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Kourogenis, Nikolaos C. (GR-ATHN2); **Papageorgiou, Nikolaos S.** (GR-ATHN2)
Nonsmooth critical point theory and nonlinear elliptic equations at resonance. (English summary)
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In this paper, the authors extend the nonsmooth critical point theory of K. C. Chang [J. Math. Anal. Appl. **80** (1981), no. 1, 102–129; MR0614246], which is based on the subdifferential of locally Lipschitz functionals, to the weaker form. They use a nonsmooth counterpart of the Cerami condition to obtain a nonsmooth C-condition, i.e. for $\phi\colon X\to\mathbf{R}$ locally Lipschitz, $c\in\mathbf{R}$, every sequence $\{x_n\}_{n\geq 1}\subset X$ such that $\phi(x_n)\to c\in\mathbf{R}$ as $n\to\infty$ and $(1+\|x_n\|)m(x_n)\to 0$ as $n\to\infty$ has a strongly convergent subsequence. Here $m(x)=\inf\{\|x^*\|\colon\ x^*\in\partial\phi(x)\}$ (extending the well-known C-condition in the "smooth" case and nonsmooth (PS)-condition of Chang). Instead of the pseudo-gradient vector field of the smooth case, they use a locally Lipschitz vector field to produce a variant of the deformation lemma. Thus they obtain a more general minimax principle. Furthermore, boundary conditions are also relaxed, namely some inequalities in the minimax principles (mountain pass theorem, saddle point theorem and linking theorem) are allowed to be nonstrict in the context of nonsmooth (similar to the well-known analogous generalizations in the context of the "smooth" case).

As for applications, the first is to use the generalized mountain pass theorem to obtain a nontrivial solution of the following nonlinear elliptic problem at resonance (P): $-\mathrm{div}(\|Dx(z)\|^{p-2}Dx(z))-\lambda_1|x(z)|^{p-2}x(z)=f(z,x(z))$ a.e. on Z, $x|_\Gamma=0$, $2\leq p<\infty$, where $Z\subset\mathbf{R}^N$ is a bounded domain with a $C^{1+\alpha}$-boundary Γ $(0<\alpha<1)$, $\lambda_1=\min\{\|Dx\|_p^p/\|x\|_p^p\colon\ x\in W_0^{1,p}(Z)\}$ and $f(z,\cdot)$ is not assumed to be continuous. So (P) need not have a nontrivial solution. To overcome this difficulty, the authors use an approximation of (P) via filling in the gaps at the discontinuity points of $f(z,\cdot)$, i.e., considering the following multivalued approximation of (P), (P′):

$$-\mathrm{div}(\|Dx(z)\|^{p-2}Dx(z))-\lambda_1|x(z)|^{p-2}x(z)\in\widehat{f}(z,x(z))$$

a.e. on Z, where $\widehat{f}(z,x)=\{y\in\mathbf{R}\colon\ f_1(z,x)\leq y\leq f_2(z,x),\ f_1(z,x)=\liminf_{x'\to x}f(z,x'),\ f_2(z,x)=\limsup_{x'\to x}f(z,x')\}$. Further, the growth condition on the nonlinearity $f(z,x)$ is not the well-known Ambrosetti-Rabinowitz condition $0<kF(z,x)\leq xf(z,x)$ $(F(z,x)=\int_0^x f(z,s)ds)$ for almost all $z\in Z$, all $|x|\geq M$ and for some $k>2$, but rather

$$\liminf_{|x|\to\infty}(f(z,x)x-pF(z,x))/|x|^\mu>0$$

uniformly for almost all $z\in Z$. Second, a multiplicity result in the presence of splitting due to Brezis-Nirenberg is used to obtain multiple solutions of (P) when $f(z,\cdot)$ is continuous. Finally, the authors use the above extended version of the nonsmooth saddle-point theorem to prove an existence theorem of nontrivial solution of (P) when $p=2$. In this particular context, the hypothesis on the discontinuity nonlinearity $f(z,\cdot)$ (even $f(z,\cdot)$ continuous) generalizes the well-known Landesman-Lazer conditions.

Without any doubt, this paper is of interest to anyone in the field of critical point theory and its applications to PDE.

Addendum (April, 2002): This paper also appears elsewhere [J. Austral. Math. Soc. Ser. A **69** (2000), no. 2, 245–271; MR1775181].

Current version of review. Go to earlier version. *Yong Qing Li*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.