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SOME RECENT DEVELOPMENTS IN THEORY OF FRACTIONAL POSITIVE AND CONE LINEAR SYSTEMS

Запропоновано огляд та деякі нові результати у теорії фракційних додатніх та конусних одновимірних (1D) та двовимірних (2D) лінійних систем. Подано рівняння стану та їхнє розв'язання для фракційних неперервних та дискретних лінійних систем. Встановлено необхідні та достатні умови для внутрішньої та зовнішньої додатності фракційних лінійних систем. Показано їхню досяжність. Запропоновано нову форму запису конусних систем та методи, придатні для комп'ютерного розрахунку таких систем. Представлено додатні фракційні 2D лінійні системи. Встановлено необхідні та достатні умови для додатності та досяжності. Теоретичні виклади проілюстровано чисельними прикладами 1D та 2D лінійних систем.

Предложены анализ состояния вопроса и некоторые новые результаты, полученные в теории фракционных положительных и конусных одномерных (1D) и двумерных (2D) линейных систем. Приведены уравнения состояния и их решения для фракционных непрерывных и дискретных линейных систем. Выявлены необходимые и достаточные условия для внутренней и внешней положительности фракционных линейных систем. Показана их достижимость. Предложена новая форма записи конусных систем, а также методы их компьютерного расчета. Представлены положительные фракционные 2D линейные системы. Установлены необходимые и достаточные условия для положительности и достижимости. Теоретические выкладки иллюстрированы численными примерами 1D и 1D линейных систем.

Introduction. In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems is given in the monographs [2, 5]. An extension of positive systems are the cone systems. The notion of cone systems was introduced in [6]. Roughly speaking cone system is a system obtained from positive one by substitution of the positive orthants of states, inputs and outputs by suitable arbitrary cones. The realization problem for cone systems has been addressed in [6].

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [21, 24, 26]. This idea has been used by engineers for modeling different processes in the late 1960s [1, 3, 25-27]. Mathematical fundamentals of fractional calculus are given in [21, 23, 24, 26]. A generalization of the Kalman filter for fractional order systems has been proposed in [27]. Fractional polynomials and nD systems have been investigated in [4]. The positive controllability of positive systems and approximate constrained controllability of mechanical systems have been investigated in [20, 21].

The aim of this paper is to give an overview of some recent developments and new results in the theory of fractional positive and cone 1D and 2D linear systems.

The paper is organized as follows. The standard and positive fractional continuous-time linear systems are addressed in section 2. Necessary and sufficient conditions for the positivity of the system are established. Similar problem for the discrete-time linear systems are considered in section 3. Section 4 is devoted to the reachability of positive fractional discrete-time linear systems. The realization problem for positive fractional continuous-time systems is addressed in section 5. The cone fractional discrete-time linear systems and their reachability are considered in section 6. Positive fractional 2D systems and their reachability are addressed in section 7. Concluding remarks and open problems are given in section 8.

The following notation will be used in this paper.

Let $\Re^{n\times m}$ be the set of $n\times m$ real matrices and $\Re^n := \Re^{n\times 1}$. The set of $m\times n$ matrices with nonnegative entries will be denoted by $\Re_+^{m\times n}$ and $\Re_+^n := \Re_+^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n\times n$ identity matrix by I_n.

Continuous-time linear systems. *1. Continuous-time fractional linear systems.* In this paper the following Caputo definition of the fractional derivative will be used [21, 24]

$$D^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}}{(t-\tau)^{\alpha+1-n}} d\tau, \qquad n-1 < \alpha \le n \in N = \{1,2,...\}, \tag{1}$$

where $\alpha \in \Re$ is the order of fractional derivative and $f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}$.

Consider the continuous-time fractional linear system described by the state equations

$$D^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha \le 1 \qquad y(t) = Cx(t) + Du(t) \ , \tag{2a,b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Theorem 1. [10] The solution of equation (2a) is given by

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \quad x(0) = x_0 \ , \tag{3}$$

where $$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \ , \qquad \Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{4,5}$$

and $E_\alpha(At^\alpha)$ is the Mittage-Leffler matrix function, $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function.

Remarks. **1**. From (4) and (5) for $\alpha = 1$ we have $$\Phi_0(t) = \Phi(t) = \sum_{k=0}^\infty \frac{(At)^k}{\Gamma(k+1)} = e^{At} \ . \tag{6}$$

2. From the Cayley-Hamilton theorem we have. If $\det[I_n s^\alpha - A] = (s^\alpha)^n + a_{n-1}(s^\alpha)^{n-1} + ... + a_1 s^\alpha + a_0$, (7)

then $$A^n + a_{n-1}A^{n-1} + ... + a_1 A + a_0 I = 0 \ . \tag{8}$$

Example. Find the solution of eq.(2a) for $0<\alpha\le 1$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ u(t) = \begin{cases} 1, \text{ for } t>0 \\ 0, \text{ for } t<0 \end{cases}$. (9)

Using (4) and (5) we obtain $$\Phi_0(t) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} = I_2 + \frac{At^\alpha}{\Gamma(\alpha+1)} \quad \Phi(t) = I_2 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}, \tag{10a,b}$$

since $A^k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for $k = 2,3,...$.

Substitution of (10) and $u(t) = 1$ into (3) yields

$$\begin{aligned} x(t) &= \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau = x_0 + \frac{Ax_0 t^\alpha}{\Gamma(\alpha+1)} + \int_0^t \left(\frac{B}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} + \frac{AB}{\Gamma(2\alpha)}(t-\tau)^{2\alpha-1} \right) d\tau = \\ &= x_0 + \frac{Ax_0 t^\alpha}{\Gamma(\alpha+1)} + \frac{Bt^\alpha}{\Gamma(\alpha+1)} + \frac{ABt^{2\alpha}}{\Gamma(2\alpha+1)} = \begin{bmatrix} 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} \end{bmatrix} \end{aligned}, \tag{11}$$

since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

2. *Positivity of continuous-time fractional systems.*

Definition 1. The fractional system (2) is called the internally positive fractional system if and only if $x(t)\in\Re_+^n$ and $y(t)\in\Re_+^p$ for $t\ge 0$ for any initial conditions $x_0\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\ge 0$.

A square real matrix $A = [a_{ij}]$ is called the Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$ [2, 5].

Lemma 1. Let $A \in \Re^{n \times n}$ and $0 < \alpha \leq 1$. Then

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \in \Re_+^{n \times n} \quad \text{for } t \geq 0, \qquad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]} \in \Re_+^{n \times n} \quad \text{for } t \geq 0, \tag{12,13}$$

if and only if A is a Metzler matrix. Proof is given in [10].

Theorem 2. The continuous-time fractional system (2) is internally positive if and only if

$$A \in M_n, \quad B \in \Re_+^{n \times m}, \quad C \in \Re_+^{p \times n}, \quad D \in \Re_+^{p \times m} \tag{14}$$

M_n is the set of Metzler matrices. Proof is given in [10].

Definition 2. The fractional system (2) is called externally positive if and only if $y(t) \in \Re_+^p, t \geq 0$ for every input $u(t) \in \Re_+^m, t \geq 0$ and $x_0 = 0$.

The impulse response $g(t)$ of single-input single-output system is called its output for the input equal to the Dirac impulse $\delta(t)$ with zero initial conditions. Assuming successively that only one input is equal to $\delta(t)$ and the remaining inputs and initial conditions are zero we may define the impulse response matrix $g(t) \in \Re^{p \times m}$ of the system (2). The impulse response matrix of the system (2) is given by

$$g(t) = C\Phi(t)B + D\delta(t) \quad \text{for } t \geq 0. \tag{15}$$

Substitution of (3) into (2b) for $x_0 = 0$ yields $y(t) = \int_0^t C\Phi(t-\tau)Bu(\tau)d\tau + Du(t), \quad t \geq 0$. (16)

The formula (15) follows from (16) for $u(t) = \delta(t)$.

Theorem 3. The continuous-time fractional system (2) is externally positive if and only if its impulse response matrix (15) is nonnegative, i.e. $g(t) \in \Re_+^{p \times m}$ for $t \geq 0$. (17)

Proof. The necessity of the condition (17) follows immediately from Definition 2. The output $y(t)$ of the system (2) with zero initial conditions for any input $u(t)$ is given by the formula

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau \tag{18}$$

which can be obtained by substitution of (15) into (16). If the condition (17) is met and $u(t) \in \Re_+^m$, then from (18) we have $y(t) \in \Re_+^p$ for $t \geq 0$. From (15) and (13) it follows that if A is a Metzler matrix and (14) holds then the impulse response matrix (15) is nonnegative. Therefore, we have the following two *corollaries*: 1. The impulse response matrix (15) of the internally positive system (2) is nonnegative.

2. Every continuous-time fractional internally positive system (2) is also externally positive.

A example of electrical circuit composed of a resistance R, capacitance C and voltage source described by fractional differential equation is given in [18].

Discrete-time linear systems. 1. *Discrete-time fractional systems*. The following definition of the fractional difference will be used $\Delta^\alpha x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j}, \ 0 < \alpha < 1$, (19)

where $\alpha \in R$ is the order of the fractional difference, and

$$\binom{\alpha}{j} = \begin{cases} 1, & \text{for } j = 0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}, & \text{for } j = 1, 2, \ldots \end{cases} \tag{20}$$

Consider the fractional discrete-time linear system, described by the state-space equations

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad u \in Z_+ \qquad y_k = Cx_k + Du_k, \tag{21a,b}$$

where $x_k \in \Re^n$, $u_k \in \Re^m$, $y_k \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$, $D \in \Re^{p \times m}$.

Using (19) we may write the equations (21) in the form

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \; k \in Z_+ \qquad y_k = Cx_k + Du_k . \tag{22a,b}$$

Definition 3. The system (22) is called the (internally) positive fractional system if and only if $x_k \in \Re_+^n$ and $y_k \in \Re_+^p$, $k \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all input sequences $u_k \in \Re_+^m$, $k \in Z_+$.

Theorem 4. [8] The solution of equation (22a) is given by
$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} B u_i , \tag{23}$$

where Φ_k is determined by the equation $\Phi_{k+1} = (A + I_n \alpha)\Phi_k + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1}$ with $\Phi_0 = I_n$. (24)

Theorem 5. Let
$$\det\left[I_n \Delta^\alpha (z^{-1}) - Az^{-1} \right] = \sum_{i=0}^{M} a_{M-i} z^{-i} \tag{25}$$

be the characteristic polynomial of the system (22). Then the matrices $\Phi_0, \Phi_1, ..., \Phi_M$ satisfy the equation

$$\sum_{i=0}^{M} a_i \Phi_i = 0 . \tag{26}$$

Proof. From definition of inverse matrix and (25) we have

$$\mathrm{Adj}\left[I_n \Delta^\alpha (z^{-1}) - Az^{-1} \right]^{-1} = \left(\sum_{i=0}^{\infty} \Phi_i z^{-i} \right) \left(\sum_{i=0}^{M} a_{m-i} z^{-i} \right), \tag{27}$$

where AdjF denotes the adjoint matrix of F.

Comparison of the coefficients at the same power z^{-M} of the equality (27) yields (26) since degree of $\mathrm{Adj}[I_n \Delta^\alpha (z^{-1}) - Az^{-1}]$ less than M.

Theorem 5 is an extension of the well-known classical Cayley-Hamilton theorem for the fractional system (20). Note that the degree M of the characteristic polynomial (25) depends on k and it increases to infinity for $k \to \infty$. In practical problems it is assumed that k is bounded by some natural number L. If $k = L$ then $M = N(L+1)$.

2. *Positivity of discrete-time fractional systems*. The following two lemmas are used in the proof of the positivity of the fractional system (23).

Lemma 2. [8] If $0 < \alpha \le 1$, then $(-1)^{i+1} \binom{\alpha}{i} > 0$ for $i = 1, 2, ...$ (28,29)

Lemma 3. [8] If (29) holds and $A + I_n \alpha \in \Re_+^{n \times n}$, then $\Phi_k \in \Re_+^{n \times n}$ for $k = 1, 2, ...$ (30,31)

Theorem 6. Let (28) be satisfied. Then the fractional system (28) is positive if and only if

$$A + I_n \alpha \in \Re_+^{n \times n}, \; B \in \Re_+^{n \times m}, \; C \in \Re_+^{p \times n}, \qquad D \in \Re_+^{p \times m}. \tag{32}$$

Proof is given in [8, 18].

Example 2. Consider the fractional system (22) for $0 < \alpha < 1$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, (n = 2)$. (33)

The fractional system is positive since $A + I_n \alpha = \begin{bmatrix} 1+\alpha & 0 \\ 0 & 0 \end{bmatrix} \in \Re_+^{2 \times 2}$.

Using (24) for $k = 0, 1, ...$ we obtain diagonal matrices of the forms

$$\Phi_1 = (A + I_n\alpha)\Phi_0 = \begin{bmatrix} 1+\alpha & 0 \\ 0 & 0 \end{bmatrix},$$
$$\Phi_2 = (A + I_n\alpha)\Phi_1 - \binom{\alpha}{2}\Phi_0 = \frac{1}{2}\begin{bmatrix} \alpha^2+5\alpha+2 & 0 \\ 0 & (1-\alpha)\alpha \end{bmatrix},$$
$$\Phi_3 = (A + I_n\alpha)\Phi_2 - \binom{\alpha}{2}\Phi_1 - \binom{\alpha}{3}\Phi_0 = \\ = \frac{1}{6}\begin{bmatrix} 3(\alpha^2+5\alpha+2)(\alpha+1)-(\alpha-1)(2\alpha+5)\alpha & 0 \\ 0 & (1-\alpha)(\alpha-2)\alpha \end{bmatrix}. \quad (34)$$

From (23) and (24) we have $$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1}\Phi_{k-i-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}u_i, \quad (35)$$

where Φ_k is given by (34).

Definition 4. The discrete-time fractional system (22) is called externally positive if $y_k \in \Re_+^p, k \in Z_+$ for every input sequence $u_k \in \Re_+^m, k \in Z_+$ and $x_0 = 0$.

Theorem 7. The discrete-time fractional system (22) is externally positive if and only if its response matrix $g_k = \begin{cases} D, & \text{for } k = 0 \\ CA^{k-1}B, & \text{for } k = 1,2,... \end{cases}$ is nonnegative, i.e. $g_k \in \Re_+^{p\times m}$ for $k \in Z_+$. (36,37)

The proof is similar to the proof of Theorem 3.

Remark 4. The impulse response matrix (36) of the internally positive system (22) is nonnegative and every discrete-time fractional internally positive system is also externally positive.

Reachability of positive fractional discrete-time systems. Consider the positive fractional discrete-time linear system (22).

Definition 5. A state $x_f \in \Re_+^n$ of the positive fractional system (22) is called reachable in q steps if there exist an input sequence $u_k \in \Re_+^m$, $k = 0,1,\dots,q-1$ which steers the state of the system from zero $(x_0 = 0)$ to the final state x_f, i.e. $x_q = x_f$.

Let e_i, $i = 1,...,n$ be the i-th column of the identity matrix I_n. A column ae_i for $a > 0$ is called a monomial column.

Theorem 8. The positive fractional system (22) is reachable in q steps if and only if the reachability matrix $$R_q := [B, \Phi_1 B, ..., \Phi_{q-1}B] \quad (38)$$
contains n linearly independent monomial columns.

Proof. Using (22) for $k = q$ and $x_0 = 0$, obtain $$x_f = x_q = \sum_{i=0}^{q-1}\Phi_{q-i-1}Bu_i = R_q\begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix}. \quad (39)$$

From Definition 5 and (39) it follows that for every $x_f \in \Re_+^n$ there exists an input sequence $u_i \in \Re_+^m$, $i = 0,1,...,q-1$ if and only if the matrix (38) contains n linearly independent monomial columns.

Example 3. Consider the positive fractional systems (22) for $0 < \alpha < 1$ with (33).Using (24) and (38) we obtain
$$R_2 = [B, \Phi_1 B] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\quad R_3 = [B, \Phi_1 B, \Phi_2 B] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0,5(1-\alpha)\alpha \end{bmatrix},$$

$$R_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{(1-\alpha)\alpha}{2} & \frac{(1-\alpha)(\alpha-2)\alpha}{6} \end{bmatrix},... \quad (40)$$

Note that the matrices (40) contain only one linearly independent monomial column. Therefore, by Theorem 8 the system (22) with (33) is unreachable.

Example 4. Consider the fractional systems (22) for $0 \le \alpha \le 1$ with $A = \begin{bmatrix} -\alpha & 0 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $(n = 2)$. (41)

The system is positive since $A + I_n\alpha = \begin{bmatrix} 0 & 0 \\ 1 & 2+\alpha \end{bmatrix} \in R^{2\times 2}$. Using (24) for $k = 0$ we obtain $\Phi_1 = (A + I_n\alpha)\Phi_0 = \begin{bmatrix} 0 & 0 \\ 1 & 2+\alpha \end{bmatrix}$. The reachability matrix (38) for $q = 2$ has the form $R_q = [B, \Phi_1 B] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

It contains two linearly independent monomial columns. Therefore, the positive fractional system with (41) is reachable in two steps.

Remark 5. From (24) and (38) it follows that the positive fractional system (22) is reachable only if the matrix

$$[B, A + I_n\alpha] \tag{42}$$

contain n linearly independent monomial columns.

The controllability of positive fractional discrete-time linear systems has been considered in [8] and the reachability of positive fractional continuous-time linear systems in [10].

Realisation problem for positive fractional continuous-time linear systems.

1. Problem formulation. Using the Laplace transform it is easy to show that the transfer matrix of the systems is given by the formula

$$T(s) = C[I_n s^\alpha - A]^{-1}B + D. \tag{43}$$

The transfer matrix is called proper if and only if $\lim_{s\to\infty} T(s) = K \in R^{p\times m}$ and it is called strictly proper if and only if $K = 0$. From (43) we have $\lim_{s\to\infty} T(s) = D$, since $\lim_{s\to\infty}[I_n s^\alpha - A]^{-1} = 0$. (44)

Definition 6. Matrices (14) are called a positive fractional realization of given transfer matrix $T(s)$ if they satisfy the equality (43). A realization is called minimal if the dimension of A is minimal among all realizations of $T(s)$.

The positive realization problem can be stated as follows. Given a proper transfer matrix $T(s)$, find its positive realizations (14). In this section sufficient conditions for the existence of positive fractional realizations will be established and procedure for computation of the positive fractional realizations will be proposed.

Problem Solution. The realization problem will be solved for single-input single-output (SISO) linear fractional systems with the proper transfer function

$$T(s) = \frac{b_n(s^\alpha)^n + b_{n-1}(s^\alpha)^{n-1} + \ldots + b_1 s^\alpha + b_0}{(s^\alpha)^n - a_{n-1}(s^\alpha)^{n-1} - \ldots - a_1 s^\alpha - a_0}. \tag{45}$$

Using (44) we obtain

$$D = \lim_{s\to\infty} T(s) = b_n \tag{46}$$

and the strictly proper transfer function has the form

$$T_{sp}(s) = T(s) - D = \frac{\overline{b}_{n-1}(s^\alpha)^{n-1} + \overline{b}_{n-2}(s^\alpha)^{n-2} + \ldots + \overline{b}_1 s^\alpha + \overline{b}_0}{(s^\alpha)^n - a_{n-1}(s^\alpha)^{n-1} - \ldots - a_1 s^\alpha - a_0}, \tag{47}$$

where

$$\overline{b}_k = b_k + a_k b_n, \quad k = 0, 1, \ldots, n-1. \tag{48}$$

From (48) it follows that if $a_k \ge 0$ and $b_k \ge 0$ for $k = 0, 1, \ldots, n$ then also $\overline{b}_k \ge 0$ for $k = 0, 1, \ldots, n-1$.

Theorem 9. There exist positive fractional minimal realizations of the forms

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ a_0 & a_1 & a_2 & \ldots & a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} \overline{b}_0 & \overline{b}_1 & \ldots & \overline{b}_{n-1} \end{bmatrix}, \quad D = b_n$$

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 & a_2 \\ 1 & 0 & \ldots & 0 & a_2 \\ 0 & 1 & \ldots & 0 & a_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} \overline{b}_0 \\ \overline{b}_1 \\ \vdots \\ \overline{b}_{n-1} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}, \quad D = b_n \tag{49a,b}$$

$$A=\begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix},\ B=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A=\begin{bmatrix} a_{n-1} & 1 & 0 & \dots & 0 \\ a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \dots & 1 \\ a_0 & 0 & 0 & \dots & 0 \end{bmatrix},\ B=\begin{bmatrix} \bar{b}_{n-1} \\ \vdots \\ \bar{b}_1 \\ \bar{b}_0 \end{bmatrix}, \tag{49c,d}$$

$$C=\begin{bmatrix} \bar{b}_{n-1} & \dots & \bar{b}_1 & \bar{b}_0 \end{bmatrix},\ D=b_n \qquad C=\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix},\ D=b_n$$

of the transfer function (45) if a) $b_k \ge 0$ for $k=0,1,\dots,n$, b) $a_k \ge 0$ for $k=0,1,\dots,n-2$ and $b_{n-1}+a_{n-1}b_n \ge 0$.

Proof. Taking into account that for (49) $\det[I_n s^\alpha - A]=(s^\alpha)^n - a_{n-1}(s^\alpha)^{n-1} - \dots - a_1 s^\alpha - a_0$ and $\mathrm{Adj}[I_n s^\alpha - A]B=[1 \quad s^\alpha \quad \dots \quad (s^\alpha)^{n-1}]$ it is easy to verify that

$$C[I_n s^\alpha - A]^{-1}B=\frac{C\mathrm{Adj}[I_n s^\alpha - A]B}{\det[I_n s^\alpha - A]}=\frac{\bar{b}_{n-1}(s^\alpha)^{n-1}+\bar{b}_{n-2}(s^\alpha)^{n-2}+\dots+\bar{b}_1 s^\alpha+\bar{b}_0}{(s^\alpha)^n - a_{n-1}(s^\alpha)^{n-1}-\dots-a_1 s^\alpha - a_0}.$$

The matrix A is Metzler matrix if and only if $a_k \ge 0$ for $k=0,1,\dots,n-2$ and arbitrary a_{n-1}. Note that the coefficient of matrices C and D are nonnegative if the conditions a) is met and $\bar{b}_{n-1}=b_{n-1}+a_{n-1}b_n \ge 0$. The proof for (49b), (49c) and (49b) are similar (dual). The matrices (49) are minimal realizations if and only if the transfer function (45) is irreducible. If the conditions of Theorem 9 are satisfied then the positive minimal realizations (49) of the transfer function (45) can be computed by use of the following procedure.

Procedure. *Step 1*. Knowing $T(s)$ and using (46) find D and the strictly proper function (47). *Step 2*. Using (49) find the desired realizations.

Example 5. Find the positive minimal fractional realizations (49) of the irreducible transfer function

$$T(s)=\frac{2(s^\alpha)^2+5s^\alpha+1}{(s^\alpha)^2+2s^\alpha-3}. \tag{50}$$

Using Procedure and (50) we obtain the following. *Step 1*. From (46) and (50) we obtain

$$D=\lim_{s\to\infty}\frac{2(s^\alpha)^2+5s^\alpha+1}{(s^\alpha)^2+2s^\alpha-3}=2 \quad \text{and} \quad T_{sp}(s)=T(s)-D=\frac{s^\alpha+7}{(s^\alpha)^2+2s^\alpha-3}. \tag{51,52}$$

Step 2. Taking into that in this case $\bar{b}_0=7,\ \bar{b}_1=1$ and using (49) we obtain the desired positive minimal fractional realizations

$$A=\begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix},\ B=\begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C=[7 \quad 1],\ D=2; \quad A=\begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix},\ B=\begin{bmatrix} 7 \\ 1 \end{bmatrix},\ C=[0 \quad 1],\ D=2; \tag{53a,b}$$

$$A=\begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix},\ B=\begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C=[1 \quad 7],\ D=2; \qquad A=\begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix},\ B=\begin{bmatrix} 1 \\ 7 \end{bmatrix},\ C=[1 \quad 0],\ D=2. \tag{53c,d}$$

An extension of this method for multi-input multi-output positive fractional continuous-time linear systems has been given in [11]. The presented method can be easily extended for positive fractional discrete-time linear systems.

Cone Fractional Discrete Time Systems And Their Reachibility.

Definition 7. Let $P=\begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in \Re^{n\times n}$ be nonsingular and p_k be the *k*-th ($k=1,\dots,n$) its row.

The set

$$\mathcal{P} := \left\{ x \in \Re^n : \bigcap_{k=1}^{n} p_k x \ge 0 \right\} \tag{54}$$

is called a linear cone generated by the matrix P.

In a similar way we may define for the inputs u the linear cone

$$\mathcal{Q} := \left\{ u \in \Re^m : \bigcap_{k=1}^{m} q_k u \ge 0 \right\} \tag{55}$$

generated by the nonsingular matrix $Q = \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \in \Re^{m\times m}$ and for the outputs y, the linear cone

$$\boldsymbol{V} := \left\{ y \in \Re^p : \bigcap_{k=1}^{p} v_k y \geq 0 \right\} \tag{56}$$

generated by the nonsingular matrix $V = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \in \Re^{p\times p}$.

Definition 8. The fractional system (22) is called ($\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{V}$) cone fractional system if $x_k \in \boldsymbol{P}$ and $y_k \in \boldsymbol{V}$, $k \in Z_+$ for every $x_0 \in \boldsymbol{P}$, $u_k \in \boldsymbol{Q}$, $k \in Z_+$.

The ($\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{V}$) cone fractional system (4) will be shortly called the cone fractional system. Note that if $\boldsymbol{P} = \Re_+^n$, $\boldsymbol{Q} = \Re_+^m$, $\boldsymbol{V} = \Re_+^p$ then the $(\Re_+^n, \Re_+^m, \Re_+^p)$ cone system is equivalent to the classical positive system [2, 5].

Theorem 10. The fractional system (22) is ($\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{V}$) cone fractional system if and only if

$$\overline{A} = PAP^{-1} \in \Re_+^{n\times n},\ \overline{B} = PBQ^{-1} \in \Re_+^{n\times m},\qquad \overline{C} = VCP^{-1} \in \Re_+^{p\times n},\ \overline{D} = VDQ^{-1} \in \Re_+^{p\times m} \tag{57}$$

Proof. Let $\quad \overline{x}_k = Px_k, \qquad \overline{u}_k = Qu_k \qquad$ and $\qquad \overline{y}_k = Vy_k,\ k \in Z_+$. (58)

From Definition 8 it follows that if $x_k \in \boldsymbol{P}$ then $\overline{x}_k \in \Re_+^n$, if $u_k \in \boldsymbol{Q}$ then $\overline{u}_k \in \Re_+^m$ and if $y_k \in \boldsymbol{V}$ then $\overline{y}_k \in \Re_+^p$. From (22) and (58) we have

$$\overline{x}_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} \overline{x}_{k-j+1} = Px_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} Px_{k-j+1} = PAx_k + PBu_k == PAP^{-1}\overline{x}_k + PBQ^{-1}\overline{u}_k = \overline{A}\overline{x}_k + \overline{B}\overline{u}_k,\ k \in Z_+$$

and $$\overline{y}_k = Vy_k = VCx_k + VDu_k = VCP^{-1}\overline{x}_k + VDQ^{-1}\overline{u}_k = \overline{C}\overline{x}_k + \overline{D}\overline{u}_k,\ k \in Z_+. \tag{59a,b}$$

It is well-known [5] that the system (59) is the positive one if and only if the conditions (57) are satisfied.

Definition 9. A state $x_f \in \boldsymbol{P}$ of the cone fractional system (22) is called reachable in q steps if there exists an input sequence $u_k \in \boldsymbol{Q}$, $k = 0,1,...,q-1$ which steers the state of the system from zero initial state $(x_0 = 0)$ to the desired state x_f, i.e. $x_q = x_f$. If every state $x_f \in \boldsymbol{P}$ is reachable in q steps then the cone fractional system is called reachable in q steps. If for every state $x_f \in \boldsymbol{P}$ there exists a natural number q such that the state is reachable in q steps then the cone fractional system is called reachable.

Theorem 11. The cone fractional system (22) is reachable in q steps if and only if the matrix

$$\overline{R}_q = [PBQ^{-1}, P\Phi_1 BQ^{-1},..., P\Phi_{q-1} BQ^{-1}] \tag{60}$$

contains n linearly independent monomial columns.

Proof. From the relations (58) it follows that if $x_k \in \boldsymbol{P}$ then $\overline{x}_k = Px_k \in \Re_+^n$ and if $u_k \in \boldsymbol{Q}$ then $\overline{u}_k = Qu_k \in \Re_+^m$ for $k \in Z_+$. Hence by Definition 8 and 9 the cone fractional system (22) is reachable in q steps if and only if the positive fractional system (59) is reachable in q steps.

Using (24) and (57) it is easy to show that $\overline{\Phi}_k$ of the system (59) with Φ_k of he system (22) are related by $$\overline{\Phi}_k = P\Phi P^{-1} \text{ for } k = 0,1,... . \tag{61}$$

Taking into account that $\quad \overline{\Phi}_k \overline{B} = P\Phi_k P^{-1} PBQ^{-1} = P\Phi_k BQ^{-1},\quad k = 1,2,...,q-1 \qquad (62)$

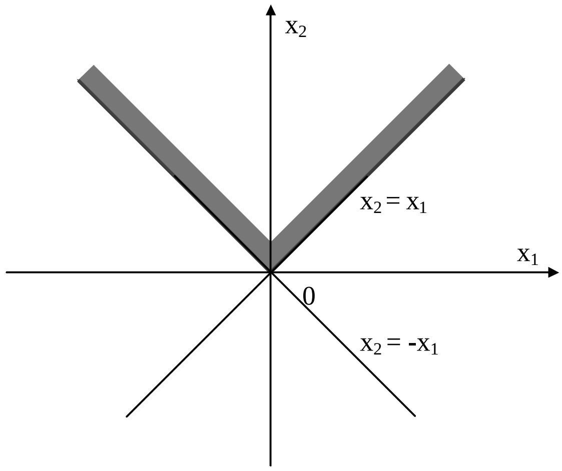
we may write
$$\begin{aligned}\bar{R}_q &= [\bar{B}, \bar{\Phi}_1\bar{B}, ..., \bar{\Phi}_{q-1}\bar{B}] = \\ &= [PBQ^{-1}, P\Phi_1 BQ^{-1}, ..., P\Phi_{q-1}BQ^{-1}]\end{aligned}. \tag{63}$$

By Theorem 8 the positive fractional system (59) is reachable in q steps if and only if the matrix (60) contain n linearly independent monomial columns.

Example 6. Consider the cone fractional system (22)

for
$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ Q = [1],$$
$$A = \begin{bmatrix} -\alpha & a \\ 1 & a-\alpha+1 \end{bmatrix},\ a > 0,\ 0 < \alpha \le 1 \tag{64}$$

and for the following two forms of the matrix B
$$B_1 = \begin{bmatrix} b \\ b \end{bmatrix},\ B_2 = \begin{bmatrix} -b \\ b \end{bmatrix},\ b > 0 .$$



The $\boldsymbol{P}$ -cone generated by the matrix P is shown in Fig.. In case 1 we shall show that the cone fractional system is not reachable.

Using (57) and (64) we obtain
$$\begin{aligned}\bar{A}_d &= P(A + I_n\alpha)P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & a \\ 1 & a+1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+1 & a \\ 1 & 0 \end{bmatrix},\ a > 0 \\ \bar{B}_1 &= PB_1Q^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} 2b \\ 0 \end{bmatrix},\ b > 0\end{aligned}. \tag{65}$$

The system (59) with matrices (65) is a positive fractional system. Using (60) for $q = 2$, (64) and taking into account that $\Phi_1 = A_d$ we obtain
$$\bar{R}_2 = [PB_1Q^{-1}, P\Phi_1B_1Q^{-1}] = P[B_1, A_dB_1] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} b & ab \\ b & (a+2)b \end{bmatrix} = \begin{bmatrix} 2b & 2(a+1)b \\ 0 & 2b \end{bmatrix}. \tag{66}$$

The matrix (66) contains only one (the first) monomial column. Thus by Theorem 11 the cone fractional system is unreachable.

In case 2 we have
$$\bar{B}_2 = PB_2Q^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -b \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2b \end{bmatrix},\ b > 0 . \tag{67}$$

The system (59) with matrices $\bar{A}_d$ and $\bar{B}_2$ given by (65) and (67) is also a positive fractional system. Using (60) and (64) we obtain the matrix
$$\bar{R}_2 = P[B_2, A_dB_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -b & ab \\ b & ab \end{bmatrix} = \begin{bmatrix} 0 & 2ab \\ 2b & 0 \end{bmatrix},\ a > 0, b > 0 , \tag{68}$$

which contains two linearly independent monomial columns. Therefore, by Theorem 11 the cone fractional system is reachable.

The controllability to zero of the cone fractional discrete-time linear systems has been considered in [9].

Positive Fractional 2d Linear Systems And Their Reachibility And Their Reachibility.

1. *Fractional 2D linear systems*. The positive fractional 2D linear systems have been introduced in [15, 16] and the positive 2D hybrid linear systems in [17].

Definition 10. The (α, β) orders fractional difference of and 2D function x_{ij} is defined by the formula
$$\Delta^{\alpha,\beta}x_{ij} = \sum_{k=0}^{i}\sum_{l=0}^{j} c_{\alpha\beta}(k,l)x_{i-k,j-l},\ , \quad n-1 < \alpha < n, \quad n-1 < \beta < n, \quad n \in N = \{1,2,...\}, \tag{69}$$

where $\Delta^{\alpha,\beta}x_{ij} = \Delta_i^{\alpha}\Delta_j^{\beta}x_{ij}$ and
$$c_{\alpha,\beta}(k,l) = \begin{cases} 1 \quad \text{for} \quad k = 0 \quad \text{or/and} \quad l = 0 \\ (-1)^{k+l}\dfrac{\alpha(\alpha-1)...(\alpha+1-k)\beta(\beta-1)...(\beta+1-l)}{k!l!} \\ \text{for} \quad k + l > 0 \end{cases}. \tag{70}$$

The justification of Definition 10 is given in [15].

Consider the (α,β) order 2D fractional linear system, described by the state equations

$$\Delta^{\alpha,\beta}x_{i+1,j+1} = A_0x_{ij} + A_1x_{i+1,j} + +A_2x_{i,j+1} + B_0u_{ij} + B_1u_{i+1,j} + B_2u_{i,j+1} \qquad y_{ij} = Cx_{ij} + Du_{ij}, \tag{71a,b}$$

where $x_{ij}\in\Re^n, u_{ij}\in\Re^m, y_{ij}\in\Re^p$ are the state, input and output vectors and $A_k\in\Re^{n\times n}$, $B_k\in\Re^{n\times m}, k=0,1,2,\; C\in\Re^{p\times n}, D\in\Re^{p\times m}$.

Using Definition 10 we may write the equation (71a) in the form

$$x_{i+1,j+1} = \bar{A}_0x_{ij} + \bar{A}_1x_{i+1,j} + \bar{A}_2x_{i,j+1} - \sum_{\substack{k=0\\k+l>0}}^{i+1}\sum_{l=0}^{j+1}c_{\alpha\beta}(k,l)x_{i-k+1,j-l+1} + B_0u_{ij} + B_1u_{i+1,j} + B_2u_{i,j+1}, \tag{72}$$

where $\bar{A}_0 = A_0 - I_n\alpha\beta$, $\bar{A}_1 = A_1 - I_n\beta$, $\bar{A}_2 = A_2 - I_n\alpha$.

From (69) it follows that the coefficients (70) in (69) strongly decrease when k and l increase. Therefore, in practical problems it is assumed that i and j are bounded by some natural numbers L_1 and L_2. In this case (72) takes the form

$$x_{i+1,j+1} = \bar{A}_0x_{ij} + \bar{A}_1x_{i+1,j} + \bar{A}_2x_{i,j+1} - \sum_{k=0}^{L_1+1}\sum_{l=0}^{L_2-k+1}c_{\alpha\beta}(k,l)x_{i-k+1,j-l+1} + B_0u_{ij} + B_1u_{i+1,j} + B_2u_{i,j+1}. \tag{73}$$

Note that the fractional systems are 2D linear systems with delays increasing with i and j.

The boundary conditions for the equation (72) and (73) are given in the form

$$x_{i0},\, i\in Z_+ \quad\text{and}\quad x_{0j},\, j\in Z_+. \tag{74}$$

Theorem 12. The solution of equation (72) with boundary conditions (74) is given by

$$x_{ij} = \sum_{p=1}^{i}T_{i-p,j-1}(\bar{A}_1x_{p0} + B_1u_{p0}) + \sum_{q=1}^{j}T_{i-1,j-q}(\bar{A}_2x_{0q} + B_2u_{0q}) + + \sum_{p=1}^{i-1}T_{i-p-1,j-1}\bar{A}_0x_{p0} +$$
$$+\sum_{q=1}^{j-1}T_{i-1,j-q-1}\bar{A}_0x_{0q} + T_{i-1,j-1}\bar{A}_0u_{00} + \sum_{p=0}^{i-1}\sum_{q=0}^{j-1}T_{i-p-1,j-q-1}B_0u_{pq} + \sum_{p=0}^{i}\sum_{q=0}^{j}(T_{i-p-1,j-q-1}B_1 + T_{i-p,j-q-1}B_2)u_{pq} \tag{75}$$

where the transition matrices T_{pq} are defined by the formula

$$T_{pq} = \begin{cases} I_n \ \text{ for } p=q=0 \\ \bar{A}_0T_{p-1,q-1} + \bar{A}_1T_{p,q-1} + \bar{A}_2T_{p-1,q} - \sum\limits_{\substack{k=0\\k+l<p+q-2}}^{p}\sum\limits_{l=0}^{q}c_{\alpha,\beta}(p-k,q-l)T_{kl} \quad \text{for} \quad p+q>0 \\ 0 \text{ (zero matrix) for } p<0 \text{ or/and } q<0 \end{cases} \tag{76}$$

Proof is given in [16]. Let

$$\bar{G}(z_1,z_2) = I_n + \sum_{k=0}^{L_1+1}\sum_{l=0}^{L_2+1}I_nc_{\alpha\beta}(k,l)z_1^{-k}z_2^{-l} - \bar{A}_0z^{-1}z_2^{-1} - \bar{A}_1z_2^{-1} - \bar{A}_2z_1^{-1} \tag{77}$$

and

$$\det\bar{G}(z_1,z_2) = \sum_{k=0}^{N_1}\sum_{l=0}^{N_1}a_{N_1-k,N_2-l}z_1^{-k}z_2^{-l}. \tag{78}$$

It is assumed that *i* and *j* are bounded by some natural numbers L_1, L_2 which determined the degrees N_1, N_2.

Theorem 13. Let (78) be the characteristic polynomial of the system (71). Then the matrices T_{kl} satisfy the equation

$$\sum_{k=0}^{N_1}\sum_{l=0}^{N_2}a_{kl}T_{kl} = 0. \tag{79}$$

Proof is given in [16].

Theorem 13 is an extension of the well-known classical Cayley-Hamilton theorem for the 2D fractional system (71).

2. *Positivity of the fractional 2D systems*

Lemma 4. [16]

a) If $0<\alpha<1$ and $1<\beta<2$ then $c_{\alpha\beta}(k,l)<0$ for $k=1,2,...;\, l=2,3,...$ (80a)

b) If $1<\alpha<2$ and $0<\beta<1$ then $c_{\alpha\beta}(k,l)<0$ for $k=2,3,...;\, l=1,2,...$. (80b)

Lemma 5. [16] If (80) is met and $\bar{A}_k \in \Re_+^{n\times n}$ for $k=0,1,2,$ (81)

then $T_{pq} \in \Re_+^{n\times n}$ for $p,q \in Z_+$. (82)

Definition 11. The system (71) is called the (internally) positive 2D fractional system if and only if $x_{ij} \in \Re_+^n$ and $y_{ij} \in \Re_+^p,\, i,j \in Z_+$ for any boundary conditions $x_{i0} \in \Re_+^n,\, i \in Z_+$ $x_{0j} \in \Re_+^n,\, i \in Z_+$ and all input sequences $u_{ij} \in \Re_+^m,\, i,j \in Z_+$.

Theorem 14. The 2D fractional system (71) for $0<\alpha<1$ and $1<\beta<2$ is positive if and only if

$$\bar{A}_k \in \Re_+^{n\times n},\, B_k \in \Re_+^{n\times m},\, k=0,1,2,\, C \in \Re_+^{p\times n}, \qquad D \in \Re_+^{p\times m}. \tag{83}$$

Proof is given in [16].

Remark 6. From (70) and (71) it follows that if $\alpha=\beta$, $0<\alpha<1$ then $c_{\alpha\beta}(k,l)<0$ for $k=l=1,2,...$ and the fractional 2D system (71) is not positive.

3. *Reachability of the positive fractional 2D systems*.

Definition 12. The positive 2D fractional system (71) is called reachable at the point $(h,k) \in Z_+ \times Z_+$ if and only if for zero boundary conditions (74) $(x_{i0}=0,\, i\in Z_+,\, x_{0j},\, j \in Z_+)$ and every vector $x_f \in \Re_+^n$ there exists a sequence of inputs $u_{ij} \in \Re_+^m$ for

$$\begin{aligned}&(i,j)\in D_{hk} = \{(i,j) \in Z_+\times Z_+ : 0\le i\le h,\\ &0\le j\le k,\, i+j\ne h+k\}\end{aligned} \tag{84}$$

such that $x_{hk}=x_f$. A vector is called monomial if and only if its one component is positive and the remaining components are zero.

Theorem 15. The positive 2D fractional system (71) is reachable at the point (h,k) if and only if the reachability matrix

$$\begin{aligned}&R_{hk} = [M_0, M_1^1,...,M_{h2}^1, M_1^2,...,M_k^2, M_{11},...,M_{1k}, M_{21},...,M_{hk}]\\ &M_0 = T_{h-1,k-1}B_0,\, M_i^1 = T_{h-i,k-1}B_1 + T_{h-i-1,k-1}B_0,\, i=1,...,h\\ &M_j^2 = T_{h-1,k-1}B_2 + T_{h-1,k-j-1}B_0,\, j=1,...,k\\ &M_{ij} = T_{h-i-1,k-j-1}B_0 + T_{h-i,k-j-1}B_1 + T_{h-i-1,k-1}B_2,\, i=1,...,h,\, j=1,...,k\end{aligned} \tag{85,86}$$

contains n linearly independent monomial columns.

Proof. Using the solution (75) for $i=h,\, j=k$ and zero boundary conditions we obtain

$$x_f = R_{hk}u(h,k), \tag{87}$$

where $$u(h,k) = [u_{00}^T, u_{10}^T,...,u_{h0}^T, u_{01}^T,...,u_{0k}^T, u_{11}^T,...,u_{1k}^T, u_{21}^T,...,u_{h,k}^T]^T \tag{88}$$

and T denotes the transpose.

For the positive 2D fractional system (71) from (86) and (85) we have $M_0 \in \Re_+^{n\times m}, M_i^1 \in \Re_+^{n\times m}$, $M_j^2 \in \Re_+^{n\times m}, M_{ij} \in \Re_+^{n\times m}, i=1,...,h,\, j=1,...,h$ and $R_{hk} \in \Re_+^{n\times[(h+1)(k+1)-1]m}$. From (87) it follows that there exists a sequence $u_{ij} \in \Re_+^m$ for $(i,j)\in D_{hk}$ for every $x_f \in \Re_+^n$ if and only if the matrix (85) contains n linearly independent monomial columns.

The following theorem gives sufficient conditions for the reachability of the positive 2D fractional system (71).

Theorem 16. The positive 2D fractional system (71) is reachable at the point (h,k) if $\operatorname{rank} R_{hk} = n$ and the right inverse R_{hk}^r of the matrix (85) has nonnegative entries

$$R_{hk}^r = R_{hk}^T [R_{hk} R_{hk}^T]^{-1} \in \Re_+^{[(h+1)(k+1)-1]m \times n}. \quad (89)$$

Proof. If $\operatorname{rank} R_{hk} = n$ then there exists the right inverse R_{hk}^r of the matrix R_{hk}. If the condition (89) is met then from (87) we obtain $u(h,k) = R_{hk}^r x_f \in \Re_+^{[(h+1)(k+1)-1]m}$ for every $x_f \in \Re_+^n$.

Example 7. Consider the positive 2D fractional system (71) with

$$\bar{A}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},. \quad (90)$$

To check the reachability at the point $(h,k) = (1,1)$ of the system we use Theorem 15. From (86) and (85) we obtain

$$M_0 = B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, M_1^1 = B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad M_1^2 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, M_{ij} = 0 \text{ for } i \ge 1, j \ge 1,$$

$$R_{11} = [M_0, M_1^1, M_1^2] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \quad (91)$$

The first two columns of (91) are linearly independent monomial columns and by Theorem 15 the positive 2D fractional system (71) with (90) is reachable at the point $(1,1)$. The sequence of inputs steering the state of the system from zero boundary conditions to an arbitrary state $x_f \in \Re_+^2$ at the point $(1,1)$ is given by $\begin{bmatrix} u_{00} \\ u_{10} \end{bmatrix} = x_f$ and $u_{01} = 0$. Using (89) and (91) we obtain

$$R_{hk}^r = R_{hk}^T [R_{hk} R_{hk}^T]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} == \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}. \quad (92)$$

From (92) it follows that the condition (89) is not satisfied in spite of the fact that the system is reachable at the point $(1,1)$. Note that the system is reachable at the point $(1,1)$ for any fractional order (α, β) $0 < \alpha < 1$, $1 < \beta < 2$ (or $1 < \alpha < 2$, $0 < \beta < 1$) and any matrices $\bar{A}_k$, $k = 0,1,2$.

Necessary and sufficient conditions for the controllability to zero of positive fractional 2D linear systems have been established in [16].

Concluding remarks and open problems. An overview of some resent developments and new results in the theory of fractional positive and cone 1D and 2D linear system have been given. The state equations and their solutions for fractional continuous-time and discrete-time linear systems have been proposed. Necessary and sufficient conditions for the internal and external positivity and reachability of the systems have been established. The realization problem for positive fractional continuous-time linear systems has been formulated and solved. A new class of cone fractional discrete-time linear systems has been introduced. The positive fractional 2D linear systems have been also introduced and their reachability has been investigated. From the long list of the open problems in the fractional systems theory the following are the natural steps:

- 1D and 2D fractional linear systems with delays,
- Positive 1D and 2D fractional linear systems with delays,
- Positive fractional 2D hybrid systems with and without delays,
- Standard and positive 2D fractional continuous-time systems,
- Standard and positive 1D and 2D nonlinear systems.

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