Schur-Finiteness (and Bass-Finiteness) Conjecture for Quadric Fibrations and Families of Sextic du Val del Pezzo Surfaces

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Abstract. Let $Q \to B$ be a quadric fibration and $T \to B$ a family of sextic du Val del Pezzo surfaces. Making use of the theory of noncommutative mixed motives, we establish a precise relation between the Schur-finiteness conjecture for Q, resp. for T, and the Schur-finiteness conjecture for B. As an application, we prove the Schur-finiteness conjecture for Q, resp. for T, when B is low-dimensional. Along the way, we obtain a proof of the Schur-finiteness conjecture for smooth complete intersections of two or three quadric hypersurfaces. Finally, we prove similar results for the Bass-finiteness conjecture.

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1 Introduction

Schur-finiteness conjecture

Let $\mathcal{C}$ be a $\mathbb{Q}$-linear, idempotent complete, symmetric monoidal category. Given a partition λ of an integer $n \geq 1$, consider the corresponding $\mathbb{Q}$-linear representation V_λ of the symmetric group $\mathfrak{S}_n$ and the associated idempotent $e_\lambda \in \mathbb{Q}[\mathfrak{S}_n]$. Under these notations, the Schur-functor $S_\lambda\colon \mathcal{C} \to \mathcal{C}$ sends an object $a \in \mathcal{C}$ to the direct summand of $a^{\otimes n}$ determined by e_λ. Following Deligne [11, §1], an object $a \in \mathcal{C}$ is called *Schur-finite* if it is annihilated by some Schur-functor. Voevodsky introduced in [39] a triangulated category of geometric mixed motives $\mathrm{DM}_{\mathrm{gm}}(k)_\mathbb{Q}$ (over a perfect base field k). By construction, this category is $\mathbb{Q}$-linear, idempotent complete, symmetric monoidal, and comes equipped with a $\otimes$-functor $M(-)_{\mathbb{Q}}\colon \mathrm{Sm}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ defined on smooth k-schemes of finite type. Given $X \in \mathrm{Sm}(k)$, an important conjecture in the theory of motives is the following:

Conjecture $\mathrm{S}(X)$: The geometric mixed motive $M(X)_{\mathbb{Q}}$ is Schur-finite.

Thanks to the (independent) work of Guletskii [12] and Mazza [28], the conjecture $\mathrm{S}(X)$ holds in the case where $\dim(X) \leq 1$. Thanks to the work of Kimura [21] and Shermenev [31], the conjecture $\mathrm{S}(X)$ also holds in the case where X is an abelian variety. Besides these cases (and some other cases scattered in the literature), the Schur-finiteness conjecture remains wide open. The main goal of this note is to prove the Schur-finiteness conjecture in the new cases of quadric fibrations and families of sextic du Val del Pezzo surfaces.

Quadric fibrations

Our first main result is the following:

Theorem 1. *Let $q\colon Q \to B$ a flat quadric fibration of relative dimension $d-2$. Assume that B and Q are k-smooth, that all the fibers of q have corank ≤ 1, and that the locus $D \subset B$ of the critical values of the fibration q is k-smooth. Under these assumptions, the following holds:*

- *(i) When d is even, we have $\mathrm{S}(Q) \Leftrightarrow \mathrm{S}(B) + \mathrm{S}(\widetilde{B})$, where $\widetilde{B}$ stands for the discriminant 2-fold cover of B (ramified over D).*
- *(ii) When d is odd and $\mathrm{char}(k) \neq 2$, we have $\{\mathrm{S}(V_i)\} + \{\mathrm{S}(\widetilde{D}_i)\} \Rightarrow \mathrm{S}(Q)$, where V_i is any affine open of B and $\widetilde{D}_i$ is any Galois 2-fold cover of $D_i := D \cap V_i$.*

To the best of the author's knowledge, Theorem 1 is new in the literature. Intuitively speaking, it relates the Schur-finiteness conjecture for the total space Q with the Schur-finiteness conjecture for certain coverings/subschemes of the base B. Among other ingredients, its proof makes use of Kontsevich's noncommutative mixed motives of twisted root stacks; consult §3-§4 below for details. Making use of Theorem 1, we are now able to prove the Schur-finiteness conjecture in new cases. Here are two low-dimensional examples:

Corollary 2 (Quadric fibrations over curves). *Let $q\colon Q \to B$ be a quadric fibration as in Theorem 1 with B a curve*[1]*. In this case, $\mathrm{S}(Q)$ holds.*

Corollary 3 (Quadric fibrations over surfaces). *Let $q\colon Q \to B$ be a quadric fibration as in Theorem 1 with B a surface and d odd. In this case, the implication $\mathrm{S}(B) \Rightarrow \mathrm{S}(Q)$ holds.*

Proof. Given a smooth k-surface X, we have $\mathrm{S}(X) \Leftrightarrow \mathrm{S}(U)$ for any open U of X. Therefore, thanks to Theorem 1(ii), the proof follows from the fact that when B is a surface, the conjectures $\{\mathrm{S}(V_i)\}$ can be replaced by the conjecture $\mathrm{S}(B)$. ☐

[1] Since B is a curve, the locus $D \subset B$ of the critical values of q is necessarily k-smooth.

Corollary 3 can be applied to the case where B is (an open subscheme of) an abelian surface or a smooth projective surface with $p_g = 0$ which satisfies Bloch's conjecture (see Guletskii-Pedrini [13, §4 Thm. 7]). Recall that Bloch's conjecture holds for surfaces not of general type (see Bloch-Kas-Leiberman [6]), for surfaces which are rationally dominated by a product of curves (see Kimura [21]), for Godeaux, Catanese and Barlow surfaces (see Voisin [40, 41]), etc.

Remark 4 (Related work). Let $q: Q \to B$ be a quadric fibration as in Theorem 1. In the particular case where Q and B are smooth *projective*, Bouali [9] and Vial [38, §4] "computed" the Chow motive $\mathfrak{h}(Q)_{\mathbb{Q}}$ of Q using smooth projective k-schemes of dimension $\leq \dim(B)$. Since the category of Chow motives (with $\mathbb{Q}$-coefficients) embeds fully-faithfully into $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ (see [39, §4]), these computations lead to an alternative "geometric" proof of Corollaries 2-3. Note that in Theorem 1 and in Corollaries 2-3 we do *not* assume that Q and B are projective; we are (mainly) interested in geometric mixed motives and *not* in pure motives.

INTERSECTIONS OF QUADRICS

Let $Y \subset \mathbb{P}^{d-1}$ be a smooth complete intersection of m quadric hypersurfaces. The linear span of these quadric hypersurfaces gives rise to a flat quadric fibration $q: Q \to \mathbb{P}^{m-1}$ of relative dimension $d-2$, with Q k-smooth. Under these notations, our second main result is the following:

THEOREM 5. *We have* $\mathrm{S}(Q) \Rightarrow \mathrm{S}(Y)$. *When* $2m \leq d$, *the converse also holds.*

By combining Theorem 5 with the above Corollaries 2-3, we hence obtain a proof of the Schur-finiteness conjecture in the following cases:

COROLLARY 6 (Intersections of two or three quadrics). *Assume that the quadric fibration* $q: Q \to \mathbb{P}^{m-1}$ *is as in Theorem 1. In this case, the conjecture* $\mathrm{S}(Y)$ *holds when* Y *is a smooth complete intersection of two, or of three odd-dimensional, quadric hypersurfaces.*

FAMILIES OF SEXTIC DU VAL DEL PEZZO SURFACES

Recall that a *sextic du Val del Pezzo surface* X is a projective k-scheme with at worst du Val singularities and ample anticanonical class such that $K_X^2 = 6$. Consider a *family of sextic du Val del Pezzo surfaces* $f: T \to B$, i.e. a flat morphism f such that for every geometric point $x \in B$ the associated fiber T_x is a sextic du Val del Pezzo surface. Following Kuznetsov [26, §5], given $d \in \{2, 3\}$, let us write $\mathcal{M}_d$ for the relative moduli stack of semistable sheaves on fibers of T over B with Hilbert polynomial $h_d(t) := (3t + d)(t + 1)$, and Z_d for the coarse moduli space of $\mathcal{M}_d$. By construction, we have finite flat morphisms $Z_2 \to B$ and $Z_3 \to B$ of degrees 3 and 2, respectively. Under these notations, our third main result is the following:

Theorem 7. *Let $f: T \to B$ be a family of sextic du Val del Pezzo surfaces. Assume that* $\mathrm{char}(k) \notin \{2,3\}$ *and that* T *is* k*-smooth. Under these assumptions, we have the equivalence of conjectures* $\mathrm{S}(T) \Leftrightarrow \mathrm{S}(B) + \mathrm{S}(Z_2) + \mathrm{S}(Z_3)$.

To the best of the author's knowledge, Theorem 7 is new in the literature. It leads to a proof of the Schur-finiteness conjecture in new cases. Here is an illustrative example:

Corollary 8 (Families of sextic du Val del Pezzo surfaces over curves). *Let $f: T \to B$ be a family of sextic du Val del Pezzo surfaces as in Theorem 7 with B a curve. In this case, the conjecture* $\mathrm{S}(T)$ *holds.*

Remark 9. Let $f: T \to B$ be a family of sextic du Val del Pezzo surfaces as in Theorem 7. To the best of the author's knowledge, the associated geometric mixed motive $M(T)_{\mathbb{Q}}$ has *not* been "computed" (in any non-trivial particular case). Nevertheless, consult Helmsauer [16] for the "computation" of the Chow motive $\mathfrak{h}(X)_{\mathbb{Q}}$ of certain *smooth* (projective) del Pezzo surfaces X.

Remark 10 (Conservativity conjecture). Given a field k equipped with a complex embedding $\sigma: k \to \mathbb{C}$, recall from Ayoub [3, Conj. 2.1] that the *conservativity conjecture* asserts that the Betti realization functor $\mathrm{B}_\sigma: \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} \to \mathcal{D}(\mathbb{Q})$ is conservative. As explained in [3, Prop. 2.26], if the conservativity conjecture holds, then every object of the category $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ is Schur-finite. In particular, the conjecture $\mathrm{S}(X)$ holds for every smooth k-scheme of finite type X (when k is equipped with a complex embedding). However, despite the (monumental) work of Ayoub [4], the conservativity conjecture remains wide open[2].

Bass-finiteness conjecture

Let k be a finite base field and X a smooth k-scheme of finite type. The Bass-finiteness conjecture $\mathrm{B}(X)$ (see [5, §9]) is one of the oldest and most important conjectures in algebraic K-theory. It asserts that the algebraic K-theory groups $K_n(X), n \geq 0$, are finitely generated. In the same vein, given an integer $r \geq 2$, we can consider the conjecture $\mathrm{B}(X)_{1/r}$, where $K_n(X)$ is replaced by $K_n(X)_{1/r} := K_n(X) \otimes \mathbb{Z}[1/r]$. Our fourth main result is the following:

Theorem 11. *The following holds:*

(i) *Theorem 1 and Corollaries 2-3 hold*[3] *similarly for the conjecture* $\mathrm{B}(-)_{1/2}$. *In Corollary 2, the groups $K_n(Q)_{1/2}, n \geq 2$, are moreover finite.*

(ii) *Theorem 5 holds similarly for the conjecture* $\mathrm{B}(-)$.

(iii) *Corollary 6 holds similarly for the conjecture* $\mathrm{B}(-)_{1/2}$. *In the case where Y is a smooth complete intersection of two quadric hypersurfaces, the groups $K_n(Y)_{1/2}, n \geq 2$, are moreover finite.*

[2] I hope that Ayoub manages to correct his work [4] in the (near) future.

[3] Corollary 3 (for the conjecture $\mathrm{B}(-)_{1/2}$) can also be applied to the case where B is (an open subscheme of) an abelian surface; see [19, Cor. 70 and Thm. 82].

(iv) Theorem 7 and Corollary 8 hold similarly for the conjecture $\mathrm{B}(-)_{1/6}$. *In Corollary 8, the groups* $K_n(T)_{1/6}, n \geq 2$, *are moreover finite.*

2 Preliminaries

In what follows, all schemes/stacks are of finite type over the perfect base field k.

Dg categories

For a survey on dg categories we invite the reader to consult [20]. In what follows, we will write $\mathrm{dgcat}(k)$ for the category of (essentially small) dg categories and dg functors. Every (dg) k-algebra A gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes/stacks. Given a k-scheme X (or stack $\mathcal{X}$), the category of perfect complexes of $\mathcal{O}_X$-modules $\mathrm{perf}(X)$ admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$; consult [20, §4.6] [27] for details. More generally, given a sheaf of $\mathcal{O}_X$-algebras $\mathcal{F}$, we can consider the dg category of perfect complexes of $\mathcal{F}$-modules $\mathrm{perf}_{\mathrm{dg}}(X;\mathcal{F})$.

Noncommutative mixed motives

For a book, resp. survey, on noncommutative motives we invite the reader to consult [33], resp. [32]. Recall from [33, §8.5.1] (see also [22, 23, 24]) the definition of Kontsevich's triangulated category of noncommutative mixed motives $\mathrm{NMot}(k)$. By construction, this category is idempotent complete, symmetric monoidal, and comes equipped with a $\otimes$-functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$. In what follows, given a k-scheme X (or stack $\mathcal{X}$) equipped with a sheaf of $\mathcal{O}_X$-algebras $\mathcal{F}$, we will write $U(X;\mathcal{F}) := U(\mathrm{perf}_{\mathrm{dg}}(X;\mathcal{F}))$.

3 Noncommutative mixed motives of twisted root stacks

Let X be a k-scheme, $\mathcal{L}$ a line bundle on X, $\sigma \in \Gamma(X,\mathcal{L})$ a global section, and $r > 0$ an integer. In what follows, we will write $D \subset X$ for the zero locus of σ. Recall from [10, Def. 2.2.1] (see also [1, Appendix B]) that the associated *root stack* $\mathcal{X}$ is defined as the following fiber-product of algebraic stacks

$$\begin{array}{ccc} \mathcal{X} := \sqrt[r]{(\mathcal{L},\sigma)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \theta_r} \\ X & \xrightarrow[(\mathcal{L},\sigma)]{} & [\mathbb{A}^1/\mathbb{G}_m]\,, \end{array}$$

where θ_r stands for the morphism induced by the r^{th} power maps on $\mathbb{A}^1$ and $\mathbb{G}_m$. A *twisted root stack* $(\mathcal{X};\mathcal{F})$ consists of a root stack $\mathcal{X}$ equipped with a sheaf of Azumaya algebras $\mathcal{F}$. In what follows, we will write s for the product of the ranks of $\mathcal{F}$ (at each one of the connected components of $\mathcal{X}$). The following result, of independent interest, will play a key role in the proof of Theorem 1.

Theorem 12. *Assume that X and D are k-smooth.*

(i) We have an isomorphism $U(\mathcal{X}) \simeq U(X) \oplus U(D)^{\oplus(r-1)}$.

(ii) Assume moreover that $\mathrm{char}(k) \neq r$ and that k contains the r^{th} roots of unity. Under these extra assumptions, $U(\mathcal{X};\mathcal{F})_{1/rs}$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_{1/rs}$ containing the noncommutative mixed motives $\{U(V_i)_{1/rs}\}$ and $\{U(\widetilde{D}_i^l)_{1/rs}\}$, where V_i is any affine open subscheme of X and $\widetilde{D}_i^l$ is any Galois l-fold cover of $D_i := D \cap V_i$ with $l \nmid r$ and $l \neq 1$.

Proof. We start by proving item (i). Following [18, Thm. 1.6], the pull-back functor p^* is fully-faithful and we have the following semi-orthogonal decomposition[4] $\mathrm{perf}(X) = \langle \mathrm{perf}(D)_{r-1}, \ldots, \mathrm{perf}(D)_1, p^*(\mathrm{perf}(X)\rangle$. All the categories $\mathrm{perf}(D)_j$ are equivalent (via a Fourier-Mukai type functor) to $\mathrm{perf}(D)$. Therefore, since the functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$ sends semi-orthogonal decompositions to direct sums, we obtain the searched direct sum decomposition $U(\mathcal{X}) \simeq U(X) \oplus U(D)^{\oplus(r-1)}$.

Let us now prove item (ii). We consider first the particular case where $X = \mathrm{Spec}(A)$ is affine and the line bundle $\mathcal{L} = \mathcal{O}_X$ is trivial. Let μ_r be the group of r^{th} roots of unity and $\chi\colon \mu_r \to k^\times$ a (fixed) primitive character. Under these notations, consider the global quotient $[\mathrm{Spec}(B)/\mu_r]$, where $B := A[t]/\langle t^r - \sigma\rangle$ and the μ_r-action on B is given by $g \cdot t := \chi(g)^{-1}t$ for every $g \in \mu_r$ and by $g \cdot a := a$ for every $a \in A$. As explained in [10, Example 2.4.1], the root stack $\mathcal{X}$ agrees, in this particular case, with the global quotient $[\mathrm{Spec}(B)/\mu_r]$. By construction, the induced map $\mathrm{Spec}(B) \to X$ is a r-fold cover ramified over $D \subset X$. Moreover, for every l such that $l \mid r$ and $l \neq 1$, the associated closed subscheme $\mathrm{Spec}(B)^{\mu_l}$ agrees with the ramification divisor $D \subset \mathrm{Spec}(B)$. Therefore, since the functor $U(-)_{1/rs}\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)_{1/rs}$ is an additive invariant of dg categories in the sense of [33, Def. 2.1] (see [33, §8.4.5]), we conclude from [36, Cor. 1.28(ii)] that, in this particular case, $U(\mathcal{X};\mathcal{F})_{1/rs}$ belongs to the smallest thick additive subcategory of $\mathrm{NMot}(k)_{1/rs}$ containing the noncommutative mixed motives $U(\mathrm{Spec}(B))^{\mu_l}_{1/rs}$ and $\{U(\widetilde{D}^l)_{1/rs}\}$, where $\widetilde{D}^l$ is any Galois l-fold cover of D with $l \nmid r$ and $l \neq 1$. Furthermore, since the geometric quotient $\mathrm{Spec}(B)/\!\!/\mu_r$ agrees with X and the latter scheme is k-smooth, [36, Thm. 1.22] implies that $U(\mathrm{Spec}(B))^{\mu_l}_{1/rs}$ is isomorphic to $U(X)_{1/rs}$. This finishes the proof of item (ii) in the particular case where X is affine and the line bundle $\mathcal{L}$ is trivial.

Let us now prove item (ii) in the general case. As explained above, given any affine open subscheme V_i of X which trivializes the line bundle $\mathcal{L}$, the noncommutative mixed motive $U(\mathcal{V}_i;\mathcal{F}_i)_{1/rs}$, with $\mathcal{V}_i := p^{-1}(V_i)$ and $\mathcal{F}_i := \mathcal{F}_{|\mathcal{V}_i}$,

[4] Consult [7, 8] for the definition of semi-orthogonal decomposition.

belongs to the smallest thick additive subcategory of $\mathrm{NMot}(k)_{1/rs}$ containing $U(V_i)_{1/rs}$ and $\{U(\widetilde{D}_i^l)_{1/rs}\}$, where $\widetilde{D}_i^l$ is any Galois l-fold cover of $D_i := D \cap V_i$ with $l \mid r$ and $l \neq 1$. Let us then choose an affine open cover $\{W_i\}$ of X which trivializes the line bundle $\mathcal{L}$. Since X is quasi-compact (recall that X is of finite type over k), this affine open cover admits a *finite* subcover. Consequently, the proof follows by induction from the $\mathbb{Z}[1/rs]$-linearization of the distinguished triangles of Lemma 13 below. □

Lemma 13. *Given an open cover $\{\mathcal{W}_1, \mathcal{W}_2\}$ of X, we have an induced Mayer-Vietoris distinguished triangle of noncommutative mixed motives*

$$U(\mathcal{X};\mathcal{F}) \longrightarrow U(\mathcal{W}_1;\mathcal{F}_1) \oplus U(\mathcal{W}_2;\mathcal{F}_2) \xrightarrow{\pm} U(\mathcal{W}_{12};\mathcal{F}_{12}) \xrightarrow{\partial} \Sigma U(\mathcal{X};\mathcal{F}), \quad (14)$$

where $\mathcal{W}_{12} := \mathcal{W}_1 \cap \mathcal{W}_2$ and $\mathcal{F}_{12} := \mathcal{F}_{|\mathcal{W}_{12}}$.

Proof. Consider the following commutative diagram of dg categories

$$\begin{array}{ccccc}
\mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F})_{\mathcal{Z}} & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F}) & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_1;\mathcal{F}_1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2)_{\mathcal{Z}} & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2) & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_{12};\mathcal{F}_{12}),
\end{array}$$

where $\mathcal{Z}$ stands for the closed complement $\mathcal{X} - \mathcal{W}_1 = \mathcal{W}_2 - \mathcal{W}_{12}$ and $\mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F})_{\mathcal{Z}}$, resp. $\mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2)_{\mathcal{Z}}$, stands for the full dg subcategory of $\mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F})$, resp. $\mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2)$, consisting of those perfect complexes of $\mathcal{F}$-modules, resp. $\mathcal{F}_2$-modules, that are supported on $\mathcal{Z}$. Both rows are short exact sequences of dg categories in the sense of Drinfeld/Keller (see [20, §4.6]) and the left vertical dg functor is a Morita equivalence. Therefore, since the functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$ is a localizing invariant of dg categories in the sense of [33, §8.1], we obtain the following morphism of distinguished triangles:

$$\begin{array}{ccccccc}
U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F})_{\mathcal{Z}}) & \longrightarrow & U(\mathcal{X};\mathcal{F}) & \longrightarrow & U(\mathcal{W}_1;\mathcal{F}_1) & \xrightarrow{\partial} & \Sigma U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F})_{\mathcal{Z}}) \\
\downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \simeq \\
U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2)_{\mathcal{Z}}) & \to & U(\mathcal{W}_2;\mathcal{F}_2) & \to & U(\mathcal{W}_{12};\mathcal{F}_{12}) & \xrightarrow{\partial} & \Sigma U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2)_{\mathcal{Z}}).
\end{array}$$

Finally, since the middle square is homotopy (co)cartesian, we hence obtain the claimed Mayer-Vietoris distinguished triangle (14). □

4 Proof of Theorem 1

Following [25, §3] (see also [2, §1.2]), let E be a vector bundle of rank d on B, $q'\colon \mathbb{P}(E) \to B$ the projectivization of E on B, $\mathcal{O}_{\mathbb{P}(E)}(1)$ the Grothendieck line bundle on $\mathbb{P}(E)$, $\mathcal{L}$ a line bundle on B, and finally

$$\rho \in \Gamma(B, S^2(E^\vee) \otimes \mathcal{L}^\vee) = \Gamma(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \mathcal{L}^\vee)$$

a global section. Given this data, recall that $Q \subset \mathbb{P}(E)$ is defined as the zero locus of ρ on $\mathbb{P}(E)$ and that $q\colon Q \to B$ is the restriction of q' to Q; note that the relative dimension of q is equal to $d-2$. Consider also the discriminant global section $\mathrm{disc}(q) \in \Gamma(B, \det(E^\vee)^{\otimes 2} \otimes (\mathcal{L}^\vee)^{\otimes d})$ and the associated zero locus $D \subset B$; note that D agrees with the locus of the critical values of q.

Recall from [25, §3.5] (see also [2, §1.6]) that when d is even, we can consider the *discriminant cover* $\widetilde{B} := \mathrm{Spec}_B(Z(\mathcal{C}l_0(q)))$ of B, where $Z(\mathcal{C}l_0(q))$ stands for the center of the sheaf $\mathcal{C}l_0(q)$ of even parts of the Clifford algebra associated to q; see [25, §3] (and also [2, §1.5]). By construction, $\widetilde{B}$ is a 2-fold cover ramified over D. Moreover, since D is k-smooth, $\widetilde{B}$ is also k-smooth.

Recall from [25, §3.6] (see also [2, §1.7]) that when d is odd and $\mathrm{char}(k) \neq 2$, we can consider the *discriminant stack* $\mathcal{X} := \sqrt[2]{(\det(E^\vee)^{\otimes 2} \otimes (\mathcal{L}^\vee)^{\otimes d}, \mathrm{disc}(q))/B}$. Since $\mathrm{char}(k) \neq 2$, $\mathcal{X}$ is a Deligne-Mumford stack with coarse moduli space B.

Proposition 15. *Under the above assumptions, the following holds:*

(i) When d is even, we have $U(Q)_{1/2} \simeq U(\widetilde{B})_{1/2} \oplus U(B)_{1/2}^{\oplus(d-2)}$.

(ii) When d is odd and $\mathrm{char}(k) \neq 2$, $U(Q)_{1/2}$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_{1/2}$ containing the noncommutative mixed motives $\{U(V_i)_{1/2}\}$ and $\{U(\widetilde{D}_i)_{1/2}\}$, where V_i is any affine open subscheme of B and $\widetilde{D}_i$ is any Galois 2-fold cover of $D_i := D \cap V_i$.

Proof. As proved in [25, Thm. 4.2] (see also [2, Thm. 2.2.1]), we have the following semi-orthogonal decomposition

$$\mathrm{perf}(Q) = \langle \mathrm{perf}(B; \mathcal{C}l_0(q)), \mathrm{perf}(B)_1, \ldots, \mathrm{perf}(B)_{d-2} \rangle ,$$

where $\mathrm{perf}(B)_j := q^*(\mathrm{perf}(B)) \otimes \mathcal{O}_{Q/B}(j)$. All the categories $\mathrm{perf}(B)_j$ are equivalent (via a Fourier-Mukai type functor) to $\mathrm{perf}(B)$. Therefore, since the functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$ sends semi-orthogonal decompositions to direct sums, we obtain the decomposition $U(Q) \simeq U(B; \mathcal{C}l_0(q))) \oplus U(B)^{\oplus(d-2)}$.

We start by proving item (i). As explained in [25, §3.5] (see also [2, §1.6]), when d is even, the category $\mathrm{perf}(B; \mathcal{C}l_0(q))$ is equivalent (via a Fourier-Mukai type functor) to $\mathrm{perf}(\widetilde{B}; \mathcal{F})$ where $\mathcal{F}$ is a certain sheaf of Azumaya algebras on $\widetilde{B}$ of rank $2^{\frac{d}{2}-1}$. This leads to an isomorphism $U(B; \mathcal{C}l_0(q)) \simeq U(\widetilde{B}; \mathcal{F})$. Making use of [37, Thm. 2.1], we hence conclude that $U(B; \mathcal{C}l_0(q))_{1/2}$ is isomorphic to $U(\widetilde{B}; \mathcal{F})_{1/2} \simeq U(\widetilde{B})_{1/2}$. Consequently, we obtain the isomorphism of item (i).

Let us now prove item (ii). As explained in [25, §3.6] (see also [2, §1.7]), when d is odd, the category $\mathrm{perf}(B; \mathcal{C}l_0(q))$ is equivalent (via a Fourier-Mukai type functor) to $\mathrm{perf}(\mathcal{X}; \mathcal{F})$ where $\mathcal{F}$ is a certain sheaf of Azumaya algebras on $\mathcal{X}$ of rank $2^{\frac{d-1}{2}}$. This leads to an isomorphism $U(B; \mathcal{C}l_0(q)) \simeq U(\mathcal{X}; \mathcal{F})$. By combining Theorem 12(ii) with the isomorphism $U(Q) \simeq U(\mathcal{X}; \mathcal{F}) \oplus U(B)^{\oplus(d-2)}$, we hence conclude that $U(Q)_{1/2}$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_{1/2}$ containing $U(B)_{1/2}$, $\{U(V_i)_{1/2}\}$, and $\{U(\widetilde{D}_i)_{1/2}\}$, where V_i is any affine open subscheme of B and $\widetilde{D}_i$ is any Galois 2-fold cover of D_i. We now claim that $U(B)_{1/2}$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_{1/2}$ containing $\{U(V_i)_{1/2}\}$; note that this would conclude the proof. Choose an affine open cover $\{W_i\}$ of B. Since B is quasi-compact (recall that B is of finite type over k), this affine open cover admits a *finite* subcover. Therefore, similarly to the proof of Theorem 12, our claim follows from an inductive argument using the $\mathbb{Z}[1/2]$-linearization of the Mayer-Vietoris distinguished triangles $U(B) \to U(W_1) \oplus U(W_2) \stackrel{\pm}{\to} U(W_{12}) \stackrel{\partial}{\to} \Sigma U(B)$. □

As proved in [34, Thm. 2.8], there exists a $\mathbb{Q}$-linear, fully-faithful, $\otimes$-functor Φ making the following diagram commute

$$\begin{array}{ccc} \mathrm{Sm}(k) & \xrightarrow{X \mapsto \mathrm{perf}_{\mathrm{dg}}(X)} & \mathrm{dgcat}(k) \\ \downarrow{\scriptstyle M(-)_{\mathbb{Q}}} & & \downarrow{\scriptstyle U(-)_{\mathbb{Q}}} \\ \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} & & \mathrm{NMot}(k)_{\mathbb{Q}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \underline{\mathrm{Hom}}(-,U(k)_{\mathbb{Q}})} \\ \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)[2]} & \xrightarrow[\Phi]{} & \mathrm{NMot}(k)_{\mathbb{Q}}\,, \end{array} \tag{16}$$

where $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)[2]}$ stands for the orbit category with respect to the Tate motive $\mathbb{Q}(1)[2]$ and $\underline{\mathrm{Hom}}(-,-)$ for the internal Hom of the monoidal structure; note that the functors $X \mapsto \mathrm{perf}_{\mathrm{dg}}(X)$ and $\underline{\mathrm{Hom}}(-,U(k)_{\mathbb{Q}})$ are contravariant. By construction, π is a faithful $\otimes$-functor. Therefore, it follows from [28, Lem. 1.11] that we have the following equivalence:

$$\mathrm{S}(X) \Leftrightarrow \text{noncommutative mixed motive } (\Phi\circ\pi)(M(X)_{\mathbb{Q}}) \text{ is Schur-finite.} \tag{17}$$

We now have all the ingredients necessary to conclude the proof of Theorem 1.

ITEM (I)

The above functors π and $\underline{\mathrm{Hom}}(-,U(k)_{\mathbb{Q}})$ are $\mathbb{Q}$-linear. Therefore, by combining Proposition 15(i) with the commutative diagram (16), we conclude that

$$(\Phi\circ\pi)(M(Q)_{\mathbb{Q}}) \simeq (\Phi\circ\pi)(M(\widetilde{B})_{\mathbb{Q}}) \oplus (\Phi\circ\pi)(M(B)_{\mathbb{Q}})^{\oplus(d-2)}\,. \tag{18}$$

Since Schur-finiteness is stable under direct sums and direct summands, the proof of the equivalence $\mathrm{S}(Q) \Leftrightarrow \mathrm{S}(B) + \mathrm{S}(\widetilde{B})$ follows then from (17)-(18).

ITEM (II)

Recall from [33, §8.5.1-8.5.2] that, by construction, $\mathrm{NMot}(k)_{\mathbb{Q}}$ is a $\mathbb{Q}$-linear closed symmetric monoidal triangulated category in the sense of Hovey [17, §6-7]. As proved in [12, Thm. 1], this implies that Schur-finiteness has the 2-out-of-3 property with respect to distinguished triangles. The functor $\underline{\mathrm{Hom}}(-,U(k)_{\mathbb{Q}})$

is triangulated. Hence, by combining Proposition 15(ii) with the commutative diagram (16), we conclude that $(\Phi\circ\pi)(M(Q)_{\mathbb{Q}})$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_{\mathbb{Q}}$ containing the noncommutative mixed motives $\{(\Phi\circ\pi)(M(V_i)_{\mathbb{Q}})\}$ and $\{(\Phi\circ\pi)(M(\widetilde{D}_i)_{\mathbb{Q}})\}$, where V_i is any affine open subscheme of B and $\widetilde{D}_i$ is any Galois 2-fold cover of D_i. Since by assumption the conjectures $\{\mathrm{S}(V_i)\}$ and $\{\mathrm{S}(\widetilde{D}_i)\}$ hold, (17) implies that the noncommutative mixed motives $\{(\Phi\circ\pi)(M(V_i)_{\mathbb{Q}})\}$ and $\{(\Phi\circ\pi)(M(\widetilde{D}_i)_{\mathbb{Q}})\}$ are Schur-finite. Therefore, making use of the 2-out-of-3 property of Schur-finiteness with respect to distinguished triangles (and of the stability of Schur-finiteness under direct summands), we conclude that $(\Phi\circ\pi)(M(Q)_{\mathbb{Q}})$ is also Schur-finite. The proof follows now from the above equivalence (17).

5 Proof of Theorem 5

Recall from the proof of Proposition 15 that we have the semi-orthogonal decomposition $\mathrm{perf}(Q)=\langle\mathrm{perf}(\mathbb{P}^{m-1};\mathcal{C}l_0(q)),\mathrm{perf}(\mathbb{P}^{m-1})_1,\ldots,\mathrm{perf}(\mathbb{P}^{m-1})_{d-2}\rangle$, and consequently the following direct sum decomposition:

$$U(Q)\simeq U(\mathbb{P}^{m-1};\mathcal{C}l_0(q))\oplus U(\mathbb{P}^{m-1})^{\oplus(d-2)}\,. \tag{19}$$

As proved in [25, Thm. 5.5] (see also [2, Thm. 2.3.7]), the following also holds:

(a) When $2m<d$, we have the following semi-orthogonal decomposition $\mathrm{perf}(Y)=\langle\mathrm{perf}(\mathbb{P}^{m-1};\mathcal{C}l_0(q)),\mathcal{O}(1),\ldots,\mathcal{O}(d-2m)\rangle$. Consequently, since the functor $U\colon\mathrm{dgcat}(k)\to\mathrm{NMot}(k)$ sends semi-orthogonal decompositions to direct sums, we obtain the following direct sum decomposition $U(Y)\simeq U(\mathbb{P}^{m-1};\mathcal{C}l_0(q))\oplus U(k)^{\oplus(d-2m)}$.

(b) When $2m=d$, the category $\mathrm{perf}(Y)$ is equivalence (via a Fourier-Mukai type functor) to $\mathrm{perf}(\mathbb{P}^{m-1};\mathcal{C}l_0(q))$. Consequently, we obtain an isomorphism of noncommutative mixed motives $U(Y)\simeq U(\mathbb{P}^{m-1};\mathcal{C}l_0(q))$.

(c) When $2m>d$, $\mathrm{perf}(Y)$ is an admissible subcategory of $\mathrm{perf}(\mathbb{P}^{m-1};\mathcal{C}l_0(q))$. Hence, $U(Y)$ is a direct summand of $U(\mathbb{P}^{m-1};\mathcal{C}l_0(q))$.

Let us now prove the implication $\mathrm{S}(Q)\Rightarrow\mathrm{S}(Y)$. If the conjecture $\mathrm{S}(Q)$ holds, then it follows from the decomposition (19), from the commutative diagram (16), from the equivalence (17), and from the stability of Schur-finiteness under direct summands, that the noncommutative mixed motive $\underline{\mathrm{Hom}}(U(\mathbb{P}^{m-1};\mathcal{C}l_0(q))_{\mathbb{Q}},U(k)_{\mathbb{Q}})$ is Schur-finite. Making use of the above descriptions (a)-(c) of $U(Y)$ and of the commutative diagram (16), we hence conclude that the noncommutative mixed motive $(\Phi\circ\pi)(M(Y)_{\mathbb{Q}})$ is also Schur-finite. Consequently, the conjecture $\mathrm{S}(Y)$ follows now from the above equivalence (17). Finally, note that when $2m\leq d$, a similar argument proves the converse implication $\mathrm{S}(Y)\Rightarrow\mathrm{S}(Q)$.

6 Proof of Theorem 7

Recall first from [26, Prop. 5.12] that since $\mathrm{char}(k) \notin \{2,3\}$ and T is k-smooth, the k-schemes B, Z_2 and Z_3 are also k-smooth.

Proposition 20. *We have* $U(T)_{1/6} \simeq U(B)_{1/6} \oplus U(Z_2)_{1/6} \oplus U(Z_3)_{1/6}$.

Proof. As proved in [26, Thm. 5.2 and Prop. 5.10], we have the semi-orthogonal decomposition $\mathrm{perf}(T) = \langle \mathrm{perf}(B), \mathrm{perf}(Z_2; \mathcal{F}_2), \mathrm{perf}(Z_3; \mathcal{F}_3)\rangle$, where $\mathcal{F}_2$ (resp. $\mathcal{F}_3$) is a certain sheaf of Azumaya algebras over Z_2 (resp. Z_3) of order 2 (resp. 3). Recall that the functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$ sends semi-orthogonal decompositions to direct sums. Hence, we obtain the direct sum decomposition:

$$U(T) \simeq U(B) \oplus U(Z_2; \mathcal{F}_2) \oplus U(Z_3; \mathcal{F}_3)\,. \tag{21}$$

Since $\mathcal{F}_2$ (resp. $\mathcal{F}_3$) is of order 2 (resp. 3), the rank of $\mathcal{F}_2$ (resp. $\mathcal{F}_3$) is necessarily a power of 2 (resp. 3). Making use of [37, Thm. 2.1], we hence conclude that the noncommutative mixed motive $U(Z_2; \mathcal{F}_2)_{1/2}$ (resp. $U(Z_3; \mathcal{F}_3)_{1/3}$) is isomorphic to $U(Z_2)_{1/2}$ (resp. $U(Z_3)_{1/3}$). Consequently, the proof follows now from the $\mathbb{Z}[1/6]$-linearization of (21). □

The functors π and $\underline{\mathrm{Hom}}(-, U(k)_{\mathbb{Q}})$ in (16) are $\mathbb{Q}$-linear. Therefore, similarly to the proof of item (i) of Theorem 1, by combining Proposition 20 with the commutative diagram (16), we conclude that

$$(\Phi\circ\pi)(M(T)_{\mathbb{Q}}) \simeq (\Phi\circ\pi)(M(\widetilde{B})_{\mathbb{Q}}) \oplus (\Phi\circ\pi)(M(Z_2)_{\mathbb{Q}}) \oplus (\Phi\circ\pi)(M(Z_3)_{\mathbb{Q}})\,. \tag{22}$$

Since Schur-finiteness is stable under direct sums and direct summands, the proof follows then from the combination of (22) with the equivalence (17).

7 Proof of Theorem 11

Item (i)

We start by proving the first claim. As explained in [33, §8.6] (see also [35, Thm. 15.10]), given $X \in \mathrm{Sm}(k)$, we have the isomorphisms of abelian groups:

$$\mathrm{Hom}_{\mathrm{NMot}(k)}(U(k), \Sigma^{-n} U(X)) \simeq K_n(X) \qquad n \in \mathbb{Z}\,. \tag{23}$$

Assume that d is even. By combining Proposition 15(i) with the $\mathbb{Z}[1/2]$-linearization of (23), we conclude that $K_n(Q)_{1/2} \simeq K_n(\widetilde{B})_{1/2} \oplus K_n(B)_{1/2}^{\oplus(d-2)}$. Therefore, since finite generation is stable under direct sums and direct summands, we obtain the equivalence $\mathrm{B}(Q)_{1/2} \Leftrightarrow \mathrm{B}(B)_{1/2} + \mathrm{B}(\widetilde{B})_{1/2}$. Assume now that d is odd and that $\mathrm{char}(k) \neq 2$. Finite generation has the 2-out-of-3 property with respect to (short or long) exact sequences and is stable under direct summands. Therefore, the proof of the following implication

$$\{\mathrm{B}(V_i)_{1/2}\} + \{\mathrm{B}(\widetilde{D}_i)_{1/2}\} \Rightarrow \mathrm{B}(Q)_{1/2}$$

follows from the combination of Proposition 15(ii) with the $\mathbb{Z}[1/2]$-linearization of (23). Finally, recall from [14, 29, 30] that the conjecture $\mathrm{B}(X)$ holds in the case where $\dim(X) \leq 1$. Therefore, the Corollaries 2-3 also hold similarly for the conjecture $\mathrm{B}(-)_{1/2}$.

We now prove the second claim. Let $q\colon Q \to B$ be a quadric fibration as in Theorem 1 with B a curve. Thanks to Corollary 2 (for the conjecture $\mathrm{B}(-)_{1/2}$), it suffices to show that the groups $K_n(Q), n \geq 2$, are torsion. Assume first that d is even. By combining Proposition 15(i) with the $\mathbb{Q}$-linearization of (23), we obtain an isomorphism $K_n(Q)_\mathbb{Q} \simeq K_n(\widetilde{B})_\mathbb{Q} \oplus K_n(B)_\mathbb{Q}^{\oplus(d-2)}$. Thanks to Proposition 24 below, we have $K_n(\widetilde{B})_\mathbb{Q} = K_n(B)_\mathbb{Q} = 0$ for every $n \geq 2$. Therefore, we conclude that the groups $K_n(Q), n \geq 2$, are torsion. Assume now that d is even and that $\mathrm{char}(k) \neq 2$. Thanks to Proposition 15(ii), $U(Q)_\mathbb{Q}$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_\mathbb{Q}$ containing the noncommutative mixed motives $\{U(V_i)_\mathbb{Q}\}$ and $\{U(\widetilde{D}_i)_\mathbb{Q}\}$, where V_i is any affine open subscheme of B and $\widetilde{D}_i$ is any Galois 2-fold cover of D_i. Moreover, $U(Q)_\mathbb{Q}$ may be explicitly obtained from $\{U(V_i)_\mathbb{Q}\}$ and $\{U(\widetilde{D}_i)_\mathbb{Q}\}$ using solely the $\mathbb{Q}$-linearization of the Mayer-Vietoris distinguished triangles. Therefore, since $K_n(V_i)_\mathbb{Q} = 0$ for every $n \geq 2$ (see Proposition 24 below) and $K_n(\widetilde{D}_i)_\mathbb{Q} = 0$ for every $n \geq 1$ (see Quillen's computation [30] of the algebraic K-theory of a finite field), an inductive argument using the $\mathbb{Q}$-linearization of (23) and the $\mathbb{Q}$-linearization of the Mayer-Vietoris distinguished triangles implies that the groups $K_n(Q), n \geq 2$, are torsion.

Proposition 24. *We have $K_n(X)_\mathbb{Q} = 0, n \geq 2$, for every smooth k-curve X.*

Proof. In the particular case where X is affine, this result was proved in [15, Cor. 3.2.3] (see also [14, Thm. 0.5]). In the general case, choose an affine open cover $\{W_i\}$ of X. Since X is quasi-compact, this affine open cover admits a *finite* subcover. Therefore, the proof follows from an inductive argument (similar to the one in the proof of Theorem 12(ii)) using the $\mathbb{Q}$-linearization of (23) and the $\mathbb{Q}$-linearization of the Mayer-Vietoris distinguished triangles. □

Item (ii)

If the conjecture $\mathrm{B}(Q)$ holds, then it follows from the decomposition (19) and from the isomorphisms (23) that the algebraic K-theory groups $K_n(\mathrm{perf}_\mathrm{dg}(\mathbb{P}^{m-1}; \mathcal{C}l_0(q))), n \geq 0$, are finitely generated. Therefore, by combining the descriptions (a)-(c) of the noncommutative mixed motive $U(Y)$ (see the proof of Theorem 5) with (23), we conclude that the conjecture $\mathrm{B}(Y)$ also holds. Note that when $2m \leq d$, a similar argument proves the converse implication $\mathrm{B}(Y) \Rightarrow \mathrm{B}(Q)$.

Item (iii)

Items (i)-(ii) of Theorem 11 imply that Corollary 6 holds similarly for the conjecture $\mathrm{B}(-)_{1/2}$. We now address the second claim. Let $q\colon Q \to \mathbb{P}^1$ be the quadric fibration associated to the smooth complete intersection Y of two quadric hypersurfaces. Thanks to item (i), the groups $K_n(Q)_{1/2}, n \geq 2$, are finite. Hence, making use of the decomposition (19), of the $\mathbb{Z}[1/2]$-linearization of (23), and of the above descriptions (a)-(c) of $U(Y)$ (see the proof of Theorem 5), we conclude that the groups $K_n(Y)_{1/2}, n \geq 2$, are also finite.

ITEM (IV)

We start by proving the first claim. By combining Proposition 20 with the $\mathbb{Z}[1/6]$-linearization of (23), we conclude that

$$K_n(T)_{1/6} \simeq K_n(B)_{1/6} \oplus K_n(Z_2)_{1/6} \oplus K_n(Z_3)_{1/6}\,.$$

Therefore, since finite generation is stable under sums and direct summands, we obtain the equivalence $\mathrm{B}(T)_{1/6} \Leftrightarrow \mathrm{B}(B)_{1/6} + \mathrm{B}(Z_2)_{1/6} + \mathrm{B}(Z_3)_{1/6}$. As mentioned in the proof of item (i), the conjecture $\mathrm{B}(X)$ holds in the case where $\dim(X) \leq 1$. Hence, Corollary 8 also holds similarly for the conjecture $\mathrm{B}(-)_{1/6}$. We now prove the second claim. Let $f\colon T \to B$ be a family of sextic du Val del Pezzo surfaces as in Theorem 7 with B a curve. Similarly to the proof of item (i) of Theorem 11, it suffices to show that the groups $K_n(T), n \geq 2$, are torsion. By combining Proposition 20 with the $\mathbb{Q}$-linearization of (23), we obtain an isomorphism $K_n(T)_\mathbb{Q} \simeq K_n(B)_\mathbb{Q} \oplus K_n(Z_2)_\mathbb{Q} \oplus K_n(Z_3)_\mathbb{Q}$. Thanks to Proposition 24, we have moreover $K_n(B)_\mathbb{Q} = K_n(Z_2)_\mathbb{Q} = K_n(Z_3)_\mathbb{Q} = 0$ for every $n \geq 2$. Therefore, we conclude that the groups $K_n(T), n \geq 2$, are torsion.

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