

(3) Since there is no single tile $t \in \Theta_{X_n}^{-1}(Q')$ so that $\{A, B, C, D\} \cap V_t \neq \emptyset$, we have that $\{(A, B), (C, D)\}$ is a split pair for $\Theta_{X_n}^{-1}(Q')$. Hence, by the same argument in (1), it is also a split pair for Q. □

Corollary 3.2.41 states that by investigating the valid and split pairs of the circle patch $\Theta_{X_n}^{-1}(Q')$ (i.e. the first circle decomposed patch of Q'), we can detect the valid and split pairs of Q. Therefore, we want the first circle decomposition step to be carefully chosen. For instance, Figure 3.75 demonstrates two different circle composition processes of the same cyclic patch which is given on the leftmost of the figure. The top row illustrates the circle composition steps by the vertices X and Y consecutively, whereas the bottom row illustrates the circle composition steps by the vertices Y and X consecutively. Suppose that we applied circle compositions by the vertices X and Y consecutively. Then the first circle decomposition step generates a circle patch with centre Y. This circle patch is the patch on the top middle of Figure 3.75. Observe that $\{(A, B), (C, D)\}$ is not a split pair for that circle patch since A, B, C, D are all isolated vertices belonging to a same tile. Therefore, applying circle compositions by the vertices X and Y consecutively is not a good choice for the circle composition process. Observe also that if we apply circle compositions by the vertices Y and X consecutively, then the first circle decomposition step provides a circle patch with centre X. This patch is illustrated in the bottom middle patch of Figure 3.75. Notice that $\{(A, B), (C, D)\}$ is a split pair for that circle patch. Hence, we can conclude by Corollary 3.2.41 that $\{(A, B), (C, D)\}$ is a split pair for the cyclic patch given in the figure.

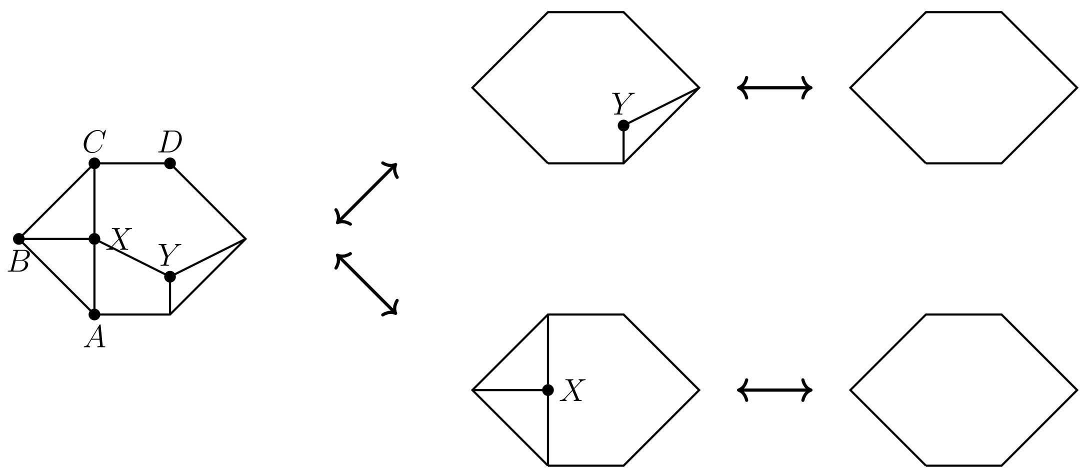


Figure 3.75: Two different circle composition steps

Finally, we are ready to prove the main result for cyclic patches.

Theorem 3.2.42. *Suppose Q is a cyclic patch consisting of convex tiles and u is a convex tile so that (Q, u) is a composition pair. Assume further, e_u is a 2-curve decoration of u with end point pairs A, B and C, D. Then the following holds:*

(1) (A', B') is a valid pair for Q for each distinct exterior vertices A', B'.

(2) $\{(A, B), (C, D)\}$ is a split pair for Q if there is no circle subpatch S of Q so that $|V_S \cap \{A, B, C, D, \}| \geq 2$. That is, $\{(A, B), (C, D)\}$ is a split pair for Q if every circle subpatch of Q contains at most one of the vertices A, B, C, D.

Proof. We will only prove part (2). The proof of part (1) is similar.

Suppose Q is a cyclic patch consisting of convex tiles and u is a convex tile such that (Q, u) is a composition pair. Suppose further e_u is a 2-curve decoration of u with end point pairs A, B and C, D. Choose a tile $t \in Q$ such that $A \in V_t$ and t has an edge $e \in E_t$ so that $e \subseteq \partial \text{supp } Q$. Apply the tree generator algorithm by the tile t. Let $G = (V, E)$ denote the tree generated.

Assume first that G is a path graph. Apply the circle composition algorithm such that t is circle composed in the final circle composition step. This can be done using Remark 3.2.36. Let $\Theta_{X_m} \circ \cdots \circ \Theta_{X_1}(Q)$ for $m \in \mathbb{Z}^+$ denote this circle composition process. Note that we have $t \in \Theta_{X_m}^{-1}(Q')$ by Remark 3.2.36, where $Q' = \Theta_{X_m} \circ \cdots \circ \Theta_{X_1}(Q)$ is the generated single tile patch. We have $A \in V_t$. So, we get $B \notin V_t$, $C \notin V_t$ and $D \notin V_t$, by the assumption given in the statement of (2). Therefore, $\{(A, B), (C, D)\}$ is a split pair for $\Theta_{X_m}^{-1}(Q')$ by Proposition 3.2.12. Thus, $\{(A, B), (C, D)\}$ is a split pair for Q by Corollary 3.2.41. By the same token, we can also conclude that (A', B') is a valid pair for Q for any given distinct exterior vertices A', B' of Q, by Corollary 3.2.41. Thus, the statements (1) and (2) hold whenever a cyclic patch corresponds to a path graph tree.

Suppose now that G is not a path graph. Let $X_1, \ldots, X_n$ for $n \in \mathbb{Z}^+$ denotes the collection of interior vertices of Q that belong to the tile t. Note that $\{X_1, \ldots, X_n\} \subseteq G$. There exists a path (sub)graph $G' \subseteq G$ such that $G' \supseteq \{X_1, \ldots, X_n\}$. Apply circle composition steps to the cyclic patch Q such that the generated patch P corresponds to the path graph G'. That is, apply circle composition steps such that the graph G reduces to the path graph G'. This can be done using Remark 3.2.36. Notice that P is a cyclic patch by Lemma 3.2.19. Therefore, by the case we just proved, $\{(A, B), (C, D)\}$ is a split pair for P by some decoration $\mathcal{C}_P$ of P.

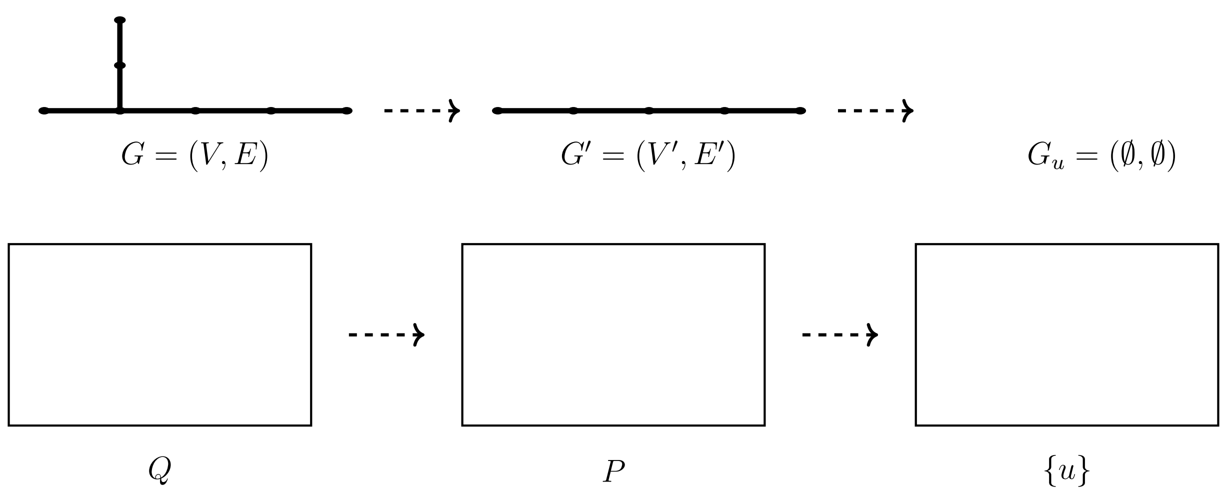


Figure 3.76: P is a composition of Q. The tree that corresponds to P is G'.

Since P is going to be circle decomposed to Q after finitely many steps, there must exist tiles of P which are going to be (circle) decomposed into subpatches of Q. Assume without loss of generality there exists only one tile $u_0 \in P$ which is going to be (circle) decomposed into a subpatch S_{u_0} of Q. Notice that S_{u_0} corresponds to the subgraph of G that does not intersect with either the vertices or the edges of G'. This is because of the fact that after circle composition applied by the vertices in S_{u_0}, we must arrive the path graph G'. In particular, S_{u_0} is a cyclic subpatch of Q by Lemma 3.2.19.

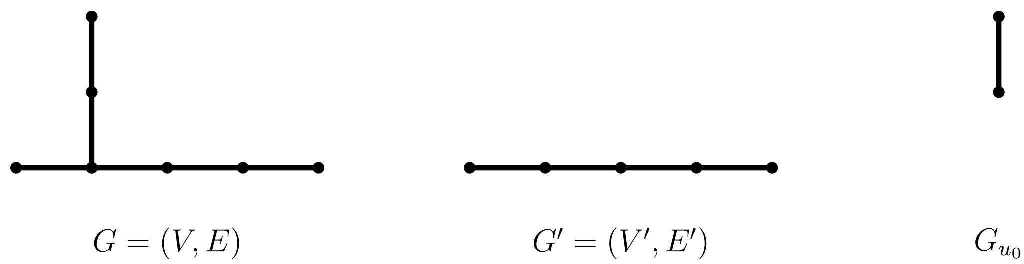


Figure 3.77: An example of G' and G_{u_0} where G_{u_0} denotes the graph corresponding to the cyclic patch S_{u_0}.

There are two cases; either the graph corresponds to S_{u_0} is a path graph or it is not a path graph. Suppose without loss of generality that S_{u_0} is a path graph. Assume further $G_{u_0} = (\{Y_1, \ldots, Y_s\}, \{[Y_i Y_{i+1}] : \ i \in \{1, \ldots, s-1\}\})$ denotes the graph corresponds to the cyclic subpatch S_{u_0} such that $s \in \mathbb{Z}^+$ and $[Y_s Z]$ is an edge of G for some vertex Z of G'.

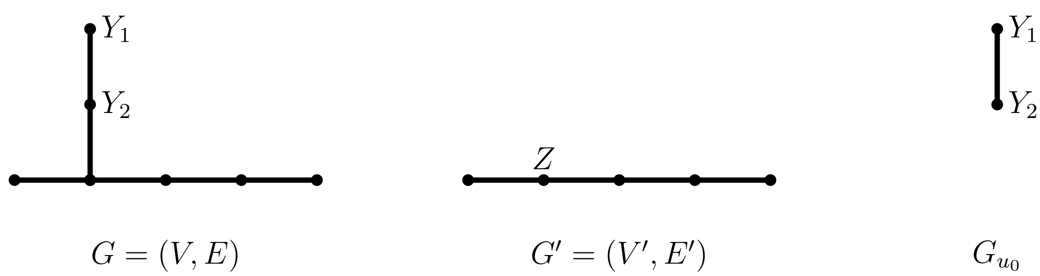


Figure 3.78: An example of G' and G_{u_0} for $s = 2$. $[Y_2 Z]$ is an edge of G and Z is a vertex of G'.

Let Z_1, Z_2 be two vertices of G' such that Z_1 is circle composed right before Z and Z_2 is circle composed right after Z. Since we circle compose by the order Z_1, Z, Z_2, we will circle decompose by the vertices Z_2, Z, Z_1, respectively. We illustrate the circle decomposition hierarchy in Figure 3.79.

Since $\{(A, B), (C, D)\}$ is a split pair for P by the curve $\mathcal{C}_P$, there exists a decoration e_{u_0} of u_0 which is induced by $\mathcal{P}$. The decoration e_{u_0} of u_0 induced by $\mathcal{C}_P$ is either a simple curve or a 2-curve. If it is a simple curve with end points X, X', then (X, X') is a valid pair for S_{u_0} because the corresponding graph for the cyclic patch S_{u_0} is a path graph. Thus, (A, B) is a valid pair for Q by Lemma 3.2.37. Suppose that e_{u_0} is a 2-curve. Note that u_0 is formed during the circle decomposition step by the vertex Z. That is, u_0 is a tile in a circle (sub)patch with centre Z, as shown in the middle patch of Figure 3.79. Using

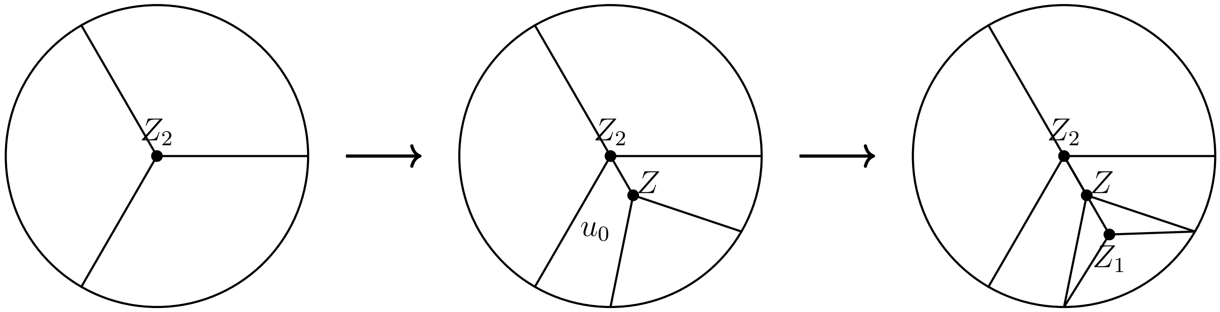


Figure 3.79: The circle decomposition hierarchy between the vertices Z, Z_1, Z_2. The tile u_0 is formed when we circle decompose by the vertex Z.

Corollary 3.2.40, we can assume without loss of generality that e_{u_0} must intersect with the centre Z of the circle patch that u_0 belongs to whenever e_{u_0} is a 2-curve decoration. That is, $Z \in e_{u_0}$. Suppose $\{u_0\} = \Theta_{Y_s} \circ \cdots \circ \Theta_{Y_1}(S_{u_0})$ denotes the circle composition steps of the cyclic patch S_{u_0} and X, X' and Y, Y' denote the end point pairs of e_{u_0} so that $\{Z\} \subseteq \{X, X', Y, Y'\}$.

The circle patch $\Theta_{Y_s}^{-1}(\{u_0\})$ contains the vertex Z as a shared exterior vertex since $[Y_s Z]$ is an edge of G, as demonstrated in Figure 3.80. Therefore, $\{(X, X'), (Y, Y')\}$ is a split pair for $\Theta_{Y_s}^{-1}(\{u_0\})$ by Proposition 3.2.12. Then $\{(X, X'), (Y, Y')\}$ is a split pair for S_{u_0} by Corollary 3.2.41 since the graph G_{u_0} corresponds to S_{u_0} is a path graph. Hence, $\{(A, B), (C, D)\}$ is a split pair for Q by Lemma 3.2.37. □

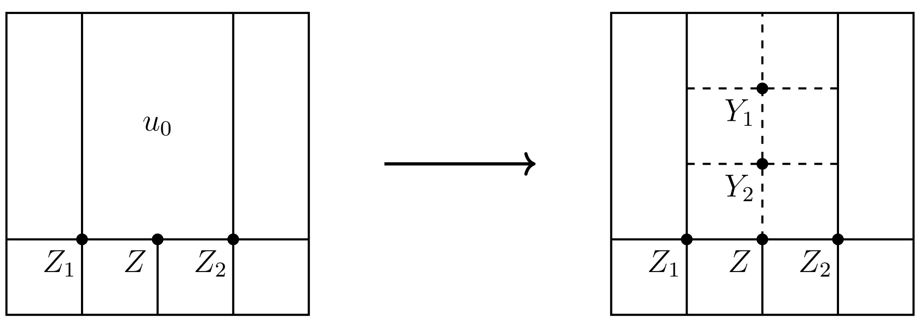


Figure 3.80: An example of the tile u_0 which is circle decomposed to a cyclic patch S_{u_0} such that $[Y_2 Z]$ is an edge of G and Z is a vertex of G'.

We are now ready to define the order structure for substitution tilings whose k-supertiles are cyclic patches for some $k \in \mathbb{Z}^+$.

Theorem 3.2.43. *Suppose $\mathcal{P}$ is a given finite collection of convex tiles, ω is a primitive substitution rule defined on $\mathcal{P}$ and T is a recognisable substitution tiling generated by the substitution rule ω. Assume further that every 1-supertile of T is a cyclic patch such that there is no circle subpatch S containing two distinct corners of the 1-supertile. Then there exist a finite collection of decorated prototiles $\mathcal{P}'$ and a primitive substitution rule ω' defined on $\mathcal{P}'$ such that the following holds:*

(1) For any $p' \in \mathcal{P}'$ there exists a unique $p \in \mathcal{P}$ such that p' is a decorated copy of p with label $l(p') = (l(p), e_p)$ for some decoration e_p for p.

(2) There exists $k \in \mathbb{Z}^+$ such that if p' is a decorated copy of p with decoration e_p, then

(i) $\text{supp } \omega_d(p') = \text{supp } \omega^k(p)$,

(ii) $\omega'(p')$ is a decorated copy of $\omega^k(p)$ with decoration $\mathcal{C}$ which has end points $\lambda^k \cdot A, \lambda^k \cdot B$ (or end point pairs $\lambda^k \cdot A, \lambda^k \cdot B$ and $\lambda^k \cdot C, \lambda^k \cdot D$) where A, B are end points of e_p (or A, B and C, D are end point pairs of e_p) and λ is the expansion factor of ω.

In particular, there exists a recognisable, primitive, self-similar substitution tiling T' with a prototile set $\mathcal{P}'$ and a substitution rule ω'.

Proof. Let $\mathcal{P}$ be a given finite collection of prototiles and ω be a primitive substitution defined on $\mathcal{P}$. Choose $q \in \mathcal{P}$ and a simple decoration e_q for q, with end points A, B. This generates a decorated prototile q' with a label set $l(q') = (l(q), e_q)$ and a support $\text{supp } q' = \text{supp } q$. Then $Q = \omega(q)$ is a cyclic patch such that no circle subpatch of it contains both vertices $\lambda \cdot A$ and $\lambda \cdot B$ together. We have that $(\lambda \cdot A, \lambda \cdot B)$ is a valid pair for the cyclic patch Q by Theorem 3.2.42. Let $\mathcal{C}$ denotes a curve that makes $(\lambda \cdot A, \lambda \cdot B)$ a valid pair. There exists a decorated copy Q' of Q with the decoration $\mathcal{C}$. Tiles in Q' have decorations induced from $\mathcal{C}$. Record Q' to be the substitute of p'.

For each prototile $p \in \mathcal{P}$ and every decoration e_p for p, construct the decorated prototile p' and the decorated 1-supertile Q'_p by the same argument applied for q' above. Record the generated decorated 1-supertiles Q'_p as the substitutions of p'. For every distinct pair of corners in each 1-supertile of T, there is no circle subpatch containing any two of the corners. Thus, for every prototile p in $\mathcal{P}$ and every decoration e_p on it, we can construct a decorated 1-supertile Q'_p such that we can regard Q'_p as the substitution of p'. This yields a collection of decorated prototiles $\overline{\mathcal{P}'}$ and a substitution rule $\overline{\omega'}$ defined on $\overline{\mathcal{P}'}$ such that

$$\overline{\mathcal{P}'} = \{p' : \ l(p') = (l(p), e_p), \ \text{supp } p' = \text{supp } p \ for \ p \in \mathcal{P} \ and \ decoration \ e_p \ of \ p\}$$

Proposition 2.2.6 assures that there exists $\mathcal{P}' \subseteq \overline{\mathcal{P}'}$ and $n \in \mathbb{Z}^+$ such that $\omega' = \overline{\omega'^n}|_{\mathcal{P}'}$ is a primitive substitution over $\mathcal{P}'$ satisfying the conditions (1) and (2). We can now generate a tiling T' from the primitive substitution ω' by applying the standard argument explained in Chapter 2. Hence, T' is recognisable, primitive and self-similar substitution tiling. □

3.3 The Algorithm

The travelling algorithm is the generalisation of the algorithms in the previous sections. Consider the substitution rule given in Figure 3.81. The substitution rule is defined on equilateral triangles of side length 1 and unit hexagons, with a fixed expansion factor $\lambda = 3$. The (primitive) substitution rule generates a recognisable primitive substitution tiling. Recognisability can be inferred by detecting the star shapes inside the 1-supertile hexagons. Observe that every triangle type k-supertile for $k \in \mathbb{Z}^+$ contains at least one slice tile. Therefore the arguments in Section 3.2 cannot be applied for this substitution rule. Though, we use a similar idea of composition for this type of substitutions as well. We first form a cyclic patch by composing the parts containing the slice tiles, into single tiles, and then apply the methods in Section 3.2. Lastly, we decompose the tiles that are composed from slice tiles, and reform the decorations appearing on the tiles into curves visiting the slice tiles.

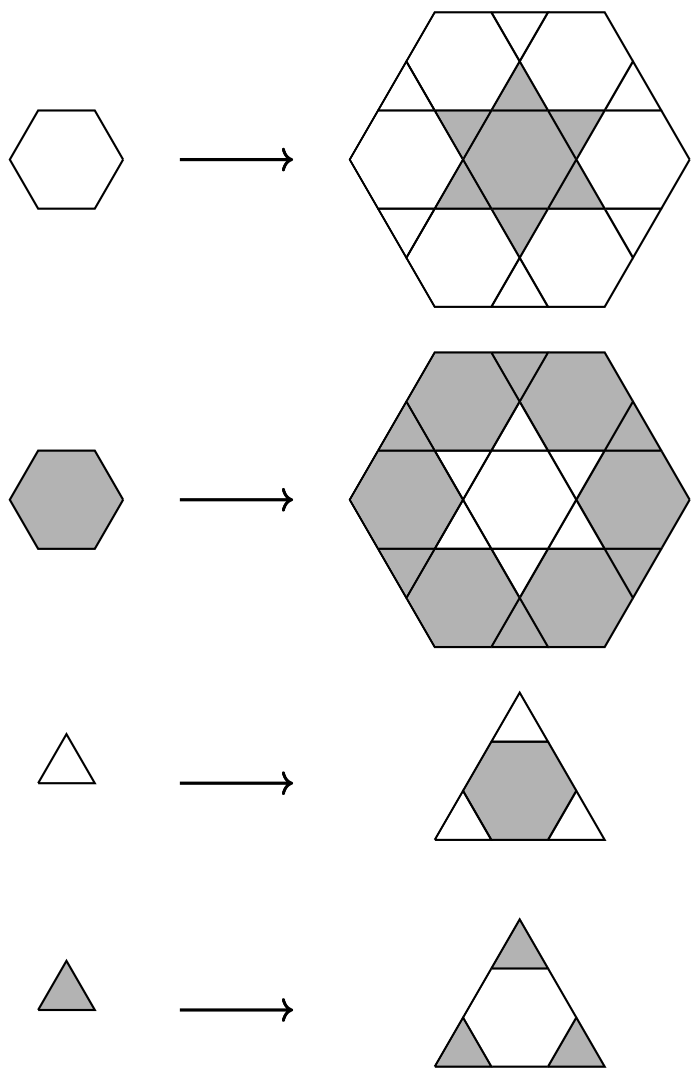


Figure 3.81: A substitution rule

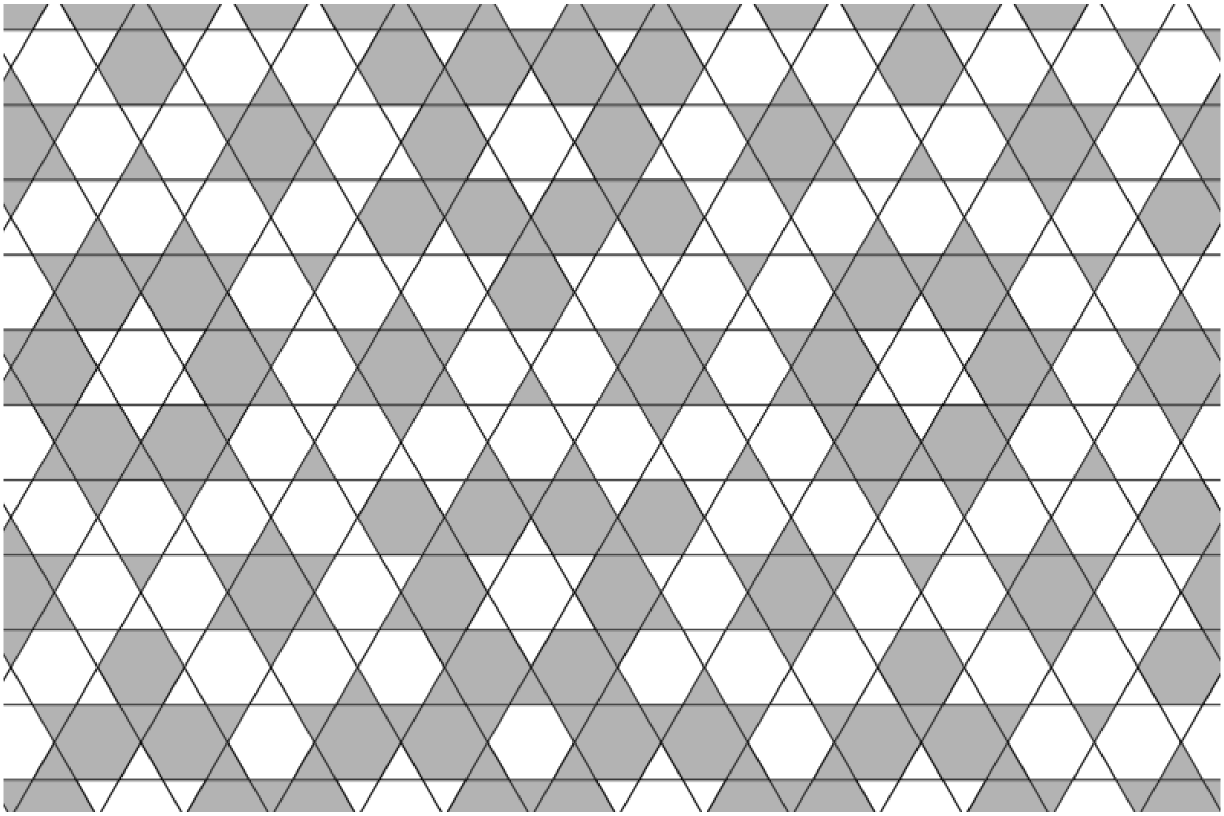


Figure 3.82: A patch of the tiling formed by the substitution rule given in Figure 3.81

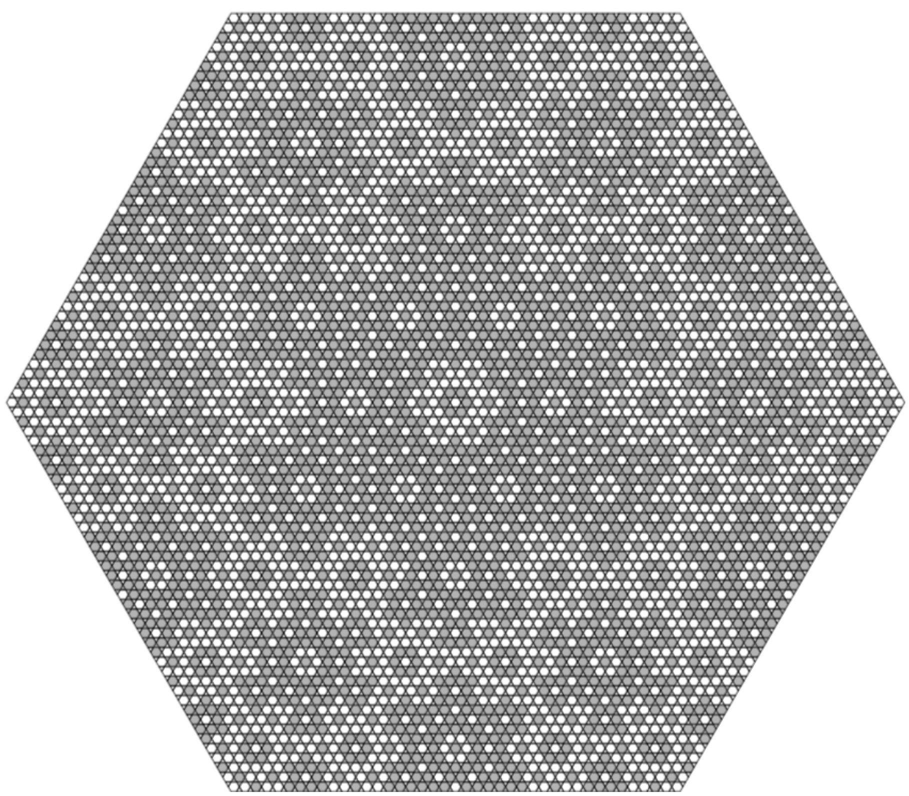


Figure 3.83: A 5-supertile formed by the white hexagon prototile in Figure 3.81

A slice tile t in a patch Q is said to be of *degree* n if int(supp $Q\backslash\{t\}$) has n connected components. For example, the tile t in the patch on the left of Figure 3.84 is a slice tile of degree 4. In order to define a decoration with end points A and B, for the patch in the figure, we have to visit the tile t three times, as demonstrated on the right side of the figure. Therefore, we cannot define a valid decoration with end points A and B, for the patch in the figure. For that reason we want to avoid getting slice tiles of degree higher than 2, in a substitution.

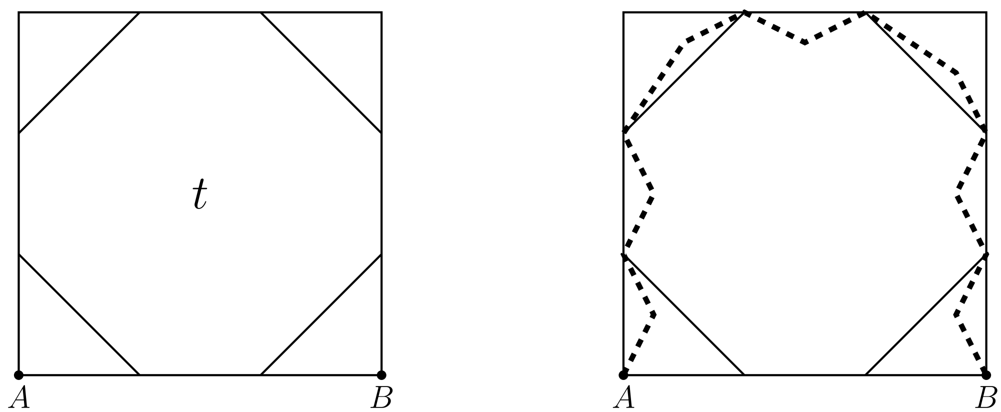


Figure 3.84: t must be visited at least three times in order to define a decoration from A to B.

In order to prevent slice tiles of degree more than two to appear in a substitution, we expand the substitution a sufficient number of times, so that the slice tiles will be pushed to the corners of the convex patch. This is explained in details in Lemma 3.3.1.

Lemma 3.3.1. *For any given substitution rule ω defined over a finite collection of convex prototile set $\mathcal{P}$, with an expansion factor $\lambda > 1$, there exists $k \in \mathbb{Z}^+$ such that every slice tile in each n-supertile for $n \geq k$ has degree 2.*

Proof. Suppose that $\mathcal{P} = \{p_1, \ldots, p_m\}$ for $m \in \mathbb{Z}^+$ denotes a finite collection of convex prototiles and ω denotes a substitution rule defined over $\mathcal{P}$ with an expansion factor $\lambda > 1$. Define $d = \min_{p\in\mathcal{P}}\{l(e) : \ l(e)$ *denotes the length of* $e \in E_p\}$ and $D = \max_{p\in\mathcal{P}}\{l(e) :$ $l(e)$ *denotes the length of* $e \in E_p\}$. Let $V_j = \{v_1^j, v_2^j, \ldots, v_{m_j}^j\}$ denotes the collection of vertices of the prototile p_j for $m_j \in \mathbb{Z}^+$ and $j \in \{1, \ldots, m\}$. For each $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, m_j\}$ define two points $u_{i,1}^j, u_{i,2}^j \in \partial$supp p_j so that $d(v_i^j, u_{i,1}^j) = d(v_i^j, u_{i,2}^j) = d/2$ where $d(v_i^j, u_{i,s}^j)$ denotes the distance between the points v_i^j and $u_{i,s}^j$ for $s = 1, 2$. For each prototile p_j for $j \in \{1, \ldots, m\}$, define the subset $S_j \subseteq$ supp p_j such that S_j is the convex hull consisting of the vertices $\{u_{1,1}^j, u_{1,2}^j, u_{2,1}^j, u_{2,2}^j, \ldots, u_{m_j,1}^j, u_{m_j,2}^j\}$. This process is illustrated in Figure 3.85 with a triangle prototile example. The shaded area in the figure represents the convex set S_j. We have that S_j separates supp p_j into m_j many triangles for each $j = 1, \ldots, m$. These triangles consist of the vertices $v_i^j, u_{j,1}^j, u_{j,2}^j$, as demonstrated in Figure 3.86. Let R_i^j for $j = 1, \ldots, m$ and $i \in \{1, \ldots, m_j\}$ denotes the triangle consisting of the vertices $v_i^j, u_{j,1}^j, u_{j,2}^j$. Then we have supp $p_j = S_j \cup \bigcup_{i=1}^{m_j} R_i^j$ for each $j = 1, \ldots, m$. Similarly, we have $\lambda^s \cdot \text{supp } p_j = (\lambda^s \cdot S_j) \cup \bigcup_{i=1}^{m_j} \left(\lambda^s \cdot R_i^j\right)$ for each $s \in \mathbb{Z}^+$ and for each $j = 1, \ldots, m$.

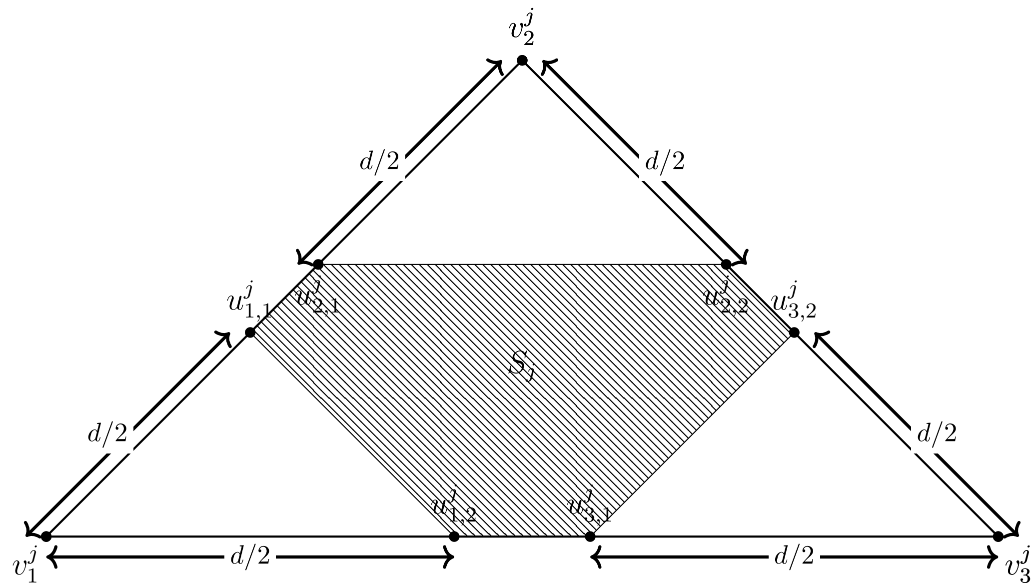


Figure 3.85: An illustration of p_j and S_j for the case p_j is a triangle with vertices v_1^j, v_2^j, v_3^j. S_j is the shaded area.

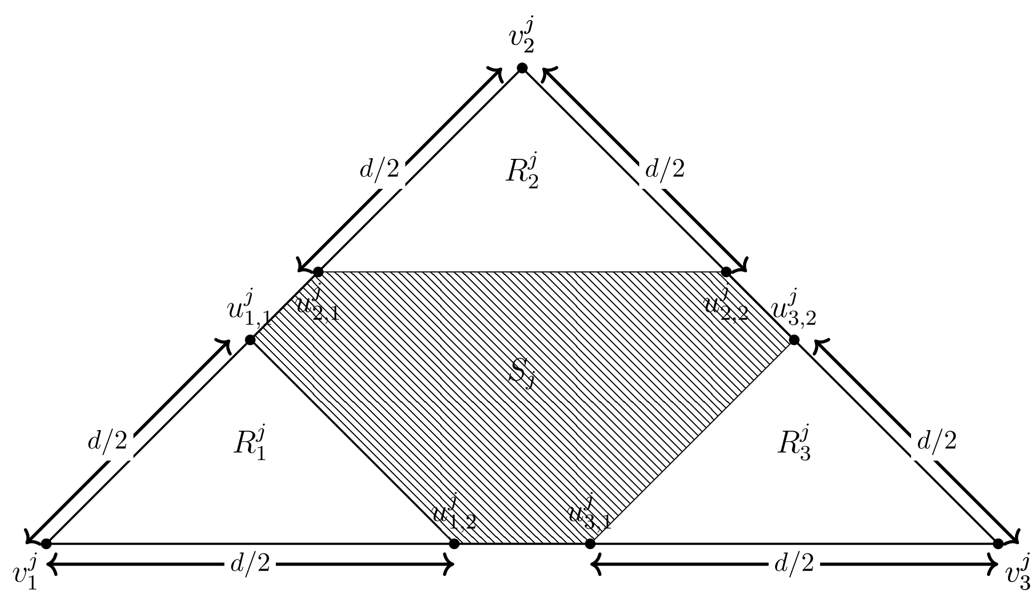


Figure 3.86: Demonstration of R_1^j, R_2^j, R_3^j for the prototile p_j given in Figure 3.85.

Let d_s^j for $j \in \{1, \ldots, m\}$ and $s \in \mathbb{Z}^+$ denote the minimum distance of a slice edge of

$\omega^s(p_j)$ that is completely contained in the convex region $\lambda^s \cdot S_j$. More precisely,

$$d_s^j = \min\{d(X,Y) : \ [XY] \cap (\lambda^s \cdot \partial S_j) \cap (\partial \text{supp } (\omega^s(p_j))) = \{X,Y\}\}$$

where $d(X,Y)$ denotes the Euclidean distance between the points X, Y and $[XY]$ denotes the line connecting the points X, Y. Choose $k \in \mathbb{Z}^+$ sufficiently large such that $d_n^j \geq D$ for each $n \geq k$. We will show that every slice tile in any n-supertile $\omega^n(p_j)$ for $n \in \mathbb{Z}^+$ and $j \in \{1, \ldots, m\}$ is a slice tile of degree 2.

Note first that every slice tile t in an r-supertile patch $\omega^r(p_j)$ for some $r \in \mathbb{Z}^+$ and $j \in \{1, \ldots, m\}$ is of degree 2 whenever supp $t \subseteq ((\lambda^r \cdot \text{supp } p_j) \backslash (\lambda^r \cdot S_j))$. That is, if t is a slice tile whose support is completely contained in one of the triangle components of $(\lambda^r \cdot \text{supp } p_j) \backslash (\lambda^r \cdot S_j)$, then t is a slice tile of degree 2. This is because of the fact that supp t is convex and the connected components of $(\lambda^r \cdot \text{supp } p_j) \backslash (\lambda^r \cdot S_j)$ are triangles. Therefore, a slice tile t in an r-supertile patch $\omega^r(p_j)$ for some $r \in \mathbb{Z}^+$ and $j \in \{1, \ldots, m\}$ is of degree greater than 2 if one of the following holds:

(i) supp $t \subseteq \lambda^r \cdot S_j$,

(ii) supp $t \subsetneq \lambda^r \cdot S_j$ and supp $t \cap \lambda^r \cdot S_j \neq \emptyset$.

In either case t must contain an edge e_t so that $l(e_t) \geq d_r^j$ where $l(e_t)$ denotes the length of the edge e_t. On the other hand, we have that $d_n^j \geq D$ for all $n \geq k$. That is, the minimum distance in the region $\lambda^n \cdot S_j$ for each $j \in \{1, \ldots, m\}$ is greater or equal than D. Therefore, no slice tile of an n-supertile patch can contain an edge e_t such that $l(e_t) \geq d_n^j$. Hence, every slice tile in an n-supertile patch $\omega^n(p_j)$ for $j \in \{1, \ldots, m\}$ is a slice tile of degree 2. □

Lemma 3.3.1 ensures that slice tiles can be arranged to be 'close' to the corners of the supertiles. Next, we show that if a substitution is expanded a sufficient number of times, then no slice tile can intersect with the corners of the supertiles (Lemma 3.3.2).

Lemma 3.3.2. *Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ for $m \in \mathbb{Z}^+$ be a finite collection of convex tiles and let ω be a substitution rule defined on $\mathcal{P}$ with a fixed expansion factor $\lambda > 1$. Suppose further $V_j = \{v_1^j, \ldots, v_{m_j}^j\}$ denotes the collection of vertices of the prototile p_j for $m_j \in \mathbb{Z}^+$ and $j \in \{1, \ldots, m\}$. Then there exists $k \in \mathbb{Z}^+$ such that if t is a slice tile in an n-supertile $\omega^n(p_j)$ for some $n \geq k$ and $j \in \{1, \ldots, m\}$, then $V_t \cap \{\lambda^n \cdot v_1^j, \ldots, \lambda^n \cdot v_{m_j}^j\} = \emptyset$. That is, the tiles in an n-supertile patch $\omega^n(p_j)$ for some $n \geq k$ and $j \in \{1, \ldots, m\}$ that intersect with any of the corners of $\omega^n(p_j)$ are necessarily non-slice tiles.*

Proof. Suppose $\mathcal{P} = \{p_1, \ldots, p_m\}$ for $m \in \mathbb{Z}^+$ is a collection of convex prototiles and ω is a substitution rule defined on $\mathcal{P}$ with an expansion factor $\lambda > 1$. Assume further $V_j = \{v_1^j, \ldots, v_{m_j}^j\}$ denotes the collection of vertices of the prototile p_j for $m_j \in \mathbb{Z}^+$

and $j \in \{1, \ldots, m\}$. Define $d = \min_{p \in \mathcal{P}}\{l(e) : \textit{ l(e) denotes the length of } e \in E_p\}$ and $D = \max_{p \in \mathcal{P}}\{l(e) : \textit{ l(e) denotes the length of } e \in E_p\}$. Choose $r \in \mathbb{Z}^+$ such that $\lambda^r \cdot d > D$. Next we will show that $k = 2 \cdot r$ satisfies the conclusion of the lemma.

Suppose that t is a slice tile in an r-supertile $\omega^r(p_j)$ such that $\lambda^r \cdot v^j_{i_0} \in V_t$ for some $j \in \{1, \ldots, m\}$ and $i_0 \in \{1, \ldots, m_j\}$. That is, the slice tile t contains a corner of the supertile $\omega^r(p_j)$. Then t contains a slice edge e_t with end points $v^j_{i_0}$ and $u^j_{i_0}$ such that $u^j_{i_0} \in \partial\text{supp }(\omega^r(p_j))$. Then the patch $\omega^k(p_j) = \omega^{2 \cdot r}(p_j)$ contains a subpatch $\omega^r(t)$ which corresponds to the substitution of t. Moreover, there exists a line $L = \lambda^r \cdot e_t$ which corresponds to the substitution of the edge e_t. Since we choose $r \in \mathbb{Z}^+$ such that $\lambda^r \cdot d > D$, the line L cannot belong to a single tile. That is, L cannot be a slice edge in the patch $\omega^k(p_j)$. This process is illustrated in Figure 3.87. The patch on the left of the figure represents the patch $\omega^r(p_j)$ for some prototile $p_j \in \mathcal{P}$. The patch on the right shows how the tile t is substituted to $\omega^r(t)$. Observe that the slice edge e_t for the patch on the left is substituted to a line that is not a slice edge for the patch on the right of the figure.

Figure 3.87

Hence, the tiles in an n-supertile patch $\omega^n(p_j)$ for some $n \geq k$ and $j \in \{1, \ldots, m\}$, and intersect with any of the corners of $\omega^n(p_j)$ are necessarily non-slice tiles. □

Corollary 3.3.3. *Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ for $m \in \mathbb{Z}^+$ be a finite collection of convex tiles and let ω be a substitution rule defined on $\mathcal{P}$ with a fixed expansion factor $\lambda > 1$. Suppose further $V_j = \{v^j_1, \ldots, v^j_{m_j}\}$ denotes the collection of vertices of the prototile p_j for $m_j \in \mathbb{Z}^+$ and $j \in \{1, \ldots, m\}$. Then there exists $k \in \mathbb{Z}^+$ such that the following holds:*

(1) if t is a tile in an n-supertile $\omega^n(p_j)$ for some $n \geq k$ and $j \in \{1, \ldots, m\}$, then $V_t \cap \{\lambda^n \cdot v^j_1, \ldots, \lambda^n \cdot v^j_{m_j}\} = \emptyset$.

(2) Every slice tile in each n-supertile for $n \geq k$ has degree 2.

Proof. The proof follows by Lemma 3.3.1 and Lemma 3.3.2. □

Finally, we are ready to group slice tiles appearing in a supertile patch, using Corollary 3.3.3. We showed in Corollary 3.3.3 that if a substitution is expanded sufficiently many times, then every slice tile is 'close to' a corner, though not touching the corner. Therefore, every slice tile separates the corners of the supertiles in two sets, one of which is a singleton. In particular, we have the following definition.

Definition 3.3.4. Suppose Q is a patch satisfying the following:

(i) supp Q is a convex n-gon with vertices $X_1, \dots, X_n$ for some $n \in \mathbb{Z}^+$ (i.e. $X_1, \dots, X_n$ are corners of the convex patch Q),

(ii) Every tile u of Q is convex,

(iii) Every slice tile u' of Q is a slice tile of degree 2 so that

(iii-a) $V_{u'} \cap \{X_1, \dots, X_n\} = \emptyset$,

(iii-b) If S_1 and S_2 are the two connected components of the collection $Q \backslash \{u'\}$, then $|V_{S_1} \cap \{X_1, \dots, X_n\}| = 1$ and $|V_{S_2} \cap \{X_1, \dots, X_n\}| = n - 1$.

Then a slice tile $t \in Q$ separates the corners of Q into two sets, one of which is a single vertex $\{X\}$. We call t the *slice tile around the vertex* X.

An example of a slice tile around a vertex is given in Figure 3.88. The slice tile t in the patch in Figure 3.88 separates the corners of the patch into two parts, one of which is a single vertex set $\{X\}$. Therefore, t is a slice tile around the vertex X. By identifying slice tiles with the corners of patches that has convex support, we can group the slice tiles at the same vertices into patches.

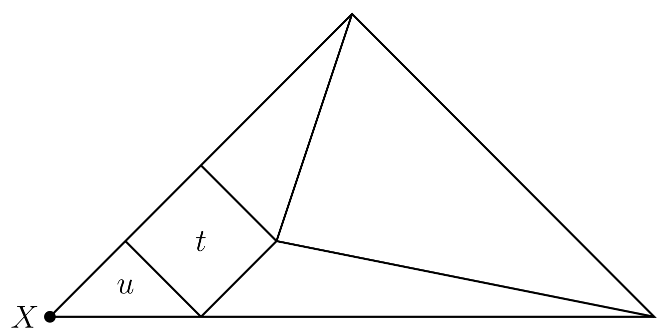


Figure 3.88: t is a slice tile around the vertex X.

Definition 3.3.5. Suppose S_X denotes the smallest collection of tiles in a given patch Q such that if S_X is composed into a single tile u_X, then $X \in V_{u_X}$ and the composed patch $Q' = (Q \backslash S_X) \cup \{u_X\}$ does not contain a slice tile around the vertex X. We call S_X the *slice cluster* of Q around X.

For example, the patch in Figure 3.88 has a slice cluster $S_X = \{t, u\}$ around the vertex X. In the next section we will examine the simplest form of slice clusters.

3.3.1 Strings

In this section we study a special class of patches called *strings*. Strings are analogous to path graphs in the graph theory (See the discussion at the end of this chapter).

Definition 3.3.6. A patch Q is called a *string* if either of the following holds:

(1) Q is a single tile patch, or

(2) Q contains exactly two non-slice tiles, and every tile t of Q has an edge e such that $e \subseteq \partial\text{supp}\ Q$.

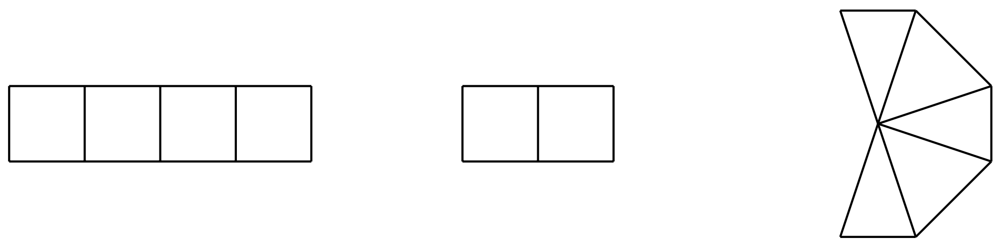


Figure 3.89: Examples of strings

Lemma 3.3.7. *No patch can only consist of slice tiles.*

Proof. Assume to the contrary. Let Q be a patch and $u_1 \in Q$ is a slice tile in Q. There exists a subpatch $Q_1 \subsetneqq Q$ such that Q_1 is one of the connected component patches that u_1 separates. Choose a tile u_2 in Q_1. Because u_2 is a slice tile in Q, there exists a subpatch Q_2 that u_2 separates so that $Q_2 \subsetneqq Q_1$. Continuing the same process leads to a contradiction since $|Q|$ is finite and can only have finitely many subpatches. □

Corollary 3.3.8. *Every patch which is not a single tile patch must contain at least two non-slice tiles.*

Proof. Let Q be a patch such that $|Q| \geq 2$. Assume to the contrary that Q contains at most one non-slice tile. Choose a slice tile u in Q. Since it separates Q, one of the component patches, which is separated by u, must consists of solely slice tiles. A contradiction, by Lemma 3.3.7. □

Corollary 3.3.9. *Every slice tile in a string is of degree* 2.

Proof. Assume to the contrary. Suppose u is a slice of degree 3 or more for a string Q. Then there are at least three disjoint subpatches that u separated. Because Q contains exactly two non-slice tiles, one of the patches that u separated does not contain a non-slice tile, which contradicts Lemma 3.3.7. □

Lemma 3.3.10. *Suppose Q is a string which is not a single tile patch and u is a non-slice tile of Q. Then we have $|\{t \in Q : \exists e \in E_u \text{ so that } e \in E_t \cap E_u\}| = 1$. That is, u can only have one 'neighbour tile' in Q which shares an edge with u.*

Proof. The statement holds for any string consisting two tiles. Suppose Q is a string consisting of at least three tiles with non-slice tiles u_1, u_2.

Assume to the contrary. Let $t_1, t_2 \in Q$ such that $E_{u_1} \cap E_{t_i} \neq \emptyset$ and $u_1 \neq t_i$ for each $i = 1, 2$. Since $u_1 \neq t_i$ for $i = 1, 2$, we have one of the following holds:

(i) $u_2 \neq t_i$ for each $i = 1, 2$,

(ii) $u_2 = t_1$ or $u_2 = t_2$.

In either case, (at least) one of t_1 or t_2 must be a slice tile for Q because u_1, u_2 are the only non-slice tiles of the string Q. Assume without loss of generality t_1 is a slice tile in Q. We have that t_1 is a slice tile of degree 2 by Corollary 3.3.9. Then t_1 must separate Q into two subpatches P_1, P_2 such that $P_1 \cup P_2 = Q \backslash \{t_1\}$ and $P_1 \cap P_2 = \emptyset$, as shown on the left side of Figure 3.90. Since t_2 and u_1 share a common edge, they must belong to the same connected subpatch, say P_1. Moreover, u_2 must belong to P_2 because P_2 cannot contain only slice tiles by Lemma 3.3.7. Since $t_2 \in P_1$ and $u_2 \in P_2$, we get $t_2 \neq u_2$. Therefore, t_2 a slice tile for Q as well. We have that $u, t_2 \in P_1$ and t_2 is a slice tile of degree 2 in Q by Corollary 3.3.9. So, t_2 separates Q into two subpatches S_1, S_2 such that $S_1 \subsetneq P_1$ and $P_2 \subsetneq S_2$, as shown on the right side of Figure 3.90.

Figure 3.90

Since u_1 and t_1 share a common edge, u_1 and t_1 must both belong to S_1. On the other hand, $t_1 \notin P_1$ because $Q \backslash \{t\} = P_1 \cup P_2$. That is, $t_1 \notin S_1$, a contradiction. □

Lemma 3.3.11. *Let Q be a string. Then for each slice tile $u \in Q$, there are exactly two arcs $\gamma_1, \gamma_2 \subseteq \partial \text{supp } u$ over the boundary of u such that $|\gamma_1 \cap \gamma_2| \leq 1$ and $|\gamma_i \cap \partial \text{supp } Q| = 2$ for $i = 1, 2$. That is, u must have exactly two arcs over its boundary that intersect but are not completely contained within the boundary $\partial \text{supp } Q$.*

Proof. Since each slice tile u of Q has degree 2 by Corollary 3.3.9, there exists two subpatches S_1, S_2 that u separates. Define $\gamma_i = \partial \text{supp } S_i \cap \partial \text{supp } u$ for $i = 1, 2$. Then γ_1, γ_2 satisfy the desired conditions. □

Remark 3.3.12. Taking Lemma 3.3.11 into account, we can identify each slice tile in a string by either a triangle or a rectangle. In particular, if u is a slice tile in a string Q, then $|V_u \cap \partial \text{supp } Q| = 3$ or $|V_u \cap \partial \text{supp } Q| = 4$, by Lemma 3.3.11.

Lemma 3.3.13. *Let Q be a string consisting at least two tiles and $u \in Q$ be a non-slice tile in Q. Then $Q \backslash \{u\}$ is a string.*

Proof. Suppose Q is a string. If Q consists of two tiles t_1, t_2 then $Q\backslash\{t_i\}$ is a single tile for each $i = 1, 2$.

Suppose Q is a string consisting of at least three tiles. Suppose further u and v denote the only non-slice tiles in Q. There exists a tile $t \in Q\backslash\{u, v\}$ such that t shares an edge with u and t is a slice tile of Q. The tile t is unique by Lemma 3.3.10. We have that $Q\backslash\{u\}$ is a well defined patch because u is a non-slice tile of Q. Moreover t and u are the only non-slice tiles in $Q\backslash\{u\}$. Thus, $Q\backslash\{u\}$ is a string. □

Lemma 3.3.14. *Suppose Q is a string and u is a tile such that* supp u *and* supp Q *intersect along a single common edge; i.e.* supp $Q \cap$ supp $u = e$ *for some* $e \in E_u \cap E_Q$. *Then* $Q' = Q \cup \{u\}$ *is a string.*

Proof. Since supp Q and supp u intersect along a single common edge e, supp Q' = supp $(Q \cup \{u\})$ is simply connected. So, Q' is a (well-defined) patch and e is a slice edge of it. Hence, Q' contains only two non-slice tiles and is a string. □

Lemma 3.3.15. *For any given circle patch Q and a tile $t \in Q$, the collection $Q\backslash\{t\}$ is a string.*

Proof. We have that supp $Q\backslash\{t\}$ is simply connected because every tile contains the centre of Q. Therefore, $Q\backslash\{t\}$ is a subpatch of Q. Suppose $Q\backslash\{t\}$ is not a single tile patch. Since every tile in a circle patch has exactly two neighbour tiles, every tile in $Q\backslash\{t\}$ must contain two neighbour tiles, except the tiles $u, u' \in Q$ which are neighbour tiles of $t \in Q$. These are the only non-slice tiles in $Q\backslash\{t\}$. Hence, $Q\backslash\{t\}$ is a string. □

Orders in Strings

Proposition 3.3.16. *Suppose Q is a string consisting of at least two tiles, u is a non-slice tile in Q and A is an isolated vertex of Q with $A \in V_u$. Then (A, B) is a valid pair for Q for any vertex $B \neq A$.*

Proof. We will prove the statement by strong induction. Assume first $Q = \{t_1, t_2\}$ is a string consisting of two tiles and A is an isolated vertex of Q with $A \in V_{t_1}$. Suppose further X, Y denote the shared exterior vertices of Q and B is a given exterior vertex of Q such that $B \neq A$. We have three cases:

(i) $B = X$ (or $B = Y$),

(ii) $B \neq X$, $B \neq Y$ and $B \in V_{t_1}$,

(iii) $B \neq X$, $B \neq Y$ and $B \in V_{t_2}$.

If $B = X$, then define decorations for t_1 and t_2 with end points A, Y and Y, X, respectively. Then (A, B) is a (simple) valid pair for Q.

If $B \neq X$, $B \neq Y$ and $B \in V_{t_1}$, then define a 2-curve decoration for t_1 with end point pairs A, X and B, Y, and a simple curve decoration for t_2 with end points X, Y. Then (A, B) is a valid pair for Q.

If $B \neq X$, $B \neq Y$ and $B \in V_{t_2}$, then define decorations for t_1 and t_2 with end points A, Y and Y, B, respectively. Then (A, B) is a (simple) valid pair for Q.

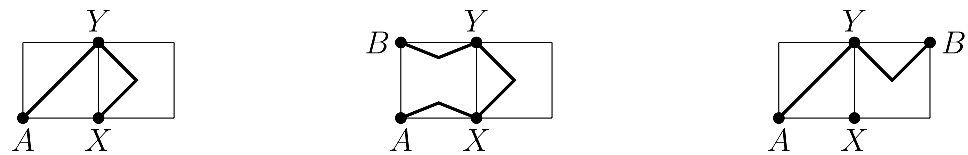


Figure 3.91: The three cases $(i) - (iii)$ are demonstrated from left to right, respectively

Thus the statement holds for every string consisting of two tiles. Suppose the statement holds for every string consisting of at most n tiles for $n \geq 2$. Assume that Q is a string with $|Q| = n + 1$, u is a non-slice tile in Q and A is an isolated vertex of Q with $A \in V_u$. We have that $Q \backslash \{u\}$ is a string by Lemma 3.3.13. Let X, Y denote the shared exterior vertices of Q such that $X, Y \in V_{Q \backslash \{u\}}$. We have two cases:

(1) X or Y (or both) is an isolated vertex for the string $Q \backslash \{u\}$,

(2) Both X and Y are shared exterior vertices for the string $Q \backslash \{u\}$.

These cases are illustrated in Figure 3.92. The middle string is an example of the case where Y and X are shared exterior vertices of $Q \backslash \{u\}$, the string on the left is an example of the case where only X is an isolated vertex of $Q \backslash \{u\}$ and the string on the right is an example for both X and Y are isolated vertices of $Q \backslash \{u\}$. Notice that Case (2) is not possible since every tile in Q must contain an edge which is completely contained over the boundary of Q. Therefore, either X or Y (or both) is an isolated vertex for the string $Q \backslash \{u\}$.

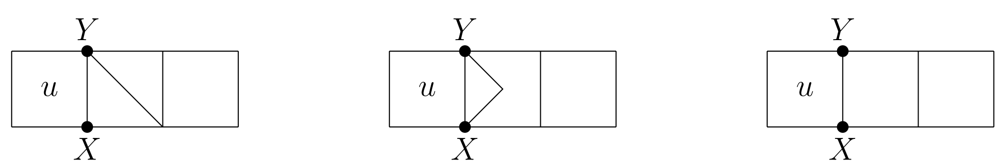


Figure 3.92

We have three subcases:

(a) $B = X$ (or $B = Y$),

(b) $B \neq X$, $B \neq Y$ and $B \in V_u$,

(c) $B \neq X$, $B \neq Y$ and $B \notin V_u$.

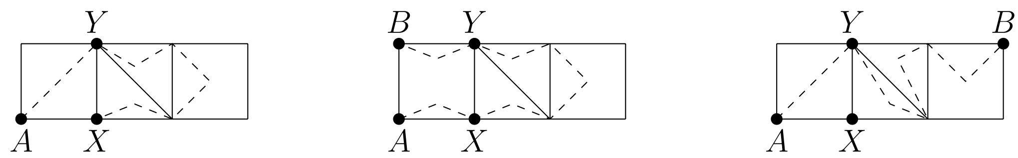


Figure 3.93: Case (a) to (c) are demonstrated from left to right, respectively

If $B = X$ (or $B = Y$), then define a decoration for u with end points A and Y (or X). Since $Q \backslash \{u\}$ is a string with X being an isolated vertex of it, (X, Y) is a valid pair for $Q \backslash \{u\}$, by the strong induction assumption. Thus, (A, B) is a valid pair for Q, as demonstrated on the leftmost patch in Figure 3.93.

If $B \neq X$, $B \neq Y$ and $B \in V_u$, then define a 2-curve decoration for u with end point pairs A, X and B, Y. Since $Q \backslash \{u\}$ is a string with X being an isolated vertex of it, (X, Y) is a valid pair for $Q \backslash \{u\}$, by the strong induction assumption. Thus, (A, B) is a valid pair for Q, as demonstrated on the middle patch in Figure 3.93.

If $B \neq X$, $B \neq Y$ and $B \notin V_u$, then define a decoration for u with end points A and Y. Once again, because (X, Y) is a valid pair for $Q \backslash \{u\}$, (A, B) is a valid pair for Q, as demonstrated on the rightmost patch in Figure 3.93. □

Corollary 3.3.17. *Suppose Q is a string and u, v are the two non-slice tiles of Q. Assume (Q, t) is a composition pair and e_t is a decoration of t with end point pairs A, B and C, D such that $A, B, C \in V_u$ and D is an isolated vertex of Q with $D \in V_v$. Then $\{(A, B), (C, D)\}$ is a split pair for Q.*

Proof. Assume without loss of generality Q have at least two tiles. Then u can have one neighbour tile in Q by Lemma 3.3.10. Therefore there are exactly two shared exterior vertices of Q that belong to u. Therefore, one of A, B, C is an isolated vertex of Q. Suppose further without loss of generality A is an isolated vertex of Q. Define $Q_1 = \{u\}$ and

$$Q_2 = \begin{cases} Q & \textit{if } C \textit{ is an isolated exterior vertex of } Q \\ Q \backslash \{u\} & \textit{if } C \textit{ is a shared exterior vertex of } Q. \end{cases}$$

We have that $Q \backslash \{u\}$ and Q are both strings by Lemma 3.3.13. Define also $\mathcal{C}_1$ to be a simple curve that connects A and B in Q_1. Then $\mathcal{C}_1$ makes (A, B) a valid pair for Q_1. Moreover, there exists a curve $\mathcal{C}_2$ that makes (C, D) a valid pair for Q_2, by Proposition 3.3.16. Since these curves can be arranged not to cross each other, $\{(A, B), (C, D)\}$ is a split pair for Q. □

Remark 3.3.18. The only order structure result we defined for strings are Proposition 3.3.16 and Corollary 3.3.17. The decorations we defined for the tiles in strings, in the proofs of the proposition and the corollary, have end points at the boundary of the string. That is, the decorations are not visiting any interior vertices in a string. Therefore, together with the comments on Remark 3.3.12, we can assume without loss of generality that all the tiles in a string is either a triangle or a rectangle.

Circle Decompositions in Strings

Lemma 3.3.19. *Suppose Q is a string consisting of rectangles and triangles and u_1, u_2 are two non-slice tiles of Q. Assume further that t is a tile of Q and Q' is a patch which is generated by finitely many circle decomposition steps over Q so that there exists a cyclic subpatch S_t of Q' with $Q' \backslash S_t = Q \backslash \{t\}$. That is, Q' is generated by decomposing the tile t in Q. Then we have:*

(1) If (A, B) is a valid pair for Q for some distinct exterior vertices A, B of Q, then (A, B) is a valid pair for Q' as well.

(2) If $\{(A, B), (C, D)\}$ is a split pair for Q for some distinct exterior vertices A, B, C, D of Q, then $\{(A, B), (C, D)\}$ is a split pair for Q' as well.

Proof. We will only prove (2). The proof of (1) is similar. Suppose $\{(A, B), (C, D)\}$ is a split pair for Q by some decoration $\mathcal{C}$. Then $\mathcal{C}$ induces a decoration e_t for t. We have two cases; either e_t is a simple decoration with end points M, N or e_t is a 2-curve decoration with end point pairs M, M' and N, N'.

Suppose that e_t is a simple decoration with end points M, N. We have that (M, N) is a valid pair for the cyclic patch S_t by (1) of Theorem 3.2.42. Therefore, $\{(A, B), (C, D)\}$ is a split pair for Q' as well, by Lemma 3.2.37.

Suppose now e_t is a 2-curve decoration with end point pairs M, M' and N, N'. We define a circle composition process for S_t, by the circle composition algorithm, such that $\{t\} = \Theta_{X_n} \circ \cdots \circ \Theta_{X_1}(S_t)$ is a single tile patch where $X_1, \ldots, X_n$ are interior vertices of S_t for some $n \in \mathbb{Z}^+$. Since the tiles of Q are either rectangles or triangles, we have that $\Theta_{X_n}^{-1}(\{t\})$ is a circle patch so that there are (at least) two tiles $u_t^1, u_t^2 \in S_t$ with $V_{u_t^i} \cap \{M, N, M', N'\} \neq \emptyset$ for $i = 1, 2$. Therefore, $\{(M, M'), (N, N')\}$ is a split pair for $\Theta_{X_n}^{-1}(\{t\})$ by Proposition 3.2.12. Thus, $\{(M, M'), (N, N')\}$ is a split pair for the cyclic patch S_t by the same argument applied in the proof of (2) of Theorem 3.2.42. Hence, $\{(A, B), (C, D)\}$ is a split pair for Q' as well, by Lemma 3.2.37. □

Corollary 3.3.20. *Suppose Q is a string consisting of rectangles and triangles and u_1, u_2 are the two non-slice tiles of Q. Assume further that Q' is a patch which is generated by finitely many circle decomposition steps over Q. Then we have:*

(1) If (A, B) is a valid pair for Q for some distinct exterior vertices A, B of Q, then (A, B) is a valid pair for Q' as well.

(2) If $\{(A, B), (C, D)\}$ is a split pair for Q for some distinct exterior vertices A, B, C, D of Q, then $\{(A, B), (C, D)\}$ is a split pair for Q' as well.

Proof. The proof follows by applying Lemma 3.3.19 for each tile t of Q. □

3.3.2 Order Systems for Substitution Tilings

The travelling algorithm provides an order system for any given substitution rule satisfying mild conditions. We explain this final step by the help of the following theorem.

Theorem 3.3.21. *Suppose Q is a patch consisting of convex tiles so that* supp Q *is convex. Assume further that $\{X_1, \dots, X_n\}$ for $n \in \mathbb{Z}^+$ denotes the collection of corners of Q and for every slice tile t of Q the following holds:*

(1) t is a slice tile of degree 2,

(2) $V_t \cap \{X_1, \dots, X_n\} = \emptyset$.

Assume further there is no circle subpatch S of Q that intersects two different slice clusters of Q. Let u be a tile so that (Q, u) is a composition pair. Then we have that

(1) If e_u is a simple decoration with end points A, B, then (A, B) is a valid pair for Q.

(2) If e_u is a 2-curve decoration with end point pairs A, B and C, D, then $\{(A, B), (C, D)\}$ is a split pair for Q.

Before giving the proof of the theorem we explain a few more results that are related with the possible decorations of tiles. These are results about simplifications of possible decorations of tiles and an observation about the decorations defined according to the proofs we provided in earlier sections. Recall that the only simplification result about 2-curve decorations we proved so far was Proposition 3.2.39 (and Corollary 3.2.40). That is, if a tile t in a circle patch Q has a 2-curve decoration induced from a decoration $\mathcal{C}$ of Q, then we can assume without loss of generality that the 2-curve decoration of t intersects with the centre X of Q (Corollary 3.2.40). This is valid for all tiles in the circle patch Q that have a 2-curve decoration induced from $\mathcal{C}$. For that, we used this result for every tile that occurs during the circle decomposition steps in the related proofs we provided earlier. We will provide two more simplification lemmas as well. The simplification lemmas we will provide now, on the other hand, are defined only for some tiles of a given cyclic patch. That is, we will only focus on possible decoration of a single tile that we choose. We will focus on modifying its decoration according to our needs. Therefore, these lemmas we provide are defined for local simplification purposes, rather than to the (global) simplification given in Proposition 3.2.39 (and Corollary 3.2.40). We will explain what we mean by local purposes with an example. Consider the decoration curves in Figure 3.94. Suppose we choose the tile t that is shown on the leftmost patch of the figure. Suppose further we want its decoration to intersects with the centre of the circle patch. The curve in the middle patch of the figure makes (A, B) a valid pair for the circle patch, and induces a simple decoration for t with end points M, N. Since the decoration of t does not intersects with X, we redefined the curve in the middle patch, which makes (A, B) a valid pair. We reform the curve in the middle patch into the curve shown on the rightmost patch in the figure. The reformed curve still makes (A, B) a valid pair, though the decoration of t induced by this curve has end points N, X. That is, we reformed the curve in the middle of the figure, which makes (A, B) a valid pair, such that the (new) simple decoration of the chosen tile t hits the centre X of the circle patch, as shown over the right side of the figure. Since we make use of the circle patches during the circle decomposition steps, we have many possible choices to pick for implementing. Due to these possible different choices we can make, we are able to define the following simplification lemmas.

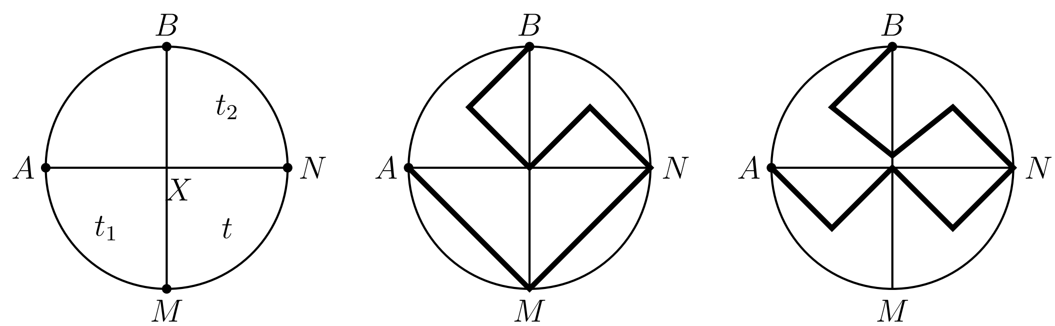


Figure 3.94

The following two lemmas (Lemma 3.3.22 and Lemma 3.3.23) are about simplifications of possible decorations appearing for some chosen tiles. The next lemma (Lemma 3.3.24) is, on the other hand, about an observation for the decorations of tiles in a cyclic patch which are defined according to the instructions in the proofs of earlier results.

Lemma 3.3.22. *Suppose Q is a circle patch with centre X, t is a tile in Q and A, B, C, D are distinct exterior vertices of Q. Suppose further Q has at least three tiles and t is a topological triangle that has end points M, N, X. Then the following holds:*

(1) *If there exists a curve $\mathcal{C}$ that makes (A, B) a valid pair for Q such that the decoration e_t of t, which is induced by $\mathcal{C}$, has end points M, N, then there exists another curve $\mathcal{C}'$ that makes (A, B) a valid pair for Q and the decoration e'_t of t, which is induced by $\mathcal{C}'$, has end points M, X or N, X. That is, the (new) decoration of t hits the centre of the circle patch.*

(2) *If there exist curves $\mathcal{C}_1, \mathcal{C}_2$ that make $\{(A,B),(C,D)\}$ a split pair for Q such that the decoration e_t of t, which is induced by $\mathcal{C}_1 \cup \mathcal{C}_2$, has end points M, N, then there exists another pair of curves $\mathcal{C}'_1, \mathcal{C}'_2$ that make $\{(A,B),(C,D)\}$ a split pair for Q and the decoration e'_t of t, which is induced by $\mathcal{C}'_1 \cup \mathcal{C}'_2$, has end points M, X or N, X. That is, the (new) decoration of t hits the centre of the circle patch.*

Proof. We prove both cases at once. Assume that Q is a circle patch with centre X that contains at least three tiles. Let t be a tile in Q with neighbour tiles $t_1, t_2 \in Q$. Suppose t is a topological triangle that has vertices M, N, X. Suppose further t_1 and t_2 can be regarded as topological triangles with vertices M, M', X and N, N', X, respectively. If t contains a simple decoration with end points M, N, then t_1 contains a decoration intersecting with M, and t_2 contains a decoration intersecting with N. Denote the decorations of t, t_1, t_2 as e_t, e_{t_1}, e_{t_2}, respectively. We have two cases; either one of e_{t_1} or e_{t_2} do not intersect with X, or $X \in e_{t_1} \cap e_{t_2}$. Notice that the latter case does not occur in Proposition 3.1.10 or Proposition 3.2.12. Therefore, we can assume without loss of generality that e_{t_1} has a simple decoration with end points M, M'. Define a decoration e'_{t_1} for t_1 with end points M' and X, as well as a decoration e'_t for t with end points X, N. Then replace the decorations e_{t_1} and e_t with e'_{t_1} and e'_t. The generated curve (or curves) satisfy the conclusions. □

Lemma 3.3.23. *Suppose Q is a circle patch with centre X and t is a tile in Q which can be defined as a topological triangle with vertices X, M, N for some shared exterior vertices M, N of Q. Assume that $|Q| > 3$ and t has neighbour tiles u_1, u_2 such that u_1 can be defined as a topological triangle with vertices X, M, M' and u_2 can be defined as a topological triangle with vertices X, N, N' where M', N' are shared exterior vertices of Q so that $\{M', N'\} \cap \{M, N\} = \emptyset$. Assume further A, B, C, D are distinct exterior vertices of Q. Then the following holds:*

(1) If (A,B) is a valid pair for Q by some curve $\mathcal{C}_1$ which is defined according to the instructions in the proof of Proposition 3.1.10 and e_1 is a 2-curve decoration of t induced by $\mathcal{C}_1$. Then e_1 cannot have end point pairs X, M and A, N.

(2) If $\{(A,B),(C,D)\}$ is a split pair for Q by some curve $\mathcal{C}_2$ according to the instructions in the proof of Proposition 3.2.12 and e_2 is a 2-curve decoration of t induced by $\mathcal{C}_2$ with end point pairs X, M and C, D ($C = N$ or $D = N$ are allowed), then there exists a another curve $\mathcal{C}'_2$ which makes $\{(A,B),(C,D)\}$ a split pair for Q such that the decoration e'_2 of t induced from $\mathcal{C}'_2$ is a simple decoration.

Proof. Suppose Q, t are a patch and a tile, respectively, that satisfy the assumptions in the lemma. Let A, B, C, D are given distinct exterior vertices of Q.

(1) Assume that (A,B) is a valid pair for Q by some curve $\mathcal{C}_1$ which induces a 2-curve decoration e_1 for t with end point pairs X, M and A, N. We have that A is an isolated

vertex of Q that belongs to the tile t, as shown in Figure 3.95. Then $B \neq N$ since e_1 has a simple curve component with end points A and N. Moreover $B \neq M$, because otherwise (A, B) would be a simple valid pair for Q, by Proposition 3.1.10. Therefore, B has to belong to another tile in Q other than t. But then, once again, (A, B) would be a simple decoration, by Proposition 3.1.10. Hence, no such 2-curve decoration exists.

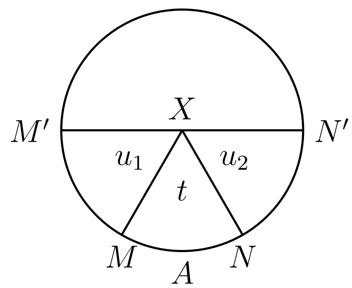


Figure 3.95

(2) We have two further cases; whether the decoration e_{u_1} of u_1 is a simple curve or not.

Suppose first e_{u_1} is a simple decoration of u_1. Since the decoration e_2 of t has end point pairs X, M and C, D, we have six possible decorations for u_1, which are demonstrated in Figure 3.96.

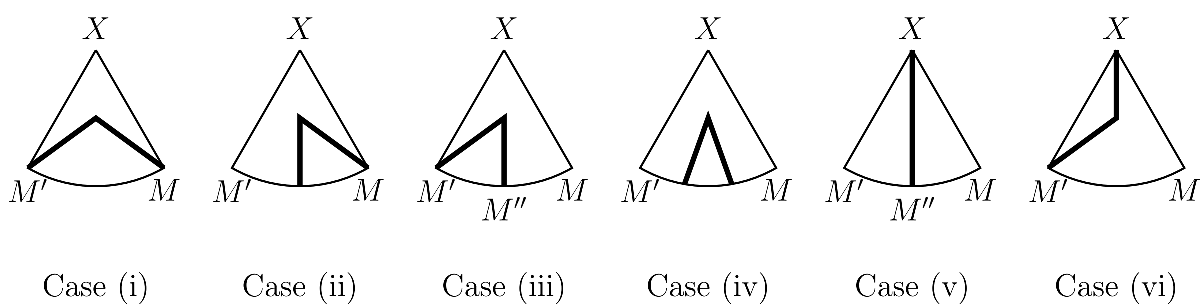


Figure 3.96: Possible simple decorations for the tile u_1

Case (i) : Let e_{u_1} be a simple decoration with end points M', M. Change the decoration of e_{u_1} of u_1 into a simple decoration e'_{u_1} with end points M' and X and change the decoration of e_2 of t into a simple decoration e'_2 with end points C and D. Then the reformed curves still make $\{(A, B), (C, D)\}$ a split pair for Q, as illustrated in Figure 3.97.

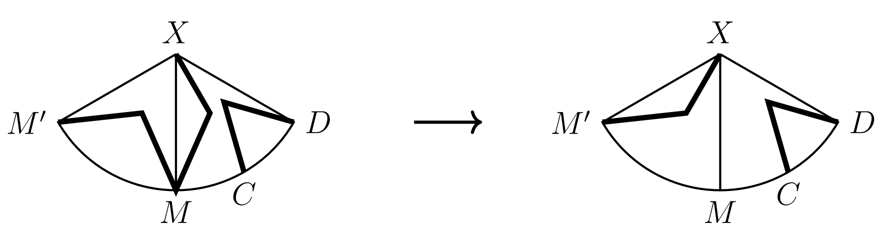


Figure 3.97: Illustration of the proof of Case (i).

Case (ii) : The proof of this case is same with Case (i).

Case (iii) : Let e_{u_1} be a simple decoration with end points M' and M'', as shown in Case (iii) in Figure 3.96. Change the decoration e_{u_1} of u_1 into a 2-curve decoration e'_{u_1} with end point pairs M', M'' and X, M. Moreover, change the decoration of e_2 of t into a simple decoration e'_2 with end points C and D. Then the reformed curves still make $\{(A, B), (C, D)\}$ a split pair for Q, as illustrated in Figure 3.98.

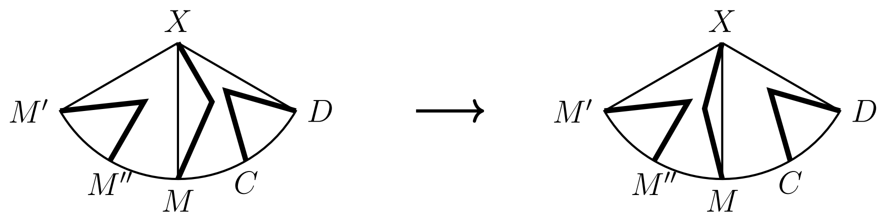


Figure 3.98: Illustration of the proof of Case (iii). The decorations of the tiles u_1 and t are reformed.

Case (iv) : The proof of this case is same with Case (iii).

Case (v) : Let e_{u_1} be a simple decoration with end points M' and X, as shown in Case (v) in Figure 3.96. Denote $e_{2,1}, e_{2,2}$ to be the simple curve components of e_2 such that $e_{2,1}$ is the simple curve with end points X, M and $e_{2,2}$ is the simple curve with end points C, D.

If e_{u_1} is followed by the curve $e_{2,1}$, then we change the decoration of u_1 into a simple curve decoration e'_{u_1} with end points M'', M and change the decoration e_2 into the simple decoration $e_{2,2}$, as illustrated in Figure 3.99. Then the reformed curves still make $\{(A, B), (C, D)\}$ a split pair for Q.

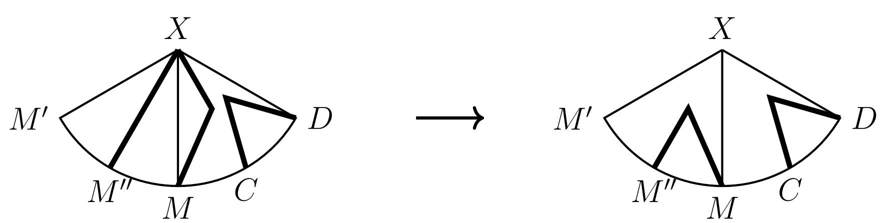


Figure 3.99: Illustration of the proof of Case (v) whenever e_{u_1} is followed by $e_{2,1}$. The decorations of the tiles u_1 and t are reformed.

If e_{u_1} is not followed by the curve $e_{2,1}$, then we must have $N \neq D$, $M = D$ and $\{M''\} \subseteq \{A, B\}$, as demonstrated with an example in Figure 3.100. Define a simple decoration e'_2 with end points C and D for the single tile patch $\{t\}$. We have that $Q \backslash \{t\}$ is a string by Lemma 3.3.15. Moreover, u_1 is a non-slice tile for the string $Q \backslash \{t\}$ and M'' is an isolated vertex for $Q \backslash \{t\}$ which belongs to u_1. Since $M'' = A$ or $M'' = B$ we have that (A, B) is a valid pair for the string $Q \backslash \{t\}$. Thus, $\{(A, B), (C, D)\}$ is a split pair for

Q by some decoration $\mathcal{C}'$ such that $\mathcal{C}'$ induces a simple curve decoration for t, with end points C and D, as demonstrated in Figure 3.101.

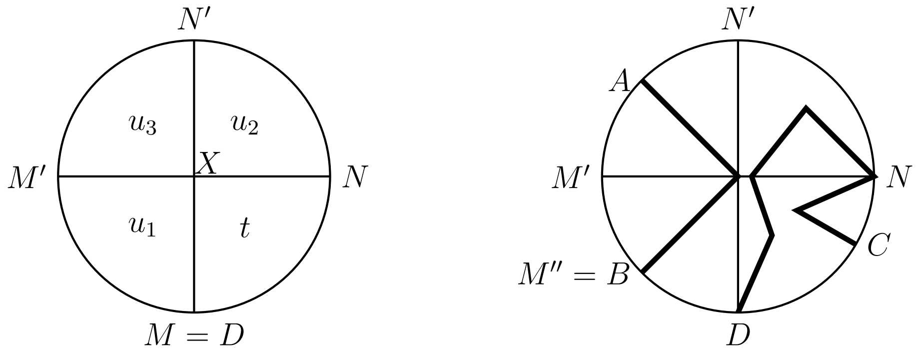


Figure 3.100: An example for the Case (v) whenever e_{u_1} is not followed by $e_{2,1}$.

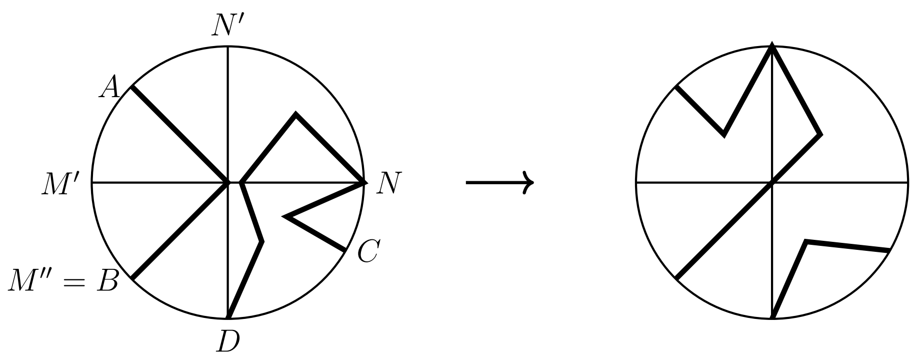


Figure 3.101

Case (vi) : The proof of this case is similar to Case (v).

Suppose now e_{u_1} is a 2-curve decoration. Since e_2 is a 2-curve decoration for t with end point pairs X, M and C, D, the possible 2-curve decorations for u_1 are shown in Figure 3.102.

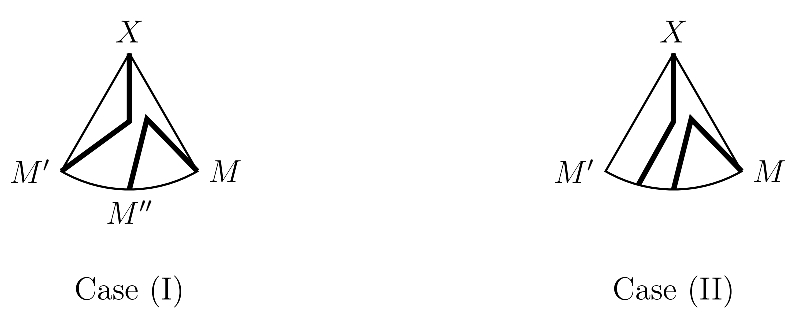


Figure 3.102: Possible 2-curve decorations for the tile u_1

We will only prove Case (I). The proof of Case (II) is similar with the proof of Case (I).

Case (I) : Let e_{u_1} be a 2-curve decoration with end point pairs M', X and M'', M, as shown in Case (I) in Figure 3.102. Denote $e_{u_1,1}, e_{u_1,2}$ to be the simple curve components of e_{u_1} such that $e_{u_1,1}$ is the simple curve with end points M', X and $e_{u_1,2}$ is the simple curve with end points M'', M. Denote also $e_{2,1}, e_{2,2}$ to be the simple curve components of e_2 such that $e_{2,1}$ is the simple curve with end points X, M and $e_{2,2}$ is the simple curve with end points C, D.

Assume first that $e_{u_1,1}$ is followed by the curve $e_{2,1}$. Change the decoration e_{u_1} of u_1 into a simple decoration e'_{u_1} which has end points M' and M''. Change also the decoration e_2 of t into the simple decoration $e_{2,2}$. Then the reformed curves still make $\{(A, B), (C, D)\}$ a split pair for Q, as shown in Figure 3.103.

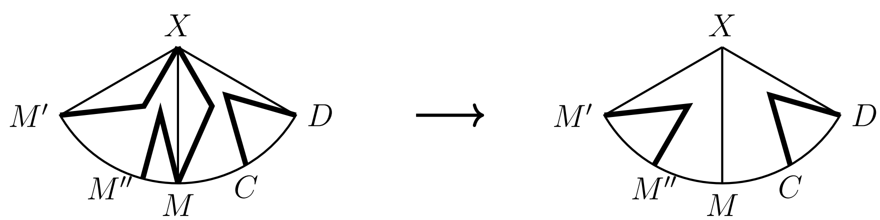


Figure 3.103: Illustration of the proof of Case (I) whenever $e_{u_1,1}$ is followed by $e_{2,1}$. The decorations of the tiles u_1 and t are reformed.

Assume now that $e_{u_1,1}$ is not followed by the curve $e_{2,1}$. Let u_3 be a tile in $Q\backslash\{u_1\}$ such that $M' \in V_{u_3}$. That is, u_3 and t are the neighbour tiles of u_1. Suppose e_{u_3} denotes the decoration of u_3 induced by $\mathcal{C}_2$. Since both e_{u_1} and e_2 - the decorations of u_1 and t, respectively- are 2-curve decorations, we have that e_{u_3} is a simple decoration. This is because of the fact that no three 'consecutive' 2-curve decorated tiles, such as u_3, u_1, t in Q, occur in the proof of Proposition 3.2.12. Let e_{u_3} denotes the simple decoration of u_3. Then e_{u_3} is a simple decoration with end points either Y, X or Y, M', where Y is an exterior vertex of Q such that $Y \neq \{M, N, M', N', X\}$, as illustrated in Figure 3.104. Moreover, e_{u_3} has to be followed by $e_{u_1,1}$, according to the instructions in the proof of Proposition 3.2.12.

Figure 3.104: Possible 2-curve decorations for the tile u_3

Subcase (I − a) : Suppose e_{u_3} is a simple decoration of u_3 with end points M', Y where

Y is an exterior vertex of Q such that $Y \notin \{M, N, M', N', X\}$. Change the decoration e_{u_3} of u_3 into a simple curve decoration e_{u_3} which has end points X and Y. Change the decoration e_{u_1} of u_1 into a simple curve decoration e'_{u_1} which has end points X and M''. Lastly, change the decoration e_2 of t into a simple curve decoration e'_2 which has end points C, D. Then the reformed curves make $\{(A, B), (C, D)\}$ a split pair for Q, as illustrated in Figure 3.105.

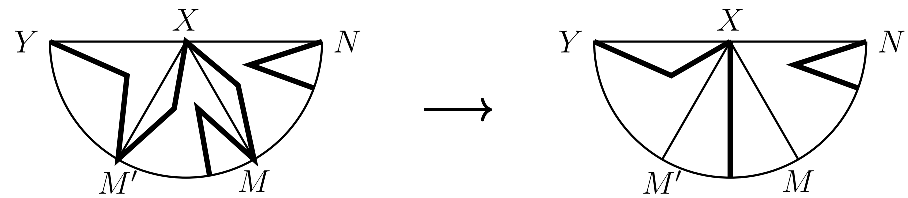


Figure 3.105: Illustration of the proof of Case (I-a). The decorations of the tiles u_3, u_1 and t are reformed. The decorations of u_3 (on the right side of the figure) is not followed by the decoration of u_1 (on the right side of the figure), even though they both intersect with X.

Subcase $(\mathbf{I-b})$: Suppose e_{u_3} is a simple decoration of u_3 with end points X, Y where Y is an exterior vertex of Q such that $Y \notin \{M, N, M', N', X\}$. Change the decoration e_{u_3} of u_3 into a simple curve decoration e_{u_3} which has end points Y and M'. Change the decoration e_{u_1} of u_1 into a simple curve decoration e'_{u_1} which has end points X and M''. Lastly, change the decoration e_2 of t into a simple curve decoration e'_2 which has end points C, D. Then the reformed curves make $\{(A, B), (C, D)\}$ a split pair for Q, as illustrated in Figure 3.106. □

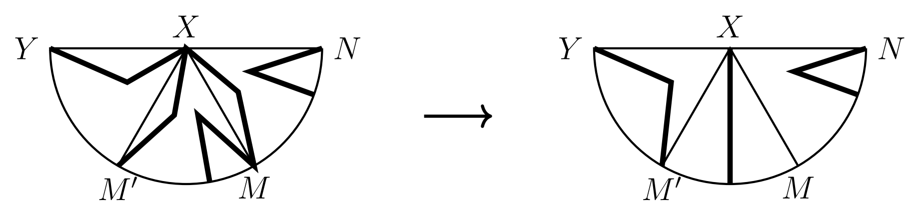


Figure 3.106: Illustration of the proof of Case (I-b). The decorations of the tiles u_3, u_1 and t are reformed. The decorations of u_3 (on the right side of the figure) is not followed by the decoration of u_1 (on the right side of the figure).

Lemma 3.3.24. *Suppose Q is a cyclic patch consisting of convex tiles and $t \in Q$ is a tile so that t has an edge e with $e \subseteq \partial\text{supp } Q$. If $\{(A, B), (C, D)\}$ is a split pair for Q by some curve $\mathcal{C}$, which is defined according to the instructions in the proof of Theorem 3.2.42, such that $\mathcal{C}$ induces a 2-curve decoration d_t for the tile t, then we have that $d_t \cap \partial\text{supp } Q \neq \emptyset$.*

Proof. The statement holds for any given circle patch. Suppose Q is a cyclic patch which is not a circle patch. Assume further $\{u\} = \Theta_{X_n} \circ \cdots \circ \Theta_{X_1}(Q)$ is a sequence of circle composition steps of Q for some $n \in \mathbb{Z}^+$. Then we have that $Q = \Theta_{X_1}^{-1} \circ \cdots \circ \Theta_{X_n}^{-1}(\{u\})$ is sequence of circle decomposition steps of the single tile patch $\{u\}$. There exists $j \in \{1, \ldots, n\}$ such that $t \in \Theta_{X_j}^{-1} \circ \cdots \circ \Theta_{X_n}^{-1}(\{u\})$ and $t \notin \Theta_{X_{j+1}}^{-1} \circ \cdots \circ \Theta_{X_n}^{-1}(\{u\})$. That is, t is formed during the circle decomposition step by the vertex X_j, and not formed in any earlier steps. The circle decomposition step by the vertex X_j can be illustrated as in Figure 3.107.

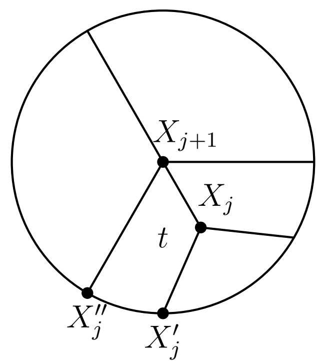


Figure 3.107: t is formed during the circle decomposition step by the vertex X_j.

Since t appears during the circle decomposition step by the vertex X_j, it is a tile in a circle patch which has a centre X_j, as shown in Figure 3.107. Suppose $\gamma = \partial\text{supp } t \cap \partial\text{supp } Q$ is the arc that corresponds to the boundary of t which is contained over the boundary of Q and has end points X'_j, X''_j. The tile t can be regarded as a topological triangle with vertices X_j, X_{j-1}, X'_j. Then we must have $d_t \cap \gamma \neq \emptyset$, by geometric reasons. □

Proof of Theorem 3.3.21. We will only prove (1). The proof of (2) is similar. Suppose Q is a patch satisfying the assumptions given in the theorem. Then we can define slice clusters for each corner of Q. Assume without loss of generality there is only one slice cluster S_X of Q where X is a corner of Q. By Corollary 3.3.20, we can further assume without loss of generality that S_X is a string. Compose the slice cluster S_X of Q into a single tile u_X. The generated composed patch Q' is a cyclic patch consisting of convex tiles such that no circle subpatch of it contains two (or more) of its corners.

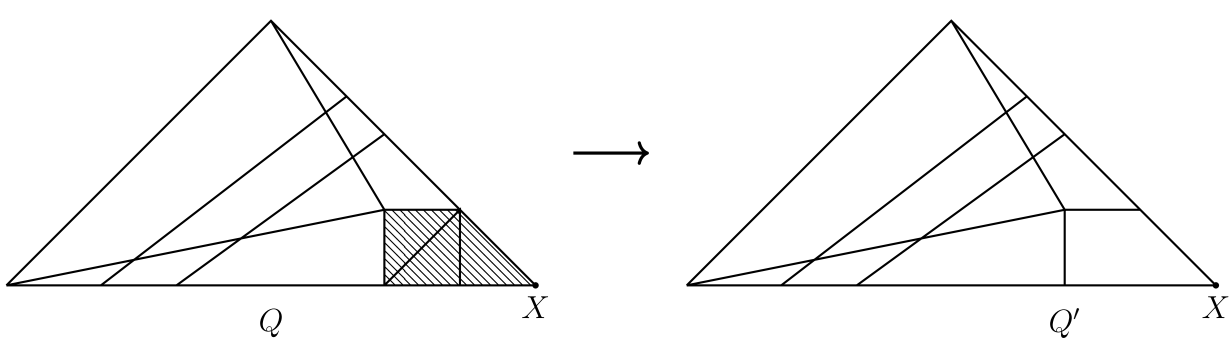


Figure 3.108: Q' is a cyclic patch and is a composition of Q

By Theorem 3.2.42, there exists a curve $\mathcal{C}$ that makes (A, B) a valid pair for Q'. We have that $\mathcal{C}$ induces a decoration for the tile u_X. Next we will decompose the decoration of u_X into a decoration for S_X with the same end points, with the help of Lemma 3.2.37. We have that u_X is formed in a circle decomposition step during the circle decomposition process of $\{u\}$ (into Q'). Identify u_X with a topological triangle with end points M, N, Y, where Y is the centre of the circle patch that u_X belongs to. Denote the decoration e_u of u_X that induces from $\mathcal{C}$. We have two cases; e_u is either a simple decoration or a 2-curve decoration.

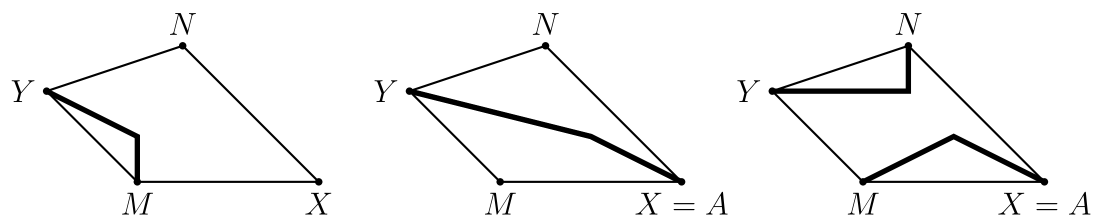


Figure 3.109: Possible decorations for u_X

Suppose first e_u is a simple decoration. Using Lemma 3.3.22 we can assume without loss of generality that e_u has end points Y, Y' for some $Y' \in V_{u_X}$. Since Y is an isolated vertex of S_X that belongs to a non-slice tile of S_X, we have that (Y, Y') is a valid pair for S_X, by Proposition 3.3.16. Thus, (A, B) is a valid pair for Q by Lemma 3.2.37.

Suppose now e_u is a 2-curve decoration of u_X. Using Lemma 3.3.23 and Lemma 3.3.24 (together with Corollary 3.2.40) the only possible 2-curve decoration u_X can have is a 2-curve decoration with end point pairs M, X and Y, N such that $X = A$ (or $X = B$). Assume that u_X is a 2-curve decoration with end point pairs M, A and N, Y, as shown on the right of Figure 3.109. Then we have that $\{(M, A), (N, Y)\}$ is a split pair for S_X by Corollary 3.3.17. Thus, once again, (A, B) is a valid pair for Q by Lemma 3.2.37. $\square$

Finally, the following theorem ensures that every substitution tiling satisfying mild conditions gives rise to a (decorated) substitution tiling whose substitution rule is compatible with the given one. In Chapter 4 we will explain in detail what we mean by compatibility with the given substitution rule.

Theorem 3.3.25. *Suppose $\mathcal{P}$ is a given finite collection of convex tiles, ω is a primitive substitution rule defined on $\mathcal{P}$ and T is a recognisable substitution tiling generated by the substitution rule ω. Then there exist a finite collection of decorated prototiles $\mathcal{P}'$ and a primitive substitution rule ω' defined on $\mathcal{P}'$ such that the following holds:*

(1) For any $p' \in \mathcal{P}'$ there exists a unique $p \in \mathcal{P}$ such that p' is a decorated copy of p with label $l(p') = (l(p), e_p)$ for some decoration e_p for p.

(2) There exists $k \in \mathbb{Z}^+$ such that if p' is a decorated copy of p with decoration e_p, then

(i) supp $\omega'(p') =$ supp $\omega^k(p)$,

(ii) $\omega'(p')$ *is a decorated copy of* $\omega^k(p)$ *with decoration* $\mathcal{C}$ *which has end points* $\lambda^k \cdot A, \lambda^k \cdot B$ *(or end point pairs* $\lambda^k \cdot A, \lambda^k \cdot B$ *and* $\lambda^k \cdot C, \lambda^k \cdot D$*) where* A, B *are end points of* e_p *(or* A, B *and* C, D *are end point pairs of* e_p*) and* λ *is the expansion factor of* ω*.*

In particular, there exists a recognisable, primitive, self-similar substitution tiling T' *with a prototile set* $\mathcal{P}'$ *and a substitution rule* ω'*.*

Proof. The proof follows by Theorem 3.3.21 together with the (same) argument explained in Theorem 3.2.43. □

Remark 3.3.26. The assumptions primitivity, recognisability and FLC in Theorem 3.3.25 are standard in order to construct the associated tiling dynamical systems. The only non-standard assumption in the theorem is the convexity of tiles which is a sufficient (but not necessary) condition in order make the desired circle composition steps well defined. On the other hand, for every 2-dimensional tiling T there exists a dual tiling T' whose tiles are convex [8]. However, T' is not necessarily a substitution tiling even if T is. We cannot make use of T' in our construction. Therefore, the convexity assumption in the statement of Theorem 3.3.25 is needed.

We outline the algorithm based on Theorem 3.3.25 as follows:

The Travelling Algorithm :

Step - 1: Check if there exists $k \in \mathbb{Z}^+$ such that every k-supertile is a cyclic patch. If this is the case, apply Theorem 3.2.43 (or even Theorem 3.1.13 if the supertiles are circle patches). Otherwise move to Step - 2.

Step - 2: Find sufficiently large $n \in \mathbb{Z}^+$ such that every n-supertile only contains a cluster of slice tiles a single tile away from a corner.

Step - 3: Compose each slice cluster into a single tile, and apply Theorem 3.2.43.

Step - 4: Decompose the decoration attached to the single tiles, which are formed by composition of slice clusters, into curves with same end points (or end point pairs) that visits the tiles in the slice clusters at most twice.

It is worth noting that although we allow tiles to be visited twice, for most of the known substitution tilings (if not all), there is a travelling algorithm that forms only simple decorations for the tiles of the tilings.

3.3.3 Relation with the Hamiltonian Path Problem

In the field of graph theory, the Hamiltonian Path Problem is a widely known problem of finding a path from a given connected graph, that visits every vertex of the graph once. We show that there is a correlation between Hamiltonian path problem and the travelling algorithm for some special cases. In fact, the travelling algorithm finds Hamiltonian paths for some connected graphs satisfying (strong) conditions (See Theorem 3.3.27 and Theorem 3.3.28).

Let Q be a given patch. Fix an interior point for each tile in Q. The collection of the fixed points define a vertex set V. For each pair of tiles that share a common vertex in Q, define an edge connecting their fixed points (vertices). This collection forms an edge set E. The graph $G = (V, E)$ admits Hamiltonian path whenever Q has exterior vertices A, B such that (A, B) is a simple valid pair for Q. This is because of the fact that simple valid pairs visit every tile exactly once, which corresponds visiting all the vertices of the graph. In particular, we have the following correlations:

patch Q	finite connected graph $G = (V, E)$
tile $t \in Q$	vertex $v \in V$
neighbourhood of t	edges intersecting with v
circle patches of n tiles	complete graphs of n vertices
strings of n rectangle tiles	path graphs of n vertices
slice tile of degree n	cut vertex of degree n

Theorem 3.3.27. *Let Q be a patch and $G = (V, E)$ is a connected finite graph defined as explained above. Then G admits a Hamiltonian path if there exists exterior vertices A, B of Q such that (A, B) is a simple valid pair of Q.*

Proof. The proof follows by the fact that tiles are visited exactly once. □

Theorem 3.3.28. *Let Q be a cyclic patch consisting of triangle tiles and $G = (V, E)$ is a connected finite graph defined as explained above. Then G admits a Hamiltonian path.*

Proof. The proof follows by the fact that triangle tiles can only have simple decorations, and thus can be visited exactly once. □

Chapter 4

Dimension Reduction

In this chapter we will use the decorations of tiles and patches, we defined in Chapter 3, as a source of order structure. We will form one dimensional tilings and one dimensional (discrete) tiling spaces. First, we construct a map that returns a one dimensional substitution tiling from any given two dimensional substitution tiling satisfying the standard conditions. Then we will prove that this map is an almost one-to-one factor map.

Suppose $\mathcal{P}$ is a finite collection of convex prototiles, ω is a primitive substitution rule on $\mathcal{P}$ and T is a recognisable self-similar singly edge-to-edge substitution tiling generated by the substitution rule ω. Applying the travelling algorithm defined in Chapter 3, we get a prototile set $\mathcal{P}_d$ which is a finite collection of decorated prototiles, a primitive substitution rule ω_d which is defined on $\mathcal{P}_d$ and a recognisable self-similar singly edge-to-edge substitution tiling T_d. In this chapter we will only consider the decorated substitution tiling T_d rather than the (non-decorated) two dimensional tiling T. Therefore, for simplicity, throughout the chapter we will denote $T_d, \mathcal{P}_d, \omega_d$ by $T, \mathcal{P}, \omega$, respectively.

Let $\mathcal{P}$ be a finite collection of convex decorated prototiles and ω be a primitive substitution rule defined on $\mathcal{P}$. We consider the decorated tilings generated by the pair $\mathcal{P}, \omega$ and decorations attached to these tilings. Suppose further that there exists a prototile $p \in \mathcal{P}$ which has a simple curve decoration. Then, we can construct a decorated tiling T from the prototile p, using the primitivity of ω. In particular, by the primitivity of ω, there exists $k \in \mathbb{Z}^+$ and $x \in \mathbb{R}^2$ such that $p-x \in \omega^k(p-x)$ and $\big(\partial \text{supp } \omega^k(p-x)\big) \cap (\text{supp } (p-x)) = \emptyset$. Thus, a tiling T can be defined by $T = \bigcup_{i=1}^{\infty} \omega^{k \cdot i}(p-x)$ as illustrated in Figure 4.1.

This generated tiling T has a single relatively dense curve attached to it, because it is generated by substituting the tile $p-x$ which has a simple decoration attached to it. The relatively dense curve is formed by the concatenation of all decoration curves appearing in its tiles. Therefore, it visits every tile of T at least once and at most twice. The generated relatively dense curve can be used as a source of total order, which will be explained later in this chapter.

The main idea for defining a one dimensional substitution tiling from a decorated two

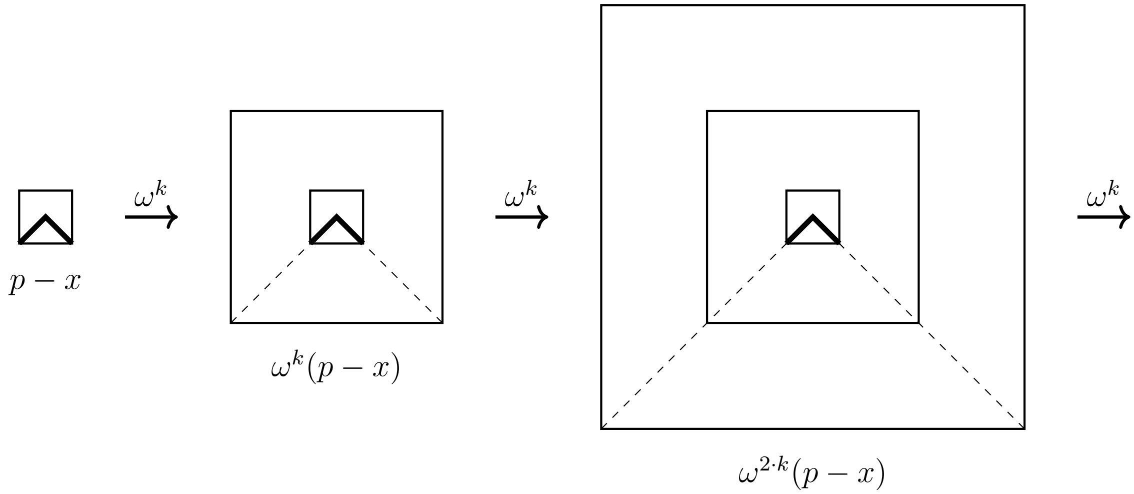


Figure 4.1: Constructing a tiling T by substituting the tile $p - x$

dimensional substitution tiling depends on the existence of such relatively dense curves. If the collection $\mathcal{P}$ contains at least one prototile with a simple curve decoration, then we can construct a tiling that has a single relatively dense curve attached to it, as explained above. On the other hand, a natural question is, can we construct a tiling with a relatively dense curve attached to it, if every prototile in $\mathcal{P}$ has a 2-curve decoration? Since then for every supertile of large order, we will have not a single curve but two curves that visit the tiles in the supertile. Therefore, it is not so obvious whether we can construct a single relatively dense curve which visits every tile at least once and at most twice. We prove in Proposition 4.0.2 that such a tiling can always be constructed even if $\mathcal{P}$ does not contain any prototile which has a simple decoration. Before proving the proposition we first illustrate the idea with an example.

Example 4.0.1. Figure 4.2 shows a substitution rule, where the labels of the prototiles are 2-curves. The prototile set in the figure consists of two decorated prototiles. Both of the prototiles have 2-curve decorations and substitute into four square (decorated) tiles. We will construct a decorated tiling T, and a (single) relatively dense curve $\mathcal{D} \subseteq \mathbb{R}^2$ such that $\mathcal{D}$ visits every tile in T exactly twice; i.e. $\mathcal{D} \cap \overline{\text{int}(\text{supp } t)}$ is a 2-curve for each $t \in T$.

Figure 4.2: A substitution rule

Since every prototile has a 2-curve decoration, every supertile has a decoration consisting of two non-crossing curves. Therefore, if we start with the prototile on the left of Figure 4.3, then the 2-supertile on the right of the figure contains 2 non-crossing curves

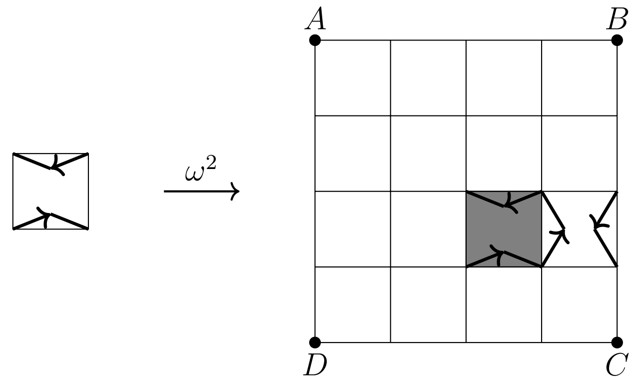


Figure 4.3: Disconnected decorations of the gray tile, are connected through a decoration curve of its neighbour tile in right patch of the Figure

that visit the tiles in the 2-supertile. These curves have end points A, B and C, D, respectively, by the geometry of the situation. Moreover, a copy of the prototile on the left of the figure appears in the 2-supertile patch, as illustrated with the highlighted tile in the figure. Although the decoration curves of the prototile (which is a supertile of level-0) in the left of Figure 4.3 are disconnected, the decoration curves of the highlighted tile in the right patch of the figure are contained in a (connected) curve in the 2-supertile patch. This is because of the fact that they are being connected by the decoration curve of the tile right next to the highlighted tile in the 2-supertile patch, as demonstrated in the figure. Similarly, the 2-supertile patch on the right of Figure 4.3 contains two non-crossing curves which is a decoration for the 2-supertile patch. These curves visit every tile in the 2-supertile patch exactly twice in total and have end points A, B and C, D, respectively. Even though these two non-crossing curves in the 2-supertile patch are not connected, they are contained in a single (connected) curve in the 4-supertile patch as illustrated in Figure 4.4. That is, the two (disconnected) curves in the 2-supertile patch are contained in a single curve in the 4-supertile patch. Therefore, by expanding these patches, we get a nested sequence of patches. Every 2n-supertile for $n \in \mathbb{Z}^+$ contains two disconnected curves as a decoration. Both of these curves are contained in one of the decoration curves of the 2n+2-supertile. Hence, this process generates a decorated tiling that has a single relatively dense curve attached, which visits every tile exactly twice.

Proposition 4.0.2. *Let $\mathcal{P}$ denote a finite collection of decorated prototiles and ω denote a primitive substitution rule defined on $\mathcal{P}$. Suppose further T is a recognisable substitution tiling with FLC that is generated by $\mathcal{P}$ and ω. Then there exists a tiling $T' \in \Omega(T)$ and a relatively dense curve $\mathcal{D} \subseteq \mathbb{R}^2$ such that $\mathcal{D} \cap \overline{\text{int}(\text{supp } t)}$ is either a simple curve or a 2-curve for each $t \in T'$.*

Before the proof of the proposition, we prove the following lemma.

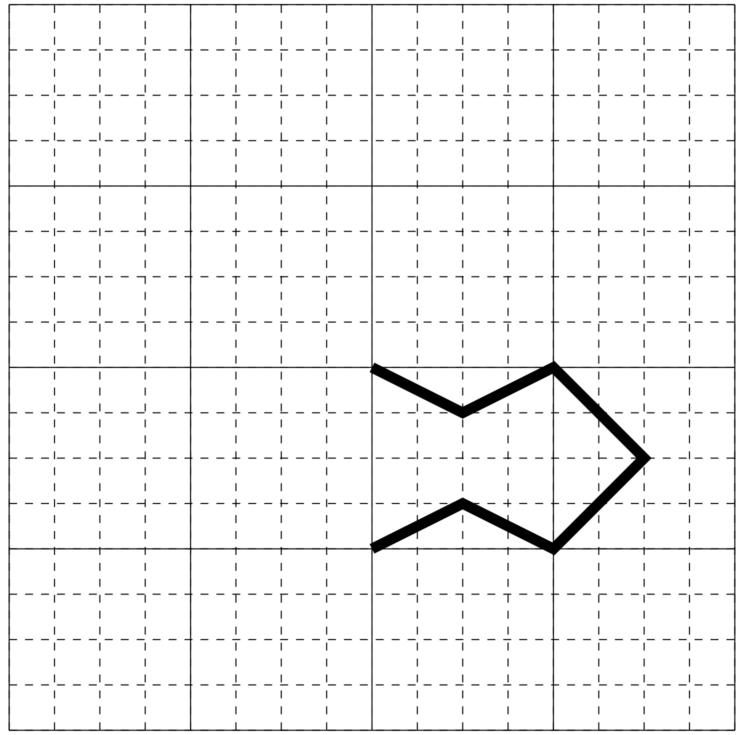


Figure 4.4: Disconnected curves in supertiles are being connected in the bigger supertiles

Lemma 4.0.3. *Suppose Q is a patch and A, B, C, D are distinct exterior vertices of Q. Assume further $\mathcal{C}$ is a decoration of Q that makes $\{(A, B), (C, D)\}$ a split pair for Q such that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1, \mathcal{C}_2$ are non-crossing curves with end points A, B and C, D, respectively. If $|V_t \cap \{A, B, C, D\}| < 2$ for each $t \in Q$, then there exists a tile $t_0 \in Q$ such that either $(\text{int } (\text{supp } t_0)) \cap \mathcal{C}_1 = \emptyset$ or $(\text{int } (\text{supp } t_0)) \cap \mathcal{C}_2 = \emptyset$. That is, there exists a tile $t_0 \in Q$ whose decoration (induced by $\mathcal{C}$) is completely contained either in $\mathcal{C}_1$ or $\mathcal{C}_2$.*

Proof. Suppose that Q is a given patch and A, B, C, D are distinct exterior vertices of Q. Suppose further $|V_t \cap \{A, B, C, D\}| < 2$ for each $t \in Q$. Let $\mathcal{C}$ denote a decoration of Q that makes $\{(A, B), (C, D)\}$ a split pair for Q such that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1, \mathcal{C}_2$ are non-crossing curves with end points A, B and C, D, respectively. The decoration $\mathcal{C}$ of Q induces a decoration e_t for each tile $t \in Q$. Assume to the contrary that for each tile $t \in Q$, $e_t \subseteq \mathcal{C}_i$ for $i = 1, 2$. Since every tile has a 2-curve decoration, each tile is visited by $\mathcal{C}_i$ exactly once for $i = 1, 2$. Let $t_A, t_B \in Q$ denote the tiles which are visited by $\mathcal{C}_1$ first and last, respectively, and let $t_C, t_D \in Q$ denote the tiles which are visited by $\mathcal{C}_2$ first and last, respectively. By the given assumption in the lemma, we have that t_A, t_B, t_C, t_D are distinct tiles in Q. Therefore, after finitely many steps of visiting t_A, $\mathcal{C}_1$ must visit t_C before visiting t_B. On the other hand, $\mathcal{C}_2$ is a curve with end points C, D and visits t_A once. Therefore, after visiting t_A and t_C, $\mathcal{C}_1$ cannot visit t_B without crossing $\mathcal{C}_2$, a contradiction. □

Proof of Proposition 4.0.2. The prototile set $\mathcal{P}$ consists of decorated prototiles with either simple curve or 2-curve decorations. If there exists a prototile $p \in \mathcal{P}$ with a simple curve decoration, then every n-supertile $\omega^n(p)$ for $n \in \mathbb{Z}^+$ contains a curve that visits every tile in the n-supertile at most twice. Thus, one can construct a tiling T', using the primitivity of ω and the prototile p, such that T' contains a relatively dense curve $\mathcal{D}$ that visits every tile in T at most twice, as desired.

Suppose now $\mathcal{P}$ consists of decorated prototiles whose decorations are solely 2-curves. Choose a prototile p. Denote the components of its 2-curve decorations with e_1, e_2. For each n-supertile $\omega^n(p)$ for $n \in \mathbb{Z}^+$ there are two curves C_1^n, C_2^n which corresponds to the substitutions of e_1, e_2, respectively. Using Lemma 4.0.3, choose a sufficiently large $k \in \mathbb{Z}^+$ so that the subcollection $S = \{t \in \omega^k(p) : \text{ supp } t \cap C_1^k \neq \emptyset\} \setminus \{t \in \omega^k(p) : \text{ supp } t \cap C_2^k \neq \emptyset\}$ is non-empty. That is, choose $k \in \mathbb{Z}^+$ sufficiently large such that there is a tile in the k-supertile that only intersects with C_1^k, and does not intersect with C_2^k. Choose $q \in S$. Let m be a sufficiently large positive integer such that $\omega^m(q)$ contains a copy of p inside. Such an m exists since ω is a primitive substitution. Then we find a copy of p inside the $k+m$-supertile $\omega^{k+m}(p)$. Next we will construct a tiling T', using the primitivity of ω. We show that this tiling contains a single relatively dense curve that visits its tiles at most twice.

Suppose $x \in \mathbb{R}^2$ such that $p+x \in \omega^{k+m}(p)$. For simplicity we will set $\omega' = \omega^{k+m}$. Then, even though the decorations $e_1 + x, e_2 + x$ of $p + x$ are disconnected, they are contained in the same connected curve in $\omega'(p)$. Similarly, $\omega'(p)$ contains two non-crossing curves C_1^{k+m}, C_2^{k+m} that are corresponding to the substitutions of the disconnected curves e_1, e_2, respectively. Once again, both of these curves are contained in the same (connected) curve in $(\omega')^2(p)$. Therefore, by the same token, all disconnected curves that are concatenation of decorations of tiles in T, are connected in a sufficiently large supertile of T. Let $\mathcal{D}$ denote the concatenation of decoration curves of the tiles. Then, by the argument above, the decorations of the tiles in T' are being visited by $\mathcal{D}$ at least once and at most twice, as desired. □

Remark 4.0.4. For any given finite collection of decorated prototiles and a primitive substitution rule defined over it, we can construct a tiling which has a single relatively dense curve attached to it (Proposition 4.0.2). We showed that even though the collection $\mathcal{P}$ consists solely of 2-curve decorated prototiles, we can construct a tiling with a single relatively dense curve attached to it.

It is also worth noting the other side of the story. Suppose $\mathcal{P}$ is a decorated prototile set of a (decorated) tiling T such that every prototile has a simple decoration on it. Then there might exist a tiling $T' \in \Omega(T)$ such that there is no single relatively dense curve $\mathcal{D}$ that can be attached to T, with the property that $\mathcal{D} \cap \text{supp } t$ is either a simple curve or 2-curve for each $t \in T'$. We explain this with an example as well.

Example 4.0.5. Consider the (Hilbert's) substitution rule in Figure 4.5 (with the expansion factor $\lambda = 2$). Suppose p is the prototile on the left of the Figure 4.6. Applying the substitution three times to the prototile p, we arrive at the 3-supertile on the right of the figure. A copy of p appears in the 3-supertile as highlighted in the figure. Therefore, there exists a point $x \in \mathbb{R}^2$ such that $p - x \in \omega^3(p - x)$ and $(\partial \text{supp } \omega^3(p-x)) \cap (\text{supp } (p-x)) = \emptyset$. Then $T = \bigcup_{i=1}^{\infty} \omega^{3 \cdot i}(p - x)$ defines a tiling of the plane. The prototile set of T consists solely of simple curve decorated prototiles, as shown in Figure 4.5. Moreover, T is a recognisable primitive substitution tiling (for recognisability, see the explanation in Example 5.2.1 and Example 5.2.2 in Chapter 5). Next we construct a tiling $A \in \Omega(T)$ such that no (single) relatively dense curve $\mathcal{D}$ is attached to A that visits every tile in A at least once and at most twice.

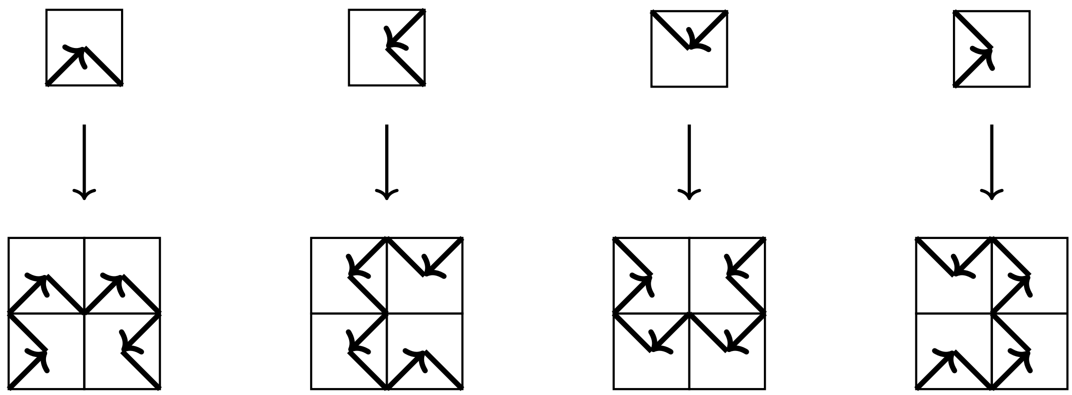


Figure 4.5: Hilbert's substitution rule

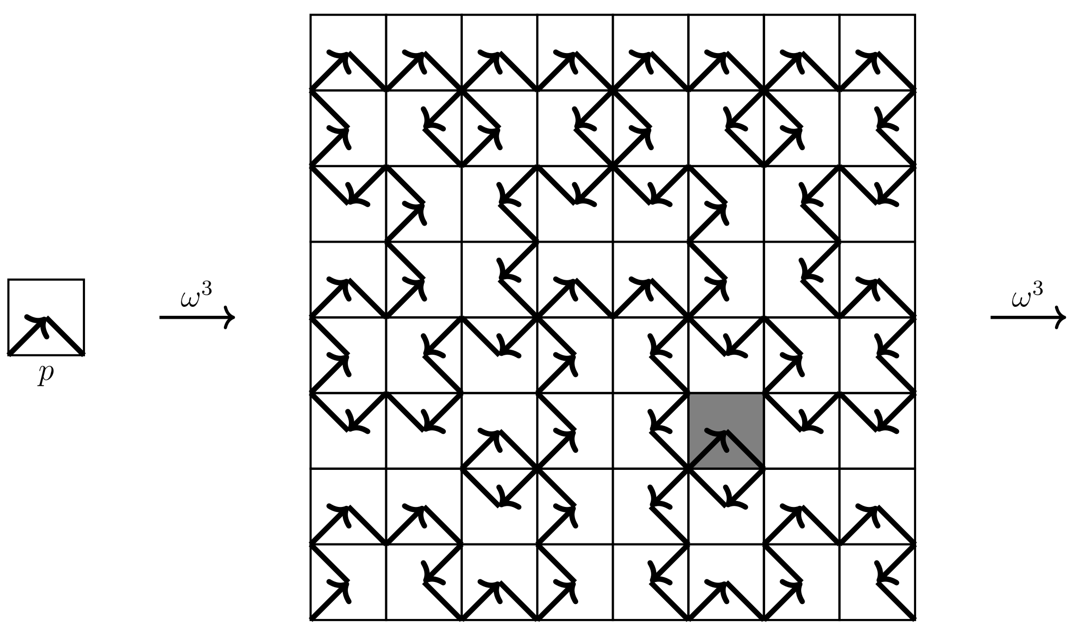


Figure 4.6: Constructing a Hilbert's tiling

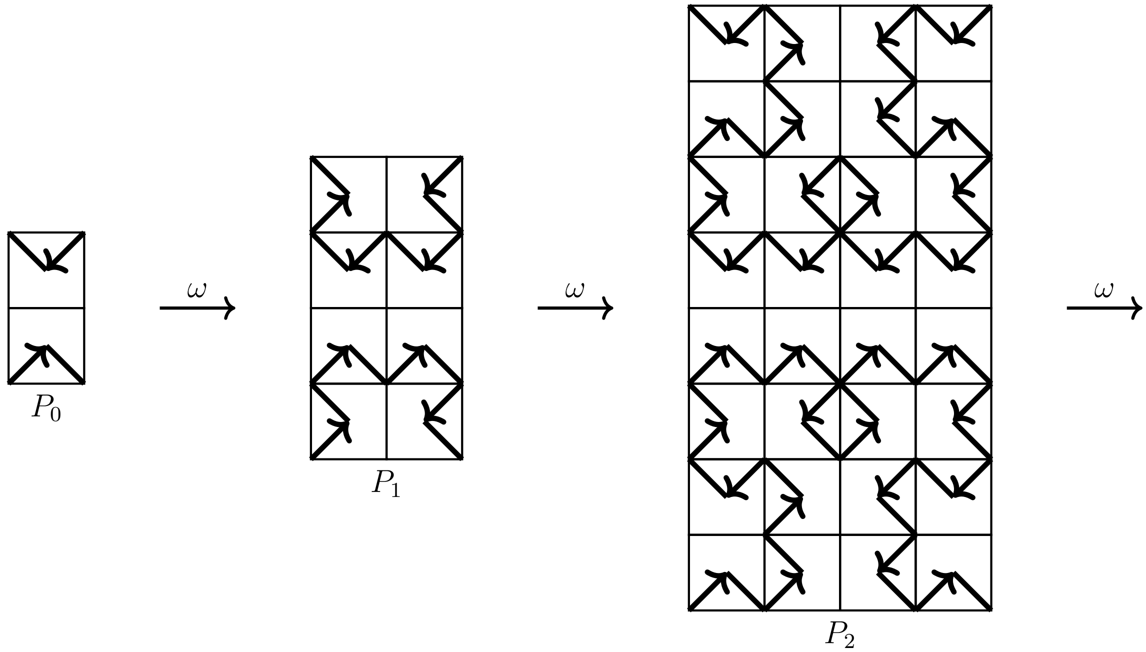


Figure 4.7: A sequence of patches generated by the substitution rules in Figure 4.5

Denote the leftmost two tile patch in Figure 4.7 with P_0. Define $P_{i+1} = \omega(P_i)$ for $i \in \{0, 1, 2, \dots\}$. These iterative steps define a sequence of patches of the form $\omega^k(P_0)$ for $k = 0, 1, 2, \dots$. The sequence is illustrated in Figure 4.7. It can be readily seen from the figure that a copy of P_i appears in P_{i+1} for each $i = 0, 1, 2, \dots$. More precisely, a copy of P_0 appears in P_1, as shown in Figure 4.7. Assume P_n appears in P_{n+1} for some $n = 0, 1, 2, \dots$. Then $P_{n+1} = \omega(P_n)$ appears in $P_{n+2} = \omega(P_{n+1})$ as well. Thus, there exists a sequence of real numbers $\{x_n\}_{n=0}^{\infty}$ such that $P_n + x_n \subseteq P_{n+1}$ for each $n = 0, 1, 2, \dots$.

Next we show that a copy of $P_{3\cdot i}$ for $i \in \{0, 1, 2, \dots\}$ appears in T. Indeed, a translate of P_0 appears in $\omega^3(p)$, as highlighted in Figure 4.8. Then $P_3 = \omega^3(P_0)$ appears in $\omega^6(p) = \omega^3(\omega^3(P_0))$ as well. In particular, by the same token, $P_{3\cdot i} = \omega^{3\cdot i}(P_0)$ appears in $\omega^{3\cdot(i+1)}(p)$ for $i \in \{0, 1, 2, \dots\}$. Hence, because $T = \bigcup_{k \in \mathbb{Z}^+} \omega^{3\cdot k}(p - x)$, a copy of $P_{3\cdot i}$ for $i \in \{0, 1, 2, \dots\}$ appears in T. Finally, we construct a tiling A as follows. Define A_n to be a tiling in $\Omega(T)$ that contains $P_{3\cdot n}$ around its origin for every $n \in \mathbb{Z}^+$. Because $\{P_n\}_n$ is a nested (increasing) sequence of patches, A_n converges to a tiling A in $\Omega(T)$. In fact, A is the tiling generated by the substitution procedure in Figure 4.7. It can be readily seen from the iterative system in Figure 4.7 that A does not contain any single relatively dense curve attached to it, which visits every tile of itself at least once and at most twice.

Lemma 4.0.6. *Decorated (substitution) tilings that are constructed according to the travelling algorithm in Chapter 3, are repetitive.*

Proof. The proof follows by (3) of Theorem 2.1.16. □

Using Proposition 4.0.2 and Lemma 4.0.6, we can assume without loss of generality,

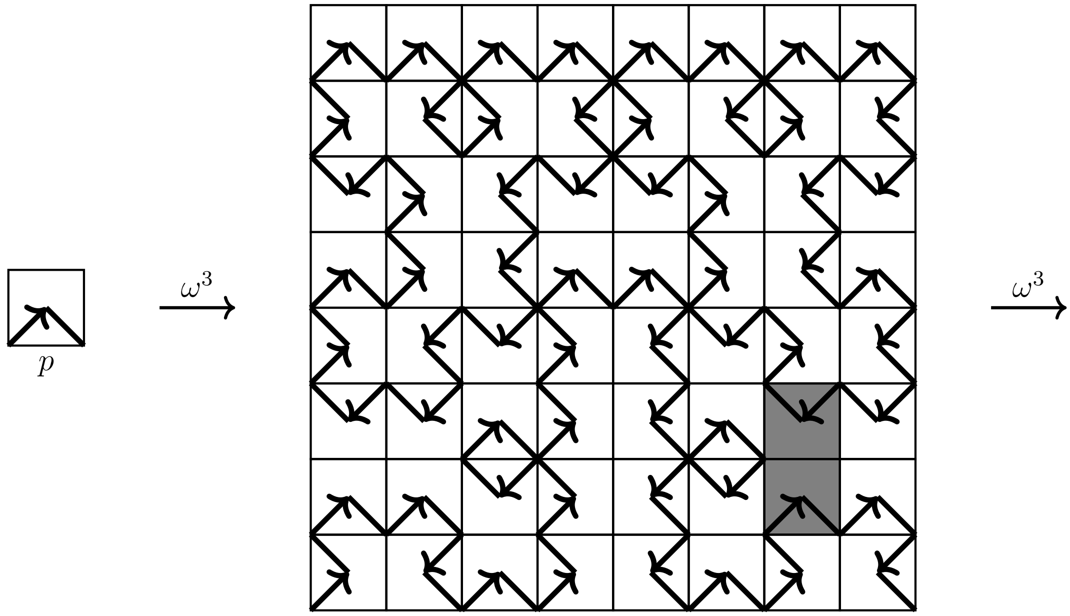


Figure 4.8: P_0 appears in $\omega^3(p)$

our decorated tilings that are constructed according to the machinery in Chapter 3 have relatively dense curves attached to them, which visit their tiles at least once and at most twice.

The Flattening Process

Suppose $\mathcal{P}$ is a finite collection of convex prototiles, ω is a substitution rule on $\mathcal{P}$ and T is a recognisable self-similar singly edge-to-edge substitution tiling generated through the process in Chapter 3. Assume without loss of generality that T contains a relatively dense curve $\mathcal{D}$ that is formed by concatenation of decorations of its tiles, and visits its tiles at most twice. We will construct a one dimensional substitution tiling V which satisfies the same standard assumptions as T.

Define $\mathcal{P}^1, \mathcal{P}^2$ to be the subcollections of $\mathcal{P}$ such that $\mathcal{P}^i$ consists of prototiles in $\mathcal{P}$ whose decorations are i-curves for $i = 1, 2$. For each $p \in \mathcal{P}^1$ we fix a point $x(p)$ to be punctured such that $x(p) \in \text{int}(\text{supp } p) \cap e_p$ where e_p is the single curve decoration of p. For each $q \in \mathcal{P}^2$, we choose two fix points $x_1(q), x_2(q)$ to be punctured such that $x_i(q) \in \text{int}(\text{supp } q) \cap e_q^i$ for $i = 1, 2$, where e_q^1, e_q^2 are the two components of the 2-curve decoration e_q of q. This defines a lattice $\Gamma \subseteq \mathbb{R}^2$. We construct a discrete tiling space $\Omega_p(T) = \overline{(T + \Gamma)}$. From now on T will denote the punctured (decorated) tiling with punctures induced from Γ.

Suppose further without loss of generality that by looking at the decorations of tiles, we can recognise the tiles. That is, for $p_1, p_2 \in \mathcal{P}$ with decorations e_{p_1}, e_{p_2}, respectively, $e_{p_1} = e_{p_2}$ if and only if $p_1 = p_2$. For example, instead of using the prototile set of 2DTM-Hilbert Substitution tiling in Figure 1.8, we reform the curves in the prototiles as demonstrated in Figure 4.9.

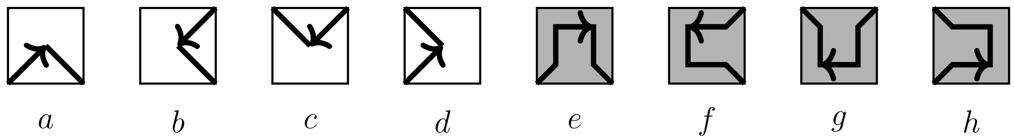


Figure 4.9: A different prototile set for 2DTM attached Hilbert

Observe that by just checking the curves on the prototiles in Figure 4.9, we are able to detect which colour label that prototile has. This is because of the fact that we used different curve labels for the different coloured unit square tiles.

We define a substitution rule on the curve decorations appearing in the prototiles, as we did in the 2DTM attached Hilbert curves example in Methodology (see Figure 1.6, Figure 1.8 and the substitution rule σ defined previous to the Figure 1.8). We first label the curves with numbers, before constructing the substitution system for the them. Let $\mathcal{W}^i = \{e_p : \ p \in \mathcal{P}^i\}$ for $i = 1, 2$ be the collection of i-curve decorations appearing in $\mathcal{P}^i$ for $i = 1, 2$. Since we have that $|\mathcal{W}^1| = |\mathcal{P}^1|$ and $|\mathcal{W}^2| = 2 \cdot |\mathcal{P}^2|$, there are bijections ϕ_1, ϕ_2 such that $\phi_1 : \mathcal{W}^1 \mapsto \{1, 2, \ldots, |\mathcal{P}^1|\}$ and $\phi_2 : \mathcal{W}^2 \mapsto \{|\mathcal{P}^1|+1, |\mathcal{P}^1|+2, \ldots, |\mathcal{P}^1|+2\cdot|\mathcal{P}^2|\}$. Define the bijection $\phi : \mathcal{W}^1 \cup \mathcal{W}^2 \mapsto \{1, 2, \ldots, |\mathcal{P}^1| + 2 \cdot |\mathcal{P}^2|\}$ so that $\phi|_{\mathcal{W}^i} = \phi_i$ for $i = 1, 2$. Then ϕ is the desired number label map for the decoration curves attached in the prototiles.

Figure 4.10: Labelling the curves with numbers

For each $e_p \in \mathcal{W}_1$, define an interval prototile q_p whose length equals to the length of e_p, and whose label is the number $\phi(e_p)$. The puncture $x(p)$, where $p \in \mathcal{P}$ is the prototile with the simple curve decoration e_p, is over the curve e_p. Therefore, the puncture $x(p)$ of p induces a puncture for q_p as well because e_p is homeomorphic to the interval q_p. Similarly, for each $e_p = e_p^1 \cup e_p^2 \in \mathcal{W}^2$, define two interval prototiles q_p^1, q_p^2 whose lengths equal to the lengths of e_p^1, e_p^2, respectively, and whose labels are the numbers $\phi(e_p^1), \phi(e_p^2)$, respectively. Furthermore, q_p^1, q_p^2 have punctures induced by the punctures over the curves e_p^1, e_p^2, respectively. For each simple curve we constructed an interval prototile. So, this defines a collection of (punctured) interval prototiles $\mathcal{Q}$ and a bijection ψ between the collection of simple curves in $\mathcal{W}^1 \cup \mathcal{W}^2$ and the prototile set $\mathcal{Q}$.

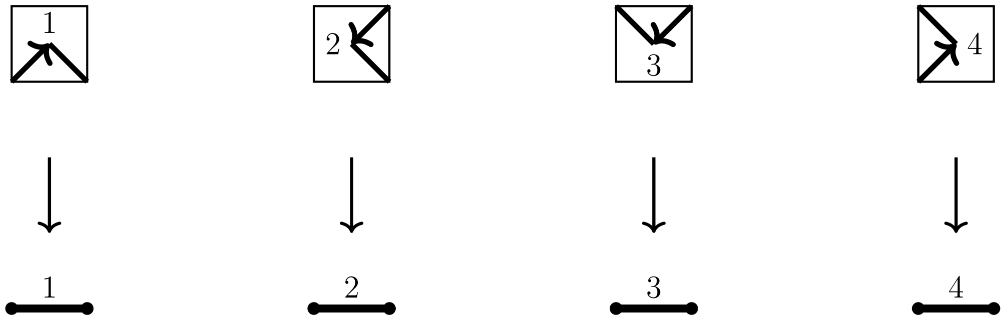


Figure 4.11: Define intervals for each simple curves

We define a substitution rule for the prototiles in $\mathcal{Q}$ by concatenating curves in 1-supertiles. If $p \in \mathcal{P}$ is a given decorated prototile with a simple decoration e_p that has end points A, B, then $\omega(p)$ is a 1-supertile with decoration $\mathcal{C}_p$ such that

(1) $\mathcal{C}_p$ is a curve with end points $\lambda \cdot A, \lambda \cdot B$, where λ is the expansion factor of ω,

(2) $\mathcal{C}_p \cap \overline{\text{int}(\text{supp } q)}$ is either a simple curve or a 2-curve for each $q \in \omega(p)$.

Let $\mathcal{C}_p$ be the concatenation of the simple curve decorations $c_1, \ldots, c_n$ for some $n \in \mathbb{Z}^+$. That is $\mathcal{C}_p = \bigcup_{i=1}^{n} c_i$ with $s(e_i) = r(e_{i-1})$ for each $i = 2, \ldots, n$. Record the collection of intervals $\{\psi(c_1), \ldots, \psi(c_n)\}$ (in the concatenation order) to be the substitution of q_p.

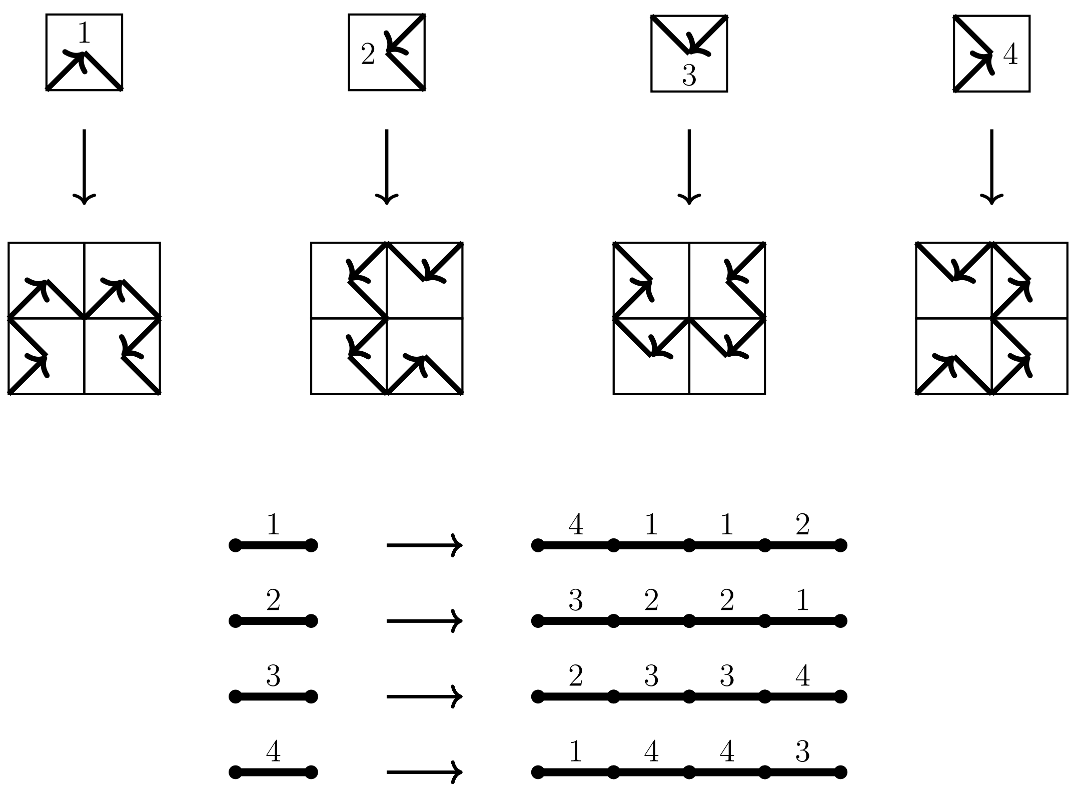


Figure 4.12: Reading the concatenation of curves in order to define a substitution rule

Similarly, if $p \in \mathcal{P}$ is a given decorated prototile with a 2-curve decoration e_p with component curves e_p^1, e_p^2, then $\omega(p)$ contains a decoration of two non-crossing curves $\mathcal{C}_p^1, \mathcal{C}_p^2$ such that $\mathcal{C}_p^i$ is a concatenation of the simple curves $c_1^i, \dots, c_{n_i}^i$ for some $n_i \in \mathbb{Z}^+$ and $i = 1, 2$. Then we can record the collection $\{\psi(c_1^i), \dots, \psi(c_{n_i}^i)\}$ to be the substitution of the simple curve e_p^i for $i = 1, 2$. Thus we formed a substitution rule ω_V defined on $\mathcal{Q}$. This substitution rule is primitive, since ω is primitive.

On the other hand, let D denote the relatively dense curve over T that passes through its tiles at most twice. Assume that $D = \bigcup_{i \in \mathbb{Z}^+} d_i$ denotes the concatenation of simple curves that are decorations of tiles in T. Let $x(d_i)$ for $i \in \mathbb{Z}^+$ denotes the puncture over the curve d_i for $i \in \mathbb{Z}^+$. Suppose further, without loss of generality, that the origin of $\mathbb{R}^2$ belongs to d_0 (i.e. $0 \in d_0$), and $s(d_i) = r(d_{i-1})$ for each $i \in \mathbb{Z}^+$. Then we define one dimensional tiling $V = \{v_i\}_{i \in \mathbb{Z}}$ such that v_i is an interval whose length equals to the length of d_i, and whose label is the number $l(v_i) = \phi(d_i)$ for $i \in \mathbb{Z}^+$. In particular, V is a substitution tiling with the substitution rule ω_V. Recognisability of V follows by the recognisability of T. Hence, V satisfies the standard conditions.

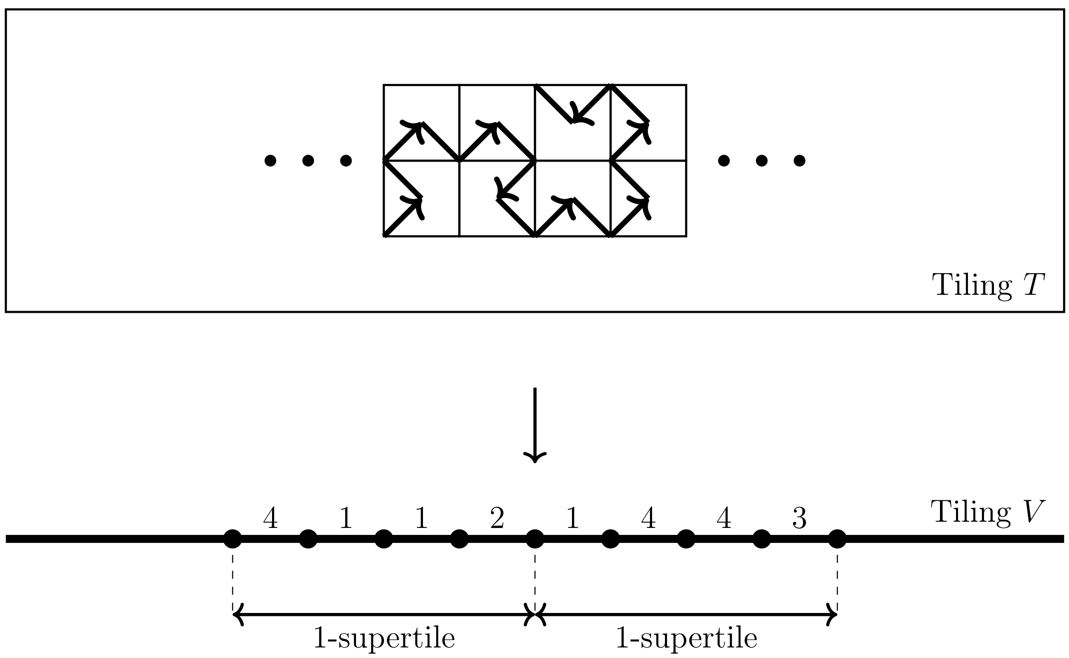


Figure 4.13: Flattened tiling V can be formed by the substitution rule ω_V. V can be decomposed into 1-supertiles defined by the substitution ω_V

Moreover, we can define punctures for V by inducing punctures of the curves d_i. Because v_i is homeomorphic to d_i for each $i \in \mathbb{Z}^+$, there exists $x(v_i)$ for each puncture $x(d_i)$ of d_i. Therefore, V can be regarded as a punctured substitution tiling with punctures induced by the punctures of the curve decorations of tiles of T.

Finally, if $A \in \Omega(T)$, then A does not necessarily contain a (single) relatively dense curve (see Example 4.0.5). However, there always exists a bi-infinite curve $\mathcal{D}_A$, which passes through its origin. Suppose $\mathcal{D}_A$ is a such bi-infinite curve passing through the origin, which is a concatenation of countably many simple curves that are either a decoration of some tile in A, or a connected component of a 2-curve decoration of some tile in A. In particular, we can define $\mathcal{D}_A = \bigcup_{i\in\mathbb{Z}} d_i$ where d_i are simple curves that are decorations or a connected part of some decorations of tiles in A. Therefore, by the same token, one can construct a one dimensional tiling V_A, for any given $A \in \Omega(T)$.

A Factor Map Between Discrete Tiling Spaces

We will use the flattening steps in order to construct an almost one-to-one factor map between discrete tiling spaces.

Theorem 4.0.7. *Let T be a decorated punctured two dimensional primitive recognisable substitution tiling with FLC. Suppose there exists a relatively dense curve $\mathcal{D}$ that visits every tile of T at least once, at most twice, and is formed by concatenation of the decorations of tiles in T. Assume further V is the punctured one dimensional tiling generated by flattening the relatively dense curve $\mathcal{D}$. Then there exists an almost one-to-one factor map $\Phi : \Omega_p(T) \mapsto \Omega_p(V)$ defined by $\Phi(A) = V_A$, where V_A is the (punctured) one dimensional tiling constructed by flattening the bi-infinite curve $\mathcal{D}_A$ as outlined previous to the theorem.*

Proof. We first prove that Φ is a well-defined map. Suppose P is patch which appears in V_A. Since flattening the curve $\mathcal{D}_A$ forms the tiling V_A, there exists a subcurve $\mathcal{D}'_A \subseteq \mathcal{D}_A$ which generates the patch P in the flattening process. Then $\mathcal{D}'_A$ is contained in a patch Q that appears in A. Since $A \in \Omega_p(T)$, Q appears in T as well. Therefore, the flattened tiling V contains a copy of P because Q contains the curve $\mathcal{D}'_A$. Thus, $V_A \in \Omega_p(V)$ by Theorem 2.1.4.

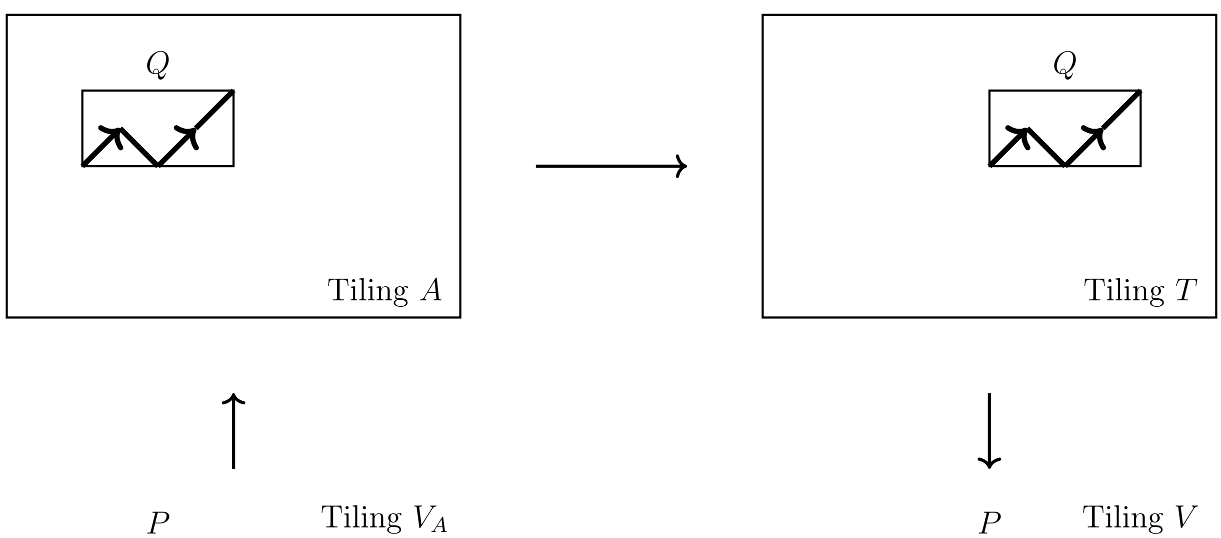


Figure 4.14: Every patch in V_A appears in V

Continuity of Φ follows from the fact that any arc connecting the origin with the boundary of a (two dimensional) ball with radius R, has length greater or equal than R. More precisely, for any given $\epsilon > 0$ and $A, B \in \Omega_p(T)$, $d_1(\Phi(A), \Phi(B)) < \epsilon$ whenever $d_2(A, B) < \epsilon$, where d_i is the i-dimensional ball for $i = 1, 2$. Thus, Φ is (uniformly) continuous.

For the surjectivity of Φ, suppose $V' \in \Omega_p(V)$ is given. Define a sequence of patches $\{P_n\}_n$ such that $P_n = V' \sqcap B(0, n)$ for $n \in \mathbb{Z}^+$. Then P_n appears in V for each $n \in \mathbb{Z}^+$, by Theorem 2.1.4. Therefore, there exists a sequence of (sub)curves $\{\mathcal{D}_n\}_n$ such that $\mathcal{D}_n \subseteq \mathcal{D}$ and the flattening process of $\mathcal{D}_n$ gives rise to the patch P_n in V, for each $n \in \mathbb{Z}^+$. Define $\{Q_n\}_n$ to be any sequence of patches in T so that Q_n contains the curve $\mathcal{D}_n$ for $n \in \mathbb{Z}^+$. Define further $E_n = \{T' \in \Omega_p(T) : \ T' \textit{ matches with } Q_n \textit{ around the origin}\}$ to be the collection of tilings in $\Omega_p(T)$ that contains Q_n around the origin. We have that E_n are non-empty compact sets for each $n \in \mathbb{Z}^+$. Moreover, since P_n is an increasing sequence of patches, $\mathcal{D}_n$ is a (nested) increasing sequence of curves. Therefore, $\{Q_n\}_n$ contains a subsequence of patches $\{Q_{n_k}\}_{n_k}$ which is increasing (Q_n is not necessarily increasing), because $\{\mathcal{D}_n\}_n$ is increasing to $\mathcal{D}$. Thus, E_{n_k} is a decreasing sequence of non-empty compact sets. Hence, by Cantor's intersection theorem, $\bigcap_{n_k} E_{n_k}$ is non-empty, and Φ is surjective.

The pre-image of V under the map Φ is the singleton $\{T\}$, by construction. In fact, any punctured tiling that is a translation of V has a singleton pre-image. That is, Φ is one-to-one over the translation orbit of T, which is dense in $\Omega_p(T)$ by Lemma 2.1.14 and Theorem 2.1.16. Hence, Φ is almost one-to-one.

Finally, $\Phi\omega = \omega_V \Phi$ holds by construction where ω and ω_V denote the substitution map of T and V, respectively. Hence, Φ is an almost one-to-one factor map. $\square$

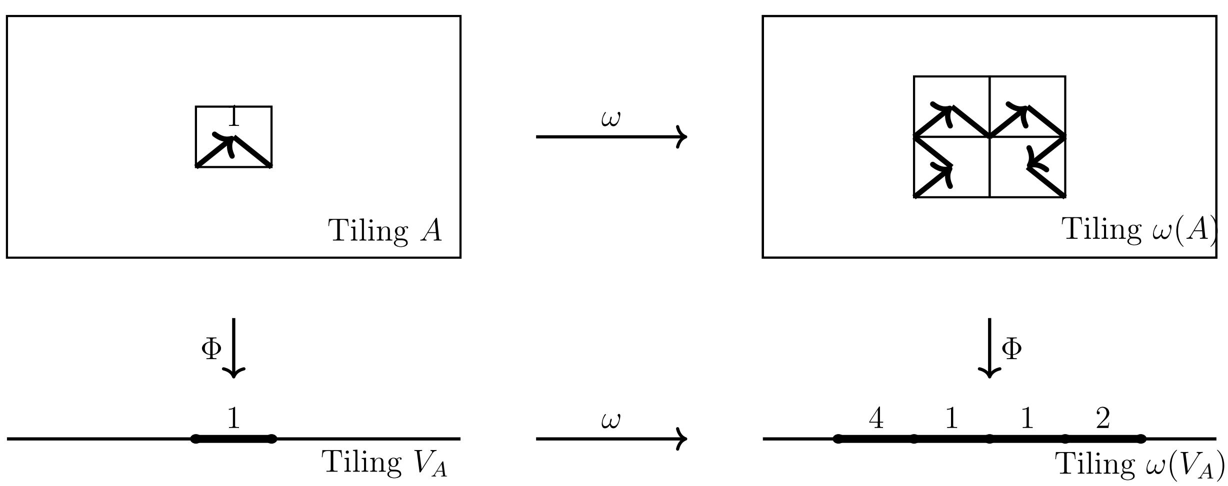


Figure 4.15: Φ is an almost one-to-one factor map

The Flattening Map Φ The factor map Φ defined in Theorem 4.0.7 can be regarded as a flattening map between 2-dimensional and 1-dimensional substitution tilings. Since 1-dimensional tilings induce total orders between their tiles, it is our interest to understand the flattening map Φ through total order systems.

Let T be a decorated substitution tiling with a relatively dense curve decoration $\mathcal{D}$ attached over it. Suppose further T is primitive, recognisable and has FLC, V is the 1-dimensional substitution tiling generated by flattening the curve $\mathcal{D}$ and $\Phi : \Omega_p(T) \mapsto \Omega_p(V)$ denote the associated almost one-to-one factor map as defined in Theorem 4.0.7. For each $t \in T$, denote the decoration of $t \in T$ with e_t. Consider the collection of tiles T^1, T^2 such that $T^1 = \{t \in T : \ e_t \text{ is a simple decoration}\}$ and $T^2 = T \backslash T^1$. Define $W^1 = \{[t, e_t] : \ t \in T^1\}$, $W^2 = \{[t, e_t^1], [t, e_t^2] : \ t \in T^2, e_t = e_t^1 \cup e_t^2 \text{ and } e_t^1 \cap e_t^2 = \emptyset\}$ and $W = W^1 \cup W^2$. The curve $\mathcal{D}$ induces a total order over the pairs in W, as explained in the introduction. More precisely, for $[t_1, e_1], [t_2, e_2] \in W$ define $[t_1, e_1] \lesssim [t_2, e_2]$ whenever e_1 comes before e_2, according to the relatively dense curve $\mathcal{D}$. This is a well-defined total order over the collection W. On the other hand, the punctured tiling V has a natural total order defined over its tiles. For $v_1, v_2 \in V$, define $v_1 \precsim v_2$ whenever $x(v_1) \leq x(v_2)$, where $x(v_1), x(v_2) \in \mathbb{R}$ are punctures of v_1, v_2, respectively. Next we show that the total orders $\lesssim$ and $\precsim$ are associated with each other through Φ.

The map Φ satisfies $\Phi(T) = V$. In particular, Φ associates every pair $[t, e_t] \in W$ with an interval $v \in V$. Let $\varphi : W \mapsto V$ denote this association map. That is, $\varphi : W \mapsto V$ is defined by $\varphi([t, e_t]) = v$ where v is the interval corresponding to the pair $[t, e_t]$ under the map Φ. Since V is generated by flattening the curve $\mathcal{D}$, we have that for all $[t_1, e_1], [t_2, e_2] \in W$,

$$[t_1, e_1] \lesssim [t_2, e_2] \Longleftrightarrow \varphi([t_1, e_1]) \precsim \varphi([t_2, e_2]). \tag{0.1}$$

If we represents the total orders as $\lesssim : W \mapsto \mathbb{Z}$ and $\precsim : V \mapsto \mathbb{Z}$, then we have that $\lesssim$ is equivalent to $\precsim \circ \varphi$. In words, Φ preserves the total order $\lesssim$ through the flattening process.

Finally, we explain how the total orders $\lesssim, \precsim$ behave under the substitutions ω, μ of T, V, respectively. We will only show the case for the total order $\precsim$. The result for the total order $\lesssim$ is similar by (0.1).

Lemma 4.0.8. *Suppose V is a 1-dimensional punctured substitution tiling with the substitution μ and $\precsim$ is the natural total order defined on the tiles of V, as defined above. Assume further $\precapprox$ denote the natural total order defined over the tiles of the 1-dimensional tiling $\mu(V)$. Let $v_1, v_2 \in V$ be given. Then $v_1 \precsim v_2$ if and only if for all $u_1 \in \mu(v_1)$ and $u_2 \in \mu(v_2)$ we have $u_1 \precapprox u_2$.*

Proof. The proof follows by the fact that supp $(\mu(v)) = \lambda_\mu \cdot \text{supp } v$ for all $v \in V$, where λ_μ is the expansion factor of μ. $\square$

Chapter 5

Space Filling Curves

Space filling curves are usually generated by substitution structures as outlined in the introduction. Furthermore, these substitution structures are defined on congruent shapes that are scaled by a fixed factor in every step. For example, Hilbert's space filling curve is formed over a substitution system that is defined upon congruent squares. These squares are scaled by a half in every iteration, ad infinitum. In this chapter we will form space filling curves from any given primitive substitution system which is defined over finite collection of convex tiles.

Hilbert's recipe of space filling curves based on two ingredients; Cantor's intersection theorem and the geometric order structure defined by an iteration process. We will provide Cantor's intersection theorem for completeness. Furthermore, we will explain the details of Hilbert's geometric construction. Finally, we will show how the travelling algorithm induces a geometric order structure which satisfies the second ingredient of Hilbert's recipe in a more broad way (Theorem 5.1.1).

Theorem 5.0.1 (Cantor's Intersection Theorem)**.** *Assume that $\{A_k\}_{k=1}^{\infty}$ is a sequence of non-empty compact subsets of $\mathbb{R}^n$ such that $A_k \supseteq A_{k+1}$ for each $k \in \mathbb{Z}^+$. Then the following holds:*

(1) $\bigcap_{k=1}^{\infty} A_k \neq \emptyset$,

(2) If, in addition, the diameters of A_k approach to zero, then $\bigcap_{k=1}^{\infty} A_k$ is a singleton.

Proof. For a proof see [2, P:56] or [4, P:88]. □

5.1 A conventional method for generating space filling curves

Hilbert's space filling curve was a milestone to construct space filling curves from geometric iterative systems. In this section we explain the details of Hilbert's construction. Moreover, we will show how the travelling algorithm generalise Hilbert's construction of space filling curves.

Define $\mathcal{I}_j^i$ to be the interval $\left[\frac{j-1}{4^{i-1}}, \frac{j}{4^{i-1}}\right]$ for $i \in \mathbb{Z}^+$ and $j \in \{1, 2, \ldots, 4^{i-1}\}$. Define also $\mathcal{J}_{j,k}^i$ to be the square $\left[\frac{j-1}{2^{i-1}}, \frac{j}{2^{i-1}}\right] \times \left[\frac{k-1}{2^{i-1}}, \frac{k}{2^{i-1}}\right]$ for $i \in \mathbb{Z}^+$ and $j, k \in \{1, 2, \ldots, 2^{i-1}\}$. Figure 5.1 illustrates these sequence of intervals and squares. For example, $\mathcal{I}_3^2$ is the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$ whereas $\mathcal{J}_{1,2}^2$ is the square $\left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right]$. Hilbert's geometric iteration system shown in Figure 5.2 induces a bijection between the intervals $\mathcal{I}_j^i$'s and the squares $\mathcal{J}_{j,k}^i$'s. For instance, $\mathcal{I}_3^2$ corresponds to the square $\mathcal{J}_{2,2}^2$. Denote this bijection by Φ. Let $x \in [0,1]$ be a given point. For each $i \in \mathbb{Z}^+$, there exists $n_i \in \{1, 2, \ldots, 4^{i-1}\}$ such that $x \in \mathcal{I}_{n_i}^i$. Note that $\bigcap_{i=1}^{\infty} \mathcal{I}_{n_i}^i = \{x\}$, by Cantor's intersection theorem. Moreover, $\bigcap_{i=1}^{\infty} \Phi(\mathcal{I}_{n_i}^i) = \{y\}$ for some $y \in [0,1] \times [0,1]$, by Cantor's intersection theorem as well. Therefore, we can define a function $F_h : [0,1] \mapsto [0,1] \times [0,1]$ by setting $F_h(x) = y$ where x, y are as defined above.

Surjectivity of F_h follows by the bijectivity of Φ. Next we show that F_h is continuous. Note first that for any given $a, b \in \mathcal{I}_j^i$ for $i \in \mathbb{Z}^+$ and $j \in \{1, 2, \ldots, 4^{i-1}\}$, we have that $|a - b| < \text{len}(\mathcal{I}_j^i) = \frac{1}{4^{i-1}}$ where $\text{len}(\mathcal{I}_j^i)$ denotes the length of the interval I_j^i. Furthermore, by the construction of F_h, we have that $d(F_h(a) - F_h(b)) < \text{diam}\left(\Phi(\mathcal{I}_j^i)\right) = \frac{\sqrt{2}}{2^{i-1}}$ where $\text{diam}\left(\Phi(\mathcal{I}_j^i)\right)$ denotes the diameter of the square $\Phi(\mathcal{I}_j^i)$ and d denotes the Euclidean distance. Let $x \in [0,1]$ be a given point. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $x_n \in [0,1]$ for each $n \in \mathbb{Z}^+$ and $x_n \to x$. We will prove that $F_h(x_n) \to F_h(x)$. Let $\epsilon > 0$ be given. Choose $i_0 \in \mathbb{Z}^+$ such that $\frac{2\sqrt{2}}{2^{i_0-1}} < \epsilon$. Subsequently, choose $N \in \mathbb{Z}^+$ sufficiently large so that $|x - x_n| < \frac{1}{4^{i_0-1}}$ for each $n \geq N$. Let $m \geq N$ be given. Since we have $|x - x_m| < \frac{1}{4^{i_0-1}}$, one of the following holds:

(1) x and x_m belong to a same interval $\mathcal{I}_{j_0}^{i_0}$ for some $j_0 \in \{1, 2, \ldots, 4^{i_0-1}\}$,

(2) $x \in \mathcal{I}_{j_0}^{i_0}$ and $x_m \in \mathcal{I}_{j_0-1}^{i_0}$ for some $j_0 \in \{1, 2, \ldots, 4^{i_0-1} - 1\}$,

(3) $x_m \in \mathcal{I}_{j_0}^{i_0}$ and $x \in \mathcal{I}_{j_0-1}^{i_0}$ for some $j_0 \in \{1, 2, \ldots, 4^{i_0-1} - 1\}$.

In all cases we get that $d\left(F_h(x) - F_h(x_m)\right) < 2 \cdot \frac{\sqrt{2}}{2^{i_0-1}} < \epsilon$ where d is the Euclidean distance. Since m was arbitrary, we arrive that F_h is (sequentially) continuous. The map F_h is called *Hilbert's space filling curve.*

Our travelling algorithm is based on a similar idea of Hilbert's space filling curve construction. Hilbert's space filling curve is formed by a substitution system which is defined on congruent squares. In addition, the curves in each iteration step passes through the common edges of tiles. Our travelling algorithm generates iterative systems where curves are reformed in each level through a (circle) decomposition step and pass through

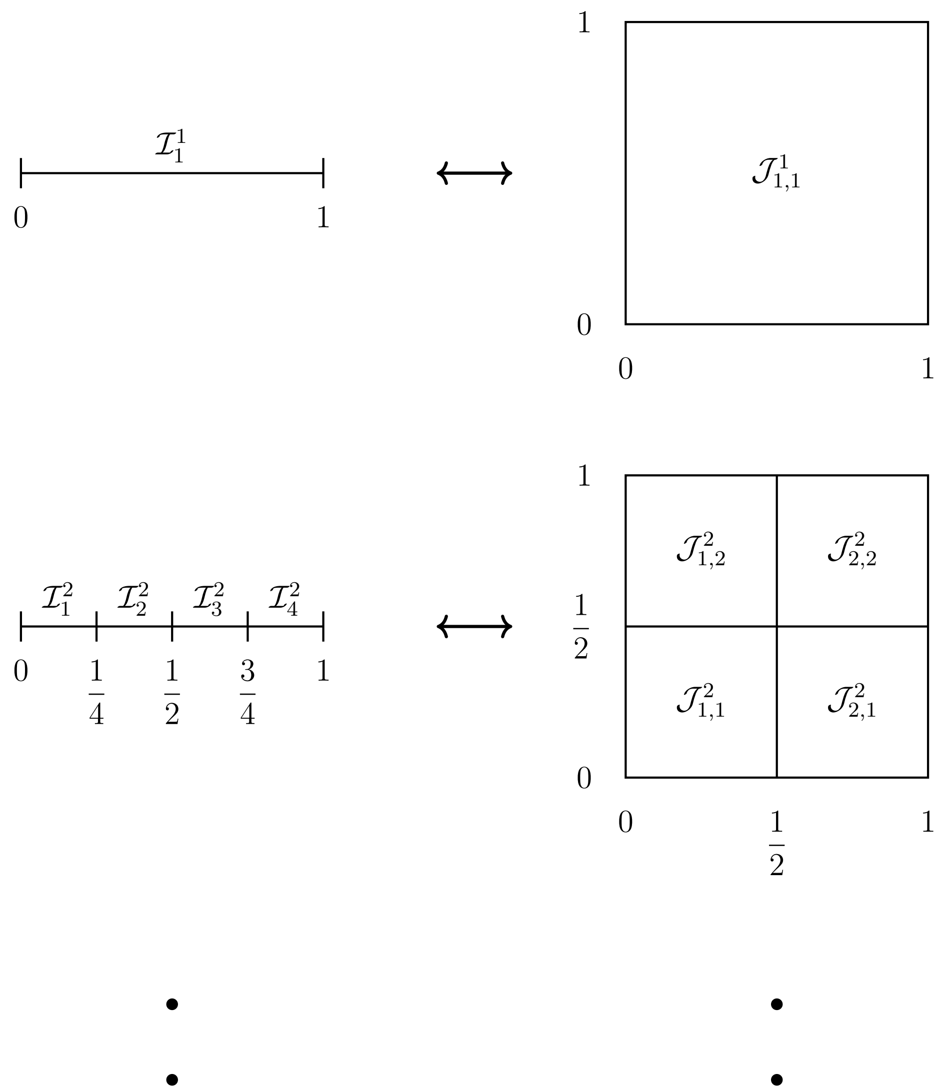


Figure 5.1: Illustration of $\mathcal{I}_j^i$'s and $\mathcal{J}_{j,k}^i$'s.

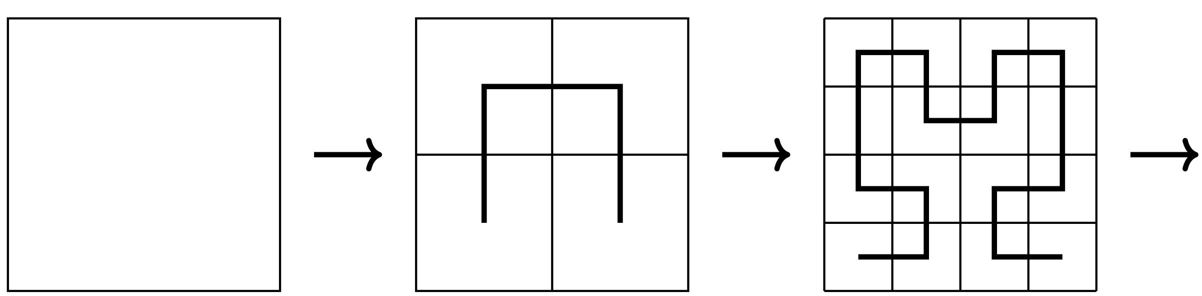


Figure 5.2: Geometry Behind the Hilbert's space filling curve

the vertices of tiles. The only differences we allow are the following:

(a) Curves move between the common vertices of tiles rather than the common edges of tiles.

(b) Substitution systems are allowed to consists of more than one type of convex shape.

So long as, the diameter of the substituted shapes approaches to zero (as the substitution is applied ad infinitum), Cantor's intersection theorem is still applicable. Hence a space filling curve can be defined through the same argument of Hilbert.

5.1.1 Substitutions to space filling curves

The following theorem ensures that we can generate infinitely many space filling curves from any given primitive substitution rule defined over finite collection of convex shapes of the plane.

Theorem 5.1.1. *Suppose $\mathcal{P}$ is a finite collection of convex tiles and ω is a primitive substitution rule defined over $\mathcal{P}$ with an expansion factor $\lambda > 1$. For each $p \in \mathcal{P}$, there exists a space filling curve $f : [0,1] \mapsto \text{supp } \omega^N(p)$ for some $N \in \mathbb{Z}^+$. In particular, for every $p \in \mathcal{P}$, there exists a sequence of (distinct) space filling curves $\{f_i\}_{i\in\mathbb{Z}^+}$ such that f_i maps the unit interval to the set $\text{supp }\left(\omega^{N\cdot i}(p)\right)$ for $i \in \mathbb{Z}^+$.*

Proof. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ for $n \in \mathbb{Z}^+$ be a finite collection of convex prototiles and let ω be a primitive substitution rule defined on $\mathcal{P}$ with an expansion factor $\lambda > 1$. By Corollary 3.3.3, there is an integer $N \in \mathbb{Z}^+$ sufficiently large enough such that each N-supertile $\omega^N(p)$ for $p \in \mathcal{P}$ satisfies the following:

(A) If there exists a slice tile in $\omega^N(p)$, then it has degree 2.

(B) There is no circle subpatch S of $\omega^N(p)$ such that $|S \cap V_c| > 1$, where V_c is the set of corners of the patch $\omega^N(p)$.

Thus, each N-supertile satisfies the conditions in Theorem 3.3.21. Therefore, for each prototile $p \in \mathcal{P}$ and each decoration e_p for p, there exists a decoration $\mathcal{C}_P$ of $\omega^N(p)$ such that

(i) If e_p has end points A and B, then $\mathcal{C}_p$ makes $(\lambda^N \cdot A, \lambda^N \cdot B)$ a valid pair for $\omega^N(p)$,

(ii) If e_p has end point pairs A, B and C, D, then $\mathcal{C}_p$ makes $\{(\lambda^N{\cdot}A, \lambda^N{\cdot}B), (\lambda^N{\cdot}C, \lambda^N{\cdot}B)\}$ a split pair for $\omega^N(p)$.

Start with the prototile $p_1 \in \mathcal{P}$ and a simple decoration e_{p_1} for p_1, with end points A, B. Then $(\lambda^N \cdot A, \lambda^N \cdot B)$ is a valid pair for $\omega^N(p_1)$, by Theorem 3.3.21. Suppose $\mathcal{C}^1$ is a curve that makes $(\lambda^N \cdot A, \lambda^N \cdot B)$ a valid pair for $\omega^N(p_1)$. Assume further $\mathcal{C}^1 = \bigcup_{j=1}^{k_1} e_j^1$ where e_j^1 is a simple curve for each $j = 1, \ldots, k_1$ so that

(1) $s(e_j^1) = X_j^1$ and $r(e_j^1) = Y_j^1$ for every $j = 1, \ldots, k_1$,

(2) $Y_j^1 = X_{j+1}^1$ for all $j = 1, \ldots, k_1 - 1$,

(3) $X_1^1 = \lambda^N \cdot A$ and $Y_{k_1}^1 = \lambda^N \cdot B$.

That is, e_j^1 for $j = 1, \ldots, k_1$ is a simple curve component of $\mathcal{C}^1$. Define $r_j^1 = \dfrac{l(e_j^1)}{l(\mathcal{C}^1)}$ for $j = 1, \ldots, k_1$ where $l(e_j^1)$ and $l(\mathcal{C}^1)$ are the lengths of the curves e_j^1 and $\mathcal{C}^1$, respectively. Partition the unit interval into k_1 subintervals of lengths $r_1^1, \ldots, r_{k_1}^1$, respectively. Denote these subintervals as $\mathcal{I}_1^1, \mathcal{I}_2^1, \ldots, \mathcal{I}_{k_1}^1$, respectively. For each $j = 1, \ldots, k_1$, there exists a tile $t_j^1 \in \omega^N(p_1)$ such that $e_j^1 \subseteq \text{supp } t_j^1$. Define a map $\psi_1 : \{\mathcal{I}_j^1 : j \in \{1, \ldots, k_1\}\} \mapsto \{\lambda^{-N} \cdot \text{supp } t_j^1 : t_j^1 \in \omega^N(p_1)\}$ such that $\psi_1(\mathcal{I}_j^1) = t_j^1$ if $e_j^1 \subseteq \text{supp } t_j^1$ for $j \in \{1, \ldots, k_1\}$. Note that ψ_1 is not necessarily bijective.

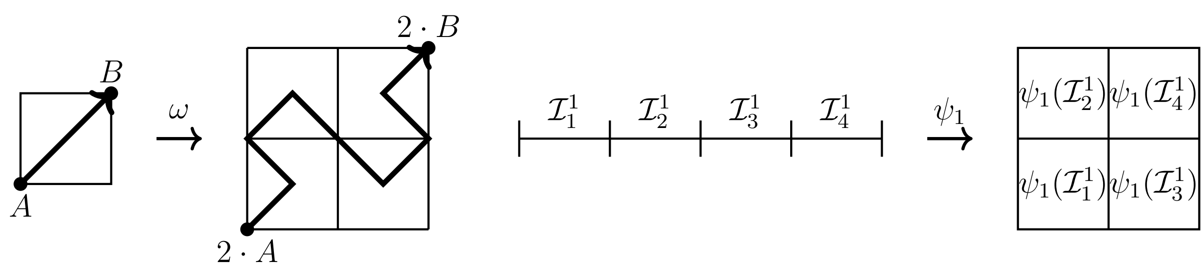


Figure 5.3: The substitution ω^N (for $N = 1$) is illustrated on the left whereas the map ψ_1 induced from ω^N is shown on the right.

Next we apply the substitution ω^N to each tile appearing in $\omega^N(p)$. Tiles in $\omega^N(p_1)$ corresponds to the N-supertiles in the 2N-supertile $\omega^{2\cdot N}(p_1)$. Each tile in $\omega^N(p_1)$ induces a decoration from $\mathcal{C}^1$. Moreover, these decorations of tiles in $\omega^N(p_1)$ (induced from $\mathcal{C}^1$) correspond to the decorations of N-supertiles in the $2N$-supertile $\omega^{2\cdot N}(p_1)$ by (i) and (ii) given above. Since $\mathcal{C}^1$ is a concatenation of decorations of tiles in $\omega^N(p_1)$, there exists a curve $\mathcal{C}^2$ which is a concatenation of decorations of N-supertiles in $\omega^{2\cdot N}(p_1)$ such that $\mathcal{C}^2$ makes $(\lambda^{2\cdot N} \cdot A, \lambda^{2\cdot N} \cdot B)$ a valid pair for $\omega^{2\cdot N}(p_1)$. Suppose $\mathcal{C}^2 = \bigcup_{j=1}^{k_2} e_j^2$ where e_j^2 is a simple curve for each $j = 1, \ldots, k_2$ so that

(1) $s(e_j^2) = X_j^2$ and $r(e_j^2) = Y_j^2$ for every $j = 1, \ldots, k_2$,

(2) $Y_j^2 = X_{j+1}^2$ for all $j = 1, \ldots, k_2 - 1$,

(3) $X_1^2 = \lambda^{2\cdot N} \cdot A$ and $Y_{k_2}^2 = \lambda^{2\cdot N} \cdot B$.

That is, e_j^2 for $j = 1, \ldots, k_2$ is a simple curve component of $\mathcal{C}^2$. Define $r_j^2 = \dfrac{l(e_j^2)}{l(\mathcal{C}^2)}$ for $j = 1, \ldots, k_2$ where $l(e_j^2)$ and $l(\mathcal{C}^2)$ are the lengths of the curves e_j^2 and $\mathcal{C}^2$, respectively.

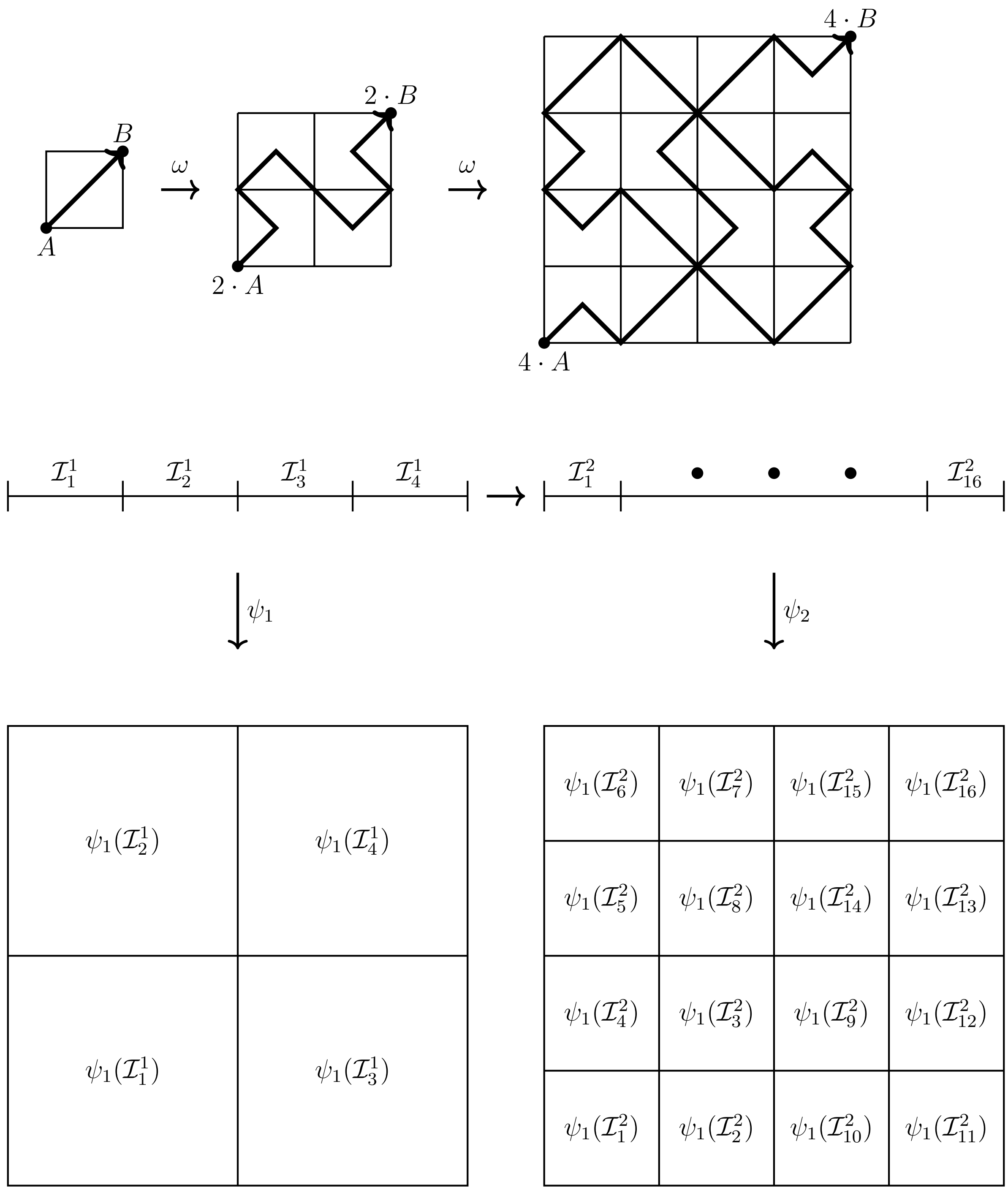


Figure 5.4: Maps ψ_1 and ψ_2 are demonstrated.

Partition the unit interval into k_2 subintervals of lengths $r_1^2, \ldots, r_{k_2}^2$, respectively. Denote these subintervals as $\mathcal{I}_1^2, \mathcal{I}_2^2, \ldots, \mathcal{I}_{k_2}^2$, respectively. For each $j = 1, \ldots, k_2$, there exists a tile $t_j^2 \in \omega^{2\cdot N}(p_1)$ such that $e_j^2 \subseteq \text{supp } t_j^2$. Define a map $\psi_2 : \{\mathcal{I}_j^2 :\ j \in \{1, \ldots, k_2\}\} \mapsto \{\lambda^{-2\cdot N} \cdot \text{supp } t_j^2 :\ t_j^2 \in \omega^{2\cdot N}(p_1)\}$ such that $\psi_2(\mathcal{I}_j^2) = t_j^2$ if $e_j^2 \subseteq \text{supp } t_j^2$ for $j \in \{1, \ldots, k_2\}$. Continue the same process ad infinitum. We get a sequence of intervals $\mathcal{I}_j^i$ for $i \in \mathbb{Z}^+$ and $j \in \{1, \ldots, k_i\}$ for some $k_i \in \mathbb{Z}^+$. Define a map ψ over the collection $\{\mathcal{I}_j^i :\ i \in \mathbb{Z}^+$ and $j \in \{1, \ldots, k_i\}$ for some $k_i \in \mathbb{Z}^+\}$ such that $\psi|_{\mathcal{I}_j^i} = \psi_i$ for each $i \in \mathbb{Z}^+$ and $j \in \{1, \ldots, k_i\}$ where $k_i \in \mathbb{Z}^+$. Then ψ corresponds each interval of the from $\mathcal{I}_j^i$ with a compact region of the plane. The first two iteration steps that are showing how ψ is defined are demonstrated with an example in Figure 5.4.

For each $i \in \mathbb{Z}^+$ define the following:

$$\delta_i = \max\left\{r_j^i :\ j \in \{1, \ldots, k_i\} \text{ for } k_i \in \mathbb{Z}^+\right\}.$$
$$\delta_i' = \min\left\{r_j^i :\ j \in \{1, \ldots, k_i\} \text{ for } k_i \in \mathbb{Z}^+\right\}.$$
$$\epsilon_i = \max\left\{\text{diam}\left(\psi(\mathcal{I}_j^i)\right) :\ j \in \{1, \ldots, k_i\} \text{ for } k_i \in \mathbb{Z}^+\right\}$$

where $\text{diam}\big(\psi(\mathcal{I}_j^i)\big)$ denotes the diameter of the compact region $\big(\psi(\mathcal{I}_j^i)\big)$. Since the substitution ω^N is primitive, both δ_i and ϵ_i are approaching to zero as i tends to infinity. Let $x \in [0,1]$ be fixed. For each $i \in \mathbb{Z}^+$ there exists an integer $n_i \in \{1, \ldots, k_i\}$ such that $x \in \mathcal{I}_{n_i}^i$. Note that $\bigcap_{i=1}^{\infty} \mathcal{I}_{n_i}^i = \{x\}$, by Cantor's intersection theorem. Moreover, $\bigcap_{i=1}^{\infty} \psi(\mathcal{I}_{n_i}^i) = \{y\}$ for some $y \in [0,1] \times [0,1]$, by Cantor's intersection theorem as well. This is because of the fact that the diameters ϵ_i approach zero as i tends to infinity. Therefore, we can define a function $F : [0,1] \mapsto \lambda^{-N} \cdot \text{supp } \omega^N(p_1)$ by setting $F(x) = y$ where x, y are as defined above.

F is surjective by construction. Next we show that F is continuous. Note first that for any given $a, b \in \mathcal{I}_j^i$ for $i \in \mathbb{Z}^+$ and $j \in \{1, 2, \ldots, k_i\}$, we have that $|a - b| < \delta_i$. Furthermore, by the construction of F, we have that $d(F(a) - F(b)) < \epsilon_i$. Let $x \in [0,1]$ be a given point. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $x_n \in [0,1]$ for each $n \in \mathbb{Z}^+$ and $x_n \to x$. We will prove that $F(x_n) \to F(x)$. Let $\epsilon > 0$ be given. Choose $i_0 \in \mathbb{Z}^+$ such that $2 \cdot \epsilon_{i_0} < \epsilon$. Choose also $N \in \mathbb{Z}^+$ sufficiently large so that $|x - x_n| < \delta_{i_0}'$ for each $n \geq N$. Let $m \geq N$ be given. Since we have $|x - x_m| < \delta_{i_0}'$, one of the following holds:

(1) x and x_m belong to a same interval $\mathcal{I}_{j_0}^{i_0}$ for some $j_0 \in \{1, 2, \ldots, k_{i_0}\}$,
(2) $x \in \mathcal{I}_{j_0}^{i_0}$ and $x_m \in \mathcal{I}_{j_0-1}^{i_0}$ for some $j_0 \in \{1, 2, \ldots, k_{i_0} - 1\}$,
(3) $x_m \in \mathcal{I}_{j_0}^{i_0}$ and $x \in \mathcal{I}_{j_0-1}^{i_0}$ for some $j_0 \in \{1, 2, \ldots, k_{i_0} - 1\}$.

In all cases we get that $d\left(F(x) - F(x_m)\right) < 2 \cdot \epsilon_{i_0} < \epsilon$. Since m was arbitrary, we arrive that F is (sequentially) continuous. □

5.1.2 Decorations of decorated tilings

Hilbert's substitution tiling T given in Example 4.0.5 has a single bi-infinite (relatively dense) curve attached over it. We showed in Example 4.0.5 that there also exists a decorated tiling $A \in \Omega(T)$ that has two bi-infinite curves attached over it (i.e. concatenation of the decorations of tiles in A form two bi-infinite curves). In this section, we further prove that for each decorated tiling $B \in \Omega(T)$ there are at most 4 bi-infinite curves attached over it. In particular, we prove the following lemma which states that for any given decorated substitution tiling U satisfying mild conditions there exists $N \in \mathbb{Z}^+$ such that each tiling $U' \in \Omega(U)$ has at most N many bi-infinite curves attached over it.

Lemma 5.1.2. *Assume that T is a recognisable primitive substitution tiling that has FLC and consists of convex tiles. Let N_X denote the number of tiles in T that contains the vertex $X \in V_T$ (i.e. X is a vertex of a tile in T). Suppose further T_d denote a decorated version of T which is generated according to the instructions in the proof of Theorem 3.3.25 and contains a single bi-infinite (relatively dense) curve attached over it. Then for every tiling $A \in \Omega(T_d)$ there are at most $2 \cdot \max_{X \in V_T} N_X$ many bi-infinite curves attached to A. If, in addition, all decorated tiles of T_d have simple decorations, then there are at most $\max_{X \in V_T} N_X$ many bi-infinite curves attached to A.*

Proof. Let T denote a recognisable primitive substitution tiling that has FLC and consists of convex tiles. Assume further T_d is a decorated version of T and is generated according to the instructions in the proof of Theorem 3.3.25. Let N_X denote the number of tiles that contains the vertex X of the tiling T (i.e. X is a vertex of a tile in T). Since T has FLC, we have that $N = 2 \cdot \max_{X \in V_T} N_X \in \mathbb{Z}^+$.

Let $A \in \Omega(T_d)$ be given. Choose a vertex $Y \in V_A$ (i.e. Y is a vertex in A). Define S_n for $n \in \mathbb{Z}^+$ to be the collection of n-supertile patches within the tiling A whose support contains the vertex Y. More precisely, for $n \in \mathbb{Z}^+$ define $S_n = \{\omega^n(t) : \ \omega^n(t) \subseteq A \text{ and } Y \in \text{supp } \omega^n(t)\}$. We have that $|S_n| = |\{u : \ u \in \omega^{-n}(A) \text{ and } Y \in \text{supp } u\}| \leq \max_{X \in V_T} N_X$, for $n \in \mathbb{Z}^+$, because T_d is recognisable. Define $A_n = \bigcup_{\omega^n(t) \in S_n} \omega^n(t)$, for $n \in \mathbb{Z}^+$. Supertiles are substituted tiles. So, every supertile has at most two curves attached over it as a decoration. Therefore, there are at most N many curves attached over the patch A_n as a decoration, for all $n \in \mathbb{Z}^+$. We have $A = \bigcup_{j=1}^{\infty} A_j$ and $A_j \subseteq A_{j+1}$ for each $j \in \mathbb{Z}^+$. The decoration of the patch A_j is contained within the decoration of the patch A_{j+1} for each $j \in \mathbb{Z}^+$. Hence, A can have at most N many bi-infinite curves attached over it, because $A = \bigcup_{j=1}^{\infty} A_j$.

If, in addition, every decorated tile has a simple decoration, then there are at most $\max_{X \in V_T} N_X$ many curves attached over the patch A_n as a decoration, for each $n \in \mathbb{Z}^+$. Hence, the conclusion follows by the same argument. □

Corollary 5.1.3. *Let T denote the Hilbert's substitution tiling defined in Example 4.0.5. For each tiling $B \in \Omega(T)$ there are at most 4 bi-infinite curves attached over it.*

Proof. The proof follows by the fact that all tiles are squares with simple decorations. □

5.1.3 A false approach to construct space filling curves

Before moving to examples of space filling curves generated by the known substitution tilings, we explain why Cantor's intersection theorem is necessary for the geometric construction of space filling curves.

Consider the substitution given in Figure 5.5. The unit square is divided into half in the first iteration. Subsequently, every rectangle is divided into half in each step as illustrated in the figure. As before, the curves in the iteration steps corresponds to the unit interval. For each point in the unit interval, pick a nested sequence of subintervals (one subinterval from each iteration step) that converges to the chosen point. This sequence of nested intervals corresponds to a sequence of nested rectangles. Notice that each rectangle appearing in an iteration step has a diameter 1. That is, diameters of the rectangles do not shrink to zero, but rather converge to 1. Therefore, we cannot apply the Hilbert's argument to construct a space filling curve. More precisely, let x be a given point in the unit interval. There exists a sequence of subintervals that contain the point x. This sequence is generated by choosing a subinterval in each iteration step such that chosen subintervals contain the point x. This sequence of subintervals corresponds a sequence of nested rectangles. Denote these rectangles by $\{E_n\}_{n\in\mathbb{Z}^+}$ such that $E_k \supseteq E_{k+1}$ for each $k \in \mathbb{Z}^+$. We have that $\bigcap_{k\in\mathbb{Z}^+} E_k \neq \emptyset$ by (1) of Theorem 5.0.1. However, because their diameters do not approach to zero, we cannot apply (2) of Theorem 5.0.1. In fact, if $(a, b) \in [0, 1]\times[0, 1]$ is given such that $(a, b) \in \bigcap_{k\in\mathbb{Z}^+} E_k$, then we have that $(a, b/n) \in \bigcap_{k\in\mathbb{Z}^+} E_k$ for every $n \in \mathbb{Z}^+$. We showed that the intersection $\bigcap_{k\in\mathbb{Z}^+} E_k$ is not a unique point unlike in Hilbert's construction. Therefore, there is not a unique candidate y that can be assigned as the image of the point x. Since the intersection is not unique, we can only define a function by this geometric argument through the axiom of choice. This can be done by fixing a point in the intersection $\bigcap_{k\in\mathbb{Z}^+} E_k$ and assigning it to the image of the point x. But then, this construction does not assure that a space filling curve can be generated. In fact, we refer the reader to [19, P: 98-99] for the numeric details of why this geometric construction can only lead to a discontinuous map, and cannot form a space filling curve.

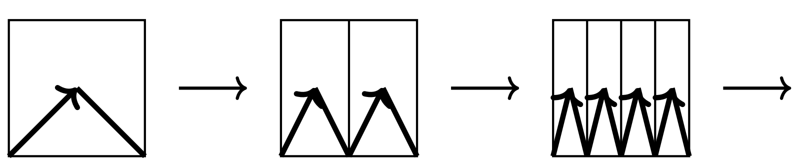


Figure 5.5: A false geometric construction

5.2 Examples

In this section, we provide some examples to observe the travelling algorithm.

Example 5.2.1 (Hilbert substitution tiling)**.** Consider the (Hilbert) substitution rule in Figure 5.6 (with expansion factor $\lambda = 2$). By adding extra curve labels to its prototiles as demonstrated in Figure 5.7, we get an isomorphic prototile set and substitution rule. Note that curves in Figure 5.7 are aligned to be concatenated in a natural way. This (primitive) substitution defines a recognisable self-similar primitive substitution tiling of the plane, and we call it *Hilbert substitution tiling.*

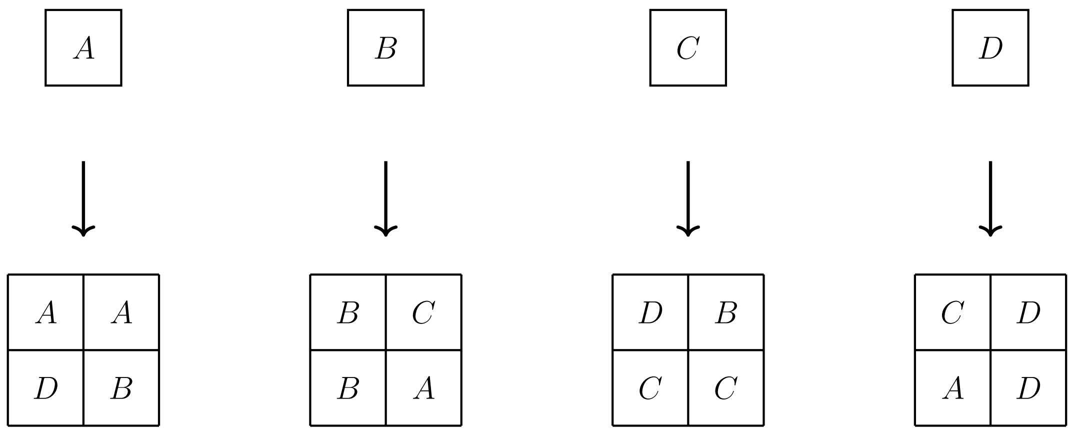


Figure 5.6: Hilbert substitution rule

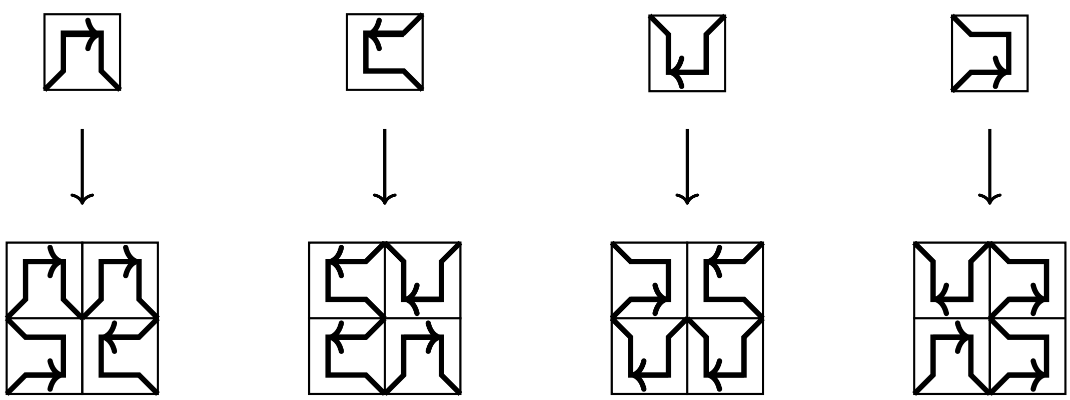


Figure 5.7: Hilbert substitution rule - with curve labels

Denote the simple curves over the prototiles in the figure as a, b, c, d, from left to right, respectively. Then, by reading the curves through their concatenation processes in the 1-supertiles, one can define the following one dimensional Hilbert substitution σ_h:

$$a \mapsto d, a, a, b, \qquad b \mapsto c, b, b, a, \qquad c \mapsto b, c, c, d, \qquad d \mapsto a, d, d, c.$$

Change the variables in the substitution as: a and c to x, and b and d to y. Then we get the substitution σ such that $\sigma(x) = y, x, x, y$ and $\sigma(y) = x, y, y, x$. Hence, σ is locally derivable from σ_h. Notice that σ is the substitution of 1DTM applied twice in the reverse order (i.e. $x \mapsto y, x$ and $y \mapsto x, y$).

Example 5.2.2 (Table substitution tiling)**.** Consider the table substitution tiling whose substitution rule is given in Figure 5.8 (with expansion factor $\lambda = 2$). By adding extra curve labels to its prototiles as demonstrated in Figure 5.9, we get an isomorphic prototile set and substitution rule. Notice that the curves in Figure 5.8 are concatenated in a natural way, which we will use to induce an order structure. In particular, we will concatenate the curves in the 1-supertiles and regard them as substitutions for the label curves in the prototiles, respectively.

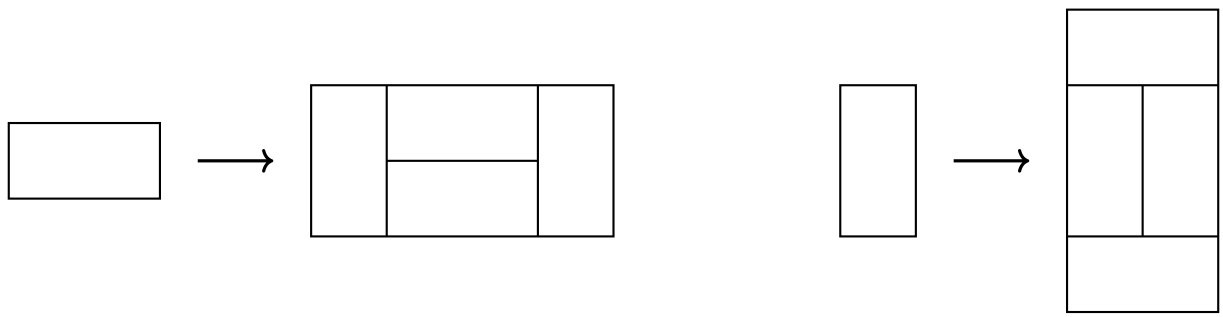


Figure 5.8: Table (tiling) substitution rule

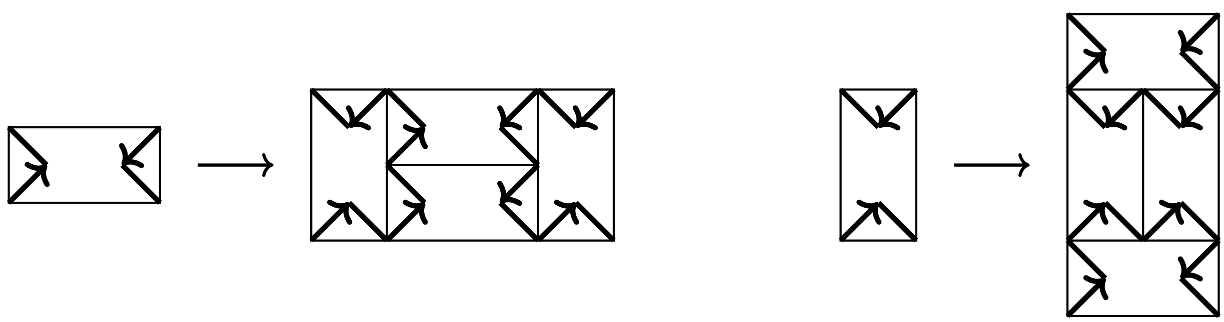


Figure 5.9: Table (tiling) substitution rule - with curve labels

Label the curves in decorations of the prototiles as a, b, c, d, where a is the left simple curve decoration for the prototile on the left of Figure 5.9, b is the right simple curve decoration for the prototile on the left of the figure, c is the top simple curve decoration for the prototile on the right of the figure and d is the bottom simple curve decoration for the prototile on the right of the figure. Then one can define the following one dimensional substitution σ_t:

$$a \mapsto d, a, a, b, \qquad b \mapsto c, b, b, a, \qquad c \mapsto b, c, c, d, \qquad d \mapsto a, d, d, c.$$

Notice that $\sigma_t = \sigma_h$. In particular, the Table substitution tiling is MLD to the (2-dimensional) Hilbert substitution tiling. The MLD relation is given by Robinson [25], by dissecting the domino rectangles into two unit squares and define a substitution rule for four unit squares as in the Hilbert substitution rule given in in Figure 5.6.

Example 5.2.3 (Check board tiling)**.** Consider the substitution rule given in Figure 5.10. For illustration purposes, we 'rounded off' the curves at the common interior vertices on the patches in the figure. The substitution rule generates the check board tiling if the curve labels are replaced with black and white colour labels. The check board tiling is a periodic tiling and its associated substitution rule is not recognisable. We apply the dimension reduction technique only for illustration purposes.

Figure 5.10

Label the curves in decorations of the prototiles as a, b, c, d, where a is the top simple curve decoration for the prototile on the left of Figure 5.10, b is the bottom simple curve decoration for the prototile on the left of the figure, c is the left simple curve decoration for the prototile on the right of the figure and d is the right simple curve decoration for the prototile on the right of the figure. Flattening the curve labels of tiles, according to the concatenation of curves shown in Figure 5.10, defines the substitution rule σ_{rs}:

$$a \mapsto d, a, c, a, \qquad b \mapsto c, b, d, b, \qquad c \mapsto c, b, c, a, \qquad d \mapsto d, a, d, b.$$

It can be readily seen that $\sigma_{rs} = \sigma^2$, where σ is the one dimensional Rudin-Shapiro substitution which is defined as:

$$\sigma(A) = D, B, \qquad \sigma(B) = C, A, \qquad \sigma(C) = C, B, \qquad \sigma(D) = D, A.$$

Example 5.2.4 (2DTM substitution tiling)**.** Consider the substitution rule given in Figure 5.11. Once again, we 'rounded off' the curves at the common interior vertices on the patches in the figure for illustration purposes. If the curve labels are replaced with colour labels, then the substitution rule is nothing but the 2DTM substitution rule.

Figure 5.11

Label the curves in decorations of the prototiles as a, b, c, d, where a is the top simple curve decoration for the prototile on the left of Figure 5.11, b is the bottom simple curve decoration for the prototile on the left of the figure, c is the left simple curve decoration for the prototile on the right of the figure and d is the right simple curve decoration for the prototile on the right of the figure. By the same argument, we generates the following one dimensional substitution σ_{TM}:

$$a \mapsto a, d, a, c, \qquad b \mapsto b, c, b, d, \qquad c \mapsto c, b, c, a, \qquad d \mapsto d, a, d, b.$$

Observe that the doubled 1DTM substitution σ, which is defined by $\sigma(x) = x, y, x, y$ and $\sigma(y) = y, x, y, x$, is locally derivable from σ_{TM}, by simple change of variables of a, b with x and c, d with y.

Example 5.2.5 (2DTM with Hilbert curves)**.** We provide the example given in the introduction for completeness. Consider the substitution rule depicted in Figure 5.12. The substitution rule defines a tiling, which we call *two dimensional Thue-Morse-Hilbert tiling* (2DTMH in short).

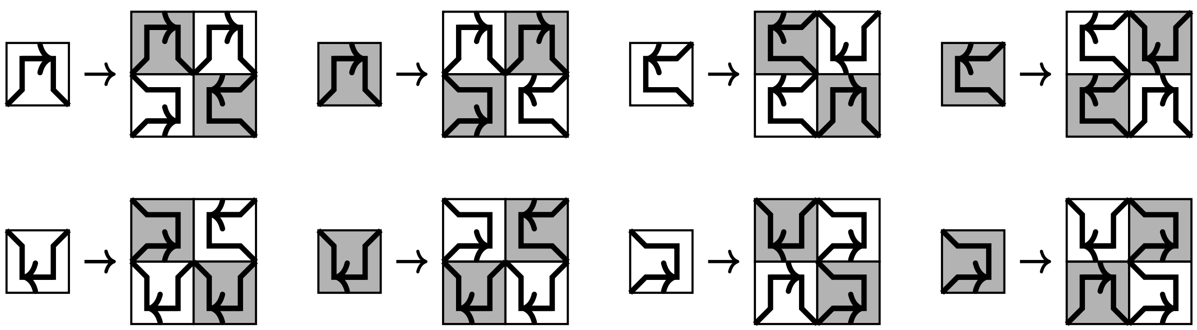
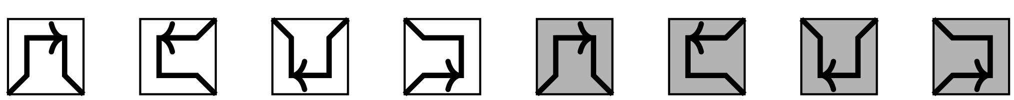


Figure 5.12

By the same token, we form the following substitution rule σ_{TMH}:

$$\sigma_{TMH}(a) = d, e, a, f, \quad \sigma_{TMH}(b) = c, f, b, e, \quad \sigma_{TMH}(c) = b, g, c, h, \quad \sigma_{TMH}(d) = a, h, d, g,$$
$$\sigma_{TMH}(e) = h, a, e, b, \quad \sigma_{TMH}(f) = g, b, f, a, \quad \sigma_{TMH}(g) = f, c, g, d, \quad \sigma_{TMH}(h) = e, d, h, c,$$

where the correspondence of prototile labels are shown in Figure 5.13.



a b c d e f g h

Figure 5.13: The correspondence between tiles

Notice that if we forget the colour labels, we get the Hilbert substitution rule. In particular, let $\phi_1 : \{a, b, c, d, e, f, g, h\} \mapsto \{A, B, C, D\}$ be a local map defined by $\phi_1(a) =$

$\phi_1(e) = A$, $\phi_1(b) = \phi_1(f) = B$, $\phi_1(c) = \phi_1(g) = C$, $\phi_1(d) = \phi_1(h) = D$. Then $\sigma_{TMH}(\phi_1) = \sigma_h$. Similarly, if we forget the curve labels, we arrive the 2DTM tiling. More precisely, let $\phi_2 : \{a, b, c, d, e, f, g, h\} \mapsto \{A, B, C, D\}$ be defined by $\phi_2(a) = \phi_2(d) = A$, $\phi_2(b) = \phi_2(c) = B$, $\phi_2(e) = \phi_2(h) = D$, $\phi_2(f) = \phi_2(g) = C$. Then $\sigma_{TMH}(\phi_2) = \sigma_{TM}$. Hence, both σ_h and σ_{TM} are locally derivable from σ_{TMH}.

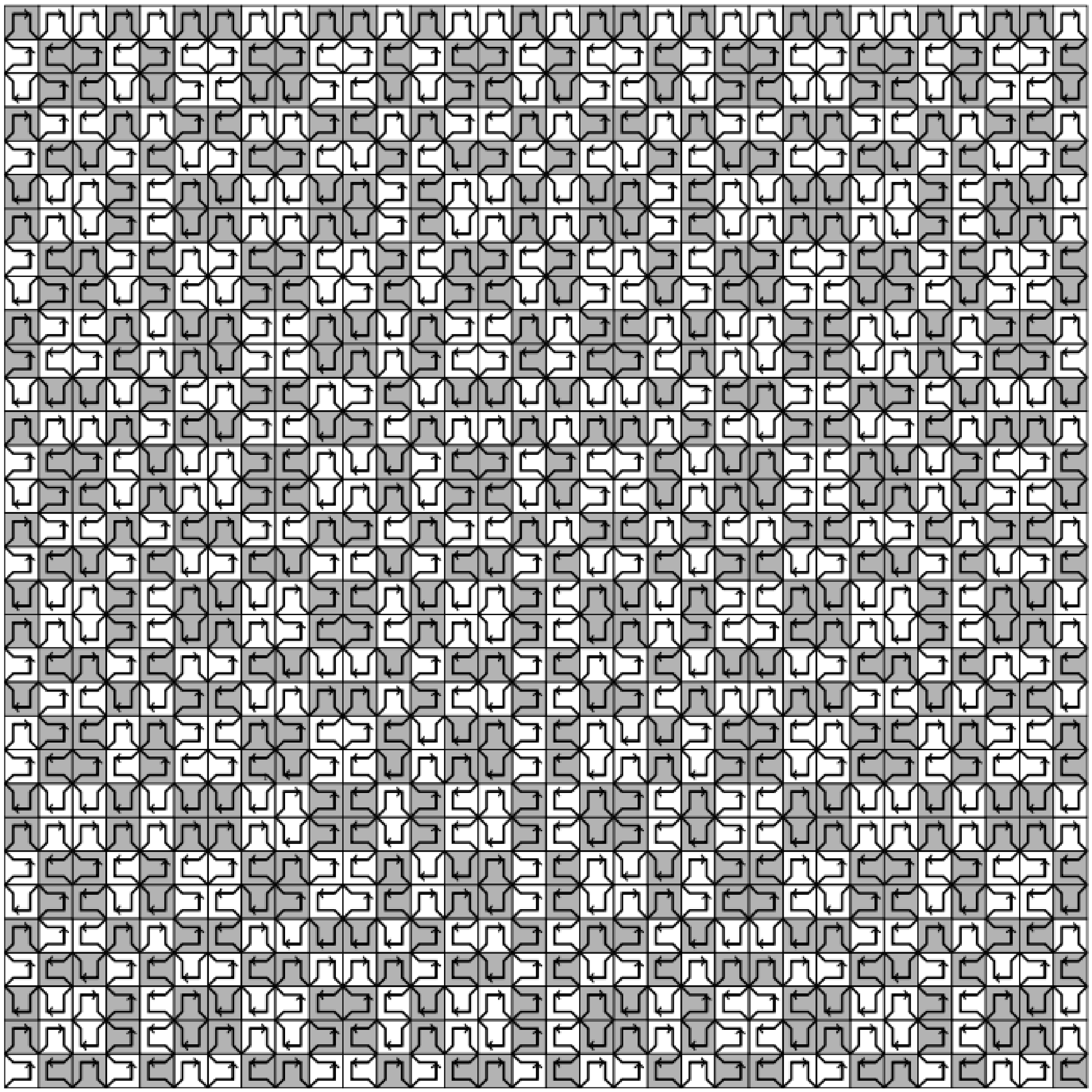


Figure 5.14: A patch of the 2DTMH tiling

Example 5.2.6 (Peano's substitution tiling)**.** Consider the (Peano's) substitution rule given in Figure 5.15 (with expansion factor $\lambda = 3$). We 'rounded off' some of the curves at the intersection points in the figure as before.

Denotes the simple curve decorations over the prototiles on the top of the figure as a, b from left to right, and the prototiles on the bottom of the figure as c, d, from left to right,

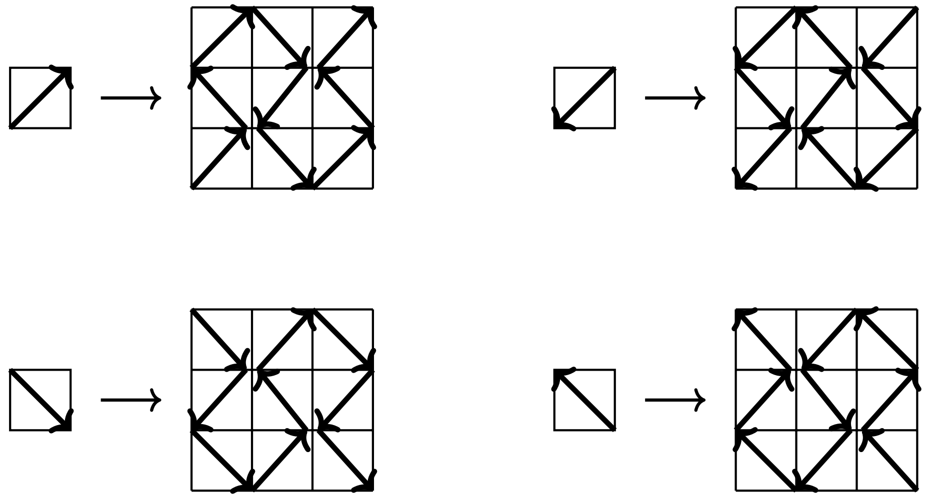


Figure 5.15: Peano's substitution rule

respectively. Then one can define the following one dimensional Peano's substitution σ_p:

$$a \mapsto a, d, a, c, b, c, a, d, a$$
$$b \mapsto b, c, b, d, a, d, b, c, b$$
$$c \mapsto c, b, c, a, d, a, c, b, c$$
$$d \mapsto d, a, d, b, c, b, d, a, d$$

It can be readily seen that $\sigma_p = \sigma^2$ where σ is the substitution defined as:

$$\sigma(a) = a, d, a, \qquad \sigma(b) = b, c, b, \qquad \sigma(c) = d, a, d \qquad \sigma(d) = c, b, c.$$

Example 5.2.7 (Square Chair substitution tiling)**.** Start with the substitution rule given in Figure 5.16. Then change the letter labels to curve labels as shown in Figure 5.17.

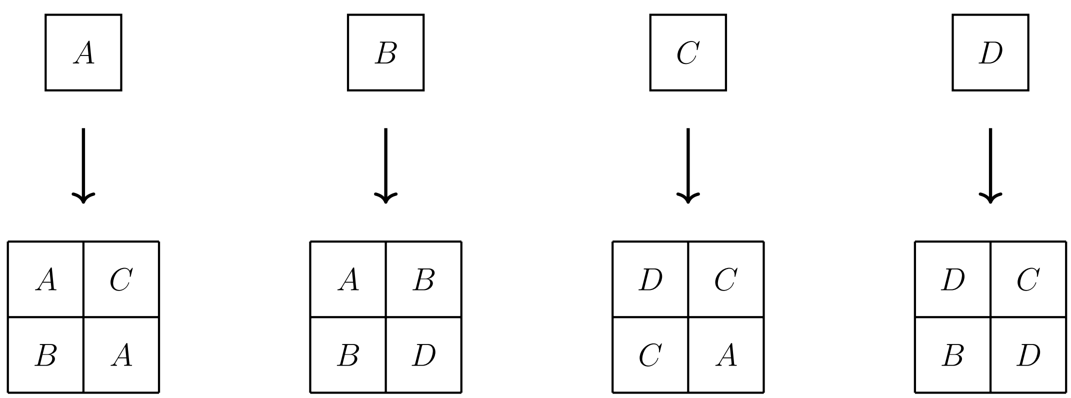


Figure 5.16: Square Chair substitution rule

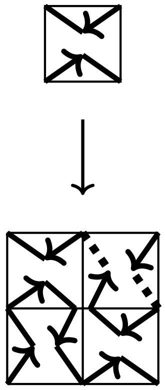


Figure 5.17: Square Chair substitution rule - with curve labels

Label the curve decorations according to Figure 5.18. The generated one dimensional substitution σ_{sc} is the following:

$$a \mapsto f, a, e, a, \qquad e \mapsto e, h, e, g,$$
$$b \mapsto c, b, d, b, \qquad f \mapsto f, a, f, b,$$
$$c \mapsto c, b, c, a, \qquad g \mapsto f, g, e, g,$$
$$d \mapsto d, g, d, h, \qquad h \mapsto c, h, d, h.$$

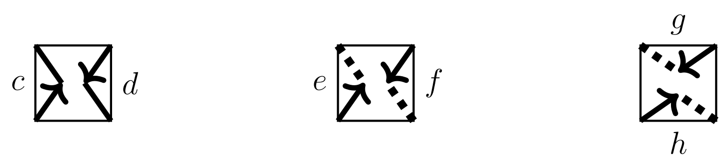


Figure 5.18: Square Chair substitution rule - labels of decorations

Define the map $\phi : \{a, b, c, d, e, f, g, h\} \mapsto \{x, y, z, t\}$ such that $\phi(a) = \phi(g) = x$, $\phi(b) = \phi(h) = y$, $\phi(c) = \phi(e) = z$ and $\phi(d) = \phi(f) = t$. Then $\sigma_{sc}(\phi)$ is the substitution defined as

$$\sigma_{sc}(\phi)(x) = t, x, z, x, \quad \sigma_{sc}(\phi)(y) = z, y, t, y, \quad \sigma_{sc}(\phi)(z) = z, y, z, x, \quad \sigma_{sc}(\phi)(t) = t, x, t, y.$$

Observe that $\sigma_{sc}(\phi) = \sigma_{rs}$, where σ_{rs} is the Rudin-Shapiro substitution rule defined in Example 5.2.3. Hence σ_{rs} is locally derivable from σ_{sc}.

Example 5.2.8 (Square Chair-Hilbert)**.** Consider the Square-Chair substitution rule with Hilbert curves attached to them, as illustrated in Figure 5.19. A patch of this tiling is illustrated in Figure 5.20.

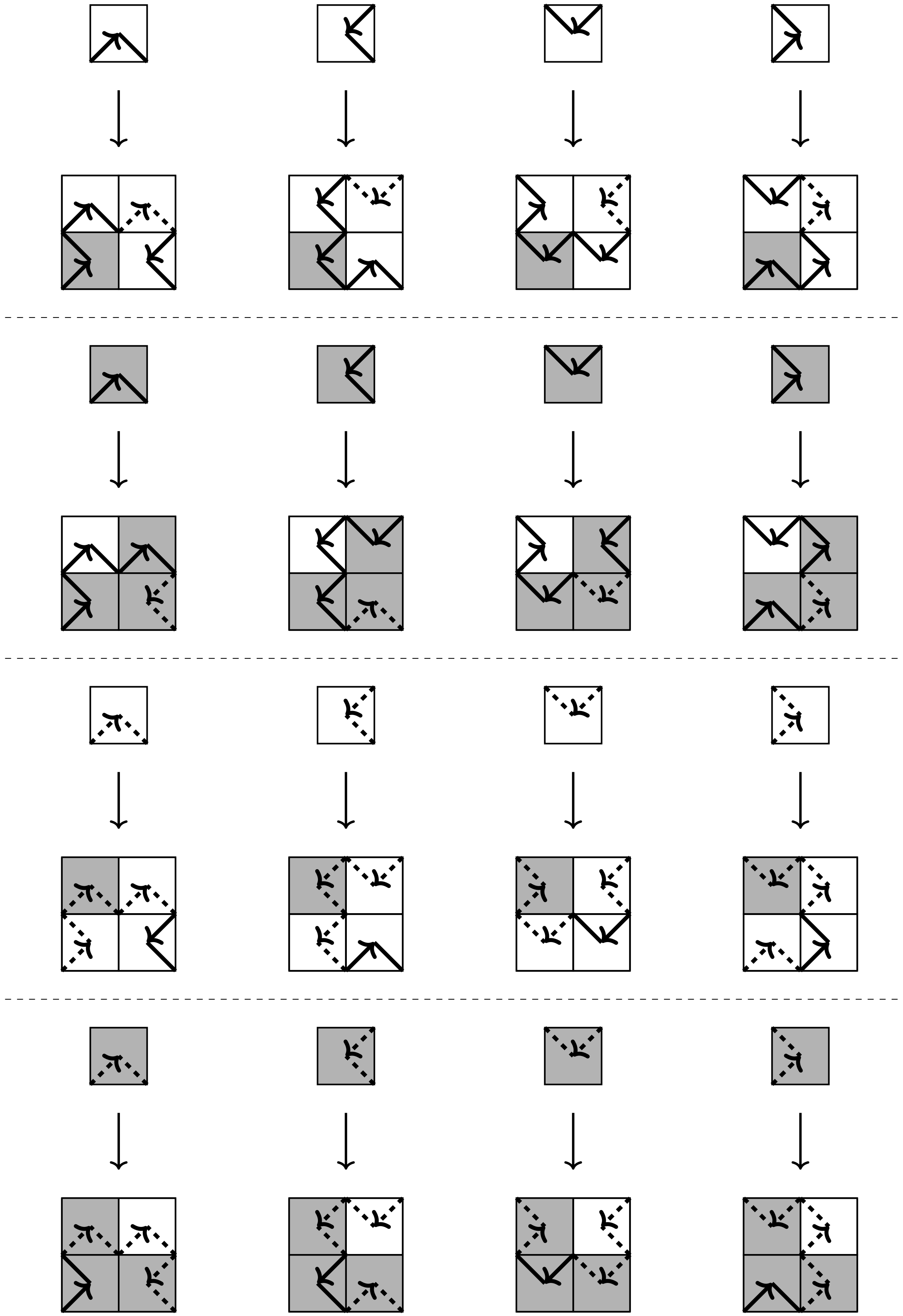


Figure 5.19: Square Chair Hilbert substitution rule

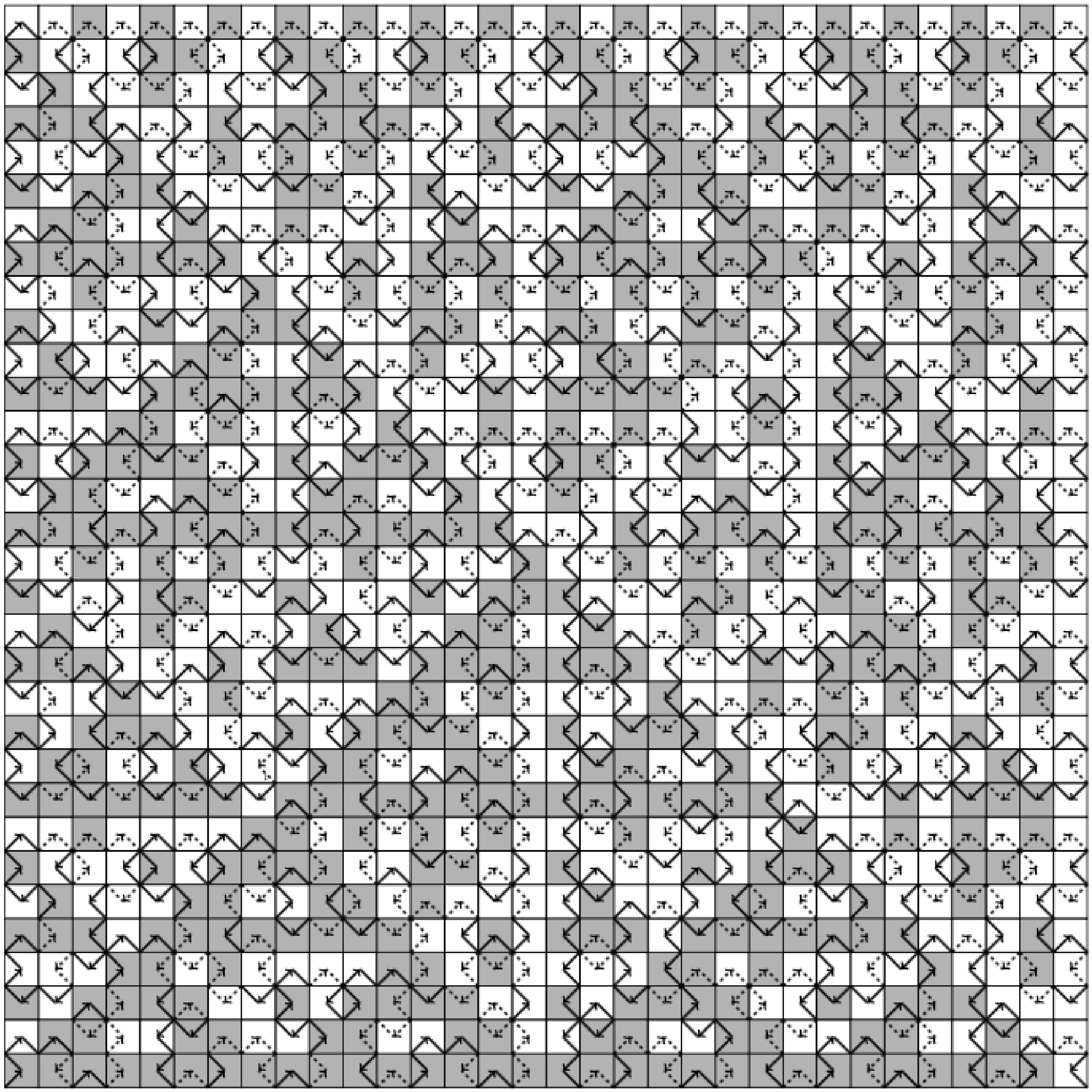


Figure 5.20: A patch of Square Chair Tiling with Hilbert Curves

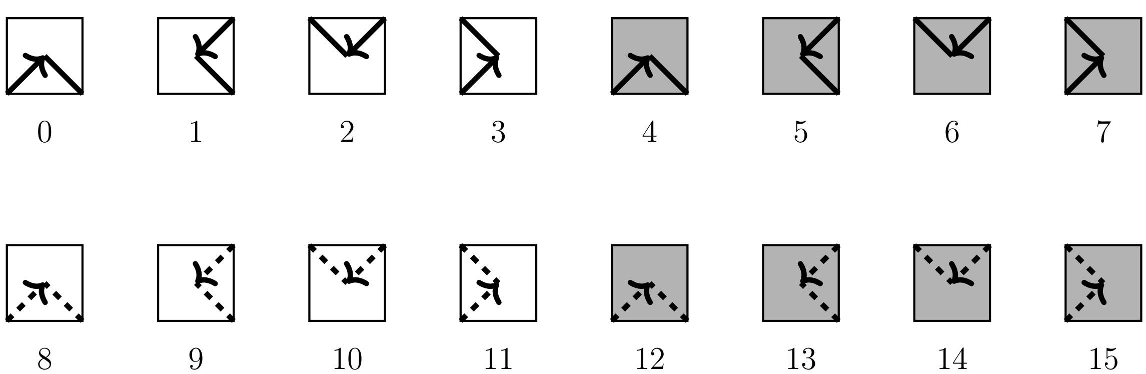


Figure 5.21: Square Chair - Hilbert substitution - labels of decorations

Label the curve decorations according to Figure 5.21. Then we get the substitution

σ_{sch}:

$$\begin{array}{llll} 0 \mapsto 7,0,8,1 & 4 \mapsto 7,0,4,13 & 8 \mapsto 11,12,8,1 & 12 \mapsto 7,12,8,13 \\ 1 \mapsto 10,1,5,0 & 5 \mapsto 6,1,5,12 & 9 \mapsto 10,13,9,0 & 13 \mapsto 10,13,5,12 \\ 2 \mapsto 9,2,6,3 & 6 \mapsto 5,14,6,3 & 10 \mapsto 9,2,10,15 & 14 \mapsto 9,14,6,15 \\ 3 \mapsto 4,3,11,2 & 7 \mapsto 4,15,7,2 & 11 \mapsto 8,3,11,14 & 15 \mapsto 4,15,11,14 \end{array}$$

Define the map $\phi : \{0, 1, \ldots, 15\} \mapsto \{0, 1, 2, 3\}$ such that $\phi(4 \cdot x + y) = y$ for $x, y \in \{0, 1, 2, 3\}$. Then $\sigma_{sch}(\phi) = \sigma_h$. Hence σ_h is locally derivable from σ_{sch}. Observe also that σ_{sc} is not locally derivable from σ_{sch}.

Example 5.2.9 (Pinwheel substitution)**.** Consider the Pinwheel substitution rule given in Figure 5.22. We attached the curve labels to its prototiles, as demonstrated in Figure 5.23. These substitution rules are isomorphic to each other.

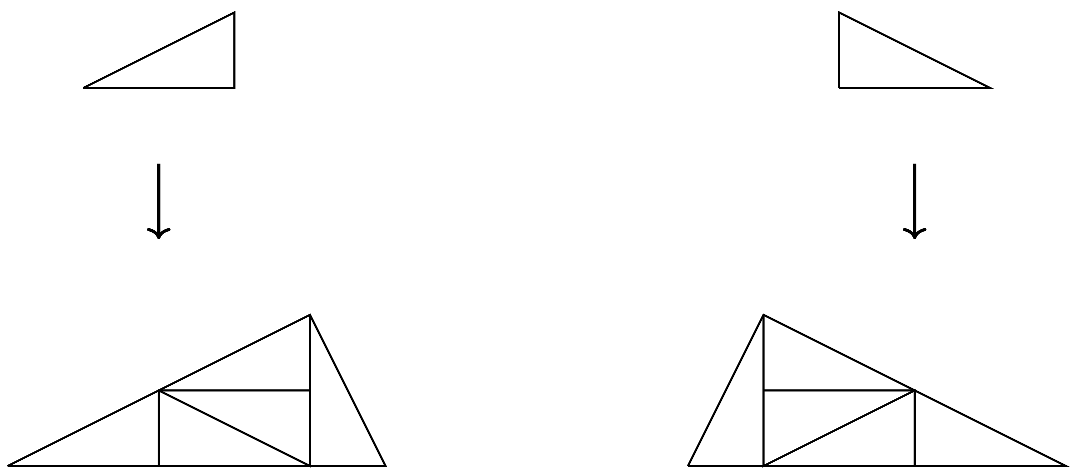


Figure 5.22: Pinwheel substitution rule

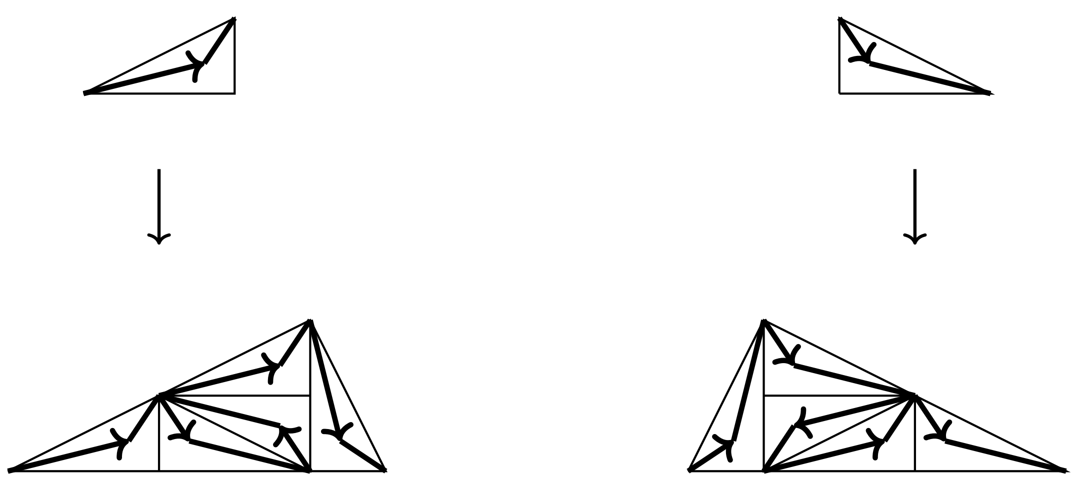


Figure 5.23: Pinwheel substitution rule - with curve labels

We construct the one dimensional substitution σ_{pw} by the same token. Label decoration curves over the prototiles as a, b, from left to right, respectively. By reading the curves through their concatenation processes we get:

$$\sigma_{pw}(a) = a, b, b, a, a, \qquad \sigma_{pw}(b) = b, b, a, a, b.$$

It should be noted that Pinwheel substitution do not satisfy FLC unless we enlarge the equivalent class of patches of translations, to translations and rotations, and allow rotations to be a congruent motion. Therefore, we only labelled the prototiles with two labels in the flattening process.

Example 5.2.10 (Penrose-Robinson substitution tiling)**.** Start with the Penrose-Robinson substitution rule shown in the Figure 5.24. Figure 5.25 presents the decorated versions of the prototiles and Figure 5.26 demonstrates their substitutions (up to scale for illustration purposes). We do not present an explicit flattened substitution for this decorated substitution, because rotation of the tiles matters and the decorated substitution therefore consists of 160 prototiles.

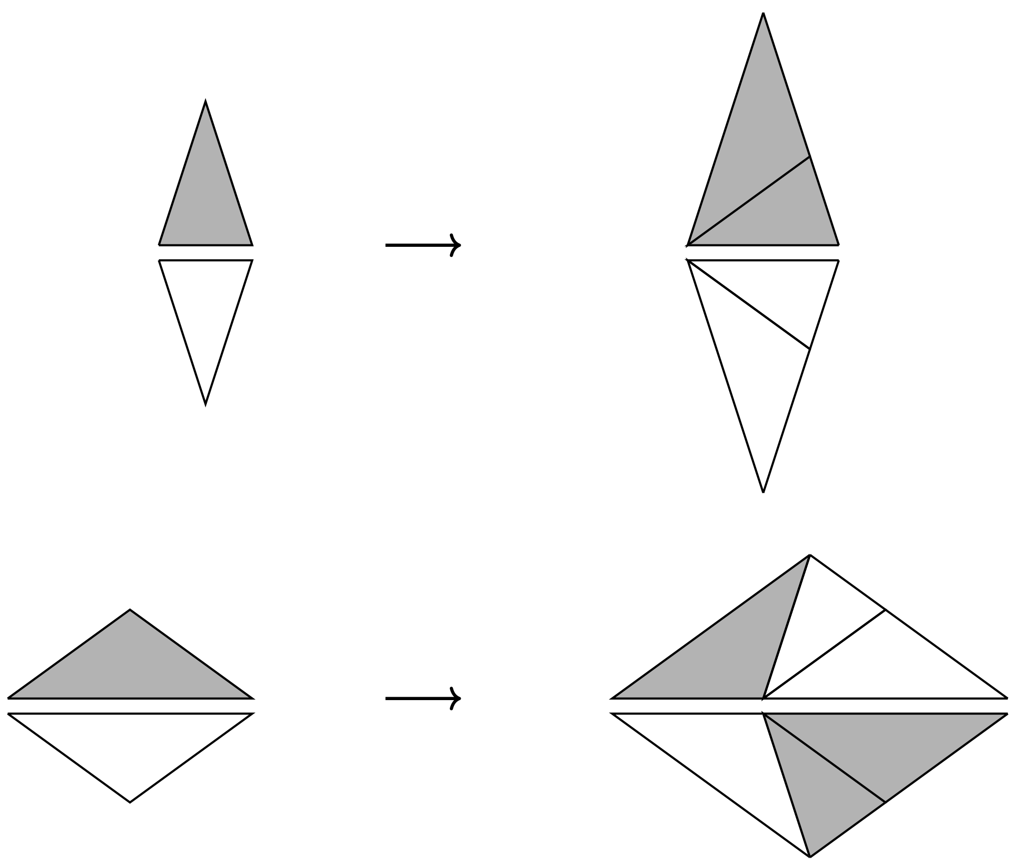


Figure 5.24: Penrose-Robinson substitution rule

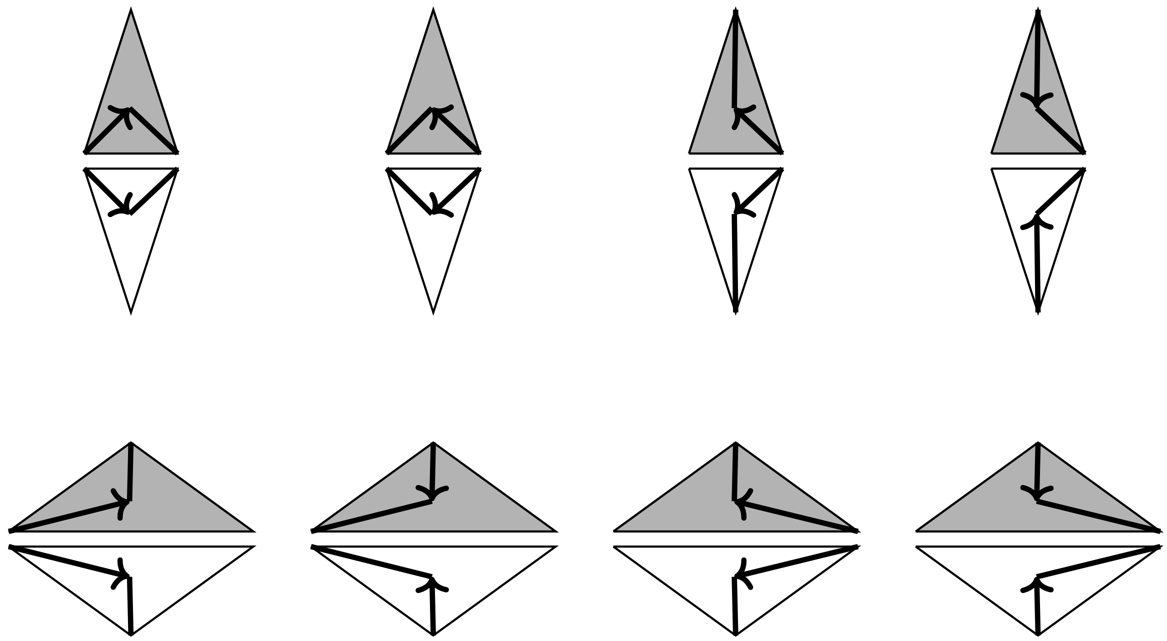


Figure 5.25: Penrose-Robinson decorated prototiles

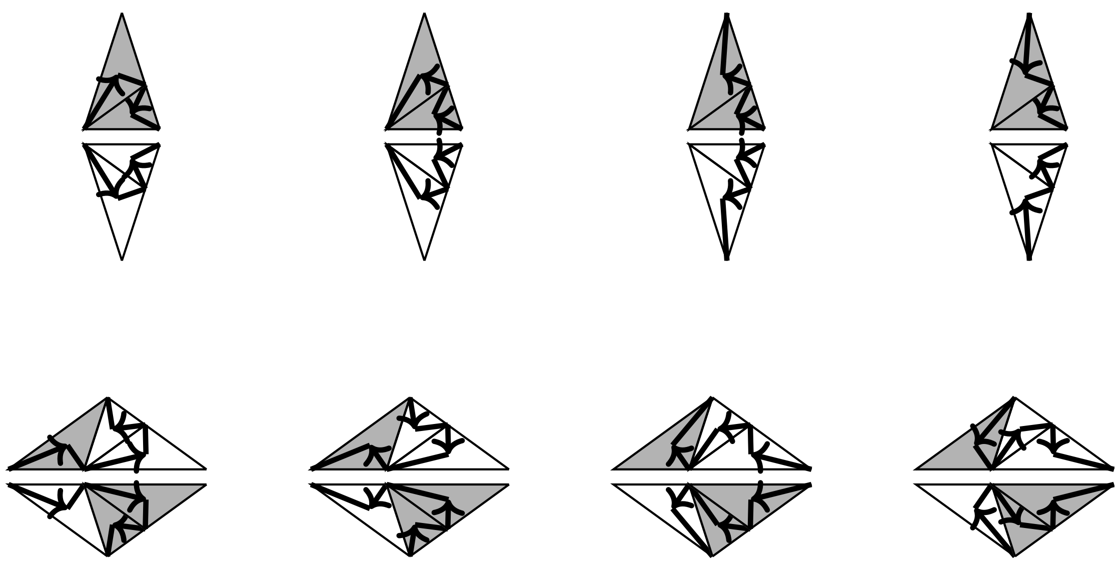


Figure 5.26: Penrose-Robinson decorated substitution rule

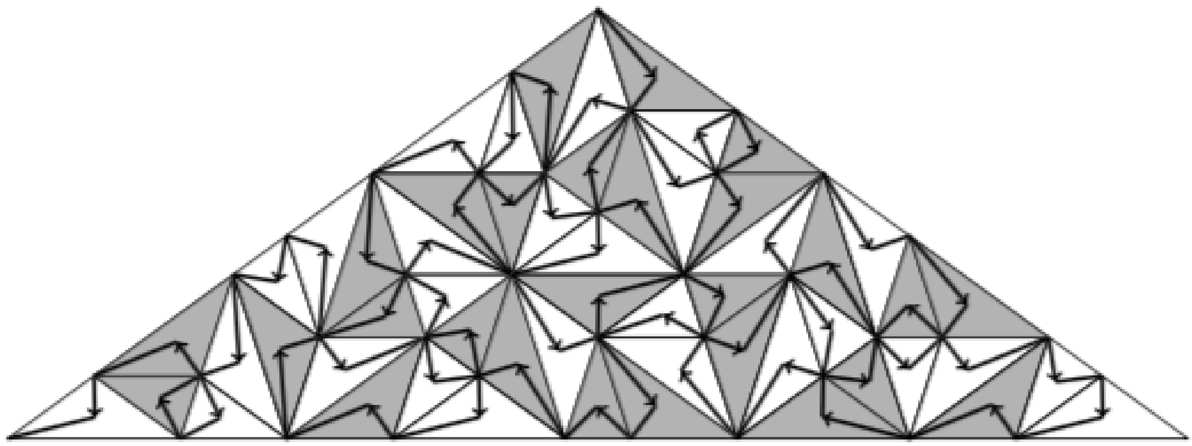


Figure 5.27: A 4th iteration of small white (decorated) triangle

5.3 Conclusion

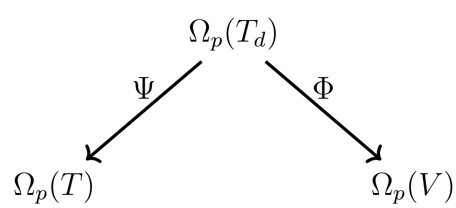
In this thesis, we propose a new approach to study 2-dimensional substitution tilings satisfying generic conditions. Our approach can be summarised within 3 steps:

(1) Given a substitution tiling T satisfying mild assumptions, construct a decorated substitution tiling T_d with a relatively dense curve $\mathcal{D}$ attached to it.

(2) Form a 1-dimensional substitution tiling V by flattening the curve $\mathcal{D}$.

(3) Generalise this construction between the associated punctured hulls of T_d and V by defining a map $\Phi : \Omega_p(T) \mapsto \Omega_p(V)$.

Note that for a decorated tiling T_d of a given tiling T, we can retrieve the tiling T by removing the decoration labels of tiles in T_d. This process can be applied for each tiling in $\Omega_p(T_d)$. That is, there exists a (factor) map $\Psi : \Omega_p(T_d) \mapsto \Omega_p(T)$ which forgets the decoration labels of tiles in decorated tilings. Since we also proved that Φ is an almost one-to-one factor map, we arrive at the following diagram where both Ψ and Φ are almost one-to-one factor maps.



For any given 2-dimensional tiling T we can construct different 1-dimensional substitution tilings by forming different space filling curves (or relatively dense curves), whenever T

satisfies the standard conditions. It is our interest to understand further how the choice of space filling curves (or relatively dense curves) affects the generated 1-dimensional tilings as well as how the dynamical or topological properties of T are inherited in V.

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