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Controllability of Matrix Sylvester System and Sylvester Integro-differential System

In the present article, Sylvester matrix first order differential system and matrix first order integro-differential system are studied. A set of sufficient conditions for controllability and complete controllability of the system is presented. As a necessary tool, a variation of parameter formula is developed for the non-linear Sylvester system.

Исследованы дифференциальные и интегро-дифференциальные матричные системы Сильвестра первого порядка. Представлен набор достаточных условий управляемости и полной управляемости систем. Как необходимый инструмент для нелинейной системы Сильвестра получена разновидность параметрической формулы.

K e y w o r d s: control function, resolvent matrix, sylvester system, integro-differential equation, Fubini's theorem, Volterra integro-differential equation, nonlinear systems.

1. Introduction. In this paper, we shall be concerned with the general first order Sylvester system

$$T'(t) = A(t)T(t) + T(t)B(t) + C(t)U(t)D(t), \tag{1}$$

where A, B are continuous $(n \times n)$ matrices on $J = [t_0, t_1]$ and C is an $(n \times m)$ continuous matrix, U is an $(m \times m)$ continuous matrix on J, and D is an $(m \times n)$ matrix. The Sylvester system (1) arises in a number of areas of control engineering, feedback systems, dynamical systems and its general form solution in terms of solution of fundamental matrix is given by Murty, Howell and Sivasundaram in 1992 [1]. Most of the results presented on control systems are new and include general first order vector system as a particular case. More specifically, the paper is organized as follows.

Section 2, presents general solution of (1) in-terms of two fundamental matrix solutions of $T' = A(t)T$ and $T' = B^*(t)T$ and then present a set of sufficient conditions for the controllability of (1).

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In section 3, we consider the general first order integro-differential equation of the form

$$T'(t)=A(t)\,T(t)+T(t)\,B(t)+\int_{t_0}^{t}[K_1(t,s)\,T(s)+T(s)\,K_2(t,s)]\,ds+F(t), \qquad (2)$$

satisfying $T(t_0)=T_0$, where $K_1(t,s)$ and $K_2(t,s)$ are continuous square matrices of the order $(n\times n)$ and $(t,\ s)\in R_+^2$ and $F\in C\,[R_+,\ R^{n\times n}]$. We first present the variation of parameters formula for (1) and then present sufficient conditions for the controllability of the integro-differential system (2). Section 4 deals with nonlinear control system. In this section, we consider the matrix integro-differential system of the form

$$T'(t)=A(t,T(t),U(t))\,T(t)+T(t)\,B(t,T(t),U(t))+\int_{t_0}^{t}K_1(t,s,T(s),U(s))\,T(s)+$$

$$+T_2(s)\,K_2(t,s,T(s),U(s))\,ds+C(t,T(t),U(t))\,U(t)\,D^*(t,T(t),U(t)) \qquad (3)$$

and present a set of sufficient conditions for its controllability, and then present sufficient conditions for complete controllability of (3). Note that the matrices involved are of appropriate dimension and are all continuous $(n\times n)$ matrices in their arguments. Here the nonlinear matrix A is such that the elements $a_{ij}=$ $=a_{ij}(t,T(t),V(t))$ are scalars. Some remarks are applicable in respect of the matrices B, K_1, K_2, C and D [2, 3].

In [1] the general solution of the homogeneous system $T'(t)=A(t)\,T(t)+$ $+T(t)\,B(t)$ is given by $T(t)=Y(t)\,CZ^*(t)$, where $Y(t)$ is a fundamental matrix solution of $T'(t)=A(t)\,T(t)$ and $Z(t)$ is a fundamental matrix solution of $T'(t)=$ $=B^*(t)\,T(t)$. In this paper, we seek a particular solution of the nonhomogeneous system.

$$T'(t)=A(t)\,T(t)+T(t)\,B(t)+F(t) \qquad (4)$$

in the form $Y(t)\,C(t)\,Z^*(t)$. It is easy to verify that

$$C(t)=\int_{t_0}^{t}Y^{-1}(s)\,F(s)\,Z^{*-1}(s)\,ds.$$

Thus a particular solution of (4) is given by

$$T_p(t)=Y(t)\left[\int_{t_0}^{t}Y^{-1}(s)\,F(s)\,Z^{*-1}(s)\,ds\right]Z^*(t),$$

and any solution of (4) is of the form

$$T(t)=Y(t)\,CZ^{*}(t)+Y(t)\left[\int_{t_0}^{t} Y^{-1}(s)\,F(s)\,Z^{*-1}(s)\,ds\right]Z^{*}(t).$$

Using this general solution, we can at once write the general solution of the control system (1) as

$$T(t)=Y(t)\,CZ^{*}(t)+Y(t)\left[\int_{t_0}^{t} Y^{-1}(s)C(s)U(s)D(s)Z^{*-1}(s)ds\right]Z^{*}(t).$$

2. Controllability of the Sylvester system. In this section, we shall be concerned with establishing a necessary and sufficient condition for the controllability and observability of the nonhomogeneous system (1) under smooth conditions. The Sylvester system (1) is said to be time invariant, if all the coefficient matrices are constant, otherwise it is called time varying.

Definition1. The linear time varying system (1) is said to be controllable on $[t_0, t_1]$, if for any initial time t_0 and any initial state $T(t_0)=T_0$ there exists a continuous input signal $U(t)$ such that the corresponding solution of (1) satisfies $T(t_1)=T_1$.

The time varying system (1) is said to be completely controllable on $[t_0, t_1]$, if it is controllable for all $t\in[t_0, t_1]$. We use the following notation $Y(t)\,Y^{-1}(t_0)=$ $=\Phi(t, t_0)$ and $Z(t)\,Z^{-1}(t_0)=\Psi(t, t_0)$.

Theorem 1. The system (1) is completely controllable, if, and only if, the $(n\times n)$ symmetric matrix

$$V(t_0, t_1)=\int_{t_0}^{t_1} \Phi(t_0, s)\,C(s)\,C^{T}(s)\,\Phi^{T}(t_0, s)\,ds, \tag{5}$$

where $\Phi(t, t_0)$ is the fundamental matrix solution $T'=A(t)T$ satisfying $\Phi(t_0, t_0)=I$ is non-singular, and that for some positive constant C, det $(UC(t_0, t_1))\geq C$. The control function U defined by

$$U(t)=-C^{T}(t)\Phi^{T}(t_0, t)V^{-1}(t_0, t_1)\lfloor T_0-\Phi(t_1, t_0)T_1\psi^{*}(t_1, t_0)\rfloor\psi^{*}(t, t_0) \tag{6}$$

transfer $T(t_0)=T_0$ into $T(t_1)=T_1$.

P r o o f. First, we suppose that $V(t_0, t_1)$ given by (5) is non-singular. Then $U(t)$ given by (6) is well defined. Further, any solution $T(t)$ of (1) satisfying $T(t_0)=T_0$ is given by

$$T(t)=\Phi(t, t_0)\,T_0\psi^{*}(t_0, t)+$$

$$+\Phi(t,t_0)\left[\int_{t_0}^{t}\Phi(t_0,s)C(s)U(s)\psi^*(t_0,s)\,ds\right]\psi^*(t_0,t)\,.$$

Substitute the general form of the control function $U(t)$ given by (6) in the above equation, we get

$$T(t_1)=\Phi(t_1,t_0)T_0\psi^*(t_0,t_1)-\Phi(t_1,t_0)\int_{t_0}^{t}\Phi(t_0,s)C(s)C^T(s)\Phi^T(t_0,s)\times$$

$$\times V^{-1}(t_0,t_1)[T_0-\Phi(t_1,t_0)T_1\psi^*(t_1,t_0)]\,ds\psi^*(t_0,t_1)=$$

$$=\Phi(t_1,t_0)T_0\Psi^*(t_0,t_1)-\Phi(t_0,t_1)V(t_0,t_1)V^{-1}(t_0,t_1)\times$$

$$\times\left\lfloor T_0-\Phi(t_1,t_0)T_1\Psi^*(t_1,t_0)\right\rfloor\Psi^*(t_0,t_1)=$$

$$=\Phi(t_1,t_0)T_0\Psi^*(t_0,t_1)-\Phi(t_1,t_0)T_0\Psi^*(t_0,t_1)+\Phi(t_0,t_1)\Phi(t_1,t_0)\times$$

$$\times T_1\,\Psi^*(t_1,t_0)\Psi^*(t_0,t_1)=\Phi(t_0,t_0)T_1\Psi^*(t_1,t_1)=T_1.$$

Thus, the control function U transforms $T(t_0)=T_0$ into $T(t_1)=T_1$ and hence the Sylvester system is controllable. This is true for all $t\in[t_0,t_1]$, it follows that the Sylvester system is completely controllable.

Conversely, suppose that the system (1) is completely controllable. Let α be an arbitrary constant matrix of the order $(n\times n)$. Since V is symmetric, we have

$$\alpha^*V\alpha=\int_{t_0}^{t_1}\alpha^*\Phi(t_0,s)C(s)C^*(s)\Phi^*(t_0,s)\alpha\,ds\ge 0\,.$$

Let $\theta^*=\alpha^*\Phi(t_0,s)C(s)$. Then

$$\alpha^*V\alpha=\int_{t_0}^{t_1}\theta^*(t_0,s)\theta(t_0,s)\,ds=\int_{t_0}^{t_1}|\theta|^2\,ds\ge 0\,. \tag{7}$$

Hence $V(t_0,t_1)$ is positive semi-definite. Suppose that there exists an $\tilde{\alpha}\ne 0$ such that $\tilde{\alpha}^T V\tilde{\alpha}=0$. Then in view of (7),

$$\int_{t_0}^{t_1}\left|\tilde{\theta}(t_0,s)\right|^2 ds=0\,,$$

which means $\widetilde{\theta} \equiv 0$ on $[t_0, t_1]$. Since the system is completely controllable, there exists a control V such that $T(t_1) = 0$ if $T(t_0) = \widetilde{\alpha}$. Hence

$$T_1 = \Phi(t_0, t_1)\left[\widetilde{\alpha} + \int_{t_0}^{t_1} \Phi(t_0, s)\, C(s, u(s), \Psi^*(s, t_0))\, ds\right] \Psi^*(t_0, t_1) = 0,$$

which implies

$$\widetilde{\alpha} = -\int_{t_0}^{t_1} \Phi(t_0, s)\, C(s, u(s), \Psi^*(s, t_0))\, ds,$$

and consequently $|\widetilde{\alpha}|^2 = \widetilde{\alpha}\, V \widetilde{\alpha} = 0$, which is a contradiction. Since $\widetilde{\alpha} \neq 0$ implies $|\widetilde{\alpha}|^2 \neq 0$. Hence the claim.

3. Integro-differential system. In this section we develop the method of variation of parameters formula for integro-differential equations of the Volterra type in terms of resolvent kernels and then offer sufficient conditions for the controllability of the Sylvester matrix integro-differential system [2]. More specifically, we consider the matrix linear integro-differential equation of the form

$$T'(t) = A(t)\, T(t) + \int_{t_0}^{t} K_1(t, s)\, T(s)\, ds + F(t), \; T(t_0) = T_0, \tag{8}$$

where $A(t)$, $K_1(t, s)$ are $(n \times n)$ continuous matrices for $t \in R_+$ and $(t, s) \in R_+^2$ and $F \in C\left\lfloor R_+, R^{n \times n} \right\rfloor$ and T is a square matrix of the order n. We get for $t_0 \leq s \leq t < \infty$

$$\Phi(t, s) = A(t) + \int_{s}^{t} K_1(t, \tau)\, d\tau,$$

$$R_1(t, s) = I + \int_{s}^{t} R_1(t, \sigma)\, \Phi(\sigma, s)\, d\sigma, \tag{9}$$

where I is the identity matrix of the order n and $K_1(t, s) = \Phi(t, s) = R_1(t, s) = 0$ for $t_0 \leq t < s$. We shall state the following result relative to the linear integro-differential equation (8).

Theorem 2. Assume that $A(t)$, $K_1(t, s)$ are $(n \times n)$ continuous matrices for $t \in R_+$, $(t, s) \in R_+^2$ and $F \in C\left\lfloor R_+, R^{n \times n} \right\rfloor$. Then the solution of (8) is given by

$$T(t) = R_1(t, t_0)\, T_0 + \int_{t_0}^{t} R_1(t, s) F(s)\, ds, \; T(t_0) = T_0,$$

where $R(t,s)$ is the unique solution of the partial differential equation

$$\frac{\partial R_1}{\partial s}(t,s)+R_1(t,s)A(s)+\int_s^t R_1(t,\sigma)K_1(\sigma,s)\,d\sigma=0 \tag{10}$$

with $R_1(t,t)=I$.

P r o o f. Since Φ is continuous, it follows that R_1 in (9) exists and subsequently $\frac{\partial R_1(t,s)}{\partial s}$ exists and satisfies (10). Let $T(t)$ be a solution of (8) for $t \ge t_0$. Then, if we set $p(s)=R_1(t,s)T(s)$, we have

$$p'(s)=\frac{\partial R_1(t,s)}{\partial s}T(s)+R_1(t,s)\left[A(s)T(s)+\int_{t_0}^{s}K_1(s,u)T(u)\,du+F(s)\right].$$

Integrating in between the limits t_0 to t yields,

$$p(t)-p(t_0)=\int_{t_0}^{t}\left[\frac{\partial R_1(t,s)}{\partial s}T(s)+R_1(t,s)A(s)T(s)+R_1(t,s)F(s)\right]ds+$$

$$+\int_{t_0}^{t}R_1(t,s)\left[\int_{t_0}^{s}K_1(s,u)T(u)\,du\right]ds.$$

Applying Fubini's theorem [4], we get

$$p(t)-R_1(t,t_0)T_0=\int_{t_0}^{t}\left[\frac{\partial R_1(t,s)}{\partial s}+R_1(t,s)A(s)+\right.$$

$$\left.+\int_s^t R_1(t,u)K_1(s,u)\,du\right]T(s)\,ds+\int_{t_0}^{t}R_1(t,s)F(s)\,ds.$$

Using (10), we get

$$p(t)=R_1(t,t_0)T_0+\int_{t_0}^{t}R_1(t,s)F(s)\,ds.$$

Since $p(t)=R_1(t,t)T(t)$ and $R_1(t,t)=I$, we have

$$T(t)=R_1(t,t_0)T_0+\int_{t_0}^{t}R_1(t,s)F(s)\,ds.$$

Now, to prove that T is a solution of (8), let $T(t)$ be the solution of (10) satisfying $T(t_0)=T_0$ existing for $t_0 \le t<\infty$. Then

$$\int_{t_0}^{t} R_1(t,s)\,T'(s)\,ds = R_1(t,t)\,T(t) - R_1(t,t_0)\,T_0 - \int_{t_0}^{t} \frac{\partial R_1(t,s)}{\partial s}\,T(s)\,ds =$$

$$= \int_{t_0}^{t} R_1(t,s)\,F(s)\,ds - \int_{t_0}^{t} \frac{\partial R_1(t,s)}{\partial s}\,T(s)\,ds\,.$$

Using (10) and Fubini's theorem, we get

$$\int_{t_0}^{t} R_1(t,s)\left[T'(s) - A(s)\,T(s) - \int_{t_0}^{s} K_1(s,u)\,T(u)\,du - F(s)\right] ds = 0\,.$$

Since $R_1(t, s)$ is a non-zero continuous matrix for $t_0 \le s \le t < \infty$, we have

$$T'(s) - A(s)\,T(s) - \int_{t_0}^{s} K_1(s,u)\,T(u)\,du - F(s) = 0\,.$$

Therefore, T is a solution of

$$T'(t) = A(t)\,T(t) + \int_{t_0}^{s} K_1(s,u)\,F(u)\,du\,.$$

Thus the proof of the theorem is complete.

Theorem 3. The matrix integro-differential system

$$T'(t) = A(t)\,T(t) + \int_{t_0}^{t} K_1(t,s)\,T(s)\,ds + C(t)\,U(t),\ \ t \in J, \tag{11}$$

$$T(t_0) = T_0$$

is completely controllable, if, and only if, the controllability matrix

$$\Phi(t_0, t_1) = \int_{t_0}^{t_1} R_1(t_0, \tau)\,C^*(s)\,C(s)\,R_1^*(t_0, s)\,ds$$

is non-singular, where $R_1(t, s)$ is the resolvent matrix. The control function

$$U(t) = -C(t)\,R_1^*(t_0, t)\,\Phi^{-1}(t_0, t_1)[T_0 - R(t_0, t_1)\,T_1]$$

defined for $t_0 < t < t_1$ transfers $T(t_0) = T_0$ to $T(t_1) = T_1$.

P r o o f. Any solution $T(t)$ of (11) is given by

$$T(t) = R(t, t_0) T_0 + \int_{t_0}^{t} R_1 (t, \tau) C^* (\tau) U(\tau) d\tau$$

and hence

$$T(t_1) = R(t_1, t_0) \left[T_0 + \int_{t_0}^{t_1} R_1 (t_0, \tau) C^* (\tau) U(\tau) d\tau \right] = R_1 (t_1, t_0) \times$$

$$\times \left[T_0 + \int_{t_0}^{t_1} R(t_0, \tau) C^* (\tau)(-C(\tau) R^* (t_0, \tau) \Phi^{-1}(t_0, t_1))(T_0 - R_1(t_0, t_1) T_1) d\tau \right] =$$

$$= R_1 (t_1, t_0)[T_0 - \Phi(t_0, t_1) \Phi^{-1}(t_0, t_1)(T_0 - R(t_0, t_1) T_1)] = T_1.$$

Converse is similar to the proof of Theorem 1.

Theorem 4. Assume that $B(t)$ and $K_2(t,s)$ are continuous $(n \times n)$ matrices for $t \in R_+, (t, s) \in R_+^2$ and $F \in C \lfloor R_+, R^{n \times n} \rfloor$. Then the solution of

$$T'(t) = T(t) B(t) + \int_{t_0}^{t} T(s) K_2(t, s) ds + F(t), \tag{12}$$

$$T(t_0) = T_0$$

is given by

$$T(t) = T_0 R_2^* (t, t_0) + \int_{t_0}^{t} F(\tau) R_2^* (t, \tau) d\tau,$$

where $R_2(t, s)$ is the resolvent kernel and is the unique solution of

$$\frac{\partial}{\partial s} R_2^* (t, s) + B(s) R_2^* (t, s) + \int_{s}^{t} K_2(\sigma, s) R_2^* (t, \sigma) d\sigma = 0 \tag{13}$$

with $R_2(t, t) = I$.

P r o o f. The proof is similar to the proof of Theorem 2.

Theorem 5. The matrix integro-differential system (12) is completely controllable, if, and only if, the controllability matrix

$$\psi(t_0, t_1) = \int_{t_0}^{t_1} R_2(t_0, \tau) D^* (\tau) D(\tau) R_2^* (t_0, \tau) d\tau$$

is non-singular, where $R_2(t, s)$ is the resolvent matrix. The control function $U(t)$ given by

$$U(t) = -D(T) R_2^*(t_0, t) \psi^{-1}(t_0, t) \lfloor T_0^* - R_2(t_0, t_1) T_1^* \rfloor$$

defined for $t_0 < t < t_1$ transfers $T(t_0) = T_0$ to $T(t_1) = T_1$.

P r o o f. The proof is similar to Theorem 1.

We shall now consider the superposition of these two systems and present a set of sufficient conditions for the complete controllability of the Sylvester integro-differential system

$$T'(t) = A(t) T(t) + T(t) B(t) + \int_{t_0}^{t} [K_1(t,s) T(s) +$$

$$+ T(s) K_2(t,s)] ds + C(t) U(t) D^*(t), \tag{14}$$

$$T(t_0) = T_0 .$$

Theorem 6. The matrix Sylvester integro-differential system (14) satisfying $T(t_0) = T_0$ has a unique solution given by

$$T(t) = R_1(t, t_0) T_0 R_2^*(t, t_0) + \int_{t_0}^{t} R_1(t, \tau) C(\tau) U(\tau) D^*(t, \tau) R_2^*(t) d\tau,$$

where R_1 and R_2 are the solutions of the partial differential equations (10) and (13), respectively.

P r o o f. The proof is similar to Theorem 2.

Theorem 7. The matrix integro-differential system (13) is completely controllable, if, and only if, the

$$\Phi(t_0, t_1) = \int_{t_0}^{t_1} R_1(t_0, s) C^*(s) C(s) R_1^*(t_0, s) ds$$

and

$$\psi(t_0, t_1) = \int_{t_0}^{t_1} R_2^*(t_0, s) D(s) D^*(s) R_2(t_0, s) ds$$

are non-singular, where $R_1(t, s)$ and $R_2(t, s)$ are resolvent matrices. The control function $U(t)$ given by

$$U(t) = -C(t) R_1^*(t_0, t) \Phi^{-1}(t_0, t_1) \lfloor T_0 - R_1(t_0, t_1) T_1 R_2^*(t_1, t_0) \rfloor \times$$

$$\times \psi^{-1}(t_0, t_1) R_2^*(t_0, t_1) D(t)$$

is defined for $t_0 < t < t_1$ transfers $T(t_0) = T_0$ to $T(t_1) = T_1$.

P r o o f. Any solution T of the matrix Sylvester integro-differential system is given by

$$T(t)=R_1(t,t_0)\,T_0R_2^*(t,t_0)+\int_{t_0}^{t}R_1(t,\tau)\,C(\tau)\,U(\tau)\,D^*(\tau)\,R_2^*(t,\tau)\,d\tau. \quad (15)$$

Substituting the general form of the control $U(t)$ in (15), we get

$$T(t_1)=R_1(t_1,t_0)\,T_0R_2^*(t_1,t_0)+$$

$$+R_1(t_1,t_0)\int_{t_0}^{t_1}R_1(t_0,\tau)\,C(\tau)\lfloor -C(\tau)\,R_1^*(t_0,\tau)\,\Phi^{-1}(t_0,t_1)[T_0-$$

$$-R_1(t_0,t_1)\,T_1R_2(t_1,t_0)]\psi^{-1}(t_0,t_1)\,R_2^*(t,t_1)\,D(\tau)\rfloor D^*(\tau)\,R_2^*(t_0,\tau)\,d\tau=$$

$$=R_1(t_1,t_0)T_0R_2^*(t_1,t_0)-R_1(t_1,t_0)\,\Phi(t_0,t_1)\,\Phi^{-1}(t_0,t_1)[T_0-$$

$$-R_1(t_0,t_1)\,T_1R_2(t_1,t_0)]\,\Psi(t_0,t_1)\,\Psi^{-1}(t_0,t_1)\,R_2^*(t_0,t_1)=$$

$$=R_1(t_1,t_0)\,T_0R_2^*(t_1,t_0)-R_1(t_1,t_0)\,T_0R_2^*(t_0,t_1)+$$

$$+R_1(t_1,t_0)\,R(t_0,t_1)\,T_1R_2^*(t_1,t_0)\,R_2^*(t_0,t_1)=R_1(t_1,t_1)\,T_1R_2^*(t_1,t_1)=T_1.$$

Conversely, suppose the system (14) is completely controllable. Then it can easily be proved as in Theorem 1, that $\Phi(t_0,t_1)$ and $\psi(t_0,t_1)$ are positive definite matrices.

4. Nonlinear control system. Observe that the system (4) is nonlinear. We aim at finding the existence of controllability conditions so that it be completely controllable. For this purpose, let $\widetilde{T}(t)$ and $\widetilde{U}(t)$ be two given $n\times n$ matrices continuous in $t\in J$. The associated linear matrix integro-differential equation with (4) is given by

$$T'(t)=A(t,\widetilde{T}(t),\widetilde{U}(t))\,T(t)+T(t)\,B(t,\widetilde{T}(t),\widetilde{U}(t))+$$

$$+\int_{t_0}^{t}K_1(t,s,\widetilde{T}(s),\widetilde{U}(s))\,T(s)+T(s)\,K_2(t,s,\widetilde{T}(s),\widetilde{U}(s))\,ds+$$

$$+C(t,\widetilde{T}(t),\widetilde{U}(t))\,U(t)\,D^*(t,\widetilde{T}(t),\widetilde{U}(t)), \quad (16)$$

$$T(t_0)=T_0,$$

where T_0 is a given $(n\times n)$ matrix. Note that (16) is similar to the system (13). Since (16) is linear, Theorems 2 and 4 are applicable. Hence the resolvent matrices $R_1(t,s,\widetilde{T}(s),\widetilde{U}(s))$, $R_2(t,s,\widetilde{T}(s),\widetilde{U}(s))$ exist for $t \in J$ and yield the solution of the initial value problem (16) and is given by

$$T(t) = R(t,t_0,\widetilde{T}(t),\widetilde{U}(t))\,T_0 R_2^*(t,t_0,\widetilde{T}(t),\widetilde{U}(t)) + \int_{t_0}^{t} R_1(t,s,\widetilde{T}(s),\widetilde{U}(s)) \times$$

$$\times C^*(s,\widetilde{T}(s),\widetilde{U}(s))\,U(s)\,D^*(t,s,\widetilde{T}(s),\widetilde{U}(s)\,R_2^*(t,s,\widetilde{T}(s),\widetilde{U}(s)))\,ds. \quad (17)$$

The representation of the solution given in (17) is a consequence of the superposition of conclusions obtained in Theorems 2, 4 and 7. Hence we have the following theorem.

Theorem 8. The matrix integro-differential system (17) is completely controllable, if, and only if, the control matrices

$$\Phi(t_0,t_1,\widetilde{T}(t_1),\widetilde{U}(t_1)) = \int_{t_0}^{t_1} R_1(t_0,s,\widetilde{T}(s),\widetilde{U}(s)) \times$$

$$\times C^*(s,\widetilde{T}(s),\widetilde{U}(s))\,C(s,\widetilde{T}(s),\widetilde{U}(s))\,R_1^*(t_0,s,\widetilde{T}(s),\widetilde{U}(s))\,ds \quad (18)$$

and

$$\psi(t_0,t_1,\widetilde{T}(t_1),\widetilde{U}(t_1)) = \int_{t_0}^{t_1} R_2^*(t_0,s,\widetilde{T}(s),\widetilde{U}(s)) \times$$

$$\times D^*(s,\widetilde{T}(s),\widetilde{U}(s))\,D^*(s,\widetilde{T}(s),\widetilde{U}(s))\,R_2(t_0,s,\widetilde{T}(s),\widetilde{U}(s))\,ds \quad (19)$$

are non-singular. The control function $U(t)$ is given by

$$U(t) = -C(t,\widetilde{T}(t),\widetilde{U}(t))\,R_1^*(t_0,t,\widetilde{T}(t),\widetilde{U}(t))\,\Phi^{-1}(t_0,t,\widetilde{T}(t_1),\widetilde{U}(t_1)) \times$$

$$\times [T_0 - R_1(t_0,t_1,\widetilde{T}(t_1),\widetilde{U}(t_1)), T_1 R_2^*(t_1,t_0,\widetilde{T}(t_1),\widetilde{U}(t_1))] \times$$

$$\times \psi^{-1}(t_0,t_1,\widetilde{T}(t_1),\widetilde{U}(t_1))\,R_2^*(t,t_1,\widetilde{T}(t_1),\widetilde{U}(t_1))\,D(t,\widetilde{T}(t_1),\widetilde{U}(t)). \quad (20)$$

Here the matrices Φ and ψ as given in (18), (19) are assumed to be non-singular.

P r o o f. Substitute (20) in (17) and prove that $T(t_1) = T_1$. Conversely, assume that the Sylvester system (16) is completely controllable. Then by following the proof of Theorem 1, we show that the control matrices Φ and ψ are positively definite. The details are omitted.

We now turn our attention to the nonlinear system given in (4). Taking the relations (17) and (20) in consideration, we let

$$F: C^{n\times n}(J) \times C^{n\times n}(J) \to C^{n\times n}(J) \times C^{n\times n}(J)$$

such that

$$F(\widetilde{T}(t),\widetilde{U}(t))=(T(t),U(t)).$$

The operator F is continuous on $C^{n\times n}(J)\times C^{n\times n}(J)$. Let there exist a closed bounded, convex set Ω in $C^{n\times n}(J)\times C^{n\times n}(J)$ such that $(T, U)=F(\widetilde{T},\widetilde{U})$ for any $\widetilde{T},\widetilde{U}$ in Ω, then from the relations (17), (20), $F(\Omega)$ is bounded. We apply Schauder's fixed point theorem and conclude that there exists at least one fixed point of F. Hence, we have the Theorem 8.

We consider a more complicated system and using Kronecker product of matrices, we obtain sufficient conditions for the system controllability.

$$T'(t)=A(t)T(t)B(t)+\int_{t_0}^{t}K_1(t,s)T(s)K_2(t,s)\,ds+F(t)$$

is equivalent to

$$V(T'(t))=(A\otimes B^*)V(T(t))+$$

$$+\int_{t_0}^{t}[K_1(t,s)\otimes K_2^*(t,s)]V(T(s))\,ds+(F(t)\otimes I_n), \tag{21}$$

where $V(T'(t))$ is an $n^2\times 1$ vector, $(A\otimes B^*)$ is an $(n^2\times n^2)$ matrix and $K_1(t,s)\otimes$ $\otimes K_2^*(t,s)$ are $(n^2\times n^2)$ continuous matrices and $\otimes$ denotes the Kronecker product. The following theorem is a simple consequence of Theorem 1 in [5].

Theorem 9. Assume that $(A\otimes B^*)$ and $K_1\otimes K_2^*$ are continuous $(n^2\times n^2)$ matrices for $t\in R_+$ and $(t,s)\in R_+^2$ and $F(t)\otimes I_n$ is an $(n^2\times n^2)$ continuous matrix. Then the solution of (21) is given by

$$V(T(t))=R_1(t,t_0)T_0+\int_{t_0}^{t}R_1(t,s)(F(s)\otimes I_n)\,ds,\ T(t_0)=T_0,$$

where $R_1(t,s)$ is the unique solution of

$$\frac{\partial}{\partial s}R_1(t,s)+R_1(t,s)(A(s)\otimes B^*(s))+$$

$$+\int_{s}^{t}R_1(t,\sigma)(K_1(\sigma,s)\otimes K_2^*(\sigma,s))(F(s)\otimes I_n)\,d\sigma$$

R_1 being $(n^2\times n^2)$ matrix and the symbol $\otimes$ means Kronecker product.

The other results like Theorem 3, 4 and 5 follow. We can consider more general systems like

$$T'(t) = F(A(t))\,T(t)\,g(B(t)) + \int_{t_0}^{t} f(K_1(t,s))\,T(s)\,g(K_2(t,s))\,ds + F(t),$$

which can be converted into a vector system by using Kroneker product of matrices as

$$V(T'(t)) = [f(A(t)) \otimes g^{*}(B(t))]V(T(t)) +$$

$$+ \int_{t_0}^{t} [f(K_1(t,s)) \otimes g^{*}(K_2(t,s))]V(T(s))\,ds + (F(t) \otimes I_n).$$

The results on controllability and observability criteria can be discussed for the above system as in linear systems [5]. In order to avoid monotony, we even omit starting those results.

Досліджено диференціальні та інтегро-диференціальні матричні системи Сильвестра першого порядку. Наведено набір достатніх умов керованості та повної керованості систем. Як необхідний інструмент для нелінійної системи Сильвестра отримано різновид параметричної формули.

REFERENCES

1. *Murty K.N., Howell G., Sivasundaram S.* Two multi-point nonlinear Lyapunov systems. Existence and uniqueness // J. Math. Anal. Appl. — 1992. —**167**. — P. 505—512.
2. *Lakshmikantham V., Deo S.G.* Method of Variation of Parameters for Dynamic Systems.— Vol. 1. — Gordon and Breach Science Publishers, 1998.
3. *Yamamoto Y.* Controllability of nonlinear systems// J. of Optim. Theory and Appl. — 1977. — **22**. — P. 41—49.
4. *Miller R.K.* On the linearization of Volterra integral equations// J. Math. Anal. Appl. — 1968. — **23**. — P. 198—206.
5. *Barnett S.* Matrix differential equations and Kronecker product//SIAM Appl. Math. — 1973. — **14**. — P. 1—5.

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