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Complex multiplication and Brauer groups of K3 surfaces

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- All the results in the thesis are original, unless otherwise stated. All the non-original results are appropriately referenced.

Abstract

One may ask how much of the classical theory of complex multiplication translates to K3 surfaces. This question looks natural and it is justified by the deep similarities between K3 surfaces and Abelian varieties, that are geometric (they are the only Calabi-Yau surfaces) or motivic (in some appropriate category, the motive of every K3 is Abelian) or moduli-space theoretical, since both objects are parametrised by Shimura varieties. The aim of this thesis is to assemble all these similarities to obtain a theory for CM K3 surfaces which bears many resemblances and yet many interesting differences to the classical one of Abelian varieties. Since our original motivation was to understand the Brauer groups of CM K3 surfaces, the results obtained will also have practical applications in this direction. In particular, given a number field K and a CM number field E, we are able to write down the finitely many Brauer groups $\mathrm{Br}(\overline{X})^{G_K}$ of any K3 surface X/K with CM by E. A second question we are going to be interested in regards fields of definition. It was known since Piatetski-Shapiro and Shafarevich that every complex K3 surface with CM can be descended over $\overline{\mathbb{Q}}$, so one would like to know if there is a natural choice for a field of definition, as it happens for elliptic curves. We show that this is true under some mild condition on the quadratic form associated to the transcendental lattice $T(X)$, and this allows us to give an elementary proof of a finiteness theorem only recently proved by Orr and Skorobogatov.

To my family, friends, and Johanna

Contents

1 Introduction

The purpose of this thesis is twofold: on the one hand, we develop a 'classical' approach to the theory of complex multiplication for K3 surfaces while, on the other, we apply our results to the study of Brauer groups, fields of definition and other inherent arithmetic properties. Let us begin with a definition:

Definition 1.0.1. A K3 surface $X/\mathbb{C}$ has CM if the Mumford-Tate group of $H^2_B(X, \mathbb{Q})$ is Abelian, i.e. a torus.

Let $\mathrm{NS}(X) \subset H^2_B(X, \mathbb{Z})(1)$ be the Neron-Severi group of X (see **??** for a definition) and $T(X) = \mathrm{NS}(X)^\perp$ be the lattice of transcendental cycles. The latter is an integral, irreducible Hodge structure of weights $(1, -1)$, $(0, 0)$ and $(-1, 1)$. Zarhin [56] showed that the definition above is equivalent to the following two properties of $T(X)$:

1. $\mathrm{End}_{\mathrm{Hdg}}(T(X)_\mathbb{Q}) = E$, a CM field, and
2. $\dim_E T(X)_\mathbb{Q} = 1$, i.e. $[E : \mathbb{Q}] = \dim_\mathbb{Q} T(X)_\mathbb{Q}$.

Therefore, complex multiplication can be read off the transcendental lattice $T(X)$ of X and, since $\dim_\mathbb{Q}(T(X)_\mathbb{Q}) \leq 20$, by point 2) in the above definition we always have $[E : \mathbb{Q}] \leq 20$. Following the results of Rizov [41] (or the Corollary 4.4 from Madapusi Pera's paper [26]) we know that the Galois action on K3 surfaces with CM is the one predicted by Deligne in the definition of the canonical models of Abelian Shimura varieties. This is an analogue of the main theorem of CM for K3 surfaces. The first examples of K3 surfaces satisfying Definition 1.0.1 occur when $X/\mathbb{C}$ has maximal Picard rank, i.e. $\rho(X) = 20$. These are called *singular* K3 surfaces and always have CM by an imaginary quadratic field, generated by the square root of the discriminant of $T(X)$. Their geometry was studied by Shioda and Inose in [44], who related them to CM elliptic curves with a construction now called Shioda-Inose structure. Another class of examples is given by $X = \mathrm{Km}(A)$, where A is an Abelian surface with CM. Note that in the instances above we always have $[E : \mathbb{Q}] = \dim_\mathbb{Q}(T(X)_\mathbb{Q}) \in \{2, 4\}$; using the Torelli Theorem for K3 surfaces and some results on rational quadratic forms, Taelman [49] was able to show that for any CM field E with $2 \leq [E : \mathbb{Q}] \leq 20$ there are infinitely many $\mathbb{C}-$isomorphism classes of K3 surfaces with CM by E, and some other examples were produced by Kondo in [20]. In recent years, much research has been published on the Brauer group of CM K3 surfaces. If X/L is a K3 surface over a number field and $\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$ its Brauer group, one has a natural filtration $\mathrm{Br}_0(X) \subset \mathrm{Br}_1(X) \subset \mathrm{Br}(X)$ given by $\mathrm{Br}_0(X) := \mathrm{Im}(\mathrm{Br}(L) \to \mathrm{Br}(X))$, the *constant* classes of $\mathrm{Br}(X)$, and $\mathrm{Br}_1(X) := \ker(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X}))$, the *algebraic* Brauer group of X. The first graded piece of this filtration, namely $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$, is naturally isomorphic to $H^1(L, \mathrm{Pic}(\overline{X}))$, which is a finite group because $\mathrm{Pic}(\overline{X}) \cong \mathrm{NS}(\overline{X})$ is a free $\mathbb{Z}$-module. It was noted that also the classes of $\mathrm{Br}(X)$ that survive in $\mathrm{Br}(\overline{X})$ can obstruct the Hasse principle, so that one is left to understand the second graded piece $\mathrm{Br}(X)/\mathrm{Br}_1(X)$, a subgroup of $\mathrm{Br}(\overline{X})^{G_L}$. Skorobogatov and Zarhin proved in [46] that $\mathrm{Br}(\overline{X})^{G_L}$ is always *finite*, and since then much work has been done to the study of these groups in some particular cases, especially when X has CM. Ieronymou, Skorobogatov and Zarhin in [17, 18, 47] have studied the cases when X is a Kummer surface, in particular associated to a product of two elliptic curves, or a diagonal quartic surface defined over $\mathbb{Q}$. Newton's paper [34] is more class field theory-oriented and her results are quite general when X is the Kummer surface associated to a product of two elliptic curves with CM. Várilli-Alvarado and Viray [52] have studied the existence of universal bounds for the growth of $\left(\mathrm{Br}(X)/\mathrm{Br}_1(X)\right)_{\text{odd}}$ under a field extension of bounded degree, for some particular Kummer surfaces. They proved the existence of such a bound when restricting to ℓ-torsion $\left(\mathrm{Br}(X)/\mathrm{Br}_1(X)\right)[\ell^\infty]$, with ℓ an odd prime number. Finally, a similar question was studied by Cadoret and Charles in [5]. Given a prime number ℓ, they prove the existence of a universal bound for $\mathrm{Br}(\overline{X})^{G_L}[\ell^\infty]$, at least when X is allowed to vary in a one-dimensional family (see Theorem 1.2.1 in *loc. cit.* for a precise statement). Before stating our results in this direction, we first need a definition.

Definition 1.0.2. Let $X/\mathbb{C}$ be a K3 surface with CM. We say that X is principal if $\mathrm{End}_{\mathrm{Hdg}}(T(X))$ is the maximal order of the CM field $\mathrm{End}_{\mathrm{Hdg}}(T(X)_{\mathbb{Q}})$.

Remark 1.0.3. One says that an Abelian variety $A/\mathbb{C}$ with CM by E is principal if $\mathrm{End}(A)$ is the ring of integers of E. It is a classical result of Shimura that isomorphism classes of principal Abelian varieties with CM by E are parametrised by $\mathrm{Cl}(E)$, so that in particular they are only finitely many. Things are different for

K3 surfaces. In Proposition 3.1.11 we prove that for any given CM number field E with $2 \leq [E : \mathbb{Q}] \leq 10$ one has infinitely many principal K3 surfaces with CM by E. It is not clear what happens when $10 < [E : \mathbb{Q}] \leq 20$, but as the proof of Proposition 3.1.11 shows, the problem of finding K3 surfaces with CM by $\mathcal{O}_E$ can be stated in purely lattice-theoretical terms.

We have (see section 4.2).

Theorem 1.0.4. *There is an algorithm that, given as a input a CM number field E and a number field L, returns a finite list of groups* $\mathrm{Br}(E, L)$ *such that, for every K3 surface X/L such that $X_{\mathbb{C}}$ is a principal K3 surface with CM by E,*

$$\mathrm{Br}(\overline{X})^{G_L} \in \mathrm{Br}(E, L).$$

This result appears in [50]. To see why it is useful, consider the case $L = E = \mathbb{Q}(i)$. Running the algorithm above we found that $\mathrm{Br}(E, L)$ consists of

$$\big\{0,\ \mathbb{Z}/3\times\mathbb{Z}/3,\ \mathbb{Z}/5,\ \mathbb{Z}/5\times\mathbb{Z}/5,\ \mathbb{Z}/2,\ \mathbb{Z}/2\times\mathbb{Z}/2,\ \mathbb{Z}/4\times\mathbb{Z}/2,\ \mathbb{Z}/4\times\mathbb{Z}/4,\ \mathbb{Z}/8\times\mathbb{Z}/4,$$

$$\mathbb{Z}/8\times\mathbb{Z}/8,\ \mathbb{Z}/3\times\mathbb{Z}/3\times\mathbb{Z}/2,\ \mathbb{Z}/3\times\mathbb{Z}/3\times\mathbb{Z}/2\times\mathbb{Z}/2,\ \mathbb{Z}/5\times\mathbb{Z}/2,\ \mathbb{Z}/5\times\mathbb{Z}/5\times\mathbb{Z}/2,$$

$$\mathbb{Z}/5\times\mathbb{Z}/2\times\mathbb{Z}/2,\ \mathbb{Z}/5\times\mathbb{Z}/5\times\mathbb{Z}/2\times\mathbb{Z}/2\big\}.$$

If one were interested in computing the Brauer-Manin obstruction for a diagonal quartic surface $X_{a,b,c}/\mathbb{Q}$ given by the equation $x^4 + ay^4 + bz^4 + cw^4 = 0$, then one would automatically know that

$$\mathrm{Br}(\overline{X_{a,b,c}})^{G_{\mathbb{Q}}} \subset \mathrm{Br}(\overline{X_{a,b,c}})^{G_L} \in \mathrm{Br}(E, L),$$

making the computations effective for every parameter $a, b, c \in \mathbb{Q}$.
In the paper [42] Schütt, building up on the results of Shioda and Inose, was able to produce lower and upper bounds on the field of definition of K3 surfaces with $\rho(X) = 20$. In another paper with Elkies, they associated an explicit K3 surface $X/\mathbb{Q}$ with Picard group defined over $\mathbb{Q}$ to any known Hecke eigenform of weight three with rational coefficients. Moreover, for any CM imaginary quadratic field of class number one, Elkies found that there exists a unique K3 surface $X/\mathbb{Q}$ with $\mathrm{Pic}(X) = \mathrm{Pic}(\overline{X})$ and CM by E, and he provided explicit equations and generators for their Picard group. This kind of results can be generalised to principal K3 surfaces with CM.
To see how, let $T(X)$ be the transcendental lattice of X, $T(X)^\vee$ be its dual lattice

and $D_X := T(X)^\vee/T(X)$ its discriminant form. It is a finite group and, since $T(X)$ is an even lattice, there is a well-defined quadratic form q on D_X with values in $\mathbb{Q}/2\mathbb{Z}$. We denote by $\mathcal{O}(q)$ the group of automorphisms of D_X that preserve q. Note that there is a natural bijection between roots of unity in E and integral Hodge isometries of $T(X)$, so that there is always a canonical map $\mu(E) \to \mathcal{O}(q)$.

Definition 1.0.5. X has big discriminant if the natural map $\mu(E) \to \mathcal{O}(q)$ is injective.

It is not difficult to show that all but finitely many ($\mathbb{C}$-isomorphism classes of) principal K3 surfaces with CM by E have big discriminant. The classification becomes explicit when $\rho(X) = 20$: in this case, there exist only two complex K3 surfaces that do not have big discriminant, and their associated quadratic forms $T(X)$ are isomorphic respectively to $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and to $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Coincidentally, these were also studied by Vinberg in [53], who called them 'the two most algebraic K3 surfaces'. The following is Theorem 5.2.3, and appeared originally in [51].

Theorem 1.0.6. *Let $X/\mathbb{C}$ be a principal K3 surface with CM by E. Assume that X has big discriminant. There exists an explicit Abelian extension $F_X =: F$ of E and a model X^{can}/F of X over F with $\rho(X^{can}) = \rho(\overline{X^{can}})$. The pair (X^{can}, F) is canonical in the following sense: if Y/L is any K3 surface with: $E \subset L$, $Y_{\mathbb{C}} \cong X$ and $\rho(Y) = \rho(\overline{Y})$, then $F \subset L$ and $X^{can}_L \cong Y$.*

Note that the condition $\rho(X^{\mathrm{can}}) = \rho(\overline{X^{\mathrm{can}}})$ can prevent F from being the 'smallest' field of definition of X, i.e. X could admit models over subfields of F. On the other hand, the difference between a smaller field of definition of X and F can be universally bounded, see Lemma 2.6, Chapter 17 of [16]. This observation, together with the finiteness of the Fourier-Mukai partners and a theorem of Stark [48], allowed us to prove the following finiteness result, see Theorem 6.0.4.

Theorem 1.0.7. *Let $N > 0$ be any given number. Then there are only finitely many $\mathbb{C}-$isomorphism classes of principal K3 surfaces with CM that can be defined over a number field K with $[K : \mathbb{Q}] < N$*

This statement was known to Shafarevich when $\rho(X) = 20$, and it was later generalised by Orr and Skorobogatov [37] to any K3 surface with CM, i.e. not only the principal ones. This also leads to a proof of Conjecture 1.10 of [52] for principal CM K3 surfaces, and we refer the reader to [38] for an account on uniformity conjectures for K3 surfaces. Theorem 1.0.6 sheds also some light on the

surfaces found by Elkies and Schütt. Indeed, if $X/\mathbb{Q}$ is a principal K3 surface with $\rho(X) = \rho(\overline{X})$ and big discriminant, then the universal property of Theorem 1.0.6 assures that $X_E \cong (X_{\mathbb{C}})^{\mathrm{can}}$.
In this last paragraph, let us spend some words on the proof of the results above. To any CM field E and ideal $I \subset \mathcal{O}_E$ we associate an abelian field extension F_I/E. By class field theory, this is equivalent to give its norm group inside $\mathbb{A}_{E,f}^{\times}$ (we consider finite idèles because E has only complex embeddings) and we put

$$S_I := \{s \in \mathbb{A}_{E,f}^{\times} : \exists\ e \in E^{\times} \text{ such that } \frac{se}{\overline{se}} \in \hat{\mathcal{O}}_E^{\times} \text{ and } \frac{se}{\overline{se}} \equiv 1 \mod I\}.$$

Since $S_I = S_{\overline{I}} = S_{I\cap\overline{I}}$, we can assume without loss of generality that $I = \overline{I}$. In sections 3.4 and 3.5 one can find a detailed study of these field extensions and a closed formula for the indices $[F_I : E]$. When E is quadratic imaginary we can easily describe the F_I's as follows: let K_I and Cl_I be respectively the ray class field and the ray class group modulo I. Then $E \subset F_I \subset K_I$ is the fixed field of $\{x \in \mathrm{Cl}_I : x = \overline{x}\} \subset \mathrm{Cl}_I \cong \mathrm{Gal}(K_I/E)$.
If $X/\mathbb{C}$ is a principal K3 surface with CM by E, there is a natural action of $\mathcal{O}_E$ on $\mathrm{Br}(X)$, and if we put $\mathrm{Br}(X)[I] := \{\alpha \in \mathrm{Br}(X) : \ \forall\ i \in I\ i\alpha = 0\}$ we have the following.

Proposition 1.0.8. *The field of moduli (over E) of the pair* $(T(X), \mathrm{Br}(X)[I])$ *corresponds to F_I.*

Here with *field of moduli* we mean the fixed field of

$$\{\sigma \in \mathrm{Aut}(\mathbb{C}/E) : \exists \text{ Hodge isometry } f : T(X) \to T(X^{\sigma}) : f_{*}\circ\sigma^{*}|_{\mathrm{Br}(X)[I]} = \mathrm{Id}\}.$$

Let us now see how the algorithm in Theorem 1.0.4 works. If X is defined over a number field L containing E, there exists a unique ideal $I \subset \mathcal{O}_E$ such that $\mathrm{Br}(\overline{X})^{G_L} = \mathrm{Br}(\overline{X})[I] \cong \mathcal{O}_E/I$, and we conclude that $F_I \subset L$ because of Proposition 4.1.3. Therefore, we can write

$$\mathrm{Br}(E, L) = \{\mathcal{O}_E/I : F_I \subset E \cdot L\},$$

or, less precisely but more efficiently,

$$\mathrm{Br}(E, L) = \{\mathcal{O}_E/I : [F_I : E] \text{ divides } [L \cdot E : E]\}.$$

Finally, to every $X/\mathbb{C}$ with CM by $\mathcal{O}_E$, we associate the *discriminant ideal* $\mathcal{D}_X \subset$

$\mathcal{O}_E$. It has the property that $\mathcal{O}_E/\mathcal{D}_X \cong D_X$ and $\overline{\mathcal{D}_X} \cong T(X)^\vee/T(X)$, and if $X/\mathbb{C}$ has big discriminant one is able to produce a unique model X^{can} over $F = F_{\mathcal{D}_X}$ satisfying the condition $\rho(X^{\mathrm{can}}/F) = \rho(X/\mathbb{C})$.
As a final application, let us point out an easy criterion to decide whether a singular, principal K3 surface $X/\mathbb{C}$ can be defined over its CM field, thus generalising the work of Elkies and Schütt.

Proposition 1.0.9. *Let $X/\mathbb{C}$ be a singular K3 surface with CM by the ring of integers of a quadratic imaginary extension E. Assume that $T(X)$ is neither isomorphic to $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ nor to $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Then X admits a model with full Picard group over E if and only if the complex conjugation acts trivially on $\mathrm{Cl}_{\mathcal{D}_X}(E)$, the ray class group modulo $\mathcal{D}_X$.*

Notations

General notations

- If K is a field, we denote by $\overline{K}$ a fixed algebraic closure and by G_K its absolute Galois group. For every scheme X/K we write $\overline{X}$ for the base change $X \times_K \overline{K}$.
- We denote by $\mathbb{A}$ the ring of adéles over $\mathbb{Q}$ and by $\mathbb{A}_f \subset \mathbb{A}$ the subring of finite adéles. Moreover, we denote by $\widehat{\mathbb{Z}} \subset \mathbb{A}_f$ the pro-finite completion of $\mathbb{Z}$, so that $\widehat{\mathbb{Z}} \otimes \mathbb{Q} = \mathbb{A}_f$.
- For any number field K, we denote by $\mathcal{O}_K$ its ring of integers, by $\mathbb{A}_K := \mathbb{A} \otimes_{\mathbb{Q}} K$ the ring of adéles over K and by $\mathbb{A}_{K,f} := \mathbb{A}_f \otimes_{\mathbb{Q}} K \subset \mathbb{A}_K$ the subring of finite adéles. We also adopt the notation $\widehat{\mathcal{O}}_E := \mathcal{O}_E \otimes \widehat{\mathbb{Z}}$.
- Ig A is a $\mathbb{Z}-$module, we write $A_{\mathbb{Q}}$ for $A \otimes_{\mathbb{Z}} \mathbb{Q}$.
- For any set S, $|S|$ will denote its cardinality, and for any two integers $a, b \in \mathbb{Z}$ we write $a|b$ for 'a divides b'.

Notations concerning K3 surfaces

- If ℓ is a prime number and K is algebraically closed with $\mathrm{char}(K) \neq \ell$, we have a natural inclusion $c_1 : \mathrm{NS}(X) \to H^2_{\text{ét}}(X, \mathbb{Z}_\ell)(1)$ and we denote by $T_\ell(X) := \{v \in H^2_{\text{ét}}(X, \widehat{\mathbb{Z}})(1) : (v, n)_X = 0 \ \forall\ n \in \mathrm{NS}(X)\}$, where $(-,-)_X$ is the pairing given by Poincaré duality.

- Similarly, if K is algebraically closed of $\mathrm{char}(K) = 0$, we let $\widehat{T}(X) \subset H^2_{\text{ét}}(X, \widehat{\mathbb{Z}})(1)$ be the orthogonal complement of $\mathrm{NS}(X)$.

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2 Basic properties of K3 surfaces with complex multiplication

2.1 Generalities on Hodge structures

We begin by reviewing the notion of integral and rational Hodge structures. We mainly follow Moonen's survey [32] and Chapter 2 in Milne's notes on Shimura varieties appearing in [1].

Definition 2.1.1. Let V be a finitely generated, free $\mathbb{Z}$-module (respectively, a finite dimensional $\mathbb{Q}$-vector space). An integral (respectively, rational) Hodge structure of weight $m \in \mathbb{Z}$ on V is a decomposition

$$V \otimes \mathbb{C} = \bigoplus_{p+q=m} V^{p,q} \tag{2.1.0.1}$$

such that $\overline{V^{p,q}} = V^{q,p}$. Here, the tensor product is taken over $\mathbb{Z}$ (respectively, over $\mathbb{Q}$), the p and q are allowed to vary in $\mathbb{Z}$, and the bar denotes the complex conjugation. One says that the Hodge structure V is of type T, where $T \subset \mathbb{Z}^2$, if $V^{p,q} \neq 0$ precisely when $(p, q) \in T$.

Remark 2.1.2. To gain more flexibility, we shall also allow direct sums of Hodge structures of different weights.

There are two other equivalent definition that are also useful. Instead of giving a decomposition like in (2.1.1), one can endow V with a *Hodge filtration*, i.e. a descending and finite filtration F^p on $V_{\mathbb{C}} := V \otimes \mathbb{C}$, such that for every $p, q \in \mathbb{Z}$ with $p + q = m + 1$ one has $F^p \cap \overline{F^q} = \{0\}$ and $F^p \oplus \overline{F^q} = V_{\mathbb{C}}$. To obtain the Hodge filtration given (2.1.1), one simply puts

$$F^p := \bigoplus_{i \geq p} V^{i,m-i}.$$

Whereas, to go the other way around, one can show that $V^{p,q} := F^p \cap \overline{F^q}$ satisfies

(2.1.1). The third definition is due to Deligne, and is phrased in the language of algebraic groups. One defines the *Deligne torus* to be the real algebraic group $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$, where 'Res' denotes the Weil restriction of scalars, so that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. The character group $X^*(\mathbb{S})$ is generated by the two characters z and $\bar{z}$, that act on the $\mathbb{R}$-points of $\mathbb{S}$, respectively, as the identity and the complex conjugation. One also has the following important characters and cocharacters:

- The *weight cocharacter* $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ given, on $\mathbb{R}-$points, by the natural inclusion $\mathbb{R}^\times \to \mathbb{C}^\times$;
- The *Norm character* $\mathrm{Nm}\colon \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$ given by $z\bar{z}$;
- The cocharacter $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_\mathbb{C}$ defined to be the only cocharacter such that $\overline{z} \circ \mu = 1$ and $z \circ \mu = \mathrm{Id}$.

With this in mind, one can define a Hodge structure on V of weight $m \in \mathbb{Z}$ as a morphism of algebraic groups

$$h\colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$$

such that $h \circ w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(V)_\mathbb{R}$ is given by $z \mapsto z^{-m}\,\mathrm{Id}$. In this case, we see that $V^{p,q}$ corresponds to

$$\{v \in V_\mathbb{C}\colon \text{ for every } (z_1, z_2) \in \mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times \text{ one has } h_\mathbb{C}(z_1, z_2) \cdot v = z_1^{-p} z_2^{-q} v\}.$$

Let us briefly explain how Hodge structures appear naturally in the context of Kähler geometry, as a consequence of Hodge's theory. Let X be a compact Kähler manifold, and consider the Hodge-de Rham spectral sequence

$$H^p(X, \Omega^q_X) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

Here, both cohomology groups refer to sheaf cohomology, the first corresponding to the sheaf of holomorphic $q-$forms, and the other to the locally constant sheaf with $\mathbb{C}$-coefficientes. Using harmonic analysis, Hodge proved that this spectral sequence degenerates at the E^1-page, so to obtain a natural isomorphism

$$\bigoplus_{p+q=n} H^p(X, \Omega^q_X) \cong H^{p+q}(X, \mathbb{C}). \tag{2.1.0.2}$$

Therefore, thanks to the de Rham isomorphism

$$H_B^{p+q}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^{p+q}(X,\mathbb{C})$$

(where the first cohomology group denotes now Betti or singular cohomolgy), one concludes that the Betti cohomology groups of Kähler maniolds are naturally endowed with a Hodge structure (once we quotient out the torsion) by putting

$$H^{p,q}(X) := H^q(X,\Omega_X^p).$$

In the next paragraph we shall only consider *integral* Hodge structures, but all the following constructions can be easily generalised to the rational case. A morphism between two Hodge structures V and W is a $\mathbb{Z}$-linear map $f : V \to W$ such that $f_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ maps $V^{p,q}$ to $W^{p,q}$. The definition readily implies that in order for a morphism to exist V and W must have the same weight. A *sub-Hodge structure* $W \subset V$ is an inclusion of $\mathbb{Z}$-modules $W \hookrightarrow V$ that is also a morphism of Hodge structures. Usually, the map $W \hookrightarrow V$ is primitive, i.e., the quotient V/W is torsion-free. One can perform some natural linear-algebra construction with Hodge structures too: if V is a Hodge structure of weight n, then one can endow the dual $V^\vee = \mathrm{Hom}(V,\mathbb{Z})$ with a natural Hodge structure of weight $-n$. Similarly, if V and W are two Hodge structures of weight n and m respectively, then also $V \otimes_{\mathbb{Z}} W$ admits a natural Hodge structure of weight $n+m$. In particular, $\mathrm{Hom}(V,W) = V^\vee \otimes_{\mathbb{Z}} W$ is a Hodge structure of weight $m-n$.
Some trivial but extremely important Hodge structures are given by the *Tate-twists*. These are denoted by $\mathbb{Z}(n)$, with $n \in \mathbb{Z}$, and consists of the $\mathbb{Z}$-module $(2\pi i)^n\mathbb{Z} \subset \mathbb{C}$ endowed with the only Hodge-structure of type $(-n,-n)$. Tate-twists allow one to shifts the weight of Hodge structures, in the sense that if V is a integral Hodge structure of weight m, then $V(n) := V \otimes_{\mathbb{Z}} \mathbb{Z}(n)$ is an integral Hodge structure of weight $m-2n$. Similarly, one can define $\mathbb{Q}(n) := \mathbb{Z}(n)_{\otimes}\mathbb{Q}$. The "$(2\pi i)$" in the definition comes from the exponential sequence

$$0 \to (2\pi i)\mathbb{Z} \to \mathbb{C} \xrightarrow{exp} \mathbb{C}^\times \to 0, \tag{2.1.0.3}$$

and plays a role mostly when computing periods.

Definition 2.1.3. (Hodge classes) Let V be a Hodge structure of weight 0. The space of Hodge classes of V is defined to be

$$\mathrm{Hdg}(V) := V \cap V^{0,0}.$$

Let us make some examples of the notions introduced until now.

Examples. 1. A complex torus is a compact Kähler manifold of the form $A = \mathbb{C}^n/\Lambda$, where $\Lambda \subset \mathbb{C}^n$ is a lattice (that is, a discrete subgroup isomorphic to $\mathbb{Z}^{2n}$). To every complex torus A one can associate a Hodge structure of weight 1, namely $H^1(A, \mathbb{Z})$, of type $(1, 0)$ and $(0, 1)$. One can check that this association defines an equivalence of categories

$$\{\text{complex tori}\} \xrightarrow{\sim} \{\text{Integral Hodge structures of type } (1,0), (0,1)\}$$

2. If X, Y are compact Kähler manifolds, then the Künneth decomposition

$$H^n(X \times Y, \mathbb{Z}) \xrightarrow{\sim} \bigoplus_{p+q=n} H^p(X, \mathbb{Z}) \otimes H^q(Y, \mathbb{Z})$$

is an isomorphism of Hodge structures (after quotienting out both sides by the torsion).

3. Let X, Y be as above, and consider a morphism $f : X \to Y$ of complex manifolds. Then the induced pullback map

$$f^* : H^n(Y, \mathbb{Z}) \to H^n(X, \mathbb{Z})$$

is a morphism of Hodge structures. In particular, $f^* \in H^n(X, \mathbb{Z})^\vee \otimes H^n(Y, \mathbb{Z})$ is a Hodge class.

4. (Hodge conjecture) Let $X/\mathbb{C}$ be a compact Kälher manifold, and let $\mathrm{CH}^n(X)$ be the Chow group of codimension-n cycles on X. We denote by

$$c_n : \mathrm{CH}^n(X) \to H^{2n}(X, \mathbb{Z})(n),$$

the cycle class map, whose image is contained in the space of Hodge classes of $H^{2n}(X, \mathbb{Z})(n)$. When $n = 1$, we have that $\mathrm{CH}^n(X) = \mathrm{Pic}(X)$, and Lefschetz proved that $c_1(\mathrm{CH}^1(X)) = \mathrm{Hdg}(H^2(X, \mathbb{Z})(1))$. In general, Groethendieck showed that the equality $c_n(\mathrm{CH}^n(X)) = \mathrm{Hdg}(H^{2n}(X, \mathbb{Z})(n))$ does not need to hold when $n > 1$, for more or less trivial reasons. Nevertheless, things change drastically when we consider rational coefficients, and the question of whether $c_n(\mathrm{CH}^n(X)) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hdg}(H^{2n}(X, \mathbb{Q})(n))$ for any X and any n is called the Hodge conjecture, one of the most important open problems in complex geometry.

There are two other notions in Hodge theory that we shall introduce before concluding this section, namely, polarisations and Mumford-Tate groups.

Definition 2.1.4. (Weil operator) Let V be a Hodge structure, the Weil operator is the morphism $C: V_{\mathbb{C}} \to V_{\mathbb{C}}$ given by multiplication by i^{p-q} on $V^{p,q}$. Since $\overline{V^{p,q}} = V^{q,p}$, one can easily check that C respects $V_{\mathbb{R}}$, i.e., it is defined over $\mathbb{R}$. Moreover, if the Hodge structure is given by $h: \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$, one can easily check that $C = h(i)$.

Note that $C^2 = (-1)^m$, where m is the weight of V.

Remark 2.1.5. The Weil operator commutes with morphisms of Hodge structures, in the sense that if $f: V \to W$ is a morphism of Hodge structures, then $f \circ C_V = C_W \circ f$, where C_V and C_W denote, respectively, the Weil operator on V and W.

Definition 2.1.6. Let V be an integral Hodge structure of weight m. A polarisation on V is a morphism of Hodge structures

$$\phi: V \otimes V \to \mathbb{Z}(-m)$$

such that the bilinear form on $V_{\mathbb{R}}$ given by $(x, y) \mapsto (2\pi i)^m \phi(Cx \otimes y)$ is symmetric and positive-definite.

Note that by Remark (2.1.5) one has the following equalities for every $x, y \in V_{\mathbb{R}}$:

$$\phi(Cx \otimes y) = C\phi(Cx \otimes y) = \phi(C^2 x \otimes Cy) = (-1)^m \phi(x, Cy).$$

Therefore, since by definition the form $(x, y) \mapsto (2\pi i)^m \phi(Cx \otimes y)$ is symmetric, we conclude that ϕ is symmetric if m is even, and alternating if m is odd. Vaguely speaking, a *polarisation* on a Hodge structure reflects the presence of an ample line bundle, in the following sense. Let $X/\mathbb{C}$ be a projective manifold of dimension $n = \dim(X)$, and let $\mathcal{L}$ be an ample line bundle on X. The Lefschetz operator

$$L: H^m(X, \mathbb{Z}) \to H^{m+2}(X, \mathbb{Z})(1)$$

is defined to be the cup-product with the class $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})(1)$. It is a morphism of Hodge structures. Working with rational coefficients, for $0 \leq m \leq n$ the

m^{th}-primitive cohomology group is defined to be

$$H^m(X,\mathbb{Q})_{\text{prim}} := \ker[L^{n-m+1} : H^m(X,\mathbb{Q}) \to H^{2n+2-m}(X,\mathbb{Q})(n-m+1)].$$

It is a sub-Hodge structure of $H^m(X,\mathbb{Q})$. One can show (see Section 7.1.2 of [54]) that the pairing $H^m(X,\mathbb{Q})_{\text{prim}} \times H^m(X,\mathbb{Q})_{\text{prim}} \to H^{2n}(X,\mathbb{Q})(n-m) \cong \mathbb{Q}(-m)$ defined by $(x,y) \mapsto (-1)^m c_1(\mathcal{L})^{n-m} \cup x \cup y$ is a polarisation. Finally, to obtain a polarisation on the whole $H^m(X,\mathbb{Q})$ one uses the Lefschets decomposition

$$H^m(X,\mathbb{Q}) = \bigoplus_{j \geq 0} c_1(\mathcal{L})^j \cup H^{m-2j}(X,\mathbb{Q})_{\text{prim}}(-j).$$

Not every Hodge structure admits a polarisation, and when it does it is said to be *polarisable*. Perhaps the best way to understand this is to look at point 1) in Examples (2.1), where we had an equivalence of categories between complex tori and integral Hodge structures of type $(1,0)$ and $(0,1)$. Under this equivalence, integral Hodge structures of type $(1,0)$ and $(0,1)$ that admit a polarisation corresponds precisely to Abelian varieties, i.e., we have an equivalence of categories

$$\{\text{Ab. varieties}\} \xrightarrow{\sim} \{\text{Integral, polarisable Hodge structures of type } (1,0),(0,1)\}.$$

The last concept we need to introduce is the one of Mumford-tate group attached to a Hodge structure. This can be defined in two different ways, either via the formalism of Tannakian categories, or in more down-to-earth terms, as we shall do. For a nice introduction to Tannakian categories and related concepts, we refer the interested reader to the relevant article by Deligne appearing in [12].

Definition 2.1.7. Let V be a rational Hodge structures, given by the morphism $h : \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$. The Mumford-Tate group of V, denoted by $\mathrm{MT}(V)$, is defined to be the smallest algebraic subgroup of $\mathrm{GL}(V)$ such that h factorises as $h : \mathbb{S} \to \mathrm{MT}(V)_{\mathbb{R}} \hookrightarrow \mathrm{GL}(V)_{\mathbb{R}}$.

Note that $\mathrm{MT}(V)$ is connected since $\mathbb{S}$ is connected and, moreover, if V is polarisable, then $\mathrm{MT}(V)$ is reductive (see Proposition 4.9. in Moonen's notes). Mumford-Tate groups allow us to detect sub-Hodge structures in tensor constructions: let $\lambda \subset \mathbb{Z}^2$ be a finite subset, $\lambda = \{(a_i,b_i)\}_{i=1,\cdots,n}$, and define

$$V^{\lambda} := \bigoplus_{i=1}^{n} V^{\otimes a_i} \otimes (V^{\vee})^{b_i}.$$

We have a natural action of $\mathrm{MT}(V)$ on V^{λ}.

Proposition 2.1.8. *A rational subspace $W \subset V^{\lambda}$ is a sub-Hodge structure if and only if it is invariant under the action of* $\mathrm{MT}(V)$*. Moreover, an element $t \in V^{\lambda}$ is a Hodge class if and only if it is fixed by* $\mathrm{MT}(V)$*.*

Definition 2.1.9. Let V be a rational Hodge structure. Following Milne, we say that V is *special* if its Mumford-Tate group is a torus.

Note that this definition is very similar to Definition 1.0.1. The only differences are some technical conditions that are automatically satisfied for K3 surfaces, but need to be imposed for general Hodge structures (see Definition 12.5 in Milne's notes). Let V be a special Hodge structure and let T be its Mumford-Tate group, that by definition is an algebraic torus defined over $\mathbb{Q}$. The cocharacter μ introduced before gives us a morphism of algebraic tori $\mu : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}$.

Definition 2.1.10. Let $h : \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ be a special Hodge structure, and let T be its Mumford-Tate group. The reflex field of V, denoted by $E(h)$, is the field of definition of the cocharacter $\mu : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}$.

Note that the reflex field is always a finite extension of $\mathbb{Q}$. Before concluding this section, let us quickly recall the statement of the Mumford-Tate conjecture. Let $K \subset \mathbb{C}$ be a number field, and consider X/K a smooth, projective, geometrically irreducible variety. For any ℓ a prime number, one has a canonical comparison isomorphism

$$H^{\bullet}_{\text{ét}}(\overline{X}, \mathbb{Z}_{\ell}) \cong H^{\bullet}_{B}(X^{an}(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} \tag{2.1.0.4}$$

between the ℓ−adic étale cohomology and the Betti cohomology of the analytic space associated to $X_{\mathbb{C}}$ (see Theorem 3.12 in [31]). Attached to $H^{\bullet}_{\text{ét}}(\overline{X}, \mathbb{Z}_{\ell})$ one has a natural Galois representation

$$\rho_{\ell} : G_K \to \mathrm{GL}(H^{\bullet}_{\text{ét}}(\overline{X}, \mathbb{Q}_{\ell})),$$

and we denote by $G := \overline{\mathrm{Im}(\rho_{\ell})}^{\circ}$ the identity component of the Zariski-closure of the image of ρ_{ℓ}.

Conjecture 2.1.1. *(Mumford-Tate conjecture) Let M be the Mumford-Tate group of $H^{\bullet}_{B}(X^{an}(\mathbb{C}), \mathbb{Q})$. Under the comparison isomorphism* (2.1.0.4)*, one has*

$$G = M \times \mathrm{Spec}(\mathbb{Q}_{\ell}).$$

2.2 Generalities on K3 surfaces

In this section we introduce K3 surfaces and recollect the basic properties that will be needed during the thesis. We mainly follow Huybrechts' book [16], especially Chapter 1.
Let K be any field. By a variety over K, we mean a separated, geometrically integral scheme of finite type defined over K.

Definition 2.2.1 (Algebraic K3 surfaces)**.** A K3 surface over K is a smooth, complete variety X/K with $\dim(X) = 2$, such that $\omega_{X/K} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. Here, $\omega_{X/K}$ denotes the canonical bundle of X/K, i.e., $\omega_{X/K} = \bigwedge^2 \Omega_{X/K}$.

Remark 2.2.2. As Remark 1.2 of Chapter 1 in [16] explains, every smooth, complete surface is automatically projective. Therefore, algebraic K3 surfaces are always projective.

Classical examples of K3 surfaces are smooth quartics in $\mathbb{P}^3_K$, complete intersections of type $(2,3)$ in $\mathbb{P}^4$, and complete intersections of type $(2,2,2)$ in $\mathbb{P}^5$. Also smooth, projective surfaces that can be realised as a $2:1$ branched covering of $\mathbb{P}^2_K$ ramified over a curve of degree 6 are K3 surfaces. Finally, one can construct K3 surfaces starting from Abelian surfaces via a process due to Kummer: let A/K be an Abelian surface, and consider the involution $\iota\colon A \to A$ given by $x \mapsto -x$. The quotient A/ι is a surface with 16 double points, and blowing them up yields a K3 surface denoted $\mathrm{Km}(A)$, called the Kummer surface associated to A. The first step to study K3 surfaces is to compute their Picard groups. To this extent, let $\mathrm{Pic}_{X/K}$ be the Picard scheme of X introduced by Grothendieck, and $\mathrm{Pic}^0_{X/K}$ the connected component of the identity. The Picard group of X/K, denoted by $\mathrm{Pic}(X)$, is nothing but the group of $K-$rational points of $\mathrm{Pic}_{X/K}$, i.e.

$$\mathrm{Pic}(X) = \mathrm{Pic}_{X/K}(K).$$

We shall denote by $\mathrm{Pic}^0(X) := \mathrm{Pic}^0_{K/K}(K)$ the subgroup of algebraically trivial line bundles, and the quotient $\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is called the Néron-Severi group of X. A famous theorem of Severi asserts that $\mathrm{NS}(X)$ is a finitely generated Abelian group. Recall that when X is a smooth, complete surface, one has a natural intersection pairing $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$, that sends $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(X)$ to the intersection number

$$(\mathcal{L}_1, \mathcal{L}_2) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}_1^*) - \chi(X, \mathcal{L}_2^*) + \chi(X, \mathcal{L}_1^* \otimes \mathcal{L}_2^*). \qquad (2.2.0.1)$$

Here, χ denotes the Euler-Poincaré characteristic and $\mathcal{L}^*$ the dual of $\mathcal{L}$. The subgroup $\mathrm{Pic}^\tau(X) \subset \mathrm{Pic}(X)$ of *numerically trivial* line bundles corresponds to the kernel of the above pairing:

$$\mathrm{Pic}^\tau(X) := \{\mathcal{L} \in \mathrm{Pic}(X) : \text{ for every } \mathcal{L}' \in \mathrm{Pic}(X) \text{ we have } (\mathcal{L}, \mathcal{L}') = 0\}.$$

One can easily show that $\mathrm{Pic}^0(X) \subset \mathrm{Pic}^\tau(X)$. Then, the quotient $\mathrm{Pic}(X)/\mathrm{Pic}^\tau(X)$ represents the *numerical classes* of line bundles, and it is denoted by $\mathrm{Num}(X)$. It is a finitely generated, free Abelian group.

Theorem 2.2.3. *Let X/K be a K3 surface. Then the quotient maps*

$$\mathrm{Pic}(X) \twoheadrightarrow \mathrm{NS}(X) \twoheadrightarrow Num(X)$$

are all isomorphisms.

In particular, $\mathrm{Pic}(X)$ is a finitely generated, free Abelian group. Its rank is denoted by $\rho(X)$ and it is called the *Picard number* of X.
The next class of invariants to compute are the *Hodge numbers* $h^{p,q} := \dim_K H^q(X, \Omega^p_{X/K})$. They are classically grouped into the Hodge diamond, that for surfaces assumes the following form:

$$\begin{matrix} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & H^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0.} & & \end{matrix}$$

Using the Hirzebruch-Riemann-Roch and the properties in the definition of K3 surfaces, one can show that if X/K is a K3 surface, then its Hodge diamond corresponds to

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1. & & \end{matrix} \tag{2.2.0.2}$$

Let us now assume that $K = \mathbb{C}$, so that we can employ techniques from complex geometry, topology and Hodge theory to study K3 surfaces. In the following, we shall identify a K3 surface $X/\mathbb{C}$ with the complex space $X^{an}(\mathbb{C})$ naturally associated to it. The first thing to notice is that in the algebraic setting, K3 surfaces are always projective (see Remark (2.2.2)). On the other hand, in complex geometry, we can give this more general definition of K3 surface.

Definition 2.2.4. A complex K3 surface is a compact, connected manifold X with $\dim(X) = 2$ such that $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

For example $X := \mathrm{Km}(A)$, where A is a complex torus that is not projective, is a complex K3 surface that is not algebraic. Every complex K3 surface is simply connected, i.e., $\pi_1(X) = 0$. The Betti-cohomology groups of a K3 surface can be computed to be $H^1_B(X, \mathbb{Z}) = H^3_B(X, \mathbb{Z}) = 0$ and $H^2_B(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$. The topological intersection form turns $H^2_B(X, \mathbb{Z})$ into a lattice, that is unimodular by Poincaré duality. The isomorphism class of this lattice does not depend on the chosen X, since it can be proven that every two K3 surfaces are deformation equivalent (see Chapter 7, Theorem 1.1. of [16]); it is usually denoted by Λ and called the *K3 lattice*. Finally, thanks to the Hodge index theorem, one knows that the signature of Λ is $(3_+, 19_-)$.
A convenient way to understand complex K3 surfaces is via their Hodge structures; as explained in the previous section, the Betti cohomology groups of smooth, complete varieties are naturally endowed with a Hodge structure. In the K3 situation, the only non-trivial Hodge structure is the one associated to $H^2_B(X, \mathbb{Z})$, with the Hodge decomposition given by

$$H^2_B(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^0(X, \Omega^2_X) \oplus H^1(X, \Omega_X) \oplus H^2(X, \mathcal{O}_X),$$

and we shall simply denote $H^q(X, \Omega^p_X)$ by $H^{p,q}(X)$. The Hodge diamond (2.2.0.2) tells us that $\dim_{\mathbb{C}} H^{2,0}(X) = \dim_{\mathbb{C}} H^{0,2}(X) = 1$ and $\dim_{\mathbb{C}} H^{1,1}(X) = 20$. If $\omega \in H^{2,0}(X) - 0$, one can show that

- $(\omega, \omega) = 0$;
- $(\omega, \overline{\omega}) > 0$;
- $(\omega, H^{1,1}(X)) = 0$.

Here, $(-, -)$ denotes the extension to $H^2(X, \mathbb{Z}) \otimes \mathbb{C}$ of the intersection form. This implies that the whole Hodge structure can be reconstructed by ω, since $H^{1,1}(X)$

corresponds to the complexification of $\langle \mathrm{Re}(\omega), \mathrm{Im}(\omega)\rangle^{\perp} \subset H^2(X, \mathbb{R})$, see Chapter 6, Proposition 1.2 of [16].

Definition 2.2.5. A Hodge structure of weight two on Λ with $\dim_{\mathbb{C}} \Lambda^{2,0} = 1$ and such that any $\omega \in \Lambda^{2,0} - 0$ satisfies the three condition above is called of *K3 type*.

The first Chern class map induces an injection $c_1 : \mathrm{Pic}(X) \cong \mathrm{NS}(X) \hookrightarrow H^2_B(X, \mathbb{Z})(1)$, and the intersection form introduced in (2.2.0.1) is nothing but the restriction of the topological intersection form to $\mathrm{NS}(X)$.

Definition 2.2.6. The transcendental lattice $T(X) \subset H^2(X, \mathbb{Z})(1)$ is defined to be the orthogonal complement of $\mathrm{NS}(X)$.

Remark 2.2.7. Clearly, $T(X) \subset H^2(X, \mathbb{Z})(1)$ is a primitive embedding, and it is possible to show that $T(X)_{\mathbb{Q}} = T(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible Hodge structure when X is projective (i.e., algebraic).

The importance of Hodge theory in the study of complex K3 surfaces lies in the global Torelli theorem (see Chapter 7, Theorem 5.3. of [16]).

Theorem 2.2.8. *Two complex K3 surfaces X and Y are isomorphic if and only if there exists a Hodge isometry $\phi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(Y, \mathbb{Z})$ (i.e., ϕ is an isomorphism of integral Hodge structures that respect the intersection form). Moreover, if ϕ sends a Kähler class of X to a Kähler class of Y, there exists a unique isomorphism $f : Y \xrightarrow{\sim} X$ such that $f^* = \phi$, where f^* denotes the induced map in cohomology.*

The theorem above shares a deep resemblance to what happens in the case of complex tori, and it tells us that all the information of a complex K3 surface is encoded in its Hodge structure. Finally, the surjectivity of the period map (Chapter 6, Remark 3.3. of *loc. cit.*) asserts that every Hodge structure of K3 type (in the sense of Definition 2.2.5) comes from a complex K3 surface.

Theorem 2.2.9. *Let us consider Λ endowed with a Hodge structure of K3 type. Then there exists a complex K3 surface X with a Hodge isometry $\Lambda \cong H^2(X, \mathbb{Z})$.*

Remark 2.2.10. As explained in the proof of Proposition 3.1.11, X is projective (i.e., algebraic) if and only if there exists a class $L \in \Lambda^{1,1} \cap \Lambda$ such that $(L, L) > 0$.

The only other ingredient we shall need from the theory of K3 surfaces is the notion of Fourier-Mukai partners.

Definition 2.2.11. Two complex K3 surfaces X and Y are said to be Fourier-Mukai partners if there exists a Hodge isometry $T(X) \cong T(Y)$.

Remark 2.2.12. This is an ad-hoc definition. The original definition comes from the theory of bounded derived categories, a subject that we should not touch upon in this thesis. The interested reader can consult Chapter 16 of Huybrechts' book [16]. The equivalence of our ad-hoc definition and the 'real' one is proved in Chapter 16, Corollary 3.7. of *loc. cit.*.

The only theorem we shall need is the following finiteness one.

Theorem 2.2.13. *Let X be a complex K3 surface. Then there exists only finitely many isomorphism classes of complex K3 surfaces Y such that X and Y are Fourier-Mukai partners.*

2.3 Absolute Hodge cycles

The idea of absolute Hodge cycle was first introduced by Deligne in his seminal paper [12], which we follow as the main reference. Absolute Hodge cycles allow one to build a meaningful category of motives, and thanks to them Deligne was able to prove one inclusion of the Mumford-Tate conjecture for Abelian motives, i.e., the ones that can be obtained by linear algebra constructions from the motives of Abelian varieties. To illustrate the main ideas, consider k an algebraically closed field of finite transcendence degree over $\mathbb{Q}$ and X/k a smooth projective variety. Every embedding $\sigma: k \hookrightarrow \mathbb{C}$ defines a variety over $\mathbb{C}$ that we denote by X^σ. Grothendieck in his paper [14] proved that there is a canonical comparison isomorphism

$$H^\bullet_{\mathrm{dR}}(X^\sigma/\mathbb{C}) \cong H^\bullet_B(X^\sigma(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C} \tag{2.3.0.1}$$

between the algebraic de Rham cohomology of X^σ and the Betti cohomology of its analytification with $\mathbb{C}$-coefficientes. Following Deligne, we put $H^\bullet_\sigma(X) := H^\bullet_B(X^\sigma(\mathbb{C}), \mathbb{Q})$. The canonical morphism of schemes

$$X^\sigma \to X,$$

induces a pullback map

$$\sigma^*: H^n_{\mathrm{dR}}(X/k) \to H^n_{\mathrm{dR}}(X^\sigma/\mathbb{C})$$

and an isomorphism

$$\sigma^* \otimes 1 =: \sigma^*_{\mathrm{dR}} : H^n_{\mathrm{dR}}(X/k) \otimes_{k,\sigma} \mathbb{C} \to H^n_{\mathrm{dR}}(X^\sigma/\mathbb{C}).$$

In étale cohomology, we have an isomorphism (since k is algebraically closed)

$$\sigma^*_{\text{ét}} : H^n_{\text{ét}}(X, \mathbb{A}_f) \to H^n_{\text{ét}}(X^\sigma, \mathbb{A}_f).$$

Here, by $H^n_{\text{ét}}(X, \mathbb{A}_f)$ we simply mean $H^n_{\text{ét}}(X, \widehat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. One now defines, for $n \geq 0$ and $m \in \mathbb{Z}$,

$$H^n_{\mathbb{A}}(X)(m) := H^n_{\mathrm{dR}}(X/k)(m) \times H^n_{\text{ét}}(X, \mathbb{A}_f)(m);$$

it is a free $(k \times \mathbb{A}_f)$-module. For every $\sigma : k \hookrightarrow \mathbb{C}$ there is a canonical diagonal embedding

$$\Delta_\sigma : H^n_\sigma(X)(m) \hookrightarrow H^n_{\mathbb{A}}(X^\sigma)(m),$$

that is constructed using (2.3.0.1) and the comparison isomorphism

$$H^\bullet_{\text{ét}}(X^\sigma, \widehat{\mathbb{Z}}) \cong H^\bullet_B(X^\sigma(\mathbb{C}), \mathbb{Z}) \otimes \widehat{\mathbb{Z}} \tag{2.3.0.2}$$

between étale cohomology and Betti cohomology, and a canonical isomorphism (the product of $\sigma^*_{\text{ét}}$ and σ^*_{dR})

$$\sigma^* : H^n_{\mathbb{A}}(X)(m) \otimes (\mathbb{C} \times \mathbb{A}_f) \to H^n_{\mathbb{A}}(X^\sigma)(m).$$

Definition 2.3.1 (Absolute Hodge)**.** An element $t \in H^{2n}_{\mathbb{A}}(X)(n)$ is said to be absolute Hodge if

1. $\sigma^*(t) \in \Delta_\sigma(H^{2n}_\sigma(X)(n))$ for every $\sigma : k \hookrightarrow \mathbb{C}$,
2. The first component of t lies in $F^0 H^{2n}_{\mathrm{dR}}(X/k)(n)$, where $F^\bullet$ denotes the Hodge filtration.

For X/k as above, denote by $C^n_{AH}(X) \subset H^{2n}_{\mathbb{A}}(X)(n)$ the subset of absolute Hodge cycles. It is a finite dimensional space over $\mathbb{Q}$.
We list now some facts about absolute Hodge cycles, whose proofs can be found in [12] (except for point 5.)

Proposition 2.3.2. *The following hold true:*

1. *The class of every algebraic cycle is absolute Hodge;*

2. *The Kunneth components of the diagonal are absolute Hodge;*

3. *The map in the hard Lefschetz theorem is absolute Hodge;*

4. *For abelian varieties, the notion of Hodge and absolute Hodge cycle coincide;*

5. *For K3 surfaces, one can prove that the Kuga-Satake map is absolute Hodge (see Deligne's paper where he proves the Weil conjectures for K3 surfaces [11]). Therefore, also for K3 surfaces the notion of Hodge and absolute Hodge coincide;*

6. *If $k \subset k'$ are algebraically closed fields of finite transcendence over $\mathbb{Q}$, and X is defined over k, then $C^{\bullet}_{AH}(X) = C^{\bullet}_{AH}(X'_k)$, where the isomorphism is given by the base-change in cohomology;*

7. *Suppose that K is a number field, and suppose that X is defined over K. Then G_K acts on $C^{\bullet}_{AH}(\overline{X})$ through a finite quotient.*

Using the last property, we can make the following definition.

Definition 2.3.3. Let X/K be a smooth projective variety over a number field K. Define $C^{\bullet}_{AH}(X) \subset C^{\bullet}_{AH}(\overline{X})$ to be

$$C^{\bullet}_{AH}(X) := C^{\bullet}_{AH}(\overline{X})^{G_K}.$$

Remark 2.3.4. If $X, Y/K$ are smooth complete varieties and

$$f : H^{\bullet}_B(X_{\mathbb{C}}, \mathbb{Q}) \to H^{\bullet}_B(Y_{\mathbb{C}}, \mathbb{Q})$$

is a correspondence whose class in $H^{\bullet}_B(X_{\mathbb{C}} \times Y_{\mathbb{C}}, \mathbb{Q})$ is absolute Hodge and defined over K, then the induced map

$$f \otimes 1 : H^{\bullet}_{\text{ét}}(\overline{X}, \hat{\mathbb{Z}}) \to H^{\bullet}_{\text{ét}}(\overline{Y}, \hat{\mathbb{Z}})$$

is G_K-invariant.

2.4 K3 surfaces with CM and their Hodge structures

Let $X/\mathbb{C}$ be a $K3$ surface and let $H^2_B(X, \mathbb{Z})$ be its second Betti cohomology. Recall that the transcendental lattice of X, denoted by $T(X)$, is defined as the orthogonal complement of NS(X) with respect to the intersection form on $H^2(X,\mathbb{Z})(1)$.

Definition 2.4.1. We say that X has complex multiplication (CM) if the Mumford-Tate group MT(X) of $T(X)_{\mathbb{Q}}$ is abelian.

Remark 2.4.2. It is easy to show that the inclusion $T(X)_{\mathbb{Q}} \subset H^2(X,\mathbb{Q})(1)$ induces an identification between the Mumford-Tate group of $T(X)_{\mathbb{Q}}$ and the one of $H^2(X,\mathbb{Q})(1)$.

In this case (see Zarhin [56]) one can prove that $E(X) := \mathrm{End}_{\mathrm{Hdg}}(T(X)_{\mathbb{Q}})$ is a CM field (where complex conjugation acts like the adjunction with respect to the intersection form) and that $\dim_{E(X)} T(X)_{\mathbb{Q}} = 1$. Since the elements of $E(X)$ are endomorphisms of Hodge structures, we obtain a natural map $\sigma_X : E(X) \to \mathrm{End}(H^{1,-1}(X)) = \mathbb{C}$. Since $T(X)_{\mathbb{Q}}$ is irreducible, Schur's lemma shows that σ_X is actually an embedding. Therefore, $E(X)$ is always canonically a subfield of $\mathbb{C}$, and in proposition 2.4.3 we shall show that it corresponds to the reflex field of the Hodge structure $T(X)_{\mathbb{Q}}$. The Hodge structure $T(X)_{\mathbb{Q}}$ can be described using the torus $\mathrm{Res}_{E(X)/\mathbb{Q}}\,\mathbb{G}_m$, whose $\mathbb{Q}$-points are naturally identified with $E(X)^\times$. If we decompose

$$(\mathrm{Res}_{E(X)/\mathbb{Q}}\,\mathbb{G}_m)(\mathbb{C}) = \bigoplus_{\sigma : E(X)\hookrightarrow\mathbb{C}} \mathbb{C}^\times_\sigma$$

where

$$\mathbb{C}^\times_\sigma := \{z \in (\mathrm{Res}_{E(X)/\mathbb{Q}}\,\mathbb{G}_m)(\mathbb{C}) : \ \forall e \in E(X),\ e\cdot z = \sigma(e)z\}$$

we have that the Hodge structure on $T(X)_{\mathbb{Q}}$ is given by the morphism of algebraic groups (defined over $\mathbb{R}$) whose action on $\mathbb{C}$-points is

$$\begin{aligned} h :\ \mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times &\to \mathbb{C}^\times_{\sigma_X} \oplus \cdots \oplus \mathbb{C}^\times_{\overline{\sigma_X}} = \mathrm{Res}_{E(X)/\mathbb{Q}}\,\mathbb{G}_m(\mathbb{C}) \subset \mathrm{GL}(T(X))(\mathbb{C}) \\ (z,w) &\mapsto (zw^{-1}, 1, \cdots, 1, wz^{-1}), \end{aligned}$$

where $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$ is the Deligne torus and σ_X is the distinguished embedding $E(X) \hookrightarrow \mathbb{C}$. Denote by $\mathrm{U}_{E(X)}$ the $E(X)$-linear unitary subgroup of $\mathrm{Res}_{E(X)/\mathbb{Q}}\,\mathbb{G}_m$, i.e. the one cut out by the equation $e\bar{e} = 1$. Zarhin in his paper [56] proved that inside $\mathrm{GL}(T(X))_{\mathbb{Q}}$ we have an identification

$$\mathrm{MT}(T(X)) = \mathrm{U}_{E(X)}\,.$$

When taking $\mathbb{C}$-points, the natural inclusion $\mathrm{U}_{E(X)} \subset \mathrm{Res}_{E(X)/\mathbb{Q}}\, \mathbb{G}_m$ becomes

$$\mathrm{U}_{E(X)}(\mathbb{C}) = \left\{ (z)_\sigma \in \bigoplus_{\sigma : E(X) \hookrightarrow \mathbb{C}} \mathbb{C}_\sigma^\times : \; z_\sigma z_{\overline{\sigma}} = 1 \right\}.$$

Therefore, the cocharacter μ associated to h is the map

$$\begin{aligned} \mu : \; \mathbb{G}_m(\mathbb{C}) &\to \mathbb{C}^\times_{\sigma_X} \oplus \cdots \oplus \mathbb{C}^\times_{\overline{\sigma_X}} \\ z &\mapsto (z, 1, \cdots, 1, z^{-1}) \end{aligned} \tag{2.4.0.1}$$

with image inside $U_{E(X)}(\mathbb{C})$.

Proposition 2.4.3. *The reflex field of the Hodge structure $T(X)_\mathbb{Q}$ is $\sigma_X(E(X)) \subset \mathbb{C}$.*

Proof. By definition, the reflex field of $T(X)_\mathbb{Q}$ is the field of definition of the cocharacter μ. By the discussion above, we see that $\tau \in \mathrm{Aut}(\mathbb{C})$ fixes μ if and only if $\tau\sigma_X = \sigma_X$, i.e. if and only if $\tau \in \mathrm{Aut}(\mathbb{C}/\sigma_X(E(X)))$. □

Remark 2.4.4. The embedding σ_X normalises the action of $E(X)$ in the sense that if $\alpha \in \sigma_X(E(X))$, then the Hodge endomorphism $\sigma_X^{-1}(\alpha)$ acts as multiplication by α on the $(1, -1)$ part of cohomology.

One can show that the CM fields E can be spanned, as $\mathbb{Q}$-vector spaces, by elements $\alpha \in E$ such that $\alpha\overline{\alpha} = 1$ (for a proof, see Proposition 4.4. in [15]). In $E(X)$, these correspond to rational Hodge isometries, since for every $v, w \in T(X)_\mathbb{Q}$ we have

$$(\alpha v, \alpha w)_X = (\alpha\overline{\alpha} v, w)_X = (v, w)_X.$$

As proved in Corollary 1.10 of [33], if $\rho(X) \geq 11$ there exist integral algebraic cycles $C_i \subseteq X \times X$ and rational numbers $q_i \in \mathbb{Q}$ for $i = 1, \cdots, n$ such that the cohomology class of α in $H^4(X \times X, \mathbb{Q})(2)$ can be expressed as

$$\alpha = \sum_i q_i [C_i].$$

(Here, we denote by $[C_i]$ the image of C_i under the cycle class map $\mathrm{CH}^2(X \times X) \to H^4(X \times X, \mathbb{Q})(2)$). Nikulin [36] improved Mukai's result to cover all the cases with $\rho(X) \geq 5$, i.e. $[E : \mathbb{Q}] \leq 16$ and, finally, Buskin [4] took care of all the remaining cases. In particular, if $X/\mathbb{C}$ has complex multiplication, the Hodge conjecture is true for $X \times X$. Therefore, if X is defined over $K \subseteq \mathbb{C}$ we can ask whether a class $\alpha \in E(X_{\mathbb{C}})$ is defined over K too.

Definition 2.4.5. Let $X/K \subseteq \mathbb{C}$ (this notation means that K is considered as a subfield of $\mathbb{C}$) with CM over $\mathbb{C}$.

1. For every $\tau \in \mathrm{Aut}(\mathbb{C})$ we define the map $\tau^{ad}: E(X_{\mathbb{C}}) \to E(X_{\mathbb{C}}^{\tau})$ as

$$\tau^{ad}(\alpha) := \sum_i q_i \tau^*[C_i],$$

   where $E(X_{\mathbb{C}}) \ni \alpha = \sum_i q_i[C_i]$ and τ^* denotes the pullback of algebraic cycles via the isomorphism of schemes $\tau: X_{\mathbb{C}}^{\tau} \to X_{\mathbb{C}}$.

2. We say that $\alpha \in E(X_{\mathbb{C}})$ is defined over K if for every $\tau \in \mathrm{Aut}(\mathbb{C}/K)$

$$\tau^{ad}(\alpha) = \alpha.$$

Definition 2.4.6. Let X be a K3 surface over a field K, and assume that K has finite transcendence degree over $\mathbb{Q}$. Like in Definition 2.3.3, we define $E(X)$ to be the subfield of $E(X_{\mathbb{C}})$ of absolute Hodge endomorphism that are defined over K. If $X_{\mathbb{C}}$ has CM, then we say that X *has CM over* K if $E(X) = E(X_{\mathbb{C}})$.

Remark 2.4.7. In order to define $E(X)$ one has to choose an embedding $K \hookrightarrow \mathbb{C}$, but one can check that $E(X)$ does not depend on the chosen embedding.

We will now give an equivalent condition for X/K to have complex multiplication over K, similar to the one for Abelian varieties.

Proposition 2.4.8. *Let X/K be as in Definition 2.4.6 such that $X\mathbb{C}$ has CM, and let $\iota: K \hookrightarrow \mathbb{C}$ be an embedding. Then X has CM over K if and only if*

$$\sigma_{X_{\mathbb{C}}}(E(X_{\mathbb{C}})) \subseteq \iota(K),$$

i.e. if and only if $\iota(K)$ contains the reflex field of $X_{\mathbb{C}}$. Moreover, the condition $\sigma_{X_{\mathbb{C}}}(E(X_{\mathbb{C}})) \subseteq \iota(K)$ does not depend on ι.

Proof. Let $\tau \in \mathrm{Aut}(\mathbb{C})$ be an automorphism of the complex numbers and consider the base change $X_{\mathbb{C}}^{\tau} := X_{\mathbb{C}} \times_{\tau} \operatorname{Spec} \mathbb{C}$. Again, we have a natural isomorphism $\tau^{\mathrm{ad}}: E(X_{\mathbb{C}}) \xrightarrow{\sim} E(X_{\mathbb{C}}^{\tau})$, given by conjugation of algebraic cycles. If $\omega \in T^{1,-1}(X_{\mathbb{C}})$ is a non-zero 2−form, we can conjugate it via τ (since it is an algebraic object) to obtain a non zero 2−form ω^{τ} on $T^{1,-1}(X_{\mathbb{C}}^{\tau})$. Now, denote by

$\sigma_X : E(X_{\mathbb{C}}) \hookrightarrow \mathbb{C}$ and by $\sigma_{X^\tau} : E(X^\tau_{\mathbb{C}}) \hookrightarrow \mathbb{C}$ the two embeddings given by evaluation on a non-zero 2−form and let $\alpha \in \sigma_X(E(X_{\mathbb{C}}))$; we have:

$$(\tau^{ad}\sigma_X^{-1}\alpha)\omega^\tau = ((\sigma_X^{-1}\alpha)\omega)^\tau = (\alpha\omega)^\tau = \tau(\alpha)\omega^\tau$$

i.e.

$$\sigma_{X^\tau} \circ \tau^{ad} = \tau \circ \sigma_X. \tag{2.4.0.2}$$

Therefore, the following diagram commutes

$$\begin{array}{ccc} E(X_{\mathbb{C}}) & \xrightarrow{\tau^{ad}} & E(X^\tau_{\mathbb{C}}) \\ \big\downarrow{\scriptstyle \sigma_X} & & \big\downarrow{\scriptstyle \sigma_{X^\tau}} \\ \mathbb{C} & \xrightarrow{\tau} & \mathbb{C}. \end{array}$$

If τ fixes K, then $X^\tau_{\mathbb{C}} = X_{\mathbb{C}}$, so that $E(X) = E(X_{\mathbb{C}})$ if and only if the map $\tau^{ad} : E(X_{\mathbb{C}}) \to E(X_{\mathbb{C}})$ is the identity. But the diagram above tells us that this happens if and only if τ fixes also $\sigma_X(E(X_{\mathbb{C}}))$. Finally, to prove that the condition $\sigma_{X_{\mathbb{C}}}(E(X_{\mathbb{C}})) \subseteq \iota(K)$ does not depend on ι, we need to show that it is true for one embedding if and only if it is true all. But if $\tau \in \mathrm{Aut}(\mathbb{C})$ is any element, equation 2.4.0.2 tells us that

$$\sigma_{X^\tau}(E(X^\tau_{\mathbb{C}})) = \tau(\sigma_X(E(X_{\mathbb{C}}))),$$

so we conclude the proof. □

Definition 2.4.9. Let $X/\mathbb{C}$ be a K3 surfaces with CM. We define the order $\mathcal{O}(X) := \mathrm{End}_{\mathrm{Hdg}}(T(X)) \subset E(X)$, and we say that X is *principal* if $\mathcal{O}(X)$ is the maximal one.

Remark 2.4.10. We do not use the full power of the Hodge conjecture in this thesis. One can define $E(X)$ and $\mathcal{O}(X)$ in the same fashion for every K3 surface $X/\mathbb{C}$ (i.e., without CM). By the results obtained in [56] by Zarhin, $E(X)$ is always going to be a field, either totally real or CM, and $\mathcal{O}(X) \subset E(X)$ an order in it. Proposition 2.3.2 tells us that every $\alpha \in E(X)$ is an absolute Hodge class, hence we can define $\tau^{ad} : E(X) \to E(X^\tau)$ to be the natural conjugation of absolute Hodge classes. In Proposition 2.4.12 we will show that τ^{ad} sends $\mathcal{O}(X)$ isomorphically to $\mathcal{O}(X^\tau)$. Working with absolute Hodge classes is particularly useful if one whishes to generalise these ideas to more general CM hyperkähler manifolds, where the Hodge conjecture for self-products is not known.

Remark 2.4.11. From now on, we will only consider $K3$ surfaces with CM that are principal.

One has to prove that the ring $\mathcal{O}(X)$ is an algebraic invariant of X, i.e. that it depends only on the scheme structure of X. What we mean by this is the following: consider X/k any K3 surface, and suppose there exists an embedding $\iota\colon k \hookrightarrow \mathbb{C}$. Base-changing X via ι, we obtain a K3 surface X^ι over $\mathbb{C}$, and we can compute the ring $\mathcal{O}(X^\iota) = \mathrm{End}_{\mathrm{Hdg}}(T(X^\iota))$. We need to prove that this ring does not depend on ι. The analogous statement for Abelian varieties is trivial, as the analogue of $\mathrm{End}_{\mathrm{Hdg}}(T(X))$ would be the endomorphism ring of the variety, and conjugation of an endomorphism is still an endomorphism. In the $K3$ surface case, though, it is not clear that if $\alpha \in \mathcal{O}(X) \subset E(X)$ then also $\tau^{ad}(\alpha) \in \mathcal{O}(X^\tau) \subset E(X^\tau)$ (we only know, so far, that $\tau^{ad}(\alpha) \in E(X^\tau)$).

Theorem 2.4.12 (Invariance of $\mathcal{O}(X)$)**.** *Let $X/\mathbb{C}$ be any K3 surface and let $\tau \in$* $\mathrm{Aut}(\mathbb{C})$*. Then the natural map $\tau^{ad}\colon E(X) \to E(X^\tau)$ sends $\mathcal{O}(X)$ isomorphically to $\mathcal{O}(X^\tau)$.*

Proof. Consider the two natural embeddings

$$\iota_B : E(X) \hookrightarrow \mathrm{Hdg}^4(X \times X) := H^{0,0}(X \times X) \cap H^4_B(X \times X, \mathbb{Q}(2))$$

$$\iota_{\text{ét}} : E(X) \hookrightarrow H^4_{\text{ét}}(X \times X, \mathbb{A}_f(2)).$$

Since for K3 surfaces every Hodge cycle is absolute Hodge, for every $\tau \in \mathrm{Aut}(\mathbb{C})$ we have a well-defined map

$$\tau_B\colon \mathrm{Hdg}^4(X \times X) \to \mathrm{Hdg}^4(X^\tau \times X^\tau)$$

and a natural inclusion

$$\mathrm{Hdg}^4(X \times X) \hookrightarrow H^4_{\text{ét}}(X \times X, \mathbb{A}_f(2))$$

such that the following commutes (note the abuse of notation in the vertical arrows)

$$\begin{array}{ccc} E(X) & \xrightarrow{\tau^{ad}} & E(X^\tau) \\ \downarrow{\scriptstyle \iota_B} & & \downarrow{\scriptstyle \iota_B} \\ \mathrm{Hdg}^4(X \times X) & \xrightarrow{\tau_B} & \mathrm{Hdg}^4(X^\tau \times X^\tau) \\ \downarrow & & \downarrow \\ H^4_{\text{ét}}(X \times X, \mathbb{A}_f(2)) & \xrightarrow{\tau^*} & H^4_{\text{ét}}(X^\tau \times X^\tau, \mathbb{A}_f(2)). \end{array}$$

where τ^* is the natural pullback in étale cohomology via the isomorphism of schemes

$\tau : X^\tau \to X$, and the composition of the vertical arrows is ι_{et}. Consider now the isomorphism of $\widehat{\mathbb{Z}}$-lattices

$$\tau^* : \widehat{T}(X) \to \widehat{T}(X^\tau)$$

and let $f \in \mathcal{O}(X)$. The commutativity of the above diagram tells us that

$$\tau^{ad}(f) = \tau^* \circ f \circ \tau^{*-1},$$

this equality happening in $H^4_{\text{ét}}(X^\tau \times X^\tau, \mathbb{A}_f(2))$. Now, $\tau^{ad}(f)(T(X^\tau)) \subset T(X^\tau)_{\mathbb{Q}}$ since $\tau^{ad}(f) \in E(X^\tau)$, and $[\tau^* \circ f \circ \tau^{*-1}]\widehat{T}(X^\tau) \subset \widehat{T}(X^\tau)$ since $\tau^* : \widehat{T}(X) \to \widehat{T}(X)$ is an isomorphism. Thus, the equality $\tau^{ad}(f) = \tau^* \circ f \circ \tau^{*-1}$ implies that $\tau^{ad}(f)(T(X^\tau)) \subset T(X^\tau)_{\mathbb{Q}} \cap \widehat{T}(X^\tau) = T(X^\tau)$, i.e. $\tau^{ad}(f) \in \mathcal{O}(X^\tau)$. Hence the map

$$\tau^{ad} : E(X) \to E(X^\tau)$$

restricts to an isomorphism between $\mathcal{O}(X)$ and $\mathcal{O}(X^\tau)$. □

2.5 Computing the order of singular K3 surfaces

In this section we will explicitly compute the order $\mathcal{O}(X)$ for every $X/\mathbb{C}$ with maximal Picard rank $\rho(X) = 20$, so to have an easy criterion to decide whether $\mathcal{O}(X)$ is principal or not. If $X/\mathbb{C}$ is a singular $K3$ surface, the order $\mathcal{O}(X) := \mathrm{End}_{\mathrm{Hdg}}(T(X))$ can be easily computed in the following standard way. Choose a $\mathbb{Z}$-basis e_1, e_2 of $T(X)$ and write the intersection matrix as

$$M = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \tag{2.5.0.1}$$

with $a, b, c \in \mathbb{Z}$ and $\Delta := b^2 - 4ac < 0$. Let $2q(x, y) := (xe_1 + ye_2, xe_1 + ye_2)_X$ be the binary quadratic form associated to $(-,-)_X$, i.e.

$$q(x, y) = ax^2 + bxy + cy^2.$$

Up to orientation, the only Hodge structure on $T(X)$ of K3 type is given by

$$T(X)^{1,-1} = \mathbb{C} \begin{bmatrix} s \\ 1 \end{bmatrix}$$

where s is a solution of $q(x,1)=0$, let's say $s=\frac{-b+\sqrt{\Delta}}{2a}$. This follows by the fact that a non-zero 2-form ω must satisfy $q(\omega,\omega)=0$. Denote by E the field $\mathbb{Q}(\sqrt{\Delta})$ and write $\Delta=f^2\Delta_E$, with Δ_E the discriminant of the field E.

Proposition 2.5.1. *The ring homomorphism*

$$\Phi:\ E\to M_{2\times 2}(\mathbb{Q})$$
$$x+y\sqrt{\Delta_E}\mapsto x\,\mathrm{Id}+\frac{y}{f}\begin{bmatrix}-b & -2c\\ 2a & b\end{bmatrix}$$

realizes E as $\mathrm{End}_{\mathrm{Hdg}}(T(X)_{\mathbb{Q}})$

Proof. The fact that the above map is a morphism of rings is an easy computation. The only thing left to check is that $\Phi(E)\subset\mathrm{End}_{\mathrm{Hdg}}(T(X)_{\mathbb{Q}})$, and this is equivalent to $\Phi(\sqrt{\Delta_E})\in\mathrm{End}_{\mathrm{Hdg}}(T(X)_{\mathbb{Q}})$. Now,

$$\Phi(\sqrt{\Delta_E})=\frac{1}{f}\begin{bmatrix}-b & -2c\\ 2a & b\end{bmatrix}$$

and we have

$$\frac{1}{f}\begin{bmatrix}-b & -2c\\ 2a & b\end{bmatrix}\begin{bmatrix}s\\ 1\end{bmatrix}=\frac{1}{f}\begin{bmatrix}\frac{\Delta-b\sqrt{\Delta}}{2a}\\ \sqrt{\Delta}\end{bmatrix}=\sqrt{\Delta_E}\begin{bmatrix}s\\ 1\end{bmatrix}$$

□

We are now ready to prove

Theorem 2.5.2. *Let $X/\mathbb{C}$ be a singular K3 surface, let $q(x,y):=ax^2+bxy+cz^2$ the quadratic form associated to a $\mathbb{Z}$-basis of $T(X)$, of discriminant $\Delta=f^2\Delta_E$, with Δ_E the discriminant of the field $E=\mathbb{Q}(\sqrt{\Delta})$. Then*

$$\mathcal{O}(X)\cong\mathbb{Z}+\frac{f}{(a,b,c)}\mathcal{O}_E.$$

In particular, X is principal if and only if $f=(a,b,c)$.

Proof. From the discussion above, we have that the order $\mathcal{O}(X)$ corresponds to

$$\mathcal{O}(X)\cong\left\{x,y\in\mathbb{Q}:\ \begin{bmatrix}x-\frac{b}{f}y & -\frac{2c}{f}y\\ \frac{2a}{f}y & x+\frac{b}{f}y\end{bmatrix}\in M_{2\times 2}(\mathbb{Z})\right\}.$$

This is equivalent to $2x \in \mathbb{Z}$, $\frac{2(a,b,c)}{f} y \in \mathbb{Z}$ and $x - \frac{b}{f} y \in \mathbb{Z}$, i.e.

$$\mathcal{O}(X) \cong \left\{ \frac{x}{2} + \frac{f y}{2(a,b,c)} \sqrt{\Delta_E} : \; x, y \in \mathbb{Z}\,,\; x + \frac{b}{(a,b,c)} y \equiv 0 \bmod 2 \right\}.$$

We also have

$$\left(\frac{b}{(a,b,c)} \right)^2 \equiv \left(\frac{f}{(a,b,c)} \right)^2 \Delta_E \mod 4$$

If $\Delta_E \equiv 0 \bmod 4$ then the above equations forces

$$\left(\frac{f}{(a,b,c)} \right)^2 \equiv \left(\frac{b}{(a,b,c)} \right)^2 \equiv 0 \mod 4$$

and $\mathcal{O}(X)$ corresponds to

$$\mathcal{O}(X) \cong \left\{ \frac{x}{2} + \frac{f y}{2(a,b,c)} \sqrt{\Delta_E} : \; x, y \in \mathbb{Z}\,,\; x \equiv 0 \mod 2 \right\} = \mathbb{Z} + \frac{f}{(a,b,c)} \mathcal{O}_E$$

If $\Delta_E \equiv 1 \mod 4$ and $\frac{f}{(a,b,c)}$ is odd, the order $\mathcal{O}(X)$ corresponds to

$$\mathcal{O}(X) \cong \left\{ \frac{x}{2} + \frac{f y}{2(a,b,c)} \sqrt{\Delta_E} : \; x, y \in \mathbb{Z}\,,\; x + y \equiv 0 \mod 2 \right\} = \mathbb{Z} + \frac{f}{(a,b,c)} \mathcal{O}_{\mathbb{Q}(\sqrt{\Delta'})}$$

And finally, if $\Delta_E \equiv 1 \mod 4$ and $\frac{f}{(a,b,c)}$ is even, $\mathcal{O}(X)$ corresponds to

$$\mathcal{O}(X) \cong \left\{ x + \frac{f y}{2(a,b,c)} \sqrt{\Delta_E} : \; x, y \in \mathbb{Z} \right\} = \mathbb{Z} + \frac{f}{2(a,b,c)} \left(\mathbb{Z} + 2\mathcal{O}_E \right) = \mathbb{Z} + \frac{f}{(a,b,c)} \mathcal{O}_E.$$

□

Corollary 2.5.3. *Let E be an imaginary quadratic extension of $\mathbb{Q}$. Then there are infinitely many $\mathbb{C}$−isomorphism classes of K3 surfaces with CM by $\mathcal{O}_E$.*

Proof. As proved in [39], K3 surfaces with maximal Picard rank correspond bijectively to isomorphism classes of positive-definite oriented even lattices of rank two, via $X \mapsto T(X)$. Let E be any imaginary quadratic field and choose a lattice M like (2.5.0.1), with $E = \mathbb{Q}(\sqrt{b^2 - 4ac})$ and $f = (a, b, c)$. Write X_M for the only K3 surface with $T(X_M) \cong M$. By Theorem (2.5.2), X_M has CM by $\mathcal{O}_E$. But for every $n \in \mathbb{Z}_{>0}$ we have that also X_{nM} has CM by $\mathcal{O}_E$ and X_M is not isomorphic to X_{nM} if $n > 1$. □

In Proposition (3.1.11) we will extend this result to all E with $[E:\mathbb{Q}] \leq 10$.

2.6 Brauer groups

In this section we briefly recall some essential facts about Brauer groups. We refer the reader to Section 4.3. of [10] for a thorough explanation. Let K be a field of characteristic zero, $\overline{K}$ a fixed algebraic closure and G_K its absolute Galois group. For any smooth, geometrically integral variety X/K let $\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$ be its Brauer group, and $\mathrm{Br}(\overline{X}) := H^2_{\text{ét}}(\overline{X}, \mathbb{G}_m)$ be the Brauer group of $\overline{X}$. Both these groups are torsion abelian groups, since X is smooth. The Kummer exact sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$$

gives rise to the short exact sequence

$$0 \to \mathrm{Pic}(\overline{X}) \otimes \mathbb{Z}/n\mathbb{Z} \to H^2_{\text{ét}}(\overline{X}, \mu_n) \to \mathrm{Br}(\overline{X})[n] \to 0,$$

which becomes

$$0 \to \mathrm{NS}(\overline{X}) \otimes \mathbb{Z}/n\mathbb{Z} \to H^2_{\text{ét}}(\overline{X}, \mu_n) \to \mathrm{Br}(\overline{X})[n] \to 0,$$

since $\mathrm{Pic}(\overline{X})$ is an extension of $\mathrm{NS}(\overline{X})$ by a divisible group. After taking projective limits, this implies that

$$\mathrm{Br}(\overline{X}) \cong \left(H^2_{\text{ét}}(\overline{X}, \hat{\mathbb{Z}})(1)/\widehat{\mathrm{NS}}(\overline{X})\right) \otimes \mathbb{Q}/\mathbb{Z}.$$

Let X/k be a K3 surface, and let $\hat{T}(X) := \widehat{\mathrm{NS}}(\overline{X})^{\perp}$ be the orthogonal complement of $\widehat{\mathrm{NS}}(\overline{X}) \subset H^2_{\text{ét}}(\overline{X}, \hat{\mathbb{Z}})(1)$. If there exists an embedding $\overline{K} \hookrightarrow \mathbb{C}$, we have a canonical comparison isomorphism $T(X_{\mathbb{C}}) \otimes \hat{\mathbb{Z}} \cong \hat{T}(\overline{X})$. The intersection pairing together with Lefschetz's $(1,1)$-Theorem leads to an isomorphism

$$\begin{aligned}(H^2_B(X_{\mathbb{C}}, \mathbb{Z})(1)/\,\mathrm{NS}(X_{\mathbb{C}}) &\xrightarrow{\sim} \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Z})\\ v + \mathrm{NS}(X_{\mathbb{C}}) &\mapsto (x \to (x, v)),\end{aligned}$$

so that we have

$$\mathrm{Br}(\overline{X}) \cong \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\hat{T}(\overline{X}), \mathbb{Q}/\mathbb{Z}).$$

Note that from the equation above one gets a natural action of $\mathcal{O}(X)$ on $\mathrm{Br}(\overline{X})$. A Hodge isometry $f: T(X_{\mathbb{C}}) \xrightarrow{\simeq} T(Y_{\mathbb{C}})$ naturally induces two maps on Brauer groups: $f^*: \mathrm{Br}(Y_{\mathbb{C}}) \to \mathrm{Br}(X_{\mathbb{C}})$ given by applying the contravariant functor $\mathrm{Hom}(-.\mathbb{Q}/\mathbb{Z})$ and $f_*: \mathrm{Br}(X_{\mathbb{C}}) \to \mathrm{Br}(Y_{\mathbb{C}})$ given by identifying

$$\mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Z}) \cong \{v \in T(X)_{\mathbb{Q}} : (v, x)_X \in \mathbb{Z} \text{ for all } x \in T(X)\}.$$

They are one the inverse of the other. Assume now that X has CM.

Definition 2.6.1. By a level structure on $T(X)$ we mean a finite subgroup $B \subset \mathrm{Br}(X)$ that is invariant under the action of $\mathcal{O}(X)$.

It is clear that level structures on $T(X)$ corresponds bijectively to free $\mathbb{Z}$-modules Λ

$$\mathrm{Hom}(T(X), \mathbb{Z}) \subset \Lambda \subset \mathrm{Hom}(T(X), \mathbb{Q})$$

that are invariant under the action of $\mathcal{O}(X)$.

Lemma 2.6.2. *Let $X/K \subset \mathbb{C}$ be a K3 surface defined over a number field K, and suppose that X has CM over K. Then* $\mathrm{Br}(\overline{X})^{G_K} \subset \mathrm{Br}(\overline{X})$ *is a level structure on* $T(X_{\mathbb{C}})$.

Proof. By the results in [46], we know that $\mathrm{Br}(\overline{X})^{G_K}$ is finite. If $X/K \subset \mathbb{C}$ has CM over K, then $Br(X_{\overline{K}})^{G_K}$ is also invariant under the $\mathcal{O}(X)$-action, since every cycle in $E(X)$ is defined over K. □

2.7 The main theorem of complex multiplication

In his paper [41], Rizov proves an analogue of the main theorem of complex multiplication for Abelian varieties, in the $K3$ case. As a matter of fact, it is a formal consequence of the fact (also proved by Rizov) that the moduli stack of polarized $K3$ surfaces over $\mathbb{Q}$ is related to the canonical model of the $K3$ Shimura variety via an étale morphism defined over $\mathbb{Q}$ (the period morphism). As pointed out by Madapusi Pera in [26], Rizov's theorem could also be proved using the theory of motives for absolute Hodge cycles, see *loc. cit.* Corollary 4.4. In this section we follow Rizov's and Milne's article 'Introduction on Shimura varieties' (appearing in [1]) notations.

2.7.1 A summary of class field theory

Before stating the main theorem of complex multiplication, let us quickly recall the main statements of class field theory. Let K be a number field. Class field theory provides us with a description of $\mathrm{Gal}(K^{ab}/K)$ by the reciprocity map, which is a surjective, continuous morphism

$$\mathrm{rec}_K : \mathbb{A}_K^\times \to \mathrm{Gal}(K^{ab}/K)$$

whose kernel contains $K^\times$. It induces an isomorphism $\widehat{K^\times\backslash\mathbb{A}_K^\times} \xrightarrow{\sim} \mathrm{Gal}(K^{ab}/K)$, where $\widehat{K^\times\backslash\mathbb{A}_K^\times}$ denotes the profinite completion of $K^\times\backslash\mathbb{A}_K^\times$. For our purposes, it is also useful to introduce the Artin map:

$$\mathrm{art}_K : \mathbb{A}_K^\times \to \mathrm{Gal}(K^{ab}/K) \xrightarrow{\sigma\mapsto\sigma^{-1}} \mathrm{Gal}(K^{ab}/K). \tag{2.7.1.1}$$

The reciprocity map enjoys the following properties

1. If L/K is an Abelian extension, we have a commutative diagram

$$\begin{array}{ccc} K^\times\backslash\mathbb{A}_K^\times & \xrightarrow{\mathrm{rec}_K} & \mathrm{Gal}(K^{ab}/K) \\ \downarrow & & \downarrow{\scriptstyle \sigma\mapsto\sigma_{|L}} \\ K^\times\backslash\mathbb{A}_K^\times/\mathrm{Nm}_{L/K}(\mathbb{A}_K^\times) & \xrightarrow{\sim} & \mathrm{Gal}(L/K). \end{array}$$

2. If v is a prime of K that is unramified in L and $\pi \in K_v$ is a prime element, then the idéle $(\cdots 1 \cdots \pi \cdots 1 \cdots)$ with π at the v-component and 1 elsewhere is sent by rec_K to the Frobenius element $(v, L/K) \in \mathrm{Gal}(L/K)$.

3. If K is totally imaginary, then the reciprocity map factors through the quotient $\mathbb{A}_K^\times \twoheadrightarrow \mathbb{A}_{K,f}^\times$.

Back to our discussion, let V be a finite dimensional $\mathbb{Q}$-vector space and let $h : \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$ be a rational Hodge structure. Suppose that h is special, i.e., that it satisfies the condition in Definition (2.1.9). In particular, there exists a torus $T \subset \mathrm{GL}(V)$ defined over $\mathbb{Q}$ such that the morphism h factors through $T_\mathbb{R}$:

$$h : \mathbb{S} \to T_\mathbb{R} \hookrightarrow \mathrm{GL}(V)_\mathbb{R}.$$

Recall that the reflex field $E(h)$ introduced in (2.1.10) is the field of definition of the composition

$$\mathbb{G}_{m,\mathbb{C}} \xrightarrow{\mu} \mathbb{S}_\mathbb{C} \xrightarrow{h} T_\mathbb{C},$$

so that we can consider the map

$$h\circ\mu\colon\ \mathbb{G}_{m,E(h)} \to T_{E(h)}.$$

By the functoriality of the Weil restriction of scalars, we also have a map

$$\mathrm{Res}_{E(h)/\mathbb{Q}}(h\circ\mu)\colon\ \mathrm{Res}_{E(h)/\mathbb{Q}}(\mathbb{G}_{m,E(h)}) \to \mathrm{Res}_{E(h)/\mathbb{Q}}(T_{E(h)}),$$

and we define the map r'_h as the composition

$$\mathrm{Res}_{E(h)/\mathbb{Q}}(\mathbb{G}_{m,E(h)}) \to \mathrm{Res}_{E(h)/\mathbb{Q}}(T_{E(h)}) \xrightarrow{\mathrm{N}} T,$$

where N is the *Norm map*, acting on $\overline{\mathbb{Q}}$-points as

$$\begin{aligned}\mathrm{Res}_{E(h)/\mathbb{Q}}(T_{E(h)})(\overline{\mathbb{Q}}) \cong \bigoplus_{\sigma\colon\ E(h)\hookrightarrow\overline{\mathbb{Q}}} T(\overline{\mathbb{Q}})_\sigma &\to T(\overline{\mathbb{Q}}) \\ (t_\sigma)_\sigma &\mapsto \prod_\sigma t_\sigma.\end{aligned}$$

Finally, we define $r_h\colon\ \mathbb{A}^\times_{E(h)} \to T(\mathbb{A}_f)$ as the composition

$$\mathbb{A}^\times_{E(h)} = \mathrm{Res}_{E(h)/\mathbb{Q}}(\mathbb{G}_{m,E(h)})(\mathbb{A}) \xrightarrow{r'_h} T(\mathbb{A}) \xrightarrow{\mathrm{proj}} T(\mathbb{A}_f).$$

In our case, where $T = U_E = \mathrm{MT}(X)$, we have

Proposition 2.7.1. *After naturally identifying* $\mathrm{MT}(X)$ *with the norm-1 torus* $\mathrm{U}_E \subset \mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_m$*, we have that the map r corresponds to*

$$\begin{aligned} r\colon\ \mathbb{A}^\times_E &\mapsto \mathbb{A}^\times_{E,f} \\ s &\to \frac{s_f}{\bar{s}_f}\end{aligned}$$

Proof. Remember that the reflex field E is naturally embedded into $\mathbb{C}$, via the evaluation map. Denote by $\tilde{E} \subset \mathbb{C}$ its Galois closure, and consider the natural embedding

$$\begin{aligned} E &\hookrightarrow E \otimes_{\mathbb{Q}} \tilde{E} \\ e &\to e \otimes 1.\end{aligned}$$

We can multiply every element $x \in E \otimes_{\mathbb{Q}} \tilde{E}$ by an element of $e \in E$ in two ways, respectively $e \cdot x$ and $x \cdot e$. Denote by $\mathcal{E} := \{\iota : E \hookrightarrow \tilde{E}\}$ the set of embeddings. The Galois group $G := \mathrm{Gal}(\tilde{E}/\mathbb{Q})$ acts transitively on S by $\iota \mapsto g\iota$. We have a decomposition

$$E \otimes_{\mathbb{Q}} \tilde{E} = \bigoplus_{\iota \in \mathcal{E}} \tilde{E}_\iota$$

where

$$\tilde{E}_\iota = \{x \in E \otimes_{\mathbb{Q}} \tilde{E} : e \cdot x = x \cdot \iota(e) \ \forall\ e \in E\}.$$

One can show that exists a unique element $1_\iota \in \tilde{E}_\iota$ such that the map $\tilde{E} \to \tilde{E}_\iota$, $\tilde{e} \mapsto 1_\iota \cdot \tilde{e}$ is an isomorphism of fields (multiplication on $\tilde{E}_\iota$ being the one induced by $E \otimes_{\mathbb{Q}} \tilde{E}$). If we let G act on the right side, i.e. $g(z \otimes w) := z \otimes g(w)$ for every $g \in G$, we have $g(1_\iota \cdot \tilde{e}) = 1_{g\iota} \cdot g(\tilde{e})$. In particular, the natural embedding $E \hookrightarrow E \otimes_{\mathbb{Q}} \tilde{E}$ becomes

$$E \hookrightarrow \bigoplus_{\iota \in \mathcal{E}} \tilde{E}_\iota \tag{2.7.1.2}$$

$$e \mapsto \oplus_\iota 1_\iota \cdot \iota(e). \tag{2.7.1.3}$$

In our case, denote by $\sigma : E \hookrightarrow \tilde{E}$ the canonical inclusion. The cocharacter is given by

$$\begin{aligned} \mu : E &\to E \otimes E \subset E \otimes \tilde{E} \\ e &\mapsto (1_\sigma \cdot \sigma(e), \cdots, \cdot 1_{\overline{\sigma}} \cdot \sigma(e)^{-1}), \end{aligned}$$

where all the other entries are 1. Denote by $S \subset G$ the stabiliser of ι, the map r' is finally given by

$$\prod_{[g] \in G/S} [g]\mu(e) = \sum_{\iota \in \mathcal{E}} 1_\iota \cdot \iota\left(\frac{e}{\bar{e}}\right) = \frac{e}{\bar{e}},$$

(note that $[g]\sigma(e)$ is well defined) where in the last equality we use the identification (2.7.1.2). □

We can now state the main theorem of CM for $K3$ surfaces:

Theorem 2.7.2 (Rizov). *Let $X/\mathbb{C}$ be a $K3$ surface with complex multiplication and let $E \subset \mathbb{C}$ be its reflex field. Let $\tau \in \mathrm{Aut}(\mathbb{C}/E)$ and $s \in \mathbb{A}^\times_{E,f}$ be a finite idèle such that $\mathrm{art}(s) = \tau_{|E^{ab}}$. There exists a unique Hodge isometry $\eta : T(X)_{\mathbb{Q}} \to T(X^\tau)_{\mathbb{Q}}$ such that the following triangle commutes*

$$\begin{array}{ccc} \widehat{T}(X)_{\mathbb{Q}} & \xrightarrow{\eta\otimes\mathbb{A}_f} & \widehat{T}(X^\tau)_{\mathbb{Q}} \\ {\scriptstyle \frac{s}{\bar{s}}}\uparrow & \nearrow{\scriptstyle \tau^*} & \\ \widehat{T}(X)_{\mathbb{Q}} & & \end{array}$$

where τ^* is the pull-back in étale cohomology of $\tau\colon\ X^\tau \to X$.

Proof. The diagram above, as found in [41], reads a bit differently:

$$\begin{array}{ccc} P_B(X,\mathbb{A}_f)(1) & \xrightarrow{\tilde{\eta}\otimes\mathbb{A}_f} & P_B(X^\tau,\mathbb{A}_f)(1) \\ {\scriptstyle r_X(s)}\uparrow & \nearrow{\scriptstyle \tau^*} & \\ P_B(X,\mathbb{A}_f)(1), & & \end{array}$$

where $P_B(X,\mathbb{A}_f)(1)$ is the primitive cohomology of X with respect to some polarisation $\ell \in \mathrm{NS}(X)$, $\tilde{\eta}\colon\ P_B(X,\mathbb{Q})(1) \to P_B(X^\tau,\mathbb{Q}(1)(1)$ is a Hodge isometry and r_X is the reciprocity map associated to the torus $\mathrm{MT}(P_B(X,\mathbb{Q})(1))$. Now, $P_B(X,\mathbb{Q})(1) = T(X)_{\mathbb{Q}} \oplus A$, where A is the rational $(0,0)$-part of $P_B(X,\mathbb{Q})(1)$, i.e. $A = \{v \in \mathrm{NS}(X)_{\mathbb{Q}} : (v,\ell) = 0\}$. It is therefore clear that the inclusion $T(X)_{\mathbb{Q}} \hookrightarrow P_B(X,\mathbb{Q})(1)$ induces an isomorphism of Mumford-Tate groups

$$\begin{array}{ccc} \mathrm{GL}(T(X)_{\mathbb{Q}}) & \hookrightarrow & \mathrm{GL}(P_B(X,\mathbb{Q})(1)) \\ \uparrow & & \uparrow \\ \mathrm{MT}(T(X)_{\mathbb{Q}}) & \xrightarrow{\cong} & \mathrm{MT}(P_B(X,\mathbb{Q})(1)). \end{array}$$

This identification implies $r_X(s) = \frac{s}{\bar{s}}$ and $\tilde{\eta} = (\eta,\tau^*)$, where $\tau^*\colon\ \mathrm{NS}(X) \to \mathrm{NS}(X^\tau)$ is the pull-back via τ. $\square$

3 Types and class field theory

3.1 Ideal lattices and idèles

Ideal lattices provide the natural framework to work with CM, polarised Hodge structures, since they allow us to faithfully translate the information contained in a polarised CM Hodge structure into some arithmetic data on the CM field. In the summary below, we mainly follow [3].

Definition 3.1.1. Let E be a CM number field, by an ideal lattice we mean the data (I, q) where $I \subset E$ is a fractional ideal and

$$q : I \times I \to \mathbb{R}$$

is a non-degenerate symmetric $\mathbb{Q}$-bilinear form such that $q(\lambda x, y) = q(x, \overline{\lambda} y)$ for every $x, y \in I$ and $\lambda \in \mathcal{O}_E$.

By the non-degeneracy of the trace, it follows that there exists $\alpha \in E$ such that $\alpha = \overline{\alpha}$ and $q(x, y) = \mathrm{tr}_{E/\mathbb{Q}}(\alpha x \overline{y})$. So that, from now on, we will denote with (I, α) the ideal lattice (I, q) with $q(x, y) = \mathrm{tr}_{E/\mathbb{Q}}(\alpha x \overline{y})$.

Definition 3.1.2. An ideal lattice (I, α) is said to be *integral* if q takes value in $\mathbb{Z}$, and *even* if $q(x, x) \in 2\mathbb{Z}$ for every $x \in I$.

Recall that the inverse different ideal $\mathcal{D}_E^{-1}$ is defined to be the maximal fractional ideal of E where $\mathrm{tr}_{E/\mathbb{Q}}$ takes integral values. Hence, if $\alpha \in E$ is like above, (I, α) is integral if and only if

$$(\alpha) I \overline{I} \subset \mathcal{D}_E^{-1}. \tag{3.1.0.1}$$

Let (I, q) be an integral ideal lattice. Its dual is defined as $(I^\vee, q)$ where

$$I^\vee = \{x \in E \ : \ q(x, I) \subset \mathbb{Z}\}. \tag{3.1.0.2}$$

Note that the quadratic form induces a natural isomorphism $I^\vee \xrightarrow{\sim} \mathrm{Hom}(I, \mathbb{Z})$ given by $x \mapsto q(x, -)$. We also have a natural inclusion $(I, q) \subset (I^\vee, q)$. From the definition, it follows that also $(I^\vee, q)$ is an ideal lattice (usually non integral) and that

$$I^\vee = \left(\alpha \overline{I} \mathcal{D}_E\right)^{-1};$$

the inclusion $I \subset \left(\alpha \overline{I} \mathcal{D}_E\right)^{-1}$ is hence also a consequence of (3.1.0.1).

Definition 3.1.3. We say that two ideal lattices (I, α) and (J, β) are equivalent, $(I, \alpha) \cong (J, \beta)$, if there exists $e \in E^\times$ such that $J = eI$ and $\alpha = e\bar{e}\beta$.

This means exactly that multiplication by e

$$e : I \to J$$

is an isometry. Note that the two lattices (I, α) and (J, β) can be isometric without being equivalent (because a general isometry between the two might not be E-linear).

Remark 3.1.4. We will prove later (see Lemma (3.1.9)) that if $(I, \alpha) \cong (J, \beta)$ via $e \in E^\times$, then $(I^\vee, \alpha) \cong (J^\vee, \beta)$ via e as well.

If (I, α) is an ideal lattice, the quotient of Abelian groups $E/I \cong I \otimes \mathbb{Q}/\mathbb{Z}$ is a torsion Abelian group, and also an $\mathcal{O}_E-$module. We now make the analogue of Definition (2.6.1).

Definition 3.1.5. By a level structure on the ideal lattice (I, α) we mean a finite, $\mathcal{O}_E$-invariant subgroup $G \subset I^\vee \otimes \mathbb{Q}/\mathbb{Z}$.

Remark 3.1.6. To give a level structure is equivalent to give a fractional ideal J such that $I^\vee \subset J$, i.e. $J = \pi^{-1}(G)$ where $\pi : E \to E/I^\vee$ is the canonical projection. From now on we will not make any distiction between one or the other definition.

We want now to extend the definition of equivalence keeping track of level structures. So let (I, α, G) and (J, β, H) be two ideal lattices with level structures. We say that $(I, \alpha, G) \cong (J, \beta, H)$ if there exists $e \in E^\times$ as before such that the map induced by multiplication by e

$$E/I^\vee \to E/J^\vee$$

restricts to an isomorphism between G and H.
In general, what we have is a way to 'multiply' an ideal lattice with a level structure

by an element $e \in E^{\times}$ by putting

$$e \cdot (I, \alpha, G) := \left(eI, \frac{\alpha}{e\bar{e}}, eG\right)$$

where eG is the image of G under the map

$$e: \; E/I^{\vee} \to E/eI^{\vee},$$

where the last equation makes sense since $eI^{\vee} = (eI)^{\vee}$ thanks to the above remark. The following facts are well-known, see Lang [21] Chapter 6 or Shimura [43] for a proof.

Proposition 3.1.7. *Let $I, J \subset E$ be fractional ideals. We have:*

1. *For all but finitely many finite places v of E, $I \otimes \mathcal{O}_{E,v} = J \otimes \mathcal{O}_{E,v}$,*
2. *$I \subset J$ if and only if $I \otimes \mathcal{O}_{E,v} \subset J \otimes \mathcal{O}_{E,v}$ for every finite place v,*
3. *If $(I_v)_v$ is a collection of $\mathcal{O}_{E,v}$–modules $I_v \subset E_v$, such that for all but finitely many v's we have that $I_v = \mathcal{O}_{E,v}$, than there exists unique a fractional ideal I such that $I \otimes \mathcal{O}_{E,v} = I_v$ for every v.*

Let now $s \in \mathbb{A}_{E,f}^{\times}$ be a finite idèle and I a fractional ideal. In virtue of the facts above, there exists a unique fractional ideal J such that

$$J_v = s_v \cdot I_v$$

since for all but finitely many v's we have that $s_v \cdot I_v = I_v$. To construct such a J, denote by $s\mathcal{O}_E$ the fractional ideal associated to s:

$$s\mathcal{O}_E = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(s)},$$

then one can see that $J = sI$. To extend the action of $E^{\times}$ on triples (I, α, G) to a subgroup of $\mathbb{A}_{E,f}^{\times}$ containing $E^{\times}$, note first that we have an isomorphism (pag. 77 in Lang's book [21])

$$E/I \cong \bigoplus E_{\mathfrak{p}}/I_{\mathfrak{p}}$$

where the sum is taken over all the prime ideals of $\mathcal{O}_E$, $E_{\mathfrak{p}}$ is the completion of E at $\mathfrak{p}$ and $I_{\mathfrak{p}} := I \otimes \mathcal{O}_{E,\mathfrak{p}}$. So we get a natural homorphism

$$\mathbb{A}_{E,f} \to E/I$$

whose kernel is exactly $\bigoplus I_{\mathfrak{p}}$. If $s \in \mathbb{A}_{E,f}^{\times}$ is an idèle, we have seen before that $J := sI$ is the only fractional ideal of E such that $J_{\mathfrak{p}} = s_{\mathfrak{p}} I_{\mathfrak{p}}$. Hence, we obtain a commutative square

$$\begin{array}{ccc} \mathbb{A}_{E,f} & \longrightarrow & E/I \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \psi} \\ \mathbb{A}_{E,f} & \longrightarrow & E/sI \end{array}$$

where ψ is given at the $\mathfrak{p}$-component by multiplication by $s_{\mathfrak{p}}$. If $G \subset E/I$ is a subgroup, we denote by $sG \subset E/sI$ the image of G under ψ in the diagram above. In order to extend the action of $E^{\times}$, we make the following definition.

Definition 3.1.8. Let $F \subset E$ be the fixed field of the complex conjugation, we define $K_E \subset \mathbb{A}_{E,f}^{\times}$ to be the kernel of

$$\mathbb{A}_{E,f}^{\times} \xrightarrow{\mathrm{Nm}_{E/F}} \mathbb{A}_{F,f}^{\times} \twoheadrightarrow C_F$$

where C_F is the idèle class group of F. Equivalently, $s \in K_E$ if and only if $s\bar{s} \in F^{\times}$

Let now (I, α) be an ideal lattice and $s \in K_E$. Define

$$s \cdot (I, \alpha) := \left(sI, \frac{\alpha}{s\bar{s}} \right).$$

If (I, α) is integral, then also $s \cdot (I, \alpha)$ is integral. We have to prove that this construction commutes with formation of duals.

Lemma 3.1.9. *Let (I, α) be an ideal lattice, and let $s \in K_E$. Then the dual of $s \cdot (I, \alpha)$ is $s \cdot (I, \alpha)^{\vee}$.*

Proof. Indeed, the dual of $s \cdot (I, \alpha)$ is

$$\left((s\bar{s})(\alpha^{-1})\mathcal{D}_E^{-1}(\bar{s}^{-1})\bar{I}^{-1}, \frac{\alpha}{s\bar{s}} \right) = \left((s)(\alpha^{-1})\mathcal{D}_E^{-1}\bar{I}^{-1}, \frac{\alpha}{s\bar{s}} \right)$$

and

$$s \cdot (I, \alpha)^{\vee} = s \cdot \left(\left(\alpha \bar{I} \mathcal{D}_E \right)^{-1}, \alpha \right) = \left((s)(\alpha^{-1})\mathcal{D}_E^{-1}\bar{I}^{-1}, \frac{\alpha}{s\bar{s}} \right).$$

□

This commutativity allows us to make the following definition.

Definition 3.1.10. Let (I, α, G) be an ideal lattice with level structure, and let $s \in K_E$. Then we define

$$s \cdot (I, \alpha, G) := \left(sI, \frac{\alpha}{s\bar{s}}, sG \right),$$

where sG is the image of G under multiplication by s

$$E/I^\vee \to E/sI^\vee = E/(sI)^\vee.$$

Before concluding this section, let us prove the following proposition.

Proposition 3.1.11. *Let E be a CM number field with $[E : \mathbb{Q}] \leq 10$. Then there are infinitely many $\mathbb{C}-$isomorphism classes of principal K3 surfaces with CM by E.*

Proof. This is basically a consequence of Corollary 1.12.3 in Nikulin's famous paper [35]. Let S be an even lattice of signature $(t_{(+)}, t_{(-)})$ and let Λ be an even unimodular lattice of signature $(\ell_{(+)}, \ell_{(-)})$. Nikulin's result says that a primitive embedding $S \hookrightarrow \Lambda$ exists if the following conditions are satisfied:

1. $\ell_{(+)} - \ell_{(-)} \equiv_8 0$;
2. $\ell_{(-)} - t_{(-)} \geq 0$ and $\ell_{(+)} - t_{(+)} \geq 0$;
3. Let g be the minimum number of generators of $S^\vee/S$. Then $\ell_{(+)} + \ell_{(-)} - t_{(+)} - t_{(-)} > g$.

Note that, in our case, where the lattices are non-degenerate, $\ell_{(+)} + \ell_{(-)} = \text{rank}(\Lambda)$ and $t_{(+)} + t_{(-)} = \text{rank}(S)$. Moreover, $g \leq \text{rank}(S)$ always. In particular, a primitive embedding exists every time that

$$\text{rank}(\Lambda) > 2 \cdot \text{rank}(S) \qquad (3.1.0.3)$$

Let us now prove the proposition. Write $[E : \mathbb{Q}] = 2n$ with $n \leq 5$. Consider $\alpha \in F^\times$ with $\alpha \mathcal{D}_E \subset \mathcal{O}_E$, so that the ideal lattice $(\mathcal{O}_E, \alpha)$ is an even integral lattice, and assume that for only one embedding $\sigma' : F \hookrightarrow \mathbb{C}$ we have $\sigma'(\alpha) > 0$. Let us denote by $\sigma : E \hookrightarrow \mathbb{C}$ an extension of σ' (the other extension will be given by $\bar{\sigma}$). This choice of α ensures that the signature of $(\mathcal{O}_E, \alpha)$ is $(2, 2n-2)$. We would like to produce an algebraic K3 surface using the surjectivity of the period map. To do so, let us write Λ for the K3 lattice, which is isomorphic to $H^2(X, \mathbb{Z})$ for any K3 surface $X/\mathbb{C}$. It is an even-unimodular lattice of rank 22 and signature $(3, 19)$, and every pure Hodge structure of weight 2 on Λ, such that $\dim_{\mathbb{C}} \Lambda^{2,0} = 1$ and for which the pairing $\Lambda \times \Lambda \to \mathbb{Z}$ induces a polarisation, comes from a complex K3 surface. Since $[E : \mathbb{Q}] \leq 10$ and by the choice of α, we easily see that conditions $1, 2, 3$ above are satisfied, so we can find a primitive embedding of lattices $(\mathcal{O}_E, \alpha) \subset \Lambda$. We want now to endow Λ with a Hodge structure which corresponds to a K3 surface with CM by $\mathcal{O}_E$. To do so, consider again the decomposition

$$\mathcal{O}_E \otimes \mathbb{C} = \bigoplus_{\tau :\ E \to \mathbb{C}} \mathbb{C}_\tau$$

and put $\Lambda^{2,0} := \mathbb{C}_\sigma$, where we consider $\mathcal{O}_E \otimes \mathbb{C} \subset \Lambda \otimes \mathbb{C}$. Let us call X the corresponding K3 surface. It is easy to show that $T(X) = (\mathcal{O}_E, \alpha)$, and that $\mathrm{End}_{\mathrm{Hdg}}(T(X)) = \mathcal{O}_E$. To show that this K3 surface is algebraic it is sufficient to find a class $L \in \mathrm{NS}(X)$ with $L^2 > 0$ by Theorem IV.6.2 of [2]. But this class must exists because the signature of $\mathrm{NS}(X)$ is $(1, 21-2n)$, so that X is algebraic. Finally, note that we can produce infinitely many α's such that the ideal lattices $(\mathcal{O}_E, \alpha)$ are pairwise non-isomorphic, so that we obtain infinitely many $\mathbb{C}-$isomorphism classes of K3 surfaces with CM by $\mathcal{O}_E$. □

Remark 3.1.12. It is not known whether the same proposition is true in any degree. The best result in this direction so far is the one of Taelman [49] already mentioned in the introduction: for every CM number field E with $[E : \mathbb{Q}] \leq 20$ there exist a complex K3 surface with CM by E. Moreover, if $[E : \mathbb{Q}] \leq 18$, there are infinitely isomorphism classes of complex K3 surfaces with CM by E.

3.2 Type of a principal K3 surface with CM

In this section we introduce the *type* of a K3 surface with CM. We have seen during the proof of Proposition (3.1.11) how to construct Hodge structures of K3 type with CM starting from an integral ideal lattice: one starts with a CM number field E and an embedding $\sigma :\ E \hookrightarrow \mathbb{C}$, and consider an ideal lattice with level structure (I, α, G), with $\alpha \in F^\times$ such that $(\alpha)\mathcal{D}_E \subset \mathcal{O}_E$ and only $\sigma, \bar{\sigma} :\ E \hookrightarrow \mathbb{C}$ satisfies $\sigma(\alpha) > 0$. To this data we can associate a polarised Hodge structure of weight zero together with a 'level structure' that we will denote by (I, α, G, σ), with $I^{1,-1} = \mathbb{C}_\sigma$. Let now (X, B, ι) be a principal CM $K3$ surface $X/\mathbb{C}$ with level structure $B \subset \mathrm{Br}(X)$ and an isomorphism $\iota :\ E \to E(X)$. Via the map ι, we consider $T(X)$ an $\mathcal{O}_E$-module.

Definition 3.2.1. We say that $(T(X), B, \iota)$ is of type (I, α, G, σ) if there exists an isomorphism of $\mathcal{O}_E$−modules

$$\Phi : T(X) \xrightarrow{\sim} I$$

such that:

1. $(v, w)_X = \mathrm{tr}_{E/\mathbb{Q}}\left(\alpha\Phi(v)\overline{\Phi(w)}\right)$ for every $v, w \in T(X)$;
2. If $\Phi^\vee : T(X)^\vee \to I^\vee$ is the induced map on dual lattices, then

   $$\Phi^\vee \otimes \mathbb{Q}/\mathbb{Z} : E/I^\vee \to \mathrm{Br}(X)$$

   sends isomorphically G to B;
3. $\sigma_X \circ \iota = \sigma$.

Remarks 3.2.2.

1. Here, with $\Phi^\vee$ we mean the induced map

   $$T(X)^\vee = \{v \in T(X)_{\mathbb{Q}} : (v, x) \in \mathbb{Z} \text{ for all } x \in T(X)\} \to I^\vee,$$

   where $I^\vee$ was defined in (3.1.0.2).
2. It may seem that fixing an abstract field E together with the maps ι and σ is redundant; to every K3 surface $X/\mathbb{C}$ with CM, we have canonically associated its reflex field E (already embedded in $\mathbb{C}$) together with an isomorphism $\sigma_X : E(X) \to E$. We chose this definition to keep track of the Aut($\mathbb{C}$)-action on $E(X)$: if $\tau \in \mathrm{Aut}(\mathbb{C})$, we put $(T(X), B, \iota)^\tau = (T(X^\tau), \tau_* B, \tau^{ad} \circ \iota)$. See Lemma (3.2.4). Also, one can check that proposition (3.2.6) would not hold without fixing such an isomorphism.
3. Every CM $K3$ surface has a type: let $E \xrightarrow{\sigma} \mathbb{C}$ be its reflex field, put $\iota := \sigma_X^{-1}$ and choose $0 \neq v \in T(X)$. The inverse image of $T(X)$ under the isomorphism $E \to T(X)_{\mathbb{Q}}$, $e \mapsto \iota(e) \cdot v$ is a lattice in E fixed by $\mathcal{O}_E$, hence a fractional ideal. By the non-degeneracy of the trace, we can find unique $\alpha \in E$ as in Definition (3.2.1).

Definition 3.2.3. Let $X, Y/\mathbb{C}$ be two principal K3 surfaces with CM. We say that the two triples $(T(X), B, \iota_X)$ and $(T(Y), C, \iota_Y)$ are isomorphic if there exists a Hodge isometry $f : T(X) \xrightarrow{\simeq} T(Y)$ such that

1. $f^{ad} \circ \iota_X = \iota_Y$, where $f^{ad} : E(X) \to E(Y)$ is the induced isomorphism and

2. $f_* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ restricts to an isomorphism between B and C, where f_* is the induced map on Brauer groups (introduced in the discussion before Definition 2.6.1).

The following is an instance of point 1) in the remark above.

Lemma 3.2.4. *Let $X/\mathbb{C}$ be a principal K3 surface with CM and let $\iota : E \to E(X)$ be an isomorphism. Let $\tau \in \mathrm{Aut}(\mathbb{C})$, and suppose that $(T(X), \iota) \cong (T(X^\tau), \tau^{ad} \circ \iota)$. Then τ fixes the reflex field of X.*

Proof. Since f is a Hodge isometry, we have that $\sigma_X = \sigma_{X^\tau} \circ f^{ad}$. During the proof of Proposition (2.4.8), we have also proved that $\sigma_{X^\tau} \circ \tau^{ad} = \tau \circ \sigma_X$. By assumption, we have $f^{ad} \circ \iota = \tau^{ad} \circ \iota$, i.e. $f^{ad} = \tau^{ad}$. Hence, $\sigma_X = \tau \circ \sigma_X$, i.e. τ fixed the reflex field of X. □

Note that if X can be defined over $\mathbb{Q}$, then $T(X) \cong T(X^\tau)$ for every $\tau \in \mathrm{Aut}(\mathbb{C})$.

Lemma 3.2.5. *Suppose that $(T(X), B, \iota)$ is of type (I, α, G, σ) and let Φ and Φ' be two maps as in Definition (3.2.1). Then there exists a root of unity $\mu \in \mathcal{O}_E^\times$ such that $\Phi = \mu \Phi'$.*

Proof. Indeed, the map $\Phi' \circ \Phi^{-1} : (I, \alpha) \to (I, \alpha)$ is an integral isometry, hence a root of unity. □

We are ready to prove the following proposition.

Proposition 3.2.6. *Let $(T(X), B, \iota_X)$ be of type (I, α, G, σ) and let $(T(Y), C, \iota_Y)$ be of type (J, β, H, θ). Then $(T(X), B, \iota_X) \cong (T(Y), C, \iota_Y)$ if and only if $(I, \alpha, G) \cong (J, \beta, H)$ and $\sigma = \theta$.*

Proof. Let us prove the implication $(T(X), B, \iota_X) \cong (T(Y), C, \iota_Y) \Rightarrow (I, \alpha, G) \cong (J, \beta, H)$ and $\sigma = \theta$. Consider the square

$$\begin{array}{ccc} T(X) & \xrightarrow{\Phi_X} & I \\ \big\downarrow{\scriptstyle f} & & \big\downarrow \\ T(Y) & \xrightarrow{\Phi_Y} & J, \end{array}$$

where f is a map as in Definition (3.2.3) and Φ_X, Φ_Y are the maps realising the types of X and Y respectively. By linearity, we see that the dashed arrow is induced by multiplication by some $e \in E^\times$, which is also an isometry between the two ideal lattices (I, α) and (J, β) , i.e. $eI = J$ and $e\bar{e}\beta = \alpha$. The induced square on Brauer groups is

$$\begin{array}{ccc} \mathrm{Br}(X) & \xrightarrow{\Phi_X^\vee} & E/I^\vee \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle e} \\ \mathrm{Br}(Y) & \xrightarrow{\Phi_Y\vee} & E/J^\vee, \end{array}$$

which implies $eG = H$, since $f_*(B) = C$, $\Phi_X^*(B) = G$ and $\Phi_Y\vee(C) = H$. By the definition of type, we have that $\sigma_X \circ \iota_X = \sigma$ and $\sigma_Y \circ \iota_Y = \theta$. Moreover, we also have $f^{ad} \circ \iota_X = \iota_Y$ (by Definition (3.2.3)) and $\sigma_X = \sigma_Y \circ f^{ad}$ (since f is a Hodge isometry). Hence, we see that $\sigma = \theta$. On the other hand, suppose that $(I, \alpha, G) \cong (J, \beta, H)$ and that $\sigma = \theta$, and let $e \in E^\times$ be an element realising the equivalence. Consider the diagram

$$\begin{array}{ccc} T(X) & \xrightarrow{\Phi_X} & I \\ \downarrow & & \downarrow{\scriptstyle e} \\ T(Y) & \xrightarrow{\Phi_Y} & J, \end{array}$$

and call f the dashed arrow. Then, f is an isometry between the lattices $T(X)$ and $T(Y)$ and satisfies condition 2 in Definition (3.2.3). We need to prove that it respects the Hodge decomposition and that $f^{ad} \circ \iota_X = \iota_Y$. Since $\sigma = \theta$, we have that $\sigma_X \circ \iota_X = \sigma_Y \circ \iota_Y$. Let $0 \neq \omega \in T^{1,-1}(X)$ be a non-zero two form, and let $x \in E$. We want to prove that $\iota_Y(x) \cdot f(\omega) = \sigma_Y(\iota_Y(x)) f(\omega)$. We have

$$\iota_Y(x) \cdot f(\omega) = \iota_Y(x) \cdot \Phi_Y^{-1}\big(e\Phi_X(\omega)\big) = \Phi_Y^{-1}\big(xe\Phi_X(\omega)\big) = \Phi_Y^{-1}\big(e\Phi_X(\iota_X(x) \cdot \omega)\big) =$$

$$= f(\iota_X(x) \cdot \omega) = f(\sigma_X(\iota_X(x))\omega) = \sigma_X(\iota_X(x)) f(\omega) = \sigma_Y(\iota_Y(x)) f(\omega).$$

Hence, f respects the Hodge decomposition. As a consequence of this, we must also have that $\sigma_X = \sigma_Y \circ f^{ad}$. Pre-composing with ι_X and using again the fact that $\sigma_X \circ \iota_X = \sigma_Y \circ \iota_Y$, we conclude. □

3.3 Main theorem of CM for K3 surfaces (after Shimura)

The next step is to translate Theorem (2.7.2) in the language of ideal lattices.

Theorem 3.3.1. *Let $X/\mathbb{C}$ be a principal $K3$ surface with complex multiplication and reflex field $E \subset \mathbb{C}$. Let $\tau \in \mathrm{Aut}(\mathbb{C}/E)$ and let $s \in \mathbb{A}_{E,f}^\times$ be a finite idèle* *such that* $\mathrm{art}(s) = \tau_{|E^{ab}}$. *Suppose that* $(T(X), B, \iota)$ *is of type* (I, α, G, σ). *Then* $(T(X^\tau), \tau_* B, \tau^{ad} \circ \iota)$ *is of type*

$$\frac{s}{\bar{s}} \cdot \big(I, \alpha, G, \sigma\big).$$

Moreover if Φ_X *is a map realising the type of* X, *there exists a unique map* Φ_{X^τ} *realising the above type of* X^τ, *such that the following commutes*

$$\begin{array}{ccc} \mathrm{Br}(X) & \xrightarrow{\Phi_X^\vee} & E/I^\vee \\ \downarrow{\scriptstyle \tau_*} & & \downarrow{\scriptstyle \frac{s}{\bar{s}}} \\ \mathrm{Br}(X^\tau) & \xrightarrow{\Phi_{X^\tau}^\vee} & E/\frac{s}{\bar{s}} I^\vee \end{array}$$

Proof. Rizov's Theorem tells us that there exists a unique Hodge isometry $\eta : T(X)_\mathbb{Q} \to T(X^\tau)_\mathbb{Q}$ such that the following diagram commutes

$$\begin{array}{ccc} \widehat{T}(X)_\mathbb{Q} & \xrightarrow{\eta \otimes \widehat{\mathbb{Z}}} & \widehat{T}(X^\tau)_\mathbb{Q} \\ \uparrow{\scriptstyle \frac{s}{\bar{s}}} & \nearrow{\scriptstyle \tau^*} & \\ \widehat{T}(X)_\mathbb{Q} & & \end{array}$$

If we consider $\widehat{T}(X) \subset \widehat{T}(X)_\mathbb{Q}$ and $\widehat{T}(X^\tau) \subset \widehat{T}(X^\tau)_\mathbb{Q}$, then the Galois action τ^* restricts to an isomorphism of $\widehat{\mathbb{Z}}$-lattices

$$\tau^* : \widehat{T}(X) \xrightarrow{\sim} \widehat{T}(X^\tau)_\mathbb{Q}.$$

This means that the two lattices $T(X^\tau)$ and $\eta\big(\frac{s}{\bar{s}}\widehat{T}(X)\big) \cap T(X^\tau)_\mathbb{Q}$ inside $\widehat{T}(X^\tau)_\mathbb{Q}$ are actually the same. Since both η and multiplication by $\frac{s}{\bar{s}}$ are isometries and since τ fixes the reflex field by assumptions, we must have that the type of $(T(X^\tau), \tau_* B, \tau^{ad} \circ \iota)$ is

$$\frac{s}{\bar{s}} \cdot \Bigg(I, \alpha, G, \sigma\Bigg).$$

Choose a map Φ'_{X^τ} realising the above type for X^τ.

Claim: there exists a unique root of unity $\mu \in \mathcal{O}_E^\times$ such that the following commute

$$\begin{array}{ccc} \widehat{T}(X) & \xrightarrow{\tau^*} & \widehat{T}(X^\tau) \\ \downarrow{\scriptstyle \Phi_X \otimes \widehat{\mathbb{Z}}} & & \downarrow{\scriptstyle \Phi'_{X^\tau} \otimes \widehat{\mathbb{Z}}} \\ I \otimes \widehat{\mathbb{Z}} & \xrightarrow[\frac{s}{\bar{s}}\mu]{} & \frac{s}{\bar{s}} I \otimes \widehat{\mathbb{Z}}. \end{array}$$

Indeed, consider the following

$$\begin{array}{ccc} T(X)_{\mathbb{Q}} & \xrightarrow{\quad\eta\quad} & T(X^{\tau})_{\mathbb{Q}} \\ \Big\downarrow{\scriptstyle \Phi_X\otimes\mathbb{Q}} & & \Big\downarrow{\scriptstyle \Phi'_{X^{\tau}}\otimes\mathbb{Q}} \\ E = I\otimes\mathbb{Q} & \dashrightarrow & \frac{s}{\bar{s}} I\otimes\mathbb{Q} = E, \end{array}$$

We can complete the dashed arrow uniquely with multiplication by some element $\mu \in E^{\times}$ with $\mu\bar{\mu} = 1$, since η is a Hodge isometry. Everything now fits into the commutative diagram

$$\begin{array}{ccccc} & & \widehat{T}(X)_{\mathbb{Q}} & & \\ & \swarrow{\scriptstyle \frac{s}{\bar{s}}} & \Big| & \searrow{\scriptstyle \tau^*} & \\ \widehat{T}(X)_{\mathbb{Q}} & & \xrightarrow{\qquad\eta\otimes\mathbb{A}_f\qquad} & & \widehat{T}(X^{\tau})_{\mathbb{Q}} \\ & & \Big\downarrow{\scriptstyle \Phi_X\otimes\mathbb{A}_f} & & \\ \Big\downarrow{\scriptstyle \Phi_X\otimes\mathbb{A}_f} & & I_{\mathbb{A}_f} & & \Big\downarrow{\scriptstyle \Phi'_{X^{\tau}}\otimes\mathbb{A}_f} \\ & \swarrow{\scriptstyle \frac{s}{\bar{s}}} & & \searrow{\scriptstyle \frac{s}{\bar{s}}\mu} & \\ I_{\mathbb{A}_f} & & \xrightarrow{\qquad\mu\qquad} & & \frac{s}{\bar{s}} I_{\mathbb{A}_f}. \end{array}$$

And we see that μ must send the $\widehat{\mathbb{Z}}-$lattice $\frac{s}{\bar{s}} I_{\widehat{\mathbb{Z}}}$ into itself, because τ^* does so. So that $\mu \in (\widehat{\mathcal{O}_E} \cap E = \mathcal{O}_E$. The condition $\mu\bar{\mu} = 1$ forces μ to be a root of unity. Now put $\Phi_{X^{\tau}} := \mu \cdot \Phi'_{X^{\tau}}$. We obtain another commutative diagram analogous to the one above

$$\begin{array}{ccccc} & & \widehat{T}(X)_{\mathbb{Q}} & & \\ & \swarrow{\scriptstyle \frac{s}{\bar{s}}} & \Big| & \searrow{\scriptstyle \tau^*} & \\ \widehat{T}(X)_{\mathbb{Q}} & & \xrightarrow{\qquad\eta\otimes\mathbb{A}_f\qquad} & & \widehat{T}(X^{\tau})_{\mathbb{Q}} \\ & & \Big\downarrow{\scriptstyle \Phi_X\otimes\mathbb{A}_f} & & \\ \Big\downarrow{\scriptstyle \Phi_X\otimes\mathbb{A}_f} & & I_{\mathbb{A}_f} & & \Big\downarrow{\scriptstyle \Phi_{X^{\tau}}\otimes\mathbb{A}_f} \\ & \swarrow{\scriptstyle \frac{s}{\bar{s}}} & & \searrow{\scriptstyle \frac{s}{\bar{s}}} & \\ I_{\mathbb{A}_f} & & \xrightarrow{\qquad 1\qquad} & & \frac{s}{\bar{s}} I_{\mathbb{A}_f}, \end{array}$$

so that $\Phi_{X^{\tau}}$ is the required map. The unicity comes from Lemma (3.2.5). $\square$

3.4 K3 class group and K3 class field

Before starting this section, let us fix some classical notations from class field theory that we are going to use through the rest of this thesis. Let E/F be a cyclic extension of number fields and write $G := \mathrm{Gal}(E/F) = \langle\sigma\rangle$. In this section, E will always be a CM field and F its maximal totally real subfield, but in Section (3.5) it will just be a general cyclic extension and most of these notations will not be used until then. Let $I \subset \mathcal{O}_E$ be an ideal, we denote by

- $\mathcal{I}_E$ the group of fractional ideals of E;
- $\mathcal{I}_E^I \subset \mathcal{I}_E$ the group of fractional ideals coprime to I;
- $E^I := \{e \in E^\times : e\mathcal{O}_E \in \mathcal{I}_E^I\}$;
- $E^{I,1} := \{e \in E^\times : v(e-1) \geq v(I)\ \forall$ finite valuations such that $v(I) > 0\}$;
- $\mathcal{O}_E^I := \mathcal{O}_E^\times \cap E^{I,1}$;
- $\mathcal{P}_E^I := \{e\mathcal{O}_E : e \in E^{I,1}\} \subset \mathcal{I}_E^I$;
- $\mathrm{Cl}_I(E) := \mathcal{I}_E^I/\mathcal{P}_E^I$ the ray class group modulo I;
- We say that the ideal I is *invariant* if $\sigma(I) = I$;
- If I is invariant, we denote by $\mathrm{Cl}'_I(E) := \mathrm{Cl}_I(E)/\mathrm{Cl}_I(E)^G$. In particular, we have $\mathrm{Cl}'(E) := \mathrm{Cl}(E)/\mathrm{Cl}(E)^G$;
- $N : E^\times \to F^\times$ the norm morphism.
- If $I \subset \mathcal{O}_E$ is a proper ideal, we will denote its support by

$$S(I) := \{p \text{ prime ideal of } E : I \subset p\}.$$

- If $\mathfrak{m}$ is a modulus for $\mathcal{O}_F$, i.e. a formal product of a proper ideal and archimedean valuations, we will denote by $e(E/F, \mathfrak{m}) := \prod_{v \nmid \mathfrak{m}} e(v)$, where the product is taken over all the places (both finite and archimedean) of F that do not divide $\mathfrak{m}$ and $e(v)$ denotes their ramification index in the field extension E/F;
- Let E be any number field, for every ideal $I \subset \mathcal{O}_E$ we denote by $\phi_E(I) := |(\mathcal{O}_E/I)^\times|$ the associated Euler's totient function.

Given a CM number field E, Theorem (3.3.1) suggests the introduction of a class group (as meant in Shimura's book [43]) which we will call the K3 class group $G_{K3}(E)$ of E, and of its related class field, an Abelian extension of E obtained via class field theory, with Galois group isomorphic to $G_{K3}(E)$. These object will be of essential use later on, especially in the computations of the fields of moduli in the next section. In order to introduce them, we recall that by

$$\mathrm{U}_E \subset \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$$

we mean E-linear unitary group, cut out by the equation $e\bar{e} = 1$.

Definition 3.4.1. Let E be a CM number field. We define the K3 class group of E to be the double coset

$$G_{K3}(E) := \mathrm{U}_E(\mathbb{Q}) \backslash \mathrm{U}_E(\mathbb{A}_f)/\tilde{\mathrm{U}},$$

where $\tilde{\mathrm{U}}$ is the subgroup generated by all the $u \in \mathrm{U}_E(\mathbb{A}_f)$ such that for every finite place v, u_v is a unit, i.e., $\tilde{\mathrm{U}} = \{u \in \mathrm{U}_E(\mathbb{A}_f) : u\mathcal{O}_E = \mathcal{O}_E\}$.

There is a canonical, continuous map from the finite idèles of E to $G_{K3}(E)$, namely

$$\begin{aligned} \mathbb{A}^{\times}_{E,f} &\to G_{K3}(E) \\ s &\mapsto \frac{s}{\bar{s}}, \end{aligned} \tag{3.4.0.1}$$

which is a surjection due to Hilbert's Theorem 90 for idèles.

Definition 3.4.2. The kernel of the above map $\mathbb{A}^{\times}_{E,f} \to G_{K3}(E)$ is denoted by S_E. We have

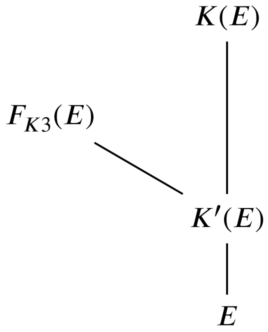
$$S_E = \{s \in \mathbb{A}^{\times}_{E,f} : \exists e \in U_E(\mathbb{Q}) : e\frac{s}{\bar{s}}\mathcal{O}_E = \mathcal{O}_E\}$$

Note that also $E^{\times} \subset S_E$.

Definition 3.4.3. The Abelian extension of E obtained via class field theory from the subgroup S_E of $\mathbb{A}^{\times}_{E,f}$ is denoted by $F_{K3}(E)$. We call it the *K3 Class Field* of E.

Understanding these class fields (the one just introduced and the others to come) will occupy the next two sections. The first step is to relate them to the Abelian extensions of E that we already know, i.e. ray class fields. As a first step, we have:

Proposition 3.4.4. *Denote by $K(E)$ the Hilbert Class field of E and by $K'(E)$ the sub-extension of $K(E)$ with Galois group $\cong \mathrm{Cl}'(E)$. We have a diagram of Abelian extensions*



with

$$\mathrm{Gal}(F_{K3}(E)/K'(E)) \cong \frac{\mathcal{O}_F^\times \cap N(E^\times)}{N(\mathcal{O}_E^\times)}$$

.

Proof. Indeed, consider the group

$$\tilde{S}_E = \{s \in \mathbb{A}_{E,f}^\times : \exists e \in E^\times : e\frac{s}{\bar{s}}\mathcal{O}_E = \mathcal{O}_E\}$$

Clearly, $S_E \subset \tilde{S}_E$. The first step is to understand the quotient $\tilde{S}_E/S_E$. Let $s \in \tilde{S}_E$ and consider $e \in E^\times$ such that $e\frac{s}{\bar{s}}\mathcal{O}_E = \mathcal{O}_E$. We must have $(e\bar{e}) = \mathcal{O}_E$, i.e. $e\bar{e} \in \mathcal{O}_F^\times \cap N(E^\times)$. If $e' \in E^\times$ is another element such that $e'\frac{s}{\bar{s}}\mathcal{O}_E = \mathcal{O}_E$, then e' and e differ by a unit, $e' = eu$ with $u \in \mathcal{O}_E^\times$, and $e'\overline{e'} = u\bar{u}e\bar{e}$. We have constructed a well-defined map

$$f : \tilde{S}_E \to \frac{\mathcal{O}_F^\times \cap N(E^\times)}{N(\mathcal{O}_E^\times)} \tag{3.4.0.2}$$
$$s \mapsto e\bar{e}.$$

Note that $\frac{\mathcal{O}_F^\times \cap N(E^\times)}{N(\mathcal{O}_E^\times)}$ is a finite $2-$torsion Abelian group. Hence it isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ for some $n \in \mathbb{N}$.
The map f is surjective: let $x \in \mathcal{O}_F^\times \cap N(E^\times)$ and write $x = y\bar{y}$ with $y \in E^\times$. By Hilbert's theorem 90 for ideals (see [9], p. 284) we can find a fractional ideal I such that $I/\bar{I} = (y)$. Pick $s \in \mathbb{A}_{E,f}^\times$ with $s\mathcal{O}_E = I$, then $f(s) = x$.

Claim: the kernel of the map (3.4.0.2) is S_E.
Indeed, $s \in \tilde{S}_E$ is in the kernel if and only if there exists $e \in E^\times$ such that $e\frac{s}{\bar{s}}\mathcal{O}_E = \mathcal{O}_E$ and $e\bar{e} = u\bar{u}$ for some $u \in \mathcal{O}_E^\times$. But consider now $e' := \frac{e}{u}$, then clearly also $e'\frac{s}{\bar{s}}\mathcal{O}_E = \mathcal{O}_E$, and moreover $e'\overline{e'} = 1$, i.e. $s \in S_E$.
The next step, and final one, is to understand the to which Abelian extension of E the group $\tilde{S}_E$ is associated. Consider the natural projection maps

$$\mathbb{A}_{E,f}^\times \twoheadrightarrow \mathrm{Cl}(E) \twoheadrightarrow \mathrm{Cl}'(E).$$

Claim: the kernel of the above composition is $\tilde{S}_E$.
Indeed, $s \in \mathbb{A}_E^\times$ lies in the kernel if and only if the frational ideals associated to s and $\bar{s}$ are the same in the class group of E, i.e. if and only if exist $e \in E^\times$ such that $e\frac{s}{\bar{s}}\mathcal{O}_E = \mathcal{O}_E$. This completes the proof. $\square$

In particular, we obtain the following equality

$$|G_{K3}(E)| = [\mathcal{O}_F^\times \cap N(E^\times) : N(\mathcal{O}_E^\times)] \cdot |\mathrm{Cl}'(E)|. \tag{3.4.0.3}$$

Remark 3.4.5. If E is imaginary quadratic, then

$$\frac{\mathcal{O}_F^\times \cap N(E^\times)}{N(\mathcal{O}_E^\times)} = 1$$

There are other class fields and class groups associated to E which are analogous to the usual ray class fields and ray class groups modulo some ideal $I \subseteq \mathcal{O}_E$. Fix an ideal $I \subseteq \mathcal{O}_E$ and denote by $\tilde{\mathrm{U}}_I$ the subgroup generated by all the $u \in \mathrm{U}_E(\mathbb{A}_f)$ such that for every finite place v, u_v is a unit and if $v(I) = n > 0$, then $v(u_v - 1) \geq n$.

Definition 3.4.6. We define the K3 class group modulo I to be the double quotient

$$G_{K3,I}(E) := \mathrm{U}_E(\mathbb{Q}) \backslash \mathrm{U}_E(\mathbb{A}_f)/\tilde{\mathrm{U}}_I,$$

and the K3 class field to be the Abelian extension $F_{K3,I}(E)$ of E associated to the surjection

$$\begin{aligned} \mathbb{A}_{E,f}^\times &\twoheadrightarrow G_{K3,I}(E) \\ s &\mapsto \frac{s}{\bar{s}} \end{aligned}$$

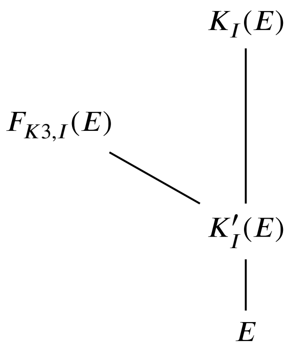
We study now these Abelian extensions. We start by noticing that if we put

$J := \mathrm{lcm}(I, \overline{I})$, then we have, straight from the definition, that

$$G_{K3,I}(E) = G_{K3,J}(E) = G_{K3,\overline{I}}(E).$$

So that, without loss of generality, we can assume that I is invariant.

Proposition 3.4.7. *Denote by $K_I(E)$ the ray Class field of E modulo I and by $K'_I(E)$ the sub-extension of $K_I(E)$ with Galois group $\cong \mathrm{Cl}'_I(E)$. We have a diagram of Abelian extension*



with

$$\mathrm{Gal}(F_{K3,I}(E)/K'_I(E) \cong \frac{\mathcal{O}_F^\times \cap N(E^{I,1})}{N(\mathcal{O}_E^I)}.$$

Proof. The first thing to understand is the kernel of the map $\mathbb{A}_{E,f}^\times \twoheadrightarrow G_{K3,I}(E)$. If we denote it by S_I, we have

$$S_I = \left\{ s \in \mathbb{A}_E^\times \,:\, \exists e \in \mathrm{U}_E(\mathbb{Q}) :\, \frac{s}{\bar{s}} e\mathcal{O}_E = \mathcal{O}_E,\, e\frac{s}{\bar{s}} \equiv 1 \mod I \right\}.$$

Using the same ideas as before, we denote by $\tilde{S}_I$ the group

$$\tilde{S}_I = \left\{ s \in \mathbb{A}_E^\times \,:\, \exists e \in E^\times \,:\, \frac{s}{\bar{s}} e\mathcal{O}_E = \mathcal{O}_E,\, e\frac{s}{\bar{s}} \equiv 1 \mod I \right\}.$$

We again have an injection

$$\tilde{S}_I / S_I \hookrightarrow \frac{\mathcal{O}_F^\times \cap N(E^{I,1})}{N(\mathcal{O}_E^I)},$$

and we need to prove surjectivity. As in the proof of Proposition (3.4.4), let $x \in \mathcal{O}_F^\times \cap N(E^{I,1})$ and let $y \in E^{I,1}$ be such that $y\bar{y} = x$ and find a fractional ideal J of E such that $J/\overline{J} = (y)$. We need J to be in $\mathcal{I}_E^I$ in order to conclude, so suppose it

is not.
Claim: there exists an invariant fractional ideal $\mathfrak{a}$ such that $\mathfrak{a}|J$ and $J/\mathfrak{a}$ is coprime to I.
Indeed, let $\mathfrak{p}$ be a prime ideal of E, suppose that $v_{\mathfrak{p}}(\gcd(I, J)) \neq 0$ and let n be the power of $\mathfrak{p}$ appearing in the factorisation of J. If $\overline{\mathfrak{p}} = \mathfrak{p}$, then the ideal $J' = J/\mathfrak{p}^n$ has still the property that we need, i.e. $J'/\overline{J'} = (y)$, and J' has no $\mathfrak{p}-$factor in common with I. If $\mathfrak{p} \neq \overline{\mathfrak{p}}$, write again $J' = J/\mathfrak{p}^n$ and consider

$$(y) = J/\overline{J} = (J'/\overline{J'})(\mathfrak{p}^n/\overline{\mathfrak{p}}^n).$$

Since by construction (y) is coprime to I and I is invariant, we must have that $\overline{\mathfrak{p}}$ divides J' exactly with the same exponent n, hence $J'' = J/(\mathfrak{p}\overline{\mathfrak{p}})^n$ is still such that $(y) = J''/\overline{J''}$ and has neither $\mathfrak{p}$ nor $\overline{\mathfrak{p}}$ factors in common with I. Doing this for every prime such that $v_{\mathfrak{p}}(\gcd(I, J)) \neq 0$, we find an ideal J coprime to I with $J/\overline{J} = (y)$, and the claim follows.
So what is left to understand is the Abelian extension of E associated to $\tilde{S}_I$. Exactly as before, we recover $\tilde{S}_I$ as the kernel of the natural projection

$$\mathbb{A}^{\times}_{E,f} \twoheadrightarrow \mathrm{Cl}'_I(E),$$

and this concludes the proof. □

Again, as a corollary, we obtain

$$|G_{K3,I}(E)| = [\mathcal{O}_F^{\times} \cap N(E^{I,1}) : N(\mathcal{O}_E^I)] \cdot |\,\mathrm{Cl}'_I(E)|. \tag{3.4.0.4}$$

Remark 3.4.8. When E is imaginary quadratic, we have the equalities $F_{K3,I}(E) = K'_I(E)$ and $G_{K3,I}(E) = \mathrm{Cl}'_I(E)$.

3.5 Invariant ideals and K3 class group

In this section we continue to study the groups $G_{K3,I}(E)$, in particular we want to compute their cardinality. By Theorem (3.4.7), we know that

$$|G_{K3,I}(E)| = \frac{|\,\mathrm{Cl}_I(E)|}{|\,\mathrm{Cl}_I(E)^G|} \cdot [\mathcal{O}_F^{\times} \cap N(E^{I,1}) : N(\mathcal{O}_E^I)].$$

When $I = 0$, we have (see Lemma 4.1 of [22])

Theorem 3.5.1. *Let E/F be a cyclic extension with Galois group G. Then*

$$|\operatorname{Cl}(E)^G| = \frac{h_F \cdot e(E/F)}{[E:F]\cdot[\mathcal{O}_F^\times : N(E^\times)\cap \mathcal{O}_F^\times]},$$

where h_F is the class number of F and

$$e(E/F) := \prod_v e(v),$$

the product of all the ramification indices over all the places of F, both finite and infinite.

Putting this together with Theorem (3.4.7) leads to

$$|G_{K3}(E)| = [E:F]\cdot \frac{h_E\cdot[\mathcal{O}_F^\times : N(\mathcal{O}_E^\times)]}{h_F\cdot e(E/F)}.$$

Using basically the same proof of [22], we compute now the cardinalities $|\operatorname{Cl}_I(E)^G|$, where I is any invariant ideal.
We are going to use the notation introduced at the beginning of the last section. Moreover, for a $G-$ module M we will denote by

$$H^i(M) := \hat{H}^i(G, M),$$

the $i-$th Tate cohomology group and by $Q(M)$ its Herbrand quotient (when defined). Let us start with a lemma.

Lemma 3.5.2. *Let I be an invariant ideal, we have*

$$Q(\mathcal{O}_E^I) = Q(\mathcal{O}_E^\times) = \frac{1}{[E:F]}e_\infty(E/F),$$

with

$$e_\infty(E/F) = \prod_{v|\infty} e(v),$$

where the product ranges over all the archimedean valuations of F.

Proof. The equality $Q(\mathcal{O}_E^I) = Q(\mathcal{O}_E^\times)$ descends from the fact that $\mathcal{O}_E^\times/\mathcal{O}_E^I$ is a finite group. The second equality of the statement follows from Corollary 2, Theorem 1, Chapter IX of [23].

□

Theorem 3.5.3. *Let $I \subset \mathcal{O}_E$ be an invariant ideal and denote by $J := I \cap \mathcal{O}_F$. We have:*

$$|\operatorname{Cl}_I(E)^G| = \frac{h_J(F) \cdot e(E/F, J) \cdot |H^1(E^{I,1})|}{[E:F][\mathcal{O}_F^J : N(E^{I,1}) \cap \mathcal{O}_F^\times]}$$

where

$$e(E/F, J) = \prod_{v \nmid J} e(v).$$

Proof. Consider the short exact sequence defining the ray class group

$$0 \to \mathcal{P}_E^I \to \mathcal{I}_E^I \to \operatorname{Cl}_I(E) \to 0,$$

taking invariants we obtain

$$0 \to \mathcal{P}_E^{I,G} \to \mathcal{I}_E^{I,G} \to \operatorname{Cl}_I(E)^G \to H^1(\mathcal{P}_E^I) \to 0,$$

since we have $H^1(\mathcal{I}_E^I) = 0$. So that

$$|\operatorname{Cl}_I(E)^G| = [\mathcal{I}_E^{I,G} : \mathcal{P}_E^{I,G}] \cdot |H^1(\mathcal{P}_E^I)|, \tag{3.5.0.1}$$

and we are going to compute the two indices on the right-hand side. We have

$$[\mathcal{I}_E^{I,G} : \mathcal{P}_E^{I,G}] = \frac{[\mathcal{I}_E^{I,G} : \mathcal{P}_F^J]}{[\mathcal{P}_E^{I,G} : \mathcal{P}_F^J]} = \frac{[\mathcal{I}_E^{I,G} : \mathcal{I}_F^J] \cdot [\mathcal{I}_F^J : \mathcal{P}_F^J]}{[\mathcal{P}_E^{I,G} : \mathcal{P}_F^J]} = \frac{e(E/F, \infty \cdot J) \cdot h_J(F)}{[\mathcal{P}_E^{I,G} : \mathcal{P}_F^J]}. \tag{3.5.0.2}$$

Taking invariants of the next exact sequence

$$0 \to \mathcal{O}_E^I \to E^{I,1} \to \mathcal{P}_E^I \to 0,$$

we obtain

$$0 \to \mathcal{O}_F^J \to F^{J,1} \to \mathcal{P}_E^{I,} \to H^1(\mathcal{O}_E^I) \to H^1(E^{I,1}).$$

Denote by $H \subset H^1(E^{I,1})$ the image of the last map, we have

$$[\mathcal{P}_E^{I,G} : \mathcal{P}_F^J] = \frac{|H^1(\mathcal{O}_E^I)|}{|H|} = \frac{|H^0(\mathcal{O}_E^I)|}{|H| \cdot Q(\mathcal{O}_E^I)}.$$

By Lemma (3.5.2) we know what $Q(\mathcal{O}_E^I)$ is, and by definition

$$|H^0(\mathcal{O}_E^I)| = [\mathcal{O}_F^J : N(\mathcal{O}_E^I)],$$

so that we have

$$[\mathcal{P}_E^{I,G} : \mathcal{P}_F^J] = \frac{|H^1(\mathcal{O}_E^I)|}{|K|} = \frac{[\mathcal{O}_F^J : N(\mathcal{O}_E^I)]}{|H| \cdot Q(\mathcal{O}_E^I)}. \quad (3.5.0.3)$$

Using the exact sequence

$$0 \to H^1(E^{I,1})/H \to H^1(\mathcal{P}_E^I) \to H^0(\mathcal{O}_E^I) \to H^0(E^{I,1}),$$

we see that

$$|H^1(\mathcal{P}_E^I)| = \frac{|H^1(E^{I,1})|}{|H|} \cdot |\ker(H^0(\mathcal{O}_E^I) \to H^0(E^{I,1}))|. \quad (3.5.0.4)$$

Now,

$$\ker(H^0(\mathcal{O}_E^I) \to H^0(E^{I,1})) \cong (N(E^{I,1}) \cap \mathcal{O}_F^\times)/N(\mathcal{O}_E^I).$$

Using the inclusions $N(\mathcal{O}_E^I) \subset N(E^{I,1}) \cap \mathcal{O}_F^\times \subset \mathcal{O}_F^J$ and putting together the equations (3.5.0.1), (3.5.0.2), (3.5.0.3) and (3.5.0.4), we conclude.

□

This, together with Theorem (3.4.7), readily implies

Theorem 3.5.4. *Let E be a CM number field and F its maximal, totally real subextension, and $I \subset \mathcal{O}_E$ an invariant ideal. Then we have*

$$|G_{K3,I}(E)| = \frac{h_I(E) \cdot [\mathcal{O}_F^J : N(\mathcal{O}_E^J)] \cdot [E : F]}{h_J(F) \cdot e(E/F, J) \cdot |H^1(E^{I,1})|} = \frac{h_E \cdot \phi_E(I) \cdot [\mathcal{O}_F^\times : N(\mathcal{O}_E^I)] \cdot [E : F]}{h_F \cdot \phi_F(J) \cdot [\mathcal{O}_E^\times : \mathcal{O}_E^I] \cdot e(E/F, J) \cdot |H^1(E^{I,1})|}.$$

Proof. This follows from Theorem (3.5.3) and Theorem (3.4.7), using the well-known fact

$$h_I(E) = h_E \frac{\phi_E(I)}{[\mathcal{O}_E^\times : \mathcal{O}_E^I]}.$$

□

The only mysterious term appearing in Theorem (3.5.4) is $|H^1(E^{I,1})|$. Note that this group is always 2−torsion and finitely generated. We have the following partial result:

Proposition 3.5.5. *In the assumptions of Theorem* (3.5.4)

1. *If* $\gcd(2, I) = (1)$. *Then* $H^1(E^{I,1}) = 0$;

2. *Write $I = I_2 \cdot I'$ with $(I', 2) = (1)$, and likewise put $J = J_2 \cdot J'$. We have a natural left exact sequence*

$$1 \to \frac{(\mathcal{O}_E/I_2)^{\times,G}}{(\mathcal{O}_F/J_2)^{\times}} \to H^1(E^{I,1}) \to \bigoplus_{q \in S(J_2)} \mathbb{Z}/e(q)\mathbb{Z}.$$

3. *If every prime ideal dividing J_2 does not ramify in E, then $H^1(E^{I,1}) = 0$.*

Proof.

1. Let $x \in E^{I,1}$ be such that $x\overline{x} = 1$. Then, if we put $y = 1/2 + x/2$, we also have $y \in E^{I,1}$ (since by assumptions 2 and I are coprime) and $y/\overline{y} = x$.

2. The first thing is to understand the quotient Q_I of

$$1 \to E^{I,1} \to E^{I',1} \to Q_I \to 1. \tag{3.5.0.5}$$

In order to do this, consider the following morphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E^{I,1} & \longrightarrow & E^{I} & \longrightarrow & (\mathcal{O}_E/I)^{\times} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & E^{I',1} & \longrightarrow & E^{I'} & \longrightarrow & (\mathcal{O}_E/I')^{\times} & \longrightarrow & 1, \end{array}$$

The following sequence

$$1 \to E^{I} \to E^{I'} \xrightarrow{\oplus v_{\mathfrak{p}}} \bigoplus_{\mathfrak{p} \in S(I_2)} \mathbb{Z} \to 0$$

is exact, due to the theorem on the independence of valuations. Via the snake lemma, we obtain the exact sequence

$$1 \to (\mathcal{O}_E/I_2)^{\times} \to Q_I \to \bigoplus_{\mathfrak{p} \in S(I_2)} \mathbb{Z} \to 0. \tag{3.5.0.6}$$

We can do the same over F, obtaining analogous results. In particular, we have the two exact sequences

$$1 \to F^{J,1} \to F^{J',1} \to Q_J \to 1$$

and

$$1 \to (\mathcal{O}_F/J_2)^{\times} \to Q_J \to \bigoplus_{q \in S(J_2)} \mathbb{Z} \to 0. \tag{3.5.0.7}$$

Taking Galois invariants of (3.5.0.5) and using the first point of this proposition, we obtain

$$1 \to F^{J,1} \to F^{J',1} \to Q_I^G \to H^1(E^{I,1}) \to 1,$$

thus, we can identify

$$H^1(E^{I,1}) \cong \operatorname{coker}(Q_J \to Q_I^G). \tag{3.5.0.8}$$

Applying the snake lemma to the following diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & (\mathcal{O}_F/J_2)^\times & \longrightarrow & Q_J & \longrightarrow & \bigoplus_{q\in S(J_2)} \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & (\mathcal{O}_E/I_2)^{\times,G} & \longrightarrow & Q_I^G & \longrightarrow & \left(\bigoplus_{\mathfrak{p}\in S(I_2)} \mathbb{Z}\right)^G, & &
\end{array}$$

we obtain

$$1 \to \frac{(\mathcal{O}_E/I_2)^{\times,G}}{(\mathcal{O}_F/J_2)^\times} \to H^1(E^{I,1}) \to \bigoplus_{q\in S(J_2)} \mathbb{Z}/e(q)\mathbb{Z}. \tag{3.5.0.9}$$

This concludes the proof of point 2.

3. Under these assumptions, we have

$$\frac{(\mathcal{O}_E/I_2)^{\times,G}}{(\mathcal{O}_F/J_2)^\times} \cong H^1(E^{I,1}).$$

However, since the primes in $S(J_2)$ do not ramify, $(\mathcal{O}_E/I_2)^{\times,G} = (\mathcal{O}_F/J_2)^\times$.

□

4 Fields of moduli and applications

4.1 Fields of moduli

In this section we compute the field of moduli of the tuple $(T(X), B, \iota)$. This should be interpreted as the field of moduli of the transcendental motive of X, together with the cycles in $E(X)$ and some additional Brauer classes.

Definition 4.1.1. The field of moduli of $(T(X), B, \iota)$ is the fixed field of

$$\{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) : (T(X), B, \iota) \cong (T(X^\sigma), \sigma_*(B), \sigma^{ad} \circ \iota)\},$$

where the isomorphism $(T(X), B, \iota) \cong (T(X^\sigma), \sigma_*(B), \sigma^{ad} \circ \iota)$ is as in Definition (3.2.3).

Remark 4.1.2. Note that if we denote by M the field of moduli of $(T(X), B, \iota)$, then we must have $E \subset M$ because of Lemma (3.2.4), so that we can 'work over E'.

Theorem 4.1.3 (Field of moduli)**.** *Let (X, B, ι) be a principal CM $K3$ surface over $\mathbb{C}$ with level structure $B \subset \mathrm{Br}(X)$ and let $E \subset \mathbb{C}$ be its reflex field. Suppose that $(T(X), B, \iota)$ is of type (I, α, J, σ) and put $I_B := I^\vee J^{-1} \subset \mathcal{O}_E$. Then the field of moduli of $(T(X), B, \iota)$ corresponds to the K3 class field $F_{K3, I_B}(E)$ modulo the ideal I_B.*

Proof. Thanks to the remark we need to compute the fixed field of

$$\{\sigma \in \mathrm{Aut}(\mathbb{C}/E) : \exists \text{ Hodge isometry } f : T(X) \to T(X^\sigma) : f_* \circ \sigma^*|_B = \mathrm{Id}\}.$$

Thanks to Proposition (3.2.6) and Theorem (3.3.1), an element $\tau \in \mathrm{Aut}(\mathbb{C}/E)$ is in the above group if and only if we can find $s \in \mathbb{A}_{E,f}^\times$ and $e \in E^\times$ such that

1. $\mathrm{art}(s) = \tau|_{E^{ab}}$;
2. $\frac{s}{\bar{s}}(I, \alpha, \sigma) \cong (I, \alpha, \sigma)$, i.e. $e\frac{s}{\bar{s}}I = I$ and $e\bar{e} = 1$;

3. The composition $E/I^\vee \xrightarrow{s/\bar{s}} E/\frac{s}{\bar{s}}I^\vee \xrightarrow{e} E/I^\vee$ restricts to the identity on $J/I^\vee$.

Via class field theory, this corresponds to

$$\{s \in \mathbb{A}_{E,f}^\times :\ \exists e \in E^\times :\ e\bar{e} = 1\,,\ e\frac{s}{\bar{s}}\mathcal{O}_E = \mathcal{O}_E\,, e\frac{s}{\bar{s}} \equiv 1 \mod I_B\,\},$$

and we recognise this group to be exactly the kernel of $\mathbb{A}_{E,f}^\times \twoheadrightarrow G_{K3,I_B}(E)$ (see the proof of Proposition (3.4.7)). □

Remark 4.1.4. If we put $B = 0$ in the above theorem, we find that the field of moduli of $(T(X), \iota)$ is $F_{K3}(E)$, the K3 class field of E. In particular, it does not depend on $T(X)$ but only on E. By the results in [37], there are only finitely many K3 surfaces with CM that can be defined over a fixed number field K. On the other hand, by Proposition (3.1.11), there are infinitely many (isomorphism classes of) principal K3 surfaces with CM by a fixed CM field E defined over $\overline{K}$ (at least when $[E : \mathbb{Q}] \leq 10$. Hence, it is interesting to notice that, even if the fields of definition of these surfaces (X, ι) are not bounded in degree, the fields of moduli of their transcendental lattices are all the same.

This theorem allows us to study the groups $\mathrm{Br}(\overline{X})^{G_K}$ where X/K is a K3 surface with CM over K and G_K is the absolute Galois group of K. Indeed, the immediate corollary we get is

Theorem 4.1.5. *Let X/K be a principal K3 surface with CM over K. There exists an ideal $I_B \subset \mathcal{O}_E$ such that*

$$\mathcal{O}_E/I_B \cong \mathrm{Br}(\overline{X})^{G_K}$$

as $\mathcal{O}_E$-modules and

$$|G_{K3,I_B}(E)| \,\Big|\, [K : E].$$

Proof. Fix an isomorphism $\iota : E \to E(X)$, and let (I, α, J, σ) be the type of $(T(X), \mathrm{Br}(\overline{X})^{G_K}, \iota)$. As usual, let $I_B = I^\vee J^{-1}$. Via the type map, we have an isomorphism of $\mathcal{O}_E-$mod

$$\mathrm{Br}(\overline{X})^{G_K} \cong \mathcal{O}_E/I_B.$$

Since $(T(X), \mathrm{Br}(\overline{X})^{G_K}, \iota)$ is defined over K, we must have

$$F_{K3,I_B}(E) \subset K.$$

□

4.2 Applications to Brauer groups

One of the consequences of the results in [37] is that for a fixed number field K, there are only finitely many groups that can appear as $\mathrm{Br}(\overline{X})^{G_K}$, where X/K is any K3 surface with CM. We shall show in this last section how the theorems in the previous ones can be applied to produce a computable bound for the Galois fixed part of Brauer groups of principal CM K3 surfaces. Indeed, what we have is an algorithm that, given as input a number field K and a CM field E, returns as output a finite set of groups $\mathrm{Br}(E, K)$ such that for every principal CM K3 surfaces X/K with reflex field E we have

$$\mathrm{Br}(\overline{X})^{G_K} \in \mathrm{Br}(E, K).$$

It works as follows:

1. Replace K by KE;
2. Find all the invariant ideals $I \subset \mathcal{O}_E$ such that

   $$|G_{K3,I}(E)| \,\Big|\, [K : E].$$

   This is possible thanks to Theorem (3.5.4) and Proposition (3.5.5), which also says that there are finitely many such ideals. Denote them $I_1, \cdots I_n$.
3. Now use Theorem (4.1.5), which says that

   $$\mathrm{Br}(\overline{X_K})^{G_K} \cong \mathcal{O}_E / I_B,$$

   with $I_B \subset \mathcal{O}_E$ an ideal dividing one of the I_i's, hence, we have an inclusion (of isomorphism classes of $\mathcal{O}_E$-modules)

   $$\{\mathrm{Br}(\overline{X})^{G_K} : X/K \text{ has CM by } \mathcal{O}_E\} \subset \{\mathcal{O}_E/I_B : I_i \subset I_B \text{ for some } i = 1, \cdots, n\},$$

   and we define the latter set to be $\mathrm{Br}(K, E)$.

Remark 4.2.1. In particular, if we put $I := \prod_i I_i$ and $C := |\mathcal{O}_E/I|$, we must have

$$|\operatorname{Br}(\overline{X})^{G_K}| \leq C$$

for every principal K3 surface X/K with CM by E over $\overline{K}$.

Let us see how this works in practice with some examples, all concerning K3 surfaces with maximal Picard rank.

1. (Gaussian integers) Let $E = \mathbb{Q}(i)$. In this case, the K3 class field of E is E itself. Put $K = E$. Every invariant ideal of E can be written as $I = (1+i)^k \cdot (n)$ with $n \in \mathbb{Z}$ and $(n, 2) = 1$, and we have to find all such I with $G_{K3,I}(E) = 1$. Decompose

$$n = p_1^{\alpha_1} \cdots p_l^{\alpha_l} \cdot q_1^{\beta_1} \cdots q_j^{\beta_j},$$

where the q's are inert (i.e. $\equiv 3 \mod 4$) and the p's are split (i.e. $\equiv 1 \mod 4$). Let us start with the cases where $k = 0$ and $n > 2$. Theorem (3.5.4) tells us that

$$|G_{K3,I}(E)| = \frac{h_E \cdot \phi_E(I) \cdot [\mathcal{O}_F^\times : N(\mathcal{O}_E^I)] \cdot [E : F]}{h_F \cdot \phi_F(J) \cdot [\mathcal{O}_E^\times : \mathcal{O}_E^I] \cdot e(E/F, J) \cdot |H^1(E^{I,1})|}.$$

If $n > 2$, then

- $[\mathcal{O}_F^\times : N(\mathcal{O}_E^I)] = 2$;
- $[\mathcal{O}_E^\times : \mathcal{O}_E^I] = 4$;
- $e(E/F, J) = 4$, since only 2 and the place at infinity ramify;
- $|H^1(E^{I,1})| = 1$, by Proposition (3.5.5).

So we obtain

$$|G_{K3,I}(E)| = \frac{\phi_K(n)}{4 \cdot \phi(n)} = \frac{1}{4} \prod p_i^{\alpha_i - 1}(p_i - 1) \cdot \prod q_i^{\beta_i - 1}(q_i + 1),$$

hence, in this case, $|G_{K3,I}(E)| = 1$ if and only if $n = 3$ or $n = 5$.
Let us assume now that $k > 0$ and that $n = 1$. We have

- $[\mathcal{O}_F^\times : N(\mathcal{O}_E^I)] = 2$;
- $e(E/F, J) = 2$;

- $[\mathcal{O}_K^\times : \mathcal{O}_K^{(1+i)^k}] = \begin{cases} 1 \text{ if } k = 1 \\ 2 \text{ if } k = 2 \\ 4 \text{ if } k > 2; \end{cases}$
- $\frac{\phi_K(1+i)^k}{\phi((1+i)^k \cap \mathbb{Z})} = 2^{\lfloor \frac{k}{2} \rfloor}$.

Since 2 ramifies in E, we have that in general the cohomology groups $H^1(E^{(1+i)^n,1})$ are not zero. However, Proposition (3.5.5) tells us that their cardinality $|H^1(E^{(1+i)^n,1})|$ always divide

$$2 \cdot [(\mathcal{O}_K/(1+i)^k)^{\times,G} : (\mathbb{Z}/(1+i)^k \cap \mathbb{Z})^\times],$$

and we compute

- $[(\mathcal{O}_K/(1+i)^k)^{\times,G} : (\mathbb{Z}/(1+i)^k \cap \mathbb{Z})^\times] = \begin{cases} 2 \text{ if } k \text{ is even,} \\ 1 \text{ if } k \text{ is odd.} \end{cases}$

Thus, putting all together, we have that if $|G_{K3,(1+i)^k}(E)| = 1$, then

$$\frac{2 \cdot 2^{\lfloor \frac{k}{2} \rfloor}}{[\mathcal{O}_E^\times : \mathcal{O}_E^{(1+i)^k}] \cdot [(\mathcal{O}_K/(1+i)^k)^{\times,G} : (\mathbb{Z}/(1+i)^k \cap \mathbb{Z})^\times]} \Bigm| 2.$$

This happens only for $k \leq 6$. Assume now $k \geq 1$ and $n > 2$. Thanks to the results above, if $|G_{K3,I}| = 1$, then $I = (1+i)^k \cdot 5^\alpha$ or $I = (1+i)^k \cdot 3^\beta$. Let us begin with the former case: we have

$$|G_{K3,I}(E)| = \frac{2^{\lfloor \frac{k}{2} \rfloor} \cdot 3^{\beta-1} \cdot 4 \cdot 4}{4 \cdot 2 \cdot |H^1|} = \frac{2^{1+\lfloor \frac{k}{2} \rfloor} \cdot 3^{\beta-1}}{|H^1|}.$$

Hence, $\beta = 1$, since $|H^1|$ is 2−torsion. As above, we see that if $|G_{K3,I}(E)| = 1$, then

$$2^{1+\lfloor \frac{k}{2} \rfloor} \Bigm| 2 \cdot [(\mathcal{O}_K/(1+i)^k)^{\times,G} : (\mathbb{Z}/(1+i)^k \cap \mathbb{Z})^\times],$$

which happens only for $k \leq 2$. The same is true for the case $I = (1+i)^k \cdot 5^\beta$. Hence, we have the following possibilities for $\mathrm{Br}(\overline{X})^{G_K}$ (as isomorphism classes of Abelian groups)

0, $\mathbb{Z}/3\times\mathbb{Z}/3$, $\mathbb{Z}/5$, $\mathbb{Z}/5\times\mathbb{Z}/5$, $\mathbb{Z}/2$, $\mathbb{Z}/2\times\mathbb{Z}/2$, $\mathbb{Z}/4\times\mathbb{Z}/2$, $\mathbb{Z}/4\times\mathbb{Z}/4$, $\mathbb{Z}/8\times\mathbb{Z}/4$,

$$\mathbb{Z}/8\times\mathbb{Z}/8,\ \mathbb{Z}/3\times\mathbb{Z}/3\times\mathbb{Z}/2,\ \mathbb{Z}/3\times\mathbb{Z}/3\times\mathbb{Z}/2\times\mathbb{Z}/2,\ \mathbb{Z}/5\times\mathbb{Z}/2,\ \mathbb{Z}/5\times\mathbb{Z}/5\times\mathbb{Z}/2,$$

$$\mathbb{Z}/5\times\mathbb{Z}/2\times\mathbb{Z}/2,\ \mathbb{Z}/5\times\mathbb{Z}/5\times\mathbb{Z}/2\times\mathbb{Z}/2.$$

This confirms the results in [17] and [18] about diagonal quartic surfaces.

2. (Eisenstein integers). Put $E = \mathbb{Q}(\sqrt{-3})$. In this case the K3 class field of E is E itself, again. Put $K = E$. The only prime of $\mathbb{Z}$ that ramifies in E is 3, with $(3) = (\sqrt{-3})^2$. In particular, since 2 does not ramify, thanks to Proposition (3.5.5) we have

$$|G_{K3,I}(E)| = \frac{h_E \cdot \phi_E(I) \cdot [\mathcal{O}_F^\times : N(\mathcal{O}_E^I)] \cdot [E : F]}{h_F \cdot \phi_F(J) \cdot [\mathcal{O}_E^\times : \mathcal{O}_E^I] \cdot e(E/F, J)} = \frac{4 \cdot \phi_E(I)}{\phi_F(J) \cdot [\mathcal{O}_E^\times : \mathcal{O}_E^I] \cdot e(E/F, J)}$$

for every invariant ideal $I \subset \mathcal{O}_E$. As before, let us proceed in computing these numbers. One can check that

$$[\mathcal{O}_E^\times : \mathcal{O}_E^I] = \begin{cases} 1 \text{ if } I = \mathcal{O}_E; \\ 2 \text{ if } I = (\sqrt{-3}); \\ 3 \text{ if } I = (2); \\ 6 \text{ otherwise.} \end{cases} \qquad (4.2.0.1)$$

Write

$$I = (\sqrt{-3})^k \cdot p_1^{\alpha_1} \cdots p_l^{\alpha_l} \cdot q_1^{\beta_1} \cdots q_j^{\beta_j},$$

where the q's are inert primes (i.e. $\equiv 2 \mod 3$) and the p's are split (i.e. $\equiv 1 \mod 3$). Hence,

$$|G_{K3,I}(E)| = 4\cdot 3^{\lfloor k/2\rfloor}\cdot \prod p_i^{\alpha_i-1}(p_i-1)\cdot \prod q_i^{\beta_i-1}(q_i+1)\cdot \frac{1}{[\mathcal{O}_E^\times : \mathcal{O}_E^I]\cdot e(E/F,J)}.$$

Using this, we see that

- If $k = 0$, then $|G_{K3,I}(E)| = 1$ if and only if $I = (2), (4), (5), (7)$;
- If $k = 1$, then $|G_{K3,I}(E)| = 1$ if and only if $I = (\sqrt{-3}), (2\sqrt{-3})$;
- if $k = 2$, then $|G_{K3,I}(E)| = 1$ if and only if $I = (3)$;
- if $k = 3$, then $|G_{K3,I}(E)| = 1$ if and only if $I = (3\sqrt{-3})$;
- if $k > 3$, then $|G_{K3,I}(E)| > 1$.

Hence, we have the following possibilities for $\mathrm{Br}(\overline{X})^{G_K}$ (as isomorphism classes of Abelian groups)

$$0,\ \mathbb{Z}/3,\ \mathbb{Z}/2\times\mathbb{Z}/2,\ \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/3,\ \mathbb{Z}/3\times\mathbb{Z}/3,\ \mathbb{Z}/4\times\mathbb{Z}/4,\ \mathbb{Z}/9\times\mathbb{Z}/3,\ \mathbb{Z}/5\times\mathbb{Z}/5,\ \mathbb{Z}/7\times\mathbb{Z}/7.$$

3. (Odd discriminant) Let E be a quadratic imaginary field with $\mu(E) = \{\pm 1\}$. Assume moreover that 2 does not ramify in E, so that we can use Proposition (3.5.5) and forget about the term $H^1(E^{I,1})$. Put $K = F_{K3}(E)$ (the smallest possible). If an invariant ideal $I \subset \mathcal{O}_E$ satisfies $G_{K3,I}(E) = 1$, a computation analogous to the ones above shows that

$$\frac{\phi_E(I)}{\phi(J)} = [\mathcal{O}_E^\times : \mathcal{O}_E^I] \cdot \frac{e(E/\mathbb{Q}, J)}{e(E/\mathbb{Q})}. \qquad (4.2.0.2)$$

The index $[\mathcal{O}_E^\times : \mathcal{O}_E^I]$ is always 2, unless $I = (2)$, in which case $[\mathcal{O}_E^\times : \mathcal{O}_E^I] = 1$. Write

$$I = \mathfrak{r}_1^{G_1} \cdots \mathfrak{r}_k^{\gamma_k} \cdot p_1^{\alpha_1} \cdots p_l^{\alpha_l} \cdot q_1^{\beta_1} \cdots q_j^{\beta_j},$$

where the p's are split primes of $\mathbb{Z}$, the q's are inert and $\mathfrak{r}_i^2 = r_i \mathcal{O}_E$ for a ramified prime r_i of $\mathbb{Z}$. We notice that the right-hand side of (4.2.0.2) is an integer if and only if at most one ramified prime divides I, i.e.

$$I = \mathfrak{r}^\gamma \cdot p_1^{\alpha_1} \cdots p_l^{\alpha_l} \cdot q_1^{\beta_1} \cdots q_j^{\beta_j}.$$

We have, in this case,

$$[\mathcal{O}_E^\times : \mathcal{O}_E^I] \cdot \frac{e(E/\mathbb{Q}, J)}{e(E/\mathbb{Q})} = \begin{cases} 1 \text{ if } \gamma > 0 \text{ or } \gamma = 0 \text{ and } I = (2); \\ 2 \text{ otherwise.} \end{cases}$$

and

$$\frac{\phi_E(I)}{\phi(J)} = r^{\gamma - 1}(r-1) \cdot \prod p^{\alpha_i - 1}(p_i - 1) \cdot \prod q^{\beta_i - 1}(q_i + 1).$$

Let us now list all the possibilities for both invariant ideals and possible Brauer groups (seen, again, as isomorphism classes of Abelian groups, i.e. forgetting the natural structure of $\mathcal{O}_E$-modules), depending on the behaviour of the primes 2 and 3 in E. There are six cases since, by assumption, 2 does not ramify.

a) Both 2 and 3 split. In this case, the only invariant ideals satisfying

(4.2.0.2) are $(2),(3),(4),(6)$. Hence, the possible Brauer groups are

$$\mathbb{Z}/4,\ \mathbb{Z}/4\times\mathbb{Z}/2,\ \mathbb{Z}/4\times\mathbb{Z}/4,\ (\mathbb{Z}/2)^{\alpha}\times(\mathbb{Z}/3)^{\beta},\ \alpha,\beta\in\{0,1,2\}.$$

b) 2 splits and 3 is inert. In this case, the only invariant ideals satisfying ((4.2.0.2)) are $(2),(4)$. Hence, the possible Brauer groups are

$$0,\ \mathbb{Z}/2,\ \mathbb{Z}/2\times\mathbb{Z}/2,\ \mathbb{Z}/4,\ \mathbb{Z}/4\times\mathbb{Z}/2,\ \mathbb{Z}/4\times\mathbb{Z}/4.$$

c) 2 splits and 3 ramifies. Write $(3)=\mathfrak{r}^2$. In this case, the only invariant ideals satisfying ((4.2.0.2)) are (2), (4), $\mathfrak{r}$, $2\mathfrak{r}$. Hence, the possible Brauer groups are

$$0,\ \mathbb{Z}/2,\ \mathbb{Z}/2\times\mathbb{Z}/2,\ \mathbb{Z}/4,\ \mathbb{Z}/4\times\mathbb{Z}/2,\ \mathbb{Z}/4\times\mathbb{Z}/4,\ \mathbb{Z}/3,\ \mathbb{Z}/3\times\mathbb{Z}/2,\ \mathbb{Z}/3\times\mathbb{Z}/2\times\mathbb{Z}/2.$$

d) 2 is inert and 3 splits. In this case, the only invariant ideal satisfying ((4.2.0.2)) is (3). Hence, the possible Brauer groups are

$$0,\ \mathbb{Z}/3,\ \mathbb{Z}/3\times\mathbb{Z}/3.$$

e) Both 2 and 3 are inert. In this case, there are no invariant ideals satisfying ((4.2.0.2)). Hence the only possible Brauer group is the trivial one.

f) 2 is inert and 3 ramifies. Write again $(3)=\mathfrak{r}^2$. In this case, the only invariant ideal satisfying ((4.2.0.2)) is $\mathfrak{r}$. Hence, the possible Brauer groups are

$$0,\ \mathbb{Z}/3.$$

Remarks 4.2.2.

- It is interesting to notice how the arithmetic properties of the field E (e.g. which primes of F ramify in E) influences the Brauer group of the principal K3 surfaces with CM by E, as the above examples show.
- We do not know whether all the groups listed above are actually achieved by some principal K3 surface X with CM.

In this last part, we shall compare Newton's work [34] on the Brauer groups of some special Kummer surfaces with our results. We briefly recall the construction of Kummer surfaces: let A/K be an Abelian surface over a number field K, and consider $G := \{\pm 1\}$ acting on A via multiplication by -1. Put $\tilde{A} := \mathrm{Bl}_{A[2]}A$, where $\mathrm{Bl}_{A[2]}A$ denoted the blow-up of A along the closed sub-scheme $A[2]$. We can extend the action of G to $\tilde{A}$ uniquely by requiring it to be the identity on the exceptional divisors. The Kummer surface associated to A is by the definition the quotient $\mathrm{Km}(A) := \tilde{A}/G$. As proved in Proposition 1.3. of [47], one has an isomorphism of G_K-modules

$$\mathrm{Br}(\overline{A}) \cong \mathrm{Br}(\overline{\mathrm{Km}(A)}), \tag{4.2.0.3}$$

Now, consider C/K an elliptic curve with CM by E, and suppose moreover that C is principal and that K contains the reflex field of C. For an Abelian group A and a prime ℓ we write $A_{\ell^\infty} := \{a \in A : \ell^n a = 0 \text{ for some } n \in \mathbb{Z}\}$. Newton's paper allows us to explicitly compute the groups $\mathrm{Br}(\overline{C} \times \overline{C})^{G_K}_{\ell^\infty}$, for every prime number ℓ and hence, thanks to equation ((4.2.0.3)), the groups $\mathrm{Br}(\overline{X})^{G_K}_{\ell^\infty}$ where $X = \mathrm{Km}(C \times C)$. Assume that $K = E(j(C)) = K(E)$, the Hilbert class field of E, and that $\mu(E) = \{\pm 1\}$. Thanks to Theorem 3.1 and Theorem 2.9 of [34], we must have that if $\ell \neq 2$ is a prime that does not ramify in E, then $\mathrm{Br}(\overline{X})^{G_K}_{\ell^\infty} = 0$. We shall show now how to prove the same result using the techniques of this thesis. Since our results are completely general, we do not need to make any assumption on the geometry of X (e.g. to be the Kummer surface of a product of two elliptic curves), on the prime ℓ or on $\mu(E)$ either. We have

Theorem 4.2.3. *Let E be a quadratic imaginary field, and let $K = K(E)$ be its Hilbert class field. Then*

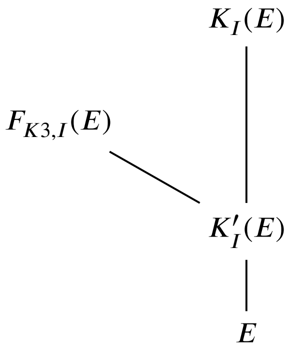
$$\mathrm{Br}(E, K(E)) = \left\{ \mathcal{O}_E/I : \ \textit{if } J = \mathrm{lcm}(I, \overline{I}) \textit{ then } \mathrm{Gal}(E/\mathbb{Q}) \textit{ acts trivially on } (\mathcal{O}_E/J)^\times/\mu(E) \right\}.$$

Remark 4.2.4. Note that, unlike the examples studied before, here the field K is not the K3 class field of E (in general) but an Abelian extension of it. We could still use our algorithm to get similar results to the one stated above, but in this case it turns out that the best strategy is to employ the (more qualitative) facts proved in Section (3.4), as the next proof shows.

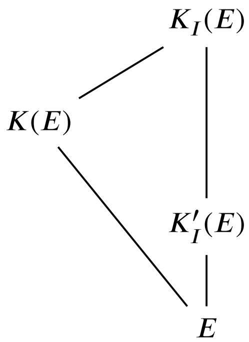
Proof. We have to find all the invariant ideals $I \subset \mathcal{O}_E$ such that

$$F_{K3,I}(E) \subset K(E). \tag{4.2.0.4}$$

By Proposition (3.4.7) we have a diagram of field extensions



however, since X is singular, we must have that $K'_I(E) = F_{K3,I}(E)$. Introducing the Hilbert class field in this diagram, we obtain



Now, we have that

$$\mathrm{Gal}(K_I(E)/K(E)) \cong \ker \pi,$$

where π is the canonical projection

$$\pi : \ \mathrm{Cl}_I(E) \to \mathrm{Cl}(E)$$

and that

$$\mathrm{Gal}(K_I(E)/K'_I(E)) \cong \mathrm{Cl}_I(E)^G,$$

hence the inclusion ((4.2.0.4)) becomes

$$\ker \pi \subset \mathrm{Cl}_I(E)^G. \tag{4.2.0.5}$$

Using the fundamental exact sequence

$$1 \to \mathcal{O}_E^I \to \mathcal{O}_E^\times \to (\mathcal{O}_E/I)^\times \to \mathrm{Cl}_I(E) \to \mathrm{Cl}(E) \to 1, \tag{4.2.0.6}$$

we see that

$$\ker \pi \cong \left(\mathcal{O}_E/I\right)^{\times}/\mu(E).$$

Hence, the inclusion ((4.2.0.5)) implies that G acts trivially on the group $\left(\mathcal{O}_E/I\right)^{\times}/\mu(E)$. □

Remarks 4.2.5.

- In particular, for every prime ideal $\mathfrak{r}$ of $\mathcal{O}_E$ that divides a ramified prime of $\mathbb{Z}$, we see that $\mathcal{O}_E/\mathfrak{r}$ is a possible Brauer group for a principal K3 surface $X/K(E)$ with CM by E.

- To obtain Newton's result, we notice that if $\mu(E) = \{\pm 1\}$ and $\ell > 3$ is a prime of $\mathbb{Z}$ that does not ramify in E, then $\mathrm{Gal}(E/\mathbb{Q})$ does not act trivially on $(\mathcal{O}_E/\ell^n)^{\times}/\{\pm 1\}$ if $n > 0$. If $\ell = 3$, then two things can happen (still assuming that it does not ramify): if 3 splits in E, then $\mathcal{O}_E/3$ is a possible Brauer group for a K3 surface $X/K(E)$ with CM by E (this does not contradict Newton's result, but it is taking into account all the other K3 surfaces X which are not the Kummer surface of a product of elliptic curves), whereas if 3 is inert, we still have that $\mathrm{Gal}(E/\mathbb{Q})$ does not act trivially on $(\mathcal{O}_E/\ell^n)^{\times}/\{\pm 1\}$ for every $n > 0$.

- The proof above can be generalised to study Brauer groups over number fields K of the form $K_I(E)$ or $F_{K3,I}(E)$, where E is a quadratic imaginary extension of $\mathbb{Q}$ and $I \subset \mathcal{O}_E$ an ideal.

5 Fields of definition

5.1 Preliminaries

In this chapter we switch focus to fields of definition. Let us recall some cohomological properties of K3 surfaces. The lattice $H^2_B(X,\mathbb{Z})(1)$ together with the intersection form is an even unimodular lattice whose isomorphism class does not depend on the chosen X. It is usually denoted by Λ and called the K3 lattice. To the Néron-Severi lattice one associates a *finite quadratic form*, i.e. a finite Abelian group A_N together with a quadratic form $q_N : A_N \to \mathbb{Q}/2\mathbb{Z}$, as follows. Consider first the dual lattice of $\mathrm{NS}(X)$:

$$\mathrm{NS}(X)^\vee := \{x \in \mathrm{NS}(X)_\mathbb{Q} : (x,v)_X \in \mathbb{Z} \text{ for all } v \in \mathrm{NS}(X)\}$$

and put $A_N := \mathrm{NS}(X)^\vee / \mathrm{NS}(X)$, under the canonical inclusion $\mathrm{NS}(X) \subset \mathrm{NS}(X)^\vee$. Then one defines a quadratic form q_N on A_N by the rule

$$q_N(x + \mathrm{NS}(X)) = (x,x)_X + 2\mathbb{Z}.$$

It takes values in $\mathbb{Q}/2\mathbb{Z}$. The primitive embedding $\mathrm{NS}(X) \hookrightarrow H^2_B(X,\mathbb{Z})(1)$ determines the lattices of transcendental cycles $T(X) := \mathrm{NS}(X)^\perp$. As above, one can associate a finite quadratic form (A_T, q_T) to $T(X)$. Nikulin proved in [35] that one has a canonical identification

$$(A_T, -q_T) \cong (A_N, q_N). \tag{5.1.0.1}$$

Definition 5.1.1. The finite quadratic form $(A_N, q_N) \cong (A_T, -q_T)$ is called the discriminant form of X, and we denote it by (D_X, q_X). The group of isomorphism of D_X preserving q_X is denoted by $O(q_X)$. We have natural maps $d_N : O(\mathrm{NS}(X)) \to O(q_X)$ and $d_T : O(T(X)) \to O(q_X)$, where the latter is constructed using the identification ((5.1.0.1)).

The classical lemma we are going to need is the following (see Corollary 1.5.2.

of [35]).

Lemma 5.1.2. *Two isometries $f_N \in O(\mathrm{NS}(X))$ and $f_T \in O(T(X))$ can be lifted to a (necessarily unique) isometry $f \in O(H^2_B(X,\mathbb{Z})(1))$ if and only if $d_N(f_N) = d_T(f_T)$.*

Remarks 5.1.3.

1. If f_T is a Hodge isometry and the lifting f exists, then f is a Hodge isometry as well;
2. It follows that one has a pull-back diagram

$$\begin{array}{ccc} O_{\mathrm{Hdg}}(H^2_B(X,\mathbb{Z})(1)) & \longrightarrow & O(\mathrm{NS}(X)) \\ \big\downarrow & & \big\downarrow{\scriptstyle d_N} \\ O_{\mathrm{Hdg}}(T(X)) & \xrightarrow{\ d_T\ } & O(q_X). \end{array}$$

3. One can reformulate Lemma (5.1.2) in the étale context as well. In this case, one considers the $\widehat{\mathbb{Z}}$-lattices $\widehat{\mathrm{NS}}(X), \widehat{T}(X) \subset H^2_{\text{ét}}(X,\widehat{\mathbb{Z}})(1)$ and carry out the very same definitions and computations, which agree to the ones above thanks to the comparison isomorphism $H^2_B(X,\mathbb{Z})(1) \otimes \widehat{\mathbb{Z}} \cong H^2_{\text{ét}}(X,\widehat{\mathbb{Z}})(1)$.

Before concluding this section, let us rephrase the main theorem of complex multiplication in a more convenient form. Let $L \subset \mathbb{C}$ be a number field and let X/L be a K3 surface with complex multiplication over L. We have a Galois representation $\rho\colon G_L \to \mathrm{Aut}(T(X)_{\mathbb{Q}})(\mathbb{A}_f)$, with image in $U_E(\mathbb{A}_f)$. Class field theory provides us with a commutative diagram

$$\begin{array}{ccc} \mathbb{A}_L^{\times} & \xrightarrow{\mathrm{art}_L} & G_L^{ab} \\ \big\downarrow{\scriptstyle \mathrm{Nm}_{L/E}} & & \big\downarrow{\scriptstyle \mathrm{res}} \\ \mathbb{A}_E^{\times} & \xrightarrow{\mathrm{art}_E} & G_E^{ab}. \end{array}$$

Note that, since E is a CM field, both the Artin maps factorise through the finite ideles. We have

Theorem 5.1.4. *Let $\tau \in G_L$, $t \in \mathbb{A}^{\times}_{L,f}$ such that* $\mathrm{art}_L(t) = \tau_{|L^{ab}}$ *and put $s := Nm_{L/E}(t) \in \mathbb{A}^{\times}_{E,f}$. There exists a unique $u \in U_E(\mathbb{Q})$ such that*

$$\rho(\tau) = u\frac{s}{\bar{s}} \in U_E(\mathbb{A}_f).$$

Remarks 5.1.5.

1. Since $\rho(\tau)$ respects the $\hat{\mathbb{Z}}$-structure, i.e. $\rho(\tau)\widehat{T}(\overline{X}) \subset \widehat{T}(\overline{X})$, also the map '$u\frac{s}{\bar{s}}$' must do so. Therefore, thanks to the remarks after Lemma

(5.1.2), it makes sense to consider the induced map $d_T\left(u\frac{s}{\bar{s}}\right) \in O(q_X)$. A direct consequence of the theorem above is that

$$d_T\left(u\frac{s}{\bar{s}}\right) = d_N(\tau^*|_{\mathrm{NS}}), \qquad (5.1.0.2)$$

where $\tau^*|_{\mathrm{NS}} : \mathrm{NS}(\overline{X}) \to \mathrm{NS}(\overline{X})$ denotes the Galois action on the Néron-Severi group.

2. From this formulation, it is clear that the image of ρ is an ádelic open subroup of $U(\widehat{\mathbb{Z}})$. An analogous statement is true for any K3 surface, see Theorem 6.6. of [6].

Definition 5.1.6 (Discriminant ideal)**.** Let (X, ι) be of type (I, α, σ). We define the *discriminant ideal* of X to be the fractional ideal

$$\mathcal{D}_X := (\alpha)I\overline{I}\mathcal{D}_E,$$

where $\mathcal{D}_E$ denoted the different ideal of the number field E.

Proposition 5.1.7. *In the situation above, we have*

1. $\mathcal{D}_X \subset \mathcal{O}_E$;
2. *The type map* $\Phi : T(X) \to I$ *induces an isomorphism between the* $\mathcal{O}_E$*-modules* D_X *and* $\mathcal{O}_E/\mathcal{D}_X$;
3. *The definition is independent of the chosen type, provided that* σ *(and hence* ι*) is fixed.*

Proof. 1. This follows from the fact that, since the quadratic form (I, α) given by

$$I \times I \to \mathbb{Q}$$
$$(x, y) \mapsto \mathrm{tr}_{E/\mathbb{Q}}(\alpha x\overline{y})$$

assumes values in $\mathbb{Z}$, the inclusion $(\alpha)I\overline{I} \subset \mathcal{D}_E^{-1}$ holds.

2. This is a direct consequence of the fact that the dual lattice of (I, α) is $\left(\left(\alpha I\overline{I}\mathcal{D}_E\right)^{-1}, \alpha\right)$.

3. If (J, β, σ) is another type of (X, ι), then by Theorem (3.2.6) there exists $e \in E^\times$ such that $\beta = e\bar{e}\alpha$ and $J = e^{-1}I$.

□

Thus, the ideal $\mathcal{D}_X$ is a well-defined invariant of (X, ι). Note that $\overline{\mathcal{D}_X} = \mathcal{D}_X$. In the following Lemma, we denote by $\mu(X)$ the group of Hodge isometries of $T(X)$.

Definition 5.1.8 (Big discriminant)**.** The kernel of the canonical map $d_T : \mu(X) \to O(q_X)$ is denoted by K_X and we say that X has *big discriminant* if $K_X = 1$.

Remarks 5.1.9.

- Thanks to the second point in Proposition (5.1.7), having big discriminant is equivalent to the injectivity of the natural map $\mu(E) \to (\mathcal{O}_E/\mathcal{D}_X)^\times$.
- There is always a natural injection $K_X \hookrightarrow \mathrm{Aut}(X)$. Indeed, for any $\mu \in K_X$, the map $(\mu, \mathrm{Id}) : T(X) \oplus \mathrm{NS}(X) \to T(X) \oplus \mathrm{NS}(X)$ can be extended to an integral Hodge isometry $H^2_B(X, \mathbb{Z})(1) \to H^2_B(X, \mathbb{Z})(1)$, which in turn is induced by a unique automorphism of X, thanks to Torelli Theorem.

Proposition 5.1.10. ***A)*** *Let E be a CM number field and let $X/\mathbb{C}$ be a K3 surface with CM by $\mathcal{O}_E$. Then*

$$\mathcal{D}_X \subset (2)^{-1}\mathcal{D}_{E/F}.$$

B) *If E is quadratic imaginary, then*

$$\mathcal{D}_X \subset \mathcal{D}_{E/\mathbb{Q}}.$$

Remark 5.1.11. In particular, if E is quadratic imaginary and the map $\mu(E) \to (\mathcal{O}_E/\mathcal{D}_{E/\mathbb{Q}})^\times$ is injective, then every X with CM by $\mathcal{O}_E$ has automatically big discriminant.

Proof. For any fractional ideal I of E, let $\mathrm{Nm}_{E/F}(I) \subset F$ be its norm, i.e. the fractional ideal of F generated by the elements $x\overline{x}$ for $x \in I$, so that we have $\mathrm{Nm}_{E/F}(I)\mathcal{O}_E = I\overline{I}$. Let (I, α) be the type of X. Every element of $\mathrm{Nm}_{E/F}(I)$ can be written as a finite sum of elements of the form $fx\overline{x}$, with $f \in \mathcal{O}_F$, and we compute

$$\mathrm{tr}_{F/\mathbb{Q}}(\alpha f x\overline{x}) = 2^{-1}\mathrm{tr}_{E/\mathbb{Q}}(\alpha f x\overline{x}) \in 2^{-1}\mathbb{Z},$$

since the quadratic form (I, α) is integral. Therefore, by the property of the discriminant ideal, we must have that

$$(\alpha)\mathrm{Nm}_{E/F}(I) \subset (2)^{-1}\mathcal{D}^{-1}_{F/\mathbb{Q}}.$$

If we 'base-change' the inclusion above to $\mathcal{O}_E$ we obtain that

$$(\alpha)I\overline{I} \subset (2)^{-1}\mathcal{D}_{F/\mathbb{Q}}^{-1}\mathcal{O}_E,$$

and point A) follows by multiplying both sides by $\mathcal{D}_E$.
To prove point B), we note that the quadratic form (I, α) is also even, so that $\mathrm{tr}_{E/\mathbb{Q}}(\alpha x\overline{x}) \in 2\mathbb{Z}$. But since E is quadratic imaginary by assumptions, we deduce that $\alpha x\overline{x} \in \mathbb{Z}$ for every $x \in I$. Consider again the fractional ideal $\mathrm{Nm}_{E/\mathbb{Q}}(I)$; every $y \in \mathrm{Nm}_{E/\mathbb{Q}}(I)$ can be written as $y = n_1x_1\overline{x}_1 + \cdots + n_kx_k\overline{x}_k$ with $n_i \in \mathbb{Z}$ and $x_i \in I$, so thanks to the computation above we conclude that $(\alpha)\mathrm{Nm}_{E/F}(I) \subset \mathbb{Z}$. Base-changing the above equation to $\mathcal{O}_E$, we obtain

$$(\alpha)I\overline{I} \subset \mathcal{O}_E,$$

and the claim follows as before.

□

5.2 Descending K3 surfaces

In this section we discuss a method to descend principal K3 surfaces with complex multiplication. Let us fix an ideal $I \subset \mathcal{D}_X$ such that

- $\overline{I} = I$;
- The map $\mu(E) \to (\mathcal{O}_E/I)^\times$ is injective.

The main theorem of this section is the following.

Theorem 5.2.1. *X admits a model X_I over $K := F_I(E)$, such that $\rho(X_I/K) = \rho(X/\mathbb{C})$ and G_K acts trivially on $T(\overline{X}_I)[I]$. Moreover, X_I/K satisfies the following universal property: if Y is a K3 surface over a number field L, with CM over L, such that $Y_\mathbb{C} \cong X$, $\rho(Y/L) = \rho(X/\mathbb{C})$ and G_L acts trivially on $T(\overline{Y})[I]$, then $F_I(E) \subset L$ and $X_{I,L} \cong Y$.*

Remark 5.2.2. Since G_K acts trivially on $T(\overline{Y})[I]$, it acts trivially also on $T(\overline{X}_I)[\mathcal{D}_X] \cong D_{\overline{X}_I}$, since $I \subset \mathcal{D}_X$.

In case X has big discriminant, we can choose $I = \mathcal{D}_X$ in the theorem above. This, together with the above remark, leads to the following corollary.

Corollary 5.2.3 (Canonical models)**.** *Let $X/\mathbb{C}$ be a K3 surface with complex multiplication by the ring on integers of a CM field and denote by $E \subset \mathbb{C}$ its reflex field. Assume that X has big discriminant. Then X admits a model X^{can} over $F_{\mathcal{D}_X}(E)$, the K3 class field of E modulo the discriminant ideal $\mathcal{D}_X$. Moreover, X^{can}/K satisfies the following universal property: if Y is a K3 surface over a number field L, with CM over L, such that $Y_{\mathbb{C}} \cong X$ and $\rho(Y/L) = \rho(X/\mathbb{C})$, then $F_{\mathcal{D}_X}(E) \subset L$ and $X^{can}_L \cong Y$.*

In order to prove the theorem, we construct a descent data using Torelli Theorem and the main theorem of complex multiplication. Before doing this, though, we need to study the field of definition of isomorphisms.

Proposition 5.2.4 (Descending isomorphisms)**.** *Let $X, Y/L \subset \mathbb{C}$ be two principal K3 surfaces with complex multiplication over a number field L, and suppose that $\overline{X}$ and $\overline{Y}$ are isomorphic. Then an isomorphism $f: \overline{X} \to \overline{Y}$ is defined over L if and only if the induced maps*

$$f^*: \mathrm{NS}(\overline{Y}) \to \mathrm{NS}(\overline{X})$$

and

$$f^*: T(\overline{Y})[I] \to T(\overline{X})[I]$$

are G_L-invariant.

Proof. The 'only if' part of the statement is trivial, so what we have to prove that if tha natural maps $\mathrm{NS}(\overline{Y}) \to \mathrm{NS}(\overline{X})$ and $T(\overline{Y})[I] \to T(\overline{X})[I]$ are Galois invariant, then f is defined over L. Recall that f is defined over L if and only if the induced map $f^*: H^2_{\text{ét}}(\overline{Y}, \widehat{\mathbb{Z}})(1) \to H^2_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}})(1)$ is G_L-invariant. To see this, one simply notes that the natural map of G_L-moules

$$\mathrm{Aut}(\overline{X}) \to \mathrm{Aut}(H^2_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}))$$

is injective (see Chapter 15, Remark 2.2. of [16]). To prove that f^* is G_L-equivariant, we break it into two components, $f^*_T: \widehat{T}(\overline{Y}) \to \widehat{T}(\overline{X})$ and $f^*_N: \mathrm{NS}(\overline{Y}) \to \mathrm{NS}(\overline{X})$. Let $\tau \in G_L$, we want prove the commutativity of the following diagram

$$\begin{array}{ccc} H^2_{\text{ét}}(\overline{Y}, \widehat{\mathbb{Z}})(1) & \xrightarrow{f^*} & H^2_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}})(1) \\ \downarrow{\tau^*_Y} & & \downarrow{\tau^*_X} \\ H^2_{\text{ét}}(\overline{Y}, \widehat{\mathbb{Z}})(1) & \xrightarrow{f^*} & H^2_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}})(1). \end{array}$$

It suffices to prove the commutativity for the following two squares:

$$\begin{array}{ccc} \widehat{T}(\overline{Y}) & \xrightarrow{f_T^*} & \widehat{T}(\overline{X}) \\ \big\downarrow{\scriptstyle \tau_Y^*|_T} & & \big\downarrow{\scriptstyle \tau_X^*|_T} \\ \widehat{T}(\overline{Y}) & \xrightarrow{f_T^*} & \widehat{T}(\overline{X}). \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{NS}(\overline{Y}) & \xrightarrow{f_N^*} & \mathrm{NS}(\overline{X}) \\ \big\downarrow{\scriptstyle \tau_Y^*|_{\mathrm{NS}}} & & \big\downarrow{\scriptstyle \tau_X^*|_{\mathrm{NS}}} \\ \mathrm{NS}(\overline{Y}) & \xrightarrow{f_N^*} & \mathrm{NS}(\overline{X}). \end{array}$$

The latter commutes by assumption, so it suffices to prove the commutativity of the former. Note that the fields E, $E(X)$ and $E(Y)$ are naturally identified. Let $s \in \mathbb{A}_E^\times$ be as in Theorem (5.1.4), and let $e, c \in U(\mathbb{Q})$ be the unique elements such that $\tau_X^* = e\frac{s}{\bar{s}}$ and $\tau_Y^* = c\frac{s}{\bar{s}}$. Note that f_T^* is $\mathbb{A}_E$-linear, so that the commutativity condition $(f_T^*)^{-1} \circ e\frac{s}{\bar{s}} \circ (f_T^*) = c\frac{s}{\bar{s}}$ amounts to $e = c$. Both $e\frac{s}{\bar{s}}$ and $c\frac{s}{\bar{s}}$ respect the $\widehat{\mathbb{Z}}$-lattice $\widehat{T}(\overline{Y})$, so e/c must do the same. This, together with the fact that $e\bar{e} = c\bar{c} = 1$, implies that e/c is a root of unity, i.e. an integral Hodge isometry of $T(\overline{Y})$. By assumptions, the induced map $T(\overline{Y})[I] \to T(\overline{X})[I]$ is Galois equivariant, therefore $e/c \equiv 1 \mod I$. Since we chose I such that $\mu(E) \to (\mathcal{O}_E/I)^\times$ is injective, we conclude that $e = c$. □

Remark 5.2.5. In case $\overline{X}$ has big discriminant, the proposition says that an isomorphism $f: \overline{X} \to \overline{Y}$ is defined over L if and only if the induced map $f^*: \mathrm{NS}(\overline{Y}) \to \mathrm{NS}(\overline{X})$ is G_L- equivariant.

The immediate corollary we get is:

Corollary 5.2.6. *Let $X, Y/L$ be two principal K3 surfaces with CM over a number field L, and suppose that $X_{\overline{L}}$ and $Y_{\overline{L}}$ are isomorphic. Suppose, moreover, the G_L-modules $\mathrm{NS}(\overline{X})$, $\mathrm{NS}(\overline{Y})$, $T(\overline{Y})[I]$ and $T(\overline{X})[I]$ are trivial. Then every isomorphism $f: \overline{X} \to \overline{Y}$ is already defined over L.*

We are now ready to prove Theorem (5.2.1).

Proof. Let $\tau \in \mathrm{Aut}(\mathbb{C}/E)$ and $s \in \mathbb{A}_{E,f}^\times$ be such that $\mathrm{art}_E(s) = \tau_{|E^{ab}}$. By the main thorem of complex multiplication, we have a unique rational Hodge isometry $\eta(s): T(X)_\mathbb{Q} \to T(X^\tau)_\mathbb{Q}$ such that the following commutes:

$$\begin{array}{ccc} \widehat{T}(X)_\mathbb{Q} & \xrightarrow{\eta(s)\otimes\mathbb{A}_f} & \widehat{T}(X^\tau)_\mathbb{Q} \\ \big\uparrow{\scriptstyle \frac{s}{\bar{s}}} & \nearrow{\scriptstyle \tau^*|_T} & \\ \widehat{T}(X)_\mathbb{Q}. & & \end{array}$$

Let now $e \in E^\times$; if we operate the substitution $s \mapsto es$, we obtain

$$\eta(s) = \frac{e}{\bar{e}}\eta(es),$$

since $\mathrm{art}_E(s) = \mathrm{art}_E(es)$. Suppose that we can find $e \in E^\times$ with

1. $\frac{es}{\overline{es}}\mathcal{O}_E = \mathcal{O}_E$
2. $\frac{es}{\overline{es}} \equiv 1 \mod I$,

and denote by $E(s) \subset E^\times$ the set of elements satisfying the above two conditions. If s is such that $E(s)$ is not empty, then the map

$$\begin{aligned} E(s) &\to U(\mathbb{Q}) \\ e &\mapsto \frac{e}{\bar{e}} \end{aligned} \tag{5.2.0.1}$$

is constant. Indeed, let $e, e' \in E(s)$ and put $x := \frac{e\overline{e'}}{\bar{e}e'}$. By the first point above, we have that $x\mathcal{O}_E = \mathcal{O}_E$, i.e. $x \in \mathcal{O}_E^\times$. Since $\bar{x} = x^{-1}$, we also have that x is a root of unity. By the second point above, we see that $x \equiv 1 \mod I$. Hence $x = 1$, since we have chosen I such that $\mu(E) \to (\mathcal{O}_E/I)^\times$ is injective. Therefore, for every element $s \in \mathbb{A}_{E,f}^\times$ such that $E(s)$ is not empty, we can associate a unique Hodge isometry

$$\eta'(s)\colon T(X)_\mathbb{Q} \to T(X^\tau)_\mathbb{Q}$$

and a unique element $\rho(s) \in U(\mathbb{A}_f)$, by putting $\eta'(s) := \eta(es)$ and $\rho(s) := \frac{es}{\overline{es}}$, for any $e \in E(s)$. By construction, they make the following diagram commute

$$\begin{array}{ccc} \widehat{T}(X)_\mathbb{Q} & \xrightarrow{\eta'(s)\otimes\mathbb{A}_f} & \widehat{T}(X^\tau)_\mathbb{Q} \\ \uparrow{\scriptstyle \rho(s)} & \nearrow{\scriptstyle \tau^*|_T} & \\ \widehat{T}(X)_\mathbb{Q}, & & \end{array} \tag{5.2.0.2}$$

and $\rho(s)\mathcal{O}_E = \mathcal{O}_E$ and $\rho(s) \equiv 1 \mod I$. Note that, since $\tau^*|_T\colon \widehat{T}(X) \xrightarrow{\sim} \widehat{T}(X^\tau)$ and $\rho(s)\colon \widehat{T}(X) \xrightarrow{\sim} \widehat{T}(X)$, we have that the rational Hodge isometry $\eta'(s)$ is actually *integral*, i.e.

$$\eta'(s)\colon T(X) \xrightarrow{\sim} T(X^\tau).$$

The elements $s \in \mathbb{A}_{E,f}^{\times}$ such that $E(s) \neq 0$ correspond to

$$\{s \in \mathbb{A}_{E,f}^{\times} :\ E(s) \neq \emptyset\} = \{s \in \mathbb{A}_{E,f}^{\times} :\ \exists e \in E^{\times} :\ \frac{es}{\overline{es}}\mathcal{O}_E = \mathcal{O}_E,\ \frac{es}{\overline{es}} \equiv 1 \mod I\}.$$

Thanks to Hilbert's Theorem 90 we can write this group as

$$\{s \in \mathbb{A}_{E,f}^{\times} :\ \exists u \in U(\mathbb{Q}) :\ u\frac{s}{\bar{s}}\mathcal{O}_E = \mathcal{O}_E,\ u\frac{s}{\bar{s}} \equiv 1 \mod I\}$$

and this is exactly the norm group S_I associated to the Abelian field extension $F_I(E)/E$. Let us denote this extension by K. Since $E \subset K \subset E^{ab}$ and K has only complex embeddings (because E is a CM number field) we have a commutative diagram

$$\begin{array}{ccc} \mathbb{A}_{K,f}^{\times} & \xrightarrow{\mathrm{art}_K} & G_K^{ab} \\ \downarrow{\scriptstyle \mathrm{Nm}_{K/E}} & & \downarrow{\scriptstyle \mathrm{res}_{K/E}} \\ S_I & \xrightarrow{\mathrm{art}_E|S_I} & G_E^{ab}. \end{array}$$

The map $\rho :\ S_I \to U(\mathbb{A}_f)$ constructed before is continuous and has the property that $\rho(E^{\times}) = 1$. Therefore, it factorises through the profinite completion of $S_I/E^{\times}$ which is canonically isomorphic to $\mathrm{art}_E(S_I) = \mathrm{res}(G_K^{ab})$. In this way, we obtain a map (that we still denote by ρ)

$$\rho :\ G_K^{ab} \to U(\mathbb{A}_f),$$

which is going to be the Galois representation associated to the model X_I. Consider again the diagram ((5.2.0.2)). We have just seen that the association $s \mapsto \rho(s)$ depends only on $\tau \in G_K^{ab}$, therefore also $\eta'(s) = \tau^*|_T \circ \rho(s^{-1})$ depends only on τ. This means that for every $\tau \in G_K^{ab}$ we have associated an element $\rho(\tau) \in U(\mathbb{A}_f)$ and an integral Hodge isometry $\eta'(\tau) :\ T(X) \to T(X^{\tau})$ such that ((5.2.0.2)) commutes. Since $\rho(\tau) \equiv 1 \mod I$, we have that $\eta'(\tau) \equiv \tau^*|_T \mod I$. Therefore, since $I \subset \mathcal{D}_X$ by assumption, $\eta'(\tau) \equiv \tau^*|_T \mod \mathcal{D}_X$ as well. This means that the Hodge isometry

$$\eta'(\tau) \oplus \tau^*|_{NS} :\ T(X) \oplus \mathrm{NS}(X) \xrightarrow{\sim} T(X^{\tau}) \oplus \mathrm{NS}(X^{\tau})$$

can be extended to an integral Hodge isometry $h(\tau) :\ H_B^2(X, \mathbb{Z})(1) \to H_B^2(X^{\tau}, \mathbb{Z})(1)$. By Torelli, we have a unique isomorphism $f(\tau) :\ X^{\tau} \to X$ that induces $h(\tau)$ in cohomology. Hence, for every $\tau \in \mathrm{Aut}(\mathbb{C}/K)$ we have constructed an isomorphism

$f_\tau : X^\tau \to X$ (Note that this makes sense, since by the main theorem of complex multiplication X^τ depends only on $\tau|_{E^{ab}}$). The assignment $\tau \mapsto f_\tau$ defines a Galois-descent data in the sense of Proposition 4.4.4. in [40]. Using Corollary 4.4.6. and Remark 4.4.8. in *loc. cit.*, we employ this descent data to build the model X_I of X over K. Since by construction $f_\tau^* \mathcal{L} = \tau^* \mathcal{L}$ for every $\mathcal{L} \in \mathrm{NS}(X)$, we conclude that G_K acts trivially on $\mathrm{NS}(\overline{X_I})$, i.e. $\rho(X_I) = \rho(X)$. In the same fashion, since $\tau^*|_T$ and $\eta'(\tau)$ agree modulo I, we have that G_K acts trivially on $T(\overline{X}_I)[I]$ as well. The universal property is a direct consequence of Proposition (5.2.4) and Corollary (5.2.6). □

Examples. 1. Let us compute the canonical model of the Fermat quartic $X/\mathbb{C}$ given by the equation $x^4 + y^4 + w^4 + z^4 = 0$. This surface has complex multiplication by $E := \mathbb{Q}(i)$. With an appropriate choice of a basis, the transcendental lattice $T(X)$ can be represented by the quadratic form $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$. One can show that the type of X is $(\mathbb{Z}[i], 4)$. Hence, the discriminant ideal of X is $8\mathbb{Z}[i]$, since the different ideal of E is $2\mathbb{Z}[i]$. The following facts can be checked by hand or using MAGMA:

- The ray class group of E modulo the ideal $8\mathbb{Z}[i]$ is isomorphic to $\mathrm{Cl}_{8\mathbb{Z}[i]}(E) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. We can choose $\mathfrak{a} := (5)$ as an order 2 generator and $\mathfrak{b} := (2i + 7)$ as an order 4 generator.
- The corresponding ray class field is $E(\alpha, \beta)$, with $\alpha^2 + 1 + i = 0$ and $\beta^4 + 2 = 0$.
- The Artin map is as follows: $\mathrm{art}(\mathfrak{a})[\alpha, \beta] = [-\alpha, \beta]$ and $\mathrm{art}(\mathfrak{b})[\alpha, \beta] = [\alpha, i\beta]$.

As shown in Proposition (3.4.7), since X has maximal Picard rank, the K3 class field of E modulo $8\mathbb{Z}[i]$ corresponds to subfield of $E(\alpha, \beta)$ that is fixed by the action of $\{x \in \mathrm{Cl}_{8\mathbb{Z}[i]}(E) : x = \overline{x}\}$. In $\mathrm{Cl}_{8\mathbb{Z}[i]}(E)$ we have $\overline{\mathfrak{a}} = \mathfrak{a}$ and $\overline{\mathfrak{b}} = \mathfrak{a}\mathfrak{b}$. Therefore, $\{x \in \mathrm{Cl}_{8\mathbb{Z}[i]}(E) : x = \overline{x}\}$ is generated by $\mathfrak{a}$ and $\mathfrak{b}^2$ and it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and $F_{K3,8\mathbb{Z}[i]}(E)$ is a quadratic extension of E which corresponds to $E(\sqrt{2}) = \mathbb{Q}(\epsilon)$, where ϵ is a primitive eight-root of unity.

Hence, $F_{K3,8\mathbb{Z}[i]}(E) = E(\sqrt{2}) = \mathbb{Q}(\epsilon)$, where ϵ is a primitive 8-th root of unity. Let us consider for a moment the model $\tilde{X}/\mathbb{Q}(\epsilon)$ of X defined by the same equations. It is a classical fact that the Picard group of the Fermat quartic is defined over $\mathbb{Q}(\epsilon)$. Therefore, we conclude that $\tilde{X}/\mathbb{Q}(\epsilon)$ is the canonical model of X. Note that $\tilde{X}(\mathbb{Q}(\epsilon)) \neq \emptyset$.

2. Let us consider a fundamental discriminant of class number one:

$$d \in \{-7, -8, -11, -19, -43, -67, -163\}.$$

For sake of simplicity, we do not consider $d = -3$ or $d = -4$. For any such a d, denote by $\mathcal{O}_d$ the ring of integers of $E_d := \mathbb{Q}(\sqrt{d})$. Let $X_d/\mathbb{C}$ be the unique (up to isomorphism) K3 surfaces of type $(\mathcal{O}(\sqrt{d}), 1)$. Its discriminant ideal is $\mathcal{D}_X = \mathcal{D}_{E_d}$, and if d is odd then $\mathcal{D}_{E_d} = (\sqrt{d})$ whereas if $d = -8$ we have $\mathcal{D}_{E_d} = (2\sqrt{-2})$. Since $\mu(E_d) = \{\pm 1\}$, we easily see that $\mu(E_d) \to \mathcal{O}_d/\mathcal{D}_d$ is injective for every such a d. Therefore, X_d admits a canonical model X_d^{can} over $F_{K3,\mathcal{D}_d}(E_d)$. Theorem (3.5.4) implies that for every d we have $F_{K3,\mathcal{D}_d}(E_d) = E_d$. Therefore, X_d^{can} can be defined over the CM field E_d. Elkies in his website listed all the K3 surfaces over $\mathbb{Q}$ with discriminant d and Néron-Severi defined over $\mathbb{Q}$. By the universal property in Theorem (5.2.3), these are our canonical models (once base-changed to E_d). Therefore, we have a list of explicit equations:

- $X_{-7}^{\text{can}} : y^2 = x^3 - 75x - (64t + 378 + 64/t)$;
- $X_{-8}^{\text{can}} : y^2 = x^3 - 675x + 27(27t - 196 + 27/t)$;
- $X_{-11}^{\text{can}} : y^2 = x^3 - 1728x - 27(27t + 1078 + 27/t)$;
- $X_{-19}^{\text{can}} : y^2 = x^3 - 192x - (t + 1026 + 1/t)$;
- $X_{-43}^{\text{can}} : y^2 = x^3 - 19200x - (t + 1024002 + 1/t)$;
- $X_{-67}^{\text{can}} : y^2 = x^3 - 580800x - (t + 170368002 + 1/t)$;
- $X_{-163}^{\text{can}} : y^2 = x^3 - 8541868800x - (t + 303862746112002 + 1/t)$.

Since K3 surfaces with big discriminant can be descended canonically, we would like to understand how strong this condition on the discriminant is. We start by considering principal K3 surfaces, i.e. with complex multiplication by the ring of integers of an imaginary quadratic field.

Theorem 5.2.7. *Let $X/\mathbb{C}$ be a principal K3 surface with complex multiplication by an imaginary quadratic field E, so that $\rho(X) = 20$. Then X has big discriminant unless*

- *$E = \mathbb{Q}(i)$ and the type of X is $(\mathcal{O}_E, 1)$ (i.e., $T(X) \cong \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$).*

- *$E = \mathbb{Q}(\sqrt{-3})$ and the type of X is $(\mathcal{O}_E, 1)$ (i.e., $T(X) \cong \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$).*

Proof. Let X having complex multiplication by the ring of integers of $\mathbb{Q}(\sqrt{-d})$, with d a square-free integer, and let (I, α) be the type of X. Suppose that $-d \equiv 2, 3 \mod 4$. In this case, $\mathcal{O}_E = \mathbb{Z}[\sqrt{-d}]$ and $\mathcal{D}_E = (2\sqrt{-d})$. Hence, having big discriminant means that the map

$$\mu(E) \to \left(\frac{\mathbb{Z}[\sqrt{-d}]}{(\alpha)I\bar{I}(2\sqrt{-d})} \right)^{\times} \tag{5.2.0.3}$$

is injective. If $d \neq -1$, then $\mu(E) = \{\pm 1\}$ and the map

$$\mu(E) \to \left(\frac{\mathbb{Z}[\sqrt{-d}]}{(2\sqrt{-d})} \right)^{\times}$$

is already injective, so that we conclude thanks to Proposition (5.1.10). If $d = -1$, then $\mu(E) = \mu_4$ and the map ((5.2.0.3)) has a kernel if and only if $(\alpha)\mathrm{Nm}_{E/\mathbb{Q}}(I) = \mathbb{Z}$. Since $\mathbb{Z}[i]$ is a UFD, every type (I, α) is equivalent to one of the form $(\mathbb{Z}[i], \alpha)$. Hence, the unique type in this case that has not big discriminant is $(\mathbb{Z}[i], 1)$.
Suppose now that $-d \equiv 1 \mod 4$, so that $\mathcal{D}_E = (\sqrt{-d})$. If $d \neq 3$, then $\mu(E) = \mu_2$. Since $(2) \nsubseteq (\sqrt{-d})$, we conclude that

$$\mu_2 \to \left(\frac{\mathcal{O}_E}{(\sqrt{-d})} \right)^{\times}$$

has trivial kernel, hence X has big discriminant. The last case left to consider is when $E = \mathbb{Q}(\sqrt{-3})$. Let $\omega := \frac{1+\sqrt{-d}}{2}$ be a primitive sixth-root of unity, so that $\mathbb{Z}[\omega]$ is the ring of integers of E. Since $\mathbb{Z}[\omega]$ is a UFD, we can suppose our type to be of the form $(\mathbb{Z}[\omega], \alpha)$ for some $\alpha \in \mathbb{Q}_{>0}$. The kernel of the map

$$\mu_6 \to \left(\frac{\mathbb{Z}[\omega]}{(\sqrt{-3})} \right)^{\times}$$

is μ_3, since $\omega^2 - 1 = \sqrt{-3}\omega$. Hence, $(\mathbb{Z}[\omega], \alpha)$ has not big discriminant if and only if $\sqrt{-3}\omega \in (\alpha\sqrt{-3})$, i.e. if and only if $\alpha = 1$. □

Therefore, there are exactly two (isomorphism classes of) complex K3 surfaces with CM by the ring of integers of a imaginary quadratic extension whose discriminant is not "big". These surfaces were studied in [53]. If the CM field is not quadratic, imaginary we have the following finiteness theorem.

Theorem 5.2.8. *Let $E \subset \mathbb{C}$ be a CM number field, and denote by $\mathcal{K}(E)$ the set of isomorphism classes of principal K3 surfaces over $\mathbb{C}$ whose reflex field equals E. Then, up to finitely many elements, every $X \in \mathcal{K}(E)$ has big discriminant.*

Proof. It is sufficient to prove that there are finitely many isomorphism classes of types without big discriminant. Indeed, the type determines the transcendental lattice of a K3, which in turn determines finitely many K3 surfaces (this is the finiteness of the Fourier-Mukai partners, see [16] p. 373, Proposition 3.10). Let $\{I_1, \cdots, I_n\}$ be the finite set of ideals for which the map $\mu(E) \to (\mathcal{O}_E/I_n)^\times$ is not injective. Denote by $\{J_1, \cdots, J_m\}$ be representatives of the elements of $\mathrm{Cl}(E)$. Every type (J, α') is equivalent to one of the form (J_i, α) for some $i \in \{1, \cdots, m\}$. Therefore, if (J_i, α) has not got big discriminant, we have that

$$(\alpha) J_i \overline{J_i} \mathcal{D}_E = I_j,$$

for some $j \in \{1, \cdots, n\}$. Fix now i and j. We want to prove that there are only finitely many isomorphism classes of types of the form (J_i, α) such that the equality

$$(\alpha) J_i \overline{J_i} \mathcal{D}_E = I_j$$

holds. To do this, suppose that both (J_i, α_1) and (J_i, α_2) have discriminant equals to I_j. In particular, we have that $(\alpha_1) = (\alpha_2)$, i.e., there exists a unit $u \in \mathcal{O}_E^\times$ such that $\alpha_1 = u\alpha_2$. Moreover, this unit is totally positive, since the signature of $T(X)$ does not depend on X (thanks to Hodge index Theorem). If we denote by U the group of totally positive units, we see that the isomorphism type of $(J_i, u\alpha)$ for $u \in U$ depends only on the image of u in the quotient $U/\mathrm{Nm}_{E/F}(\mathcal{O}_E^\times)$, where F denotes the maximal totally real subfield of E. Since the group $U/\mathrm{Nm}_{E/F}(\mathcal{O}_E^\times)$ is finite, we conclude the proof. □

5.3 Almost-canonical models and the general case

In the previous section, we have shown that when the map

$$\mu(E) \to (\mathcal{O}_E/\mathcal{D}_X)^\times$$

is injective, X admits a model X^{can} over $K := F_{\mathcal{D}_X}(E)$ with $\rho(X^{\mathrm{can}}/K) = \rho(X/\mathbb{C})$, and that the pair (K, X^{can}) satisfies a universal property. In case when X has not big discriminant, we solved our problem by fixing a level structure (determined by an ideal $I \subset \mathcal{D}_X$) in such a way that the map

$$\mu(E) \to (\mathcal{O}_E/I)^\times$$

is injective. To the pair (X, I) we associated a model X_I over $K := F_I(E)$, which satisfies a universal property analogous to the one of X^{can}. It could happen, though, that even if X has not got big discriminant, it still admits a model over $F_{\mathcal{D}_X}(E)$ with full Picard rank. Our aim in this section is to explain why and when this happens.

Definition 5.3.1. Let $X/\mathbb{C}$ be as usual, and fix an ideal $I \subset \mathcal{D}_X$. We say that the pair (X, I) admits an almost-canonical model if there exists a model Y of X over $K := F_I(E)$ satisfying the following properties:

1. $\rho(Y) = \rho(X)$;
2. $T(\overline{Y})^{G_K} = T(\overline{Y})[I]$;
3. Let $G_K^{ab} \xrightarrow{\mathrm{res}_{K/E}} G_E^{ab}$ be the canonical map. Then the Galois representation $\rho\colon G_K^{ab} \to U(\mathbb{A}_f)$ associated to Y is trivial on $\ker(\mathrm{res}_{K/E})$.

Remarks 5.3.2.

- The condition $I \subset \mathcal{D}_X$ is necessary, and a consequence of the fact that $\rho(Y/K) = \rho(X/\mathbb{C})$. Indeed, by the main theorem of complex multiplication, we have that $\mathrm{Nm}_{K/E}(\mathbb{A}_{K,f}^\times) \subset S_{\mathcal{D}_X}$;
- Condition 3) is a technical condition that is going to be essential in the proof of Theorem (5.3.7). Suppose that we had an Y/K satisfying only conditions 1) and 2) above and let $\tau \in \ker(\mathrm{res}_{K/E})$. By the main theorem of complex multiplication and point 2) above, we have that $\rho(\tau) \in K_I$, since we can choose $s = 1$ in Theorem (5.1.4). Therefore, we see that condition 3) is automatically satisfied by the canonical models constructed during the previous section.

We provide necessary and sufficient conditions on (X, I) to ensure the existence of an almost-canonical model. Moreover, in case these conditions are met, we prove that these models are only finitely many and characterise them in terms of their Hecke characters. Consider the set-up of Theorem (5.1.4): $t \in \mathbb{A}_L^\times$, $s := \mathrm{Nm}_{L/E} t$ and $\tau \in G_L$ such that $\tau_{|L^{ab}} = \mathrm{art}_L(t)$. By Theorem (5.1.4), we have unique $u \in U(\mathbb{Q})$ such that the following commutes

$$\begin{array}{ccc} \widehat{T}(\overline{X})_{\mathbb{Q}} & \xrightarrow{\quad u \quad} & \widehat{T}(\overline{X})_{\mathbb{Q}} \\ {\scriptstyle \frac{s_f}{\bar{s}_f}} \Big\uparrow & \overset{\rho(\tau)}{\nearrow} & \\ \widehat{T}(\overline{X})_{\mathbb{Q}}. & & \end{array} \tag{5.3.0.1}$$

The inclusion $E \subset \mathbb{C}$ induces an archimedean absolute value on E. For every $s \in A_E^\times$ let us denote by s_∞ the component of s corresponding to this archimedean absolute value.

Definition 5.3.3. We define the Hecke character of X/L to be the map

$$\begin{aligned} u_X \colon \mathbb{A}_L^\times &\longrightarrow \mathbb{C}^\times \\ t &\mapsto u \cdot \left(\frac{\mathrm{Nm}_{L/E} t}{\overline{\mathrm{Nm}_{L/E} t}} \right)_\infty \end{aligned}$$

Proposition 5.3.4. *In the situation above, we have:*

1. $u_X(L^\times) = 1$;
2. *The map u_X is continuous;*
3. *Let $\mathfrak{P}$ be a prime of L and let $t \in L_{\mathfrak{P}}^\times \hookrightarrow \mathbb{A}_L^\times$. Suppose that ℓ is a prime of $\mathbb{Q}$ such that $\mathfrak{P}$ does not divide ℓ. Then on $T_\ell(\overline{X}) = T(\overline{X}) \otimes \mathbb{Z}_\ell$ we have* $\mathrm{art}_L(t)^* = u_X(t)$.
4. *Let $\mathfrak{P}$ be a prime of L. The Galois representation $\rho \colon G_L \to U_E(\mathbb{A}_f)$ is unramified at $\mathfrak{P}$ if and only if $u_X(\mathcal{O}_{\mathfrak{P}}^\times) = 1$;*
5. *If t is a finite idele and $\frac{s}{\bar{s}} \mathcal{O}_E = \mathcal{O}_E$, then $u_X(t) \in \mu(E)$;*
6. *If t is a finite idele, $\frac{s}{\bar{s}} \mathcal{O}_E = \mathcal{O}_E$ and $\frac{s}{\bar{s}} \equiv 1 \mod \mathcal{D}_X$, then $d_T(u_X(t)) =$* $d_N(\tau^*)$;

Proof. The proof of most of these facts is identical to the one found in Silverman (Chapter II, Section 9 of [45]);

1. Clear from the definition.

2. Consider the Galois representation $\rho\colon G_L \to \mathrm{U}_E(\mathbb{A}_f)$ associated to X. We know that ρ has open image. Let $m \geq 3$ be an integer and denote by $U_m := \ker(\mathbb{A}_L^\times \xrightarrow{\mathrm{art}_L} G_L^{ab} \xrightarrow{\rho} \mathrm{Aut}(T(\overline{X}) \otimes \mathbb{Z}/m\mathbb{Z}))$. This is an open subgroup of $\mathbb{A}_L^\times$. Consider now the open subgroup of $\mathbb{A}_E^\times$ given by

$$W_m := \{s \in \mathbb{A}_E^\times : \frac{s_f}{\bar{s}_f}\mathcal{O}_E = \mathcal{O}_E \text{ and } \frac{s_f}{\bar{s}_f} \equiv 1 \mod m\}$$

   and put

$$V_m := U_m \cap \mathrm{Nm}_{L/E}^{-1}(W_m).$$

   Let $t \in V_m$ and $s := \mathrm{Nm}_{L/E}t$. The commutative diagram ((5.2.0.2)) together with the condition $\frac{s_f}{\bar{s}_f}\mathcal{O}_E = \mathcal{O}_E$ imply that e is a root of unity. Moreover, since $\frac{s_f}{\bar{s}_f} \equiv 1 \mod m$ and $t \in U_m$, we also have that $e \equiv 1 \mod m$, i.e. $e = 1$. Therefore, for every $t \in W_m$, we have the identity

$$u_X(t) = \frac{s_\infty}{\bar{s}_\infty}$$

   which proves the continuity of u_X on V_m and, since V_m is open in $\mathbb{A}_L^\times$, on all of $\mathbb{A}_L^\times$.

3. By assumptions, the idéle $s := \mathrm{Nm}_{L/E}t$ has component 1 at every finite place not divided by $\mathfrak{P}$ and every archimedean place. Thus, again from the diagram ((5.2.0.2)) we deduce that

$$\tau^*|_{T_\ell(\overline{X})} = u_X(t)|_{T_\ell(\overline{X})}.$$

4. This follows directly from the point above and class field theory.

5. This fact is implicit in the proof of point 2. Indeed, since t is finite, $u_X(t) = e$ with $e\bar{e} = 1$. Moreover, $\tau^*, \frac{s}{\bar{s}} \in \widehat{\mathcal{O}_E}^\times$, hence $e \in \widehat{\mathcal{O}_E}^\times \cap E^\times = \mathcal{O}_E^\times$. The condition $e\bar{e} = 1$ now forces e to be a root of unity.

□

Definition 5.3.5. Let $I \subset \mathcal{O}_E$ be an ideal. We put $\tilde{S}_I := S_I/E^\times$, $W_I := \{(s,u) \in \mathbb{A}^\times_{E,f} \times U(\mathbb{Q}) : u\frac{s}{\bar{s}}\mathcal{O}_E = \mathcal{O}_E \text{ and } u\frac{s}{\bar{s}} \equiv 1 \mod I\}$ and $\tilde{W}_I := W_I/E^\times$, where we consider $E^\times \subset W_I$ via the map $e \mapsto (e, \overline{e}/e)$. Finally, let us put $K_I := \ker(\mu(E) \to (\mathcal{O}_E/I)^\times)$. These groups are related to each-other by the following short exact sequence

$$1 \to K_I \to \tilde{W}_I \to \tilde{S}_I \to 1 \tag{5.3.0.2}$$

where the second map is induced by the projection $\mathbb{A}^\times_{E,f} \times U(\mathbb{Q}) \to \mathbb{A}^\times_{E,f}$ and the first map by the inclusion $\mu(E) \subset U(\mathbb{Q})$.

Let us prove the following lemma

Lemma 5.3.6. *Assume (X, I) admits an almost-canonical model. Then the short exact sequence*

$$1 \to K_I \to \tilde{W}_I \to \tilde{S}_I \to 1$$

splits.

Proof. Assume we have an almost-canonical model Y/K, with $K = F_I(E)$ and consider the associated Galois representation $\rho : G^{ab}_K \to U(\mathbb{A}_f)$. By point 3) in Definition (5.3.1), we have a factorisation $\rho : G^{ab}_K \to \mathrm{res}_{K/E}(G^{ab}_K) \to U(\mathbb{A}_f)$ and, by class field theory, $\mathrm{res}_{K/E}(G^{ab}_K) = \mathrm{art}_E(S_I)$, so that we obtain a map $\rho' : S_I \to U(\mathbb{A}_f)$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{A}^\times_{K,f} & \xrightarrow{\mathrm{art}_K} & G^{ab}_K \\ \downarrow{\scriptstyle \mathrm{Nm}_{K/E}} & & \downarrow{\scriptstyle \rho} \\ S_I & \xrightarrow{\rho'} & U(\mathbb{A}_f). \end{array}$$

By the main theorem of CM, for every $s \in S_I$ there exists a unique $u(s) \in U(\mathbb{Q})$ such that

$$\rho'(s) = \frac{s}{\bar{s}}u(s).$$

Moreover, by point 2) in Definition (5.3.1) we must have that $\rho'(s) \equiv 1 \mod I$. Clearly $\rho'(E^\times) = 1$, so we see that the map $S_I \to W_I$ given by $s \mapsto (s, u(s)$ descends to a splitting of $1 \to K_I \to \tilde{W}_I \to \tilde{S}_I \to 1$. □

The next theorem says that this condition is also sufficient:

Theorem 5.3.7. *Let $X/\mathbb{C}$ be a K3 surface with CM by the ring of integers of $E \subset \mathbb{C}$ and let $I \subset \mathcal{D}_X$ be an ideal. Then (X, I) admits an almost-canonical model Y if* *and only if the sequence*

$$1 \to K_I \to \tilde{W}_I \to \tilde{S}_I \to 1 \tag{5.3.0.3}$$

splits. If $\mathscr{s} : \tilde{S}_I \to \tilde{W}_I$ *is a splitting, there exists an almost-canonical model* $X_{\mathscr{s}}$ *whose Hecke character induces the splitting* $\mathscr{s}$*, in the sense of Lemma* (5.3.6)*. The association* $\mathscr{s} \mapsto X_{\mathscr{s}}$ *is one-to-one between the splittings of* $\tilde{S}_I \to \tilde{W}_I$ *and the almost-canonical models of* X*, up to* $F_I(E)$*-isomorphism.*

Proof. The crucial point in the proof of Theorem ((5.2.1)) was that the map ((5.2.0.1)) took constant values. This relies on the big discriminant condition, and it fails to be true if $K_I \neq 0$. The splitting $\mathscr{s}$ comes into play allowing us to choose one value of ((5.2.0.1)), in the following way. Let us consider once again the commutative diagram

$$\begin{array}{ccc} \mathbb{A}_{K,f}^{\times} & \xrightarrow{\mathrm{art}_K} & G_K^{ab} \\ \downarrow{\scriptstyle \mathrm{Nm}_{K/E}} & & \downarrow{\scriptstyle \mathrm{res}} \\ S_I & \xrightarrow{\mathrm{art}_E|S_I} & G_E^{ab}. \end{array}$$

Let $s \in S_I$ and write $\mathscr{s}([s]) = [(s,u)] \in \tilde{W}_I$. The product $\frac{s}{\bar{s}} \cdot u$ is a well-defined element of $U(\mathbb{A}_f)$, and we denote it by $\rho'(s)$. The map $\rho' : S_I \to U(\mathbb{A}_f)$ has $E^{\times}$ is its kernel, so just like in Theorem (5.2.1) it gives us another map $\rho : G_K^{ab} \to U(\mathbb{A}_f)$, which is going to be the Galois representation associated to $X_{\mathscr{s}}$. The construction of $X_{\mathscr{s}}$ proceeds now exactly like in Theorem (5.2.1). To prove the second part of the theorem, let Y/K be another almost canonical model. By Lemma (5.3.6), Y/K induces a splitting of the short exact sequence ((5.3.0.3)). □

Remark 5.3.8. The condition on the splitting of $1 \to K_I \to \tilde{W}_I \to \tilde{S}_I \to 1$ is more theoretical than practical. Nevertheless, it clarifies what happens when X has big discriminant: in this case, if we choose $I = \mathcal{D}_X$, the short exact sequence boils down to an isomorphism $\tilde{W}_{\mathcal{D}_X} \xrightarrow{\sim} \tilde{S}_{\mathcal{D}_X}$ and therefore admits only one splitting which determines a unique (hence canonical) model over $F_{\mathcal{D}_X}(E)$.

5.4 On the Picard group of canonical models

Let X/K be the canonical model of a K3 surface with big discriminant. Since $\rho(X/K) = \rho(\overline{X}/\overline{K})$, it seems natural to ask whether also the equality $\mathrm{Pic}(X) = \mathrm{Pic}(\overline{X})$ holds. The explicit examples provided by Elkies show that this is indeed the case, when E is quadratic imaginary with class number one. In general, one has a spectral sequence

$$E_2^{p,q} := H^p(K, H^q_{\text{ét}}(\overline{X}, \mathbb{G}_m) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathbb{G}_m)$$

which induces an exact sequence

$$0 \to \mathrm{Pic}(X) \xrightarrow{\delta} \mathrm{Pic}(\overline{X})^{G_K} \to \mathrm{Br}(K) \to \mathrm{Br}(X). \tag{5.4.0.1}$$

The group $\delta(\mathrm{Pic}(\overline{X})^{G_K}) = \ker(\mathrm{Br}(K) \to \mathrm{Br}(X))$ is called the *Amitsur* group of X and it is denoted by $\mathrm{Am}(X)$. It is a finite Abelian group. In order to study some basic properties of $\mathrm{Am}(X)$ let us recall the definition of index of a variety.

Definition 5.4.1. Let X an algebraic variety over a field K. The index of X/K is

$$\delta(X/K) := \gcd\{[L : K] : [L : K] < \infty \text{ and } X(L) \neq \emptyset\}.$$

Proposition 5.4.2. *Let X/K be a smooth projective and geometrically irreducible variety. Then*

$$\delta(X/K) \cdot Am(X) = \{0\}.$$

Proof. This follows from the functoriality of ((5.4.0.1)) and by a restriction-corestriction argument. □

If K is a number field, more can be said. For every place v of K consider the local index $\delta(X_v/K_v)$ of the base change of X to the completion K_v of K at v.

Corollary 5.4.3. *If every local index of X is one, the map* $\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{G_K}$ *is an isomorphism.*

Proof. This follows from Proposition (5.4.2) and the short exact sequence

$$0 \to \mathrm{Br}(K) \to \bigoplus_v \mathrm{Br}(K_v) \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0.$$

□

In particular, if X has a point everywhere locally, then the map $\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{G_K}$ is an isomorphism. Before studying the index of the canonical models, let us say something about the existence of local points. Since K is a CM field, we only need to consider finite places. For every finite place v of K let us denote by $\pi \in K_v$ a local parameter. If $u_X : \mathbb{A}_K^\times \to \mathbb{C}^\times$ is the Hecke character of X and ℓ a prime number invertible in $\mathcal{O}_v$, we have that $u_X(\pi)$ belongs to the reflex field $E \subset \mathbb{C}$ and acts as Frob_v^* on $T_\ell(\overline{X})$.

Proposition 5.4.4. *Assume that X has good reduction at $\mathfrak{P}$ and let $\mathfrak{p}$ the prime ideal $\mathfrak{P} \cap \mathcal{O}_E$, Moreover, let $q := |\mathcal{O}_K/\mathfrak{P}|$ and write X_{red} for the reduction of X modulo $\mathfrak{P}$.*

1. *If $\mathfrak{p}$ is inert or ramified over F (the maximal totally real subfield of E) then $X(K_{\mathfrak{P}}) \neq \emptyset$. Moreover, X_{red} is supersingular and* $\mathrm{Pic}(X_{red}) = \mathrm{Pic}(\overline{X_{red}})$.
2. *If $\mathfrak{p}$ is split over F and $[E : \mathbb{Q}] \leq 12$ or $q \geq 18$, then $X(K_{\mathfrak{P}}) \neq \emptyset$.*

Proof. 1. Let $t \in E_{\mathfrak{p}}$ be a local parameter, so that $\mathrm{Nm}_{K/E}(\pi) = t^n$, where $n = e(\mathfrak{P}/\mathfrak{p}) \cdot f(\mathfrak{P}/\mathfrak{p})$. Since $\mathfrak{p}$ is inert or ramified over F by assumption, we have that $(t) = \overline{(t)}$, i.e. $t/\bar{t}$ is a unit. Since the Galois action on the Picard group is trivial and $K = F_{\mathcal{D}_X}$, by point 6) in Proposition (5.3.4) we conclude that $u_X(\pi) = 1$. Hence, the Frobenius acts trivially on $T_\ell(\overline{X})$ and therefore on the whole cohomology group $H^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)(1)$. By the Lefschetz fixed point formula, we obtain that

$$|X(\mathcal{O}_K/\mathfrak{P})| = q^2 + 22q + 1,$$

so that by Hensel lemma we conclude that $X(K_{\mathfrak{P}}) \neq \emptyset$. As the Tate conjecture is true for X_{red} (see [26], [19] and [7]), we see that $H^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)(1)$ is spanned by algebraic classes, and that X is super-singular. But then the equality $\rho(X_{\text{red}}) = \rho(\overline{X_{\text{red}}})$ and the short exact sequence ((5.4.0.1)) implies that $\mathrm{Pic}(X_{\text{red}}) = \mathrm{Pic}(\overline{X_{\text{red}}})$, since the Brauer group of a finite field is always zero.

2. Let us write $u := u_X(t)$ and $\rho := \rho(X)$. Since $\mathfrak{p}$ is split over F, we see that u is never in $\mathcal{O}_E$. By the Lefschetz fixed point formula we have

$$|X(\mathcal{O}_K/\mathfrak{P})| = q^2 + q\rho + q\,\mathrm{tr}_{E/\mathbb{Q}}(u) + 1.$$

Let us fix a CM type $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$. We have

$$\mathrm{tr}_{E/Q}(u) = \sum_{\sigma :\ E \hookrightarrow \mathbb{C}} \sigma(u) = \sum_{\sigma \in \Phi} \sigma(u) + \overline{\sigma(u)} = 2 \sum_{\sigma \in \Phi} \mathrm{Re}\sigma(u).$$

Clearly, for any $x \in \mathbb{C}$ we have $-|x| \leq \mathrm{Re}(x) \leq |x|$. Since

$$|\sigma(u)| = \sigma(u)\overline{\sigma(u)} = \sigma(u)\sigma(\overline{u}) = 1,$$

we conclude that

$$-[E: \mathbb{Q}] \leq \mathrm{tr}_{E/Q}(u) \leq [E: \mathbb{Q}]$$

and the claim follows by $q \geq 2$ and $\rho + [E: \mathbb{Q}] = 22$.

□

The estimates above are really rough, nevertheless they apply to many cases of interests, like Kummer surfaces with CM (where $[E: \mathbb{Q}]$ is either 2 or 4) and singular K3 surfaces. The following proposition shows us that, at least at the level of cohomology, the canonical models have good reduction properties:

Proposition 5.4.5. *The Galois representation $H^2_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell)(1)$ is unramified at $\mathfrak{P}$.*

Proof. Let $t \in \mathcal{O}^\times_{\mathfrak{P}}$, so that $\mathrm{art}_K(t) \in I_{\mathfrak{P}}$. Since t is a unit and $\rho(X) = \rho(\overline{X})$, we conclude again by point 6) of Proposition (5.3.4). □

Unfortunately, a criterion analogous to the Néron-Ogg-Shafarevich one is false for K3 surfaces. Nevertheless, much work has been done to that direction, see for example [8], [24] and [28]. Let us explain briefly their results before applying them to our questions. Let $\mathcal{O}_K$ be a local Henselian DVR, K its field of fractions and k is residue field of characteristic $p \geq 0$. We make the following assumption

Assumption 5.4.6 ($\star$)**.** Let X/K be a K3 surface over a local field. We say that X satisfies assumption ($\star$) if there exists a finite field extension L/K such that X_L admits a model $\mathcal{X} \to \mathcal{O}_L$ that is a regular algebraic space with trivial canonical sheaf $\omega_{X/\mathcal{O}_L}$, and whose geometric special fiber is a normal crossing divisor.

This assumption holds in the equal characteristic case ($p = 0$) and it is expected to be true in mixed characteristic.

Definition 5.4.7. Let k be a field. A K3 surface X/k with at worst RDP singularities is a proper and geometrically-irreducible surface such that $X_{\overline{K}}$ has at worst rational double point singularities, and whose resolution is a K3 surface.

The main result we are interested in is Theorem 1.3 of [24].

Theorem 5.4.8. *Let X be a K3 surface over a local field K and assume that X satisfies $(\star)$. If the Galois representation $H^2_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell)$ is unramified, then there exists a model $\mathcal{X} \to \mathcal{O}_K$ which is a regular projective scheme, and whose special fibre X_k is a K3 surface with at worst RDP singularities.*

In order to apply this result (and similars) to our question about the surjectivity of the map $\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{G_K}$, we need a way to compare the index of X_K to the index of X_k. This is accomplished by the following Theorem:

Theorem 5.4.9 ([13])**.** *Let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ be a proper flat morphism, with $\mathcal{X}$ regular and irreducible. Let us write the special fibre $\sum_i r_i \Gamma_i$ as a divisor on $\mathcal{X}$, where G_i is irreducible and of multiplicity r_i. Then*

$$\delta(X_K/K) = \gcd_i\{r_i \delta(\Gamma_i^{reg}/k)\},$$

where Γ_i^{reg} denotes the regular locus of Γ_i.

In our situation, k is a finite field, so that $\delta(X/k) = 1$ for any geometrically irreducible algebraic variety X/k. Combining Theorems (5.4.9) and (5.4.8) together with Proposition (5.4.5), we obtain

Corollary 5.4.10. *Assume that for every place v of K, X_v/K_v satisfies $(\star)$. Then* $\mathrm{Pic}(X) = \mathrm{Pic}(\overline{X})$.

Let us now finish this section with some unconditional results.

Theorem 5.4.11.
1. *Let X/K be the canonical model of a Kummer surface associated to an Abelian variety with complex multiplication. There exists an Abelian variety A/K such that $X = Km(A)$. In particular, $X(K) \neq \emptyset$, hence the equality* $\mathrm{Pic}(X) = \mathrm{Pic}(\overline{X})$ *holds. Moreover,* $A[2]$ *is $K-$rational and X has good reduction at every finite place of K that does not divide* 2.
2. *Let X/K be the canonical model of a singular K3 surface. Then $X(K) \neq \emptyset$, hence* $\mathrm{Pic}(X) = \mathrm{Pic}(\overline{X})$. *Moreover, X has potentially good reduction at every place of K which does not divide* 2 *or* 3.

Proof. 1. Since X has full Picard rank, all the sixteen exceptional lines $E_1, \cdots, E_{16}$ are defined over K. This is because they are rigid in their linear system. There exists a reduced divisor $D \subset \overline{X}$ such that, in $\mathrm{Pic}(\overline{X})$, we have $2[D] = E_1 + \cdots + E_{16}$. Consider now the short exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X}) \to \mathrm{Br}(K) \to \mathrm{Br}(X).$$

Since X has a $K-$point, we have that the last map has a section, hence $\mathrm{Pic}(X) = \mathrm{Pic}(\overline{X})$, and we can assume that D is defined over K. Let $\phi : Y \to X$ be a 2-covering associated to D. It follows that the ramification locus of ϕ can be written as $R_\phi := \sum_i C_i$, where each C_i is a (-1) curve. From the arithmetic version of Castelnuovo's contractibility criterion (see [25], Theorem 3.7 p. 416) we can contract these curves to obtain a smooth surface A, and let us denote $\pi : Y \to A$ the contraction morphism. We see that $A(K) \neq \emptyset$, since $\pi(C_i)$ is a $K-$point. Let us denote by $O := \phi(C_1) \in A(K)$. This very same procedure carried out over $\overline{K}$ tells us that $(A_{\overline{K}}, O)$ must be an Abelian surface such that $X_{\overline{K}} \cong \mathrm{Km}(A_{\overline{K}})$. Therefore, (A, O) is an Abelian surface too and $X \cong \mathrm{Km}(A)$. The fact that the full $2-$torsion is defined over K follows by the same statement about the lines C_i. The statement about the good reduction properties follow the results proved in the unpublished master thesis of Tetsushi Ito, that can be found in the appendix of [29].

2. Since $X_{\overline{K}}$ is a singular K3 surface, it admits an elliptic fibration $X_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$ with two singular fibres of type II^* in Kodadira's classification (see Shioda-Inose). It follows immediately that there exist two (-2)-curves C_1 and C_2 on $X_{\overline{K}}$ such that $(C_1, \cdot C_2) = 1$. Therefore, since once again C_1 and C_2 are defined over K, we conclude that their intersection is a $K-$rational point. The good reduction properties follows from Theorem Theorem 0.1 of [29].

□

6 Some applications

In this last section, let us illustrate a couple of applications of the results above. The first is related to Schütt and Elkies' work on the field of definition of singular K3 surfaces, as explained in the introduction. Let $X/\mathbb{C}$ be a singular K3 surface with CM by the ring of integers of a quadratic imaginary extension E. Assume that $T(X)$ is neither isomorphic to $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ nor to $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Theorem 6.0.1. *X admits a model with full Picard group over E if and only if the complex conjugation acts trivially on $\mathrm{Cl}_{\mathcal{D}_X}(E)$, the ray class group modulo $\mathcal{D}_X$.*

Proof. By Theorem (5.2.7), we know that X has big discriminant. Moreover, by Theorem (5.4.11) and the universal property of Theorem (5.2.1), we see that such a model exists if and only if $F_{\mathcal{D}_X}(E) = E$. By remark (3.4.8), this happens if and only if the complex conjugation acts trivially on $\mathrm{Cl}_{\mathcal{D}_X}(E)$. □

We are now in the position to generalise this to any K3 surface with CM. Let us recall some notation introduced at the end of section 1. If E is a CM number field, let $F \subset E$ be the maximal totally real subfield. Remember that for any ideal $I \subset \mathcal{O}_E$, we denote by $E^{I,1} = \{e \in E^\times : \operatorname{ord}_\mathfrak{p}(e-1) \geq \operatorname{ord}_\mathfrak{p} I \ \forall \ \mathfrak{p}|I\}$ and let us put $\mathcal{O}_E^I := \mathcal{O}_E^\times \cap E^{I,1}$. The proof of the following is identical to the one above.

Theorem 6.0.2. *Let $X/\mathbb{C}$ by a K3 surface with CM by the ring of integers of a CM number field E, and assume that X has big discriminant. Then X admits a model with full Picard rank over E if and only if*

1. *The complex conjugation acts trivially on $\mathrm{Cl}_{\mathcal{D}_X}(E)$ and*
2. *The natural inclusion $Nm_{E/F}(\mathcal{O}_E^{\mathcal{D}_X}) \subset \mathcal{O}_F^\times \cap Nm_{E/F}(E^{\mathcal{D}_X,1})$ is an isomorphism (see (3.4.7)).*

In both cases, a necessary condition is that the complex conjugation acts trivially on $\mathrm{Cl}_{\mathcal{D}_X}(E)$, therefore also on $\mathrm{Cl}(E)$.
The second application we have in mind concerns the asymptotic growth of the

fields of definition. Let us fix a CM number field $E \subset \mathbb{C}$ and let us consider the set $\mathcal{K}(E)$ introduced in Theorem (5.2.8). We know that the set $\mathcal{K}(E)$ is infinite, at least when $[E:\mathbb{Q}] \leq 10$ ([ref]). Denote by $\mathcal{K}_b(E) := \{X \in \mathcal{K}(E): X \text{ has big discriminant}\}$. Thanks to Theorem (5.2.8) we know that $\mathcal{K}(E) - \mathcal{K}_b(E)$ is a finite set. For any $X \in \mathcal{K}_b(E)$ let us denote by $F_X := \min\{[K: E\}: X \text{ has a model over } K\}$. We have

$$\frac{1}{C}[F_{\mathcal{D}_X}(E): E] \leq F_X \leq [F_{\mathcal{D}_X}(E): E],$$

where $C > 0$ was defined in the remark after Theorem A in the introduction. The numbers $[F_{\mathcal{D}_X}(E): E]$ are explicitly computed in Theorem (3.5.4). Let us write G for the subgroup of $\mathrm{Aut}(E)$ generated by the complex conjugation, $d_X := \mathcal{D}_X^G = \mathcal{D}_X \cap F$ and $H^i(M) := H^i(G, M)$ for any $G-$module M. Finally, let $n := [F:\mathbb{Q}]$ and let $e(d_X)$ be the product of the ramification indices of all the places of F in E that are coprime to the ideal $d_X \subset \mathcal{O}_F$. We have

$$[F_{\mathcal{D}_X}(E): E] = \frac{2 \cdot h_E \cdot \phi_E(\mathcal{D}_X) \cdot [\mathcal{O}_F^\times : \mathrm{Nm}_{E/F}(\mathcal{O}_E^{\mathcal{D}_X})]}{h_F \cdot \phi_F(d_X) \cdot [\mathcal{O}_E^\times : \mathcal{O}_E^{\mathcal{D}_X}] \cdot e(E/F, d_X) \cdot |H^1(E^{\mathcal{D}_X,1})|},$$

where h_F, h_E are the Hilbert class numbers of F and E respectively, $\phi_E(\mathcal{D}_X) = |(\mathcal{O}_E/\mathcal{D}_X)^\times|$ and $\phi_F(d_X) = |(\mathcal{O}_F/d_X)^\times|$. Let us first consider the term

$$\frac{[\mathcal{O}_F^\times : \mathrm{Nm}_{E/F}(\mathcal{O}_E^{\mathcal{D}_X})]}{[\mathcal{O}_E^\times : \mathcal{O}_E^{\mathcal{D}_X}]}.$$

In the following short exact sequence, we note that the first vertical arrow is surjective

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}_E^{\mathcal{D}_X} & \longrightarrow & \mathcal{O}_E^\times & \longrightarrow & \mathcal{O}_E^\times/\mathcal{O}_E^{\mathcal{D}_X} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Nm}_{E/F}} & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Nm}_{E/F}(\mathcal{O}_E^{\mathcal{D}_X}) & \longrightarrow & \mathcal{O}_F^\times & \longrightarrow & \mathcal{O}_F^\times/\mathrm{Nm}_{E/F}(\mathcal{O}_E^{\mathcal{D}_X}) & \longrightarrow & 1,
\end{array}$$

so that we obtain the following exact sequence

$$1 \to \mu(E)/K_X \to \mathcal{O}_E^\times/\mathcal{O}_E^{\mathcal{D}_X} \to \mathcal{O}_F^\times/\mathrm{Nm}_{E/F}(\mathcal{O}_E^{\mathcal{D}_X}) \to \mathcal{O}_F^\times/\mathrm{Nm}_{E/F}(\mathcal{O}_E^\times) \to 1,$$

that implies

$$\frac{[\mathcal{O}_F^\times : \mathrm{Nm}_{E/F}(\mathcal{O}_E^{\mathcal{D}_X})]}{[\mathcal{O}_E^\times : \mathcal{O}_E^{\mathcal{D}_X}]} = \frac{[\mathcal{O}_F^\times : \mathrm{Nm}_{E/F}(\mathcal{O}_E^\times)]}{[\mu(E): K_X]}. \tag{6.0.0.1}$$

Moreover,

$$1 \leq [\mathcal{O}_F^\times : \mathrm{Nm}_{E/F}(\mathcal{O}_E^\times)] \leq [\mathcal{O}_F^\times : \mathcal{O}_F^{\times 2}] = 2^{[F:\mathbb{Q}]}$$

and

$$1 \leq [\mu(E) : K_X] \leq |\mu(E)| \leq N,$$

where N is the biggest integer such that $\phi(N) \leq [E : \mathbb{Q}]$. Let us now consider the term $e(d_X)$. By Proposition (5.1.10) we have that $\mathcal{D}_X \subset (2)^{-1}\mathcal{D}_{E/F}$, and since the primes the divide $\mathcal{D}_{E/F} \cap F$ are exactly the ones that ramify in E, we see that in the product $e(d_X)$, only the places at infinity appear, and at most the places over 2, so that $e(d_X) \leq 2^{2[F:\mathbb{Q}]}$. Finally, the term $H^1(E^{\mathcal{D}_X,1})$ is described in Proposition (3.5.5), and it also can universally bounded for any CM field E with $[E : \mathbb{Q}] \leq 20$. Therefore, there are constants $A, B > 0$ such that for *any* principal K3 surface X with CM

$$A \leq \frac{2 \cdot [\mathcal{O}_F^\times : \mathrm{Nm}_{E/F}(\mathcal{O}_E^{\mathcal{D}_X})]}{[\mathcal{O}_E^\times : \mathcal{O}_E^{\mathcal{D}_X}] \cdot e(E/F, d_X) \cdot |H^1(E^{\mathcal{D}_X,1})|} \leq B. \qquad (6.0.0.2)$$

As a consequence of this analysis, we have the following Theorem:

Theorem 6.0.3 (Asymptotic growth). *Assume that E is a CM number field with $[E : \mathbb{Q}] \leq 20$. For X varying in $\mathcal{K}(E)$ we have*

$$F_X \sim \frac{\phi_E(\mathcal{D}_X)}{\phi_F(d_X)},$$

where $\sim$ means "up to multiplicative constants", i.e. there exist $A, B > 0$ such that

$$A\frac{\phi_E(\mathcal{D}_X)}{\phi_F(d_X)} \leq F_X \leq B\frac{\phi_E(\mathcal{D}_X)}{\phi_F(d_X)}.$$

In [55], Shafarevich proved that, for a given natural number $N > 0$, there are only finitely many $\mathbb{C}-$isomorphism classes of singular K3 surfaces that can be defined over a number field of degree N. Later, Orr and Skorobogatov in [37] proved that the same is true for any K3 surface with CM. We are now in the position to prove it for principal K3 surfaces:

Theorem 6.0.4. *Let $N > 0$ be an integer. There are only finitely many $\mathbb{C}-$isomoprhism classes of K3 surfaces with CM by the ring of integers of a CM number field E that can be defined over an algebraic extension $K/\mathbb{Q}$ of degree at most N.*

Before proving the theorem, we need a lemma.

Lemma 6.0.5. *Let $N > 0$ be an integer. Then there are only finitely many CM fields $\{E_1, \cdots, E_n\}$ such that if X/K is a K3 surface over a number field K with $[K : \mathbb{Q}] \leq N$ and $\overline{X}$ has CM by the ring of integers of a CM field E, then $E = E_i$ for some $1 \leq i \leq n$.*

Proof. We consider all the fields embedded into $\mathbb{C}$. Let X/K be any K3 surface such that $\overline{X}$ has CM by the ring of integers of a CM field E, with $[K : \mathbb{Q}] \leq N$. Let K' be the extension given by the kernel of the Galois representation associated to $\mathrm{NS}(\overline{X})$. Again, there is a universal constant C such that $[K' : K] \leq C$. In particular, we also have that $[K' \cdot E : \mathbb{Q}] \leq 20CN$. The degree of $F_{\mathcal{D}_X}(E)$ over E satisfies

$$[F_{\mathcal{D}_X}(E) : E] = \frac{2 \cdot h_E \cdot \phi_E(\mathcal{D}_X) \cdot [\mathcal{O}_F^\times : \mathrm{Nm}_{E/F}(\mathcal{O}_E^{\mathcal{D}_X})]}{h_F \cdot \phi_F(d_X) \cdot [\mathcal{O}_E^\times : \mathcal{O}_E^{\mathcal{D}_X}] \cdot e(E/F, d_X) \cdot |H^1(E^{\mathcal{D}_X,1})|},$$

and the term

$$\frac{2 \cdot [\mathcal{O}_F^\times : \mathrm{Nm}_{E/F}(\mathcal{O}_E^{\mathcal{D}_X})]}{[\mathcal{O}_E^\times : \mathcal{O}_E^{\mathcal{D}_X}] \cdot e(E/F, d_X) \cdot |H^1(E^{\mathcal{D}_X,1})|},$$

obeys to the same bound of (6.0.0.2). In particular, since $\phi_E(\mathcal{D}_X) \geq \phi_F(d_X)$, there exists a constant A such that

$$A\frac{h_E}{h_F} \leq [F_{\mathcal{D}_X}(E) : E] \leq 20NC.$$

The same result of Stark employed before tells us that only finitely many CM number fields satisfy the above inequality for a fixed N. □

Let us now prove Theorem (6.0.4).

Proof. By the lemma above and Theorem (5.2.8), it is enough to prove the theorem only for X with big discriminant. Again,there are constants $A, B > 0$ such that for any K3 surface X with big discriminant and CM by the ring of integers of E, we have

$$A \cdot \frac{h_E}{h_F} \cdot \frac{\phi_E(\mathcal{D}_X)}{\phi_F(d_X)} \leq F_X \leq B \cdot \frac{h_E}{h_F} \cdot \frac{\phi_E(\mathcal{D}_X)}{\phi_F(d_X)}.$$

Therefore, we need only to prove that for a given $N > 0$ there are only finitely many $\mathbb{C}-$isomoprhism classes of K3 surfaces with CM by the ring of integers of a

CM number field E such that

$$\frac{h_E}{h_F} \cdot \frac{\phi_E(\mathcal{D}_X)}{\phi_F(d_X)} \leq N. \tag{6.0.0.3}$$

Again by the results obtained in Stark's paper [48], there are only finitely many CM number fields $E \subset \mathbb{C}$ such that

$$\frac{h_E}{h_F} \leq N.$$

Let them be $E_1, E_2, \cdots, E_n$. Therefore, if X is such that (6.0.0.3) holds, we see that X have CM by one of the $E_i's$. For any $1 \leq i \leq n$ there are only finitely many ideals $I \subset \mathcal{O}_{E_i}$ such that

$$\frac{\phi_E(I)}{\phi_F(I \cap F)} \leq N,$$

let them be $I_{i,k}$ for $1 \leq k \leq k_i$. Finally, using the same argument as in the proof of Theorem (5.2.8), we see that for any $I_{i,k}$ there are only finitely many $\mathbb{C}-$isomorphism classes of K3 surfaces with CM by $\mathcal{O}_{E_i}$ and $\mathcal{D}_X = I_{i,k}$. □

7 Conclusion

We would like to conclude this thesis with some open questions and observations.

1. For a CM number field with $10 < [E : \mathbb{Q}] \leq 20$ one would like to know which ideal lattices (I, α) satisfying the properties at the beginning of (3.2) admit a primitive embedding $(I, \alpha) \hookrightarrow \Lambda$, where Λ is the K3 lattice. By the work of Taelman [49] we know that if $[E : \mathbb{Q}] < 20$, for every $\alpha \in F^{\times}$ of the right signature we can embed $(I, \alpha) \otimes \mathbb{Q}$ into $\Lambda \otimes \mathbb{Q}$, thus producing infinitely many K3 surfaces with CM by E, but it is not clear how proceed to construct principal K3 surfaces with CM by E starting from this.

2. It would be interesting to generalise Section (4.2) and Theorem (1.0.6) to non-principal $K3$ surfaces, i.e. dropping the condition on the maximality of $\mathrm{End}_{\mathrm{Hdg}}(T(X))$. This would lead to an elementary proof of Theorem (1.0.7) (and hence to a completely elementary proof of Conjecture 10.1 of [52]). As in the theory of Abelian varieties with CM, such a generalisation should be more a matter of number theory than arithmetic geometry.

3. The next class of varieties that could be studied is given by CM hyperkähler manifolds. Indeed, they share many properties of $K3$'s, especially when their H^2-motive has been proven to be Abelian (either in the absolute Hodge or in the André sense). A class of such hyperkähler manifolds includes those that are of $K3^{[m]}$-type, i.e. deformation equivalent to the Hilbert scheme of m-points of a $K3$ surface. It would be interesting to generalise Theorem (1.0.4) and (1.0.6) to these varieties. In order to use the same idea in Theorem (1.0.6) one would need a strong version of the Torelli theorem, i.e. on the lines of *every Hodge isometry respecting two polarisations is induced by a unique isomorphism*, because the descent data used to produce the canonical models is constructed cohomologically. This is known for example when $m = p^n + 1$ with p a prime number, see Theorem 1.3 of [27]. Such a generalisation would in particular imply a finiteness result analogue to Theorem (1.0.7) for

hyperkähler varieties, and therefore an analogue of Conjecture 10.1 of [52] in this setting.

4. Even if it is known that the analogue of the Néron-Ogg-Shafarevich criterion is false for $K3$ surfaces, one may ask whether or not the canonical models enjoy good reduction everywhere over their field of definition (i.e., over the K3 class field modulo the discriminant ideal). Some (insufficient) evidence is given by the results obtained at the end of (5.4), and it would be interesting to consider the case of Kummer surfaces first. By Proposition (5.4.11) we already know that the canonical models of a Kummer surface enjoys good reduction over every prime that does not lie over 2. Due to the nature of the Kummer construction, studying their good reductions at primes over 2 is a subtle question. Indeed, consider the example of the Fermat quartic $x^4 + y^4 + z^4 + w^4 = 0$ over $\mathbb{Q}(\mu_8)$, the K3 class field modulo its discriminant ideal (see example (5.2)). It is somehow baffling that nothing is known about its reduction at the only prime lying over 2!

5. Finally, we observe that some of our results could help spot principal K3 surfaces with CM in some particular families. Consider $X \to S$ a (non-isotrivial) smooth family of $K3$ surfaces over a number field L, and suppose that S is a rational (quasi-projective) curve. By our results, there are only finitely many $L-$points of S that gives principal CM specialisations. If the generic Picard rank ρ is odd, then the subset of CM points $S_{CM}(L) \subset S(L)$ must be contained in the $L-$points of the Noether-Lefschetz locus of $X \to S$, i.e. the locus where the Picard rank jumps, because $K3$ surfaces with CM have even Picard number. There are some other 'numerical' constrains on $L-$points of S that give principal CM specialisation that could be exploited. Suppose that a specialisation X_t, with $t \in S(L)$, has CM by $\mathcal{O}_E$, with E a CM number field of degree $[E : \mathbb{Q}] = \rho + 1$. A Galois-cohomology arguments shows that there exists a quadratic extension L'/L such that $\mathrm{NS}(X_t \times L') = \mathrm{NS}(\overline{X}_t)$. Therefore, if $\mathcal{D} \subset \mathcal{O}_E$ is the discriminant ideal of X, we must have $F_{K3,\mathcal{D}}(E) \subset EL'$, and therefore that $[F_{K3,\mathcal{D}}(E) : E]$ divides $2[L : \mathbb{Q}]$. There are finitely many pairs $(E, \mathcal{D})$ that satisfy the divisibility condition. Therefore, there are only finitely many values $|\mathcal{O}_{E_t}/\mathcal{D}_t|$ for the cardinality of the discriminant form of a CM specialisation. Usually, one has some control on the discriminant of X_t, so that one can look at those finitely many $t's$ where the Picard jumps and the discriminant of the intersection form on

$\mathrm{NS}(X_t)$ is one of those $|\mathcal{O}_{E_t}/\mathcal{D}_t|$.

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