A Parametric Rule for Planning and Management of Multiple Reservoir Systems

Appendix (Microform Supplement)

I. Nalbantis and D. Koutsoyiannis

Department of Water Resources, Faculty of Civil Engineering,

National Technical University, Athens, Greece

A1. Derivation of the rule for the restriction of losses

Given that the losses due to leakage and evaporation from reservoir i are $l_i(S_i)$, to minimize the total system losses we demand that

$$\text{minimize} \quad L = \sum_{i=1}^{N} l_i(S_i) \tag{A1}$$

under the constraints

$$S_i \geq 0, \; i = 1, \ldots, N \qquad \sum_{i=1}^{N} S_i = V \tag{A2}$$

We remind that S_i denotes the storage of the reservoir i and V denotes the total storage of the system.

To incorporate the first constraint in the objective function we use the auxiliary variables x_i such as $S_i = x_i^2$. The handling of the second constraint can be done either by a Lagrange multiplier or by expressing one of the variables, say x_1, in terms of the other variables. In the latter case, which we have adopted, the objective function becomes

$$\min h(x_2, \ldots, x_N) = \sum_{i=2}^{N} l_i(x_i^2) + l_1(V - \sum_{i=2}^{N} x_i^2) \tag{A3}$$

Taking the partial derivatives of h with respect to x_i (for i = 2, …, N) and equating them to zero, we get

$$\frac{\partial h}{\partial x_i} = 2\,x_i \left[l_i'(x_i^2) - l_1'(V - \sum_{i=2}^{N} x_i^2) \right] = 0 \tag{A4}$$

where $l_i'(\,)$ denotes the first derivative of the function $l_i(\,)$. The second derivatives of h are

$$\frac{\partial^2 h}{\partial x_i^2} = 2 \left[l_i'(x_i^2) - l_1'(V - \sum_{i=2}^{N} x_i^2) \right] + 4\,x_i^2 \left[l_i''(x_i^2) + l_1''(V - \sum_{i=2}^{N} x_i^2) \right] \tag{A5}$$

$$\frac{\partial^2 h}{\partial x_i \partial x_j} = 4\,x_i\,x_j\,l_1''(V - \sum_{i=2}^{N} x_i^2) \tag{A6}$$

where $l_i''(\,)$ denotes the second derivative of the function $l_i(\,)$.

To continue our proof, we observe that generally the functions $l_i(S_i)$ are increasing and concave (or equivalently increasing with first derivative decreasing). To justify this, we assume that the area - elevation relationship of a reservoir is approximated by power relation $A \propto z^{\beta}$, where A and z denote the area and elevation (above reservoir bottom), respectively, and β is a parameter, typically greater than 2 (the value 2 corresponds to a shape of the area scaling linearly with z). Then the storage - elevation relationship will be $S \propto z^{\beta+1}$. Thus, we can write $z \propto S^{1/(\beta+1)}$ and $A \propto S^{\beta/(\beta+1)}$. The first result means that, under the commonly met condition that the loss due to leakage is proportional to z, or even to a power of z up to $z^{\beta+1}$ (e.g., for β = 2, up to z^3), this loss will be a concave function of z. The second result means that the evaporation loss, which is proportional to A, is always a concave function of z (note that $0 < \beta / (\beta + 1) < 1$). Hence the sum of these two concave functions will be a concave function, too.

Clearly, (A4) has two solutions for x_i, the first being $x_i = 0$ and the second $x_i \neq 0$. From (A5) we obtain that the second solution corresponds to

$$\frac{\partial^2 h}{\partial x_i^2} = 0 + 4\,x_i^2 \left[l_i''(x_i^2) + l_1''(V - \sum_{i=2}^{N} x_i^2) \right] < 0 \tag{A7}$$

since both terms in the square brackets are negative because of the concavity of the functions $l_i(\ )$. Thus, this solution corresponds to a maximum, rather than a minimum of the objective function (A3). For a geometrical explanation of this consider that h is a $(N-1)$-dimensional hypersurface defined as the intersection of the N-dimensional hypersurface L (Equation (A1)) and a hyperplane (second restriction in (A2)). As the hypersurface L is concave, so will be h, which means that both cannot have minima at any point except for some of the corners of the hypercube they are defined on. Indeed, let us consider the corner ($S_1 = V$, $S_2 = \cdots = S_N = 0$), which corresponds to $(x_2, x_3, \ldots, x_N) = (0, 0, \ldots, 0)$. We assume that the indexes i are assigned so that reservoir 1 is that corresponding to the minimum value of the loss rate at the origin, that is

$$l_1'(0) \le l_i'(0),\ i = 2, \ldots, N \tag{A8}$$

Moreover, given that $l_1(\ )$ is concave we will have

$$l_1'(V) \le l_1'(0) \le l_i'(0),\ i = 2, \ldots, N \tag{A9}$$

From (A4) we obtain that all first derivatives at the point $(0, 0, \ldots, 0)$ are zero and from (A5) and (A6) we obtain that the second derivatives are

$$\frac{\partial^2 h}{\partial x_i^2} = 2\,[(l_i'(0) - l_1'(V)] \ge 0 \qquad \frac{\partial^2 h}{\partial x_i\,\partial x_j} = 0 \tag{A10}$$

where we have combined (A9) to obtain the above inequality. Moreover, because of the zero derivatives with respect to x_i and x_j, all Hessian determinants at the origin, related to the optimization, are

$$\left| \left(\frac{\partial^2 h}{\partial x_i\,\partial x_j} \right) \right|_{k \times k} = \frac{\partial^2 h}{\partial x_2^2} \frac{\partial^2 h}{\partial x_3^2} \cdots \frac{\partial^2 h}{\partial x_{k+1}^2} \ge 0 \tag{A11}$$

for any $k = 2, \ldots, N-1$. All the above inequalities ensure that the point (0, 0, ..., 0) corresponds to a local minimum of the function h. Hence the function L has a local minimum at the point $(V, 0, 0, \ldots, 0)$, which equals $l_1(V)$.

It is possible that assignment of the index 1 to another reservoir may result in a local minimum at the point (0, 0, ..., 0) as well, since it can be $l_1'(V) \le l_i'(0)$ for all i and thus (A10) be still valid. Thus, we have to examine all possible minima, i.e., the values of $l_i(V)$ for all i, and keep the smallest. Theoretically, for different values of V, it is possible that a different reservoir i may have the minimum losses $l_i(V)$. In practice, however, it is expected that the reservoir with the smallest loss rate at the origin will also have less losses for any volume V.

To further generalize the above result, we observe that, in order to prove that the origin is a point of local minimum, we needed only the condition $l_1'(V) \le l_i'(0)$ to be valid. This does not necessarily require that all $l_i(\ )$ are concave. However, at least $l_1(\ )$ should be concave, because otherwise it can be proved that there may be a local minimum for nonzero S_1 with a value of L smaller than $l_1(V)$.

Finally, we observe that in the limit case where the loss is proportional to storage, i.e., $l_i(S_i) = \lambda_i S_i$ the above analysis remains valid. Obviously, in that case the total loss is minimum when we store all water at the reservoir with the minimum loss rate λ_i.

A2. Explanations for the adjusting procedure of the linear rule

Having modified the linear rule from form (4) to form (12), in order to obey the physical constraints, the S'^*_i no longer add up to V, as they should. To reestablish the additive property we must distribute the departure $V - \sum_{i=1}^{N} S'^*_i$ among the different S'^*_i and get some new target storages S''^*_i satisfying $\sum_{i=1}^{N} S''^*_i = V$. The transformation $S'^*_i \to S''^*_i$ must not affect the full or empty reservoirs. That is, $(S'^*_i = 0)$ should map to $(S''^*_i = 0)$, and $(S'^*_i = K_i)$ should map to $(S''^*_i = K_i)$. The easiest way to do so this is to distribute the departure $V - \sum_{i=1}^{N} S'^*_i$ in proportion to the quantity $S'^*_i\,(1 - S'^*_i/K_i)$, that is

$$S''^*_i - S'^*_i = \phi\, S'^*_i\,(1 - S'^*_i/K_i) \qquad \text{(A12)}$$

where ϕ is constant for all reservoirs. Adding equations (A12) for all i, equating $\sum_{i=1}^{N} S''^*_i = V$, and solving for ϕ we get

$$\phi = \frac{V - \sum\limits_{i=1}^{N} S'^*_i}{\sum\limits_{i=1}^{N} S'^*_i(1 - S'^*_i/K_i)} \tag{A13}$$

It is easily shown that as long as $-1 \le \phi \le 1$ the adjusted target storage

$$S''^*_i = S'^*_i + \phi\, S'^*_i\,(1 - S'^*_i/K_i) = S'^*_i \left[1 + \phi\,(1 - S'^*_i/K_i) \right] \tag{A14}$$

remains within the interval $[0, K_i]$, as it should. Indeed, for $\phi \ge -1$, since $0 \le 1 - S'^*_i/K_i \le 1$, we will have $\phi\,(1 - S'^*_i/K_i) \ge -1 + S'^*_i/K_i$, and $1 + \phi\,(1 - S'^*_i/K_i) \ge 0$, which proves that $S''^*_i \ge 0$. Similarly, for $\phi \le 1$, since $0 \le S'^*_i/K_i \le 1$ we will have $\phi\, S'^*_i \le K_i$ and $\phi\, S'^*_i(1 - S'^*_i/K_i) \le K_i\,(1 - S'^*_i/K_i) = K_i - S'^*_i$, which proves that $S''^*_i = S'^*_i + \phi\, S'^*_i\,(1 - S'^*_i/K_i) \le K_i$.

However, (A13) does not ensure that the value of ϕ will be within the interval $[-1, 1]$, which means that possibly the new target storage S''^*_i may violate the physical constraint $0 \le S''^*_i \le K_i$. If this happens, the following iterative algorithm fixes the problem:

1. Calculate ϕ using (A13).
2. Calculate S''^*_i using (A14) for all i.
3. If $(-1 \le \phi \le 1)$ or $(0 \le S''^*_i \le K_i)$ for all i, then go to step 7, otherwise continue with step 4.
4. For those i with $S''^*_i < 0$ replace S'^*_i with 0.
5. For those i with $S''^*_i > K_i$ replace S'^*_i with K_i.
6. Go to step 1.
7. Done.

For the complete presentation of the algorithm, we note that the denominator in (A13) can be zero if all S'^*_i are either zero or equal to K_i. If the nominator is also zero, then there is no problem, because the target storages do already add up to V. Otherwise, we can arbitrarily modify S'^{*}_i (e.g., by setting $S'^{*}_i = K_i / 2$) and let the iterative adjusting algorithm determine the final target storages.

A3. Quadratic, linear, and homogeneous linear rules

In this section we explore a quadratic rule of the form

$$S_i^{*} = a_i' + b_i' \, V + c'_i \, V^2 \tag{A15}$$

where a_i', b_i', c_i' are parameters for each reservoir i, and compare it with the linear rule of equation (4) in both its complete and homogeneous ($a_i = 0$) form. The quadratic rule comprises $3N$ parameters for a system of N reservoirs. Because of (2) we have three constraints on the parameters, i.e.,

$$\sum_{i=1}^{N} a_i' = 0, \qquad \sum_{i=1}^{N} b_i' = 1, \qquad \sum_{i=1}^{N} c_i' = 0 \tag{A16}$$

and thus the number of unknown parameters is finally $3(N-1)$. Furthermore, in order for the rule to have physical meaning, all S_i^{*} in (A15) must be increasing functions of V in the interval $[0, K]$, where $K = \sum_{i=1}^{N} K_i$. Taking the first derivatives of (A15) and constraining them to be nonnegative we get

$$b_i' \geq 0, \qquad c_i' \geq -b_i' / 2K \tag{A17}$$

Combining (A16) and (A17) we get

$$0 \leq b_i' \leq 1, \quad -b_i' / 2K \leq c_i' \leq (1 - b_i') / 2K \tag{A18}$$

Hence, the curvature of the quadratic low cannot be arbitrary high, as the maximum value of c_i' is $1 / 2K$ (for $b_i' = 0$) and the minimum value is $-1 / 2K$ (for $b_i' = 1$).

Let us experiment numerically with the quadratic rule in our study reservoir system. To get the highest possible departure from the linear rule we set for one reservoir, say reservoir 3 (Iliki), $c_3' = 1 / 2K = 3.75 \times 10^{-4}$ hm^{-3} and $b_3' = 0$. For each of the other two reservoirs we set

c_i' equal to its lower bound, i.e. $-b_i' / 2K$. Choosing one of parameters b_i' and two of a_i', in a manner that all three quadratic laws have reasonable and rather extreme appearance, we got the curves shown in Figure A1(a). The parameter vectors for these curves are $\mathbf{a}' = (-148.1, 321.5, -173.4)^T$ hm^3, $\mathbf{b}' = (0.444, 0.556, 0)^T$, and $\mathbf{c}' = (-1.67 \times 10^{-4}, -2.08 \times 10^{-4}, 3.75 \times 10^{-4})^T$ hm^{-3}.

It is clear from Figure A1(a) that the quadratic rules with the above parameters violate the physical constraints $0 \leq S_i^* \leq K_i$ in large parts of their domain. Thus, we have applied the correction procedure described above (which, notably, can also be used for any rule, linear or nonlinear) and obtained the final adjusted curves shown in Figure A1(b).

Now, let us compare the above quadratic rule to the linear rule of equation (4) with parameters a_i and b_i. Experimenting with different sets of parameters a_i and b_i we determined a parameter set of this linear rule such that the final laws (after introducing corrections for constraints) of both the linear and quadratic rules are very close to each other. This parameter set is $\mathbf{a} = (-58.0, 438.2, -380.2)^T$ hm^3 and $\mathbf{b} = (0.199, 0.244, 0.558)^T$. These linear laws are plotted in Figure A1(a) together with the quadratic laws. We observe that the linear laws depart somehow from their corresponding quadratic laws, with their overall root mean square error, based on the departures between all pairs of curves after normalization by the respective reservoir capacity, being 22%. However, when we applied the correction procedure and got the final curves shown in Figure A1(b), this error became as low as 0.1%. In Figure A1(b) the curves originating from the linear rule are practically indistinguishable from those originating from the quadratic rule.

It is interesting to compare the above quadratic rule with the homogeneous linear rule, i.e., that with $a_i = 0$. Experimenting, as above, with different sets of parameters b_i we resulted in a parameter set of this homogeneous rule such that the final laws (after introducing corrections for constraints) of both the homogeneous and quadratic form are close to each other. This parameter set is $\mathbf{b} = (0.049, 0.951, 0)^T$ (note the zero value of b_3). The homogeneous lines are plotted in Figure A2(a) together with the quadratic curves. We observe that the homogeneous laws depart significantly from their corresponding quadratic laws, with their overall root mean square error (as previously defined) being 52%. However, when we applied the correction procedure and got the final curves shown in Figure A2(b), this error became 6.8%. We observe in Figure A2(b) that the curves originating from the homogeneous rule agree well with those originating from the quadratic rule.

As a final experiment, we have attempted to approximate the space rule (shown in Figure 4(c) of the paper) with a homogeneous rule. We remind that the space rule results in a law for the Evinos reservoir (reservoir 1) that passes very far from the origin (intersects the V axis at V = 1010 hm^3). This is expected to create inaccuracy in approaching the law with a homogeneous line. Working as above, we fitted the vector $\mathbf{b}$ = $(0.018, 0.583, 0.399)^T$ describing the homogeneous rule. The lines of the homogeneous rule are plotted in Figure A3(a) together with those of the space rule. We observe that the homogeneous lines depart significantly from their corresponding complete linear forms, with the overall root mean square error (as previously defined) being 92%. However, when we applied the correction procedure and got the final curves shown in Figure A3(b), this error became 9.2%. In Figure A3(b) the curves originating from the homogeneous rule agree well with those originating from the complete linear rule.

In conclusion, the above results indicate that, given a quadratic rule, it can be approximated almost perfectly by a linear rule. Furthermore, we can obtain good approximations of either a quadratic and a linear rule by a homogeneous rule.

Figures

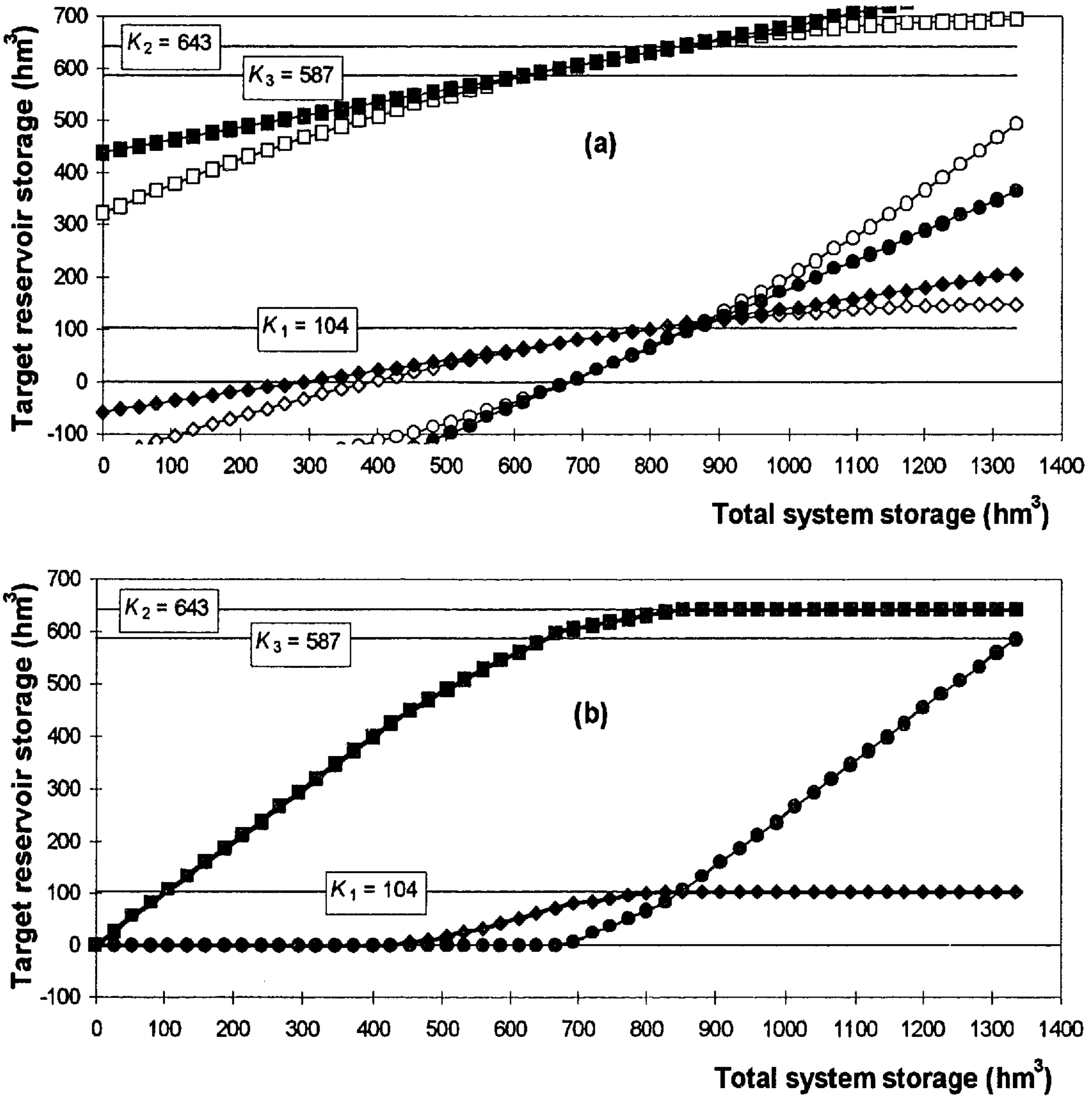


Figure A1 Approximation of a quadratic rule with a linear rule for the reservoir system of the Athens water supply: (a) initial forms of rules, and (b) final adjusted (corrected) forms of rules. Rhombi, squares and circles correspond to reservoirs 1, 2 and 3 (Evinos, Mornos and Iliki), respectively. Empty and solid symbols correspond to the quadratic and the linear rules, respectively. In (b) the curves corresponding to both rules are indistinguishable.

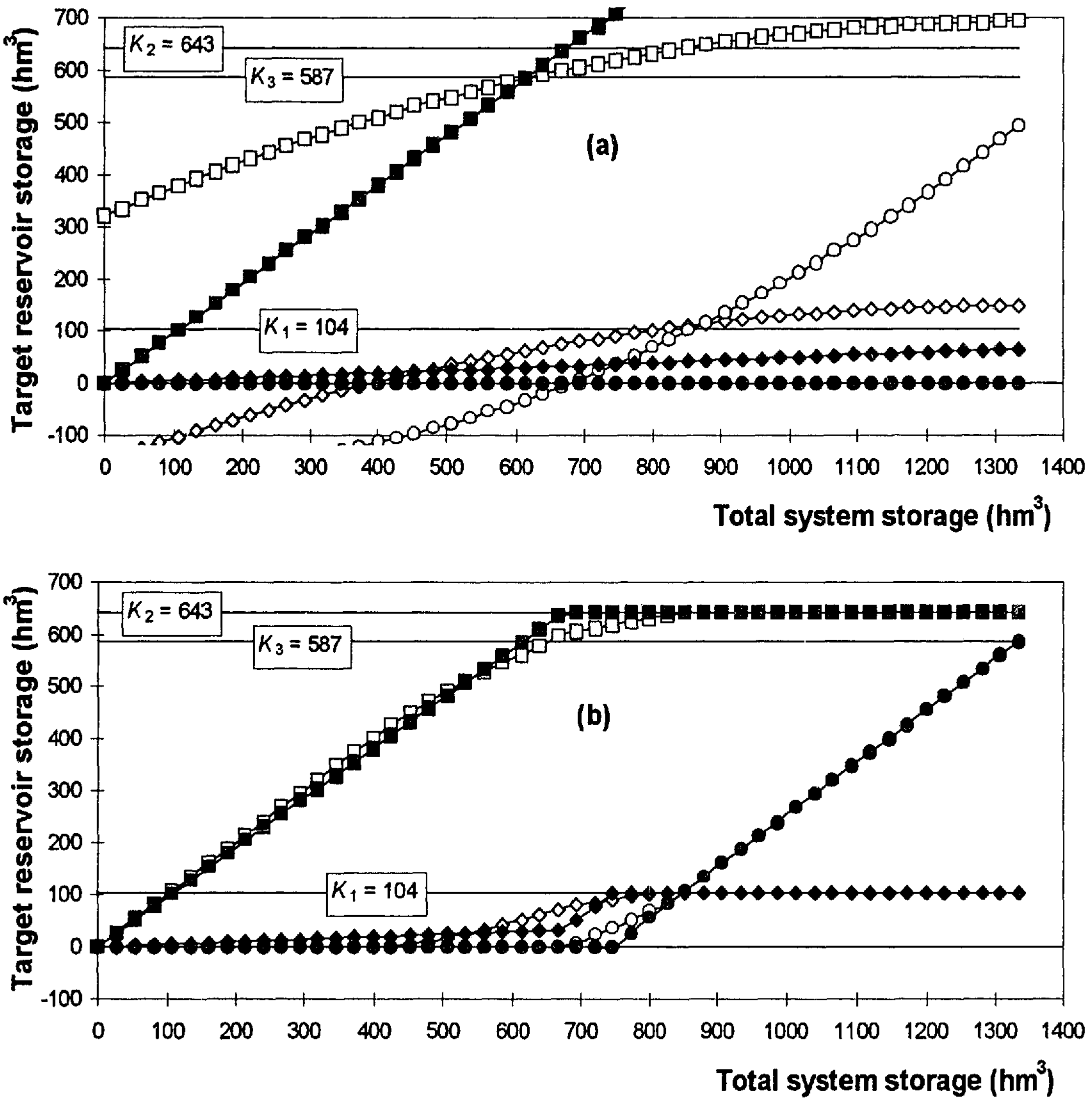
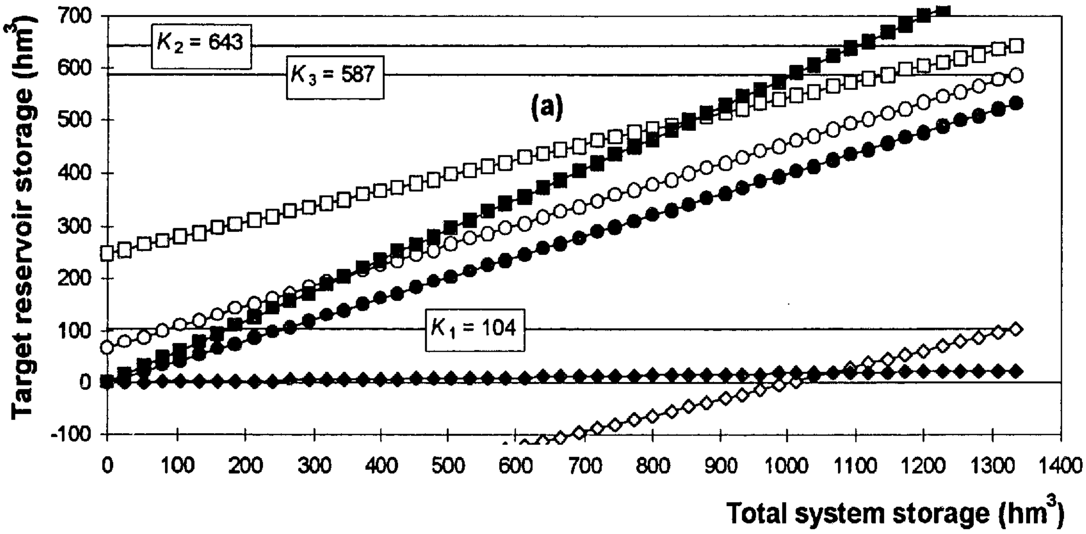


Figure A2 Approximation of a quadratic rule with a homogeneous linear rule for the reservoir system of the Athens water supply: (a) initial forms of rules, and (b) final adjusted (corrected) forms of rules. Rhombi, squares and circles correspond to reservoirs 1, 2 and 3 (Evinos, Mornos and Iliki), respectively. Empty and solid symbols correspond to the quadratic and the homogeneous rules, respectively.



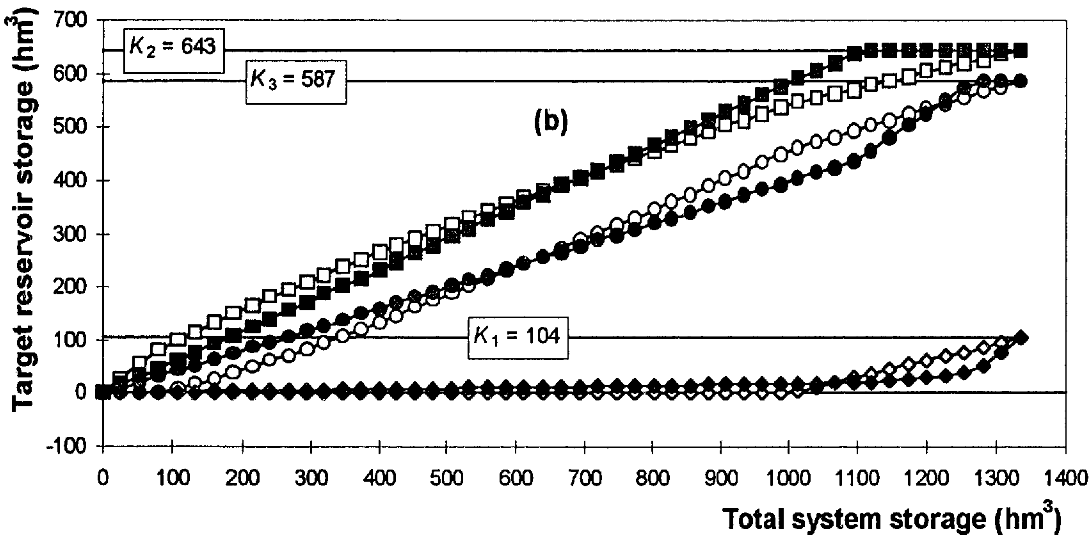


Figure A3 Approximation of the space rule with a homogeneous linear rule for the reservoir system of the Athens water supply: (a) initial forms of rules, and (b) final adjusted (corrected) forms of rules. Rhombi, squares and circles correspond to reservoirs 1, 2 and 3 (Evinos, Mornos and Iliki), respectively. Empty and solid symbols correspond to the space rule and the homogeneous rule, respectively.